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# CAMBRIAN HOPF ALGEBRAS

GRÉGORY CHATEL AND VINCENT PILAUD

Abstract. Cambrian trees are oriented and labeled trees which fulfill local conditions around each node generalizing the conditions for classical binary search trees. Based on a natural surjection from signed permutations to Cambrian trees, we define the Cambrian Hopf algebra extending J.-L. Loday and M. Ronco's algebra on binary trees. We describe combinatorially the products and coproducts of both the Cambrian algebra and its dual in terms of operations on Cambrian trees. We then construct the Baxter-Cambrian algebra which extends S. Law and N. Reading's Baxter Hopf algebra on rectangulations and S. Giraudo's equivalent Hopf algebra on twin binary trees.

The background of this paper is the fascinating interplay between the combinatorial, geometric and algebraic structures of permutations, binary trees and binary sequences (see Table 1):

- ⋆ **Combinatorially**, the descent map from permutations to binary sequences factors via binary trees through the BST insertion and the canopy map. These maps define lattice homomorphisms from the weak order via the Tamari lattice to the boolean lattice.
- ⋆ **Geometrically**, the permutahedron is contained in Loday's associahedron [Lod04] which is in turn contained in the parallelepiped generated by the simple roots. These polytopes are just obtained by deleting inequalities from the facet description of the permutahedron. See Figure 1.
- ⋆ **Algebraically**, these maps translate to Hopf algebra inclusions from M. Malvenuto and C. Reutenauer's algebra on permutations [MR95] via J.-L. Loday and M. Ronco's algebra on binary trees [LR98] to I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh and J.-Y. Thibon's Hopf algebra on binary sequences [GKL+95].

| Combinatorics | Permutations | Binary trees | Binary sequences |
|---|---|---|---|
| Geometry | Permutahedron $\mathrm{conv}(\mathfrak{S}_n)$ | Loday's associahedron [Lod04] | Parallelepiped generated by $e_{i+1} - e_i$ |
| Algebra | Malvenuto-Reutenauer Hopf algebra [MR95] | Loday-Ronco Hopf algebra [LR98] | descent Hopf algebra [GKL+95] |

Table 1. Related combinatorial, geometric and algebraic structures.

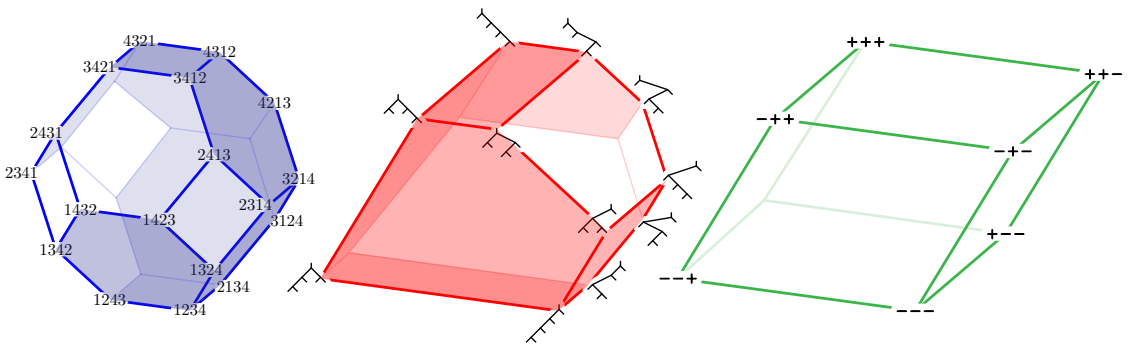


Figure 1. The 3-dimensional permutahedron (blue, left), Loday's associahedron (red, middle), and parallelepiped (green, right). Shaded facets are preserved to get the next polytope.

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These structures and their connections have been partially extended in several directions in particular to the Cambrian lattices of N. Reading [Rea06, RS09] and their polytopal realizations by C. Hohlweg, C. Lange, and H. Thomas [HL07, HLT11], to the graph associahedra of M. Carr and S. Devadoss [CD06, Dev09], the nested complexes and their realizations as generalized associahedra by A. Postnikov [Pos09] (see also [PRW08, FS05, Zel06]), or to the $m$-Tamari lattices of F. Bergeron and L.-F. Préville-Ratelle [BPR12] (see also [BMFPR11, BMCPR13]) and the Hopf algebras on these $m$-structures recently constructed by J.-C. Novelli and J.-Y. Thibon [NT14, Nov14].

This paper explores combinatorial and algebraic aspects of Hopf algebras related to the type $A$ Cambrian lattices. N. Reading provides in [Rea06] a procedure to map a signed permutation of $\mathfrak{S}_n$ into a triangulation of a certain convex $(n+3)$-gon. The dual trees of these triangulations naturally extend rooted binary trees and were introduced and studied as "spines" [LP13] or "mixed cobinary trees" [IO13]. We prefer here the term "Cambrian trees" in reference to N. Reading's work. The map $\mathbf{ct}$ from signed permutations to Cambrian trees is known to encode combinatorial and geometric properties of the Cambrian structures: the Cambrian lattice is the quotient of the weak order under the fibers of $\mathbf{ct}$, each maximal cone of the Cambrian fan is the incidence cone of a Cambrian tree T and is refined by the braid cones of the permutations in the fiber $\mathbf{ct}^{-1}(\mathrm{T})$, etc.

