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# BRICK POLYTOPES, LATTICE QUOTIENTS, AND HOPF ALGEBRAS

VINCENT PILAUD

Abstract. This paper is motivated by the interplay between the Tamari lattice, J.-L. Loday's realization of the associahedron, and J.-L. Loday and M. Ronco's Hopf algebra on binary trees. We show that these constructions extend in the world of acyclic $k$-triangulations, which were already considered as the vertices of V. Pilaud and F. Santos' brick polytopes. We describe combinatorially a natural surjection from the permutations to the acyclic $k$-triangulations. We show that the fibers of this surjection are the classes of the congruence $\equiv^k$ on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule $UacV_1b_1\cdots V_kb_kW \equiv^k UcaV_1b_1\cdots V_kb_kW$ for letters $a < b_1, \dots, b_k < c$ and words $U, V_1, \dots, V_k, W$ on $[n]$. We then show that the increasing flip order on $k$-triangulations is the lattice quotient of the weak order by this congruence. Finally, we use this surjection to define a Hopf subalgebra of C. Malvenuto and C. Reutenauer's Hopf algebra on permutations, indexed by acyclic $k$-triangulations, and to describe the product and coproduct in this algebra in term of combinatorial operations on acyclic $k$-triangulations.

Keywords. Brick polytopes – Multitriangulations – Pipe dreams – Combinatorial Hopf algebras – Lattice quotients

## Introduction

The motivation of this paper comes from relevant combinatorial, geometric, and algebraic structures on permutations, binary trees and binary sequences. Classical surjections from permutations to binary trees (BST insertion) and from binary trees to binary sequences (canopy) yield:

- lattice morphisms from the weak order, via the Tamari lattice, to the boolean lattice;
- normal fan coarsenings from the permutahedron, via J.-L. Loday's associahedron [Lod04], to the parallelepiped generated by the simple roots $\mathbf{e}_{i+1} - \mathbf{e}_i$;
- Hopf algebra refinements from C. Malvenuto and C. Reutenauer's algebra [MR95], via J.-L. Loday and M. Ronco's algebra [LR98], to L. Solomon's descent algebra [Sol76].

These fascinating connections were widely extended by N. Reading in his work on "Lattice congruences, fans and Hopf algebras" [Rea05]. In particular, he proves that any lattice congruence $\equiv$ of the weak order on the permutations of $\mathfrak{S}_n$ (see Section 1.6 for proper definitions) defines a complete simplicial fan $\mathcal{F}_\equiv$ refined by the Coxeter fan, and he characterizes in terms of simple rewriting rules the families $(\equiv_n)_{n\in\mathbb{N}}$ of lattice congruences of the weak orders on $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ which yield Hopf subalgebras of C. Malvenuto and C. Reutenauer's algebra on permutations. His work opens two natural questions. On the geometric side, it is not clear which of the fans $\mathcal{F}_\equiv$ are actually normal fans of polytopes, as in the previous example of the associahedron. On the algebraic side, this construction produces a combinatorial Hopf algebra whose basis is indexed by the congruence classes of $(\equiv_n)_{n\in\mathbb{N}}$. However, the product and coproduct in this Hopf algebra are performed extrinsically: the algebra is embedded in C. Malvenuto and C. Reutenauer's algebra on permutations and the computations are performed at that level. The remaining challenge is to realize the resulting Hopf algebra intrinsically by attaching a combinatorial object to each congruence class of $(\equiv_n)_{n\in\mathbb{N}}$ and working out the rules for product and coproduct directly on these combinatorial objects. The present paper answers these two questions for a relevant family of lattice congruences of the weak order, generalizing the classical sylvester congruence [HNT05].

Our starting point is the world of acyclic multitriangulations. A *$k$-triangulation* of a convex $(n+2k)$-gon is a maximal set of diagonals such that no $k+1$ of them are pairwise crossing. Multitriangulations were introduced by V. Capoyleas and J. Pach [CP92] in the context of extremal theory for geometric graphs and studied for their rich combinatorial properties [Nak00, DKM02, Jon05, PS09]. Using classical point-line duality, V. Pilaud and M. Pocchiola interpreted $k$-triangulations of the $(n+2k)$-gon as pseudoline arrangements on $n$-level sorting networks [PP12], which can also be seen more combinatorially as beam arrangements in a trapezoidal shape [Pil10, Section 4.1.4]. In this paper, we call these specific arrangements *$(k,n)$-twists*. As observed in [Stu11, SS12],

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this connects multitriangulations to (specific) pipe dreams studied in [BB93, KM05]. Motivated by possible polytopal realizations of the simplicial complex of $(k+1)$-crossing-free sets of diagonals of a convex $(n+2k)$-gon, V. Pilaud and F. Santos defined in [PS12] the brick polytope of a sorting network, whose vertices correspond to certain *acyclic* pseudoline arrangements on the network. When $k=1$, the pseudoline arrangements on the trapezoidal network correspond to the triangulations of the $(n+2)$-gon. They are all acyclic and the brick polytope coincides with J.-L. Loday's associahedron [Lod04]. The goal of this paper is to explore further properties of the acyclic $(k,n)$-twists for arbitrary $k$ and $n$.

Section 1 is devoted to the combinatorics of acyclic $(k,n)$-twists. We present a purely combinatorial description of the natural map $\mathsf{ins}^k$ from the permutations of $\mathfrak{S}_n$ to the acyclic $(k,n)$-twists in terms of successive insertions in a $k$-twist. Extending the sylvester congruence of [HNT05], we show that the fibers of this map are the classes of a congruence $\equiv^k$ defined as the transitive closure of the rewriting rule $UacV_1b_1V_2b_2\cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2\cdots V_kb_kW$ where $a,b_1,\dots,b_k,c\in[n]$ are such that $a<b_i<c$ for all $i\in[k]$ and $U,V_1,\dots,V_k,W$ are words on $[n]$. This congruence is a lattice congruence of the weak order, so that the increasing flip order on the acyclic $(k,n)$-twists defines a lattice, generalizing the Tamari lattice. We also define a canopy map $\mathsf{can}^k$ from the acyclic $(k,n)$-twists to the acyclic orientations of the graph $\mathrm{G}^k(n)=([n],\{\{i,j\}\mid |i-j|\le k\})$. For $\tau\in\mathfrak{S}_n$, the composition $\mathsf{rec}^k(\tau)=\mathsf{can}^k\big(\mathsf{ins}^k(\tau)\big)$ records the relative positions in $\tau$ of any entries $i$ and $j$ such that $|i-j|\le k$, thus generalizing the recoils of the permutation $\tau$. Note that this generalization of recoils was already considered by J.-C. Novelli, C. Reutenauer and J.-Y. Thibon in [NRT11] with a slightly different presentation. To sum up, at the combinatorial level, we obtain a commutative triangle of lattice homomorphisms from the weak order, via the increasing flip order on the acyclic $(k,n)$-twists, to the boolean lattice on acyclic orientations of $\mathrm{G}^k(n)$.

In Section 2, we survey and partially revisit the geometric aspects of these combinatorial maps. We recall the definitions of the classical permutahedron $\mathsf{Perm}^k(n)$, of the brick polytope $\mathsf{Brick}^k(n)$ of the $n$-level trapezoidal network [PS12], and of the zonotope $\mathsf{Zono}^k(n)$ of the graph $\mathrm{G}^k(n)$. Their vertices correspond to the permutations of $\mathfrak{S}_n$, to the acyclic $(k,n)$-twists, and to the acyclic orientations of $\mathrm{G}^k(n)$, respectively. When oriented in the direction $(n-1,n-3,\dots,1-n)$, their 1-skeleta are the Hasse diagrams of the weak order on permutations of $\mathfrak{S}_n$, of the increasing flip lattice on acyclic $(k,n)$-twists, and of the boolean lattice on acyclic orientations of $\mathrm{G}^k(n)$. The maps $\mathsf{ins}^k$, $\mathsf{rec}^k$ and $\mathsf{can}^k$ can be read as inclusions of normal cones of vertices of $\mathsf{Perm}^k(n)$, $\mathsf{Brick}^k(n)$ and $\mathsf{Zono}^k(n)$. The reader can already glance at Figure 8 on page 13 for an illustration of the geometric situation. Although most results in this section are not new, they provide the geometric side of the picture.

Section 3 finally presents the new algebraic construction which motivated this paper. We consider C. Malvenuto and C. Reutenauer's Hopf algebra on permutations [MR95], that we denote by $\mathsf{FQSym}$. We consider the subspace $\mathsf{Twist}^k$ generated by the sums of the elements of $\mathsf{FQSym}$ over the fibers of $\mathsf{ins}^k$. Since these fibers are classes of a congruence $\equiv^k$ which satisfy standard compatibility conditions with the shuffle and the standardization [Rea05, Hiv04, HN07, Pri13], this subspace automatically defines a Hopf subalgebra of $\mathsf{FQSym}$. Our approach with $k$-twists provides a combinatorial interpretation for this subalgebra, and it is interesting to describe the product and the coproduct directly on acyclic $k$-twists. Note that our Hopf algebra $\mathsf{Twist}^k$ on acyclic $k$-twists is sandwiched in between C. Malvenuto and C. Reutenauer's Hopf algebra $\mathsf{FQSym}$ on permutations [MR95] and J.-C. Novelli, C. Reutenauer and J.-Y. Thibon's $k$-recoil Hopf algebra $\mathsf{Rec}^k$ on acyclic orientations of $\mathrm{G}^k(n)$ [NRT11]. We finally briefly study further algebraic properties of $\mathsf{Twist}^k$: we define multiplicative bases and study their indecomposable elements, we connect it to integer point transforms of the normal cones of the brick polytope, and we define a natural extension of dendriform structures in the context of $k$-twists.

The results of this paper can be extended in three independent directions: on Cambrian acyclic twists (parametrized by a sequence of signs [Rea06, HL07, PS12], similar to [CP17, Part 1]), on tuples of Cambrian twists (similar to [LR12, Gir12] and [CP17, Part 2]), and on Schröder twists (corresponding to the faces of the brick polytope, similar to [Cha00] and [CP17, Part 3]). These extensions are skipped here to keep this paper short, but details can be found in [Pil15].

## 1. Combinatorics of twists

1.1. **Pipe dreams and twists.** A *pipe dream* is a filling of a triangular shape with crosses $+$ and elbows $\lrcorner\!\!\ulcorner$ so that all pipes entering on the left side exit on the top side. These objects were studied in the literature, under different names including "pipe dreams" [KM05], "RC-graphs" [BB93], "beam arrangements" [Pil10, PS12]. This paper is mainly concerned with the following specific family of pipe dreams which already appeared under different names in [Pil10, PP12, Stu11].

**Definition 1** ([Pil10, PP12, Stu11])**.** *For $k, n \in \mathbb{N}$, a $(k, n)$-twist (we also use just $k$-twist, or even just twist) is a pipe dream with $n + 2k$ pipes such that*

- *no two pipes cross twice (the pipe dream is reduced),*
- *the pipe which enters in row $i$ exits in column $i$ if $k \leq i \leq n+k$, and in column $n + 2k + 1 - i$ otherwise. Here and throughout the paper, rows are indexed from bottom to top and columns are indexed from left to right.*

*Besides the first $k$ and last $k$ trivial pipes, a $(k, n)$-twist has $n$ relevant pipes, labeled by $[n]$ from bottom to top, or equivalently from left to right. In other words, the $p$th pipe enters in row $p + k$ and exits at column $p + k$ of the $(n + 2k) \times (n + 2k)$-triangular shape. We denote by $\mathcal{T}^k(n)$ the set of $(k, n)$-twists.*

**Definition 2** ([PS12])**.** *The contact graph of a $(k, n)$-twist $\mathrm{T}$ is the directed multigraph $\mathrm{T}^{\#}$ with vertex set $[n]$ and with an arc from the SE-pipe to the WN-pipe of each elbow in $\mathrm{T}$ involving two relevant pipes. We say that a twist $\mathrm{T}$ is acyclic if its contact graph $\mathrm{T}^{\#}$ is (no oriented cycle), and we then let $\lhd_T$ be the transitive closure of $\mathrm{T}^{\#}$. We denote by $\mathcal{AT}^k(n)$ the set of acyclic $(k, n)$-twists.*

Figure 1 illustrates examples of $(k, 5)$-twists and their contact graphs for $k = 0, 1, 2, 3$. The first two are acyclic, the last two are not. Except in Figure 2, we only represent the $n$ relevant pipes of the $(k, n)$-twists and hide the other $2k$ trivial pipes (the first $k$ and last $k$ pipes).

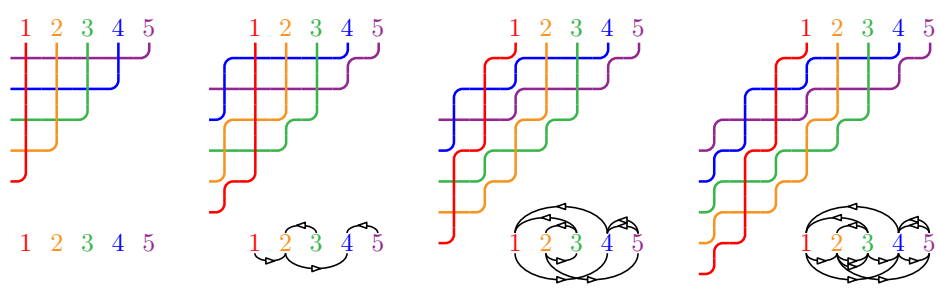


Figure 1. $(k, 5)$-twists (top) and their contact graphs (bottom) for $k = 0, 1, 2, 3$.

**Example 3** (1-twists, binary trees and triangulations)**.** As already observed by different authors [Woo04, Pil10, PP12, Stu11], $(1, n)$-twists are in bijective correspondence with triangulations of a convex $(n + 2)$-gon. This bijection is illustrated in Figure 2. It has been extended to a correspondence between $(k, n)$-twists and $k$-triangulations of a convex $(n+2k)$-gon [PP12, Stu11].

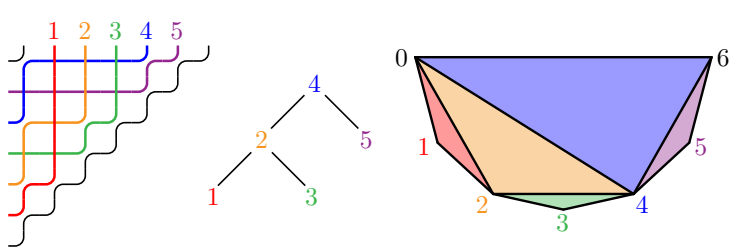


Figure 2. A $(1, n)$-twist $\mathrm{T}$ (left), its contact graph $\mathrm{T}^{\#}$ (middle), and its dual triangulation $\mathrm{T}^{\star}$ of the $(n + 2)$-gon (right).

1.2. **Elementary properties of pipes and twists.** We now give some elementary properties of pipes and twists needed in the next sections. For a given pipe in a twist, we call:

- WE-*crosses* its horizontal crosses ┼, and SN-*crosses* its vertical crosses ┼,
- SE-*elbows* its elbows ╭ (*aka.* "peaks") and WN-*elbows* its elbows ╯ (*aka.* "valleys"),
- *internal steps* the segments between two consecutive elbows and *external steps* the first and last steps of the pipe.

The following statements gather some elementary properties of pipes in $k$-twists already observed in earlier works, see *e.g.* [Pil10, Section 4.1.4]. Detailed proofs can be found in [Pil15].

**Lemma 4.** *The following properties hold for any $k, n \in \mathbb{N}$:*

*(i) A $(k,n)$-twist has $\binom{n}{2}$ crosses ┼ and $kn$ elbows ┘╭.*

*(ii) The $p$th pipe of a $(k,n)$-twist has*
- *$n-1$ crosses: $p-1$ WE-crosses ┼ and $n-p$ SN-crosses ┼,*
- *$2k+1$ elbows: $k$ SE-elbows ╭ and $k+1$ WN-elbows ╯,*
- *$2k+2$ steps: $k+1$ vertical steps and $k+1$ horizontal steps.*

*(iii) The interior of the rectangle defined by two consecutive steps of a pipe contains no elbow.*

**Lemma 5.** *For two pipes $\mathrm{p}, \mathrm{p}'$, write $\mathrm{p} < \mathrm{p}'$ when $\mathrm{p}$ starts below and ends to the left of $\mathrm{p}'$.*

*(i) If $\mathrm{p} < \mathrm{p}'$ are two pipes of a twist $\mathrm{T}$ such that the crossing between $\mathrm{p}$ and $\mathrm{p}'$ is on an internal step of $\mathrm{p}'$, then there is a path from $\mathrm{p}$ to $\mathrm{p}'$ in the contact graph $\mathrm{T}^{\#}$ of $\mathrm{T}$.*

*(ii) If $\mathrm{p} < \mathrm{p}'$ are two pipes of a twist $\mathrm{T}$ and are incomparable in the contact graph $\mathrm{T}^{\#}$ of $\mathrm{T}$, then the last vertical step of $\mathrm{p}$ crosses the first horizontal step of $\mathrm{p}'$.*

*(iii) If a twist $\mathrm{T}$ is acyclic, then no two pipes of $\mathrm{T}$ cross at internal steps of both.*

When $\mathrm{T}$ is acyclic, we denote by $\lhd_{\mathrm{T}}$ the transitive closure of its contact graph $\mathrm{T}^{\#}$. Note the difference between our notations $\mathrm{p} < \mathrm{p}'$ (meaning that $\mathrm{p}$ starts below and ends to the left of $\mathrm{p}'$) and $\mathrm{p} \lhd_{\mathrm{T}} \mathrm{p}'$ (meaning that $\mathrm{p}$ is smaller than $\mathrm{p}$ in the contact graph $\mathrm{T}^{\#}$).

1.3. **Pipe insertion and deletion.** We now define the pipe insertion and deletion, two reverse operations on $k$-twists: an insertion transforms a $(k,n)$-twist into a $(k,n+1)$-twist by inserting a single pipe while a deletion transforms a $(k,n+1)$-twist into a $(k,n)$-twist by deleting a single pipe. Insertions are always possible (see Definition 6), while only certain pipes are allowed to be deleted (see Definition 9). We start with pipe insertions.

