Teaching Mathematics and Algorithmics with Recreational Problems: The Liber Abaci of Fibonacci

Marc Moyon

TEACHING MATHEMATICS AND ALGORITHMICS WHITH RECREATIONAL PROBLEMS

The *Liber Abaci* of Fibonacci

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ABSTRACT

I propose an empirical study on theintroduction of an historical perspective onmathematics education at different levels in the French secondary school curriculum (11-18 years old). First of all, I present, in the historical contexts of the twelfth and thirteenth centuries, elements of the biography of Leonardo da Pisa, better known as Fibonacci, and of his mathematical work. I pay close attention to the mathematics of the Islamic countries that had largely fed his thinking. I dedicate an important part of our contribution to excerpts from theLiber Abaci to understandbetterhow his work might contribute to today's classroom. All selected extracts belong to the category of so-called 'recreational' problems. They were chosen for their algorithmic structure that allows us to work with pupils on, among other themes, algebra and algorithmics (including coding). Finally, I give details of mathematical and historical extensions.

1 Introduction

The work presented in this paper[1]is part of a global project introducing an historical perspective on mathematics education developed in the IREM (Institute of Research in Mathematics Education) of Limoges under my supervision[2] and following the works of the French Inter-IREM Commission on history and epistemology of mathematics.[3] With secondary school mathematics teachers, I experimented with activities in classroom and analysed pupils' work.[4] More precisely, my purpose, here, isto take advantage of the algorithmic nature of a 'recreational' problem from medieval mathematics. I think that it is a suitable way to introduce, practice and clarifyalgorithmics in classroom and also to improve pupils' abilities to code computer programs (with Scratch[5]).

I focus on the so-called 'Apple Orchard Problem' as it was presented by Leonardo Fibonacci (13th c.) in his well-known Liber Abaci. Since the historical context is important to understand the genesis and the development of mathematical ideas, I present briefly, in the first part, a bibliographical overview of Fibonacci. Then, I explain the main reasons which make me consider this problem as a recreational one. Furthermore, I

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[1] I partly base our contribution on (Moyon, 2019). I give here new extensions by taking into account others historical sources and new mathematical developments. Iwould liketo express mysincere thanks to Maurice OReilly for his diligent proofreading of this paper. Nevertheless, I am the only one responsible for the flaws or errors that remain.

[2] http://www.irem.unilim.fr/recherche/algorithmique-histoire-des-mathematiques/ (Accessed: 20 August 2018). For this contribution, I worked with Valérie Fréty (Collège Maurice-Genevoix, Couzeix) and Julie Pousse (Collège Louise-Michel, Saint-Junien). I thank them for their confidence and their availability.

[3] http://www.univ-irem.fr/spip.php?rubrique15 (Accessed: 20 August 2018).

[4] In the same way, the IREM of Limoges participated to the "Passerelles" project with the edition of a book (Moyon & Tournès, 2017) and the construction of a companion website providing complementary documents http://www.univ-irem.fr/spip.php?rubrique505 (Accessed: 20 August 2018).

[5] Scratch is used (https://scratch.mit.edu. Accessed: 20 August 2018) because it is provided free of charge and it corresponds to the most used in French classroom.

analyse[6] several different pupils' work (13 to 15 years old) from mathematical activity to coding. In the last part, I give, for the same problem, mathematical extensions, for more advanced pupils, involvingsequences satisfying a recurrence relation. Finally, I give extracts from another medieval Latin text in the same genre.

2 Fibonacci and his *Liber Abaci*

In this part, I limit my remarks to necessary historical contents[7] because my goal is not to offer a complete work on Fibonacci (even if that were possible[8]) but it is rather an opportunity to put in context the mathematical content that follows. In addition, this part could be useful for mathematics teachers or teacher educators who would like to reproduce my experiments in their own practice.

2.1 Fibonacci: some biographical elements

Leonardo of Pisa was born in the last third of the twelfth century and died after 1241. He belonged to the merchant elite of Pisa, a very important maritime republic on the Tuscan coast. He was the son of Guglielmo Bonacii, hence his nickname Fibonacci, contraction of 'filius Bonacci'. He was also known as Leonardo *Bigollo* (the traveller or the wanderer) for his numerous travels in the Mediterranean Basin. He visited several regions such as Egypt, Syria, Constantinople, Sicily and Provence. His most important journey was to Bugia (now Bejaïa in Algeria) where his father worked as a 'scriberesponsible for Customs' (Caianiello, 2013, 241). In thisMediterranean port city under Pisan rules, he was initiated to Arabic mathematics and, especially, to the Hindu-Arabic numeral system and algebra. Fibonacci largely benefitted from the cultural and scientific contexts of the Mediterranean Basin of the thirteenth century with important commercial and diplomatic relations between the north and the south, in particular with the fourth, fifth and sixth crusades, contemporary with Fibonacci. Probably in his native city, he met the Emperor Frederick IIwho appreciated the sciences in general.[9] It is well-known that the court of the Holy Roman Emperor comprised various multilingual scholars from all over the Mediterranean Basin. As Latin, Greek and Arabic were spoken, numerous written sources circulated. Fibonacci seemed to have scientific and friendly links with many of these scholars such as Michael Scot, John of Palermo and Theodore of Antioch.

2.2 The works of Leonardo

Fibonacci is nowadays well-known thanks to the famous sequence defined such as every number after the first two is the sum of the two preceding ones. Nevertheless, his work should not be reduced to it (Figure 2.1). Two books of fundamental importance were: his*Liber Abaci* for arithmetic and algebra (Fibonacci, 1857) and his

[6] The analysis is based both on the observation of students' activity during the research and on the examination of written documents (taking into accounts general and mathematical abilities). See (Moyon, 2019, 248).

[7]All this part is extracted from (Moyon & Spiesser, 2015) and (Moyon, 2016).

[8]For more details, see (Caianiello, 2012, 2013; Høyrup, 2016).

[9]Fibonacci begins his *Flos* – a short text on algebra with problems involving several unknowns written between 1226 and 1230 (Picutti, 1983) – with: "While I was in Pisa in the presence of your Majesty, very glorious Prince Frederick, Master Jean of Palermo, your philosopher, discussed with me many questions about numbers and among other things proposed me two problems concerning both geometry and number" (Fibonacci, 1854, 2; Moyon, 2016, 7).

Practica geometriae for geometry (Fibonacci, 1862).[10] He authored also three other writings: the *Liber quadratorum*, the aforesaid *Flos*, and the *Epistola ad magistrum Theodorum*.[11] Considering the breadth of his work, Fibonacci seems to have mastered the scholarly mathematics of his time, as well as the practical problems related to it. After the vast movement of translation from Arabic into Latin in the twelfth century, he himself contributed to the spread throughout Europe of Islamic mathematics with their new theoretical ideas and problems. Below are three basic examples.

