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Smooth Duals of Inner Forms of GL_n and SL_n

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Abstract. Let F be a non-archimedean local field. We prove that every Bernstein component in the smooth dual of each inner form of the general linear group $\mathrm{GL}_n(F)$ is canonically in bijection with the extended quotient for the action, given by Bernstein, of a finite group on a complex torus. For inner forms of $\mathrm{SL}_n(F)$ we prove that each Bernstein component is canonically in bijection with the associated twisted extended quotient. In both cases, the bijections satisfy naturality properties with respect to the tempered dual, parabolic induction, central character, and the local Langlands correspondence.

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Introduction

Let X be a complex affine variety. Denote the coordinate algebra of X by $\mathcal{O}(X)$. The Hilbert Nullstellensatz asserts that $X \mapsto \mathcal{O}(X)$ is an equivalence of categories

$$\left(\begin{array}{l}\text{unital commutative finitely}\\ \text{generated nilpotent-free}\\ \mathbb{C}\text{-algebras}\end{array}\right) \sim \left(\begin{array}{l}\text{affine complex}\\ \text{algebraic varieties}\end{array}\right)^{\mathrm{op}}$$
$$\mathcal{O}(X) \mapsto X$$

where op denotes the opposite category.
A finite type algebra is a $\mathbb{C}$-algebra A with a given structure as an $\mathcal{O}(X)$-module such that A is finitely generated as an $\mathcal{O}(X)$-module. A compatibility is required between the algebra structure of A and the given action of $\mathcal{O}(X)$ on A. However, A is not required to be unital. Due to the above equivalence of categories, any finite type algebra can be viewed as a slightly non-commutative affine algebraic variety. This will be the point of view of the paper.
Following this point of view, each Bernstein component in the smooth dual of any reductive p-adic group G is a non-commutative affine algebraic variety. In a series of papers we have examined the question "What is the geometric structure of any given Bernstein component?" Our proposed answer to this (which has been verified for all the classical split reductive p-adic groups) is based on the notion of *extended quotient*. In the present paper we show that for inner forms of SL_n it is necessary to use a *twisted extended quotient* (see the Appendix). The twisting is given by a family of 2-cocycles.
Let F be a non-archimedean local field. Let D be a central simple F-algebra with $\dim_F(D) = d^2$. Then $\mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_{md}(F)$ and the derived group $\mathrm{GL}_m(D)_{\mathrm{der}}$ is an inner form of $\mathrm{SL}_{md}(F)$. The main result of this paper is

Theorem 1. *Let G be an inner form of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$. Let $\mathbf{Irr}^{\mathfrak{s}}(G)$ be any Bernstein component in the smooth dual of G. Let $T_{\mathfrak{s}}//W_{\mathfrak{s}}$ and $(T_{\mathfrak{s}}//W_{\mathfrak{s}})_\natural$ be the appropriate extended quotient and twisted extended quotient. Then:*

(1) *If G is an inner form of $\mathrm{GL}_n(F)$ there exists a bijection*
$$\mathbf{Irr}^{\mathfrak{s}}(G) \longleftrightarrow T_{\mathfrak{s}}//W_{\mathfrak{s}}.$$

(2) *If G is an inner form of $\mathrm{SL}_n(F)$ there exists a family of 2-cocycles $\natural$ and a bijection*
$$\mathbf{Irr}^{\mathfrak{s}}(G) \longleftrightarrow (T_{\mathfrak{s}}//W_{\mathfrak{s}})_\natural.$$

(3) *In either case, the bijection satisfies naturality properties with respect to the tempered dual, parabolic induction, and central character.*

We remark that
• $T_{\mathfrak{s}}//W_{\mathfrak{s}}$ is the non-commutative affine variety whose coordinate algebra is the crossed product algebra $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ where $T_{\mathfrak{s}}, W_{\mathfrak{s}}$ are respectively the complex torus and finite group acting on the torus which Bernstein assigns to $\mathbf{Irr}^{\mathfrak{s}}(G)$. The crossed product algebra $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ is a finite $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$-algebra.
• $(T_{\mathfrak{s}}//W_{\mathfrak{s}})_\natural$ is the non-commutative affine variety whose coordinate algebra is the twisted crossed product algebra $\mathcal{O}(T_{\mathfrak{s}}) \rtimes_\natural W_{\mathfrak{s}}$. The twisted crossed product algebra $\mathcal{O}(T_{\mathfrak{s}}) \rtimes_\natural W_{\mathfrak{s}}$ is a finite $\mathcal{O}(T_{\mathfrak{s}}/W_{\mathfrak{s}})$-algebra. Example 7.7 shows that for inner forms of SL_5 there are Bernstein components where the twisting is non-trivial.
We observe that it is somewhat remarkable that many of the subtleties of the representation theory of $\mathrm{GL}_m(D)$, $\mathrm{SL}_m(D)$ are captured by the algebras

$$\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \qquad \mathcal{O}(T_{\mathfrak{s}}) \rtimes_\natural W_{\mathfrak{s}}$$

and their geometric realizations $T_{\mathfrak{s}}//W_{\mathfrak{s}}, (T_{\mathfrak{s}}//W_{\mathfrak{s}})_\natural$.

The proof of Theorem 1 uses a weakening of Morita equivalence called *stratified equivalence*, see [ABPS7]. The proof is achieved by combining the theory of types with an analysis of the structure of Hecke algebras.
Outline of the proof.
The proof of Theorem 1 for $\mathrm{GL}_m(D)$ consists of two steps.

- Step 1: For each point $\mathfrak{s}$ in the Bernstein spectrum $\mathfrak{B}(\mathrm{GL}_m(D))$, the ideal $\mathcal{H}(G)^{\mathfrak{s}}$ in the Hecke algebra $\mathcal{H}(\mathrm{GL}_m(D))$ is Morita equivalent to an affine Hecke algebra, see [ABPS4].

- Step 2: Using the Lusztig asymptotic algebra, a stratified equivalence is constructed between this affine Hecke algebra and the associated crossed product algebra. The irreducible representations of the crossed product algebra (in a canonical way) are the required extended quotient.

The proof of Theorem 1 for $\mathrm{SL}_m(D)$ is achieved by introducing an intermediate group $\mathrm{SL}_m(D) \cdot Z(\mathrm{GL}_m(D))$:

$$\mathrm{SL}_m(D) \subset \mathrm{SL}_m(D) \cdot Z(\mathrm{GL}_m(D)) \subset \mathrm{GL}_m(D),$$

where $Z(\mathrm{GL}_m(D))$ is the center of $\mathrm{GL}_m(D)$. It consists in two analogous steps.

- Step 1: For each point $\mathfrak{s}$ in the Bernstein spectrum $\mathfrak{B}(\mathrm{SL}_m(D))$, the ideal $\mathcal{H}(\mathrm{SL}_m(D))^{\mathfrak{s}}$ in the Hecke algebra $\mathcal{H}(\mathrm{SL}_m(D))$ is Morita equivalent to a twisted affine Hecke algebra.

- Step 2: Using the Lusztig asymptotic algebra, a stratified equivalence is constructed between this twisted affine Hecke algebra and the associated twisted crossed product algebra. The irreducible representations of the twisted crossed product algebra (in a canonical way) are the required twisted extended quotient.

In Section 7, we prove that the bijections of Theorem 1 are compatible with the local Langlands correspondence, in the sense below. Let G be an inner form of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$, and let $\check{G}$ denote $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{PGL}_n(\mathbb{C})$, respectively. The set of $\check{G}$-conjugacy classes of Langlands parameters (resp. enhanced Langlands parameters) for G is denoted by $\Phi(G)$ (resp. $\Phi_e(G)$). We may identify $\Phi_e(G)$ with $\Phi(G)$ when G is an inner form of $\mathrm{GL}_n(F)$.
Let $\mathcal{L}$ be a set of representatives for the conjugacy classes of Levi subgroups of G. For L in $\mathcal{L}$, we denote by $\mathbf{Irr}_{\mathrm{cusp}}(L)$ the set of isomorphism classes of supercuspidal irreducible representations of L and we let $\Phi(L)_{\mathrm{cusp}}$ be its image in $\Phi_e(L)$. The group $W(G,L) = \mathrm{N}_G(L)/L$, quotient by L of the normalizer of L in G, acts naturally on both sets. Thus we may consider the extended quotients $\mathbf{Irr}_{\mathrm{cusp}}(L)//W(G,L)$ and $\Phi(L)_{\mathrm{cusp}}//W(G,L)$, as well as their twisted versions. It follows directly from the definitions that

$$(\mathbf{Irr}_{\mathrm{cusp}}(L)//W(G,L))_\natural = \bigsqcup_{\mathfrak{s}} (T_{\mathfrak{s}}//W_{\mathfrak{s}})_\natural,$$

where the disjoint union runs over all $\mathfrak{s} \in \mathfrak{B}(G)$ coming from supercuspidal L-representations.

Theorem 2. *The following statements hold:*

- *If $G = \mathrm{GL}_m(D)$ there exists a canonical, bijective, commutative diagram*

$$\begin{tikzcd}
\mathbf{Irr}(G) \arrow[r, leftrightarrow] & \Phi(G) \\
\bigsqcup_{L\in\mathcal{L}} \mathbf{Irr}_{\mathrm{cusp}}(L)//W(G,L) \arrow[r, leftrightarrow] \arrow[u, leftrightarrow] & \bigsqcup_{L\in\mathcal{L}} \Phi(L)_{\mathrm{cusp}}//W(G,L). \arrow[u, leftrightarrow]
\end{tikzcd}$$

- *If $G = \mathrm{SL}_m(D)$ there exists a family of 2-cocycles $\natural$ and a (canonical up to permutations within L-packets) bijective commutative diagram*

$$\begin{tikzcd}
\mathbf{Irr}(G) \arrow[r, leftrightarrow] & \Phi_e(G) \\
\bigsqcup_{L\in\mathcal{L}} (\mathbf{Irr}_{\mathrm{cusp}}(L)//W(G,L))_\natural \arrow[r, leftrightarrow] \arrow[u, leftrightarrow] & \bigsqcup_{L\in\mathcal{L}} (\Phi(L)_{\mathrm{cusp}}//W(G,L))_\natural. \arrow[u, leftrightarrow]
\end{tikzcd}$$

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1 Preliminaries

We start with some generalities, to fix the notations.
Let G be a connected reductive group over a local non-archimedean field F of residual characteristic p. All our representations are assumed to be smooth and over the complex numbers. We write $\mathrm{Rep}(G)$ for the category of such G-representations and $\mathbf{Irr}(G)$ for the collection of isomorphism classes of irreducible representations therein. Let P be a parabolic subgroup of G with Levi factor L. The Weyl group of L is $W(G,L) = N_G(L)/L$. It acts on equivalence classes of L-representations π by

$$(w \cdot \pi)(g) = \pi(\bar{w} g \bar{w}^{-1}),$$

where $\bar{w} \in N_G(L)$ is a chosen representative for $w \in W(G,L)$. We write

$$W_\pi = \{w \in W(G,L) \mid w \cdot \pi \cong \pi\}.$$

Let ω be an irreducible supercuspidal L-representation. The inertial equivalence class $\mathfrak{s} = [L,\omega]_G$ gives rise to a category of smooth G-representations $\mathrm{Rep}^{\mathfrak{s}}(G)$ and a subset $\mathbf{Irr}^{\mathfrak{s}}(G) \subset \mathbf{Irr}(G)$. Write $X_{\mathrm{nr}}(L)$ for the group of unramified characters $L \to \mathbb{C}^\times$. Then $\mathbf{Irr}^{\mathfrak{s}}(G)$ consists of all irreducible constituents of the parabolically induced representations $I_P^G(\omega \otimes \chi)$ with $\chi \in X_{\mathrm{nr}}(L)$. We note that I_P^G always means normalized, smooth parabolic induction from L via P to G.
The set $\mathbf{Irr}^{\mathfrak{s}_L}(L)$ with $\mathfrak{s}_L = [L,\omega]_L$ can be described explicitly, namely by

$$X_{\mathrm{nr}}(L,\omega) = \{\chi \in X_{\mathrm{nr}}(L) : \omega \otimes \chi \cong \omega\}, \tag{1}$$
$$\mathbf{Irr}^{\mathfrak{s}_L}(L) = \{\omega \otimes \chi : \chi \in X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)\}. \tag{2}$$

Several objects are attached to the Bernstein component $\mathbf{Irr}^{\mathfrak{s}}(G)$ of $\mathbf{Irr}(G)$ [BeDe]. Firstly, there is the torus

$$T_{\mathfrak{s}} := X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega),$$

which is homeomorphic to $\mathbf{Irr}^{\mathfrak{s}_L}(L)$. Secondly, we have the groups

$$\begin{aligned} N_G(\mathfrak{s}_L) =& \{g \in N_G(L) \mid g \cdot \omega \in \mathbf{Irr}^{\mathfrak{s}_L}(L)\} \\ =& \{g \in N_G(L) \mid g \cdot [L,\omega]_L = [L,\omega]_L\}, \\ W_{\mathfrak{s}} :=& \{w \in W(G,L) \mid w \cdot \omega \in \mathbf{Irr}^{\mathfrak{s}_L}(L)\} = N_G(\mathfrak{s}_L)/L. \end{aligned}$$

Of course $T_{\mathfrak{s}}$ and $W_{\mathfrak{s}}$ are only determined up to isomorphism by $\mathfrak{s}$, actually they depend on $\mathfrak{s}_L$. To cope with this, we tacitly assume that $\mathfrak{s}_L$ is known when considering $\mathfrak{s}$.
The choice of $\omega \in \mathbf{Irr}^{\mathfrak{s}_L}(L)$ fixes a bijection $T_{\mathfrak{s}} \to \mathbf{Irr}^{\mathfrak{s}_L}(L)$, and via this bijection the action of $W_{\mathfrak{s}}$ on $\mathbf{Irr}^{\mathfrak{s}_L}(L)$ is transferred to $T_{\mathfrak{s}}$. The finite group $W_{\mathfrak{s}}$ can be thought of as the "Weyl group" of $\mathfrak{s}$, although in general it is not generated by reflections.

Let $C_c^\infty(G)$ be the vector space of compactly supported locally constant functions $G \to \mathbb{C}$. The choice of a Haar measure on G determines a convolution product $*$ on $C_c^\infty(G)$. The algebra $(C_c^\infty(G), *)$ is known as the Hecke algebra $\mathcal{H}(G)$. There is an equivalence between $\mathrm{Rep}(G)$ and the category $\mathrm{Mod}(\mathcal{H}(G))$ of $\mathcal{H}(G)$-modules V such that $\mathcal{H}(G) \cdot V = V$. We denote the collection of inertial equivalence classes for G by $\mathfrak{B}(G)$, the Bernstein spectrum of G. The Bernstein decomposition

$$\mathrm{Rep}(G) = \prod\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}^{\mathfrak{s}}(G)$$

induces a factorization in two-sided ideals

$$\mathcal{H}(G) = \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

From now on we discuss things that are specific for $G = \mathrm{GL}_m(D)$, where D is a central simple F-algebra. We write $\dim_F(D) = d^2$. Every Levi subgroup L of G is conjugate to $\prod_j \mathrm{GL}_{\tilde{m}_j}(D)$ for some $\tilde{m}_j \in \mathbb{N}$ with $\sum_j \tilde{m}_j = m$. Hence every irreducible L-representation ω can be written as $\otimes_j \tilde{\omega}_j$ with $\tilde{\omega}_j \in \mathbf{Irr}(\mathrm{GL}_{\tilde{m}_j}(D))$. Then ω is supercuspidal if and only if every $\tilde{\omega}_j$ is so. As above, we assume that this is the case. Replacing (L, ω) by an inertially equivalent pair allows us to make the following simplifying assumptions:

Condition 1.1.

- *if $\tilde{m}_i = \tilde{m}_j$ and $[\mathrm{GL}_{\tilde{m}_j}(D), \tilde{\omega}_i]_{\mathrm{GL}_{\tilde{m}_j}(D)} = [\mathrm{GL}_{\tilde{m}_j}(D), \tilde{\omega}_j]_{\mathrm{GL}_{\tilde{m}_j}(D)}$, then $\tilde{\omega}_i = \tilde{\omega}_j$;*
- *$\omega = \bigotimes_i \omega_i^{\otimes e_i}$, such that ω_i and ω_j are not inertially equivalent if $i \neq j$;*
- *$L = \prod_i L_i^{e_i} = \prod_i \mathrm{GL}_{m_i}(D)^{e_i}$, embedded diagonally in $\mathrm{GL}_m(D)$ such that factors L_i with the same (m_i, e_i) are in subsequent positions;*
- *as representatives for the elements of $W(G, L)$ we take permutation matrices;*
- *P is the parabolic subgroup of G generated by L and the upper triangular matrices;*
- *if $m_i = m_j, e_i = e_j$ and ω_i is isomorphic to $\omega_j \otimes \gamma$ for some character γ of $\mathrm{GL}_{m_i}(D)$, then $\omega_i = \omega_j \otimes \gamma\chi$ for some $\chi \in X_{\mathrm{nr}}(\mathrm{GL}_{m_i}(D))$.*

Most of the time we will not need the conditions for stating the results, but they are useful in many proofs. Under Conditions 1.1 we consider

$$M_i = \mathrm{GL}_{m_i e_i}(D) \text{ naturally embedded in } Z_G\big(\prod\nolimits_{j \neq i} L_j^{e_j}\big). \tag{3}$$

Then $\prod_i M_i$ is a Levi subgroup of G containing L. For $\mathfrak{s} = [L, \omega]_G$ we have

$$W_{\mathfrak{s}} = N_{\prod_i M_i}(L)/L = \prod\nolimits_i N_{M_i}(L_i^{e_i})/L_i^{e_i} \cong \prod\nolimits_i S_{e_i}, \tag{4}$$

a direct product of symmetric groups. Writing $\mathfrak{s}_i = [L_i, \omega_i]_{L_i}$, the torus associated to $\mathfrak{s}$ becomes

$$T_{\mathfrak{s}} = \prod_i (T_{\mathfrak{s}_i})^{e_i} = \prod_i T_i, \tag{5}$$

$$T_{\mathfrak{s}_i} = X_{\mathrm{nr}}(L_i)/X_{\mathrm{nr}}(L_i, \omega_i). \tag{6}$$

By our choice of representatives for $W(G, L)$, $\omega_i^{\otimes e_i}$ is stable under $N_{M_i}(L_i^{e_i})/L_i^{e_i} \cong S_{e_i}$. Let $R_i \subset X_*(\prod_i Z(L_i)^{e_i})$ denote the coroot system of $(M_i, Z(L_i)^{e_i})$ and recall the meaning of the notation $W(R_i)$. Then we can identify S_{e_i} with $W(R_i)$. The action of $W_{\mathfrak{s}}$ on $T_{\mathfrak{s}}$ is just permuting coordinates in the standard way and

$$W_{\mathfrak{s}} = W_\omega. \tag{7}$$

The reduced norm map $D \to F$ gives rise to a group homomorphism $\mathrm{Nrd} : G \to F^\times$. We denote its kernel by $G^\sharp$, so $G^\sharp$ is also the derived group of G. For subgroups $H \subset G$ we write

$$H^\sharp = H \cap G^\sharp.$$

In [ABPS4] we determined the structure of the Hecke algebras associated to types for $G^\sharp$, starting with those for G. As an intermediate step, we did this for the group $G^\sharp Z(G)$, where $Z(G) \cong F^\times$ denotes the centre of G. The advantage is that the comparison between $G^\sharp$ and $G^\sharp Z(G)$ is easy, while $G^\sharp Z(G) \subset G$ can be treated as an extension of finite index. In fact it is a subgroup of finite index if p does not divide md. In case p does divide md, the quotient $G/G^\sharp Z(G)$ is compact and similar techniques can be applied.

For an inertial equivalence class $\mathfrak{s} = [L, \omega]_G$ we define $\mathbf{Irr}^{\mathfrak{s}}(G^\sharp)$ as the set of irreducible $G^\sharp$-representations that are subquotients of $\mathrm{Res}^G_{G^\sharp}(\pi)$ for some $\pi \in \mathbf{Irr}^{\mathfrak{s}}(G)$, and $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$ as the collection of $G^\sharp$-representations all of whose irreducible subquotients lie in $\mathbf{Irr}^{\mathfrak{s}}(G^\sharp)$. We want to investigate the category $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$. It is a product of finitely many Bernstein blocks for $G^\sharp$, see [ABPS4, Lemma 2.2]:

$$\mathrm{Rep}^{\mathfrak{s}}(G^\sharp) = \prod_{\mathfrak{t}^\sharp \prec \mathfrak{s}} \mathrm{Rep}^{\mathfrak{t}^\sharp}(G^\sharp). \tag{8}$$

We note that the Bernstein components $\mathbf{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ which are subordinate to one $\mathfrak{s}$ (i.e., such that $\mathfrak{t}^\sharp \prec \mathfrak{s}$) form precisely one class of L-indistinguishable components: every L-packet for $G^\sharp$ which intersects one of them intersects them all.