**Definition 6.** *Consider a $(k,n)$-twist $\mathrm{T}$ with an increasing relabeling $\lambda : [n] \to \mathbb{N}$ of its relevant pipes and an integer $q \in \mathbb{N}$. Let $p \in \{0, \dots, n\}$ be such that $\lambda(p) \le q < \lambda(p+1)$, where we set the convention $\lambda(0) = -\infty$ and $\lambda(n+1) = +\infty$. The* pipe insertion *of $q$ in the relabeled $(k,n)$-twist $\mathrm{T}$ produces the relabeled $(k,n+1)$-twist $\mathrm{T} \Cap q$ obtained by:*

- *inserting a row and a column in the triangular shape between $p+k-1$ and $p+k$,*
- *filling in with elbows ┘╭ the new boxes $(p+k,p), (p+k+1,p+1), \dots, (p+2k,p+k)$,*
- *filling in with crosses ┼ all other new boxes $(p+k,j)$ for $j < p$ and $(i,p+k)$ for $i > p+2k$,*
- *relabeling the $r$th relevant pipe of the resulting twist by $\lambda(r)$ if $r < p$, by $q$ if $r = p$, and by $\lambda(r-1)$ if $r > p$.*

Figure 3 illustrates the insertion of 4 in the $(k,5)$-twists of Figure 1 relabeled by $[2,3,6,8,9]$. The following statement is an immediate consequence of the definition of pipe insertion.

**Lemma 7.** *The contact graph $(\mathrm{T} \Cap q)^{\#}$ is obtained from $\mathrm{T}^{\#}$ by connecting to some existing nodes the new node corresponding to the inserted pipe $q$. In particular, the node corresponding to the inserted pipe is a source of the contact graph $(\mathrm{T} \Cap q)^{\#}$.*

**Example 8** (Insertion in 1-twists, triangulations and binary search trees)**.** The following operations are equivalent under the bijections between 1-twists, triangulations, and binary search trees (see Example 3 and Figure 2):

- the pipe insertion of $q$ in the (relabeled) 1-twist $\mathrm{T}$,
- the triangle insertion of $q$ in the (relabeled) triangulation $\mathrm{T}^{\star}$,
- the node insertion $q$ in the (relabeled) binary search tree $\mathrm{T}^{\#}$.

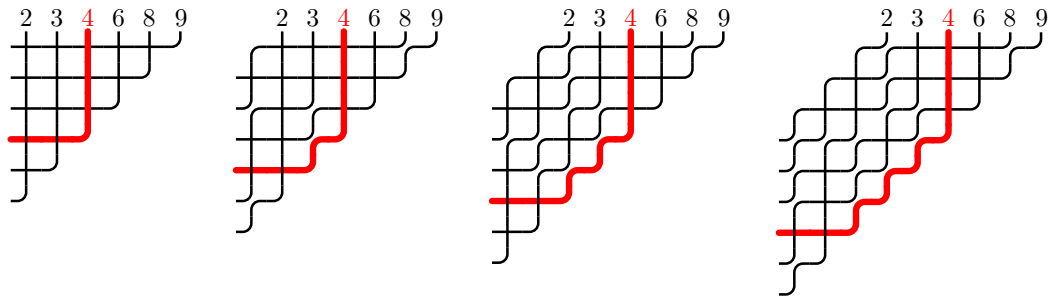


FIGURE 3. Inserting 4 in the $(k, 5)$-twists of Figure 1. The inserted pipe is in bold red.

We now define the deletion, which just erases a pipe from a $(k, n+1)$-twist.

**Definition 9.** *Consider a $(k, n+1)$-twist $\mathrm{T}$ with an increasing relabeling $\lambda : [n+1] \to \mathbb{N}$ of its relevant pipes. Assume that the $p$th pipe of $\mathrm{T}$, labeled by $\lambda(p) = q$, is a source of the contact graph $\mathrm{T}^{\#}$. Then the* pipe deletion *of $q$ in the relabeled $(k, n+1)$-twist $\mathrm{T}$ produces the relabeled $(k, n)$-twist $\mathrm{T} \vee\!\!\!\!\vee q$ obtained by:*

- *deleting the $(p+k)$th row and column of $\mathrm{T}$,*
- *relabeling the $r$th relevant pipe of the resulting twist by $\lambda(r)$ if $r < p$ and by $\lambda(r+1)$ if $r \geq p$.*

The following statements are immediate consequences of the definitions.

**Lemma 10.** *The contact graph $(\mathrm{T} \vee\!\!\!\!\vee q)^{\#}$ is obtained from $\mathrm{T}^{\#}$ by deleting the node corresponding to the delete pipe $q$.*

**Lemma 11.** *For any $(k, n)$-twist $\mathrm{T}$ relabeled by $\lambda : [n] \to \mathbb{N}$ and any integer $q \in \mathbb{N}$, we have*

- *$(\mathrm{T} \wedge\!\!\!\!\wedge q) \vee\!\!\!\!\vee q = \mathrm{T}$, and*
- *$(\mathrm{T} \vee\!\!\!\!\vee q) \wedge\!\!\!\!\wedge q = \mathrm{T}$ as soon as $q \in \lambda([n])$ labels a source of $\mathrm{T}$.*

1.4. **$k$-twist correspondence.** We now present a natural surjection from permutations to acyclic $k$-twists. It relies on an insertion operation on pipe dreams similar to the insertion in binary search trees (see Example 13 for details). It is motivated by the geometry of the normal fan of the corresponding brick polytope (see Section 2 and [PS12]).

We now describe this algorithm. From a permutation $\tau := [\tau_1, \ldots, \tau_n]$ (written in one-line notation), we construct a $(k, n)$-twist $\mathsf{ins}^k(\tau)$ obtain from the $(k, 0)$-twist by successive pipe insertions of the entries $\tau_n, \ldots, \tau_1$ of $\tau$ read from right to left. Equivalently, starting from the empty triangular shape, we insert the pipes $\tau_n, \ldots, \tau_1$ of the twist such that each new pipe is as northwest as possible in the space left by the pipes already inserted. This procedure is illustrated in Figure 4 for the permutation 31542 and different values of $k$.

**Proposition 12.** *For any $(k, n)$-twist $\mathrm{T}$, the permutations $\tau \in \mathfrak{S}_n$ such that $\mathsf{ins}^k(\tau) = \mathrm{T}$ are precisely the linear extensions of the contact graph of $\mathrm{T}$. In particular, $\mathsf{ins}^k$ is a surjection from the permutations of $\mathfrak{S}_n$ to the acyclic $(k, n)$-twists.*

*Proof.* We prove the result by induction on $n$. Consider a permutation $\tau = [\tau_1, \ldots, \tau_n] \in \mathfrak{S}_n$, and let $\tau' = [\tau_2, \ldots, \tau_n]$. By definition, we have $\mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau') \wedge\!\!\!\!\wedge \tau_1$. By induction hypothesis, $\tau'$ is a linear extension of the contact graph $\mathsf{ins}^k(\tau')^{\#}$ and Lemma 7 ensures that $\tau_1$ is a source of $\mathsf{ins}^k(\tau)^{\#} = (\mathsf{ins}^k(\tau') \wedge\!\!\!\!\wedge \tau_1)^{\#}$. It follows that $\tau$ is a linear extension of the contact graph $\mathsf{ins}^k(\tau)^{\#}$. Conversely, assume that $\tau$ is a linear extension of the contact graph of a $(k, n)$-twist $\mathrm{T}$. Since $\tau_1$ is a source of $\mathrm{T}^{\#}$, the twist $\mathrm{T} \vee\!\!\!\!\vee \tau_1$ is well-defined, and $\tau'$ is a linear extension of $(\mathrm{T} \vee\!\!\!\!\vee \tau_1)^{\#}$. By induction, we have $\mathsf{ins}^k(\tau') = \mathrm{T} \vee\!\!\!\!\vee \tau_1$ and thus $\mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau') \wedge\!\!\!\!\wedge \tau_1 = (\mathrm{T} \vee\!\!\!\!\vee \tau_1) \wedge\!\!\!\!\wedge \tau_1 = \mathrm{T}$ by Lemma 11. $\square$

**Example 13** (1-twist correspondence)**.** The contact graph of the 1-twist $\mathsf{ins}^1(\tau)$ is the binary search tree obtained by the successive insertions of the entries of $\tau$ read from right to left.

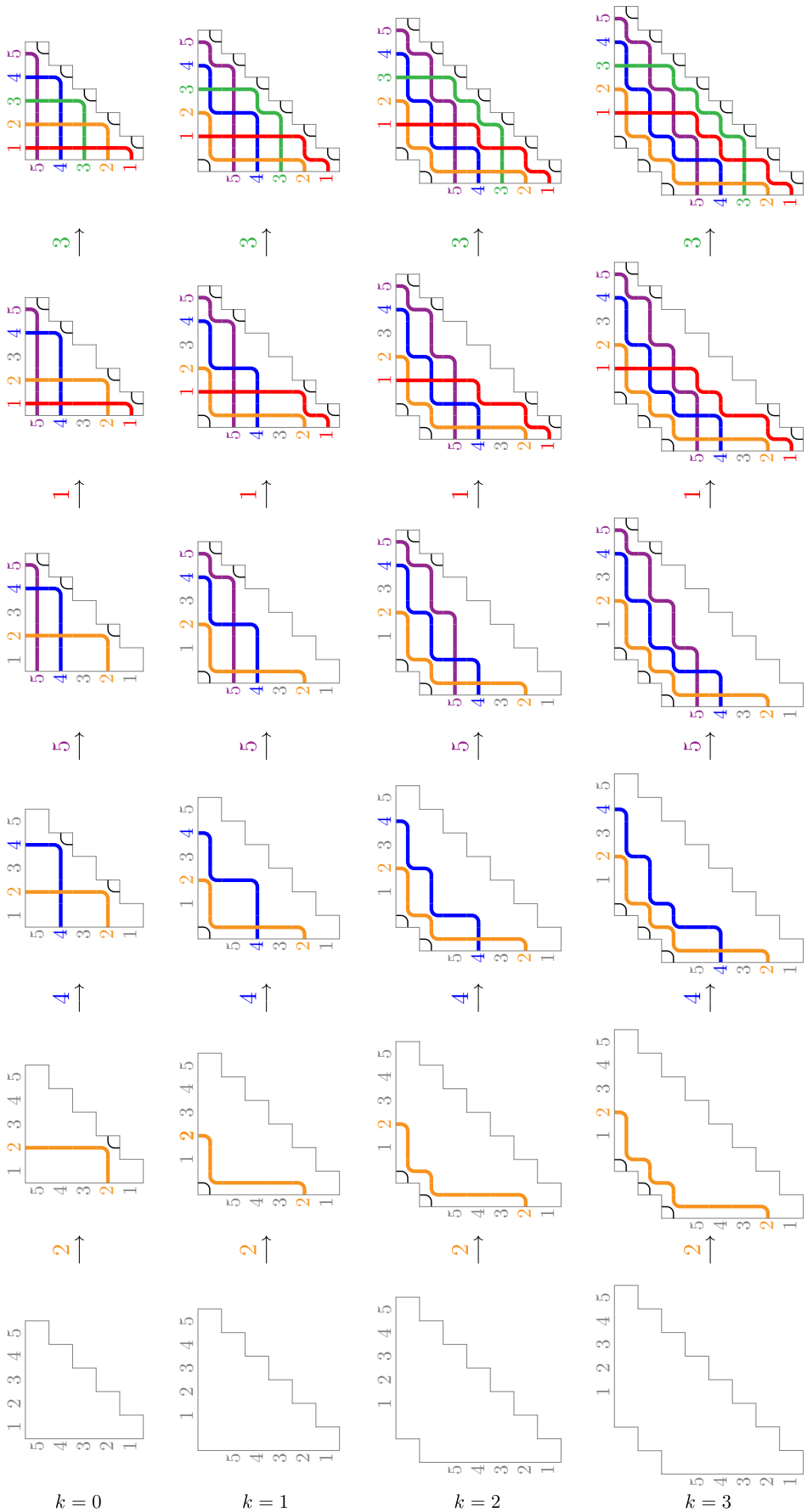


FIGURE 4. Insertion of the permutation 31542 in $k$-twists for $k = 0, 1, 2, 3$.

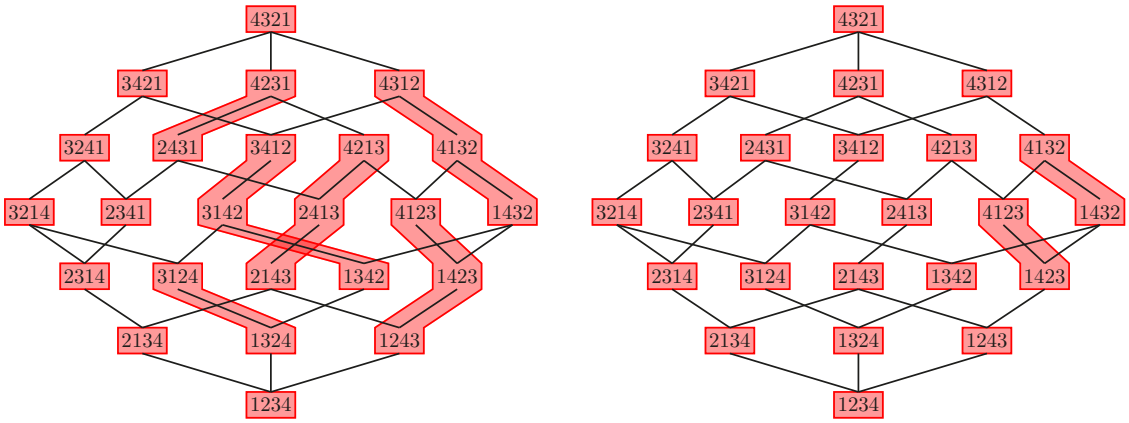


FIGURE 5. The $k$-twist congruence classes on $\mathfrak{S}_4$ for $k = 1$ (left) and $k = 2$ (right).

1.5. **$k$-twist congruence.** We now characterize the fibers of $\mathsf{ins}^k$ as classes of a congruence $\equiv^k$ defined by a simple rewriting rule, similar to the sylvester congruence [HNT05].

**Definition 14.** *Write the permutations of $\mathfrak{S}_n$ as words in one-line notation. The $k$-twist congruence is the equivalence relation $\equiv^k$ on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule*

$$UacV_1b_1V_2b_2 \cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2 \cdots V_kb_kW \qquad \text{if } a < b_i < c \text{ for all } i \in [k],$$

*where $a, b_1, \ldots, b_k, c$ are elements of $[n]$ while $U, V_1, \ldots, V_k, W$ are (possibly empty) words on $[n]$. We say that $b_1, \ldots, b_k$ are $k$-twist congruence witnesses for the exchange of $a$ and $c$. See Figure 5.*

**Proposition 15.** *For any $\tau, \tau' \in \mathfrak{S}_n$, we have $\tau \equiv^k \tau' \iff \mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau')$. In other words, the fibers of $\mathsf{ins}^k$ are precisely the $k$-twist congruence classes.*

*Proof.* From Proposition 12, each fiber of $\mathsf{ins}^k$ gathers the linear extensions of a $k$-twist. Since the set of linear extensions of a poset is connected by simple transpositions, we just need to show that $\tau \equiv^k \tau' \iff \mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau')$ for any two permutations $\tau = UacV$ and $\tau' = UcaV$ of $\mathfrak{S}_n$ which differ by the inversion of two consecutive values.

Let $\mathrm{T} = \mathsf{ins}^k(V)$ denote the $k$-twist obtained after the insertion of $V$. The positions where $a$ and $c$ will be inserted in T are separated by the letters $b$ in $V$ such that $a < b < c$. Therefore, if there exists at least $k$ such letters, the pipes $a$ and $c$ are not comparable in $(\mathrm{T} \Cap c) \Cap a = (\mathrm{T} \Cap a) \Cap c$ and we have $\mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau')$. Conversely, if there are strictly less than $k$ such letters, then $a$ is below $c$ in $(\mathrm{T} \Cap c) \Cap a$ while $a$ is above $c$ in $(\mathrm{T} \Cap a) \Cap c$, and thus we get $\mathsf{ins}^k(\tau) \neq \mathsf{ins}^k(\tau')$. □

**Example 16** (1-twist congruence and sylvester congruence)**.** The 1-twist congruence coincides with the sylvester congruence defined in [HNT05] as the transitive closure of the rewriting rule $UacVbW \equiv UcaVbW$ for $a < b < c$ elements of $[n]$ while $U, V, W$ are (possibly empty) words on $[n]$.

1.6. **Lattice congruences of the weak order.** In this section, we remind results of N. Reading [Rea04, Rea06, Rea05] concerning lattice congruences of the weak order.

Remember first that the (right) weak order on $\mathfrak{S}_n$ is defined as the inclusion order of (right) inversions, where a (right) inversion of $\tau \in \mathfrak{S}_n$ is a pair of values $i, j \in \mathbb{N}$ such that $i < j$ while $\tau^{-1}(i) > \tau^{-1}(j)$. See *e.g.* Figure 5 for the Hasse diagram of the weak order on $\mathfrak{S}_4$.

A *lattice congruence* is an equivalence relation $\equiv$ on a lattice $L$ compatible with meets and joins: for any $x \equiv x'$ and $y \equiv y'$, we have $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. This implies in particular that each equivalence class under $\equiv$ is an interval of $L$. Consider now the poset quotient $L/\!\equiv$ on the equivalence classes of $\equiv$ defined by $X \leq Y$ in $L/\!\equiv$ iff there exists representatives $x \in X$ and $y \in Y$ such that $x \leq y$ in $L$. It inherits a lattice structure where the meet $X \wedge Y$ (resp. the join $X \vee Y$) of two congruence classes $X$ and $Y$ is the congruence class of $x \wedge y$ (resp. of $x \vee y$) for arbitrary representatives $x \in X$ and $y \in Y$.

It turns out that the $k$-twist congruence already appeared in the work of N. Reading [Rea05].