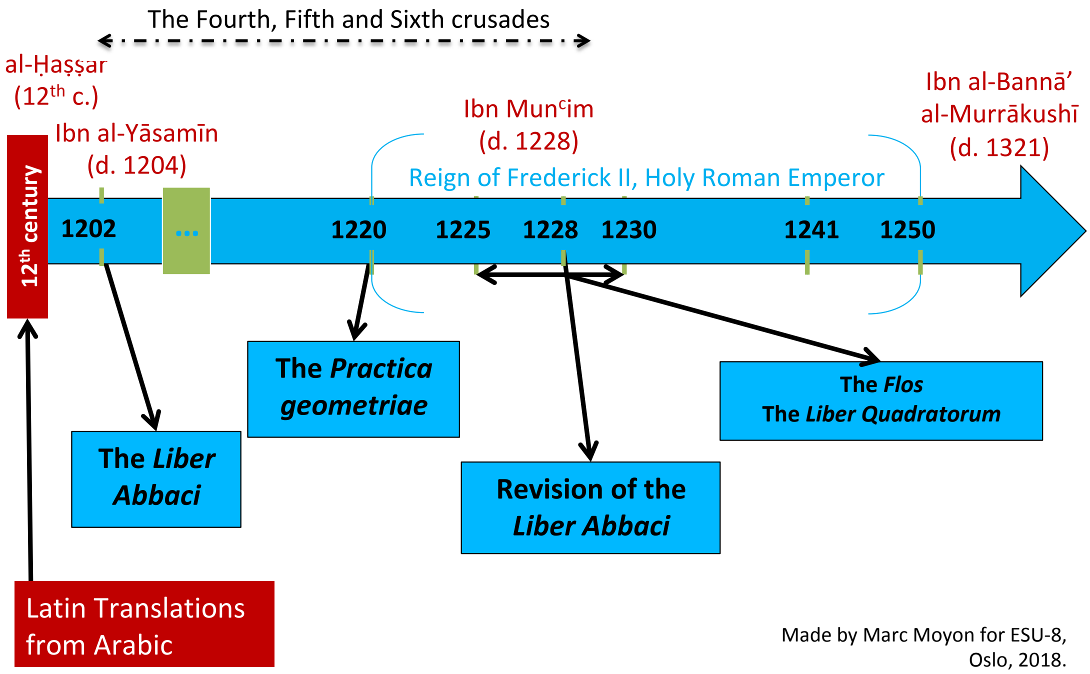


Figure 2.1

The first example is the Hindu-Arabic numerals that are presented at the beginning of the first chapter of the *Liber Abaci*:[12]

> *The nine Indians figures are:*
>
> *9 8 7 6 5 4 3 2 1*
>
> *With these nine figures and with the sign 0 which the Arabs call zephyr, any number whatsoever is written.* (Fibonacci, 1857, 2; Moyon, 2016, 12)

And he follows explaining the writing of any number, however big:

> *A number is a collection of units, which increases indefinitely by its own orders. The first [order] is composed of the units from one to ten. The second is composed of the tens, from ten to one hundred. The third is composed of the hundreds, from one hundred to one thousand. The fourth is composed of the thousands, from one thousand to ten thousand and so one for any of the following orders until infinity, each order being tenfold from the previous one.* (Fibonacci, 1857, 2; Moyon, 2016, 12)

An important part of the *Liber Abaci* (chapter 5, 6 and 7) is dedicated to fractions –

[10] Both are translated into English (Sigler, 2002; Hughes, 2008). Nevertheless, the Latin readings are really necessary because both suffer from various failings. See, for example, (Rommevaux 2008; Høyrup 2009) for the *Practica Geometriae*.

[11] Fibonacci mentions two other texts but they are nowadays considered lost.

[12] In the following, all the original extracts from the *Liber Abaci* are framed.

with the horizontal bar – and to computations with them. This part is largely (but not exclusively) inspired by Maghrebian mathematics. Following al-Ḥaṣṣār (12th c.) and Ibn al-Yāsamīn (d.1204), Fibonacci identified fractions to solve various types of problems (Moyon & Spiesser, 2015).

The second example is related to algebra detailed in the last two chapters of his *Liber Abaci* but also widely used in his *Practica geometriae* to solve geometrical problems of measurement (Miura, 1981; Hughes, 2004; Moyon, 2012). This new discipline was born in Bagdad between 813 and 833 with *al-kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala* [The Compendious Book on Calculation by Restoration and Comparison] written by al-Khwārizmī.Fibonacci demonstrates an excellent knowledge of the Arabic corpus, in particular, the seminal text of al-Khwārizmī(d.*ca*.850) and, among others,*al-Kitāb al-jabr wa-l-muqābala* [Book on Restoration and Comparison] of Abū Kāmil (d. *ca*.930). For numerous problems, he proposes both algorithmic and algebraic procedures.

The third example focuses on practical geometry. Inheritor of the Islamic tradition of *ᶜilm al-misāḥa* [science of measurement], we must consider him as a major player in the beginning of the geometry of measurement in Latin Europe. With the *Practica geometriae*, the reader is taught, among other things, to measure plane figures, to divide them into different other figures giving geometrical constraintsand to measure inaccessible distances (Moyon, 2017, 111–113, 577–616).

Finally, Fibonacci is one of the greatest medieval Western mathematicians. However, it is also necessary to consider him as a crucial link between Islamic countries and Latin Europe.

3 'Recreational problems' in the medieval history of mathematics

3.1 Generalities

The focus is on problems that I now consider part of recreational mathematics even tough they are not always defined as such by their authors. It is useful to consider Høyrup's comment on this matter when he explains that recreational problems:

> *... are pure in the sense that they do not deal with real applications, however much they speak in the idiom of everyday* [...]. *Nonetheless, their social basis is in the world of know-how, not that of know-why* [...]. *The distinction between these two orientations of knowledge is of general validity but has particular implications for mathematics.* [...]
> *Here, techniques and methods are by necessity primary, and problems are secondary, derived from the techniques which are to be trained. Anybody familiar with schoolbooks on arithmetic will recognize the situation, and scholasticized systems are indeed those where problems constructed for training purposes dominate. Apprenticeship-based systems, for their part, tend to train as much as possible on real, albeit simple tasks*. (Høyrup, 2008, 1252–1253)

Recreational problems (or mathematical riddles) are, in general, easy to state, tempting to work on and, sometimes, annoyingly difficult to solve. What we categorize today assuch problems are very ancient and omnipresent in many mathematical practicesworldwide. They travel between different cultures.[13] The same problem can even be found in Mesopotamian, Chinese, Greek, Sanskrit, Arabic, and Latin sources (Heeffer, 2014). Historians of mathematics, such as Hermelink (1978), consider the analysis of this corpus as an approach to studying the transmission of knowledge from one culture to another. Following this kind of project, Singmaster wrote in 1988:

> *I think that the temporal and geographical distribution of the sources suggests that recreational mathematics owes a much greater debt to China and India than to Greece, and that there must have been earlier and more extensive communication from India to Europe via the Arabs than is presently known.* (Singmaster, 1988, 195)

Recreational problems seem to be more and more important in the history of mathematics, especially in the context of medieval teaching from as far back as the *Propositiones acuendo ad juvenes* written by Alcuin of York in the ninth century.[14] In addition, as Sesiano (2014b) excellently showed, the medieval texts are historically important. They widely feed the later collections edited in the Renaissance. Furthermore, Sesiano mentioned about the *Liber Mahameleth* that "many problems tend to be of a recreational nature" adding that "this had become traditional at the time for mathematical treatises" (Sesiano, 2014a, xvii). As he made clear in his *Introduction to the History of Algebra* that the "area of recreational mathematics [is] a domain that was to grow considerably during the Middle Ages to the point where it became a standard component of works on algebra" (Sesiano, 2009, 25). In this context, Fibonacci is one of the most influential medieval authors thanks to his *Liber Abaci*.

3.2 Recreational Problems in the *Liber Abaci*

The *Liber Abaci* is divided into fifteen chapters.[15] The first seven chapters deal with numeration involving integers, fractions and operations on them. The following chapters, from the eighth to the eleventh, focus on problems linked to commercial rules with special emphasis on conversion of currency, allocations of profit, alloying of currencies where ratio and proportions (with the rule of three among others) are very important. The twelfth and thirteenth chapter of *Liber Abaci*, the longest part (almost half of the book) are devoted to various methods of solving recreational problems.[16] Fibonacci named them with the expression *erraticae questiones*.[17] In the last two chapters, the Pisan mathematician presents computations on radical numbers (based on Euclid's *Elements*) and, as already mentioned, algebra.

The problems of chap. 12 are really different, but they all seem to be common application problems with improbable and even absurd conditions. Moreover they all reveal the Fibonacci's passion for numbers, for algorithms and for mathematical reasoning. Here are some typical problems where, to the reader (or learner)'s amusement,

[13]The most complete reference is (Singmaster 2013) with a detailed annoted bibliography. The reader can also refer to (Tropfke, 1980, 573–660).