Analogously we define $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp Z(G))$, and we obtain

$$\mathrm{Rep}^{\mathfrak{s}}(G^\sharp Z(G)) = \prod_{\mathfrak{t} \prec \mathfrak{s}} \mathrm{Rep}^{\mathfrak{t}}(G^\sharp Z(G)),$$

where the $\mathfrak{t}$ are inertial equivalence classes for $G^\sharp Z(G)$.

The restriction of $\mathfrak{t}$ to $G^\sharp$ is a single inertial equivalence class $\mathfrak{t}^\sharp$, and by [ABPS4, (43)]:

$$T_{\mathfrak{t}^\sharp} = T_{\mathfrak{t}}/X_{\mathrm{nr}}(\mathrm{Nrd}(Z(G))). \tag{9}$$

For $\pi \in \mathbf{Irr}(G)$ we put

$$X^G(\pi) := \{\gamma \in \mathbf{Irr}(G/G^\sharp) : \gamma \otimes \pi \cong \pi\}.$$

The same notation will be used for representations of parabolic subgroups of G which admit a central character. For every $\gamma \in X^G(\pi)$ there exists a nonzero intertwining operator

$$I(\gamma, \pi) \in \mathrm{Hom}_G(\pi \otimes \gamma, \pi) = \mathrm{Hom}_G(\pi, \pi \otimes \gamma^{-1}), \tag{10}$$

which is unique up to a scalar. As $G^\sharp \subset \ker(\gamma)$, $I(\gamma, \pi)$ can also be considered as an element of $\mathrm{End}_{G^\sharp}(\pi)$. As such, these operators determine a 2-cocycle κ_π by

$$I(\gamma, \pi) \circ I(\gamma', \pi) = \kappa_\pi(\gamma, \gamma') I(\gamma\gamma', \pi). \tag{11}$$

By [HiSa, Lemma 2.4] they span the $G^\sharp$-intertwining algebra of π:

$$\mathrm{End}_{G^\sharp}(\mathrm{Res}^G_{G^\sharp} \pi) \cong \mathbb{C}[X^G(\pi), \kappa_\pi], \tag{12}$$

where the right hand side denotes the twisted group algebra of $X^G(\pi)$. Furthermore by [HiSa, Corollary 2.10]

$$\mathrm{Res}^G_{G^\sharp} \pi \cong \bigoplus_{\rho \in \mathbf{Irr}(\mathbb{C}[X^G(\pi), \kappa_\pi])} \mathrm{Hom}_{\mathbb{C}[X^G(\pi), \kappa_\pi]}(\rho, \pi) \otimes \rho \tag{13}$$

as representations of $G^\sharp \times X^G(\pi)$.
The analogous groups for $\mathfrak{s} = [L, \omega]_G$ and $\mathfrak{s}_L = [L, \omega]_L$ are

$$\begin{aligned} X^L(\mathfrak{s}) &:= \{\gamma \in \mathbf{Irr}(L/L^\sharp Z(G)) : \gamma \otimes \omega \in [L, \omega]_L\}, \\ X^G(\mathfrak{s}) &:= \{\gamma \in \mathbf{Irr}(G/G^\sharp Z(G)) : \gamma \otimes I_P^G(\omega) \in \mathrm{Rep}^{\mathfrak{s}}(G)\}. \end{aligned}$$

The role of the group $W_{\mathfrak{s}}$ for $\mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$ is played by

$$W^\sharp_{\mathfrak{s}} := \{w \in W(G, L) \mid \exists \gamma \in \mathbf{Irr}(L/L^\sharp Z(G)) \text{ such that } w(\gamma \otimes \omega) \in [L, \omega]_L\}$$

By [ABPS4, Lemma 2.3]

$$W^\sharp_{\mathfrak{s}} = W_{\mathfrak{s}} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \text{ , where } \mathfrak{R}^\sharp_{\mathfrak{s}} = W^\sharp_{\mathfrak{s}} \cap N_G(P \cap \textstyle\prod_i M_i)/L. \tag{14}$$

while [ABPS4, Lemma 2.4.d] says that

$$X^G(\mathfrak{s})/X^L(\mathfrak{s}) \cong \mathfrak{R}^\sharp_{\mathfrak{s}}. \tag{15}$$

For another way to view $X^G(\mathfrak{s})$, we start with

$$\mathrm{Stab}(\mathfrak{s}) := \{(w, \gamma) \in N_G(L)/L \times \mathbf{Irr}(L/L^\sharp Z(G)) \mid w(\gamma \otimes \omega) \in [L, \omega]_L\}.$$

The normal subgroup $W_{\mathfrak{s}}$ has a complement:

$$\mathrm{Stab}(\mathfrak{s}) = \mathrm{Stab}(\mathfrak{s}, P \cap \prod_i M_i) \ltimes W_{\mathfrak{s}} := \mathrm{Stab}(\mathfrak{s})^+ \ltimes W_{\mathfrak{s}}$$

$$\mathrm{Stab}(\mathfrak{s})^+ := \{(w,\gamma) \in N_G(P \cap \prod_i M_i)/L \times \mathbf{Irr}(L/L^\sharp Z(G)) \mid w(\gamma \otimes \omega) \in [L,\omega]_L\}$$

By [ABPS4, Lemma 2.4.a] projection of $\mathrm{Stab}(\mathfrak{s})$ on the second coordinate gives an isomorphism

$$X^G(\mathfrak{s}) \cong \mathrm{Stab}(\mathfrak{s})/W_{\mathfrak{s}} \cong \mathrm{Stab}(\mathfrak{s})^+ \tag{16}$$

In particular

$$\mathrm{Stab}(\mathfrak{s})^+/X^L(\mathfrak{s}) \cong \mathfrak{R}^\sharp_{\mathfrak{s}}. \tag{17}$$

As in [ABPS4, (159)–(161)] we choose $\chi_\gamma \in X_{\mathrm{nr}}(L)^{W_{\mathfrak{s}}}$ for $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})^+$, such that

$$w(\omega) \otimes \gamma \cong \omega \otimes \chi_\gamma. \tag{18}$$

Notice that χ_γ is unique up to $X_{\mathrm{nr}}(L,\omega)$. Furthermore we choose an invertible

$$J(\gamma, \omega \otimes \chi_\gamma^{-1}) \in \mathrm{Hom}_L(\omega \otimes \chi_\gamma^{-1}, w^{-1}(\omega) \otimes \gamma^{-1}). \tag{19}$$

This generalizes (10) in the sense that

$$J(\gamma, \omega \otimes \chi_\gamma^{-1}) = I(\gamma,\omega) \quad \text{if} \quad \gamma \in X^L(\omega) \text{ and } \chi_\gamma = 1.$$

Let V_ω denote the vector space underlying ω. We may assume that

$$\chi_\gamma = \gamma \quad \text{and} \quad J(\gamma, \omega \otimes \chi_\gamma^{-1}) = \mathrm{id}_{V_\omega} \quad \text{if} \quad \gamma \in X_{\mathrm{nr}}(L/L^\sharp Z(G)). \tag{20}$$

2 Bernstein tori

We will determine the Bernstein tori for $G^\sharp Z(G)$ and $G^\sharp$, in terms of those for G. The group $X^L(\mathfrak{s})$ acts on $T_{\mathfrak{s}} = \mathbf{Irr}^{\mathfrak{s}_L}(L)$ by $\pi \mapsto \pi \otimes \gamma$. By [ABPS4, Proposition 2.1] $\mathrm{Res}^L_{L^\sharp}(\omega)$ and $\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi)$ with $\chi \in X_{\mathrm{nr}}(L)$ have a common irreducible subquotient if and only if there is a $\gamma \in X^L(\mathfrak{s})$ such that $\omega \otimes \chi \cong \omega \otimes \chi_\gamma$. As in (19) we choose a nonzero

$$J(\gamma,\omega) \in \mathrm{Hom}_L(\omega, \omega \otimes \chi_\gamma \gamma^{-1}) = \mathrm{Hom}_L(\omega \otimes \gamma, \omega \otimes \chi_\gamma).$$

Then $J(\gamma,\omega) \in \mathrm{Hom}_{L^\sharp}(\omega, \omega \otimes \chi_\gamma)$ and for every irreducible subquotient $\sigma^\sharp$ of $\mathrm{Res}^L_{L^\sharp}(\omega)$

$$\gamma * (\sigma^\sharp \otimes \chi) : m \mapsto J(\gamma,\omega) \circ (\sigma^\sharp \otimes \chi)(m) \circ J(\gamma,\omega)^{-1} \tag{21}$$

is an irreducible subquotient representation of

$$\mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi\chi_\gamma) = \mathrm{Res}^L_{L^\sharp}(\omega \otimes \chi\chi_\gamma\gamma^{-1}).$$

This prompts us to consider

$$X^L(\mathfrak{s}, \sigma^\sharp) := \{\gamma \in X^L(\mathfrak{s}) \mid \gamma * \sigma^\sharp \cong \sigma^\sharp \otimes \chi_\gamma\}. \tag{22}$$

By [ABPS4, Lemma 4.14]

$$\sigma^\sharp \otimes \chi \cong \sigma^\sharp \text{ for all } \chi \in X_{\mathrm{nr}}(L,\omega). \tag{23}$$

Hence the group (22) is well-defined, that is, independent of the choice of the χ_γ. For $\gamma \in X^L(\omega)$ (21) reduces to $\sigma^\sharp \otimes \chi$, so $\gamma \in X^L(\mathfrak{s},\sigma^\sharp)$. By (20) the same goes for $\gamma \in X_{\mathrm{nr}}(L/L^\sharp Z(G))$, so there is always an inclusion

$$X^L(\omega) X_{\mathrm{nr}}(L/L^\sharp Z(G)) \subset X^L(\mathfrak{s},\sigma^\sharp). \tag{24}$$

We gathered enough tools to describe the Bernstein tori for $G^\sharp$ and $G^\sharp Z(G)$. Recall that $\mathfrak{s}_L = [L,\omega]_L, T_{\mathfrak{s}} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)$. Let $T_{\mathfrak{s}}^\sharp$ be the restriction of $T_{\mathfrak{s}}$ to $L^\sharp$, that is,

$$T_{\mathfrak{s}}^\sharp := T_{\mathfrak{s}}/X_{\mathrm{nr}}(G) = T_{\mathfrak{s}}/X_{\mathrm{nr}}(L/L^\sharp) \cong X_{\mathrm{nr}}(L^\sharp)/X_{\mathrm{nr}}(L,\omega), \tag{25}$$

where $X_{\mathrm{nr}}(L/L^\sharp)$ denotes the group of unramified characters of L which are trivial on $L^\sharp$.

Proposition 2.1. *Let $\sigma^\sharp$ be an irreducible subquotient of $\mathrm{Res}^L_{L^\sharp}(\omega)$ and write $\mathfrak{t} = [L^\sharp Z(G), \sigma^\sharp]_{G^\sharp Z(G)}$ and $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$.*

(a) *$X^L(\mathfrak{s},\sigma^\sharp)$ depends only on $\mathfrak{s}_L$, not on the particular $\sigma^\sharp$.*

(b) *$X_{\mathrm{nr}}(L,\omega)\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s},\sigma^\sharp)\}$ is a subgroup of $X_{\mathrm{nr}}(L)$ which contains $X_{\mathrm{nr}}(L/L^\sharp Z(G))$.*

(c) *$T_{\mathfrak{t}} \cong T_{\mathfrak{s}}/\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s},\sigma^\sharp)\} \cong X_{\mathrm{nr}}(L^\sharp Z(G))/X_{\mathrm{nr}}(L,\omega)\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s},\sigma^\sharp)\}$.*

(d) *$T_{\mathfrak{t}^\sharp} \cong T_{\mathfrak{s}}^\sharp/\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s},\sigma^\sharp)\} \cong X_{\mathrm{nr}}(L^\sharp)/X_{\mathrm{nr}}(L,\omega)\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s},\sigma^\sharp)\}$.*

Proof. (a) By [ABPS4, Proposition 2.1] every two irreducible subquotients of $\mathrm{Res}^L_{L^\sharp}(\omega)$ are direct summands and are conjugate by an element of L. Given $\gamma \in X^L(\mathfrak{s})$, pick $m_\gamma \in L$ such that

$$\gamma * \sigma^\sharp \cong (\omega(m_\gamma)^{-1} \circ \sigma^\sharp \circ \omega(m_\gamma)) \otimes \chi_\gamma = (m_\gamma \cdot \sigma^\sharp) \otimes \chi_\gamma.$$

For any other irreducible summand $\tau = m_\tau \cdot \sigma^\sharp$ of $\mathrm{Res}^L_{L^\sharp}(\omega)$ we compute

$$\begin{aligned}
\gamma * \tau = \gamma * (m_\tau \cdot \sigma^\sharp) &= J(\gamma,\omega) \circ \omega(m_\tau)^{-1} \circ \sigma^\sharp \circ \omega(m_\tau) \circ J(\gamma,\omega)^{-1} \\
&= (\chi_\gamma \gamma^{-1} \otimes \omega)(m_\tau^{-1}) \circ J(\gamma,\omega) \circ \sigma^\sharp \circ J(\gamma,\omega)^{-1} \circ (\chi_\gamma \gamma^{-1} \otimes \omega)(m_\tau) \\
&\cong \omega(m_\tau^{-1}) \circ (m_\gamma \cdot \sigma^\sharp) \otimes \chi_\gamma \circ \omega(m_\tau) \\
&\cong (m_\tau m_\gamma \cdot \sigma^\sharp) \otimes \chi_\gamma.
\end{aligned}$$

As $L/L^\sharp$ is abelian, we find that $m_\tau m_\gamma \cdot \sigma^\sharp \cong m_\gamma m_\tau \cdot \sigma^\sharp$ and that

$$\gamma * \tau \cong (m_\gamma m_\tau \cdot \sigma^\sharp) \otimes \chi_\gamma = m_\gamma \cdot \tau \otimes \chi_\gamma.$$

Writing $L_\tau = \{m \in L \mid m \cdot \tau \cong \tau\}$, we deduce the following equivalences:

$$\gamma * \sigma^\sharp \cong \sigma^\sharp \otimes \chi_\gamma \Leftrightarrow m_\gamma \in L_{\sigma^\sharp} \Leftrightarrow m_\gamma \in m_\tau L_{\sigma^\sharp} m_\tau^{-1} = L_\tau \Leftrightarrow \gamma * \tau \cong \tau \otimes \chi_\gamma .$$

This means that $X^L(\mathfrak{s}, \sigma^\sharp) = X^L(\mathfrak{s}, \tau)$.
(b) By (20) and (24)

$$X_{\mathrm{nr}}(L/L^\sharp Z(G)) \subset \{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}.$$

In view of the uniqueness property of χ_γ the map

$$X^L(\mathfrak{s}) \to X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega) : \gamma \mapsto \chi_\gamma$$

is a group homomorphism with kernel $X^L(\omega)$. Hence the χ_γ form a subgroup of $X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L, \omega)$, isomorphic to $X^L(\mathfrak{s})/X^L(\omega)$.
(c) Consider the family of $L^\sharp Z(G)$-representations

$$\{\sigma^\sharp \otimes \chi \mid \chi \in X_{\mathrm{nr}}(L)\}.$$

We have to determine the χ for which $\sigma^\sharp \otimes \chi \cong \sigma^\sharp \in \mathbf{Irr}(L^\sharp Z(G))$. From [ABPS4, Lemma 4.14] we see that this includes all the elements of $X_{\mathrm{nr}}(L, \omega) X_{\mathrm{nr}}(L/L^\sharp Z(G))$. By [ABPS4, Proposition 2.1.b] and part (a), all the remaining χ come from $\{\chi_\gamma \mid \gamma \in X^L(\mathfrak{s}, \sigma^\sharp)\}$. This gives the first isomorphism, and the second follows with part (b).
(d) This is a consequence of part (c) and (9). □

Proposition 2.1 entails that for every inertial equivalence class

$$\mathfrak{t} = [L^\sharp Z(G), \sigma^\sharp]_{G^\sharp Z(G)} \prec \mathfrak{s} = [L, \omega]_G$$

the action (21) of $X^L(\mathfrak{s}, \sigma^\sharp)$ leads to

$$T_{\mathfrak{t}} \cong T_{\mathfrak{s}} / X^L(\mathfrak{s}, \sigma^\sharp).$$

However, some of the tori

$$T_{\mathfrak{t}} = T_{\mathfrak{t}_L} = \mathbf{Irr}^{[L^\sharp Z(G), \sigma^\sharp]_{L^\sharp Z(G)}}(L^\sharp Z(G))$$

associated to inequivalent $\sigma^\sharp \subset \mathrm{Res}^L_{L^\sharp}(\omega)$ can coincide as subsets of $\mathbf{Irr}(L^\sharp Z(G))$. This is caused by elements of $X^L(\mathfrak{s}) \setminus X^L(\mathfrak{s}, \sigma^\sharp)$ via the action (21). With (23), (22) and (13) we can write

$$\mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp Z(G)) = \bigcup\nolimits_{\mathfrak{t}_L \prec \mathfrak{s}_L} T_{\mathfrak{t}_L} = \big(T_{\mathfrak{s}} \times \mathbf{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega])\big)/X^L(\mathfrak{s}), \tag{26}$$

where $(\omega \otimes \chi, \rho) \in T_{\mathfrak{s}} \times \mathbf{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega])$ corresponds to

$$\mathrm{Hom}_{\mathbb{C}[X^L(\omega), \kappa_\omega]}(\rho, \omega \otimes \chi) \in \mathbf{Irr}(L^\sharp Z(G)).$$

With (9) we can deduce a similar expression for $L^\sharp$:

$$\begin{aligned}\mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp) = \bigcup_{\mathfrak{t}_L^\sharp \prec \mathfrak{s}_L} T_{\mathfrak{t}_L^\sharp} &= \big(T_{\mathfrak{s}}^\sharp \times \mathbf{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega])\big)/X^L(\mathfrak{s}) \\ &= \big(T_{\mathfrak{s}} \times \mathbf{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega])\big)/X^L(\mathfrak{s}) X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp).\end{aligned} \tag{27}$$

In the notation of (26) and (27) the action of $\gamma \in X^L(\mathfrak{s})$ becomes

$$\gamma \cdot (\omega \otimes \chi, \rho) = (\omega \otimes \chi\chi_\gamma, \phi_{\omega,\gamma}\rho), \tag{28}$$

where $\phi_{\omega,\gamma}$ is yet to be determined. Any $\gamma \in X^L(\omega)$ can be adjusted by an element of $X_{\mathrm{nr}}(L, \omega)$ to achieve $\chi_\gamma = 1$. Then (23) shows that $\phi_{\omega,\gamma}\rho \cong \rho$ for all $\gamma \in X^L(\omega)$.

Lemma 2.2. *For $\gamma \in X^L(\mathfrak{s})$, $\phi_{\omega,\gamma}\rho$ is ρ tensored with a character of $X^L(\omega)$, which we also call $\phi_{\omega,\gamma}$. Then*

$$X^L(\mathfrak{s}) \to \mathbf{Irr}(X^L(\omega)) : \gamma \mapsto \phi_{\omega,\gamma}$$

is a group homomorphism.