**Proposition 17** ([Rea05])**.** *The $k$-twist congruence $\equiv^k$ is a lattice congruence of the weak order.*

**Corollary 18.** *The following combinatorial objects are in explicit bijection:*

- *acyclic $(k,n)$-twists,*
- *$k$-twist congruence classes of $\mathfrak{S}_n$,*
- *permutations of $\mathfrak{S}_n$ avoiding $1(k+2)-(\sigma_1+1)-\ldots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$ (maximums),*
- *permutations of $\mathfrak{S}_n$ avoiding $(k+2)1-(\sigma_1+1)-\ldots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$ (minimums).*

**Remark 19** ($13-2$ versus $1-3-2$ avoiding permutations)**.** It is easy to see that a permutation avoids $13-2$ if and only if it avoids $1-3-2$. This property fails for larger values of $k$. For example, the permutation 13524 avoids $14-2-3$ but not $1-4-2-3$. Here, we deal with permutations avoiding the pattern $1(k+2)-(\sigma_1+1)-\ldots-(\sigma_k+1)$, where 1 and $(k+2)$ are consecutive.

1.7. **Increasing flip lattice.** We now recall the notion of flips in pipe dreams and study the graph of increasing flips in acyclic $k$-twists.

**Definition 20.** *An* elbow flip *(or just* flip*) in a $k$-twist is the exchange of an elbow ┘┌ between two relevant pipes $\mathrm{p}, \mathrm{p}'$ with the unique crossing ┼ between $\mathrm{p}$ and $\mathrm{p}'$. The flip is* increasing *if the initial elbow is located (largely) south-west of the final elbow.*

We are interested in the graph of increasing flips, restricted to the acyclic $(k,n)$-twists. See Figure 6 for an illustration when $k=2$ and $n=4$. Note that although the graph of flips is regular, its restriction to acyclic twists is not anymore regular in general: the first example appears for $k=2$ and $n=5$. It is known (see *e.g.* via subword complexes in [PS13] or via multitriangulations in [PS09]) that this graph is acyclic and it has a unique source (resp. sink) given by the $(k,n)$-twist where all relevant elbows are in the first $k$ columns (resp. last $k$ rows) while all crosses are on the last $n$ columns (resp. first $n$ rows).

We call *increasing flip order* the transitive closure of the increasing flip graph on acyclic $k$-twists. Be aware that it is strictly contained in the restriction to acyclic $k$-twists of the transitive closure of the increasing flip graph on all $k$-twists: namely, there are pairs of acyclic $k$-twists so that any path of increasing flips between them passes through a cyclic $k$-twist.

**Example 21** (Tamari lattice)**.** When $k=1$, the increasing flip lattice is the classical Tamari lattice [MHPS12].

**Proposition 22.** *The following posets are all isomorphic:*

- *the increasing flip order on acyclic $k$-twists,*
- *the quotient lattice of the weak order by the $k$-twist congruence $\equiv^k$,*
- *the subposet of the weak order induced by the permutations of $\mathfrak{S}_n$ avoiding the pattern $1(k+2)-(\sigma_1+1)-\cdots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$,*
- *the subposet of the weak order induced by the permutations of $\mathfrak{S}_n$ avoiding the pattern $(k+2)1-(\sigma_1+1)-\cdots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$.*

*Proof.* Consider two distinct $k$-twists $\mathrm{T}, \mathrm{T}'$ and their $k$-twist congruence classes $C, C'$. If there exist representatives $\tau = UijV \in C$ and $\tau' = UjiV \in C'$ adjacent in weak order, then $\mathrm{T} = \mathsf{ins}^k(\tau)$ and $\mathrm{T}' = \mathsf{ins}^k(\tau')$ differ by the flip of the first/last elbow between the $i$th and $j$th pipes, by definition of the map $\mathsf{ins}^k$. Conversely, if $\mathrm{T}$ and $\mathrm{T}'$ differ by the flip of the first/last elbow between the $i$th and $j$th pipes, then $i$ and $j$ are connected in the contact graph $\mathrm{T}^\#$, so that there exists a linear extension $\tau = UijV$ of $\mathrm{T}^\#$ where $i$ and $j$ are consecutive. Let $\tau' = UjiV$ be the permutation obtained by the switch of $i$ and $j$ in $\tau$. By definition of the map $\mathsf{ins}^k$, the twist $\mathsf{ins}^k(\tau')$ is obtained by flipping the first elbow between the $i$th and $j$th pipes in $\mathsf{ins}^k(\tau)$. The representatives $\tau \in C$ and $\tau' \in C'$ are thus adjacent in weak order. This proves that $\mathsf{ins}^k$ induces an isomorphism from the quotient lattice of the weak order by the $k$-twist congruence to the increasing flip order on acyclic $k$-twists. In turn, since the $k$-twist congruence is a lattice congruence, this quotient lattice is isomorphic to the subposet of the weak order induced by the minimal (resp. maximal) elements of the classes. See [Rea04, Rea06] for further details on quotient lattices. □

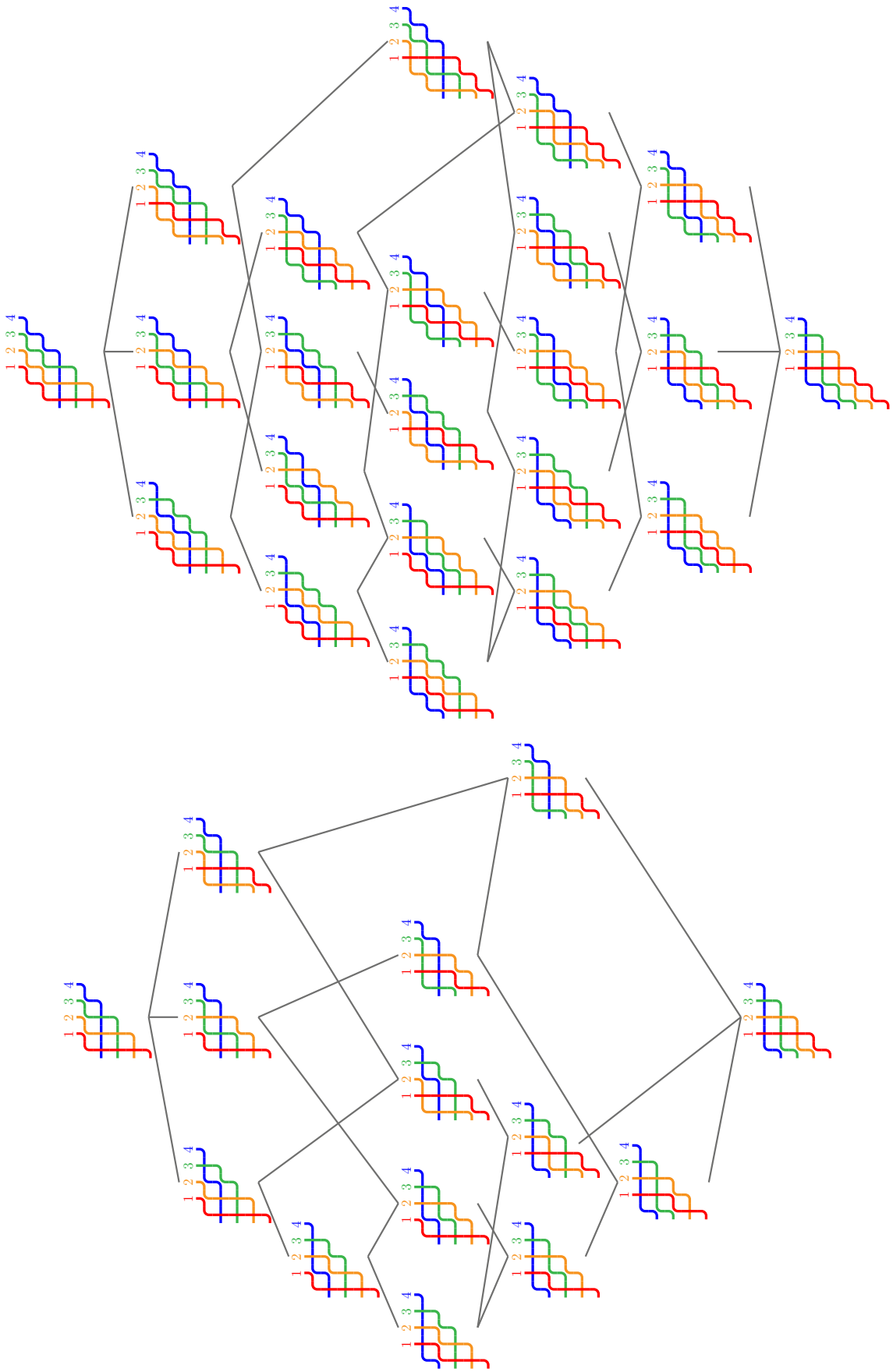

FIGURE 6. The increasing flip lattice on $(k, 4)$-twists for $k = 1$ (left) and $k = 2$ (right).

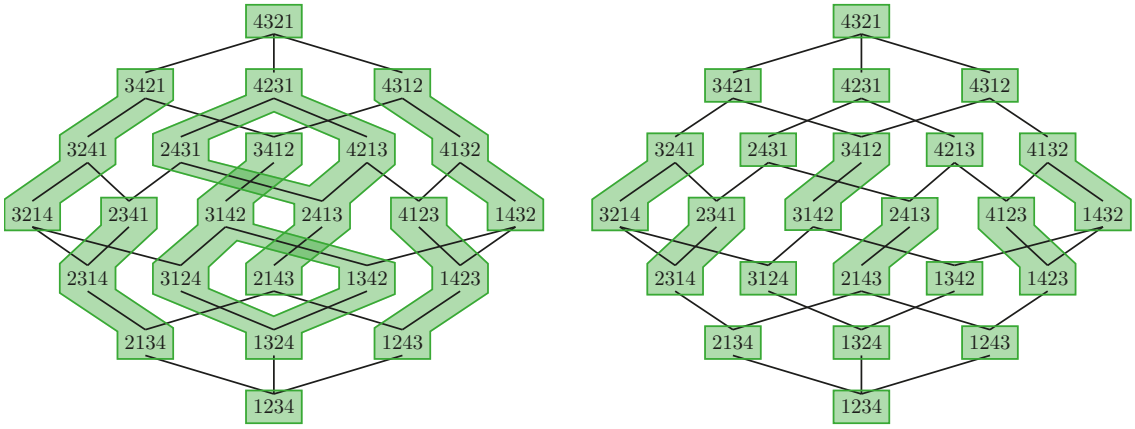


FIGURE 7. The $k$-recoil congruence classes on $\mathfrak{S}_4$ for $k = 1$ (left) and $k = 2$ (right).

1.8. **$k$-recoil schemes.** To prepare the definition of the $k$-canopy of an acyclic $k$-twist, we now briefly recall the notion of $k$-recoil schemes of permutations, which was already defined by J.-C. Novelli, C. Reutenauer and J.-Y. Thibon in [NRT11]. We use a description in terms of acyclic orientations of a certain graph as it is closer to the description of the vertices of the zonotope that we will use later in Section 2. We skip the proofs as they appear in [NRT11, Pil15]

A *recoil* in a permutation $\tau \in \mathfrak{S}_n$ is a position $i \in [n-1]$ such that $\tau^{-1}(i) > \tau^{-1}(i+1)$ (in other words, it is a descent of the inverse of $\tau$). The *recoil scheme* of $\tau \in \mathfrak{S}_n$ is the sign vector $\mathsf{rec}(\tau) \in \{-,+\}^{n-1}$ defined by $\mathsf{rec}(\tau)_i = -$ if $i$ is a recoil of $\tau$ and $\mathsf{rec}(\tau)_i = +$ otherwise.

To extend this definition to general $k$, we consider the graph $\mathrm{G}^k(n)$ with vertex set $[n]$ and edge set $\big\{\{i,j\} \in [n]^2 \mid i < j \le i+k\big\}$. For example, when $k = 1$, the graph $G^1(n)$ is just the $n$-path. We denote by $\mathcal{AO}^k(n)$ the set of acyclic orientations of $\mathrm{G}^k(n)$ (*i.e.* with no oriented cycle).

**Proposition 23** ([NRT11, Prop. 2.1])**.** *The number of acyclic orientations of* $\mathrm{G}^k(n)$ *is*

$$|\mathcal{AO}^k(n)| = \begin{cases} n! & \text{if } n \le k, \\ k!\,(k+1)^{n-k} & \text{if } n \ge k. \end{cases}$$

We use these acyclic orientations to define the $k$-recoil scheme of a permutation and the corresponding $k$-recoil congruence.

**Definition 24.** *The* $k$-recoil scheme *of a permutation* $\tau \in \mathfrak{S}_n$ *is the orientation* $\mathsf{rec}^k(\tau) \in \mathcal{AO}^k(n)$ *with an edge* $i \to j$ *for all* $i, j \in [n]$ *such that* $|i-j| \le k$ *and* $\tau^{-1}(i) < \tau^{-1}(j)$*. We call* $k$-recoil map *the map* $\mathsf{rec}^k : \mathfrak{S}_n \to \mathcal{AO}^k(n)$*.*

**Proposition 25.** *For* $\mathrm{O} \in \mathcal{AO}^k(n)$*, the fiber of* $\mathrm{O}$ *by the* $k$*-recoil map is the set of linear extensions of the transitive closure of* $\mathrm{O}$*.*

**Definition 26.** *The* $k$-recoil congruence $\approx^k$ *on* $\mathfrak{S}_n$ *is the transitive closure of the rewriting rule*

$$UijV \approx^k UjiV \qquad \text{if } i + k < j,$$

*where* $i, j$ *are elements of* $[n]$ *while* $U, V$ *are (possibly empty) words on* $[n]$*. See Figure 7.*

**Proposition 27** ([NRT11, Prop. 2.2])**.** *For any* $\tau, \tau' \in \mathfrak{S}_n$*, we have* $\tau \approx^k \tau' \iff \mathsf{rec}^k(\tau) = \mathsf{rec}^k(\tau')$*. In other words, the fibers of* $\mathsf{rec}^k$ *are precisely the* $k$*-recoil congruence classes.*

**Definition 28.** *A* direction flip *(or just* flip*) in an acyclic orientation is the switch of the direction of an edge of* $\mathrm{G}^k(n)$*. The flip is* increasing *if the initial direction was increasing. Define the* increasing flip order *on* $\mathcal{AO}^k(n)$ *to be the transitive closure of the increasing flip graph on* $\mathcal{AO}^k(n)$*.*

**Proposition 29.** *The $k$-recoil congruence $\approx^k$ is a lattice congruence of the weak order. The $k$-recoil map $\mathsf{rec}^k$ defines an isomorphism from the quotient lattice of the weak order by the $k$-recoil congruence $\approx^k$ to the increasing flip lattice on the acyclic orientations of $\mathrm{G}^k(n)$.*

1.9. **$k$-canopy schemes.** We recall that the *canopy* of a binary tree T with $n$ nodes is the sign vector $\mathsf{can}(\mathrm{T}) \in \{-,+\}^{n-1}$ defined by $\mathsf{can}(\mathrm{T})_i = -$ if the node $i$ of T is above the node $i+1$ of T and $\mathsf{can}(\mathrm{T})_i = +$ otherwise. This map was already used *e.g.* in [LR98, Lod04, Vie07]. The binary search tree insertion map and the canopy map factorize the recoil map: $\mathsf{can} \circ \mathsf{ins} = \mathsf{rec}$. This combinatorial fact can also be understood on the geometry of the normal fans of the permutahedron, the associahedron and the cube, see Section 2.2.

We now define an equivalent of the canopy map for general $k$. To ensure that Definition 31 is valid, we need the following simple observation on comparisons of closed pipes in a $k$-twist.

**Lemma 30.** *If $|i-j| \le k$, the $i$th and $j$th pipes in an acyclic $k$-twist T are comparable for $\lhd_T$.*

*Proof.* By Lemma 5 (ii), if $i < j$ and the $i$th and $j$th pipes are incomparable, then the last vertical step of $i$ crosses the first horizontal step of $j$. Since each pipe has $k$ horizontal steps by Lemma 4 (ii), it ensures that $j > i + k$. □

**Definition 31.** *The $k$-canopy scheme of a $(k,n)$-twist T is the orientation $\mathsf{can}^k(\mathrm{T}) \in \mathcal{AO}^k(n)$ with an edge $i \to j$ for all $i,j \in [n]$ such that $|i-j| \le k$ and $i \lhd_{\mathrm{T}} j$. It indeed defines an acyclic orientation of $\mathrm{G}^k(n)$ by Lemma 30. We call $k$-canopy the map $\mathsf{can}^k : \mathcal{AT}^k(n) \to \mathcal{AO}^k(n)$.*

**Proposition 32.** *The maps $\mathsf{ins}^k$, $\mathsf{can}^k$, and $\mathsf{rec}^k$ define the following commutative diagram of lattice homomorphisms:*

$$\begin{tikzcd}
\mathfrak{S}_n \arrow[rr, two heads, "\mathsf{rec}^k"] \arrow[dr, two heads, "\mathsf{ins}^k"'] & & \mathcal{AO}^k(n) \\
& \mathcal{AT}^k(n) \arrow[ur, two heads, "\mathsf{can}^k"'] &
\end{tikzcd}$$

*Proof.* Consider a permutation $\tau$ and let $i,j \in [n]$ with $|i-j| < k$ such that $\tau^{-1}(i) < \tau^{-1}(j)$. By definition, there is an arc $i \to j$ in $\mathsf{rec}^k(\tau)$. Moreover, the $i$th pipe is inserted after the $j$th pipe in $\mathsf{ins}^k(\tau)$, so that $i \lhd_{\mathsf{ins}^k(\tau)} j$ and there is also an arc $i \to j$ in $\mathsf{can}^k \circ \mathsf{ins}^k(\tau)$. □

See also Section 2.2 for a geometric interpretation of Proposition 32.

**Remark 33** (Combinatorial inclusions)**.** When $k > \ell$, the $k$-twist congruence $\equiv^k$ refines the $\ell$-twist congruence $\equiv^\ell$ (meaning that $\tau \equiv^k \tau'$ implies $\tau \equiv^\ell \tau'$) and the $k$-recoil congruence $\approx^k$ refines the $\ell$-recoil congruence $\approx^\ell$. We can thus define surjective restriction maps $\mathsf{res}^{k\to\ell} : \mathcal{AT}^k(n) \to \mathcal{AT}^\ell(n)$ and $\mathsf{res}^{k\to\ell} : \mathcal{AO}^k(n) \to \mathcal{AO}^\ell(n)$ by:

- for an acyclic $k$-twist T, the $\ell$-twist $\mathsf{res}^{k\to\ell}(\mathrm{T})$ is obtained by insertion of any linear extension of $\mathrm{T}^\#$ (it is independent of the choice of this linear extension),
- for an acyclic orientation $\theta \in \mathcal{AO}^k(n)$, the sign vector $\mathsf{res}^{k\to\ell}(\theta) \in \mathcal{AO}^\ell(n)$ is obtained by restriction of $\theta$ to the edges of $G^\ell(n)$.