[14] The edition of the Latin text is made by Folkerts (1978), the English translation by Hadley and Singmaster (1992). For an extended and corrected version, see (Hadley & Singmaster, 1995).

[15]For details, read (Moyon & Spiesser, 2015, 421–422).

[16]In order to better understand Fibonacci's conception, Hannah (2011) analyses in detail three collections of problems: men giving and taking, men finding a purse, and men wishing to buy a horse.

[17]For a discussion on the meaning of this expression, read (Moyon, 2019).

animals are seen reasoning with integers, fractions and proportionality.[18]

> *A certain lion is in a certain pit, the depth of which is 50 palms, and he ascends daily 1/7 of a palm, and descends 1/9. It is sought in how many days will he leave the pit?* (Fibonacci, 1857, 177; Sigler 2002, 273; Moyon, 2016, 28)

And

> *A certain lion eats one sheep in 4 hours, and a leopard eats one sheep in 5 hours, and a bear eats one sheep in 6 hours; it is sought, if one sheep is thrown to them, how many hours it will take them together to devour it?* (Fibonacci, 1857, 182; Sigler, 2002, 279–280; Moyon, 2016, 24–26)

And finally,

> *Two ants are on the ground 100 paces apart, and they move in the same direction towards a single point; the first of them advances daily 1/3 of a pace and retreats 1/4; the other advances 1/5 and retreats 16; it is sought in how many days they will meet?* (Fibonacci, 1857, 182; Sigler 2002, 280; Moyon, 2016, 27)

It is good to finish any discussion of animal problems with the famous 'Rabbits problem' which gave birth to the well-known 'Fibonacci sequence'. It is a very suitable example of mathematical modelling.

> *How many pairs of Rabbits are created by one pair in one year?*
> *A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.* (Fibonacci, 1857, 283; Sigler 2002, 404; Moyon, 2016, 29–32)

Here, for our purpose – to introduce historical perspective in maths education –, I chose another problem from the twelfth chapter with an algorithmic structure. I call it the 'Apple Orchard Problem'.[19]

3.3 The 'Apple Orchard Problem'

> *A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected.* (Fibonacci, 1857, 278; Sigler 2002, 397; Moyon, 2016, 33–36)

I give the entire problem (with the solution) in the first appendix. Fibonacci, as often, proposed two solutions. The first one can be translated into modern notation as follows:

$$1 \to 1+1=2 \to 2\times 2=4$$
$$4 \to 4+1=5 \to 5\times 2=10$$
$$10 \to 10+1=11 \to 11\times 2=22$$
$$22 \to 22+1=23 \to 23\times 2=46$$

[18]These problems are all extracted from (Moyon, 2016) where I also give solutions with commentary (in French).
[19]Singmaster (2013) mentioned this problem in the chapter 'Monkey and Coconuts problems'under the heading 'Arithmetic& Number-Theoretic Recreation'.

$$46 \to 46+1=47 \to 47\times 2=94$$
$$94 \to 94+1=95 \to 95\times 2=190$$
$$190 \to 190+1=191 \to 191\times 2=382$$

Fibonacci ended with: "this total is the number of apples; and thus reversing the order that was proposed you will be able to solve any similar problem". This first method is based on the inversion of the algorithm: Fibonacci proposed to work backwards and each operation is replaced by its inverse. This 'method of inversion' is well-known and we have, among others, different Indian sources mentioning it. For example, Āryabhaṭa (fifth-sixth century) in the *Gaṇitapāda*, the mathematical part of the *Āryabhaṭīya*, wrote the following verse: "In a reversed [operation], multipliers become divisors and divisors, multipliers. And an additive [quantity] becomes a subtractive [quantity], a subtractive [quantity] an additive [quantity]" (Keller, 2006, 118). Another example comes from Brahmagupta. In his *Brāhmasphuṭasiddhānta*, writtenin the seventh century, this rule can be read: "beginning from the end, make the multiplier divisior, the divisor multiplier; [make] addition subtraction and subtraction addition; [make] square square-root, and square-root square; this gives the required quantity" (Datta & Singh, 2004, 232).

The second solution is an algebraic one. In modern notation, let *x* be the number of apples initially picked, then the linear equation to solve is:

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}x-1\right)-1\right)-1\right)-1\right)-1\right)-1\right)-1=1$$

or, step by step (or door by door), following the original approach,

After the first door, there remain$\frac{1}{2}x-1$apples,

After the second door, there remain$\frac{1}{2}\left(\frac{1}{2}x-1\right)-1=\frac{1}{4}x-\frac{3}{2}$ apples,

After the third door, there remain$\frac{1}{2}\left(\frac{1}{4}x-\frac{3}{2}\right)-1=\frac{1}{8}x-\frac{7}{4}$ apples, and so one

$$\to \frac{1}{2}\left(\frac{1}{8}x-\frac{7}{4}\right)-1=\frac{1}{16}x-\frac{15}{16}$$
$$\to \frac{1}{2}\left(\frac{1}{16}x-\frac{15}{16}\right)-1=\frac{1}{32}x-\frac{31}{32}$$
$$\to \frac{1}{2}\left(\frac{1}{32}x-\frac{31}{32}\right)-1=\frac{1}{64}x-\frac{63}{64}$$

After the seventh door, there remain$\frac{1}{2}\left(\frac{1}{64}x-\frac{63}{64}\right)-1=\frac{1}{128}x-\frac{127}{128}$ apples, and indeed, it corresponds to only 1 apple, thus, $\frac{1}{128}x-\frac{127}{128}=1$ and so$x=382$.

4 The 'Apple Orchard Problem': Experiments in French Secondary Schools (14-16 years old)

Secondary school mathematics teachers, involved in the experiments with me, judged 'Apple Orchard Problem'and the Fibonacci's original text mathematically and pedagogically interesting for pupils aged 14 to 16 years old. Taking into account several theoretical studies on the history and pedagogy of mathematics (Fried, 2001; Fried, Guillemette & Jahnke, 2016; Jankvist, 2009), my colleague and I decided to base our activity on using original sources in classroom. I know, from my own experience,[20] that the whole enterprise of reading a source is a really difficult and

[20] For example, (Moyon, 2013, 2017).

time-consuming task but we believed that the pupils were able to work with the extract of the *Liber Abaci*. We agree with the idea that “reading a source deepens the mathematical understanding on both levels, on that of doing mathematics and on that of reflecting about mathematics” (Fried, Guillemette & Jahnke, 2016, 219). And, for us, it was a major purpose to make pupils think about the algorithmic structure of the problem, the generalization indicated by Fibonacci[21] and the modern use of coding. Thus, reading an original source (translated into French) adapted to the expected educational level, is another manner of applying new concepts, quite different from usual exercises. Thus we followed the basic guidelines of the hermeneutic procedure such as presented by Jahnke in (Fried, Guillemette & Jahnke, 2016, 216).

To help pupils read and comment the text, we presented the text and the author into their historical context (see part 2 of this contribution) and we gave them a questionnaire in two stages (each of one hour).[22] The aim of the first stage was to read the problem, understand it and work on it so as to solve it in their own words. At this stage, they did not have Fibonacci’s solutions.

1. Individual work: Identify the different steps of the statement in your rough work.
2. Collective work (pupils are separated in small groups of 4): Determine the number of apples sought by a method agreed by the group.
3. On the answer sheet, write your solution to the ‘apple orchard problem’ (one sheet per group) that can be presented to the class.

The second stage consists in reading the algorithmic solution. It was stilla collective work. The supporting questions were:

4. Read Fibonacci’s solution. Did you find the same number of apples?
5. How could you describe simply the solution given by the Pisan mathematician?
6. Translate it into today’s mathematical language.
7. According to you, what does it mean “you will be able to solve any similar problem”?

Several procedures were tested by the pupils. Most are quite explicit. I analyse below the main results (all the works have been anonymized) regarding the method used.