Proof. Let $N_{\gamma'}$ be a standard basis element of $\mathbb{C}[X^L(\omega), \kappa_\omega]$. In view of (21) $\phi_{\omega,\gamma}\rho$ is given by

$$N_{\gamma'} \mapsto J(\gamma, \omega) I(\gamma', \omega) J(\gamma, \omega)^{-1} \in \mathrm{Hom}_L(\omega \otimes \gamma'\chi_\gamma, \omega \otimes \chi_\gamma). \tag{29}$$

Since these are irreducible L-representations, there is a unique $\lambda \in \mathbb{C}^\times$ such that

$$\begin{aligned} &J(\gamma, \omega) I(\gamma', \omega) J(\gamma, \omega)^{-1} = \lambda^{-1} I(\gamma', \omega \otimes \chi_\gamma), \\ &(\phi_{\omega,\gamma}\rho)(N_{\gamma'}) = \rho(\lambda I(\gamma', \omega)) = \lambda \rho(N_{\gamma'}). \end{aligned}$$

Moreover the relation

$$I(\gamma_1', \omega \otimes \chi_\gamma) I(\gamma_2', \omega \otimes \chi_\gamma) = \kappa_{\omega \otimes \chi_\gamma}(\gamma_1', \gamma_2') I(\gamma_1'\gamma_2', \omega \otimes \chi_\gamma) \tag{30}$$

also holds with $J(\gamma, \omega) I(\gamma_i', \omega) J(\gamma, \omega)^{-1}$ instead of $I(\gamma_i', \omega)$– a basic property of conjugation. It follows that $\gamma' \mapsto \lambda$ defines a character of $X^L(\omega)$ which implements the action $\rho \mapsto \phi_{\omega,\gamma}\rho$. As $\phi_{\omega,\gamma}$ comes from conjugation by $J(\gamma, \omega \otimes \chi)$ and by (30), $\gamma \mapsto \phi_{\omega,\gamma}$ is a group homomorphism. □

A straightforward check, using the above proof, shows that

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}[X^L(\omega), \kappa_\omega]}(\rho, \omega \otimes \chi) & \to & \mathrm{Hom}_{\mathbb{C}[X^L(\omega), \kappa_\omega]}(\phi_{\omega,\gamma}\rho, \omega \otimes \chi\chi_\gamma) \\ f & \mapsto & J(\gamma, \omega \otimes \chi) \circ f \end{array} \tag{31}$$

is an isomorphism of $L^\sharp Z(G)$-representations.

3 Hecke algebras

We will show that the algebras $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ and $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ are stratified equivalent [ABPS7] with much simpler algebras. In this section we recall the final results of [ABPS4], which show that up to Morita equivalence these algebras are closely related to affine Hecke algebras. In section 4 we analyse the latter algebras in the framework of [ABPS7].
Our basic affine Hecke algebra is called $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$. By definition [ABPS4, (119)] it has a $\mathbb{C}$-basis $\{\theta_x[w] : x \in X^*(T_{\mathfrak{s}}), w \in W_{\mathfrak{s}}\}$ such that

- the span of the θ_x is identified with the algebra $\mathcal{O}(T_{\mathfrak{s}})$ of regular functions on $T_{\mathfrak{s}}$;
- the span of the $[w]$ is the finite dimensional Iwahori–Hecke algebra $\mathcal{H}(W_{\mathfrak{s}}, q_{\mathfrak{s}})$;
- the multiplication between these two subalgebras is given by
$$f[s] - [s](s \cdot f) = (q_{\mathfrak{s}}(s) - 1)(f - (s \cdot f))(1 - \theta_{-\alpha})^{-1} \qquad f \in \mathcal{O}(T_{\mathfrak{s}}), \quad (32)$$
for a simple reflection $s = s_\alpha$;
- the algebra is well-defined for any array of parameters $q_{\mathfrak{s}} = (q_{\mathfrak{s},i})_i$ in $\mathbb{C}^\times$. The parameters $q_{\mathfrak{s},i}$ that we will use are given explicitly in [Séc, Théorème 4.6].

Thus $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ is a tensor product of affine Hecke algebras of type GL_e, but written in such a way that the torus $T_{\mathfrak{s}}$ appears canonically in it (i.e. independent of the choice of a base point of $T_{\mathfrak{s}}$). Sécherre and Stevens [Séc, SéSt2] showed that
$$\mathcal{H}(G)^{\mathfrak{s}} \text{ is Morita equivalent with } \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}). \quad (33)$$
From [SéSt1] we know that there exists a simple type (K, λ) for $[L, \omega]_M$, and in [SéSt2] it was shown to admit a G-cover (K_G, λ_G). We denote the associated central idempotent of $\mathcal{H}(K)$ by e_λ, and similarly for other irreducible representations. Then $V_\lambda = e_\lambda V_\omega$.
For the restriction process we need an idempotent that is invariant under $X^G(\mathfrak{s})$. To that end we replace λ_G by the sum of the representations $\gamma \otimes \lambda_G$ with $\gamma \in X^G(\mathfrak{s})$, which we call μ_G. In [ABPS4, (91)] we constructed an idempotent $e_{\mu_G} \in \mathcal{H}(G)$ which is supported on the compact open subgroup $K_G \subset G$. It follows from the work of Sécherre and Stevens [SéSt2] that $e_{\mu_G}\mathcal{H}(G)e_{\mu_G}$ is Morita equivalent with $\mathcal{H}(G)^{\mathfrak{s}}$.
In [ABPS4, (128)] we defined a finite dimensional subspace
$$V_\mu := \sum\nolimits_{\gamma \in X^L(\mathfrak{s})} e_{\gamma \otimes \lambda} V_\omega$$

of V_ω which is stable under the operators $I(\gamma,\omega)$ with $\gamma \in X^L(\mathfrak{s})$. By [Séc] and [ABPS4, Theorem 4.5.d]

$$e_{\mu_G}\mathcal{H}(G)e_{\mu_G} \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes_\mathbb{C} \mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s}). \tag{34}$$

The groups $X^G(\mathfrak{s})$ and $X_{\mathrm{nr}}(G)$ act on $e_{\mu_G}\mathcal{H}(G)e_{\mu_G}$ by pointwise multiplication of functions $G \to \mathbb{C}$ with characters of G. However, for technical reasons we use the action

$$\alpha_\gamma(f)(g) = \gamma^{-1}(g)f(g) \qquad f \in \mathcal{H}(G), \gamma \in \mathbf{Irr}(G/G^\sharp), g \in G. \tag{35}$$

The action on the right hand side of (34) preserves the tensor factors, and on $\mathrm{End}_\mathbb{C}(\mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s})$ it is the natural action of $X^G(\mathfrak{s})/X^L(\mathfrak{s}) \cong \mathfrak{R}^\sharp_\mathfrak{s}$.
Although e_{μ_G} looks like the idempotent of a type, it is not clear whether it is one, because the associated K_G-representation is reducible and no more suitable compact subgroup of G is in sight. Let $e_{\mu_{G^\sharp}}$ (respectively $e_{\mu_{G^\sharp Z(G)}}$) be the restriction of $e_{\mu_G} : G \to \mathbb{C}$ to $G^\sharp$ (resp. $G^\sharp Z(G)$). We normalize the Haar measure on $G^\sharp$ (resp. $G^\sharp Z(G)$) such that it becomes an idempotent in $\mathcal{H}(G^\sharp)$ (resp. $\mathcal{H}(G^\sharp Z(G))$).
In [ABPS4, Lemma 3.3] we constructed a certain finite set $[L/H_\lambda]$, consisting of representatives for a normal subgroup $H_\lambda \subset L$. Consider the elements

$$\begin{aligned} e^\sharp_{\lambda_G} &:= \textstyle\sum_{a \in [L/H_\lambda]} a e_{\mu_G} a^{-1} \in \mathcal{H}(G), \\ e^\sharp_{\lambda_{G^\sharp Z(G)}} &:= \textstyle\sum_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp Z(G)}} a^{-1} \in \mathcal{H}(G^\sharp Z(G)), \\ e^\sharp_{\lambda_{G^\sharp}} &:= \textstyle\sum_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp}} a^{-1} \in \mathcal{H}(G^\sharp). \end{aligned} \tag{36}$$

It follows from [ABPS4, Lemma 3.12] that they are again idempotent. Notice that $e^\sharp_{\lambda_G}$ detects the same category of G-representations as e_{μ_G}, namely $\mathrm{Rep}^\mathfrak{s}(G)$. In the proof of [ABPS4, Proposition 3.15] we established that (34) extends to an isomorphism

$$e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G} \cong \mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu \otimes_\mathbb{C} \mathbb{C}\mathfrak{R}^\sharp_\mathfrak{s}) \otimes M_{[L:H_\lambda]}(\mathbb{C}). \tag{37}$$

Theorem 3.1. [ABPS4, Theorem 4.13] *Let $G = \mathrm{GL}_m(D)$ be an inner form of $\mathrm{GL}_n(F)$. Then for any $\mathfrak{s} \in \mathfrak{B}(G)$:*

(a) *$\mathcal{H}(G^\sharp Z(G))^\mathfrak{s}$ is Morita equivalent with its subalgebra*

$$e^\sharp_{\lambda_{G^\sharp Z(G)}}\mathcal{H}(G^\sharp Z(G))e^\sharp_{\lambda_{G^\sharp Z(G)}} = \bigoplus_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp Z(G)}} a^{-1}\mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1}$$

(b) *Each of the algebras $a e_{\mu_{G^\sharp Z(G)}} a^{-1}\mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1}$ is isomorphic to*

$$\big(\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, q_\mathfrak{s}) \otimes \mathrm{End}_\mathbb{C}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_\mathfrak{s}. \tag{38}$$

(c) *Under these isomorphisms the action of* $X_{\mathrm{nr}}(G)$ *on* $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ *becomes the action of* $X_{\mathrm{nr}}(L/L^\sharp) \cong X_{\mathrm{nr}}(G)$ *on* (38) *via translations on* $T_{\mathfrak{s}}$.

Recall from (25) that $T^\sharp_{\mathfrak{s}}$ is the restriction of $T_{\mathfrak{s}}$ to $L^\sharp$. With this torus we build an affine Hecke algebra $\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ for $G^\sharp$.

Theorem 3.2. [ABPS4, Theorem 4.15]

(a) $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ *is Morita equivalent with*

$$e^\sharp_{\lambda_{G^\sharp}} \mathcal{H}(G^\sharp) e^\sharp_{\lambda_{G^\sharp}} = \bigoplus\nolimits_{a \in [L/H_\lambda]} a e_{\mu_{G^\sharp}} a^{-1} \mathcal{H}(G^\sharp) a e_{\mu_{G^\sharp}} a^{-1}$$

(b) *Each of the algebras* $a e_{\mu_{G^\sharp}} a^{-1} \mathcal{H}(G^\sharp) a e_{\mu_{G^\sharp}} a^{-1}$ *is isomorphic to*

$$\big(\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

Let us describe the above actions of the group $X^G(\mathfrak{s})$ explicitly. The action on

$$a e_{\mu_{G^\sharp Z(G)}} a^{-1} \mathcal{H}(G^\sharp Z(G)) a e_{\mu_{G^\sharp Z(G)}} a^{-1} \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu). \tag{39}$$

does not depend on $a \in [L/H_\lambda]$ because

$$\alpha_\gamma(a f a^{-1}) = a(\alpha_\gamma(f)) a^{-1} \qquad f \in \mathcal{H}(G).$$

The isomorphism (17) yields an action α of $\mathrm{Stab}(\mathfrak{s})^+$ on (39).

Theorem 3.3. [ABPS4, Lemmas 3.5 and 4.11]

(a) *The action of* $\mathrm{Stab}(\mathfrak{s})^+$ *on* $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ *in Theorem 3.1 preserves both tensor factors. On* $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ *it is given by*

$$\alpha_{(w,\gamma)}(\theta_x[v]) = \chi_\gamma^{-1}(x) \theta_{w(x)}[wvw^{-1}] \qquad x \in X^*(T_{\mathfrak{s}}), v \in W_{\mathfrak{s}},$$

and on $\mathrm{End}_{\mathbb{C}}(V_\mu)$ *by*

$$\alpha_{(w,\gamma)}(h) = J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}.$$

(b) *The subgroup of elements that act trivially is*

$$X^L(\omega, V_\mu) = \{\gamma \in X^L(\omega) \mid I(\gamma, \omega)|_{V_\mu} \in \mathbb{C}^\times \mathrm{id}_{V_\mu}\}.$$

Its cardinality equals $[L : H_\lambda]$.

(c) *Part (a) and Theorem 3.1.c also describe the action of* $\mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(G)$ *on* $\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ *in Theorem 3.2. The subgroup of elements that act trivially on this algebra is*

$$X^L(\omega, V_\mu) X_{\mathrm{nr}}(G) = X^L(\omega, V_\mu) X_{\mathrm{nr}}(L/L^\sharp).$$

4 Spectrum preserving morphisms and stratified equivalences

We will show that the Hecke algebras obtained in Theorems 3.1 and 3.2 fit in the framework of spectrum preserving morphisms and stratified equivalence of finite type algebras, see [ABPS7]. First we exhibit an algebra that interpolates between

$$(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}$$

and $(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}$. Recall that Conditions 1.1 are in force and write

$$T_{\mathfrak{s}} = \prod\nolimits_i T_i,\ R_{\mathfrak{s}} = \bigsqcup\nolimits_i R_i,\ W_{\mathfrak{s}} = \prod\nolimits_i W(R_i) = \prod\nolimits_i S_{e_i}.$$

Let q_i be the restriction of $q_{\mathfrak{s}} : X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \to \mathbb{R}_{>0}$ to $X^*(T_i) \rtimes W(R_i)$. Recall Lusztig's asymptotic Hecke algebra $J(X^*(T_i) \rtimes W(R_i))$ from [Lus2, Lus3]. We remark that, although in [Lus2] it is supposed that the underlying root datum is semisimple, this assumption is shown to be unneccesary in [Lus3]. This algebra is unital and of finite type over $\mathcal{O}(T_i)^{W(R_i)}$. It has a distinguished $\mathbb{C}$-basis $\{t_{xv} \mid x \in X^*(T_i), v \in W(R_i)\}$ and the t_x with $x \in X^*(T_i)^{W(R_i)}$ are central. We define

$$J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) = \bigotimes\nolimits_i J(X^*(T_i) \rtimes W(R_i)).$$

This is a unital finite type algebra over $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$, in fact for several different $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$-module structures.
Lusztig [Lus3, §1.4] defined injective algebra homomorphisms

$$\mathcal{H}(T_i, W(R_i), q_i) \xrightarrow{\phi_{i,q_i}} J(X^*(T_i) \rtimes W(R_i)) \xleftarrow{\phi_{i,1}} \mathcal{O}(T_i) \rtimes W(R_i) \tag{40}$$

with many nice properties. Among these, we record that

$$\phi_{i,q_i} \text{ and } \phi_{i,1} \text{ are the identity on } \mathbb{C}[X^*(T_i)^{W(R_i)}] \cong \mathcal{O}(X_{\mathrm{nr}}(Z(M_i))). \tag{41}$$

There exist $\mathcal{O}(T_i)^{W(R_i)}$-module structures on $J(X^*(T_i) \rtimes W(R_i))$ for which the maps (40) are $\mathcal{O}(T_i)^{W(R_i)}$-linear, namely by letting $\mathcal{O}(T_i)^{W(R_i)}$ act via the map ϕ_{i,q_i} or via $\phi_{i,1}$. Taking tensor products over i in (40) and with the identity on $\mathrm{End}_{\mathbb{C}}(V_\mu)$ gives injective algebra homomorphisms

$$\begin{array}{rrcl} \phi_{q_{\mathfrak{s}}} : & \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) & \to & J(X(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu), \\ \phi_1 : & \mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) & \to & J(X(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu). \end{array} \tag{42}$$

The maps $\phi_{q_{\mathfrak{s}}}$ and ϕ_1 are $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$-linear with respect to the appropriate module structure on $J(X(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}})$.

Lemma 4.1. *Via (42) the action of* $\mathrm{Stab}(\mathfrak{s})^+$ *on* $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ *from Theorem 3.3 extends canonically to an action on* $J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, *which stabilizes the subalgebra* $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$.

Proof. In $X^*(T_{\mathfrak{s}}) \rtimes W(G,L)$ every $w \in \mathfrak{R}^\sharp_{\mathfrak{s}}$ normalizes the subgroup $X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$. The group automorphism

$$xv \mapsto wxvw^{-1} \quad \text{of} \quad X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \tag{43}$$

only permutes the subgroups $X^*(T_i) \rtimes W(R_i)$. In particular it stabilizes the set of simple reflections. With the canonical representatives for $\mathfrak{R}^\sharp_{\mathfrak{s}}$ in G from (14), conjugation by w stabilizes the type for $\mathfrak{s}_L$ (it is a product of simple types in the sense of [Séc, §4]). Hence (43) preserves $q_{\mathfrak{s}}$. Thus (43) can be factorized as

$$\prod_j \omega_j \text{ with } \omega_j \in \mathrm{Aut}\big(\prod_{i : R_i = R_j, q_i = q_j} X^*(T_i) \rtimes W(R_i) \big) \tag{44}$$

The function $q_{\mathfrak{s}}$ takes the same value on all simple (affine) roots associated to the group for one j in (44), so the algebra

$$\bigotimes\nolimits_{i | R_i = R_j, q_i = q_j} J(X^*(T_i) \rtimes W(R_i)) \tag{45}$$

is of the kind considered in [Lus3, §1]. Then ω_j is an automorphism which fits in a group called Ω in [Lus3, §1.1], so it gives rise to an automorphism of the algebra (45). In this way the group $\mathfrak{R}^\sharp_{\mathfrak{s}} \cong \mathrm{Stab}(\mathfrak{s})^+ / X^L(\mathfrak{s})$ acts naturally on $J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}})$.
Since $T_{\mathfrak{s}}^{W_{\mathfrak{s}}}$ is central in $T_{\mathfrak{s}} \rtimes W_{\mathfrak{s}}$, every $\chi \in T_{\mathfrak{s}}^{W_{\mathfrak{s}}}$ gives rise to an algebra automorphism of $J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}})$:

$$t_{xv} \mapsto \chi(x) t_{xv} \qquad x \in X^*(T_{\mathfrak{s}}), v \in W_{\mathfrak{s}}. \tag{46}$$

Thus we can make $\mathrm{Stab}(\mathfrak{s})^+$ act on $J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}})$ by

$$(w, \gamma) \cdot t_{xv} = \chi_\gamma^{-1}(x) t_{wxvw^{-1}} \qquad x \in X^*(T_{\mathfrak{s}}), v \in W_{\mathfrak{s}}.$$

The action of $\mathrm{Stab}(\mathfrak{s})^+$ on $\mathrm{End}_{\mathbb{C}}(V_\mu)$ may be copied to this setting, so we can define the following action on $J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$:

$$\alpha_{(w,\gamma)}(t_{xv} \otimes h) = \chi_\gamma^{-1}(x) t_{wxvw^{-1}} \otimes J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}.$$

Of course the above also works with the label function 1 instead of $q_{\mathfrak{s}}$. That yields a similar action of $\mathrm{Stab}(\mathfrak{s})^+$ on $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, namely

$$\alpha_{(w,\gamma)}(xv \otimes h) = \chi_\gamma^{-1}(x)\, wxvw^{-1} \otimes J(\gamma, \omega \otimes \chi_\gamma^{-1}) \circ h \circ J(\gamma, \omega \otimes \chi_\gamma^{-1})^{-1}, \tag{47}$$

where $xv \in X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$. It follows from [Lus3, §1.4] that $\phi_{q_{\mathfrak{s}}}$ and ϕ_1 are now $\mathrm{Stab}(\mathfrak{s})^+$-equivariant. □

Lemma 4.2. *The $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$-algebra homomorphisms $\phi_{q_{\mathfrak{s}}}$ and ϕ_1 from (42) are spectrum preserving with respect to filtrations, in the sense of [ABPS7].*

Proof. It suffices to consider the map $\phi_{q_{\mathfrak{s}}}$, for the same reasoning will apply to ϕ_1. Our argument is a generalization of [BaNi, Theorem 10], which proves the analogous statements for $J(X^*(T_i) \rtimes W(R_i))$. Recall the function

$$a : X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \to \mathbb{Z}_{\geq 0} \tag{48}$$

from [Lus3, §1.3]. For fixed $n \in \mathbb{Z}_{\geq 0}$, the subspace of $J(X^*(T_i) \rtimes W(R_i))$ spanned by the t_{xv} with $a(xv) = n$ is a two-sided ideal, let us call it $J^{i,n}$. Then

$$J(X^*(T_i) \rtimes W(R_i)) = \bigoplus\nolimits_{n \geq 0} J^{i,n}$$

and the sum is finite by [Lus1, §7]. Moreover

$$\mathcal{H}^{i,n} := \phi_{i,q_i}^{-1}\big(\bigoplus\nolimits_{k \geq n} J^{i,k}\big)$$

is a two-sided ideal of $\mathcal{H}(T_i, W(R_i), q_i)$. According to [Lus2, Corollary 3.6] the morphism of $\mathcal{O}(T_i)^{W(R_i)}$-algebras

$$\mathcal{H}^{i,n}/\mathcal{H}^{i,n+1} \to J^{i,n} \text{ induced by } \phi_{i,q_i}$$

is spectrum preserving. For any irreducible $J^{i,n}$-module M_J^i the $\mathcal{H}^{i,n}$-module $\phi_{q_i,i}^*(M_J^i)$ has a distinguished quotient $M_{\mathcal{H}}^i$, which is an irreducible $\mathcal{H}^{i,n}/\mathcal{H}^{i,n+1}$-module.
Let $\mathbf{n}$ be a vector with coordinates $n_i \in \mathbb{Z}_{\geq 0}$ and put $|\mathbf{n}| = \sum_i n_i$. We write $\mathbf{n} \leq \mathbf{n}'$ if $n_i \leq n_i'$ for all i. We define the two-sided ideals

$$\begin{array}{lcll} J^{\mathbf{n}} & = & \bigotimes_i J^{i,n_i} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) & \subset \quad J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu), \\ \mathcal{H}^{\mathbf{n}} & = & \bigotimes_i \mathcal{H}^{i,n_i} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) & \subset \quad \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu), \\ \mathcal{H}^{\mathbf{n}+} & = & \sum_{\mathbf{n}' \geq \mathbf{n}, |\mathbf{n}'| = |\mathbf{n}|+1} \mathcal{H}^{\mathbf{n}'}. & \end{array}$$

It follows from the above that the morphism of $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}}}$-algebras

$$\bigotimes\nolimits_i (\mathcal{H}^{i,n_i}/\mathcal{H}^{i,n_i+1}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \cong \mathcal{H}^{\mathbf{n}}/\mathcal{H}^{\mathbf{n}+} \to J^{\mathbf{n}} \tag{49}$$

induced by $\phi_{q_{\mathfrak{s}}}$ is spectrum preserving, and that every irreducible $J^{\mathbf{n}}$-module M_J has a distinguished quotient $M_{\mathcal{H}}$ which is an irreducible $\mathcal{H}^{\mathbf{n}}/\mathcal{H}^{\mathbf{n}+}$-module. Next we define, for $n \in \mathbb{Z}_{\geq 0}$:

$$J^n := \bigoplus_{|\mathbf{n}|=n} J^{\mathbf{n}}, \ \mathcal{H}^n := \bigoplus_{|\mathbf{n}|=n} \mathcal{H}^{\mathbf{n}}. \tag{50}$$

The aforementioned properties of the map (49) are also valid for

$$\mathcal{H}^n/\mathcal{H}^{n+1} \to J^n, \tag{51}$$

which shows that $\phi_{q_{\mathfrak{s}}}$ is spectrum preserving with respect to the filtrations $(\mathcal{H}^n)_{n \geq 0}$ and $(\oplus_{m \geq n} J^m)_{n \geq 0}$. □

By Lemma 4.1 and (17)

$$\begin{aligned}&\big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}},\\ &\big(J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}},\\ &\big(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}\end{aligned} \tag{52}$$

are unital finite type $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$-algebras, while $\phi_{q_{\mathfrak{s}}}$ and ϕ_1 provide morphisms between them.