We therefore obtain the following commutative diagram of lattice homomorphisms:

$$\begin{tikzcd}
\mathcal{AT}^k(n) \arrow[rr, two heads, "\mathsf{res}^{k\to\ell}"] \arrow[dd, two heads, "\mathsf{can}^k"'] & & \mathcal{AT}^\ell(n) \arrow[dd, two heads, "\mathsf{can}^\ell"] \\
& \mathfrak{S}_n \arrow[ul, "\mathsf{ins}^k"] \arrow[ur, "\mathsf{ins}^\ell"'] \arrow[dl, "\mathsf{rec}^k"'] \arrow[dr, "\mathsf{rec}^\ell"] & \\
\mathcal{AO}^k(n) \arrow[rr, two heads, "\mathsf{res}^{k\to\ell}"'] & & \mathcal{AO}^\ell(n)
\end{tikzcd}$$

## 2. Geometry of acyclic twists

This section is devoted to the polyhedral geometry of permutations of $\mathfrak{S}_n$, acyclic twists of $\mathcal{AT}^k(n)$, and acyclic orientations of $\mathcal{AO}^k(n)$. It is mainly based on properties of brick polytopes of sorting networks, defined and studied by V. Pilaud and F. Santos in [PS12]. To keep this section short, we skip all proofs of its statements as they follow directly from [PS12]. This section should be seen as a brief geometric motivation for the combinatorial and algebraic construction of this paper. The reader familiar with the geometry of the brick polytope is invited to proceed directly with Section 3.

2.1. **Permutahedra, brick polytopes, and zonotopes.** We first recall the definition of three families of polytopes, which are illustrated in Figure 8. We refer to [Zie98, Lectures 0 to 2] for background on polytopes. We denote by $(\mathbf{e}_i)_{i\in[n]}$ the canonical basis of $\mathbb{R}^n$ and let $\mathbb{1} := \sum_{i\in[n]} \mathbf{e}_i$.

Permutahedra The permutahedron is a classical polytope [Zie98, Lecture 0] whose geometric and combinatorial properties reflect that of the symmetric group $\mathfrak{S}_n$.

**Definition 34.** *The* permutahedron $\mathsf{Perm}(n)$ *is the* $(n-1)$*-dimensional polytope*

$$\mathsf{Perm}(n) \;:=\; \operatorname{conv}\{\mathbf{x}(\tau) \mid \tau \in \mathfrak{S}_n\} \;=\; \mathbf{H}^{=}([n]) \cap \bigcap_{\varnothing \neq I \subsetneq [n]} \mathbf{H}^{\geq}(I) \;=\; \mathbb{1} + \sum_{1\leq i<j\leq n} [\mathbf{e}_i, \mathbf{e}_j],$$

*defined equivalently as*

- *the convex hull of the points* $\mathbf{x}(\tau) := [\tau^{-1}(i)]_{i\in[n]} \in \mathbb{R}^n$ *for all permutations* $\tau \in \mathfrak{S}_n$*,*
- *the intersection of the hyperplane* $\mathbf{H}^{=}([n]) := \big\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i\in[n]} x_i = \binom{n+1}{2}\big\}$ *with the half-spaces* $\mathbf{H}^{\geq}(I) := \big\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i\in I} x_i \geq \binom{|I|+1}{2}\big\}$ *for all proper non-empty subset* $\varnothing \neq I \subsetneq [n]$*,*
- *the Minkowski sum of the point* $\mathbb{1}$ *with the segments* $[\mathbf{e}_i, \mathbf{e}_j]$ *for* $1 \leq i < j \leq n$*.*

We consider a dilated and translated copy of the permutahedron $\mathsf{Perm}(n)$, which will fit better the other two families of polytopes defined later (see Remark 41 for a precise statement). Namely, we set

$$\mathsf{Perm}^k(n) \;:=\; k\,\mathsf{Perm}(n) - \frac{k(n+1)}{2}\,\mathbb{1}.$$

Observe that $\mathsf{Perm}^k(n)$ now lies in the hyperplane $\mathbb{H} := \big\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i\in[n]} x_i = 0\big\}$.

Brick polytopes To define the brick polytope, we essentially follow [Pil10, PS12] except that we apply again a translation in direction $\mathbb{1}$ to obtain a polytope in the hyperplane $\mathbb{H}$.

**Definition 35.** *We call* bricks *the squares* $[i, i+1] \times [j, j+1]$ *of the triangular shape. The* brick area *of a pipe* $\mathrm{p}$ *is the number of bricks located below* $\mathrm{p}$ *but inside the axis-parallel rectangle defined by the two endpoints of* $\mathrm{p}$*. The* brick vector *of a* $k$*-twist* $\mathrm{T}$ *is the vector* $\mathbf{x}(\mathrm{T}) \in \mathbb{R}^n$ *whose* $i$*th coordinate is the brick area of the* $i$*th pipe of* $\mathrm{T}$*, minus* $\frac{k(n+k)}{2}$*. The* brick polytope $\mathsf{Brick}^k(n)$ *is the polytope defined as the convex hull of the brick vectors of all* $(k,n)$*-twists.*

As for the permutahedron described above, we know three descriptions of the brick polytopes: its vertex description, its hyperplane description, and a Minkowski sum description. These properties are proved in [PS12] (modulo our translation in the direction $\mathbb{1}$).

**Proposition 36** ([PS12])**.** *The brick polytope* $\mathsf{Brick}^k(n)$ *has the following properties for any* $n, k \in \mathbb{N}$*.*

*(i) It lies in the hyperplane* $\mathbb{H} := \big\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i\in[n]} x_i = 0\big\}$ *and has dimension* $n-1$*.*

*(ii) The vertices of* $\mathsf{Brick}^k(n)$ *are precisely the brick vectors of the acyclic* $(k,n)$*-twists.*

*(iii) The normal vectors of the facets of* $\mathsf{Brick}^k(n)$ *are given by the* proper $k$-connected $\{0,1\}$-sequences *of size* $n$*, i.e. the sequences of* $\{0,1\}^n$ *distinct from* $0^n$ *and* $1^n$ *and which do not*

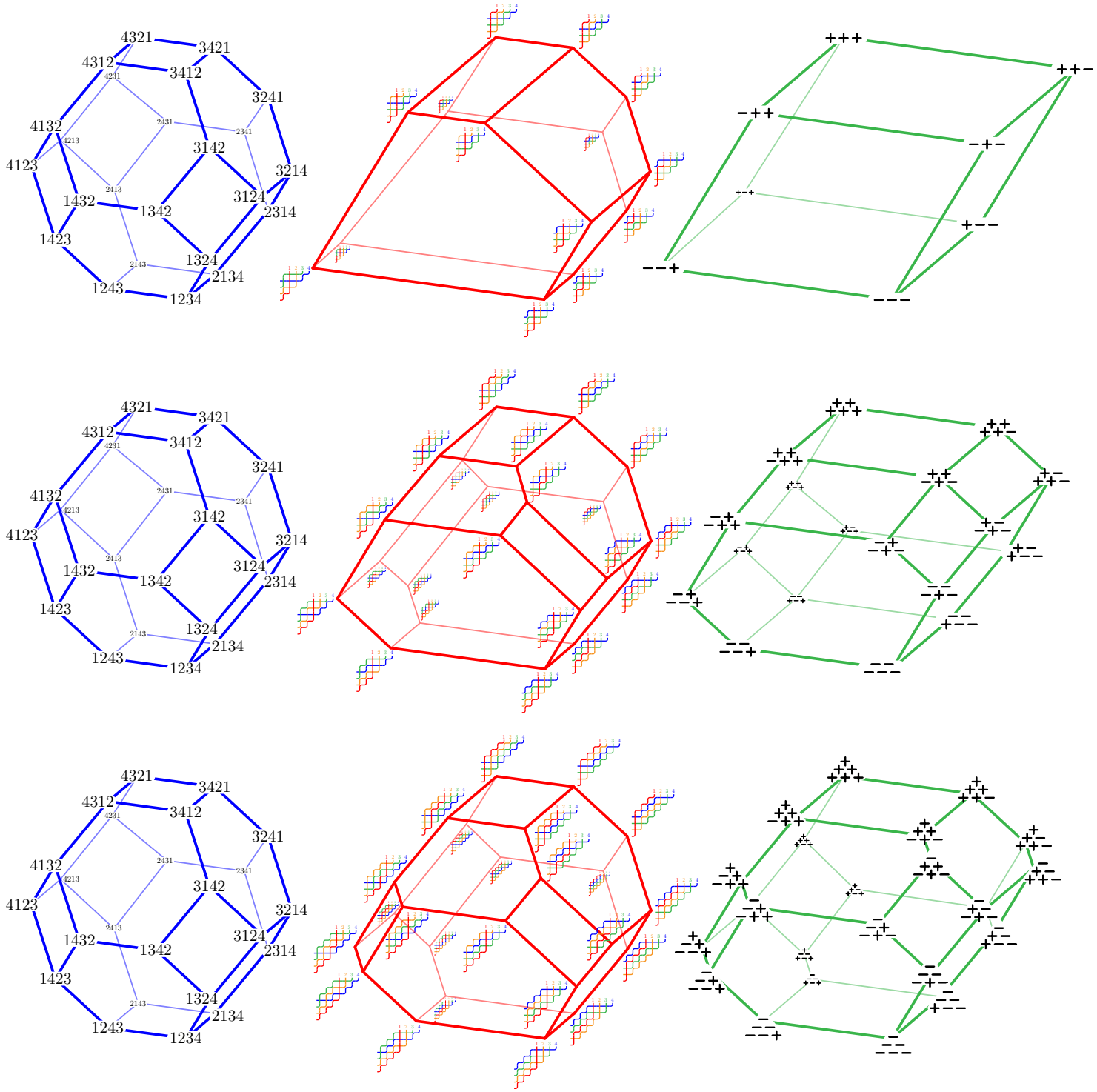


FIGURE 8. The permutahedron $\mathsf{Perm}^k(4)$ (left), the brick polytope $\mathsf{Brick}^k(4)$ (middle) and the zonotope $\mathsf{Zono}^k(4)$ (right) for $k = 1$ (top), $k = 2$ (middle) and $k = 3$ (bottom). For readability, we represent orientations of $\mathrm{G}^k(n)$ by pyramids of signs.

*contain a subsequence* $10^\ell 1$ *for* $\ell \geq k$*. The number of facets of* $\mathsf{Brick}^k(n)$ *is therefore the coefficient of* $t^n$ *in*

$$\frac{t^2(2-t^k)}{(1-2t+t^{k+1})(1-t)}.$$

*(iv) For a brick* $b$*, let* $\mathbf{x}_b(\mathrm{T})$ *denote the characteristic vector of the pipes of a* $(k,n)$*-twist* $\mathrm{T}$ *whose brick area contain* $b$*, and define* $\mathsf{Brick}_b^k(n)$ *to be the convex hull of the vectors* $\mathbf{x}_b(\mathrm{T})$ *for all* $(k,n)$*-twists* $\mathrm{T}$*. Then, up to the translation of vector* $\frac{k(n+k)}{2}\mathbb{1}$*, the brick polytope* $\mathsf{Brick}^k(n)$ *is also the Minkowski sum of the polytopes* $\mathsf{Brick}_b^k(n)$ *over all bricks* $b$*.*

**Example 37** (J.-L. Loday's associahedron)**.** When $k=1$, the brick polytope $\mathsf{Brick}^1(n)$ coincides (up to translation) with J.-L. Loday's associahedron [Lod04]. See Figure 8 (top center).

ZONOTOPES Zonotopes are particularly important polytopes which are constructed equivalently as projections of cubes, or as Minkowski sums of segments. The combinatorics of a zonotope is completely determined by the matroid of the vector configuration defined by these summands. We refer to [Zie98, Lecture 7] for a presentation of these polytopes and their relations to oriented matroids. Notable examples are graphical zonotopes, defined as follows.

**Definition 38.** *The* graphical zonotope $\mathsf{Zono}(\mathrm{G})$ *of a graph* $\mathrm{G}$ *is the Minkowski sum of the segments* $[\mathbf{e}_i, \mathbf{e}_j]$ *for all edges* $\{i,j\}$ *of* $\mathrm{G}$*.*

The following classical statement gives the vertex and facet descriptions of graphical zonotopes.

**Proposition 39.** *The graphical zonotope* $\mathsf{Zono}(\mathrm{G})$ *has the following properties for any graph* $\mathrm{G}$*.*

*(i) The dimension of* $\mathsf{Zono}(\mathrm{G})$ *is the number of edges of a maximal cycle-free subgraph of* $\mathrm{G}$*.*
*(ii) The vertices of* $\mathsf{Zono}(\mathrm{G})$ *correspond to the acyclic orientations of* $\mathrm{G}$*. The* $i$*th coordinate of the vertex* $\mathbf{x}(\mathrm{O})$ *of* $\mathsf{Zono}(\mathrm{G})$ *corresponding to an acyclic orientation* $\mathrm{O}$ *of* $\mathrm{G}$ *is the indegree of vertex* $i$ *in* $\mathrm{O}$*.*
*(iii) The facets of* $\mathsf{Zono}(\mathrm{G})$ *correspond to minimal cuts of* $\mathrm{G}$*.*

For example, the permutahedron $\mathsf{Perm}(n)$ is the graphical zonotope of the complete graph, its vertices correspond to permutations of $[n]$ (acyclic tournaments), and its facets correspond to proper subsets of $[n]$ (minimal cuts). Here, we focus on the zonotope of $\mathrm{G}^k(n)$ whose vertices correspond to the acyclic orientations in $\mathcal{AO}^k(n)$ and whose facets correspond to minimal cuts of $\mathrm{G}^k(n)$. As for the previous polytopes, we perturb this zonotope to fit the other two polytopes better (see Remark 41 for a precise statement), and thus define

$$\mathsf{Zono}^k(n) := \sum_{1\leq i<j\leq n} \lambda(i,j,k,n)\cdot[\mathbf{e}_i,\mathbf{e}_j] \;-\; \frac{(n-1)(n+3k-2)}{6}\,\mathbb{1},$$

where

$$\lambda(i,j,k,n) := \begin{cases} n+k-2|i-j| & \text{if } |i-j|<k,\\ \min(i,n+1-j)\big(n+k-1-\min(i,n+1-j)\big) & \text{if } |i-j|=k,\\ 0 & \text{if } |i-j|>k.\end{cases}$$

Note that this perturbation is only cosmetic and preserves the combinatorics. Indeed, observe that for all $1\leq i<j\leq n$, we have $\lambda_{i,j,k,n}\neq 0$ if and only if $|i-j|\leq k$. Therefore, the zonotopes $\mathsf{Zono}^k(n)$ and $\mathsf{Zono}(\mathrm{G}^k(n))$ have the same normal fan (see Section 2.2) and thus the same face lattice. The translation ensures that $\mathsf{Zono}^k(n)$ lies in the hyperplane $\mathbb{H} := \big\{\mathbf{x}\in\mathbb{R}^n \mid \sum_{i\in[n]} x_i = 0\big\}$.

**Example 40** (Cube)**.** When $k=1$, the zonotope $\mathsf{Zono}^1(n)$ coincides (up to translation and scaling) with the parallelotope generated by the simple roots $\mathbf{e}_{i+1}-\mathbf{e}_i$. It has the same combinatorics as the $n$-dimensional cube. See Figure 8 (top right).

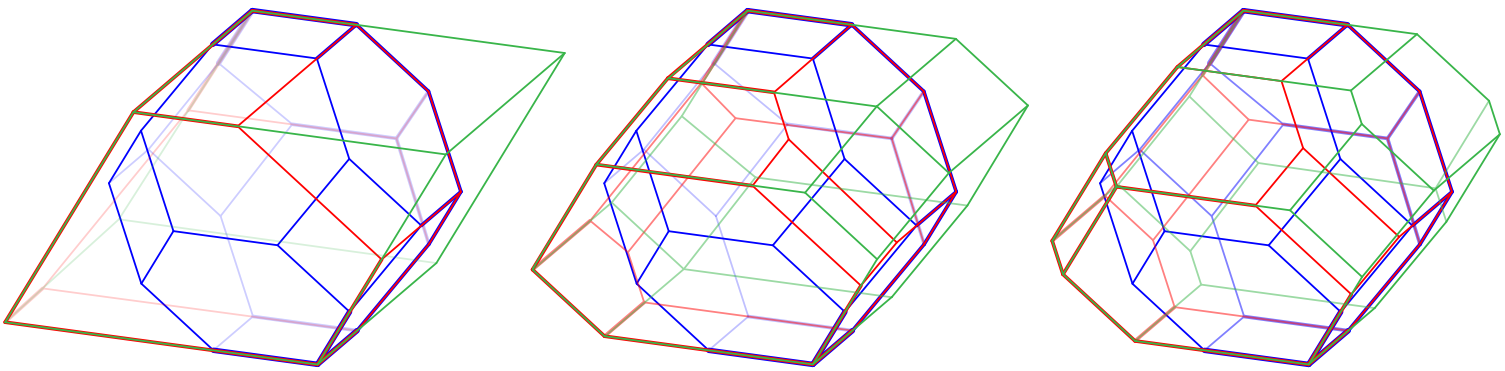

FIGURE 9. The inclusions $\mathsf{Perm}^k(n) \subseteq \mathsf{Brick}^k(n) \subseteq \mathsf{Zono}^k(n)$ for $k = 1$ (left), $k = 2$ (middle) and $k = 3$ (right). The permutahedron $\mathsf{Perm}^k(n)$ is in blue, the brick polytope $\mathsf{Brick}^k(n)$ in red and the zonotope $\mathsf{Zono}^k(n)$ in green.