4.1 Mathematical Workings

4.1.1 By an algebraic method

Several pupils (about 30%) used algebra to solve the problem taking the number of apples picked as the unknown. A major error arose: the confusion between the remaining fruits $\left(\frac{1}{2}x-1\right)$ and the number of fruits given at the doorkeeper$\left(\frac{1}{2}x+1\right)$.

[21] “et sic revertendo, secundum quod propositum fuerit, in ordinem retro, **poteris quamlibet similium positionum reperire**” (Fibonacci, 1857, 278), bolded by me.

[22] My colleague and I planned a third stage to work on the algebraic solution but unfortunately we ran out of time. The questions were: 1) Read the second solution proposed by Fibonacci. How is it different from the first? 2) What is, for Fibonacci, the “thing”? What kind of mathematics is he using? 3) Translate this solution into today mathematical language.

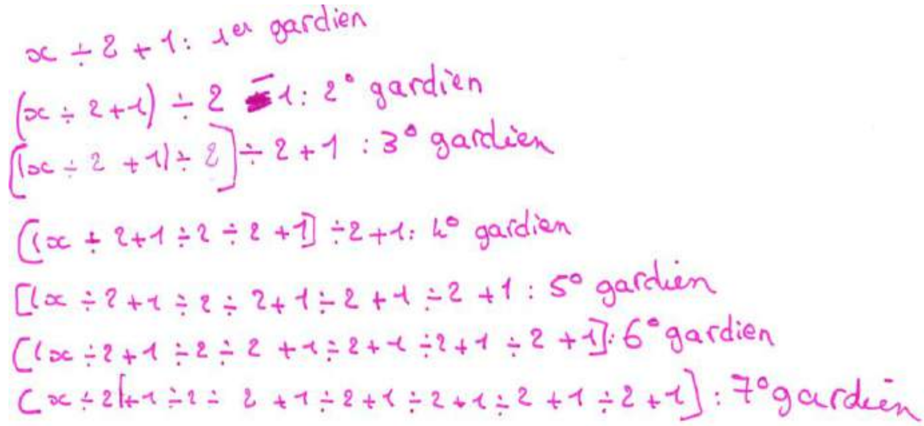


Figure 4.1: Work of Elisa, Nina and Juliette

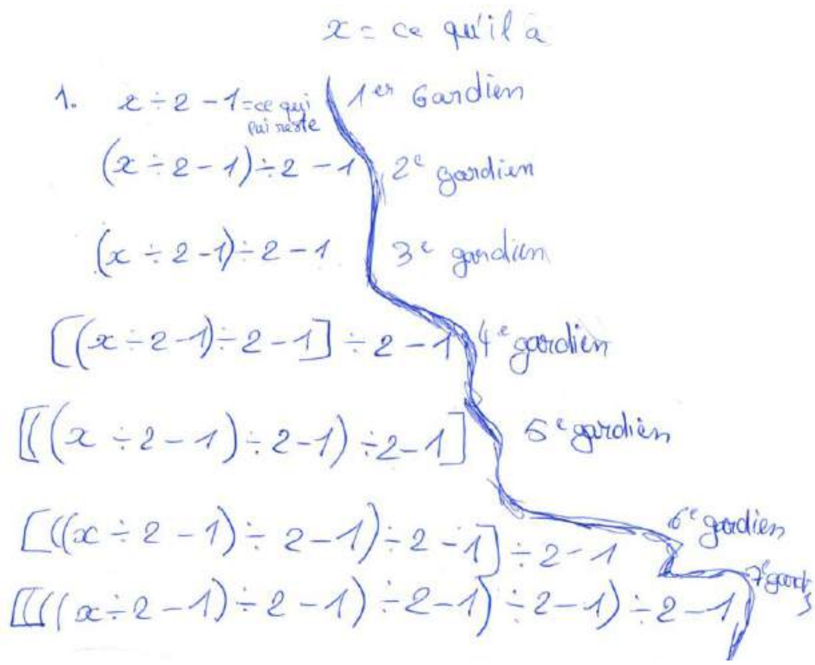


Figure 4.2: Work of Mathilde and Mélissandre

Only half students of this group came up with an equation. And, in this case (Figure 4.3), as they did not reduce the fractions as soon as possible, they gave up because they could not solve the equation.Here, a direct approach involves a complicated equation, by the standards of their mathematical ability.It follows that, algebra seems not to be the better strategy for 14-16 years old pupils.

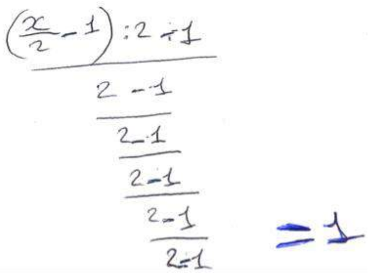


Figure 4.3: Algebraic solution written by Fayza and Germain (first attempt, see fig. 4.5).

4.1.2 By a trial and error procedure

Other pupils (about 20%) tried to solve the problem by trial and error. They took a random number and they executed the program repeating the process until they decided to stop trying. Several attempts were made. They generally stopped when they obtained a negative or a decimal number.

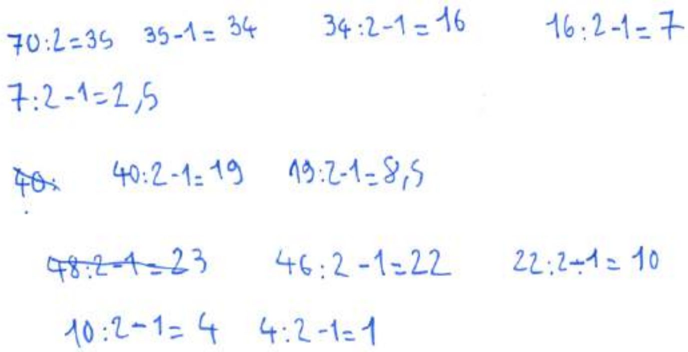


Figure 4.4: Here, in the work of Elvis and Dorian,four different numbers were tried: 70, 40, 48 and 46 (the last ‘successfully’)

Unfortunately, only few pupils (less than a third) reasoned from the results found to improve the choice of the initial number to be tested. No pupil solved the problem completely by following this procedure.

4.1.3 By the method of inversion or the working backwards strategy

The problem proceeds from complex, initially,to simple at the end. And, as it involves a sequence of reversible actions, the work backwards strategy isprobably the best procedure to perform.

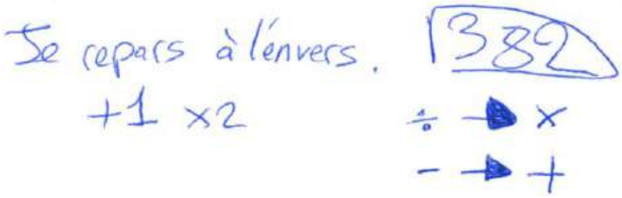


Figure 4.5: Second attempt of Fayza and Germain (see fig. 4.3): explanation of the algorithm of inversion (quite similar to Āryabhaṭa and Brahmagupta!)