Theorem 4.3. (*a*) *The above morphisms between the* $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$*-algebras* (52) *are spectrum preserving with respect to filtrations.*

(*b*) *The same holds for the three algebras of* (52) *with* $T^\sharp_{\mathfrak{s}}$ *instead of* $T_{\mathfrak{s}}$.

Proof. (a) We use the notations from the proof of Lemma 4.2. Since Lusztig's a-function is constant on two-sided cells [Lus3, §1.3] and conjugation by elements of $\mathfrak{R}^\sharp_{\mathfrak{s}}$ preserves the set of simple (affine) reflections in $X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$:

$$a(wxvw^{-1}) = a(xv) \text{ for all } x \in X^*(T_{\mathfrak{s}}), v \in W_{\mathfrak{s}}, w \in \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

Hence J^n and $\mathcal{H}^n$ are stable under the respective actions α and (51) is $\mathrm{Stab}(\mathfrak{s})^+$-equivariant. Let M_J be an irreducible $J^{\mathbf{n}}$-module and regard it as a $(J^n)^{X^L(\mathfrak{s})}$-module via the map $J^n \to J^{\mathbf{n}}$ from (50). By Clifford theory (see [RaRa, Appendix]) its decomposition is governed by a twisted group algebra of the stabilizer of M_J in $X^L(\mathfrak{s})$. Since (51) is $X^L(\mathfrak{s})$-equivariant and $M_{\mathcal{H}}$ is a quotient of M_J, the decomposition of $M_{\mathcal{H}}$ as module over $(\mathcal{H}^n/\mathcal{H}^{n+1})^{X^L(\mathfrak{s})}$ is governed by the same twisted group algebra in the same way. Therefore (51) restricts to a spectrum preserving morphism of $\mathcal{O}(T_{\mathfrak{s}})^{W_{\mathfrak{s}} \times X^L(\mathfrak{s})}$-algebras

$$(\mathcal{H}^n/\mathcal{H}^{n+1})^{X^L(\mathfrak{s})} \to (J^n)^{X^L(\mathfrak{s})}.$$

Now a similar argument with Clifford theory for crossed product algebras shows that

$$(\mathcal{H}^n/\mathcal{H}^{n+1})^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \to (J^n)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}$$

is a spectrum preserving morphism of $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$-algebras. By definition [BaNi, §5], this means that the map

$$\begin{aligned}\phi'_{q_{\mathfrak{s}}} : \big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \to\\ \big(J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}\end{aligned} \tag{53}$$

induced by $\phi_{q_{\mathfrak{s}}}$ is spectrum preserving with respect to filtrations.
The same reasoning is valid with $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ instead of $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ – it is simply the case $q_{\mathfrak{s}} = 1$ of the above.

(b) Recall that $T_{\mathfrak{s}} \cong X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\omega)$. The torus $T_{\mathfrak{s}}/X_{\mathrm{nr}}(L/L^\sharp Z(G))$ can be identified with

$$X_{\mathrm{nr}}(L^\sharp Z(G))/X_{\mathrm{nr}}(L,\omega). \tag{54}$$

Since the elements of $X_{\mathrm{nr}}(L,\omega)$ are trivial on $Z(L) \supset Z(G)$ and $L^\sharp \cap Z(G) \cong \mathfrak{o}_F^\times$ is compact, (54) factors as

$$X_{\mathrm{nr}}(L^\sharp)/X_{\mathrm{nr}}(L,\omega) \times X_{\mathrm{nr}}(Z(G)) = T_{\mathfrak{s}}^\sharp \times X_{\mathrm{nr}}(Z(G)).$$

By Theorem 3.1 the action of $X_{\mathrm{nr}}(L/L^\sharp Z(G)) \subset X^L(\mathfrak{s})$ on the algebras (52) comes only from its action on the torus $T_{\mathfrak{s}}$. Hence these three algebras do not change if we replace $T_{\mathfrak{s}}$ by (54). Equivalently, we may replace $T_{\mathfrak{s}}$ by $T_{\mathfrak{s}}^\sharp \times X_{\mathrm{nr}}(Z(G))$. It follows that

$$\big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \cong \\ \big(\mathcal{O}(X_{\mathrm{nr}}(Z(G))) \otimes \mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp.$$

The action of $\mathrm{Stab}(\mathfrak{s})^+$ fixes $\mathcal{O}(X_{\mathrm{nr}}(Z(G)))$ pointwise, so this equals

$$\big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \otimes \mathcal{O}(X_{\mathrm{nr}}(Z(G))).$$

The other two algebras in (52) can be rewritten similarly. By (41) the morphisms $\phi_{q_{\mathfrak{s}}}$ and ϕ_1 fix the respective subalgebras $\mathcal{O}(X_{\mathrm{nr}}(Z(G)))$ pointwise. It follows that (53) decomposes as

$$\phi_{q_{\mathfrak{s}}}^\sharp \otimes \mathrm{id} : \big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \otimes \mathcal{O}(X_{\mathrm{nr}}(Z(G))) \to \\ \big(J(X^*(T_{\mathfrak{s}}^\sharp) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \otimes \mathcal{O}(X_{\mathrm{nr}}(Z(G))),$$

and similarly for ϕ_1'. From part (a) we know that $\phi_{q_{\mathfrak{s}}}' = \phi_{q_{\mathfrak{s}}}^\sharp \otimes \mathrm{id}$ and $\phi_1' = \phi_1^\sharp \otimes \mathrm{id}$ are spectrum preserving with respect to filtrations. So $\phi_{q_{\mathfrak{s}}}^\sharp$ and

$$\phi_1^\sharp : \big(\mathcal{O}(T_{\mathfrak{s}}^\sharp) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \to \\ \big(J(X^*(T_{\mathfrak{s}}^\sharp) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp$$

have that property as well. □

With Theorem 4.3 we can show that the Hecke algebras for $G^\sharp$ and for $G^\sharp Z(G)$ are stratified equivalent (see [ABPS7]) to much simpler algebras. Recall the subgroup $H_\lambda \subset L$ from [ABPS4, Lemma 3.3].

Theorem 4.4. (*a*) *The algebra* $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ *is stratified equivalent with*

$$\bigoplus_1^{[L:H_\lambda]} \big(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp.$$

Here the action of $w \in W_{\mathfrak{s}}^\sharp$ *is* $\alpha_{(w,\gamma)}$ *(as in Theorem 3.3.a) for any* $\gamma \in \mathbf{Irr}(L/L^\sharp Z(G))$ *such that* $(w,\gamma) \in \mathrm{Stab}(\mathfrak{s})$.

(*b*) *The algebra* $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$ *is stratified equivalent with*

$$\bigoplus_1^{[L:H_\lambda]} \big(\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp,$$

with respect to the same action of $W_{\mathfrak{s}}^\sharp$.

Remark. In principle one could factorize the above algebras according to single Bernstein components for $G^\sharp Z(G)$ and $G^\sharp$. However, this would result in less clear formulas.

Proof. (a) Recall from Theorem 3.1 that $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ is Morita equivalent with

$$\bigoplus_1^{[L:H_\lambda]} \big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp. \tag{55}$$

Consider the sequence of algebras

$$\begin{aligned} &\big(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \\ &\to \big(J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \\ &= \big(J(X^*(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \\ &\leftarrow \big(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp. \end{aligned} \tag{56}$$

In Theorem 4.3.a we proved that the map between the first two lines is spectrum preserving with respect to filtrations. The equality sign does nothing on the level of $\mathbb{C}$-algebras, but we use it to change the $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$-module structure, such that the map from

$$\big(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}} \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp \tag{57}$$

becomes $\mathcal{O}(T_{\mathfrak{s}})^{\mathrm{Stab}(\mathfrak{s})}$-linear. By Theorem 4.3.a that map is also spectrum preserving with respect to filtrations.
Every single step in the above sequence is an instance of stratified equivalence from [ABPS7] (the second step by definition), so $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ is stratified equivalent with a direct sum of $[L : H_\lambda]$ copies of (57). Since $\chi_\gamma \in T_{\mathfrak{s}}$ in (47) is $W_{\mathfrak{s}}$-invariant, the actions of $X^L(\mathfrak{s})$ and $W_{\mathfrak{s}}$ on $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ commute. This observation and (14) allow us to identify (57) with

$$\big(\big(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}\big) \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp = \big(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp. \tag{58}$$

The description of the action of $W_{\mathfrak{s}}^\sharp$ can be derived from Theorem 3.3.
(b) This follows from Theorem 3.2 and the same proof as for part (a). □

5 Extended quotients for inner forms of GL_n

It turns out that via (33) any Bernstein component for G can be described in a canonical way with an extended quotient. Before we prove that, we recall the parametrization of irreducible representations of $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$.
Let $\check{G}_{\mathfrak{s}}$ be the complex reductive group with root datum $(X^*(T_{\mathfrak{s}}), R_{\mathfrak{s}}, X_*(T_{\mathfrak{s}}), R_{\mathfrak{s}}^\vee)$, it is isomorphic to $\prod_i \mathrm{GL}_{e_i}(\mathbb{C})$, embedded in $\check{G} = \mathrm{GL}_{md}(\mathbb{C})$ as

$$\check{G}_{\mathfrak{s}} = Z_{\check{G}}(\check{L}) = Z_{\mathrm{GL}_{md}(\mathbb{C})}\Big(\prod\nolimits_i \mathrm{GL}_{m_i d}(\mathbb{C})^{e_i}\Big).$$

Recall that a Kazhdan–Lusztig triple for $\check{G}_{\mathfrak{s}}$ consists of:

- a unipotent element $u = \prod_i u_i \in \check{G}_{\mathfrak{s}}$;
- a semisimple element $t_q \in \check{G}_{\mathfrak{s}}$ with $t_q u t_q^{-1} = u^{q_{\mathfrak{s}}} := \prod_i u_i^{q_i}$;
- a representation $\rho_q \in \mathbf{Irr}(\pi_0(Z_{\check{G}_{\mathfrak{s}}}(t_q, u)))$ which appears in the homology of variety of Borel subgroups of $\check{G}_{\mathfrak{s}}$ containing $\{t_q, u\}$.

Typically such a triple is considered up to $\check{G}_{\mathfrak{s}}$-conjugation, we denote its equivalence class by $[t_q, u, \rho_q]_{\check{G}_{\mathfrak{s}}}$. These equivalence classes parametrize $\mathbf{Irr}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}))$ in a natural way, see [KaLu]. We denote that by

$$[t_q, u, \rho_q]_{\check{G}_{\mathfrak{s}}} \mapsto \pi(t_q, u, \rho_q). \tag{59}$$

Recall from [ABPS6, §7] that an affine Springer parameter for $\check{G}_{\mathfrak{s}}$ consists of:

- a unipotent element $u = \prod_i u_i \in \check{G}_{\mathfrak{s}}$;
- a semisimple element $t \in Z_{\check{G}_{\mathfrak{s}}}(u)$;
- a representation $\rho \in \mathbf{Irr}(\pi_0(Z_{\check{G}_{\mathfrak{s}}}(t, u)))$ which appears in the homology of variety of Borel subgroups of $\check{G}_{\mathfrak{s}}$ containing $\{t, u\}$.

Again such a triple is considered up to $\check{G}_{\mathfrak{s}}$-conjugacy, and then denoted $[t, u, \rho]_{\check{G}_{\mathfrak{s}}}$. Kato [Kat] established a natural bijection between such equivalence classes and $\mathbf{Irr}(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}})$, say

$$[t, u, \rho]_{\check{G}_{\mathfrak{s}}} \mapsto \tau(t, u, \rho). \tag{60}$$

For a more explicit description, we note that $Z_{\check{G}_{\mathfrak{s}}}(t)$ is a connected reducitive group with Weyl group $W_{\mathfrak{s},t}$, and that (u, ρ) represents a Springer parameter for $W_{\mathfrak{s},t}$. Via the classical Springer correspondence

$$(u, \rho) \text{ determines an irreducible } W_{\mathfrak{s},t}\text{-representation } \pi(u, \rho). \tag{61}$$

Then [Kat] and (60) work out to

$$\tau(t, u, \rho) = \mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s},t}}^{\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}} (\mathbb{C}_t \otimes \pi(u, \rho)). \tag{62}$$

From [KaLu, §2.4] we get a canonical bijection between Kazhdan–Lusztig triples and affine Springer parameters:

$$[t_q, u, \rho_q]_{\check{G}_{\mathfrak{s}}} \longleftrightarrow [t, u, \rho]_{\check{G}_{\mathfrak{s}}}. \tag{63}$$

Basically it adjusts t_q in a minimal way so that it commutes with u, and then there is only one consistent way to modify ρ_q to ρ.
Via Lemma 4.2 the algebra homomorphisms (42) give rise to a bijection

$$\mathbf{Irr}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})) \longleftrightarrow \mathbf{Irr}(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}). \tag{64}$$

We showed in [ABPS6, (90)] that (64) is none other than the composition of (63) with (60) and the inverse of (59):

$$\pi(t_q, u, \rho_q) \longleftrightarrow \tau(t, u, \rho). \tag{65}$$

Theorem 5.1. *The Morita equivalence $\mathcal{H}(G)^{\mathfrak{s}} \sim_M \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ and (64) give rise to a bijection*

$$\mathbf{Irr}^{\mathfrak{s}}(G) \longleftrightarrow T_{\mathfrak{s}} // W_{\mathfrak{s}} \tag{66}$$

with the following properties:

1. *Let $T_{\mathfrak{s},\mathrm{un}}$ be the maximal compact subtorus of $T_{\mathfrak{s}}$ and let $\mathbf{Irr}_{\mathrm{temp}}(G) \subset \mathbf{Irr}(G)$ be the subset of tempered representations. Then (66) restricts to a bijection $\mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G) \longleftrightarrow T_{\mathfrak{s},\mathrm{un}} // W_{\mathfrak{s}}$.*

2. *(66) can be obtained from its restriction to tempered representations by analytic continuation, as in [ABPS1]. For instance, suppose that $\sigma \in \mathbf{Irr}^{\mathfrak{s}_{L'}}_{\mathrm{temp}}(L')$ for some standard parabolic $P' = L'U' \supset P = LU$, and that $I^G_{P'}(\sigma \otimes \chi)$ is mapped to $\tau(t\chi, u, \rho)$ for almost all unitary $\chi \in X_{\mathrm{nr}}(L')$. Then, whenever $\chi_{\mathrm{nr}}(L')$ and $I^G_{P'}(\sigma \otimes \chi)$ is irreducible, it is mapped to $\tau(t\chi, u, \rho)$.*

3. *If $\pi \in \mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G)$ is mapped to $[t, \rho'] \in T_{\mathfrak{s},\mathrm{un}} // W_{\mathfrak{s}}$ and has cuspidal support $W_{\mathfrak{s}}\sigma \in T_{\mathfrak{s}}/W_{\mathfrak{s}}$, then $W_{\mathfrak{s}}t$ is the unitary part of $W_{\mathfrak{s}}\sigma$, with respect to the polar decomposition*

$$T_{\mathfrak{s}} = T_{\mathfrak{s},\mathrm{un}} \times \mathrm{Hom}_{\mathbb{Z}}(X^*(T_{\mathfrak{s}}), \mathbb{R}_{>0}).$$

4. *In the notation of (3), suppose that the parameter of $\rho' \in \mathbf{Irr}(W_{\mathfrak{s},t})$ in the classical Springer correspondence (61) involves a unipotent class $[u]$ which is distinguished in a Levi subgroup $\check{M} \subset Z_{\check{G}_{\mathfrak{s}}}(t)$. Then $\pi = I^G_{PM}(\delta)$, where $M \supset L$ is the unique standard Levi subgroup of G corresponding to $\check{M}$ and $\delta \in \mathbf{Irr}^{[L,\omega]_M}_{\mathrm{temp}}(M)$ is square-integrable modulo centre.*

Moreover (66) is the unique bijection with the properties (1)–(4).