**Remark 41** (Geometric inclusions)**.** We have chosen our normalizations (dilations and translations) so that the polytopes $\mathsf{Perm}^k(n)$, $\mathsf{Brick}^k(n)$ and $\mathsf{Zono}^k(n)$ all leave in the hyperplane $\mathbb{H} := \left\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i\in[n]} x_i = 0\right\}$ and fulfill the following inclusions:

$$\begin{array}{ccccc}
\frac{1}{k}\mathsf{Brick}^k(n) & & \subseteq & & \frac{1}{\ell}\mathsf{Brick}^\ell(n) \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\rotatebox{90}{$\subseteq$} & & \mathsf{Perm}(n) & & \rotatebox{90}{$\subseteq$} \\
& \rotatebox{45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
\frac{1}{k}\mathsf{Zono}^k(n) & & \subseteq & & \frac{1}{\ell}\mathsf{Zono}^\ell(n)
\end{array}$$

These inclusions are illustrated in Figures 8, 9 and 10. Compare to Remark 33.

Observe moreover that

$$\mathsf{Zono}^k(n) = \mathsf{Zono}^{n-1}(n) + (k+1-n)\,\mathsf{Perm}^1(n)$$

for all $k \geq n-1$. We therefore obtain that the rescaled polytopes $\frac{1}{k}\mathsf{Brick}^k(n)$ and $\frac{1}{k}\mathsf{Zono}^k(n)$ both converge to $\frac{1}{k}\mathsf{Perm}^k(n) = \mathsf{Perm}^1(n)$ when $k$ tend to $\infty$ (see Figure 10).

2.2. **The geometry of the surjections $\mathsf{ins}^k$, $\mathsf{can}^k$, and $\mathsf{rec}^k$.** Besides Remark 41, the main geometric connection between the three polytopes $\mathsf{Perm}^k(n)$, $\mathsf{Brick}^k(n)$ and $\mathsf{Zono}^k(n)$ is given by their normal fans. Remember that a *polyhedral fan* is a collection of polyhedral cones of $\mathbb{R}^n$ closed under faces and which intersect pairwise along faces, see *e.g.* [Zie98, Lecture 7]. The (outer) *normal cone* of a face $F$ of a polytope $P$ is the cone generated by the outer normal vectors of the facets of $P$ containing $F$. Finally, the (outer) *normal fan* of $P$ is the collection of the (outer) normal cones of all its faces.

The *incidence cone* $\mathrm{C}(\lhd)$ and the *braid cone* $\mathrm{C}^\diamondsuit(\lhd)$ of an poset $\lhd$ are the polyhedral cones defined by

$$\mathrm{C}(\lhd) := \operatorname{cone}\{\mathbf{e}_i - \mathbf{e}_j \mid \text{for all } i \lhd j\} \quad \text{and} \quad \mathrm{C}^\diamondsuit(\lhd) := \{\mathbf{x} \in \mathbb{H} \mid x_i \leq x_j \text{ for all } i \lhd j\}.$$

These two cones lie in the space $\mathbb{H}$ and are polar to each other. For a permutation $\tau \in \mathfrak{S}_n$ (resp. a twist $\mathrm{T} \in \mathcal{AT}^k(n)$, resp. an orientation $\mathrm{O} \in \mathcal{AO}^k(n)$), we slightly abuse notation to write $\mathrm{C}(\tau)$ (resp. $\mathrm{C}(\mathrm{T})$, resp. $\mathrm{C}(\mathrm{O})$) for the incidence cone of the chain $\tau_1 \lhd \cdots \lhd \tau_n$ (resp. of the transitive closure $\lhd$ of the contact graph $\mathrm{T}^\#$, resp. of the transitive closure of O). We define similarly the braid cone $\mathrm{C}^\diamondsuit(\tau)$ (resp. $\mathrm{C}^\diamondsuit(\mathrm{T})$, resp. $\mathrm{C}^\diamondsuit(\mathrm{O})$). These cones (together with all their faces) form the normal fans of the polytopes of Section 2.1.

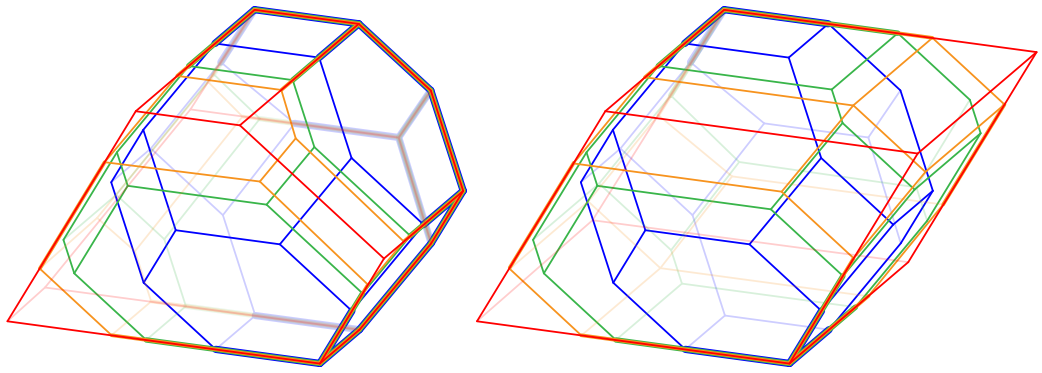

FIGURE 10. The inclusions of the brick polytopes $\frac{1}{k}\mathsf{Brick}^k(4)$ (left) and of the zonotopes $\frac{1}{k}\mathsf{Zono}^k(4)$ (right) for $k = 1$ (red), $k = 2$ (orange) and $k = 3$ (green). Both tend to the classical permutahedron $\mathsf{Perm}^1(4)$ (blue) when $k$ tends to $\infty$.

**Proposition 42.** *The collections of cones*

$$\big\{\mathrm{C}^\diamondsuit(\tau) \mid \tau \in \mathfrak{S}_n\big\}, \qquad \big\{\mathrm{C}^\diamondsuit(\mathrm{T}) \mid \mathrm{T} \in \mathcal{AT}^k(n)\big\} \qquad \text{and} \qquad \big\{\mathrm{C}^\diamondsuit(\mathrm{O}) \mid \mathrm{O} \in \mathcal{AO}^k(n)\big\},$$

*together with all their faces, are the normal fans of the permutahedron* $\mathsf{Perm}^k(n)$*, the brick polytope* $\mathsf{Brick}^k(n)$ *and the zonotope* $\mathsf{Zono}^k(n)$ *respectively.*

Observe moreover that the normal fan of $\mathsf{Perm}^k(n)$ is also the collection of chambers of the Coxeter arrangement given by all hyperplanes $\{\mathbf{x} \in \mathbb{H} \mid x_i = x_j\}$ for all $i, j \in [n]$. Similarly, the normal fan of $\mathsf{Zono}^k(n)$ is also the collection of chambers of the graphical arrangement given by the hyperplanes $\{\mathbf{x} \in \mathbb{H} \mid x_i = x_j\}$ for all edges $\{i, j\}$ in $\mathrm{G}^k(n)$.

Using these normal fans, one can interpret geometrically the maps $\mathsf{ins}^k$, $\mathsf{can}^k$, and $\mathsf{rec}^k$ as follows.

**Proposition 43.** *The insertion map* $\mathsf{ins}^k : \mathfrak{S}_n \to \mathcal{AT}^k(n)$*, the* $k$*-canopy* $\mathsf{can}^k : \mathcal{AT}^k(n) \to \mathcal{AO}^k(n)$ *and the* $k$*-recoil map* $\mathsf{rec}^k : \mathfrak{S}_n \to \mathcal{AO}^k(n)$ *are characterized by*

$$\begin{aligned}
\mathrm{T} = \mathsf{ins}^k(\tau) \iff \mathrm{C}(\mathrm{T}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamondsuit(\mathrm{T}) \supseteq \mathrm{C}^\diamondsuit(\tau),\\
\mathrm{O} = \mathsf{can}^k(\mathrm{T}) \iff \mathrm{C}(\mathrm{O}) \subseteq \mathrm{C}(\mathrm{T}) \iff \mathrm{C}^\diamondsuit(\mathrm{O}) \supseteq \mathrm{C}^\diamondsuit(\mathrm{T}),\\
\mathrm{O} = \mathsf{rec}^k(\tau) \iff \mathrm{C}(\mathrm{O}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamondsuit(\mathrm{O}) \supseteq \mathrm{C}^\diamondsuit(\tau).
\end{aligned}$$

Finally, the lattices studied in Section 1 also appear naturally in the geometry of the polytopes $\mathsf{Perm}^k(n)$, $\mathsf{Brick}^k(n)$ and $\mathsf{Zono}^k(n)$. Denote by $U$ the vector

$$U := (n, n-1, \dots, 2, 1) - (1, 2, \dots, n-1, n) = \sum_{i \in [n]} (n + 1 - 2i)\, \mathbf{e}_i.$$

**Proposition 44.** *When oriented in the direction* $U$*, the* 1*-skeleton of the permutahedron* $\mathsf{Perm}^k(n)$ *(resp. of the brick polytope* $\mathsf{Brick}^k(n)$*, resp. of the zonotope* $\mathsf{Zono}^k(n)$*) is the Hasse diagram of the weak order on permutations (resp. of the increasing flip lattice on acyclic* $(k, n)$*-twists, resp. of the increasing flip lattice on acyclic orientations of* $\mathrm{G}^k(n)$*).*

## 3. Algebra of acyclic twists

Motivated by the Hopf algebra on binary trees constructed by J.-L. Loday and M. Ronco [LR98] as a subalgebra of the Hopf algebra on permutations of C. Malvenuto and C. Reutenauer (see also [HNT05, AS06]), we define a Hopf algebra with bases indexed by acyclic $k$-twists. We then give combinatorial interpretations of the product and coproduct of this algebra and its dual in terms of $k$-twists. We finally conclude with further algebraic properties of this algebra.

### 3.1. Hopf algebras FQSym and FQSym*.

We briefly recall here the definition and some elementary properties of C. Malvenuto and C. Reutenauer's Hopf algebra on permutations [MR95]. We denote this algebra by FQSym to stress out its connection to free quasi-symmetric functions. We will however not use this connection in this paper. We denote by $\mathfrak{S} := \bigsqcup_{n\in\mathbb{N}} \mathfrak{S}_n$ the set of all permutations, of arbitrary size.

For $n, n' \in \mathbb{N}$, let

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau_1 < \cdots < \tau_n \text{ and } \tau_{n+1} < \cdots < \tau_{n+n'}\}$$

denote the set of permutations of $\mathfrak{S}_{n+n'}$ with at most one descent, at position $n$. The *shifted concatenation* $\tau\bar{\tau}'$, the *shifted shuffle* $\tau \bar{\sqcup\!\sqcup} \tau'$, and the *convolution* $\tau \star \tau'$ of two permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ are classically defined by

$$\tau\bar{\tau}' := [\tau_1, \ldots, \tau_n, \tau'_1 + n, \ldots, \tau'_{n'} + n] \in \mathfrak{S}_{n+n'},$$
$$\tau \bar{\sqcup\!\sqcup} \tau' := \left\{(\tau\bar{\tau}') \circ \pi^{-1} \mid \pi \in \mathfrak{S}^{(n,n')}\right\} \qquad \text{and} \qquad \tau \star \tau' := \left\{\pi \circ (\tau\bar{\tau}') \mid \pi \in \mathfrak{S}^{(n,n')}\right\}.$$

For example,

$$12 \bar{\sqcup\!\sqcup} 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\},$$
$$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}.$$

**Definition 45.** *We denote by* FQSym *the Hopf algebra with basis* $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}}$ *and whose product and coproduct are defined by*

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \bar{\sqcup\!\sqcup} \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau\star\tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}.$$

*This algebra is graded by the size of the permutations.*

**Proposition 46.** *A product of weak order intervals in* FQSym *is a weak order interval: for any two weak order intervals* $[\mu, \omega]$ *and* $[\mu', \omega']$*, we have*

$$\Big(\sum_{\mu\le\tau\le\omega} \mathbb{F}_\tau\Big) \cdot \Big(\sum_{\mu'\le\tau'\le\omega'} \mathbb{F}_{\tau'}\Big) = \sum_{\mu\bar{\mu}'\le\sigma\le\bar{\omega}'\omega} \mathbb{F}_\sigma.$$

**Corollary 47.** *For* $\tau \in \mathfrak{S}_n$*, define*

$$\mathbb{E}^\tau = \sum_{\tau\le\tau'} \mathbb{F}_{\tau'} \qquad \text{and} \qquad \mathbb{H}^\tau = \sum_{\tau'\le\tau} \mathbb{F}_{\tau'}$$

*where* $\le$ *is the weak order on* $\mathfrak{S}_n$*. Then* $(\mathbb{E}_\tau)_{\tau\in\mathfrak{S}}$ *and* $(\mathbb{H}_\tau)_{\tau\in\mathfrak{S}}$ *are multiplicative bases of* FQSym*:*

$$\mathbb{E}^\tau \cdot \mathbb{E}^{\tau'} = \mathbb{E}^{\tau\backslash\tau'} \qquad \text{and} \qquad \mathbb{H}^\tau \cdot \mathbb{H}^{\tau'} = \mathbb{H}^{\tau/\tau'},$$

*where* $\tau\backslash\tau' = \tau\bar{\tau}'$ *and* $\tau/\tau' = \bar{\tau}'\tau$*. A permutation* $\tau \in \mathfrak{S}_n$ *is* $\mathbb{E}$*-decomposable (resp.* $\mathbb{H}$*-decomposable) if and only if there exists* $k \in [n-1]$ *such that* $\tau([k]) = [k]$ *(resp. such that* $\tau([k]) = [n] \smallsetminus [k]$*). Moreover,* FQSym *is freely generated by the elements* $\mathbb{E}^\tau$ *(resp.* $\mathbb{H}^\tau$*) for the* $\mathbb{E}$*-indecomposable (resp.* $\mathbb{H}$*-indecomposable) permutations.*

We will also consider the dual Hopf algebra of FQSym, defined as follows.

**Definition 48.** *We denote by* FQSym* *the Hopf algebra with basis* $(\mathbb{G}_\tau)_{\tau\in\mathfrak{S}}$ *and whose product and coproduct are defined by*

$$\mathbb{G}_\tau \cdot \mathbb{G}_{\tau'} = \sum_{\sigma \in \tau\star\tau'} \mathbb{G}_\sigma \qquad \text{and} \qquad \triangle \mathbb{G}_\sigma = \sum_{\sigma \in \tau \bar{\sqcup\!\sqcup} \tau'} \mathbb{G}_\tau \otimes \mathbb{G}_{\tau'}.$$

*This algebra is graded by the size of the permutations.*

3.2. **Subalgebra of FQSym.** We denote by $\mathsf{Twist}^k$ the vector subspace of $\mathsf{FQSym}$ generated by the elements

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S} \\ \mathsf{ins}^k(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T}^\#)} \mathbb{F}_\tau,$$

for all acyclic $k$-twists T. For example, for the $(k,5)$-twists of Figure 4, we have

$= \sum_{\tau \in \mathfrak{S}_5} \mathbb{F}_\tau$

$= \mathbb{F}_{13542} + \mathbb{F}_{15342} + \mathbb{F}_{31542} + \mathbb{F}_{51342} + \mathbb{F}_{35142} + \mathbb{F}_{53142} + \mathbb{F}_{35412} + \mathbb{F}_{53412}$

$= \mathbb{F}_{31542} + \mathbb{F}_{35142}$

$= \mathbb{F}_{31542}.$

**Theorem 49.** $\mathsf{Twist}^k$ *is a Hopf subalgebra of* $\mathsf{FQSym}$.

*Proof.* This statement is a particular case of the results of [Rea05]. Alternatively, we could also invoke the formalism of [Hiv04, HN07, Pri13] and just observe that the $k$-twist congruence $\equiv^k$ is compatible with the standardization and the restriction to intervals. A detailed proof can also be found in [Pil15]. □

**Example 50** (J.-L. Loday and M. Ronco's algebra)**.** The bijection given in Example 3 (see also Figure 2) defines an isomorphism from the 1-twist algebra $\mathsf{Twist}^1$ to M. Ronco and J.-L. Loday's Hopf algebra $\mathsf{PBT}$ on planar binary trees [LR98, HNT05].

We now aim at understanding the product and the coproduct in $\mathsf{Twist}^k$ directly on $k$-twists. Although they are not always as satisfactory, our descriptions naturally extend classical results on the binary tree Hopf algebra $\mathsf{PBT}$ described in [LR98, AS06, HNT05].

PRODUCT To describe the product in $\mathsf{Twist}^k$, we need the following notation, which is illustrated in Figure 11. For a $(k,n)$-twist T and a $(k,n')$-twist T′, we denote by $\mathrm{T}\backslash\mathrm{T}'$ the $(k,n+n')$-twist obtained by inserting T in the first rows and columns of T′ and by $\mathrm{T}/\mathrm{T}'$ the $(k,n+n')$-twist obtained by inserting T′ in the last rows and columns of T.

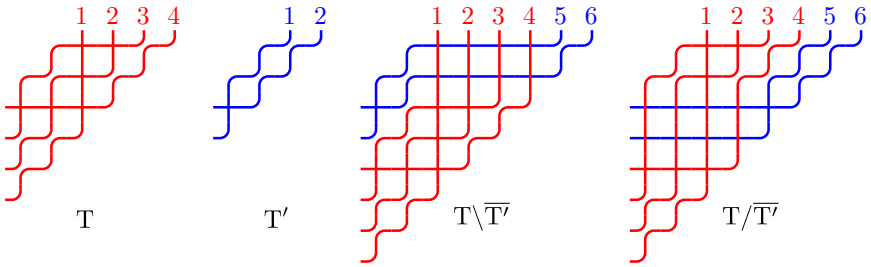


FIGURE 11. Two twists $\mathrm{T}, \mathrm{T}'$ (left) and the two twists $\mathrm{T}\backslash\mathrm{T}'$ and $\mathrm{T}/\mathrm{T}'$ (right).