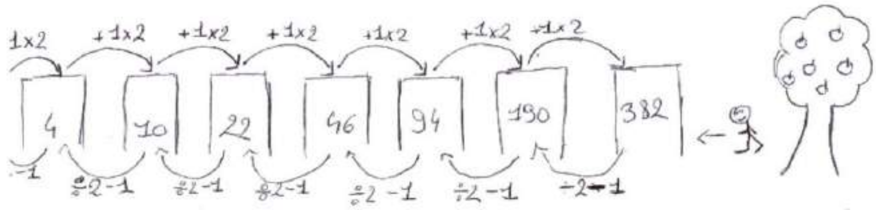


Figure 4.6: The algorithm in both directions: forward and backward (work of Julie, Lisa, Lola and Mélanie)

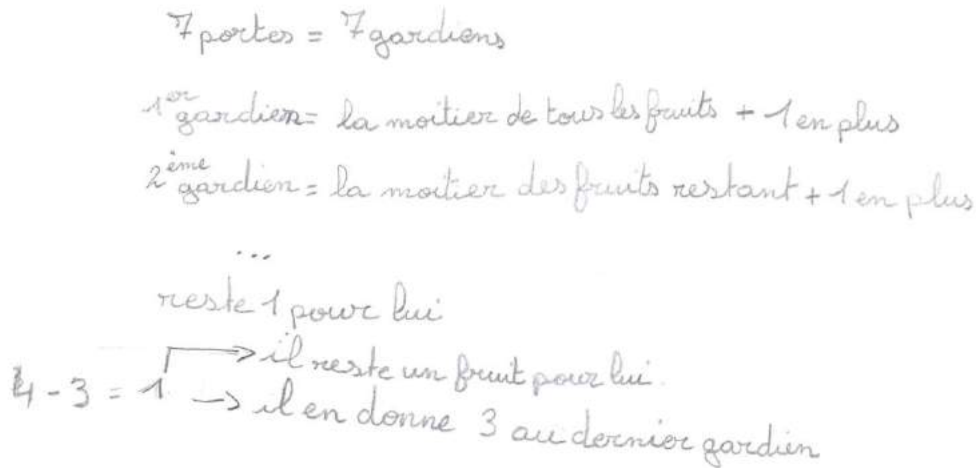


Figure 4.7: Reformulation of the problem (solution of Alice, Mina and Lilou)

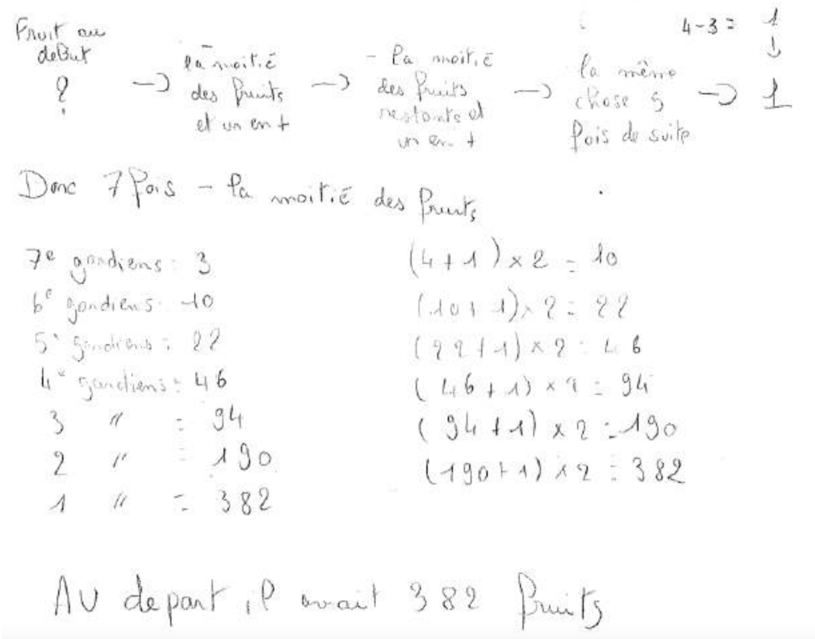


Figure 4.8: Schematization of the problem and complete resolution made by Annabelle and Glwadys.

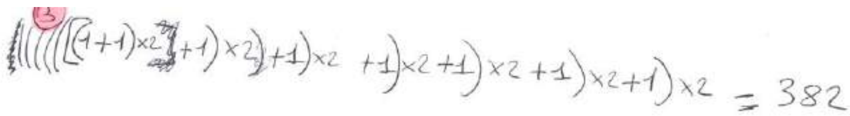


Figure 4.9: With the help of the calculator (final solution of Fayza and Germain, see fig. 4.3 and fig. 4.5)

Nevertheless, several difficulties appeared. The first one is just at the beginning to understand the exact order of the operations, i.e. $(1+1)\times 2=4$ or $1\times 2+1=3$. After that, the main problems arising are the misuse of parentheses or the distributivity of multiplication over addition. Those difficulties were expected by our *a priori* analysis. It was thus necessary to work on it.

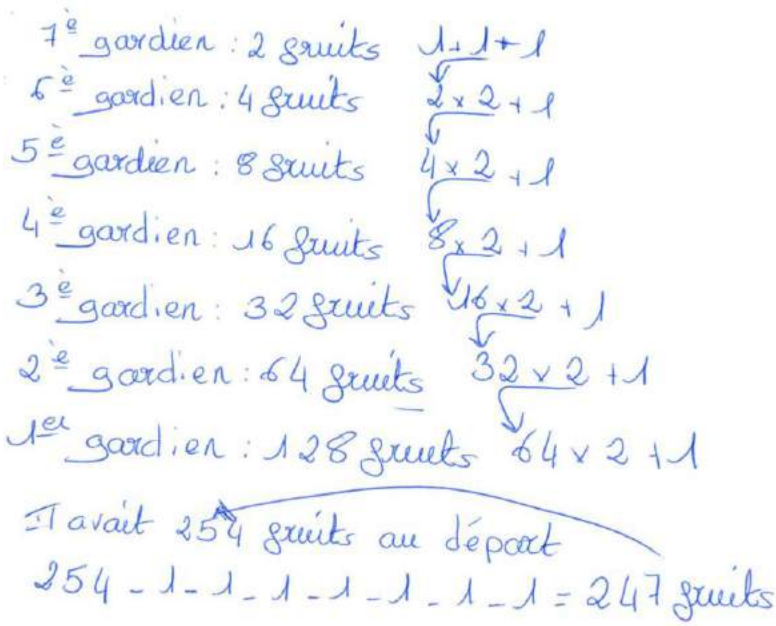
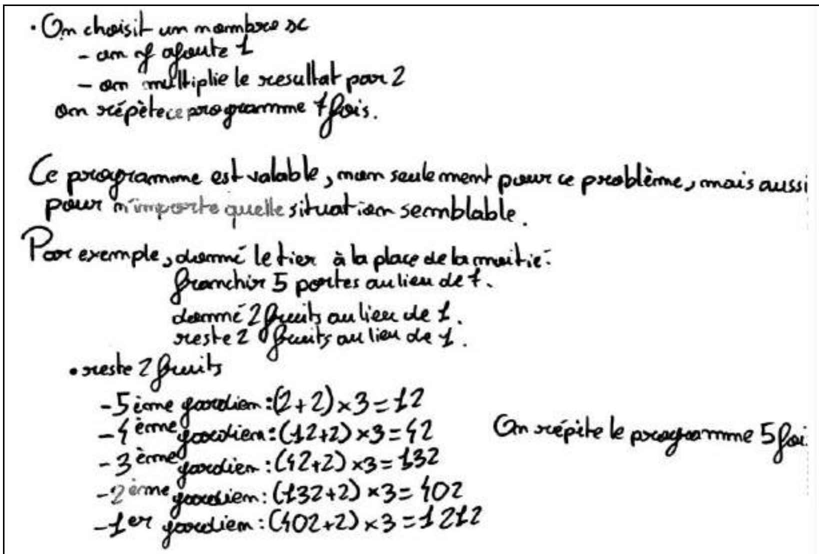


Figure 4.10: Attempt of Baptiste, Manon and Francisco.

4.2 Generalization and coding

You will be able to solve any similar problem. This is Fibonacci's conclusion following his first solution. Pupils understand what kind of similarity (and generality) Fibonacci mentioned. They discuss on the mathematical parameters of the problem: the number of doors, the number of apples given to the doorkeepers, number of apples remaining.For example:



We choose a number x:
– we add to it 1,
– we multiply the result by 2,
We repeat this program 7 times.

This program is valid not only for this problem but also for any similar situation.
For example, give the third rather than half; pass through 5 doors rather than 7; give 2 fruit rather than 1; 2 fruit remain rather than 1. [...]

Figure 4.11: Proposition from the work of Mathéo, Louis, Léo and Maxence.