Proof. The Morita equivalence (33) gives a bijection

$$\mathbf{Irr}^{\mathfrak{s}}(G) \longleftrightarrow \mathbf{Irr}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})). \tag{67}$$

Via Lemmas 4.2 and A.1 the right hand side is in bijection with

$$\mathbf{Irr}(\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}) \cong T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}}. \tag{68}$$

In this way we define the map (66).
(1) It is easy to check from [Séc, Théorème 4.6] and [SéSt2, Theorem C] that the Morita equivalence $\mathcal{H}(G)^{\mathfrak{s}} \sim_M \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$ preserves the canonical involution and trace (maybe up to a positive scalar). Accepting that, [DeOp, Theorem 10.1] says that the ensueing bijection between $\mathbf{Irr}^{\mathfrak{s}}(G)$ and $\mathbf{Irr}(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}))$ respects the subsets of tempered representations. By [ABPS6, Proposition 9.3] the latter subset corresponds to the set of Kazhdan–Lusztig triples such that the t in (63) lies in $T_{\mathfrak{s},\mathrm{un}}$.
(2) Consider the bijection (64) and its formulation (65). Here the representations are tempered if and only if $t \in T_{\mathfrak{s}}$ is unitary. Thus (65) for tempered representations determines the bijection (64), by analytic continuation (in the parameters t and t_q) of the formula.
The relation between $\mathbf{Irr}^{\mathfrak{s}}(G)$ and $\mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G)$ is similar, see [ABPS1, Proposition 2.1]. Hence (67) is can also be deduced from its restriction to tempered representations, with the method from [ABPS1, §4].
(3) In [Séc, Théorème 4.6] a $\mathfrak{s}_L$-type (K_L, λ_L) is constructed, with

$$e_{\lambda_L} \mathcal{H}(L) e_{\lambda_L} \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda).$$

It [SéSt2] it is shown that it admits a cover (K_G, λ_G) with

$$e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda),$$

see also [ABPS4, §4.1]. It follows from [ABPS4, Proposition 3.15] that (67) arises from this cover of a $\mathfrak{s}_L$-type. With [BuKu, §7] this implies that (66) translates the cuspidal support of a $(\pi, V_\pi) \in \mathbf{Irr}^{\mathfrak{s}}(G)$ to the unique $W_{\mathfrak{s}} t_q \in T_{\mathfrak{s}}/W_{\mathfrak{s}}$ such that $e_{\lambda_G} V_\pi$ is a subquotient of $\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})}(\mathbb{C}_{t_q}) \otimes V_\lambda$. It follows from [ABPS6, (33) and Lemma 7.1] that the bijection (65) sends any tempered irreducible subquotient of $\mathrm{ind}_{\mathcal{O}(T_{\mathfrak{s}})}^{\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})}(\mathbb{C}_{t_q})$ to an irreducible $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$-representation with $\mathcal{O}(T_{\mathfrak{s}})$-weights $W_{\mathfrak{s}}(t_q \, |t_q|^{-1})$. The associated element of $T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}}$ is then $[t = t_q \, |t_q|^{-1}, \rho]$ with $\rho \in \mathbf{Irr}(W_{\mathfrak{s},t})$.
(4) Since $W_{\mathfrak{s},t}$ is a direct product of symmetric groups, the representations ρ' of the component groups in the Springer parameters are all trivial. By (61), (62) and (65) the $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}})$-representation associated to $[t, \rho']$ is $\pi(t_q, u, \mathrm{triv})$. Then (t_q, u, triv) is also a Kazhdan–Lusztig triple for $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s},M}, q_{\mathfrak{s}})$ and by [KaLu, §7.8]

$$\pi(t_q, u, \mathrm{triv}) = \mathrm{ind}_{\mathcal{H}^M}^{\mathcal{H}} \pi_M(t_q, u, \mathrm{triv}).$$

By [ABPS6, Proposition 9.3] (see also [KaLu, Theorem 8.3]) $\pi_M(t_q, u, \mathrm{triv})$ is essentially square-integrable and tempered, that is, square-integrable modulo centre.
Since (K_G, λ_G) is a a cover of a $\mathfrak{s}_L$-type (K_L, λ_L), there is a a $[L, \omega]_M$-type (K_M, λ_M) which covers (K_L, λ_L) and is covered by (K_G, λ_G). By [ABPS6, Proposition 16.6] $\pi_M(t_q, u, \mathrm{triv})$ corresponds to a M-representation δ which is square-integrable modulo centre. By [BuKu, Corollary 8.4] the bijection (67) respects parabolic induction, so $\pi(t_q, u, \mathrm{triv})$ corresponds to $I_{PM}^G(\delta)$.
Now we check that (66) is canonical in the specified sense. By (1) and (2) it suffices to do so for tempered representations. For $\pi \in \mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G)$, property (3) determines the $W_{\mathfrak{s}}$-orbit $W_{\mathfrak{s}} t$. Fix a t in this orbit. By a result of Harish-Chandra [Wal, Proposition III.4.1] there are a Levi subgroup $M \subset G$ containing L and a square-integrable (modulo centre) representation $\delta \in \mathbf{Irr}(M)$ such that π is a subquotient of $I_{PM}^G(\delta)$. Moreover (M, δ) is unique up to conjugation.
For $t \in T_{\mathfrak{s},\mathrm{un}}$, $W_{\mathfrak{s},t}$ is a product of symmetric groups S_e and $Z_{\check{G}_{\mathfrak{s}}}(t)$ is a product of groups of the form $\mathrm{GL}_e(\mathbb{C})$. Hence the Springer correspondence for $W_{\mathfrak{s},t}$ is a bijection between $\mathbf{Irr}(W_{\mathfrak{s},t})$ and unipotent classes in $Z_{\check{G}_{\mathfrak{s}}}(t)$. Every Levi subgroup of $\mathrm{GL}_e(\mathbb{C})$ has a unique distinguished unipotent class, and these exhaust the unipotent classes in $\mathrm{GL}_e(\mathbb{C})$. Hence $\mathbf{Irr}(W_{\mathfrak{s},t})$ is also in canonical bijection with the set of conjugacy classes of Levi subgroups $\check{M} \subset Z_{\check{G}_{\mathfrak{s}}}(t)$.
Viewed in this light, properties (3) and (4) entail that for every pair $(\check{M}, t)$ as above there is precisely one square-integrable modulo centre $\delta \in \mathbf{Irr}(M)$ such that $W_{\mathfrak{s}} t$ is the unitary part of the cuspidal support of $I_{PM}^G(\delta)$. Thus (3) and (4) determine the (tempered) G-representation associated to $[t, \rho] \in T_{\mathfrak{s},\mathrm{un}} // W_{\mathfrak{s}}$. $\square$

6 Twisted extended quotients for inner forms of SL_n

Twisted extended quotients appear naturally in the description of the Bernstein components for $L^\sharp Z(G)$ and $L^\sharp$.

Lemma 6.1. *Let $\mathfrak{s}_L = [L, \omega]_L$ and define a two-cocycle κ_ω by (11).*

(*a*) *Equation (13) for L determines bijections*

$$(T_{\mathfrak{s}} // X^L(\mathfrak{s}))_{\kappa_\omega} \to \mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp Z(G)),$$
$$(T_{\mathfrak{s}} // X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} = (T_{\mathfrak{s}} // X^L(\mathfrak{s}))_{\kappa_\omega} / X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp) \to \mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp).$$

(*b*) *The induced maps*

$$\mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp Z(G)) \to T_{\mathfrak{s}} / X^L(\mathfrak{s}) \quad \textit{and} \quad \mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp) \to T_{\mathfrak{s}} / X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)$$

are independent of the choice of κ_ω.

(*c*) *Let $T_{\mathfrak{s},\mathrm{un}}$ be the real subtorus of unitary representations in $T_{\mathfrak{s}}$. The subspace of tempered representations $\mathbf{Irr}^{\mathfrak{s}_L}_{\mathrm{temp}}(L^\sharp Z(G))$ corresponds to $(T_{\mathfrak{s},\mathrm{un}} // X^L(\mathfrak{s}))_{\kappa_\omega}$. Similarly $\mathbf{Irr}^{\mathfrak{s}_L}_{\mathrm{temp}}(L^\sharp)$ is obtained by restricting the second line of part (a) to $T_{\mathfrak{s},\mathrm{un}}$.*

Proof. (a) Apart from the equality, this is a reformulation of the last page of Section 2. For the equality, we note that by (27) the action of

$$X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp) \cong X_{\mathrm{nr}}(L/L^\sharp)/(X^L(\mathfrak{s}) \cap X_{\mathrm{nr}}(L/L^\sharp))$$

on $(T_{\mathfrak{s}}//X^L(\mathfrak{s}))_{\kappa_\omega}$ is free. Hence the isotropy groups for the action of $X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)$ are the same as for $X^L(\mathfrak{s})$, and we can use the same 2-cocycle κ_ω to construct a twisted extended quotient.
(b) By (13) a different choice of κ_ω in part (a) would only lead to the choice of another irreducible summand of $\mathrm{Res}^G_{G^\sharp}(\pi)$ for $\pi \in T_{\mathfrak{s}}$, and similarly for $G^\sharp Z(G)$.
(c) Since ω is supercuspidal, the set of tempered representations in $T_{\mathfrak{s}} = \mathbf{Irr}^{\mathfrak{s}_L}(L)$ is $T_{\mathfrak{s},\mathrm{un}}$. In the decomposition (13), an irreducible representation of $L^\sharp$ or $L^\sharp Z(G)$ is tempered if and only if it is contained in a tempered L-representation $\omega \otimes \chi$. This proves the statement for L. The claim for $L^\sharp$ follows upon dividing out the free action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$. □

The subgroup $X^L(\omega, V_\mu)$ acts trivially on $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, and for that reason it can be pulled out of the extended quotient from Lemma 6.1.

Lemma 6.2. *There are bijections*

$$(T_{\mathfrak{s}}//X^L(\mathfrak{s}))_{\kappa_\omega} \longleftrightarrow (T_{\mathfrak{s}}//X^L(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \times \mathbf{Irr}(X^L(\omega, V_\mu)),$$
$$(T_{\mathfrak{s}}//X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} \longleftrightarrow (T_{\mathfrak{s}}//X)_{\kappa_\omega} \times \mathbf{Irr}(X^L(\omega, V_\mu)),$$

where $X = X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)/X^L(\omega, V_\mu)$. They fix the coordinates in $T_{\mathfrak{s}}$.

Proof. In Lemma 6.1.a we saw that $(T_{\mathfrak{s}}//X^L(\mathfrak{s}))_{\kappa_\omega}$ is in bijection with $\mathbf{Irr}(\mathcal{H}(L^\sharp Z(G))^{\mathfrak{s}_L})$. By [ABPS4, (169)] $\mathcal{H}(L^\sharp Z(G))^{\mathfrak{s}_L}$ is Morita equivalent with $(\mathcal{H}(L)^{\mathfrak{s}_L})^{X^L(\mathfrak{s})}$ and with the subalgebra

$$\begin{aligned} e_L^{\mathfrak{s}} \mathcal{H}(L)^{X^L(\mathfrak{s})} e_L^{\mathfrak{s}} &\cong (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(e_L^{\mathfrak{s}} V_\omega))^{X^L(\mathfrak{s})} \\ &\cong \bigoplus\nolimits_{a \in [L/H_\lambda]} (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})/X^L(\omega, V_\mu)}. \end{aligned} \tag{69}$$

Here the $X^L(\mathfrak{s})$-action on the middle term comes from an isomorphism

$$\mathrm{End}_{\mathbb{C}}(e_L^{\mathfrak{s}} V_\omega) \cong \mathrm{End}_{\mathbb{C}}(V_\mu) \otimes \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}[L/H_\lambda]^*.$$

We recall that by [ABPS4, Lemma 3.5] there is a group isomorphism

$$L/H_\lambda \cong \mathbf{Irr}(X^L(\omega, V_\mu)). \tag{70}$$

By the above Morita equivalences, Lemma 6.1.a and Clifford theory, the set

$$\{\rho \in \mathbf{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega]) : \rho|_{X^L(\omega, V_\mu)} = \mathrm{triv}\} \tag{71}$$

parametrizes the irreducible representations of (69) associated to a fixed $\chi \in T_{\mathfrak{s}}$ and the trivial character of $X^L(\omega, V_\mu)$.

For any $\chi \in T_{\mathfrak{s}}$, the $X^L(\mathfrak{s})/X^L(\omega, V_\mu)$-stabilizer of the irreducible representation $\mathbb{C}_\chi \otimes V_\mu$ of $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}(V_\mu)$ is $X^L(\omega)/X^L(\omega, V_\mu)$. It follows that the irreducible representations of (69) with fixed $a \in [L/H_\lambda]$ and fixed $\mathcal{O}(T_{\mathfrak{s}})$-character χ are in bijection with $\mathbf{Irr}(\mathrm{End}_{\mathbb{C}}(V_\mu)^{X^L(\omega)/X^L(\omega,V_\mu)})$. Comparing with (71), we see that every irreducible representation of $\mathbb{C}[X^L(\omega)/X^L(\omega, V_\mu), \kappa_\omega]$ appears in V_μ. This is equivalent to each irreducible representation of $\mathrm{End}_{\mathbb{C}}(V_\mu) \rtimes X^L(\omega)/X^L(\omega, V_\mu)$ having nonzero vectors fixed by $X^L(\omega)/X^L(\omega, V_\mu)$. Thus Lemma A.2 can be applied to $X^L(\omega)/X^L(\omega, V_\mu)$ acting on $\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, and it shows that the irreducible representations on the right hand side of (69) are in bijection with

$$(T_{\mathfrak{s}} /\!/ X^L(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \times \mathbf{Irr}(X^L(\omega, V_\mu)).$$

The second bijection follows by dividing out the free action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$, as in the proof of Lemma 6.1.a. $\square$

As a result of the work in Section 4, twisted extended quotients can also be used to describe the spaces of irreducible representations of $G^\sharp Z(G)$ and $G^\sharp$. Let us extend κ_ω to a two-cocycle of $\mathrm{Stab}(\mathfrak{s})$, trivial on the normal subgroup $W_{\mathfrak{s}} \times X^L(\omega, V_\mu)$, by

$$J(\gamma, \omega) J(\gamma', \omega) = \kappa_\omega(\gamma, \gamma') J(\gamma\gamma', \omega) \qquad \gamma, \gamma' \in X^G(\mathfrak{s}). \tag{72}$$

Theorem 6.3. (*a*) *Lemmas A.1 and A.2 gives rise to bijections*

$$(T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \to \mathbf{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp\big),$$
$$(T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)/X^L(\omega, V_\mu))_{\kappa_\omega} \to \mathbf{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp\big).$$

(*b*) *The stratified equivalences from 4.4 provide bijections*

$$(T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s}))_{\kappa_\omega} \to (T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \times \mathbf{Irr}(X^L(\omega, V_\mu)) \to \mathbf{Irr}^{\mathfrak{s}}(G^\sharp Z(G)),$$
$$(T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} \to (T_{\mathfrak{s}} /\!/ S)_{\kappa_\omega} \times \mathbf{Irr}(X^L(\omega, V_\mu)) \to \mathbf{Irr}^{\mathfrak{s}}(G^\sharp),$$

where $S = \mathrm{Stab}(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)/X^L(\omega, V_\mu)$.

(*c*) *In part (b)* $\mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G^\sharp Z(G))$ *(respectively* $\mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G^\sharp)$*) corresponds to the same extended quotient, only with* $T_{\mathfrak{s},\mathrm{un}}$ *instead of* $T_{\mathfrak{s}}$.

Proof. In each of the three parts the second claim follows from the first upon dividing out the action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$, like in Lemma 6.1.a
(a) In the proof of Lemma 6.2 we exhibited a bijection

$$(T_{\mathfrak{s}} /\!/ \mathrm{Stab}(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \longleftrightarrow \mathbf{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})}\big).$$

With Lemma A.2 we deduce a Morita equivalence

$$(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \sim_M (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes (X^L(\mathfrak{s})/X^L(\omega, V_\mu)). \tag{73}$$

In the notation of (98) this means that $p := p_{X^L(\mathfrak{s})/X^L(\omega,V_\mu)}$ is a full idempotent in the right hand side of (73), that is, the two-sided ideal it generates is the entire algebra. Then p is also full in

$$(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes (\mathrm{Stab}(\mathfrak{s})/X^L(\omega, V_\mu)), \tag{74}$$

which implies that (74) is Morita equivalent with

$$p\big((\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes (\mathrm{Stab}(\mathfrak{s})/X^L(\omega, V_\mu))\big)p \cong \\ (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})/X^L(\omega,V_\mu)} \rtimes (\mathrm{Stab}(\mathfrak{s})/X^L(\mathfrak{s})).$$

As a direct consequence of (14), (16) and (17),

$$\mathrm{Stab}(\mathfrak{s})/X^L(\mathfrak{s}) \cong W_{\mathfrak{s}}^\sharp.$$

In this way we reach the algebra featuring in part (a). By the above Morita equivalence, its irreducible representations are in bijection with those of (74). Apply Lemma A.1.a to the latter algebra.
(b) All the morphisms in (56) are spectrum preserving with respect to filtrations. In combination with the other remarks in the proof of Theorem 4.4.a this gives a bijection

$$\mathbf{Irr}^{\mathfrak{s}}(G^\sharp Z(G)) \to \mathbf{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp\big) \times [L/H_\lambda]. \tag{75}$$

By part (a) and (70) the right hand side of (75) is in bijection with

$$(T_{\mathfrak{s}}//\mathrm{Stab}(\mathfrak{s})/X^L(\omega, V_\mu))_{\kappa_\omega} \times \mathbf{Irr}(X^L(\omega, V_\mu)). \tag{76}$$

The group $X^L(\mathfrak{s})$ acts on $\mathbb{C}[L]$ by pointwise multiplication of functions on L. That gives rise to actions on $\mathbb{C}[L/H_\lambda]$ and on

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}[L/H_\lambda]) \cong \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}[L/H_\lambda]^*.$$

Regarding $\mathbb{C}[L/H_\lambda]$ as the algebra $\bigoplus_{a \in [H/H_\lambda]} \mathbb{C}$, (70) leads to an isomorphism

$$\begin{aligned} &(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp \otimes \mathbb{C}[L/H_\lambda] \cong \\ &(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp. \end{aligned} \tag{77}$$

We note that (76) is also the space of irreducible representations of (77). In the proof of Lemma 6.2 we encountered a bijection

$$(T_{\mathfrak{s}}//X^L(\mathfrak{s}))_{\kappa_\omega} \longleftrightarrow \mathbf{Irr}\big((\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})}\big).$$

It implies a Morita equivalence

$$(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \sim_M (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda])) \rtimes X^L(\mathfrak{s}). \blacksquare$$

Just as in the proof of part (a), this extends to

$$\begin{aligned}(\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{s}}^\sharp \sim_M \\ (\mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda])) \rtimes \mathrm{Stab}(\mathfrak{s}).\end{aligned} \tag{78}$$

Finally we apply Lemma A.1.a to the right hand side and we combine it with (78), (77) and (75).

(c) The first bijection in part (b) obviously preserves the subspaces associated to $T_{\mathfrak{s},\mathrm{un}}$. We need to show that the second bijection sends them to $\mathbf{Irr}^{\mathfrak{s}}_{\mathrm{temp}}(G^\sharp Z(G))$. This is a property of the geometric equivalences in Theorem 4.4, as we will now check.

We may and will assume that ω is unitary, or equivalently that it is tempered. The Morita equivalence between $\mathcal{H}(G^\sharp Z(G))^{\mathfrak{s}}$ and (55) is induced by an idempotent $e^\sharp_{\lambda_{G^\sharp Z(G)}} \in \mathcal{H}(G^\sharp Z(G))$, see Theorem 3.1. Its construction (which starts around (34)) shows that eventually it comes from a central idempotent in the algebra of a profinite group, so it is a self-adjoint element. Hence, by [BHK, Theorem A] this Morita equivalence preserves temperedness. The notion of temperedness in [BHK] agrees with temperedness for representations of affine Hecke algebras (see [Opd]) because both are based on the Hilbert algebra structure and the canonical tracial states on these algebras.

The sequence of algebras (56) is derived from its counterpart for $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$. By Theorem 5.1 that one matches tempered representations with $T_{\mathfrak{s},\mathrm{un}}//W_{\mathfrak{s}}$. By Clifford theory any irreducible representation π of

$$(\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}} \tag{79}$$

is contained in a sum of irreducible representations $\tilde{\pi}$ of $\mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$, which are all in the same $\mathrm{Stab}(\mathfrak{s})$-orbit. Temperedness of π depends only on the action of the subalgebra $\mathcal{O}(T_{\mathfrak{s}}) \cong \mathbb{C}[X^*(T_{\mathfrak{s}})]$, and in fact can already be detected on $\mathbb{C}[X]$ for any finite index sublattice $X \subset X^*(T_{\mathfrak{s}})$. The analogous statement for (79) holds as well, with $X = X^*(T_{\mathfrak{s}}/X^L(\mathfrak{s}))$, and it is stable under the action of $\mathrm{Stab}(\mathfrak{s})$. Consequently π is tempered if and only if $\tilde{\pi}$ is tempered.

These observations imply that the sequence of algebra homomorphisms (56) preserves temperedness of irreducible representations, and that it maps such representations of (79) to irreducible representations of (58) with $\mathcal{O}(T_{\mathfrak{s}}/X^L(\mathfrak{s}))$-weights in $T_{\mathfrak{s},\mathrm{un}}/X^L(\mathfrak{s})$.

Now we invoke this property for every $a \in L/H_\lambda \cong \mathbf{Irr}(X^L(\omega, V_\mu))$ and we deduce that the second map in part (b) has the required property with respect to temperedness. □

We will work out what Theorem 6.3 says for a single Bernstein component of $G^\sharp$. To this end, we first analyse what parabolic induction from $L^\sharp$ to $G^\sharp$ looks like in the setting of Theorem 3.2.