**Proposition 51.** *For any acyclic $k$-twists $\mathrm{T} \in \mathcal{AT}^k(n)$ and $\mathrm{T}' \in \mathcal{AT}^k(n')$, the product $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}$ is given by*

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}},$$

*where* S *runs over the interval between* $\mathrm{T}\backslash\mathrm{T}'$ *and* $\mathrm{T}/\mathrm{T}'$ *in the* $(k, n+n')$*-twist lattice. See Figure 12.*

*Proof.* Consider two acyclic $k$-twists $\mathrm{T}, \mathrm{T}'$. By Proposition 17, their fibers under $\mathsf{ins}^k$ are intervals of the weak order, which we denote by $[\mu, \omega]$ and $[\mu', \omega']$ respectively. By Proposition 46, the product $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}$ is therefore the weak order interval $[\mu\bar{\mu}', \bar{\omega}'\omega]$. Theorem 49 ensures that this interval is partitioned into various fibers of $\mathsf{ins}^k$. In particular, the fiber of $\mathrm{T}\backslash\mathrm{T}'$ contains $\mu\bar{\mu}'$ while the fiber of $\mathrm{T}/\mathrm{T}'$ contains $\bar{\omega}'\omega$. Proposition 17 finally ensures that $[\mu\bar{\mu}', \bar{\omega}'\omega]$ is precisely the union of the fibers of the increasing flip interval $[\mathrm{T}\backslash\mathrm{T}', \mathrm{T}/\mathrm{T}']$. □

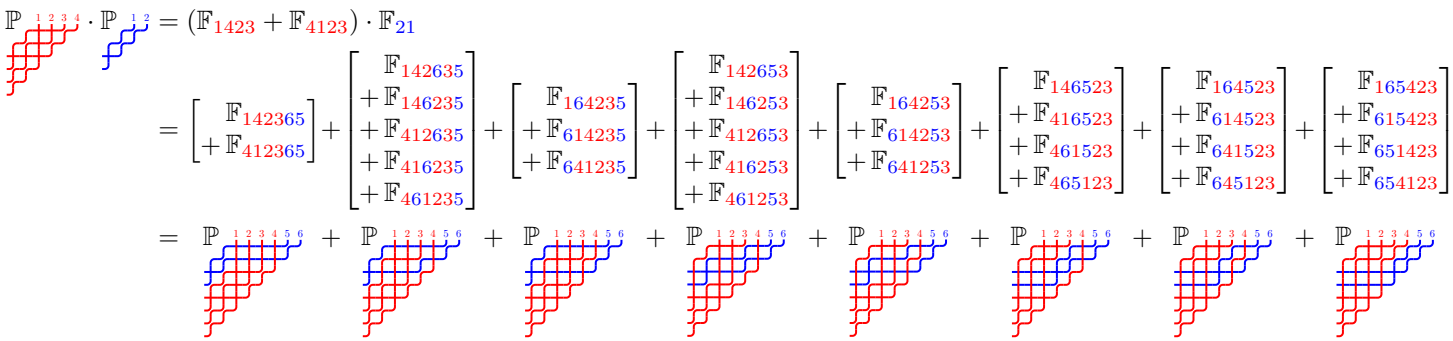


FIGURE 12. An example of product in the 2-twist algebra $\mathsf{Twist}^2$.

$\boxed{\text{COPRODUCT}}$ Our description of the coproduct in $\mathsf{Twist}^k$ is unfortunately not as simple as the coproduct in $\mathsf{PBT}$. It is very closed to the description of the direct computation using the coproduct of $\mathsf{FQSym}$. We need the following definition. A *cut* in a $k$-twist $\mathrm{S}$ is a set $\gamma$ of edges of the contact graph $\mathrm{S}^\#$ such that any path in $\mathrm{S}^\#$ from a leaf to the root contains precisely one edge of $\gamma$. We then denote by $A^\#(\mathrm{S},\gamma)$ (resp. $B^\#(\mathrm{S},\gamma)$) the restriction of the contact graph $\mathrm{S}^\#$ to the nodes above (resp. below) $\gamma$. Moreover, $A^\#(\mathrm{S},\gamma)$ is the contact graph of the $k$-twist $A(\mathrm{S},\gamma)$ obtained from $\mathrm{S}$ by deleting all pipes below $\gamma$ in $\mathrm{S}^\#$. Nevertheless, note that $B^\#(\mathrm{S},\gamma)$ is not *a priori* the contact graph of a $k$-twist.

**Proposition 52.** *For any acyclic $k$-twist $\mathrm{S} \in \mathcal{AT}^k(m)$, the coproduct $\triangle\mathbb{P}_\mathrm{S}$ is given by*

$$\triangle\mathbb{P}_\mathrm{S} = \sum_\gamma \left( \sum_\tau \mathbb{P}_{\mathsf{ins}^k(\tau)} \right) \otimes \mathbb{P}_{A(\mathrm{S},\gamma)},$$

*where $\gamma$ runs over all cuts of $\mathrm{S}$ and $\tau$ runs over a set of representatives of the $k$-twist congruence classes of the linear extensions of $B^\#(\mathrm{S},\gamma)$. See Figure 13.*

*Proof.* By Theorem 49, any element of $\triangle\mathrm{S}$ is of the form $\mathsf{ins}^k(\tau) \otimes \mathsf{ins}^k(\tau')$ for some permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ such that $\tau \star \tau'$ contains a linear extension $\sigma$ of $\mathrm{S}^\#$. Let $\gamma$ denote the cut of $\mathrm{S}$ that separates $\sigma(\{1,\dots,n\})$ from $\sigma(\{n+1,\dots n+n'\})$. Then $\tau$ and $\tau'$ are linear extensions of $B^\#(\mathrm{S},\gamma)$ and $A^\#(\mathrm{S},\gamma)$ respectively, so that $\mathsf{ins}^k(\tau) \otimes \mathsf{ins}^k(\tau')$ indeed appear in the sum on the right hand side. Conversely, for any cut $\gamma$ of $\mathrm{S}$ and linear extensions $\tau$ of $B^\#(\mathrm{S},\gamma)$ and $\tau'$ of $A^\#(\mathrm{S},\gamma)$, there is a linear extension $\sigma$ of $\mathrm{S}^\#$ in $\tau \star \tau'$, so that $\mathsf{ins}^k(\tau) \otimes A(\mathrm{S},\gamma) = \mathsf{ins}^k(\tau) \otimes \mathsf{ins}^k(\tau')$ appears in $\triangle\mathrm{S}$. Finally, we have to prove that the coproduct is boolean, meaning that only $0/1$ coefficients may appear: this follows from the fact that we can reconstruct the cut $\gamma$ from $A(\mathrm{S},\gamma)$ and the $k$-twist congruence class of $\tau$ from $\mathsf{ins}^k(\tau)$. $\square$

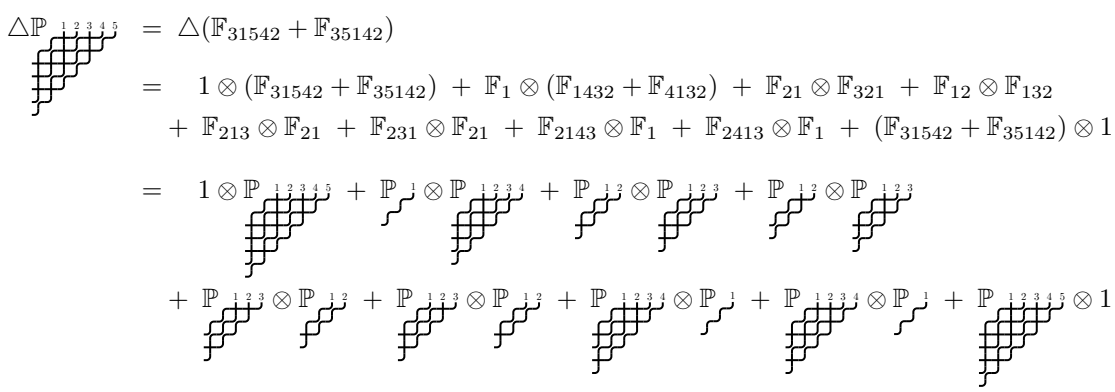


FIGURE 13. An example of coproduct in the 2-twist algebra $\mathsf{Twist}^2$.

$\boxed{\text{Matriochka algebras}}$ We now connect the twist algebra to the $k$-recoil algebra $\mathsf{Rec}^k$ defined as the Hopf subalgebra of $\mathsf{FQSym}$ generated by the elements

$$\mathbb{X}_{\mathrm{O}} := \sum_{\substack{\tau \in \mathfrak{S} \\ \mathsf{rec}^k(\tau) = \mathrm{O}}} \mathbb{F}_\tau,$$

for all acyclic orientations O of the graph $\mathrm{G}^k(n)$ for all $n \in \mathbb{N}$. This algebra was first defined by J.-C. Novelli, C. Reutenauer and J.-Y. Thibon in [NRT11] (the dual of $\mathsf{Rec}^k$ is denoted $\mathsf{DSym}^k$ in their paper). The commutative diagram of Proposition 32 ensures that

$$\mathbb{X}_{\mathrm{O}} = \sum_{\substack{\mathrm{T} \in \mathcal{AT}^k \\ \mathsf{can}^k(\mathrm{T}) = \mathrm{O}}} \mathbb{P}_{\mathrm{T}},$$

and thus that $\mathsf{Rec}^k$ is a Hopf subalgebra of $\mathsf{Twist}^k$.

**Remark 53** (Algebraic inclusions)**.** Following Remarks 33 and 41, note that we have in fact the following inclusions of subalgebras for $k > \ell$:

$$\begin{array}{ccccc}
\mathsf{Twist}^k & & \supseteq & & \mathsf{Twist}^\ell \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & \\
\rotatebox{-90}{$\subseteq$} & & \mathsf{FQSym} & & \rotatebox{-90}{$\subseteq$} \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} & \\
\mathsf{Rec}^k & & \supseteq & & \mathsf{Rec}^\ell
\end{array}$$

We informally call this picture the diagram of Matriochka algebras.

3.3. **Quotient algebra of $\mathsf{FQSym}^*$.** The following statement is automatic from Theorem 49.

**Theorem 54.** *The graded dual $\mathsf{Twist}^{k*}$ of the $k$-twist algebra is the quotient of $\mathsf{FQSym}^*$ under the $k$-twist congruence $\equiv^k$. The dual basis $\mathbb{Q}_{\mathrm{T}}$ of $\mathbb{P}_{\mathrm{T}}$ is expressed as $\mathbb{Q}_{\mathrm{T}} = \pi(\mathbb{G}_\tau)$, where $\pi$ is the quotient map and $\tau$ is any permutation such that $\mathsf{ins}^k(\tau) = \mathrm{T}$.*

Similarly as in the previous section, we try to describe combinatorially the product and coproduct of $\mathbb{Q}$-basis elements of $\mathsf{Twist}^{k*}$ in terms of operations on Cambrian trees.

$\boxed{\text{Product}}$ Once more, our description of the product in the dual twist algebra is not as simple as the coproduct in $\mathsf{PBT}$, and is very closed to the description of the direct computation using the coproduct of $\mathsf{FQSym}$. We use the following notation. For $X = \{x_1 < \cdots < x_n\} \in \binom{[n+n']}{n}$, $\tau \in \mathfrak{S}_n$, and $\mathrm{T}' \in \mathcal{AT}^k(n')$, we denote by $\mathrm{T}' \mathbin{\hat{\wedge}} (\tau \cdot X)$ the result of iteratively inserting $x_{\tau_n}, \ldots, x_{\tau_1}$ in the $k$-twist $\mathrm{T}'$ relabeled increasingly by $[n+n'] \smallsetminus X$.

**Proposition 55.** *For any acyclic $k$-twists $\mathrm{T} \in \mathcal{AT}^k(n)$ and $\mathrm{T}' \in \mathcal{AT}^k(n')$, the product $\mathbb{Q}_{\mathrm{T}} \cdot \mathbb{Q}_{\mathrm{T}'}$ is given by*

$$\mathbb{Q}_{\mathrm{T}} \cdot \mathbb{Q}_{\mathrm{T}'} = \sum_X \mathbb{Q}_{\mathrm{T}' \mathbin{\hat{\wedge}} (\tau \cdot X)}$$

*where $X$ runs over $\binom{[n+n']}{n}$ and $\tau$ is an arbitrary permutation such that $\mathsf{ins}^k(\tau) = \mathrm{T}$. See Figure 14.*

*Proof.* Consider $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ such that $\mathsf{ins}^k(\tau) = \mathrm{T}$ and $\mathsf{ins}^k(\tau') = \mathrm{T}'$, and a permutation $\sigma$ in the convolution $\tau \star \tau'$. Let $X$ denote the first $n$ values in $\sigma$. Since the relative order of the last $n'$ entries in $\sigma$ is that of the entries of $\tau'$, the insertion of the last $n'$ values creates a copy of $\mathrm{T}'$. The remaining entries are then inserted in this copy of $\mathrm{T}'$ at the positions given by $X$ according to the order given by of $\tau$. The result immediately follows. $\square$

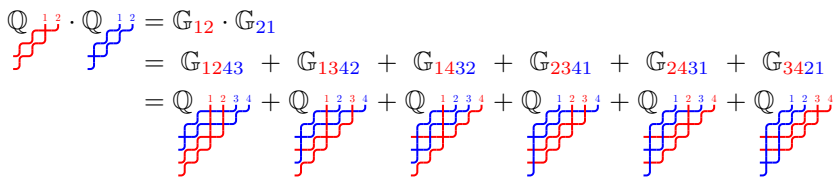


FIGURE 14. An example of product in the dual 2-twist algebra $\mathsf{Twist}^{2*}$.

COPRODUCT Our description of the coproduct is more satisfactory. It is a special case of a coproduct on arbitrary pipe dreams studied by N. Bergeron and C. Ceballos [BC14]. We need the following notations, illustrated in Figure 15. For an acyclic $(k, m)$-twist S and a position $p \in \{0, \dots, m\}$, we define two $k$-twists $L(\mathrm{S}, p) \in \mathcal{AT}^k(p)$ and $R(\mathrm{S}, p) \in \mathcal{AT}^k(m-p)$ as follows. The twist $L(\mathrm{S}, p)$ is obtained by erasing the last $m-p$ pipes in S and glide the elbows of the remaining pipes as northwest as possible. More precisely, each elbow e of one of the first $p$ pipes is translated one step north (resp. west) for each of the last $m-p$ pipes passing north (resp. west) of e. The definition is similar for $R(\mathrm{S}, p)$, except that we erase the first $p$ pipes instead of the last $m-p$ pipes.

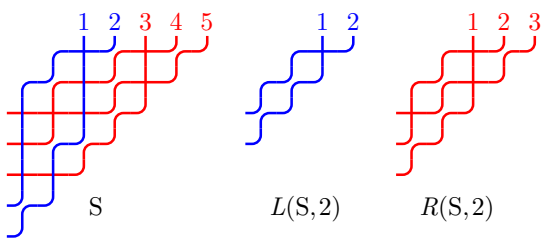


FIGURE 15. A twist S (left) and the two twists $L(\mathrm{S}, p)$ (middle) and $R(\mathrm{S}, p)$ (right).

**Proposition 56.** *For any acyclic $k$-twist $\mathrm{S} \in \mathcal{AT}^k(m)$, the coproduct $\triangle\mathbb{Q}_{\mathrm{S}}$ is given by*

$$\triangle\mathbb{Q}_{\mathrm{S}} = \sum_{p \in \{0,\dots,m\}} \mathbb{Q}_{L(\mathrm{S},p)} \otimes \mathbb{Q}_{R(\mathrm{S},p)}.$$

*See Figure 16.*

*Proof.* Consider $\sigma \in \mathfrak{S}_m$ such that $\mathsf{ins}^k(\sigma) = \mathrm{S}$, let $p \in \{0, \dots, m\}$, and let $\tau \in \mathfrak{S}_p, \tau' \in \mathfrak{S}_{m-p}$ be the two permutations such that $\sigma \in \tau \,\overline{\sqcup}\, \tau'$. By definition, $\tau$ (resp. $\tau'$) is given by the relative order of the first $p$ (resp. last $m-p$) values of $\sigma$. It is immediate to see that the insertion process then gives $\mathsf{ins}^k(\tau) = L(\mathrm{S}, p)$ and $\mathsf{ins}^k(\tau') = R(\mathrm{S}, p)$. The result follows. □

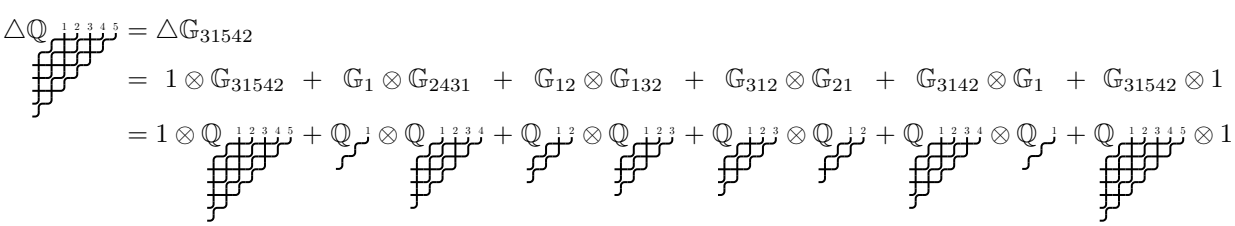


FIGURE 16. An example of coproduct in the dual 2-twist algebra $\mathsf{Twist}^{2*}$.

SELF-DUALITY Note that the contact graph of any $k$-twist has a unique sink. It follows that the last entry of the permutations in a $\equiv^k$-congruence class is constant. S. Giraudo [Gir11, Prop. 5.2.11]

proved that this property implies that the $k$-twist algebra admits a bidendriform structure (see also Section 3.6) and is therefore self-dual, free and cofree (see also Section 3.4). This intriguing self-duality property deserves further study. In particular, the map defined by $\Phi(\mathbb{P}_{\mathrm{T}}) = \sum_{\tau \in \mathcal{L}(\mathrm{T}^{\#})} \mathbb{Q}_{\sigma^{-1}}$ would be a natural candidate for an explicit self-duality [HNT05].