When pupils all understood the algorithmic nature of the problem and its parameters by following a collective exchange in their proper studyinggroup, we asked to them: "So, how many apples were picked if there remains only one apple after 457 doors?" All (except one girl who wanted to work 'in her own head') experimented with Scratch (in a French environment) to code the solution. They done it alone or in pairs and 73% of our pupils successfully completed this work.

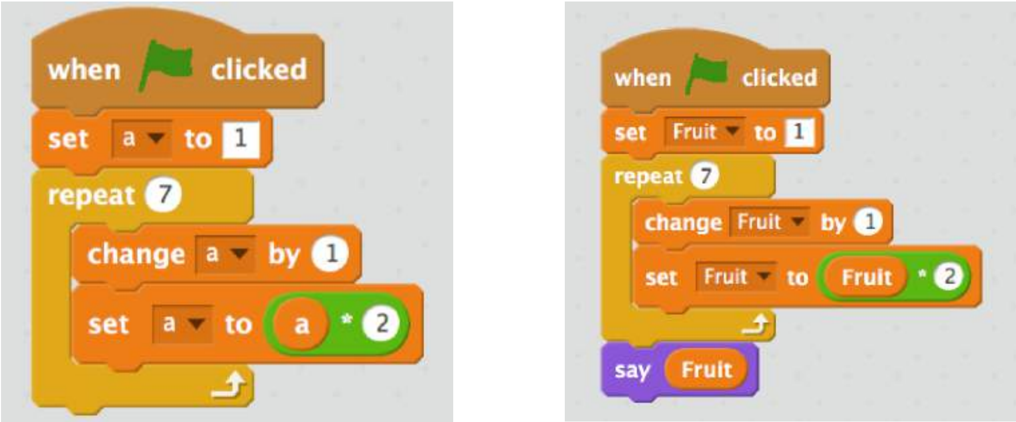


Figure 4.12: First examples – works of Leo and Maxence (see fig. 4.11) and work of Annabelle and Gwladys (see fig. 4.8) –with two successive operations (adding 1 and multiplying by 2)

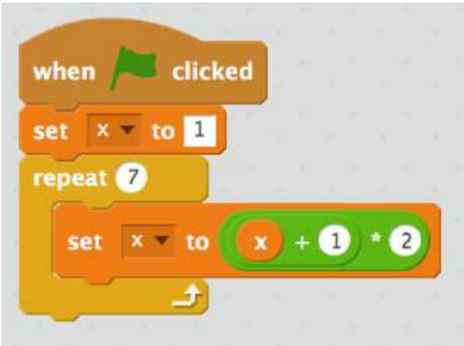


Figure 4.13: The work of Fayza and Germain (see fig. 4.3, 4.5, 4.9) where operations are reduced in one line

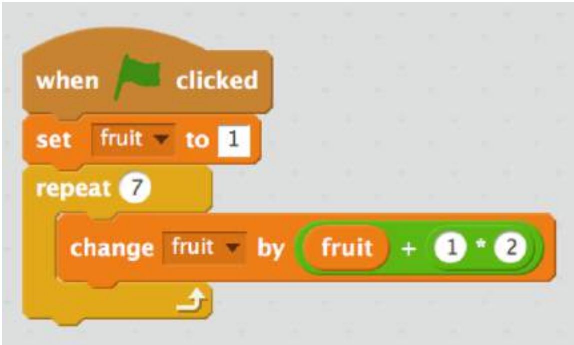


Figure 4.14:Alice's work (see fig. 4.7) : misuse of parentheseswith the Scratch conventions

In a final step, all class wrote a collective program taking into account several mathematical parameters. Some parameters wererequested from the user and others were modifiable directly in the program. Thus, pupils learn the notion of 'variable' in computer science, distinct from the mathematical notion (with the necessity, in programming, to initialize the variable in order to define it). After that, the math teacher checked this learning with several exercises in individual work.

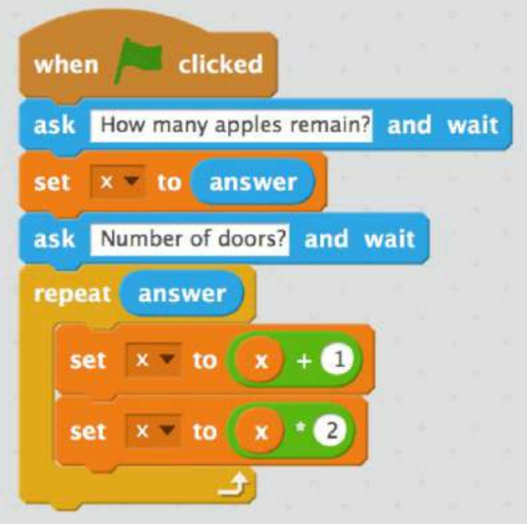


Figure 4.15: The collective coding written with the help of the math teacher

Finally, pupils were invited to think about the mathematical links between the ‘apple orchard problem’ and the following one.

A certain man went on business to Lucca to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and spent 12 denari. Then he returned to Pisa, doubled his money and it is proposed that he had nothing left. It is sought how much he had at the beginning.(Fibonacci, 1857, 258–259; Sigler 2002, 372–373; Moyon, 2016, 54–56)

They easily produced, individually, the following program taking into account the previous one. They understood the generality of the method of inversion proposed by Fibonacci and the category of problems of which these are examples.

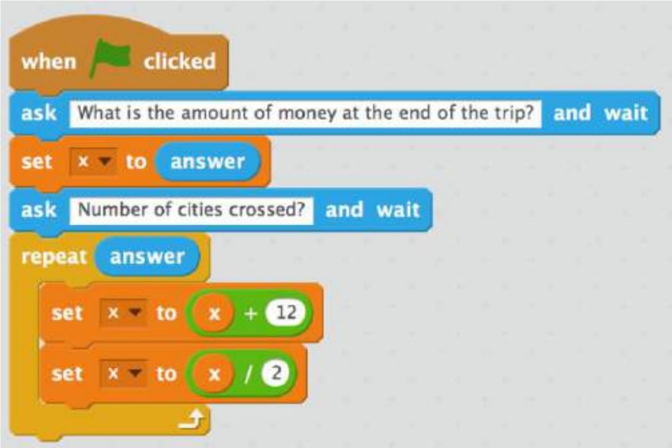


Figure 4.16: The Maxence’s coding

5 Some extensions

5.1 For High-Schools curriculum

The mathematical reasoning of Fibonacci to solve the ‘apple orchard problem’ can be reworkedwith the notion of sequence for more advanced pupils (16 to 18 years old in France). Here, it is possible to consider the extract of the *Liber Abaci* as a starting point to write mathematics at a higher level than did Fibonacci himself.

5.1.1 First sequence working the algorithm backwards

Let be $(\boldsymbol{u_n})_{\boldsymbol{n\geq 0}}$ the sequence defined recursivelyrepresenting the number of apples remaining after the $\boldsymbol{n^{th}}$ door, $\boldsymbol{u_0}$ being the number of applesinside the first door. Thus, we have:

$$\forall n \in \mathbb{N}, \begin{cases} u_0 = ? \\ u_{n+1} = \frac{1}{2}u_n - 1 \end{cases}$$

It is an arithmetico-geometric sequence and its general term[23]is given by:

$$\forall n \in \mathbb{N}^*, u_n = \frac{1}{2^n}(u_0 + 2) - 2$$

Thus, in Fibonacci's conditions, we have:

$$u_7 = \frac{1}{128}(u_0 + 2) - 2 = 1$$
$$\frac{1}{128}(u_0 + 2) = 3$$
$$u_0 = 3 \times 128 - 2$$
$$u_0 = 382$$

5.1.2 Second sequence using algebra

If *x* is the number of applesinitially picked, a new sequence $(v_n)_{n\geq 1}$can be defined giving the number of applesremaining after the n^{th} door by:

$$\forall n \geq 1, \qquad v_n = \frac{1}{2^n}x - \left(1 + \frac{2^{n-1} - 1}{2^{n-1}}\right)$$

Or, after reduction,

$$\forall n \geq 1, \qquad v_n = \frac{1}{2^n}x - \frac{2^n - 1}{2^{n-1}}$$

An elementary mathematical induction can prove it. Thus, in Fibonacci's conditions, we have:

$$v_7 = \frac{1}{128}x - \frac{127}{64} = 1$$
$$\frac{1}{128}x - \frac{127}{64} = 1$$
$$\frac{1}{128}x = 1 + \frac{127}{64}$$
$$x = 128 \times \left(1 + \frac{127}{64}\right) = 382$$

5.2 Another solution method and Other problems from the *Liber augmenti et diminutionis*

The *Liber augmenti et diminutionis* [Book on increase and decrease] is an anonymous text from the twelfth century.[24] Hughes (2001, 107) described itas "a book that reads like a set of lectures on how to solve problemsinvolving numbers". The author's main purpose is to teach the method of double false position. One of the nine chapters of the book is about "apples stolen from an orchard". It containssix problemsdivided into three groups, whose

[23]In France, students do not know the formula for the general term. They do derive it from the recurrence relation or with help from the teacher.