Theorem 6.4. (*a*) *There exist idempotents* $e_L^{\mathfrak{s}} \in \mathcal{H}(L), e_{L^\sharp}^{\mathfrak{s}} \in \mathcal{H}(L^\sharp)$ *such that* $\mathcal{H}(L^\sharp)^{\mathfrak{s}}$ *is Morita equivalent with*

$$e_{L^\sharp}^{\mathfrak{s}} \mathcal{H}(L^\sharp) e_{L^\sharp}^{\mathfrak{s}} \cong e_L^{\mathfrak{s}} \mathcal{H}(L)^{X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)} e_L^{\mathfrak{s}} \cong \big(\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]) \big)^{X^L(\mathfrak{s})}$$
$$\cong \bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal{O}(T_{\mathfrak{s}}^\sharp) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \big)^{X^L(\mathfrak{s})/X^L(\omega, V_\mu)}.$$

(*b*) *Under the equivalences from part (a) and Theorem 3.2, the normalized parabolic induction functor*

$$I_{P^\sharp}^{G^\sharp} : \mathrm{Rep}^{\mathfrak{s}_L}(L^\sharp) \to \mathrm{Rep}^{\mathfrak{s}}(G^\sharp)$$

corresponds to induction from the last algebra in part (a) to

$$\bigoplus\nolimits_{a \in [L/H_\lambda]} \big(\mathcal{H}(T_{\mathfrak{s}}^\sharp, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu) \big)^{X^L(\mathfrak{s})/X^L(\omega, V_\mu)} \rtimes \mathfrak{R}_{\mathfrak{s}}^\sharp.$$

Proof. Part (a) is a consequence of [ABPS4, (169)] and [ABPS4, Lemma 4.8], which shows that

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}[L/H_\lambda]) \cong \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}[X^L(\omega, V_\mu)].$$

The analogue of part (b) for L and G says that

$$I_P^G : \mathrm{Rep}^{\mathfrak{s}_L}(L) \to \mathrm{Rep}^{\mathfrak{s}}(G)$$

corresponds to induction from

$$e_L^{\mathfrak{s}} \mathcal{H}(L) e_L^{\mathfrak{s}} \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda]) \quad \text{to}$$
$$e_{\lambda_G}^\sharp \mathcal{H}(G) e_{\lambda_G}^\sharp \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu \otimes \mathbb{C}[L/H_\lambda] \otimes \mathbb{C}\mathfrak{R}_{\mathfrak{s}}^\sharp).$$

To see that it is true, we reduce with [ABPS4, Theorem 4.5] to the algebras

$$e_{\lambda_L} \mathcal{H}(L) e_{\lambda_L} \cong \mathcal{O}(T_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda),$$
$$e_{\lambda_G} \mathcal{H}(G) e_{\lambda_G} \cong \mathcal{H}(T_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\lambda).$$

Then we are in the situation where (K_G, λ_G) is a cover of a type (K_L, λ_L), and the statement about the induction functors follows from [ABPS4, (126)] and [BuKu, Corollary 8.4].
We note that here, for a given algebra homomorphism $\phi : A \to B$, we must use induction in the version $\mathrm{Ind}_A^B(M) = \mathrm{Hom}_A(B, M)$. However, in all the cases we encounter B is free of finite rank as a module over A and it is endowed with a canonical anti-involution

$$f \mapsto [f^\vee : g \mapsto \overline{f(g^{-1})}].$$

Hence we may identify $\mathrm{Hom}_A(B, M) \cong B^* \otimes_A M \cong B \otimes_A M$. This shows the desired claim for I_P^G.

Since $G^\sharp/P^\sharp \cong G/P$, $I_P^G = I_{P^\sharp}^{G^\sharp}$ on $\mathrm{Rep}^{\mathfrak{s}_L}(L)$. In particular

$$\mathrm{Res}_{G^\sharp}^G \circ I_P^G = I_{P^\sharp}^{G^\sharp} \circ \mathrm{Res}_{L^\sharp}^L \quad \text{as functors} \quad \mathrm{Rep}^{\mathfrak{s}_L}(L) \to \mathrm{Rep}^{\mathfrak{s}}(G^\sharp). \tag{80}$$

The functor $\mathrm{Res}_{L^\sharp}^L$ corresponds to $\mathrm{Res}_{e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}}^{e^\sharp_L\mathcal{H}(L)e^\sharp_L}$, and $\mathrm{Res}_{G^\sharp}^G$ to restriction from $e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}$ to $e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}}$, which is the algebra appearing in the statement of part (b).
The set $\mathcal{H}(W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathbb{C}[\mathfrak{R}^\sharp_{\mathfrak{s}}]$ forms a basis for $e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}$ as a module over $e^\sharp_L\mathcal{H}(L)e^\sharp_L$ and for $e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}}$ as a module over $e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}$. It follows that

$$\mathrm{Res}_{e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}}}^{e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}} \circ \mathrm{ind}_{e^\sharp_L\mathcal{H}(L)e^\sharp_L}^{e^\sharp_{\lambda_G}\mathcal{H}(G)e^\sharp_{\lambda_G}} = \mathrm{ind}_{e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}}^{e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}}} \circ \mathrm{Res}_{e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}}^{e^\sharp_L\mathcal{H}(L)e^\sharp_L}. \tag{81}$$

Comparing (80) and (81), we find that

$$I_{P^\sharp}^{G^\sharp} \circ \mathrm{Res}_{L^\sharp}^L \quad \text{corresponds to} \quad \mathrm{ind}_{e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}}^{e^\sharp_{\lambda_{G^\sharp}}\mathcal{H}(G^\sharp)e^\sharp_{\lambda_{G^\sharp}}} \circ \mathrm{Res}_{e^{\mathfrak{s}}_{L^\sharp}\mathcal{H}(L^\sharp)e^{\mathfrak{s}}_{L^\sharp}}^{e^\sharp_L\mathcal{H}(L)e^\sharp_L} \tag{82}$$

under the Morita equivalences from Theorems 3.2 and 6.4.a. Here both inductions can be constructed entirely in $\mathcal{H}(G^\sharp)$. The two sides of (82) are then related by applications of the idempotents $e^{\mathfrak{s}}_{L^\sharp}$ and $e^\sharp_{\lambda_{G^\sharp}}$ (which are supported on $G^\sharp$). Hence the correspondence (82) preserves $L^\sharp$-subrepresentations. Since every irreducible $L^\sharp$-representation appears as a summand of an L-representation, this implies the analogue of (82) on the whole of $\mathrm{Rep}^{\mathfrak{s}_L}(L^\sharp)$. □

Let $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$ be an inertial equivalence class for $G^\sharp$, with $\mathfrak{t}^\sharp \prec \mathfrak{s} = [L, \omega]_G$. We abbreviate $\phi_{\omega, X^L(\mathfrak{s})} = \{\phi_{\omega,\gamma} : \gamma \in X^L(\mathfrak{s})\}$, where $\phi_{\omega,\gamma}$ is as in Lemma 2.2. By (27) there is a unique $X^L(\mathfrak{s})$-orbit

$$\phi_{\omega, X^L(\mathfrak{s})}\, \rho \ \subset\ \mathbf{Irr}(\mathbb{C}[X^L(\omega), \kappa_\omega]) \tag{83}$$

such that $T_{\mathfrak{t}^\sharp} = (T^\sharp_{\mathfrak{s}} \times \phi_{\omega, X^L(\mathfrak{s})}\rho)/X^L(\mathfrak{s})$. Then $\phi_{\omega, X^L(\mathfrak{s})}\rho$ determines a unique summand $\mathbb{C}a$ of $\mathbb{C}[L/H_\lambda] \cong \bigoplus_{[L/H_\lambda]} \mathbb{C}$, namely the irreducible representation of $X^L(\omega, V_\mu)$ obtained by restricting ρ. Let $V_{\sigma^\sharp}$ be the intersection of V_μ with the subspace of V_ω on which $\sigma^\sharp$ is defined, and let $\mathfrak{R}_{\mathfrak{t}^\sharp}$ be its stabilizer in $\mathfrak{R}^\sharp_{\mathfrak{s}}$. Then $\mathfrak{R}_{\mathfrak{t}^\sharp}$ is also the stabilizer of $\mathfrak{t}^\sharp$ in $\mathfrak{R}^\sharp_{\mathfrak{s}}$ and

$$W_{\mathfrak{t}^\sharp} = W_{\mathfrak{s}} \rtimes \mathfrak{R}_{\mathfrak{t}^\sharp}, \tag{84}$$

by [ABPS4, Lemma 2.3]. Via the formula (72) the operators $J(\gamma, \omega)|_{V_{\sigma^\sharp}}$ determine a 2-cocycle κ'_ω of the group

$$W' = \{(w, \gamma) \in \mathrm{Stab}(\mathfrak{s}) : w \in W_{\mathfrak{t}^\sharp}\}. \tag{85}$$

Since (72) is 1 on $W_{\mathfrak{s}}$, so is κ'_ω. By (15) $W'/X^L(\mathfrak{s}) \cong W_{\mathfrak{t}^\sharp}$. As $V_{\sigma^\sharp}$ is associated to the single $X^L(\mathfrak{s})$-orbit (83), $\kappa'_\omega((w,\gamma),(w',\gamma'))$ depends only on (w,w'). Thus it determines a 2-cocycle $\kappa_{\sigma^\sharp}$ of $W_{\mathfrak{t}^\sharp}$, which factors through $\mathfrak{R}_{\mathfrak{t}^\sharp} \cong W_{\mathfrak{t}^\sharp}/W_{\mathfrak{s}}$.

Lemma 6.5. (*a*) *The bijections in Theorem 6.3 restrict to*

$$\mathbf{Irr}^{\mathfrak{t}^\sharp}(G^\sharp) \longleftrightarrow (T_{\mathfrak{t}^\sharp}//W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}},$$
$$\mathbf{Irr}^{\mathfrak{t}^\sharp}_{\mathrm{temp}}(G^\sharp) \longleftrightarrow (T_{\mathfrak{t}^\sharp,\mathrm{un}}//W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}},$$

where $T_{\mathfrak{t}^\sharp,\mathrm{un}}$ denotes the space of unitary representations in $T_{\mathfrak{t}^\sharp}$.

(*b*) *Suppose $\pi \in \mathbf{Irr}^{\mathfrak{t}^\sharp}_{\mathrm{temp}}(G^\sharp)$ corresponds to $[t,\rho]$ and has cuspidal support $W_{\mathfrak{t}^\sharp}(\chi \otimes \sigma^\sharp) \in T_{\mathfrak{t}^\sharp}/W_{\mathfrak{t}^\sharp}$. Then $W_{\mathfrak{t}^\sharp}t$ is the unitary part of $\chi \otimes \sigma^\sharp$, with respect to the polar decomposition*

$$T_{\mathfrak{t}^\sharp} = T_{\mathfrak{t}^\sharp,\mathrm{un}} \times \mathrm{Hom}_{\mathbb{Z}}(X^*(T_{\mathfrak{t}^\sharp}), \mathbb{R}_{>0}).$$

Proof. (a) Recall that $\mathbf{Irr}^{\mathfrak{t}^\sharp}(G^\sharp)$ consists of those irreducible representations that are contained in $I_{P^\sharp}^{G^\sharp}(\chi \otimes \sigma^\sharp)$ for some $\chi \otimes \sigma^\sharp \in T_{\mathfrak{t}^\sharp}$. In Theorem 6.4.b we translated $I_{P^\sharp}^{G^\sharp}$ to induction between two algebras. The first one, Morita equivalent with $\mathcal{H}(L^\sharp)^{\mathfrak{s}_L}$, was

$$\mathbb{C}[L/H_\lambda] \otimes \big(\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})}.$$

The second algebra, Morita equivalent with $\mathcal{H}(G^\sharp)^{\mathfrak{s}}$, was

$$\mathbb{C}[L/H_\lambda] \otimes \big(\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}.$$

As above, the $L^\sharp$-representation $\sigma^\sharp$ determines a summand $\mathbb{C}a$ of $\mathbb{C}[L/H_\lambda]$ and a $X^L(\mathfrak{s})$-stable subspace $V_{\sigma^\sharp} \subset V_\mu$. Consequently $\mathcal{H}(L^\sharp)^{[L^\sharp,\sigma^\sharp]}$ is Morita equivalent with the two algebras

$$\big(\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp})\big)^{X^L(\mathfrak{s})} \quad \text{and} \quad \big(\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}} \cdot V_{\sigma^\sharp})\big)^{X^L(\mathfrak{s})}. \tag{86}$$

In these terms Theorem 6.4 shows that $\mathcal{H}(G^\sharp)^{\mathfrak{t}^\sharp}$ is is Morita equivalent with

$$\big(\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}} \cdot V_{\sigma^\sharp})\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}^\sharp_{\mathfrak{s}}. \tag{87}$$

Here the subspaces $wV_{\sigma^\sharp}$ with $w \in \mathfrak{R}^\sharp_{\mathfrak{s}}$ are permuted transitively by $\mathfrak{R}^\sharp_{\mathfrak{s}}$, so upon taking $\mathfrak{R}^\sharp_{\mathfrak{s}}$-invariants only the $\mathfrak{R}_{\mathfrak{t}^\sharp}$ on $V_{\sigma^\sharp}$ survives. Recall that, for any finite group Γ and any Γ-algebra A:

$$A \rtimes \Gamma \cong (A \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}\Gamma))^\Gamma. \tag{88}$$

Applying (88) to (87) first with $\Gamma = \mathfrak{R}^\sharp_{\mathfrak{s}}$ and subsequently with $\Gamma = \mathfrak{R}_{\mathfrak{t}^\sharp}$ (in the opposite direction, taking the above transitivity into account), we find that (87) is Morita equivalent with

$$\big(\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp})\big)^{X^L(\mathfrak{s})} \rtimes \mathfrak{R}_{\mathfrak{t}^\sharp}. \tag{89}$$

The constructions in Section 4 restrict to stratified equivalences between (87) and

$$\begin{aligned} &(\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}\mathfrak{R}^\sharp_{\mathfrak{s}} \cdot V_{\sigma^\sharp}))^{X^L(\mathfrak{s})} \rtimes W^\sharp_{\mathfrak{s}}, \\ &(\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}))^{X^L(\mathfrak{s})} \rtimes W_{\mathfrak{t}^\sharp}. \end{aligned} \tag{90}$$

By (86)

$$\mathbf{Irr}\big((\mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}))^{X^L(\mathfrak{s})}\big) \cong T_{\mathfrak{t}^\sharp}. \tag{91}$$

As explained above with (85), the 2-cocycle κ_ω of $\mathrm{Stab}(\mathfrak{s})$ reduces to the 2-cocycle $\kappa_{\sigma^\sharp}$ for the action of $W_{\mathfrak{t}^\sharp}$ in (90). Now we apply Lemma A.1.a to (90) and we find the first bijection. To obtain the second bijection, we use Theorem 6.3.c.

(b) For the stratified equivalence between

$$\mathcal{H}(T^\sharp_{\mathfrak{s}}, W_{\mathfrak{s}}, q_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}) \text{ and } \mathcal{O}(T^\sharp_{\mathfrak{s}}) \otimes \mathrm{End}_{\mathbb{C}}(V_{\sigma^\sharp}) \rtimes W_{\mathfrak{s}}$$

the analogous claim about the cuspidal support is property (3) of Theorem 5.1. Clifford theory relates the irreducible representations of these algebras to those of (87) and (90), in a way already discussed after (79). This implies that the desired property of the cuspidal support persists to the stratified equivalence between (87) and (90), which underlies part (a). □

7 Relation with the local Langlands correspondence

We show how the local Langlands correspondence (LLC) for G and $G^\sharp$ can be reconstructed in terms of twisted extended quotients.

Let $\mathbf{W}_F$ be the Weil group of the local non-archimedean field F. Recall that the Langlands dual group of $G = \mathrm{GL}_m(D)$ is $\check{G} = \mathrm{GL}_{md}(\mathbb{C})$. A Langlands parameter for G is continuous group homomorphism $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \check{G}$ such that:

- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is a homomorphism of algebraic groups.
- $\phi(\mathbf{W}_F)$ consists of semisimple elements.
- ϕ is relevant for G: if $\check{L}$ is a Levi subgroup of $\check{G}$ which contains $\mathrm{im}(\phi)$ and is minimal for that property, then (the conjugacy class of) $\check{L}$ corresponds to (the conjugacy class of) a Levi subgroup of G.

We denote the collection of Langlands parameters for G, modulo conjugation by $\check{G}$, by $\Phi(G)$.
Every smooth character of G is of the form $\nu \circ \mathrm{Nrd}$, with ν a smooth character of $F^\times$. Via Artin reciprocity it determines a Langlands parameter (trivial on $\mathrm{SL}_2(\mathbb{C})$)

$$\hat{\nu} : \mathbf{W}_F \to \mathbb{C}^\times \cong Z(\mathrm{GL}_{md}(\mathbb{C})). \tag{92}$$

For any $\phi \in \Phi(G)$, $\phi\hat{\nu}$ is a well-defined element of $\Phi(G)$ because the image of $\hat{\nu}$ is central in $\check{G}$.

Theorem 7.1. *The local Langlands correspondence for G is a canonical bijection*

$$\mathrm{rec}_{D,m} : \mathbf{Irr}(G) \to \Phi(G)$$

with the following properties:

(*a*) $\pi \in \mathbf{Irr}(G)$ *is tempered if and only if* $\mathrm{rec}_{D,m}(\pi)$ *is bounded, that is, if* $\mathrm{rec}_{D,m}(\pi)(\mathbf{W}_F)$ *is a bounded subset of* $\check{G}$.

(*b*) *The L-packet* $\Pi_\phi(G)$ *is the single representation* $\mathrm{rec}_{D,m}^{-1}(\phi)$.

(*c*) $\mathrm{rec}_{D,m}$ *is equivariant for the two actions of* $\mathbf{Irr}(G/G^\sharp)$*: on* $\mathbf{Irr}(G)$ *by twisting with smooth characters and on* $\Phi(G)$ *by multiplication with central Langlands parameters as in* (92).

Proof. For the bijection and part (a) see [HiSa, §11] and [ABPS3, §2]. Ultimately it relies on the Jacquet–Langlands correspondence from [DKV, Bad].
(b) This is a direct consequence of the bijectivity.
(c) Since $\mathrm{rec}_{D,m}$ is determined completely by its behaviour on essentially square integrable representations of Levi subgroups of G [ABPS3, (13)], it suffices to prove (c) for such representations. Via the Jacquet–Langlands correspondence the issue can be transferred to $\mathbf{Irr}(\mathrm{GL}_n(F))$ with $n \leq md$. For general linear groups (c) is a well-known property of the LLC, and in fact constitutes a starting point of the construction, confer [Hen, 1.2]. □

For $\mathfrak{s} = [L, \omega]_G$ we define $\Phi(G)^{\mathfrak{s}}$ as the image of $\mathbf{Irr}^{\mathfrak{s}}(G)$ under the bijection $\mathrm{rec}_{D,m}$. Similarly we define $\Phi(L)^{\mathfrak{s}_L} \subset \Phi(L)$.

Lemma 7.2. *The* LLC *for G fits in a commutative diagram of canonical bijections*

$$\begin{array}{ccc} \mathbf{Irr}^{\mathfrak{s}}(G) & \xrightarrow{\mathrm{rec}_{D,m}} & \Phi(G)^{\mathfrak{s}} \\ \updownarrow & & \updownarrow \\ T_{\mathfrak{s}}//W_{\mathfrak{s}} & \longleftrightarrow & \Phi(L)^{\mathfrak{s}_L}//W_{\mathfrak{s}} \end{array}$$

Here the bottom map comes from the LLC *for* $\mathbf{Irr}^{\mathfrak{s}_L}(L)$ *and the left hand side comes from Theorem 5.1.*

(a) *Suppose that* $[\phi_L] \in \Phi(L)^{\mathfrak{s}_L}$ *and that* $\rho \in \mathbf{Irr}(W_{\mathfrak{s},\phi_L})$ *has as Springer parameter a unipotent class* $[u] \in Z_{\check{G}_{\mathfrak{s}}}(\phi_L)$. *Then there is a representative* u *such that the right hand side sends* $[\phi_L, \rho]$ *to a Langlands parameter* ϕ *with* $\phi|_{\mathbf{W}_F} = \phi_L|_{\mathbf{W}_F}$ *and* $\phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = \phi_L(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))u$.