3.4. **Multiplicative bases and irreducible elements.** In this section, we define multiplicative bases of $\mathsf{Twist}^k$ and study the indecomposable elements of $\mathsf{Twist}^k$ for these bases. For an acyclic $(k,n)$-twist T, we define

$$\mathbb{E}^{\mathrm{T}} := \sum_{\mathrm{T} \le \mathrm{T}'} \mathbb{P}_{\mathrm{T}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} := \sum_{\mathrm{T}' \le \mathrm{T}} \mathbb{P}_{\mathrm{T}'},$$

where $\le$ denotes the increasing flip lattice on acyclic $(k,n)$-twists. As the elements $\mathbb{E}^{\mathrm{T}}$ and $\mathbb{H}^{\mathrm{T}}$ have symmetric properties, we focus our analysis on $\mathbb{E}^{\mathrm{T}}$. The reader is invited to translate the statements and proofs below to $\mathbb{H}^{\mathrm{T}}$. We first observe that these elements can also be seen as elements of the multiplicative bases $(\mathbb{E}^{\tau})_{\tau \in \mathfrak{S}}$ and $(\mathbb{H}^{\tau})_{\tau \in \mathfrak{S}}$ of $\mathsf{FQSym}$.

**Lemma 57.** *For any acyclic $k$-twist T, we have $\mathbb{E}^{\mathrm{T}} = \mathbb{E}^{\mu}$ and $\mathbb{H}^{\mathrm{T}} = \mathbb{H}^{\omega}$, where $\mu$ and $\omega$ respectively denote the weak order minimal and maximal permutations in the fiber of T under $\mathsf{ins}^k$.*

*Proof.* We directly obtain from the definition that

$$\mathbb{E}^{\mathrm{T}} = \sum_{\mathrm{T} \le \mathrm{T}'} \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{T} \le \mathrm{T}'} \sum_{\substack{\tau' \in \mathfrak{S}_n \\ \mathsf{ins}^k(\tau') = \mathrm{T}'}} \mathbb{F}_{\tau'} = \sum_{\substack{\tau' \in \mathfrak{S}_n \\ \mathrm{T} \le \mathsf{ins}^k(\tau')}} \mathbb{F}_{\tau'} = \sum_{\substack{\tau' \in \mathfrak{S}_n \\ \mu \le \tau'}} \mathbb{F}_{\tau'} = \mathbb{E}^{\mu}. \qquad \square$$

To describe the product of two elements of the $\mathbb{E}$- or $\mathbb{H}$-basis, remember that the twist $\mathrm{T} \backslash \mathrm{T}'$ (resp. $\mathrm{T}/\mathrm{T}'$) is obtained by inserting T in the first rows and columns of $\mathrm{T}'$ (resp. $\mathrm{T}'$ in the last rows and columns of T). Examples are given in Figure 11.

**Proposition 58.** *$(\mathbb{E}^{\mathrm{T}})_{\mathrm{T} \in \mathcal{AT}^k}$ and $(\mathbb{H}^{\mathrm{T}})_{\mathrm{T} \in \mathcal{AT}^k}$ are multiplicative bases of $\mathsf{Twist}^k$:*

$$\mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'} = \mathbb{E}^{\mathrm{T} \backslash \mathrm{T}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} \cdot \mathbb{H}^{\mathrm{T}'} = \mathbb{H}^{\mathrm{T}/\mathrm{T}'}.$$

*Proof.* Let $\mu$ and $\mu'$ respectively denote the minimal elements of the fibers of T and $\mathrm{T}'$ under $\mathsf{ins}^k$. Using Lemma 57 and the fact that $\mathsf{ins}^k(\mu \backslash \mu') = \mathrm{T} \backslash \mathrm{T}'$ and $\mu \backslash \mu'$ is minimal in its $k$-twist congruence class, we write

$$\mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'} = \mathbb{E}^{\mu} \cdot \mathbb{E}^{\mu'} = \mathbb{E}^{\mu \backslash \mu'} = \mathbb{E}^{\mathrm{T} \backslash \mathrm{T}'}. \qquad \square$$

We now consider multiplicative decomposability. We call *cut* of an acyclic oriented graph any partition $(X \parallel Y)$ of its vertices such that all edges between $X$ and $Y$ are oriented from $X$ to $Y$.

**Proposition 59.** *The following properties are equivalent for an acyclic $k$-twist S:*

*(i) $\mathbb{E}^{\mathrm{S}}$ can be decomposed into a product $\mathbb{E}^{\mathrm{S}} = \mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'}$ for non-empty acyclic $k$-twists $\mathrm{T}, \mathrm{T}'$;*
*(ii) $([k] \parallel [n] \smallsetminus [k])$ is a cut of $\mathrm{S}^{\#}$ for some $k \in [n-1]$;*
*(iii) at least one linear extension $\tau$ of $\mathrm{S}^{\#}$ is $\mathbb{E}$-decomposable, i.e. $\tau([k]) = [k]$ for some $k \in [n]$;*
*(iv) the weak order minimal linear extension of $\mathrm{S}^{\#}$ is $\mathbb{E}$-decomposable.*

*The $k$-twist S is then called $\mathbb{E}$-decomposable. Otherwise, it is called $\mathbb{E}$-indecomposable, and we denote by $\mathcal{IAT}^k(n)$ the set of $\mathbb{E}$-indecomposable acyclic $(k,n)$-twists.*

*Proof.* The equivalence (i) $\iff$ (ii) is an immediate consequence of the description of the product $\mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'} = \mathbb{E}^{\mathrm{T} \backslash \mathrm{T}'}$ in Proposition 58. The implication (ii) $\implies$ (iii) follows from the fact that for any cut $(X \parallel Y)$ of a directed acyclic graph $G$, there exists a linear extension of $G$ which starts with $X$ and finishes with $Y$. The implication (iii) $\implies$ (iv) follows from the fact that the $\mathbb{E}$-indecomposable permutations form an up ideal of the weak order. Finally, if $\tau$ is a decomposable linear extension of S, then the insertion algorithm on $\tau$ first creates a twist labeled by $[n] \smallsetminus [k]$ and then inserts the pipes labeled by $[k]$. Any arc between $[k]$ and $[n] \smallsetminus [k]$ in $\mathrm{S} = \mathsf{ins}^k(\tau)$ will thus be directed from $[k]$ to $[n] \smallsetminus [k]$. $\square$

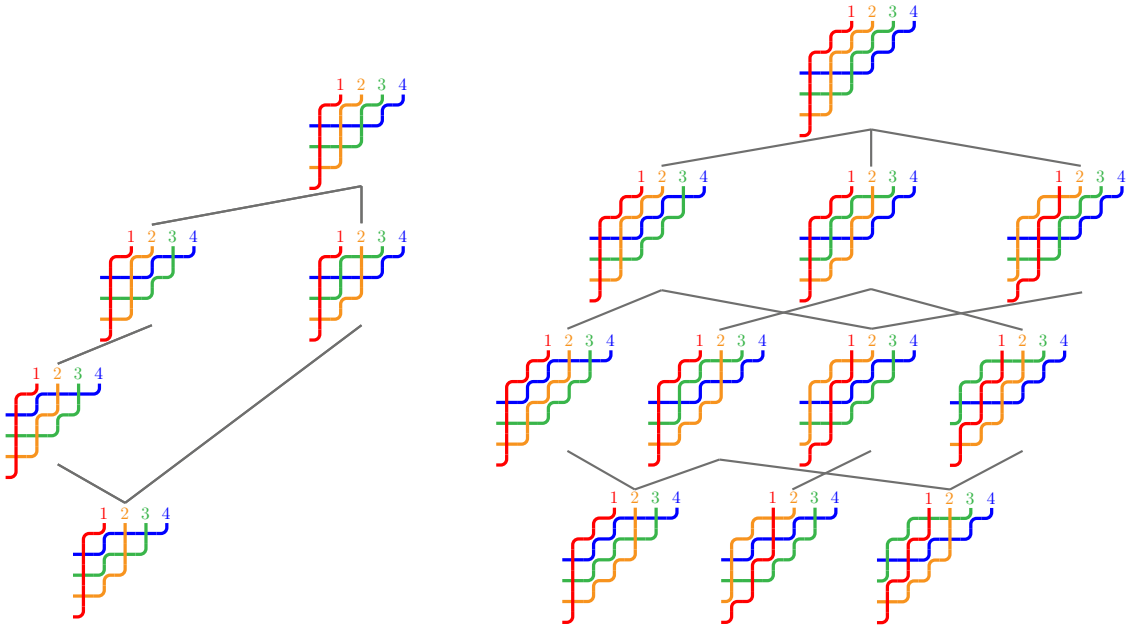

FIGURE 17. The $\mathbb{E}$-indecomposable acyclic $(k,4)$-twists for $k=1,2$.

**Example 60** (Right-tilting $k$-twists)**.** Say that a $k$-twist is *right-tilting* when it has no elbow in its first column. When $k=1$, the $\mathbb{E}$-indecomposable 1-twists are precisely the right-tilting 1-twists. Therefore, the number of $\mathbb{E}$-indecomposable $(1,n)$-twists is the Catalan number $C_{n-1}$, and the $\mathbb{E}$-indecomposable $(1,n)$-twists form a principal ideal of the increasing flip lattice. When $k \geq 2$, right-tilting $k$-twists are $\mathbb{E}$-indecomposable, but are not the only ones. The $\mathbb{E}$-indecomposable $(k,n)$-twists form an upper ideal of the increasing flip lattice, but this ideal is not principal. Figure 17 illustrates the $\mathbb{E}$-indecomposable acyclic $(k,4)$-twists for $k=1,2$.

**Proposition 61.** *The $k$-twist algebra is freely generated by the elements $\mathbb{E}^{\mathrm{T}}$ such that $\mathrm{T}$ is $\mathbb{E}$-indecomposable.*

*Proof.* Let $\mathrm{T}$ be an acyclic $k$-twist and let $\mu$ be the weak order minimal permutation such that $\mathsf{ins}^k(\mu) = \mathrm{T}$. Decompose $\mu = \mu_1 \backslash \ldots \backslash \mu_p$ into $\mathbb{E}$-indecomposable permutations $\mu_1, \ldots, \mu_p$. For $i \in [p]$, define $\mathrm{T}_i := \mathsf{ins}^k(\mu_i)$. Since $\mu_i$ avoids the patterns $(k+2)1-(\sigma_1+1)-\cdots-(\sigma_k+1)$ for all $\sigma \in \mathfrak{S}_k$ (because $\mu$ avoids these patterns), it is the weak order minimal permutation in the fiber of $\mathrm{T}_i$. Since $\mu_i$ is $\mathbb{E}$-indecomposable, we get by Proposition 59 (iv) that $\mathrm{T}_i$ is $\mathbb{E}$-indecomposable. Using Lemma 57, we thus obtained a decomposition $\mathbb{E}^{\mathrm{T}} = \mathbb{E}^{\mu} = \mathbb{E}^{\mu_1} \cdots \mathbb{E}^{\mu_p} = \mathbb{E}^{\mathrm{T}_1} \cdots \mathbb{E}^{\mathrm{T}_p}$ of $\mathrm{T}$ into $\mathbb{E}$-indecomposable $k$-twists $\mathrm{T}_1, \ldots, \mathrm{T}_p$.

Now, there is no relation between the elements $\mathbb{E}^{\tau}$ of $\mathsf{FQSym}$ corresponding to the $\mathbb{E}$-indecomposable permutations. Hence, by Lemma 57 and Proposition 58, there is no relation between the elements $\mathbb{E}^{\mathrm{T}}$ of $\mathsf{Twist}^k$ corresponding to the $\mathbb{E}$-indecomposable $k$-twists. $\square$

**Corollary 62.** *The generating functions of the numbers of $\mathbb{E}$-indecomposable acyclic $(k,n)$-twists and of the numbers of all acyclic $(k,n)$-twists are related by*

$$\frac{1}{1-\sum_{n\in\mathbb{N}} |\mathcal{IAT}^k(n)|\, t^n} = \sum_{n\in\mathbb{N}} |\mathcal{AT}^k(n)|\, t^n.$$

Again, the $\mathbb{E}$-indecomposable 1-twists are precisely the right-tilting 1-twists, and are therefore counted by the Catalan number $C_{n-1}$. Analogous results for $k \geq 2$ remain to be found.

3.5. **Integer point transform.** In this section, we observe that the product in the $k$-twist Hopf algebra $\mathsf{Twist}^k$ can be interpreted in terms of the integer point transforms of the normal cones of the brick polytope $\mathsf{Brick}^k(n)$. To make this statement precise, we introduce some notations.

**Definition 63.** *The* integer point transform $\mathbb{Z}_S$ *of a subset $S$ of $\mathbb{R}^n$ is the multivariate generating function of the integer points inside $S$:*

$$\mathbb{Z}_S(t_1, \dots, t_n) = \sum_{(i_1,\dots,i_n) \in \mathbb{Z}^n \cap S} t_1^{i_1} \cdots t_n^{i_n}.$$

For a poset $\lhd$, we denote by $\mathbb{Z}_\lhd$ the integer point transform of the cone

$$\mathrm{C}^{\blacklozenge}(\lhd) := \left\{ \mathbf{x} \in \mathbb{R}^n_+ \;\middle|\; \begin{array}{l} x_i \le x_j \text{ for all } i \lhd j \text{ with } i < j \\ x_i < x_j \text{ for all } i \lhd j \text{ with } i > j \end{array} \right\}.$$

Note that this cone differs in two ways from the cone $\mathrm{C}^{\lozenge}(\lhd)$ defined in Section 2.2: first it leaves in $\mathbb{R}^n_+$ and not in $\mathbb{H}$, second it excludes the facets of $\mathrm{C}^{\lozenge}(\lhd)$ corresponding to the decreasing relations of $\lhd$ (*i.e.* the relations $i \lhd j$ with $i > j$).

Following the notations of Section 2.2, we denote by $\mathbb{Z}_\tau$ the integer point transform of the chain $\tau_1 \lhd \cdots \lhd \tau_n$ for a permutation $\tau \in \mathfrak{S}_n$. The following statements are classical.

**Proposition 64.** *(i) For any permutation $\tau \in \mathfrak{S}_n$, the integer point transform $\mathbb{Z}_\tau$ is given by*

$$\mathbb{Z}_\tau(t_1, \dots, t_n) = \prod_{\substack{i \in [n-1] \\ \tau_i > \tau_{i+1}}} t_{\tau_i} \cdots t_{\tau_n} \Big/ \prod_{i \in [n]} \big(1 - t_{\tau_i} \cdots t_{\tau_n}\big).$$

*(ii) The integer point transform of an arbitrary poset $\lhd$ is given by*

$$\mathbb{Z}_\lhd = \sum_{\tau \in \mathcal{L}(\lhd)} \mathbb{Z}_\tau,$$

*where the sum runs over the set $\mathcal{L}(\lhd)$ of linear extensions of $\lhd$.*

*(iii) The product of the integer point transforms $\mathbb{Z}_\tau$ and $\mathbb{Z}_{\tau'}$ of two permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ is given by the shifted shuffle*

$$\mathbb{Z}_\tau(t_1, \dots, t_n) \cdot \mathbb{Z}_{\tau'}(t_{n+1}, \dots, t_{n+n'}) = \sum_{\sigma \in \tau \,\overline{\sqcup}\, \tau'} \mathbb{Z}_\sigma(t_1, \dots, t_{n+n'}).$$

*In other words, the linear map from $\mathsf{FQSym}$ to the rational functions defined by $\Psi : \mathbb{F}_\tau \mapsto \mathbb{Z}_\tau$ is an algebra morphism.*

*Proof.* For Point (i), we just observe that the cone $\big\{\mathbf{x} \in \mathbb{R}^n_+ \;\big|\; x_{\tau_i} \le x_{\tau_{i+1}} \text{ for all } i \in [n-1]\big\}$ is generated by the vectors $\mathbf{e}_{\tau_i} + \cdots + \mathbf{e}_{\tau_n}$, for $i \in [n]$, which form a (unimodular) basis of the lattice $\mathbb{Z}^n$. A straightforward inductive argument shows that the integer point transform of the cone $\big\{\mathbf{x} \in \mathbb{R}^n_+ \;\big|\; x_{\tau_i} \le x_{\tau_{i+1}} \text{ for all } i \in [n-1]\big\}$ is thus given by $\prod_{i \in [n]} \big(1 - t_{\tau_i} \cdots t_{\tau_n}\big)^{-1}$. The numerator of $\mathbb{Z}_\tau$ is then given by the facets which are excluded from the cone $\mathrm{C}^{\blacklozenge}(\tau)$.

Point (ii) follows from the fact that the cone $\mathrm{C}^{\blacklozenge}(\lhd)$ is partitioned by the cones $\mathrm{C}^{\blacklozenge}(\tau)$ for the linear extensions $\tau$ of $\lhd$.

Finally, the product $\mathbb{Z}_\tau(t_1, \dots, t_n) \cdot \mathbb{Z}_{\tau'}(t_{n+1}, \dots, t_{n+n'})$ is the integer point transform of the poset formed by the two disjoint chains $\tau$ and $\bar{\tau}'$, whose linear extensions are precisely the permutations which appear in the shifted shuffle of $\tau$ and $\tau'$. This shows Point (iii). □

For an acyclic $k$-twist $\mathrm{T}$, we denote by $\mathbb{Z}_\mathrm{T}$ the integer point transform of the transitive closure of the contact graph $\mathrm{T}^\#$. It follows from Proposition 64 that the product of the integer point transforms of two acyclic $k$-twists behaves as the product in the $k$-twist algebra $\mathsf{Twist}^k$.