[24]I am preparing a new critical edition of the text with a French translation of the whole text.

statements may besummarized as follows:

1. There are 3 doors; the man gives half and 2 fruit more; 1 fruit remainat the exit.
2. There are 3 doors; the man gives half and 4 (resp. 6, 8) fruit more to the first (resp. second, third) doorkeeper; no fruit remains at the exit.
3. There are 3 doors; the man gives half and the first (resp. second, third) doorkeeper gives back to the man 2 (resp. 4, 6) fruit; the man has 10 fruit at the exit.

In appendix 2, I give the entire textof the first problem (the mostsimilar to the ‘apple orchard problem’ of the *Liber Abaci*). I detailhere the method of double false position used in the *Liber augmenti et diminutionis*.

Let be $n_1 = 100$ a first number (first false position). After the third doorkeeper, exactly 9 apples remain. So, the differencebetween 9 and 1, the number of applesthat should remain, is (an excess of) 8. Let be $n_2 = 200$a second number (second false position). After the third doorkeeper, exactly 21.5 apples remain. So, the difference between 21.5 and 1 is (an excess of) 20.5. Thus, we have two errors[25]: $e_1 = +8$ and $e_2 = +20.5$ (both excesses), correspondingto the numbers initially chosen: $n_1 = 100$ and $n_2 = 200$. Here, the solution is given by the following relation:

$$\frac{n_1 \times e_2 - n_2 \times e_1}{e_2 - e_1} = \frac{100 \times 20.5 - 200 \times 8}{20.5 - 8} = 36$$

The method of double false position is adapted for linear problems that, in algebraic form, reduce to the solution of equations of the type: $ax + b = cx + d$, such as in this case (Chabert, 1999, 83–112). In high-school the problem of proving the correctness of the method may be given as an exercise.

6 Elements of conclusion

In this emprirical contribution, I showed one more time that the history of mathematics may support mathematics education using medieval texts (*Liber Abaci* or *Liber augmenti et diminutionis*) inthe classroom.And, I also highlighted that historycreatesa suitable context to think about mathematical concepts and procedures. This is the main purpose of my contribution.

When pupils engaged with the source, most of them had questions (on terminology or on mathematical procedures) similar to those a professional historian of mathematics would ask, especially when they compared different historical solutions with their own. It is, for me, a great opportunity to develop the critical thinkingof pupils.

Furthermore, I focused only on problems today characterized as ‘recreational’ because I consider them as important sources for problem solving (for different educational levels, even in the context of initial teacher education as I experiment it in the university of Limoges). It is really interesting toappreciatethe richness of these sources.

Finally, I easily integratedan importantaspect ofliteracy in today’s society, namelythe ability to code computer programs (here, using Scratch). In spite of the antiquity of the mathematical source, it was so easy –without hard efforts – for the pupils to engage with these problems, thanks to their algorithmic nature.

[25]That’s why this method is called *ḥisāb al-khata’ayn* [calculation of the two errors] in the Arabic tradition. Fibonacci transliterated the expression by *elchatayn* in his *Liber Abaci* (Fibonacci, 1857, 318).

Appendix 1: the 'Apple Orchard Problem' in the *Liber Abaci*

Here is the text of Fibonacci translatedin English by Sigler (2002, 397) from the edition made by Boncompagni (Fibonacci, 1857).

A certain man entered a certain pleasure garden through 7 doors, and he took from there a number of apples; when he wished to leave he had to give the first doorkeeper half of all the apples and one more; to the second doorkeeper he gave half of the remaining apples and one more. He gave to the other 5 doorkeepers similarly, and there was one apple left for him. It is sought how many apples there were that he collected; you do thus: for the one apple which remained for him you keep 1 to which you add the one apple that he gave to the last doorkeeper; there will be 2 that you double; there will be 4, and he had this many when he came to the last doorkeeper; to this you add the apple that he gave to the sixth doorkeeper; there will be 5 that you double; there will be 10, and this many remained after he left 5 doors; to this you add the one apple of the fifth doorkeeper; there will be 11 that you double; there will be 22 to which you add 1 for the apple that he gave the fourth doorkeeper; there will be 23 that you double; there will be 46 to which you add 1 for the apple that he gave to the third doorkeeper; there will be 47 that you double; there will 94; to his you add 1 for the apple that he gave the second doorkeeper; there will be 95 that you double; there will be 190 to which you add the 1 that he gave at the first door, and you double this amount; there will be 382, and this total is the number of apples; and thus reversing the order that was proposed you will be able to solve any similar problem.

In another way you put the number of collected apples to be the thing from which he gave at the first door $\frac{1}{2}$ of it and 1 apple. There remained therefore $\frac{1}{2}$ thing minus 1 from which he gave one half and one apple at the second door; therefore there remained for him one quarter thing minus $\frac{1}{2}1$[26] apples from which he gave at the third door one half and 1 apple. Therefore there remained for him $\frac{1}{8}$ thing minus $\frac{3}{4}1$ apples, half of which and one apple, he gave at the fourth door, and thus there remained for him $\frac{1}{16}$ thing minus $\frac{7}{8}1$ apples; of this half and one apple more he gave at the fifth door; there remained for him $\frac{1}{32}$ thing minus $\frac{15}{16}1$ apples of which half and one apple more he gave at the sixth door; there remained for him $\frac{1}{64}$ thing minus $\frac{31}{32}1$ apples; of this still he gave at the seventh door half and one apple more; there remained for him $\frac{1}{128}$ thing minus $\frac{63}{64}1$ apples which is equal to one apple; this is namely the one which remained after his passing the seven doors. If $\frac{63}{64}1$ apples are commonly added, then it will result that $\frac{1}{128}$ thing is equal to $\frac{63}{64}$ 2 apples. Therefore you multiply the $\frac{63}{64}2$ by the 128; there will be similarly 382 apples.

[26]Inspired by the Maghrebian mathematics' texts, the writing $\frac{a}{b}n$ represents the sum $n+\frac{a}{b}$.

Appendix 2: the ‘Apple Orchard Problem’in the *Liber augmenti et diminutionis*

I give below an English translation of the Latin text edited by Libri (1838, 336–339) and Hughes (2001, 140).

A certain man went into an orchard and picked some apples. The orchard had three gates each guarded by a bailiff. So that man gave the first bailiff half of what he picked plus two apples more. He gave the second bailiff half [of what remained] and two more apples. He gave the third half [of what remained] and two apples more. The man was left with one [apple]. How many apples did he pick?

The procedure consists in taking a platter[27] with one hundred. You give half and two more to the first [bailiff]. You still have forty-eight [apples]. You give half and two more to the second. You still have twenty-two. You give half and two more to the third. You still have nine. So compare this with the one [apple] which was left. Thus now the error is eight by excess, that is the first error.