(b) *Two elements* $[t, \rho]$ *and* $[t', \rho']$ *of* $T_{\mathfrak{s}}//W_{\mathfrak{s}}$ *map to the same* G*-representation if and only if there exists a* $w \in W_{\mathfrak{s}}$ *such that* $wt = t'$ *and the* $W_{\mathfrak{s},t}$*-representations* $\rho, w\rho'$ *have Springer parameters involving the same unipotent class.*

Proof. The top horizontal and left vertical maps have already been established as bijective and canonical. The LLC for L is the Cartesian product of the LLCs for the factors of L. Hence it is $W(G, L)$-equivariant, and its restriction to $T_{\mathfrak{s}} \longleftrightarrow \Phi(L)^{\mathfrak{s}_L}$ is $W_{\mathfrak{s}}$-equivariant. The canonicity and bijectivity of the LLC for L are inherited by the bottom horizontal map in the diagram. This leaves a unique, canonical way to complete the commutative diagram.
(a) To work out the map on the right hand side, it suffices to consider

$$L = \prod\nolimits_i L_i^{e_i} \quad \text{and} \quad \omega = \prod\nolimits_i \omega_i^{e_i}$$

such that (L_i, ω_i) is not isomorphic to (L_j, ω_j) for $i \neq j$. Let $\phi_i : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{m_i d}(\mathbb{C})$ be a Langlands parameter for ω_i. Then

$$\phi_L = \prod\nolimits_i \phi_i^{e_i} : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \prod\nolimits_i \mathrm{GL}_{m_i d}(\mathbb{C})^{e_i}$$

is a Langlands parameter for ω. We have $W_{\mathfrak{s},\phi_L} = \prod_i S_{e_i}$, where S_{e_i} is embedded in $N_{\mathrm{GL}_{e_i d_i m}(\mathbb{C})}(\mathrm{GL}_{m_i d}(\mathbb{C})^{e_i})$ as permutation matrices. The unipotent class

$$[u] = [\prod\nolimits_i u_i] \in \prod\nolimits_i \mathrm{GL}_{e_i m_i d}(\mathbb{C}) \subset Z_{\check{G}_{\mathfrak{s}}}(\phi_L)$$

is determined by the standard Levi subgroup in which it is distinguished, say

$$\check{M} = \prod\nolimits_{i,j} \mathrm{GL}_{b_{ij} m_i d}(\mathbb{C})^{c_{ij}} \quad \text{with} \quad \sum\nolimits_j c_{ij} b_{ij} = e_i.$$

Assume for the moment that ω is tempered. By Theorem 5.1 $[\omega, \rho] \in T_{\mathfrak{s},\mathrm{un}}//W_{\mathfrak{s}}$ corresponds to $I_{PM}^G(\delta)$, where

$$\delta = \prod\nolimits_{i,j} \delta_{ij}^{c_{ij}} \in \mathbf{Irr}_{\mathrm{temp}}^{[L,\omega]_M}(M)$$

is the unique square-integrable modulo centre representation such that $W_{\mathfrak{s},M}\omega$ is the unitary part of the cuspidal support of δ. By construction [ABPS4, §2] the Langlands parameter ϕ of $I_{PM}^G(\delta)$ is the same as that of δ, namely $\phi = \prod_{i,j} \phi_{ij}^{c_{ij}}$ with $\phi_{ij}|_{\mathbf{W}_F} = \phi_i^{b_{ij}}|_{\mathbf{W}_F}$ and

$$\phi_{ij}(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = \phi_i(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))^{b_{ij}} u_{ij}$$

where u_{ij} is a distinguished unipotent element in $Z_{\mathrm{GL}_{b_{ij} m_i d}(\mathbb{C})}(\mathrm{GL}_{m_i d}(\mathbb{C})^{b_{ij}})$. Thus $\phi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ is distinguished in $\check{M}$ and ϕ has the asserted shape.
The general case, where ω is not necessarily tempered, follows from the tempered case. The reason is that all the maps in the commutative diagram (a priori except the right hand side) can be obtained from their tempered parts by some kind of analytic continuation, as in [ABPS1] and Theorem 5.1.
(b) By Theorem 7.1.b the elements of $T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}}$ are in bijection with the L-packets in $\mathbf{Irr}^{\mathfrak{s}}(G)$. Two elements $[t, \rho]$ and $[t', \rho']$ are equal if and only if there is a $w \in W_{\mathfrak{s}}$ such that $wt' = t$ and $w \cdot \rho' = \rho$. We note also that for every $t \in T_{\mathfrak{s}}$ the group $W_{\mathfrak{s},t} = W(R_{\mathfrak{s},t})$ is product of symmetric groups. Hence all irreducible representations of $W_{\mathfrak{s},t}$ are parametrized by different unipotent classes in a connected complex reductive group with maximal torus $T_{\mathfrak{s}}$ and root system $R_{\mathfrak{s},t}$. So the condition becomes that ρ and $w \cdot \rho'$ have the same unipotent class as Springer parameter. □

Let $\mathbf{Irr}_{\mathrm{cusp}}(L)$ be the space of supercuspidal L-representations and let $\Phi(L)_{\mathrm{cusp}}$ be its image in $\Phi(L)$. The Weyl group

$$W(G, L) = N_G(L)/L \cong N_{\check{G}}(\check{L})/\check{L}$$

acts naturally on both sets.

Theorem 7.3. *Let $\mathcal{L}$ be a set of representatives for the conjugacy classes of Levi subgroups of G. The maps from Lemma 7.2 combine to a commutative diagram of canonical bijections*

$$\begin{array}{ccc} \mathbf{Irr}(G) & \xrightarrow{\mathrm{rec}_{D,m}} & \Phi(G) \\ \updownarrow & & \updownarrow \\ \bigsqcup_{L \in \mathcal{L}} \mathbf{Irr}_{\mathrm{cusp}}(L) /\!/ W(G, L) & \longleftrightarrow & \bigsqcup_{L \in \mathcal{L}} \Phi(L)_{\mathrm{cusp}} /\!/ W(G, L) \end{array}$$

Here the tempered representations correspond to the bounded Langlands parameters.

Proof. The action of $W(G, L)$ on L is simply by permuting some direct factors of L, and the same for $\check{L}$. Hence the canonical bijection $\mathbf{Irr}(L) \leftrightarrow \Phi(L)$ is $W(G, L)$-equivariant. The group $W_{\mathfrak{s}}$ is defined as the stabilizer in $W(G, L)$ of $T_{\mathfrak{s}} = \mathbf{Irr}^{\mathfrak{s}_L}(L)$, and by the above equivariance it is also the stabilizer of $\Phi^{\mathfrak{s}_L}(L)$. Consequently

$$\begin{aligned} \mathbf{Irr}_{\mathrm{cusp}}(L) /\!/ W(G, L) &\cong \bigsqcup\nolimits_{\mathfrak{s} = [L, \omega]_G} T_{\mathfrak{s}} /\!/ W_{\mathfrak{s}}, \\ \Phi(L)_{\mathrm{cusp}} /\!/ W(G, L) &\cong \bigsqcup\nolimits_{\mathfrak{s} = [L, \omega]_G} \Phi^{\mathfrak{s}_L}(L) /\!/ W_{\mathfrak{s}}. \end{aligned}$$

Now we simply take the union of the commutative diagrams of Lemma 7.2. The characterization of temperedness and boundedness comes from Theorems 7.1.a and 6.3.c. □

To formulate the LLC for $G^\sharp$, we need enhanced Langlands parameters. In fact these are already present in the LLC for G, but there the enhancement can be neglected without any problems.
Recall that a Langlands parameter for $G^\sharp = \mathrm{GL}_m(D)_{\mathrm{der}}$ is a homomorphism $\phi : \mathbf{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_{md}(\mathbb{C})$ subject to the same requirements as a Langlands parameter for G. The set of such parameters modulo conjugation by $\check{G}^\sharp = \mathrm{PGL}_{md}(\mathbb{C})$ is denoted $\Phi(G^\sharp)$. We note that the simply connected cover $\mathrm{SL}_{md}(\mathbb{C})$ of $\mathrm{PGL}_{md}(\mathbb{C})$ also acts by conjugation on Langlands parameters for $G^\sharp$.
An enhancement of ϕ is an irreducible representation ρ of $\pi_0(Z_{\mathrm{SL}_{md}(\mathbb{C})}(\phi))$. In order that (ϕ, ρ) is relevant for $G^\sharp$, an extra condition is needed. For this we have to regard D as part of the data of $G^\sharp$, in other words, we must consider not just the inner form $G^\sharp$ of $\mathrm{SL}_{md}(F)$, but even the inner twist determined by $(G^\sharp, D)$. The Hasse invariant of D gives a character χ_D of $Z(\mathrm{SL}_{md}(\mathbb{C})) \cong \mathbb{Z}/md\mathbb{Z}$ with kernel $m\mathbb{Z}/md\mathbb{Z}$. Notice that, by Schur's lemma, every enhancement ρ of ϕ determines a character of $Z(\mathrm{SL}_{md}(\mathbb{C}))$. We define an enhanced Langlands parameter for $G^\sharp = \mathrm{GL}_m(D)_{\mathrm{der}}$ as a pair (ϕ, ρ) such that $\rho|_{Z(\mathrm{SL}_{md}(\mathbb{C}))} = \chi_D$. The collection of these, modulo conjugation by $\mathrm{SL}_{md}(\mathbb{C})$, is denoted $\Phi_e(G^\sharp)$.
The LLC for $G^\sharp$ [ABPS3] is a bijection

$$\Phi_e(G^\sharp) \longleftrightarrow \mathbf{Irr}(G^\sharp) : (\phi, \rho) \mapsto \pi(\phi, \rho). \tag{93}$$

such that

(i) if ϕ lifts to a Langlands parameter $\tilde{\phi}$ for G, then $\pi(\phi, \rho)$ is a direct summand of $\mathrm{Res}^G_{G^\sharp}(\mathrm{rec}^{-1}_{D,m}(\tilde{\phi}))$,

(ii) $\pi(\phi, \rho)$ is tempered if and only if ϕ is bounded,

(iii) the L-packet

$$\Pi_\phi(G^\sharp) = \{\pi(\phi, \rho) : \rho \in \mathbf{Irr}(\pi_0(Z_{\mathrm{SL}_{md}(\mathbb{C})}(\phi))), \rho|_{Z(\mathrm{SL}_{md}(\mathbb{C}))} = \chi_D\}$$

is canonically determined.

As $\mathbf{Irr}^{\mathfrak{s}}(G^\sharp)$ is defined in terms of restriction from $\mathbf{Irr}^{\mathfrak{s}}(G)$, it is a union of L-packets for $G^\sharp$. With (i) it canonically determines a set $\Phi_e(G^\sharp)^{\mathfrak{s}}$ of enhanced Langlands parameters for $G^\sharp$.
In the same way as for G, the LLC for a Levi subgroup $L^\sharp = L \cap G^\sharp$ follows from that for $L = \prod_i \mathrm{GL}_{m_i}(D)$. It involves enhancements from the action of

$$(\check{L}^\sharp)_{sc} = \mathrm{SL}_{md}(\mathbb{C}) \cap \prod\nolimits_i \mathrm{GL}_{m_i d}(\mathbb{C}).$$

Given $\mathfrak{s}_L = [L, \omega]_L$, $\mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp)$ is a union of L-packets for $L^\sharp$. Hence the corresponding set $\Phi_e(L^\sharp)^{\mathfrak{s}_L}$ of enhanced Langlands parameters is well-defined.

Lemma 7.4. *The* LLC *for* $G^\sharp$ *and the maps from Lemma 6.1 and Theorem 6.3.b fit in the following commutative bijective diagram:*

$$\begin{array}{ccc}
\mathbf{Irr}^{\mathfrak{s}}(G^\sharp) & \longleftrightarrow & \Phi_e(G^\sharp)^{\mathfrak{s}} \\
\updownarrow & & \updownarrow \\
(\mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp)/\!/W_{\mathfrak{s}}^\sharp)_{\kappa_\omega} & \longleftrightarrow & (\Phi_e(L^\sharp)^{\mathfrak{s}_L}/\!/W_{\mathfrak{s}}^\sharp)_{\kappa_\omega} \\
\updownarrow & & \updownarrow \\
\big((T_{\mathfrak{s}}/\!/X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega}/\!/W_{\mathfrak{s}}^\sharp\big)_{\kappa_\omega} & \longleftrightarrow & \big((\Phi(L)^{\mathfrak{s}_L}/\!/X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega}/\!/W_{\mathfrak{s}}^\sharp\big)_{\kappa_\omega} \\
\updownarrow & & \updownarrow \\
\big(T_{\mathfrak{s}}/\!/\mathrm{Stab}(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)\big)_{\kappa_\omega} & \longleftrightarrow & \big(\Phi(L)^{\mathfrak{s}_L}/\!/\mathrm{Stab}(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)\big)_{\kappa_\omega} \\
\updownarrow & & \updownarrow \\
\big((T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})/\!/\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)\big)_{\kappa_\omega} & \longleftrightarrow & \big((\Phi(L)^{\mathfrak{s}_L}/\!/W_{\mathfrak{s}})/\!/\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)\big)_{\kappa_\omega}
\end{array}$$

All these maps are canonical up to permutations within L-packets. In the last row the collection of L-packets is in bijection with $(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})/\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)$ *and with* $(\Phi(L)^{\mathfrak{s}_L}/\!/W_{\mathfrak{s}})/\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)$.

Proof. The bijection between the first and the fourth set on the left hand side is given by Theorem 6.3.b. Then Corollary A.4 and (78) give bijections to the third and fifth sets on the left, as the 2-cocycle κ_ω is by construction (72) trivial on $W_{\mathfrak{s}}$. The bijection between the second and third sets on the left comes from Lemma 6.1.a. By Lemma 6.1.b it is canonical up to permutations within L-packets.

The LLC for L is equivariant for permutations of the direct factors of L and for twisting with characters of L (because the LLC for $\mathrm{GL}_m(D)$ is so). This gives the three lower horizontal bijections. Applying Corollary A.4 to the three lower terms on the right hand side gives bijections between them, and shows that the two lower squares in the diagram are canonical and commutative.

Similarly the LLC for $L^\sharp$ is equivariant for the action of $W_{\mathfrak{s}}^\sharp$, which leads to the second horizontal bijection. We define the upper two maps on the right hand side as the unique bijections that make the diagram commute. Since all the other maps in the upper two squares are canonical up to permutations within L-packets, so are the last two.

An L-packet for $G^\sharp$ consists of the irreducible $G^\sharp$-constituents of an irreducible G-representation. In view of Lemma 7.2, the collection of L-packets in $\mathbf{Irr}^{\mathfrak{s}}(G^\sharp)$ is canonically in bijection with $T_{\mathfrak{s}}/\!/W_{\mathfrak{s}}$. From (56) we can see how

$$\big((T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})/\!/\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)\big)_{\kappa_\omega}$$

is constructed on the level of representations. We take an element $\pi \in \mathbf{Irr}^{\mathfrak{s}}(G)$ and transform it to an irreducible representation of $\mathcal{O}(T_{\mathfrak{s}}) \rtimes W_{\mathfrak{s}}$ by a geometric

equivalence. Then we form the twisted extended quotient by $\mathrm{Stab}(\mathfrak{s})^+$, using Lemmas A.1 and A.2, which corresponds to identifying π with π' if they have the same restriction to $G^\sharp Z(G)$, and decomposing π in irreducible $G^\sharp Z(G)$-subrepresentations. Finally we divide out the action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$, thus identifying the $G^\sharp Z(G)$-representations with the same restriction to $G^\sharp$. The implies the description of the L-packets in the lower left term of the commutative diagram, and hence also in the lower right term. $\square$

The bijection between the upper and the lower term on the right hand side of Lemma 7.4 can also be obtained as follows. First apply the recipe from Lemma 7.2 to $\Phi(L)^{\mathfrak{s}_L}/\!/W_{\mathfrak{s}}$, then take the twisted extended quotient with respect to $\mathrm{Stab}(\mathfrak{s})^+$, and finally divide out the free action of $X_{\mathrm{nr}}(L^\sharp Z(G)/L^\sharp)$ to reach $\Phi_e(G^\sharp)^{\mathfrak{s}_L}$.
Let $R_{\mathfrak{t}^\sharp,t}$ be the root system associated to $(G^\sharp, t)$ be Harish-Chandra, by means of zeros of the μ-function [Wal, §V.2]. Recall that the classical Springer correspondence was extended to Weyl groups of disconnected complex reductive groups in [ABPS5, Theorem 4.4].

Lemma 7.5. *Let $\mathfrak{t}^\sharp = [L^\sharp, \sigma^\sharp]_{G^\sharp}$ be an inertial equivalence class subordinate to $\mathfrak{s} = [L, \omega]_G$. Lemma 6.5.a and the* LLC *for $G^\sharp$ and for $L^\sharp$ provide a commutative, bijective diagram*

$$\begin{array}{ccc} \mathbf{Irr}^{\mathfrak{t}^\sharp}(G^\sharp) & \longleftrightarrow & \Phi_e(G^\sharp)^{\mathfrak{t}^\sharp} \\ \updownarrow & & \updownarrow \\ (T_{\mathfrak{t}^\sharp}/\!/W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}} & \longleftrightarrow & (\Phi_e(L^\sharp)^{[L^\sharp,\sigma^\sharp]_{L^\sharp}}/\!/W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}} \end{array}$$

Two elements $[t,\rho], [t',\rho'] \in (T_{\mathfrak{t}^\sharp}/\!/W_{\mathfrak{t}^\sharp})_{\kappa_{\sigma^\sharp}}$ are mapped to $G^\sharp$-representations in the same L-packet if and only if

- *$wt' = t$ for some $w \in W_{\mathfrak{t}^\sharp}$;*
- *the $W_{\mathfrak{t}^\sharp,t}$-representations ρ and $w \cdot \rho'$ have parameters (in the Springer correspondence for possibly disconnected complex reductive groups) with the same unipotent class, in the complex reductive group with maximal torus $T_{\mathfrak{t}^\sharp}$, root system $R_{\mathfrak{t}^\sharp,t}$ and Weyl group $W_{\mathfrak{t}^\sharp,t}$.*

Proof. The commutative diagram is obtained from Lemma 7.4, taking (27) into account. To see whether $[t,\rho]$ and $[t',\rho']$ belong to the same L-packet, Lemma 7.4 says that it suffices to look at their images in $(T_{\mathfrak{s}}/\!/W_{\mathfrak{s}})/\mathrm{Stab}(\mathfrak{s})^+ X_{\mathrm{nr}}(L/L^\sharp)$. Let $\tilde{t} \in T_{\mathfrak{s}}$ be a lift of t. Then $W_{\mathfrak{t}^\sharp,t}$ is the isotropy group of $X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)(\tilde{\sigma}^\sharp) \in T_{\mathfrak{t}^\sharp}$ in $W_{\mathfrak{s}}^\sharp$. Here $\sigma^\sharp$ is a projective representation of

$$(X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\tilde{t}} = X^L(\omega).$$

With Lemma A.1 we get

$$\sigma^\sharp \rtimes \rho \in \mathbf{Irr}(\mathbb{C}[(\mathrm{Stab}(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp)_{\tilde{t}}, \kappa_\omega]).$$

The intersection of $(\mathrm{Stab}(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)_{\tilde t}$ with $W_{\mathfrak{s}}$ is $W_{\mathfrak{s},\tilde t} = W(R_{\mathfrak{s},\tilde t})$. Since $W_{\mathfrak{s}}$ commutes with $X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)$, the restriction of $\sigma^\sharp \rtimes \rho$ to $W_{\mathfrak{s},\tilde t}$ is $\dim(\sigma^\sharp)$ times $\rho|_{W_{\mathfrak{s},\tilde t}}$. We want to show that

$$R_{\mathfrak{t}^\sharp,t} = R_{\mathfrak{s},\tilde t}, \tag{94}$$

although in general $W_{\mathfrak{t}^\sharp,t}$ is strictly larger than $W_{\mathfrak{s},\tilde t}$. Both root systems can be defined in terms of zeros of Harish-Chandra μ-functions associated to roots $\alpha \in R_{\mathfrak{s}}$. The function μ_α (for G) is defined via intertwining operators betweeen G-representations, see [Wal, §IV.3 and §V.2]. These remain well-defined as intertwining operators between $G^\sharp$-representations, which implies that μ_α factors through $T_{\mathfrak{s}} \to T_{\mathfrak{t}^\sharp}$ and in this way gives the function μ_α for $G^\sharp$. By [Sil2, Theorem 1.6] all zeros of μ_α are fixed points of the reflection $s_\alpha \in W_{\mathfrak{s}}$. Hence $\mu_\alpha(t) \neq 0$ if $s_\alpha(\tilde t) \neq \tilde t$, proving (94).
It follows that $[t,\rho]$ maps to $[\tilde t, \rho|_{W(R_{\mathfrak{t}^\sharp,t})}$ in $(T_{\mathfrak{s}}//W_{\mathfrak{s}})/\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)$, and similarly for $[t',\rho']$. The $\mathrm{Stab}(\mathfrak{s})^+X_{\mathrm{nr}}(L/L^\sharp)$-orbits of $[\tilde t, \rho|_{W(R_{\mathfrak{t}^\sharp,t})}$ and $[\tilde t', \rho|_{W(R_{\mathfrak{t}^\sharp,t'})}$ are equal if and only if

$$\text{there is a } w \in W_{\mathfrak{t}^\sharp} \text{ such that } wt' = t \text{ and } (w\rho')|_{W(R_{\mathfrak{t}^\sharp,t})} = \rho|_{W(R_{\mathfrak{t}^\sharp,t})}.$$

By Lemma 7.2.b the last condition is equivalent to $w\rho'$ and ρ having the same unipotent class as Springer parameter. Because w is only determined up to $W_{\mathfrak{t}^\sharp,t}$, these unipotent classes must be considered in the complex reductive group with maximal torus $T_{\mathfrak{t}^\sharp}$, root system $R_{\mathfrak{t}^\sharp,t}$ and Weyl group $W_{\mathfrak{t}^\sharp,t}$. □

As before, let $\mathcal{L}$ be a set of representatives for the conjugacy classes of Levi subgroups of G. Then $\{L^\sharp : L \in \mathcal{L}\}$ is a set of representatives for the conjugacy classes of Levi subgroups of $G^\sharp$.