**Corollary 65.** *For any two acyclic $k$-twists $\mathrm{T}, \mathrm{T}'$, we have*

$$\mathbb{Z}_\mathrm{T}(t_1, \dots, t_n) \cdot \mathbb{Z}_{\mathrm{T}'}(t_{n+1}, \dots, t_{n+n'}) = \sum_{\mathrm{T} \backslash \mathrm{T}' \le \mathrm{S} \le \mathrm{T}/\mathrm{T}'} \mathbb{Z}_\mathrm{S}(t_1, \dots, t_{n+n'}).$$

*Proof.* Hiding the variables $(t_1, \dots, t_{n+n'})$ for concision, we have

$$\mathbb{Z}_{\mathrm{T}} \cdot \mathbb{Z}_{\mathrm{T}'} = \Psi(\mathbb{P}_{\mathrm{T}}) \cdot \Psi(\mathbb{P}_{\mathrm{T}'}) = \Psi(\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}) = \Psi\Big(\sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}\Big) = \sum_{\mathrm{S}} \Psi(\mathbb{P}_{\mathrm{S}}) = \sum_{\mathrm{S}} \mathbb{Z}_{\mathrm{S}},$$

where the sums run over the $k$-twists S of the increasing flip lattice interval $[\mathrm{T}\backslash\mathrm{T}', \mathrm{T}/\mathrm{T}']$. □

3.6. **$k$-twistiform algebras.** In this section, we extend the notion of dendriform algebras to $k$-twistiform algebras. Dendriform algebras were introduced by J.-L. Loday in [Lod01, Chap. 5]. In a dendriform algebra, the product $\cdot$ is decomposed into two partial products $\prec$ and $\succ$ satisfying:

$$\mathsf{x} \prec (\mathsf{y} \cdot \mathsf{z}) = (\mathsf{x} \prec \mathsf{y}) \prec \mathsf{z}, \qquad \mathsf{x} \succ (\mathsf{y} \prec \mathsf{z}) = (\mathsf{x} \succ \mathsf{y}) \prec \mathsf{z}, \qquad \mathsf{x} \succ (\mathsf{y} \succ \mathsf{z}) = (\mathsf{x} \cdot \mathsf{y}) \succ \mathsf{z}.$$

In our context, we will still decompose the product of $\mathsf{FQSym}$ (and of $\mathsf{Twist}^k$) into partial products, but we will use $2^k$ partial products satisfying $3^k$ relations. In this paper, we just give the definition and observe that the algebras $\mathsf{FQSym}$ and $\mathsf{Twist}^k$ are naturally endowed with a $k$-twistiform structure, as they motivated the definition. A detailed study of combinatorial and algebraic properties of $k$-twistiform algebras and operads is in progress in a joint work with F. Hivert [HP15].

We need to fix some natural notations on words. We denote by $|W|$ the length of a word $W$. For a subset $P$ of positions in $W$ and a subset $L$ of letters of $W$, we denote by $W_P$ the subword of $W$ consisting only of the letters at positions in $P$ and by $W^L$ the subword of $W$ consisting only of the letters which belong to $L$.

**Definition 66.** *A $k$-twistiform algebra is a vector space $\mathsf{Alg}$ endowed with a collection $\mathfrak{B} := \{\prec, \asymp, \succ\}^k$ of $3^k$ bilinear operations which satisfy the following $k3^{k-1} + 3^k$ relations:*

**Split relations:** *For any $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \asymp, \succ\}^*$ with $|\mathfrak{b}| + |\mathfrak{b}'| = k-1$, the operation $\mathfrak{b}\asymp\mathfrak{b}' \in \mathfrak{B}$ decomposes into the operations $\mathfrak{b}\prec\mathfrak{b} \in \mathfrak{B}$ and $\mathfrak{b}\succ\mathfrak{b} \in \mathfrak{B}$:*

$$\mathsf{x}\ \mathfrak{b}\asymp\mathfrak{b}'\ \mathsf{y} \ = \ \mathsf{x}\ \mathfrak{b}\prec\mathfrak{b}'\ \mathsf{y} \ + \ \mathsf{x}\ \mathfrak{b}\succ\mathfrak{b}'\ \mathsf{y} \qquad \textit{for all } \mathsf{x}, \mathsf{y} \in \mathsf{Alg}.$$

**Associativity relations:** *For any $W \in \{x, y, z\}^k$, the operations $\mathfrak{b}_W, \mathfrak{b}'_W, \mathfrak{b}''_W, \mathfrak{b}'''_W \in \mathfrak{B}$ defined by*

$$(\mathfrak{b}_W)_p := \begin{cases} \prec & \textit{if } W_p = x \\ \succ & \textit{if } W_p \in \{y, z\} \end{cases} \qquad (\mathfrak{b}'_W)_p := \begin{cases} \prec & \textit{if } |W^{\{y,z\}}| \geq p \textit{ and } (W^{\{y,z\}})_p = y \\ \succ & \textit{if } |W^{\{y,z\}}| \geq p \textit{ and } (W^{\{y,z\}})_p = z \\ \asymp & \textit{otherwise} \end{cases}$$

$$(\mathfrak{b}''_W)_p := \begin{cases} \prec & \textit{if } |W^{\{x,y\}}| \geq p \textit{ and } (W^{\{x,y\}})_p = x \\ \succ & \textit{if } |W^{\{x,y\}}| \geq p \textit{ and } (W^{\{y,z\}})_p = y \\ \asymp & \textit{otherwise} \end{cases} \qquad (\mathfrak{b}'''_W)_p := \begin{cases} \prec & \textit{if } W_p \in \{x, y\} \\ \succ & \textit{if } W_p = z \end{cases}$$

*satisfy the associativity relation*

$$\mathsf{x}\ \mathfrak{b}_W\ (\mathsf{y}\ \mathfrak{b}'_W\ \mathsf{z}) = (\mathsf{x}\ \mathfrak{b}''_W\ \mathsf{y})\ \mathfrak{b}'''_W\ \mathsf{z} \qquad \textit{for all } \mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{Alg}.$$

**Example 67** (1- and 2-twistiform algebras)**.** 1-twistiform algebras are precisely dendriform algebras, *i.e.* vector spaces endowed with three operations $\prec$, $\asymp$, $\succ$ which fulfill the 4 relations:

$$\mathsf{x} \asymp \mathsf{y} \ = \ \mathsf{x} \prec \mathsf{y} \ + \ \mathsf{x} \succ \mathsf{y},$$
$$\mathsf{x} \prec (\mathsf{y} \asymp \mathsf{z}) = (\mathsf{x} \prec \mathsf{y}) \prec \mathsf{z}, \qquad \mathsf{x} \succ (\mathsf{y} \prec \mathsf{z}) = (\mathsf{x} \succ \mathsf{y}) \prec \mathsf{z}, \qquad \mathsf{x} \succ (\mathsf{y} \succ \mathsf{z}) = (\mathsf{x} \asymp \mathsf{y}) \succ \mathsf{z}.$$

2-twistiform algebras are vector spaces endowed with 9 operations $\prec\prec$, $\prec\asymp$, $\prec\succ$, $\asymp\prec$, $\asymp\asymp$, $\asymp\succ$, $\succ\prec$, $\succ\asymp$, $\succ\succ$ which satisfy the following 15 relations:

$$\mathsf{x} \asymp\prec \mathsf{y} \ = \ \mathsf{x} \prec\prec \mathsf{y} \ + \ \mathsf{x} \succ\prec \mathsf{y}, \qquad \mathsf{x} \asymp\asymp \mathsf{y} \ = \ \mathsf{x} \prec\asymp \mathsf{y} \ + \ \mathsf{x} \succ\asymp \mathsf{y}, \qquad \mathsf{x} \asymp\succ \mathsf{y} \ = \ \mathsf{x} \prec\succ \mathsf{y} \ + \ \mathsf{x} \succ\succ \mathsf{y},$$
$$\mathsf{x} \prec\asymp \mathsf{y} \ = \ \mathsf{x} \prec\prec \mathsf{y} \ + \ \mathsf{x} \prec\succ \mathsf{y}, \qquad \mathsf{x} \asymp\asymp \mathsf{y} \ = \ \mathsf{x} \asymp\prec \mathsf{y} \ + \ \mathsf{x} \asymp\succ \mathsf{y}, \qquad \mathsf{x} \succ\asymp \mathsf{y} \ = \ \mathsf{x} \succ\prec \mathsf{y} \ + \ \mathsf{x} \succ\succ \mathsf{y},$$
$$\mathsf{x} \prec\prec (\mathsf{y} \asymp\asymp \mathsf{z}) = (\mathsf{x} \prec\prec \mathsf{y}) \prec\prec \mathsf{z}, \qquad \mathsf{x} \prec\succ (\mathsf{y} \prec\asymp \mathsf{z}) = (\mathsf{x} \prec\succ \mathsf{y}) \prec\prec \mathsf{z}, \qquad \mathsf{x} \prec\succ (\mathsf{y} \succ\asymp \mathsf{z}) = (\mathsf{x} \prec\asymp \mathsf{y}) \prec\succ \mathsf{z},$$
$$\mathsf{x} \succ\prec (\mathsf{y} \prec\asymp \mathsf{z}) = (\mathsf{x} \succ\prec \mathsf{y}) \prec\prec \mathsf{z}, \qquad \mathsf{x} \succ\succ (\mathsf{y} \prec\prec \mathsf{z}) = (\mathsf{x} \succ\succ \mathsf{y}) \prec\prec \mathsf{z}, \qquad \mathsf{x} \succ\succ (\mathsf{y} \prec\succ \mathsf{z}) = (\mathsf{x} \succ\asymp \mathsf{y}) \prec\succ \mathsf{z},$$
$$\mathsf{x} \succ\prec (\mathsf{y} \succ\asymp \mathsf{z}) = (\mathsf{x} \prec\asymp \mathsf{y}) \succ\prec \mathsf{z}, \qquad \mathsf{x} \succ\succ (\mathsf{y} \succ\prec \mathsf{z}) = (\mathsf{x} \succ\asymp \mathsf{y}) \succ\prec \mathsf{z}, \qquad \mathsf{x} \succ\succ (\mathsf{y} \succ\succ \mathsf{z}) = (\mathsf{x} \asymp\asymp \mathsf{y}) \succ\succ \mathsf{z}.$$

**Remark 68.** Adding up all associativity relations, one obtains that

$$\mathsf{x} \asymp^k (\mathsf{y} \asymp^k \mathsf{z}) = (\mathsf{x} \asymp^k \mathsf{y}) \asymp^k \mathsf{z} \qquad \text{for all } \mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{Alg},$$

so that the $k$-twistiform algebra $(\mathsf{Alg}, \{\prec, \asymp, \succ\}^k)$ defines in particular a structure of associative algebra $(\mathsf{Alg}, \asymp^k)$. Reciprocally, we say that an associative algebra $(\mathsf{Alg}, \cdot)$ admits a *$k$-twistiform structure* if it is possible to split the product $\cdot$ into $3^k$ operations $\mathfrak{B} := \{\prec, \asymp, \succ\}^k$ defining a $k$-twistiform algebra on $\mathsf{Alg}$.

We now show that C. Malvenuto and C. Reutenauer's Hopf algebra on permutations $\mathsf{FQSym}$ can be endowed with a structure of $k$-twistiform algebra. For an operation $\mathfrak{b} \in \mathfrak{B}$ and two words $X := x\underline{X}$ and $Y := y\underline{Y}$, we define

$$X \mathbin{\mathfrak{b}} Y = \begin{cases} X \sqcup\!\sqcup Y & \text{if } \mathfrak{b} = \varnothing, \\ x(\underline{X} \mathbin{\underline{\mathfrak{b}}} Y) & \text{if } \mathfrak{b} = \prec\underline{\mathfrak{b}}, \\ x(\underline{X} \mathbin{\underline{\mathfrak{b}}} Y) \ \cup \ y(X \mathbin{\underline{\mathfrak{b}}} \underline{Y}) & \text{if } \mathfrak{b} = \asymp\underline{\mathfrak{b}}, \\ y(X \mathbin{\underline{\mathfrak{b}}} \underline{Y}) & \text{if } \mathfrak{b} = \succ\underline{\mathfrak{b}} \end{cases}$$

with the initial conditions $X \succ\underline{\mathfrak{b}} \varnothing = \varnothing \prec\underline{\mathfrak{b}} Y = 0$.

In other words, we consider the shuffle of $X$ and $Y$, except that the $i$th letter of $X \mathbin{\mathfrak{b}} Y$ is forced to belong to $X$ (resp. to $Y$) if the $i$th letter of $\mathfrak{b}$ is $\prec$ (resp. is $\succ$). For example, when $k = 1$, the three operators are given by

$$X \prec Y = x(\underline{X} \sqcup\!\sqcup Y), \qquad X \asymp Y = X \sqcup\!\sqcup Y, \qquad X \succ Y = y(X \sqcup\!\sqcup \underline{Y}),$$

with the initial conditions $X \succ \varnothing = \varnothing \prec Y = 0$.

Now for an operation $\mathfrak{b} \in \mathfrak{B}$ and two permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, we define $\tau \mathbin{\mathfrak{b}} \tau' = \tau \mathbin{\mathfrak{b}} \bar{\tau}'$, where $\bar{\tau}'$ is the permutation $\tau'$ shifted by the length $n$ of $\tau$. Equivalently, $\tau \mathbin{\mathfrak{b}} \tau'$ is the set of permutations $\sigma \in \tau \mathbin{\bar{\sqcup\!\sqcup}} \tau'$ such that for all $i \in [k]$, we have $\sigma_i \le n$ if $\mathfrak{b}_i = \prec$ while $\sigma_i > n$ if $\mathfrak{b}_i = \succ$. Finally, we define the operations $\mathfrak{B}$ on the Hopf algebra $\mathsf{FQSym}$ itself by

$$\mathbb{F}_\tau \mathbin{\mathfrak{b}} \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \mathbin{\mathfrak{b}} \tau'} \mathbb{F}_\sigma.$$

**Proposition 69.** *The Hopf algebra $\mathsf{FQSym}$, endowed with the operations $\mathfrak{B}$ described above, defines a $k$-twistiform algebra. The product of $\mathsf{FQSym}$ is then given by $\cdot = \asymp^k$.*

*Proof.* We have to show that the operations defined above on $\mathsf{FQSym}$ indeed satisfy the $k3^{k-1} + 3^k$ relations of Definition 66.

- **Split relations:** Let $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \asymp, \succ\}^*$ with $|\mathfrak{b}| + |\mathfrak{b}'| = k - 1$. It is immediate from the definitions that $X \mathbin{\mathfrak{b}\asymp\mathfrak{b}'} Y = (X \mathbin{\mathfrak{b}\prec\mathfrak{b}'} Y) \cup (X \mathbin{\mathfrak{b}\succ\mathfrak{b}'} Y)$ for any two words $X, Y$.
- **Associativity relations:** Let $W \in \mathfrak{B}$. It follows from the definition of the operations $\mathfrak{b}_W$, $\mathfrak{b}'_W$, $\mathfrak{b}''_W$ and $\mathfrak{b}'''_W$ that for any words $X, Y, Z$

  $$X \mathbin{\mathfrak{b}_W} (Y \mathbin{\mathfrak{b}'_W} Z) = (X \mathbin{\mathfrak{b}''_W} Y) \mathbin{\mathfrak{b}'''_W} Z$$

  is the set of all words in $X \sqcup\!\sqcup Y \sqcup\!\sqcup Z$ whose $p$th letter is in the word $X$ if $W_p = x$, in the word $Y$ if $W_p = y$ and in the word $Z$ if $W_p = z$.

These equalities of sets then translate to the desired linear relations on the corresponding operations in $\mathsf{FQSym}$. □

We say that the operations $\mathfrak{B}$ define the *forward $k$-twistiform structure* on $\mathsf{FQSym}$. There is also a *backward $k$-twistiform structure* on $\mathsf{FQSym}$ which considers the last $k$ letters rather than the first $k$ ones. Namely, for each operation $\mathfrak{b} \in \mathfrak{B}$, define an operation $\mathfrak{b}^\bullet$ by $V \mathbin{\mathfrak{b}^\bullet} W = (V^\bullet \mathbin{\mathfrak{b}} W^\bullet)^\bullet$ where $W^\bullet = w_n \cdots w_1$ denotes the mirror of a word $W = w_1 \cdots w_n$. Clearly, the operations $\mathfrak{b}^\bullet$ for $\mathfrak{b} \in \mathfrak{B}$ still fulfill the relations of Definition 66. We have chosen to define the forward $k$-twistiform structure as it leads to a simpler presentation, but we need this backward $k$-twistiform structure in the next statement to be coherent with the insertion in $k$-twists (whose direction was chosen consistently with J.-L. Loday and M. Ronco's conventions).

**Proposition 70.** *The subalgebra* $\mathsf{Twist}^k$ *of* $\mathsf{FQSym}$ *is stable by the operations* $\mathfrak{b}^\bullet$ *for* $\mathfrak{b} \in \mathfrak{B}$ *and therefore inherits a* $k$*-twistiform structure.*

*Proof.* Let T be an acyclic $k$-twist. We claim that the last $k$ entries are the same in all linear extensions of $\mathrm{T}^\#$. Indeed, pick a linear extension $\tau$ of $\mathrm{T}^\#$ and let $\mathrm{T}' := \varnothing \Cap \tau_n \Cap \cdots \Cap \tau_{n-k+1}$ denote the $k$-twist obtain after the insertion of the last $k$ values of $\tau$. All pipes of $\mathrm{T}'$ are then comparable by Lemma 30 and will be comparable to all other pipes in $\mathrm{T} = \mathrm{T}' \Cap \tau_{n-k} \Cap \cdots \Cap \tau_1$. Thus the last $k$ values of $\tau$ form a chain at the end of the contact graph $\mathrm{T}^\#$.

It then follows that the operation $\mathfrak{b}^\bullet$ stabilizes $\mathsf{Twist}^k$ for any $\mathfrak{b} \in \mathfrak{B}$. Indeed, for any two acyclic twists $\mathrm{T} \in \mathcal{AT}^k(n)$ and $\mathrm{T}' \in \mathcal{AT}^k(n')$, we have

$$\mathbb{P}_\mathrm{T}\ \mathfrak{b}^\bullet\ \mathbb{P}_{\mathrm{T}'} = \sum_\mathrm{S} \mathbb{P}_\mathrm{S}$$

where the sum runs over all acyclic twists $\mathrm{S} \in \mathcal{AT}^k(n+n')$ such that $\mathrm{T}\backslash\mathrm{T}' \leq \mathrm{S} \leq \mathrm{T}/\mathrm{T}'$ and $\sigma_{n+n'+1-i} \leq n$ if $\mathfrak{b}_i = \prec$ and $\sigma_{n+n'+1-i} > n$ if $\mathfrak{b}_i = \succ$ for any linear extension $\sigma$ of $\mathrm{S}^\#$. □

**Remark 71.** One can also define similarly $k$-cotwistiform coalgebras, and such a structure on both $\mathsf{FQSym}$ and $\mathsf{Twist}^k$. Details will be given in [HP15].

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