Then take a second platter which is two hundred. And give half and two more to the first [bailiff]. You still have ninety-eight [apples]. And give half and two more to the second. You still have forty-seven. And give half and two more to the third. You still have twenty-one and a half. So compare this with the one [apple] which was left. Thus now the error is twenty and a half by excess, that is the second error.

So multiply the first platter which is one hundred, by the error of the second platter which is twenty-one and a half. It results two thousands and fifty. Then multiply the second platter by the error of the first platter, that is to say multiply two hundred by eight and it results one thousand and six hundred. So take off the smaller of the two numbers from the larger, i.e. diminish one thousand and six hundred from two thousands and fifty. It remains four hundred and fifty. Then subtract one of both errors from the other, i.e. take off eight from twenty-one and a half. It remains twelve and a half. Then divide four hundred and fifty by it, and it results thirty-six. This is the number of apples picked.

REFERENCES

Caianiello, E. (2012). La vita e l’opera di Leonardo Pisano, Appendice 1. In G. Matino (Ed.), *Forme e modi delle linguedei testi tecnici antichi* (pp.54-60). Naples: D'Auria.

Caianiello, E. (2013). Leonardo of Pisa and the Liber Abaci. Biographical elements and the project of the work. In A. Bernard, & C. Proust (Eds.) *Scientific sources and teaching contexts throughout history: Problems and perspectives* (pp. 217-246). Dordrecht: Springer.

Chabert, J.-L. (Ed.) (1999). *A History of Algorithms. From the Pebble to the Microchip*. Paris: Belin.

Datta, B., & Singh, A.N. (2004). *History of Hindu Mathematics*. (vol. 1). Delhi: Bharatiya Kala Prakashan.

Fibonacci, L. (1854). *Tre scritti inediti di Leonardo Pisano*. B. Boncompagni (Ed.). Firenze: Tipografia Galileina di M. Cellini.

Fibonacci, L. (1857). *Il liber abaci*. B. Boncompagni (Ed.). Roma: Tipografia della scienze matematiche e fisiche.

Fibonacci, L. (1862). Practica geometriae. B. Boncompagni (Ed.). Roma: Tipografia della scienze matematiche e fisiche.

Folkerts, M. (1978). *Die älteste mathematische Aufgabensammlung in lateinischer Sprache: die Alkuin*

[27]The Latin word is *lanx*. It refers to the maghebian terminology of the ‘method of the Scales’ (Chabert & al., 1999, 101–103).

zugeschriebenen "Propositiones ad acuendos iuvenes". Vienna: Springer Verlag.
Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education*, *10*(4), 391–408.
Fried, M.N., Guillemette, D. & Jahnke, H.N., (2016). Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting of the International Study Group on the Relations Between the History and Pedagogy of Mathematics* (pp.211–230). Montpellier: IREM de Montpellier.
Hadley, J., & Singmaster, D. (1992). Problems to Sharpen the Young. The Mathematical Gazette, *76*(475), 102–126.
Hadley, J.& Singmaster, D. (1995). *Problems to Sharpen the Young*. London: South Bank University.
Hannah, J. (2011). Conventions for recreational problems in Fibonacci's Liber Abbaci. *Archive for History of Exact Sciences*, *65*(2), 155–180.
Heeffer, A. (2014). How algebra spoiled recreational problems: A case study in the cross-cultural dissemination of mathematics. *Historia Mathematica*, *41*, 400–437.
Hermelink, H. (1978). Arabische Unterhaltungsmathematik als Spiegel Jahrtausendealter Kulturbeziehungen zwischen Ost und West. *Janus*, *65*, 105–117.
Høyrup, J. (2009). Fibonacci's De Practica Geometrie, Edited by Barnabas Hughes. *Historia Mathematica*, *36*, 73–89.
Høyrup, J. (2008). Mathematics, Practical and Recreational. In H. Selin (Ed.), *Encyclopaedia of the History of Science, Technology, and Medicine in non-Western Cultures* (pp.1352–1359). Dordrecht: Springer.
Høyrup, J. (2016). Fibonacci, Leonardo. In *The Encyclopaedie of Islam* (vol.3, pp.58–61). Leiden-Boston: Brill.
Hughes, B. (2001). A Treatise on Problem Solving From Early Medieval Latin Europe. *Mediaeval Studies*, *63*, 107–141.
Hughes, B. (2004). Fibonacci, Teacher of Algebra: An Analysis of Chapter 15.3 of Liber Abbaci. *Mediaeval Studies*, *66*, 313–361.
Hughes, B. (2008). *Fibonacci's De Practica Geometrie*. New York: Springer.
Libri, G. (1838). *Histoire des sciences mathématiques en Italie, depuis la Renaissance des Lettres jusqu'à la fin du XVIIe siècle*. (vol. 1). Paris: J. Renouard.
Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics*, *71*, 235–261.
Miura, N. (1981). The Algebra in the Liber Abaci of Leonardo Pisano. *Historia Scientiarum*, *21*, 57–65.
Moyon, M. (2012). Algèbre & Practica geometriae en Occident médiéval latin: Abū Bakr, Fibonacci et Jean de Murs. In S. Rommevaux, M. Spiesser, & M.-R. Massa Esteve (Eds.), *Pluralité de l'algèbre à la Renaissance* (pp. 33–65). Paris: Honoré Champion.
Moyon, M. (2013). Diviser en multipliant les approches... Quand les mathématiques remontent aux sources. *Repères-IREM*, *93*, 47–77.
Moyon, M. (2016). *Fibonacci: extraits du Liber Abaci*. Paris: ACL-Les éditions du Kangourou.
Moyon, M. (2017). *La géométrie de la mesure dans les traductions arabo-latines médiévales*. Turnhout: Brepols.
Moyon, M. (2019). Récréations mathématiques et algorithmiques dans le *Liber Abaci* de Fibonacci (XIII^{e} siècle): une mise en oeuvre pédagogique. In N. Chevalarias, M. Gandit, M. Morales, & D. Tournès (Eds.), *Mathématiques récréatives: éclairages historiques et épistémologiques* (pp. 225–251). Grenoble: UGA Éditions.
Moyon, M., & Spiesser, M. (2015). L'arithmétique des fractions dans l'œuvre de Fibonacci: fondements & usages. *Archive for History of Exact Sciences*, *69*(4), 391–427.
Moyon, M. (2017). Dividing a triangle in the Middle Ages: an example from Latin works on practical

geometry. In É. Barbin (Ed.). *Let History into Mathematics Classroom* (pp.17–29). Cham: Springer.

Moyon, M., & Tournès, D. (Eds.). (2018). *Passerelles: enseigner les mathématiques par leur histoire au cycle 3*, Bouc-Bel-Air: ARPEME.

Picutti, E. (1983). Il Flos di Leonardo Pisano.Traduzioni e commenti. *Physis*, *25*. 293–387.

Rommevaux, S. (2008). Fibonacci's De Practica Geometrie, Edited by Barnabas Hughes. *Aestimatio*, *5*, 152–156.

Sesiano, J. (2009). *An introduction to the history of algebra: solving equations from Mesopotamian times to the Renaissance*. Providence: American Mathematical Society.

Sesiano, J. (2014a). *The* Liber mahameleth*. A 12th-century mathematical treatise*. Cham: Springer.

Sesiano, J. (2014b). *Récréations mathématiques au Moyen Âge*. Lausanne: Presses polytechniques et universitaires romandes.

Sigler, L. (2002). *Fibonacci's Liber Abaci*. New York: Springer.

Singmaster, D., (1988). Some early sources in recreational mathematics. In C. Hay (Ed.), *Mathematics from Manuscript to Print 1300–1600*, (pp.195–208). Oxford: Clarendon Press.

Singmaster, D., (2013). *Assorted Articles on Recreational Mathematics and the History of Mathematics*. Available at https://www.puzzlemuseum.com/singma/singma-index.htm Accessed: 22 August 2018.

Tropfke, J., Vogel, K., Reich, K., & Gericke, H. (1980). *Geschichte der Elementarmathematik* (4. Aufl.). Berlin: de Gruyter.