Theorem 7.6. *The maps from Lemma 7.4 combine to a commutative diagram of bijections*

$$\begin{array}{ccc}
\mathbf{Irr}(G^\sharp) & \longleftrightarrow & \Phi_e(G^\sharp) \\
\updownarrow & & \updownarrow \\
\bigsqcup_{L\in\mathcal{L}} \big(\mathbf{Irr}_{\mathrm{cusp}}(L^\sharp)//W(G^\sharp,L^\sharp)\big)_\natural & \longleftrightarrow & \bigsqcup_{L\in\mathcal{L}} \big(\Phi(L^\sharp)_{\mathrm{cusp}}//W(G^\sharp,L^\sharp)\big)_\natural \\
\updownarrow & & \updownarrow \\
\bigsqcup_{L\in\mathcal{L}} \big(\mathbf{Irr}_{\mathrm{cusp}}(L)//\mathbf{Irr}(L/L^\sharp)W(G,L)\big)_\natural & \longleftrightarrow & \bigsqcup_{L\in\mathcal{L}} \big(\Phi(L)_{\mathrm{cusp}}//\mathbf{Irr}(L/L^\sharp)W(G,L)\big)_\natural
\end{array}$$

Here the family of 2-cocycles $\natural$ restricts to κ_ω on $\mathbf{Irr}^{[L,\omega]_L}(L)$. The tempered representations correspond to the bounded enhanced Langlands parameters and the entire diagram is canonical up to permutations within L-packets.

Proof. The upper square follows quickly from Lemma 7.4, in the same way as Theorem 7.3 followed from Lemma 7.2.
Recall from Lemma 6.1 that

$$\mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp) \text{ is in bijection with } (T_{\mathfrak{s}} /\!/ X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega}.$$

Here $X^L(\mathfrak{s})$ is the stabilizer of $\mathfrak{s}_L = [L,\omega]_L$ in $\mathbf{Irr}(L/L^\sharp Z(G))$. A character of $L/L^\sharp$ which is ramified on $Z(G)$ cannot stabilize $\mathfrak{s}_L$, so $X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)$ is the stabilizer of $\mathfrak{s}_L$ in $\mathbf{Irr}(L/L^\sharp)$. By Theorem 7.1 the LLC for L is bijective and $\mathbf{Irr}(L/L^\sharp)$-equivariant, so $X^L(\mathfrak{s})X_{\mathrm{nr}}(L/L^\sharp)$ is also the stabilizer of $\Phi(L)^{\mathfrak{s}_L}$ in $\mathbf{Irr}(L/L^\sharp)$. This implies

$$\begin{aligned}(\mathbf{Irr}_{\mathrm{cusp}}(L) /\!/ \mathbf{Irr}(L/L^\sharp))_\natural &\cong \bigsqcup_{\mathfrak{s}_L = [L,\omega]_L} (\mathbf{Irr}^{\mathfrak{s}_L}(L) /\!/ X^L(\mathfrak{s}) X_{\mathrm{nr}}(L/L^\sharp))_{\kappa_\omega} \\ &\cong \bigsqcup_{\mathfrak{s}_L = [L,\omega]_L} \mathbf{Irr}^{\mathfrak{s}_L}(L^\sharp) = \mathbf{Irr}_{\mathrm{cusp}}(L^\sharp),\end{aligned}$$

and similarly for Langlands parameters. These bijections are equivariant for permutations of the direct factors of L, so applying $(- /\!/ W(G,L))_{\kappa_\omega}$ to all of them produces a commutative square as in the theorem, but with lower row

$$\begin{aligned}&\bigsqcup_{L \in \mathcal{L}} \big((\mathbf{Irr}_{\mathrm{cusp}}(L) /\!/ \mathbf{Irr}(L/L^\sharp))_\natural /\!/ W(G,L) \big)_\natural \longleftrightarrow \\ &\qquad\qquad \bigsqcup_{L \in \mathcal{L}} \big((\Phi(L)_{\mathrm{cusp}} /\!/ \mathbf{Irr}(L/L^\sharp))_\natural /\!/ W(G,L) \big)_\natural .\end{aligned}$$

We apply Corollary A.4 to get the row in the theorem. The canonicity of the thus obtained commutative diagram is a consequence of the analogous property in Lemma 7.4. The temperedness/boundedness correspondence follows from the properties of the local Langlands correspondences for $G, G^\sharp, L$ and $L^\sharp$. □

Example 7.7. Let $G = \mathrm{SL}_5(D)$. Let V_4 denote the non-cyclic group of order 4. Let $\mathbf{W}_F$ denote the Weil group of F. There exists a classical Langlands parameter ϕ which factors through V_4:

$$\phi \colon \mathbf{W}_F \to V_4 \to \mathrm{PGL}_2(\mathbb{C}) \tag{95}$$

Let τ be the cuspidal representation of $D^\times$ which has, as its Langlands parameter, a lift of ϕ to $\mathrm{GL}_2(\mathbb{C})$. Consider the group of characters χ for which $\chi\tau \cong \tau$. This group is isomorphic to V_4 and comprises the four characters $\{1, \gamma, \eta, \gamma\eta\}$, where γ, η are quadratic characters. Let

$$\begin{aligned} L &= (D^\times)^5 \cap \mathrm{SL}_5(D) \\ \sigma &= \tau \otimes 1 \otimes \gamma \otimes \eta \otimes \gamma\eta \in \mathbf{Irr}(L) \\ \mathfrak{s} &= [L,\sigma]_G \end{aligned}$$

Twisting by η corresponds to the permutation (13)(24), twisting by $\gamma\eta$ corresponds to the permutation (14)(23). The Bernstein finite group $W^{\mathfrak{s}}$ is isomorphic to V_4. The corresponding Bernstein variety is the quotient $T^{\mathfrak{s}}/V_4$, where $T^{\mathfrak{s}}$ has the structure of a complex torus of dimension 4.

The summand $\mathcal{H}^{\mathfrak{s}}(G)$ of the Hecke algebra $\mathcal{H}(G)$ is Morita equivalent to the twisted crossed product

$$\mathcal{O}(T^{\mathfrak{s}}) \rtimes_{\natural} V_4,$$

where $\natural$ is the 2-cocycle associated to the above projective representation of V_4. Following [ABPS3], $\mathbf{Irr}(\mathcal{O}(T^{\mathfrak{s}}) \rtimes_{\natural} V_4)$ is the twisted extended quotient $(T^{\mathfrak{s}} // V_4)_{\natural}$. Now consider the standard projection

$$\pi^{\mathfrak{s}} : (T^{\mathfrak{s}} // V_4)_{\natural} \to T^{\mathfrak{s}} / V_4.$$

Let $(V_4)_t$ denote the isotropy group of $t \in T^{\mathfrak{s}}$. Let

$$X = \{t \in T^{\mathfrak{s}} : |(V_4)_t| = 2\},$$
$$Y = \{t \in T^{\mathfrak{s}} : (V_4)_t = V_4\}.$$

We note that

- on the complement of $(X \cup Y)/V_4$ the fibre of $\pi^{\mathfrak{s}}$ has cardinality 1: the corresponding (parabolically) induced representation is irreducible
- on X/V_4 the fibre of $\pi^{\mathfrak{s}}$ has cardinality 2: the corresponding induced representation has two inequivalent irreducible constituents
- on Y/V_4 the fibre of $\pi^{\mathfrak{s}}$ has cardinality 1, because V_4 admits a unique irrreducible projective representation with cocycle $\natural$: the corresponding induced representation has two *equivalent* constituents.

The geometric structure of $\mathbf{Irr}^{\mathfrak{s}}(\mathrm{SL}_5(D))$ is the variety $T^{\mathfrak{s}}/V_4$ with doubling on X/V_4.

A Twisted extended quotients

Let Γ be a group acting on a topological space X. In [ABPS6, §2] we studied various extended quotients of X by Γ. In this paper we need the most general version, the twisted extended quotients.
Let $\natural$ be a given function which assigns to each $x \in X$ a 2-cocycle

$$\natural_x : \Gamma_x \times \Gamma_x \to \mathbb{C}^\times, \text{ where } \Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

It is assumed that $\natural_{\gamma x}$ and $\gamma_* \natural_x$ define the same class in $H^2(\Gamma_x, \mathbb{C}^\times)$, where $\gamma_* : \Gamma_x \to \Gamma_{\gamma x}$ sends α to $\gamma \alpha \gamma^{-1}$. Define

$$\widetilde{X}_{\natural} := \{(x, \rho) : x \in X, \rho \in \mathbf{Irr}\, \mathbb{C}[\Gamma_x, \natural_x]\}.$$

We require, for every $(\gamma, x) \in \Gamma \times X$, a definite algebra isomorphism

$$\phi_{\gamma,x} : \mathbb{C}[\Gamma_x, \natural_x] \to \mathbb{C}[\Gamma_{\gamma x}, \natural_{\gamma x}]$$

such that:

- $\phi_{\gamma,x}$ is inner if $\gamma x = x$;
- $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma, x \in X$.

We call these maps connecting homomorphisms, because they are reminiscent of a connection on a vector bundle. Then we can define a Γ-action on $\widetilde{X}_\natural$ by

$$\gamma \cdot (x, \rho) = (\gamma x, \rho \circ \phi_{\gamma,x}^{-1}).$$

We form the *twisted extended quotient*

$$(X//\Gamma)_\natural := \widetilde{X}_\natural / \Gamma.$$

We note that this reduces to the extended quotient of the second kind $X//\Gamma$ from [ABPS6, §2] if $\natural_x$ is trivial for all $x \in X$ and $\phi_{\gamma,x}$ is conjugation by γ.
The map

$$\widetilde{X}_\natural \to X, \quad (x, \rho) \mapsto x$$

induces a map

$$\pi_1 : (X//\Gamma)_\natural \to X/\Gamma$$

which we will call the *standard projection*.
Such twisted extended quotients typically arise in the following situation. Let A be a $\mathbb{C}$-algebra such that all irreducible A-modules have countable dimension over $\mathbb{C}$. Let Γ be a group acting on A by automorphisms and form the crossed product $A \rtimes \Gamma$.
Let $X = \mathbf{Irr}(A)$. Now Γ acts on $\mathbf{Irr}(A)$ and we get $\natural$ as follows. Given $x \in \mathbf{Irr}(A)$ choose an irreducible representation (π_x, V_x) whose isomorphism class is x. For each $\gamma \in \Gamma$ consider π_x twisted by γ:

$$\gamma \cdot \pi_x : a \mapsto \pi_x(\gamma^{-1} a \gamma).$$

Then $\gamma \cdot x$ is defined as the isomorphism class of $\gamma \cdot \pi_x$. Since $\gamma \cdot \pi_x$ is equivalent to $\pi_{\gamma x}$, there exists a nonzero intertwining operator

$$T_{\gamma,x} \in \mathrm{Hom}_A(\gamma \cdot \pi_x, \pi_{\gamma x}). \tag{96}$$

By Schur's lemma (which is applicable because $\dim V_x$ is countable) $T_{\gamma,x}$ is unique up to scalars, but in general there is no preferred choice. For $\gamma, \gamma' \in \Gamma_x$ there exists a unique $c \in \mathbb{C}^\times$ such that

$$T_{\gamma,x} \circ T_{\gamma',x} = c T_{\gamma\gamma',x}.$$

We define the 2-cocycle by

$$\natural_x(\gamma, \gamma') = c.$$

Let $N_{\gamma,x}$ with $\gamma \in \Gamma_x$ be the standard basis of $\mathbb{C}[\Gamma_x, \natural_x]$. The algebra homomorphism $\phi_{\gamma,x}$ is essentially conjugation by $T_{\gamma,x}$, but we must be careful if some of the T_γ coincide. The precise definition is

$$\phi_{\gamma,x}(N_{\gamma',x}) = \lambda N_{\gamma\gamma'\gamma^{-1},\gamma x} \quad \text{if} \quad T_{\gamma,x} T_{\gamma',x} T_{\gamma,x}^{-1} = \lambda T_{\gamma\gamma'\gamma^{-1},\gamma x}, \lambda \in \mathbb{C}^\times. \tag{97}$$

Notice that (97) does not depend on the choice of $T_{\gamma,x}$.
Suppose that Γ_x is finite and $(\tau, V_\tau) \in \mathbf{Irr}(\mathbb{C}[\Gamma_x, \natural_x])$. Then $V_x \otimes V_\tau^*$ is an irreducible $A \rtimes \Gamma_x$-module, in a way which depends on the choice of intertwining operators $T_{\gamma,x}$.

Lemma A.1. [ABPS6, Lemma 2.3]
Let A and Γ be as above and assume that the action of Γ on $\mathbf{Irr}(A)$ has finite isotropy groups.

(*a*) *There is a bijection*

$$\begin{array}{ccc} (\mathbf{Irr}(A)/\!/\Gamma)_\natural & \longleftrightarrow & \mathbf{Irr}(A \rtimes \Gamma) \\ (\pi_x, \tau) & \mapsto & \pi_x \rtimes \tau := \mathrm{Ind}_{A \rtimes \Gamma_x}^{A \rtimes \Gamma}(V_x \otimes V_\tau^*). \end{array}$$

(*b*) *If all irreducible A-modules are one-dimensional, then part (a) becomes a natural bijection*

$$\mathbf{Irr}(A)/\!/\Gamma \longleftrightarrow \mathbf{Irr}(A \rtimes \Gamma).$$

Via the following result twisted extended quotients also arise from algebras of invariants.

Lemma A.2. *Let Γ be a finite group acting on a $\mathbb{C}$-algebra A. There is a bijection*

$$\begin{array}{ccc} \{V \in \mathbf{Irr}(A \rtimes \Gamma) : V^\Gamma \neq 0\} & \longleftrightarrow & \mathbf{Irr}(A^\Gamma) \\ V & \mapsto & V^\Gamma. \end{array}$$

If all elements of $\mathbf{Irr}(A)$ have countable dimension, it becomes

$$\begin{array}{ccc} \{(\pi_x, \tau) \in (\mathbf{Irr}(A)/\!/\Gamma)_\natural : \mathrm{Hom}_{\Gamma_x}(V_\tau, V_x) \neq 0\} & \longleftrightarrow & \mathbf{Irr}(A^\Gamma) \\ (\pi_x, \tau) & \mapsto & \mathrm{Hom}_{\Gamma_x}(V_\tau, V_x). \end{array}$$

Proof. Consider the idempotent

$$p_\Gamma = |\Gamma|^{-1} \sum\nolimits_{\gamma \in \Gamma} \gamma \in \mathbb{C}[\Gamma]. \tag{98}$$

It is well-known and easily shown that

$$A^\Gamma \cong p_\Gamma (A \rtimes \Gamma) p_\Gamma$$

and that the right hand side is Morita equivalent with the two-sided ideal

$$I = (A \rtimes \Gamma) p_\Gamma (A \rtimes \Gamma) \subset A \rtimes \Gamma.$$

The Morita equivalence sends a module V over the latter algebra to

$$p_\Gamma (A \rtimes \Gamma) \otimes_{(A \rtimes \Gamma) p_\Gamma (A \rtimes \Gamma)} V = V^\Gamma.$$

As I is a two-sided ideal,

$$\mathbf{Irr}(I) = \{V \in \mathbf{Irr}(A \rtimes \Gamma) : I \cdot V \neq 0\} = \{V \in \mathbf{Irr}(A \rtimes \Gamma) : p_\Gamma V = V^\Gamma \neq 0\}$$

This gives the first bijection. From Lemma A.1.a we know that every such V is of the form $\pi_x \rtimes \tau$. With Frobenius reciprocity we calculate

$$(\pi_x \rtimes \tau)^\Gamma = \left(\mathrm{Ind}_{A \rtimes \Gamma_x}^{A \rtimes \Gamma}(V_x \otimes V_\tau^*)\right)^\Gamma \cong (V_x \otimes V_\tau^*)^{\Gamma_x} = \mathrm{Hom}_{\Gamma_x}(V_\tau, V_x).$$

Now Lemma A.1.a and the first bijection give the second. □

Let A be a commutative $\mathbb{C}$-algebra all whose irreducible representations are of countable dimension over $\mathbb{C}$. Then $\mathbf{Irr}(A)$ consists of characters of A and is a T_1-space. Typical examples are $A = C_0(X)$ (with X locally compact Hausdorff), $A = C^\infty(X)$ (with X a smooth manifold) and $A = \mathcal{O}(X)$ (with X an algebraic variety).
As a kind of converse to Lemmas A.1 and A.2, we show that every twisted extended quotient of $\mathbf{Irr}(A)$ appears as the space of irreducible representations of some algebras. With small modifications, the argument also works for smooth manifolds and algebraic varieties.
Let Γ be a group acting on A by algebra automorphisms, such that Γ_x is finite for every $x \in \mathbf{Irr}(A)$. Recall that every 2-cocycle $\natural$ of Γ arises from a projective Γ-representation (μ, V_μ) by

$$\mu(\gamma)\mu(\gamma') = \natural(\gamma, \gamma')\mu(\gamma, \gamma').$$

Let Γ act on $A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)$ by

$$\gamma \cdot (a \otimes h) = \gamma(a) \otimes \mu(\gamma) h \mu(\gamma)^{-1}.$$

Lemma A.3. *There are bijections*

$$\begin{array}{lcl} \mathbf{Irr}((A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes \Gamma) & \longleftrightarrow & (\mathbf{Irr}(A)//\Gamma)_\natural, \\ \mathbf{Irr}((A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))^\Gamma) & \longleftrightarrow & \{[x, \rho] \in (X//\Gamma)_\natural : \rho \text{ appears in } V_\mu\}. \end{array}$$

Proof. We can identify $\mathbf{Irr}(A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu))$ with $\{\mathbb{C}_x \otimes V_\mu : x \in \mathbf{Irr}(A)\}$. It follows directly from (96) that we can take $T_{\gamma,x} = \mu(\gamma)$ for all $\gamma \in \Gamma$ and $x \in \mathbf{Irr}(A)$. Thus the first bijection is an instance of Lemma A.1.a.
Let $x \in \mathbf{Irr}(A)$ and $(\tau, V_\tau) \in \mathbf{Irr}(\mathbb{C}[\Gamma_x, \natural])$. Then

$$\mathrm{Hom}_{\Gamma_x}(\tau, \mathbb{C}_x \otimes V_\mu) = \mathrm{Hom}_{\Gamma_x}(\tau, V_\mu),$$

and this is nonzero if and only if τ appears in V_μ. Now an application of Lemma A.2 proves the second bijection. □

Corollary A.4. *In the above setting, suppose that $\Gamma = \Gamma_1 \rtimes \Gamma_2$ is a semidirect product. Then there is a canonical bijection*

$$(\mathbf{Irr}(A)//\Gamma)_\natural \longleftrightarrow ((\mathbf{Irr}(A)//\Gamma_1)_\natural//\Gamma_2)_\natural.$$

Proof. The bijection is obtained from Lemma A.3 and

$$(A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes \Gamma = ((A \otimes \mathrm{End}_{\mathbb{C}}(V_\mu)) \rtimes \Gamma_1) \rtimes \Gamma_2$$

It is canonical because the same 2-cocycle is used on both sides. □

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