In this paper, we use this map $\mathbf{ct}$ for algebraic purposes. In the first part, we introduce the Cambrian Hopf algebra $\mathsf{Camb}$ as a subalgebra of the Hopf algebra $\mathsf{FQSym}_\pm$ on signed permutations, and the dual Cambrian algebra $\mathsf{Camb}^*$ as a quotient algebra of the dual Hopf algebra $\mathsf{FQSym}_\pm^*$. Their bases are indexed by all Cambrian trees. Our approach extends that of F. Hivert, J.-C. Novelli and J.-Y. Thibon [HNT05] to construct J.-L. Loday and M. Ronco's Hopf algebra on binary trees [LR98] as a subalgebra of C. Malvenuto and C. Reutenauer's Hopf algebra on permutations [MR95]. We also use this map $\mathbf{ct}$ to describe both the product and coproduct in the algebras $\mathsf{Camb}$ and $\mathsf{Camb}^*$ in terms of simple combinatorial operations on Cambrian trees.

In the second part of this paper, we study Baxter-Cambrian structures, extending in the Cambrian setting the constructions of S. Law and N. Reading on rectangulations [LR12] and that of S. Giraudo on twin binary trees [Gir12]. We define Baxter-Cambrian lattices as quotients of the weak order under the intersections of two opposite Cambrian congruences. Their elements can be labeled by pairs of twin Cambrian trees, *i.e.* Cambrian trees with opposite signatures whose union forms an acyclic graph. We study in detail the number of such pairs of Cambrian trees for arbitrary signatures. Following [LR12], we also observe that the Minkowski sums of opposite associahedra of C. Hohlweg and C. Lange [HL07] provide polytopal realizations of the Baxter-Cambrian lattices. Finally, we introduce the Baxter-Cambrian Hopf algebra $\mathsf{BaxCamb}$ as a subalgebra of the Hopf algebra $\mathsf{FQSym}_\pm$ on signed permutations and its dual $\mathsf{BaxCamb}^*$ as a quotient algebra of the dual Hopf algebra $\mathsf{FQSym}_\pm^*$. Their bases are indexed by pairs of twin Cambrian trees, and it is also possible to describe both the product and coproduct in the algebras $\mathsf{BaxCamb}$ and $\mathsf{BaxCamb}^*$ in terms of simple combinatorial operations on Cambrian trees.

Finally, the ideas of this paper can be extended further to construct Hopf algebras on tuples of Cambrian trees (generalizing the pairs of twin Cambrian trees), and on Schröder-Cambrian trees (corresponding to all faces of all C. Hohlweg and C. Lange's associahedra [HL07]). We skip these constructions to keep this version short, but the interested reader can find details in [CP14].

# Part 1. The Cambrian Hopf Algebra

## 1.1. Cambrian trees

In this section, we recall the definition and properties of "Cambrian trees", generalizing standard binary search trees. They were introduced independently by K. Igusa and J. Ostroff in [IO13] as "mixed cobinary trees" in the context of cluster algebras and quiver representation theory and by C. Lange and V. Pilaud in [LP13] as "spines" (*i.e.* oriented and labeled dual trees) of triangulations of polygons to revisit the multiple realizations of the associahedron of C. Hohlweg and C. Lange [HL07]. Here, we use the term "Cambrian trees" to underline their connection with the type $A$ Cambrian lattices of N. Reading [Rea06].

1.1.1. **Cambrian trees.** Consider a directed tree T and a vertex $v$ of T. We call *children* (resp. *parents*) of $v$ the sources of the incoming arcs (resp. the targets of the outgoing arcs) at $v$ and *descendant* (resp. *ancestor*) *subtrees* of $v$ the subtrees attached to them. We focus on the following trees, which generalize standard binary search trees, see Figure 2 (left).

**Definition 1.** *A* Cambrian tree *is a directed tree* T *on* $[n]$ *such that for each* $v \in [n]$,

*(i)* $v$ *has either one parent and two children (its descendant subtrees are called* left and right subtrees*) or one child and two parents (its ancestor subtrees are called* left and right subtrees*);*

*(ii) all vertices are smaller (resp. larger) than* $v$ *in the left (resp. right) subtree of* $v$.

*The* signature *of* T *is the* $n$*-tuple* $\varepsilon(\mathrm{T}) \in \pm^n$ *defined by* $\varepsilon(\mathrm{T})_v = -$ *if* $v$ *has two children and* $\varepsilon(\mathrm{T})_v = +$ *if* $v$ *has two parents. Denote by* $\mathrm{Camb}(\varepsilon)$ *the set of Cambrian trees with signature* $\varepsilon$*, by* $\mathrm{Camb}(n) = \bigsqcup_{\varepsilon \in \pm^n} \mathrm{Camb}(\varepsilon)$ *the set of all Cambrian trees on* $[n]$*, and by* $\mathrm{Camb} := \bigsqcup_{n \in \mathbb{N}} \mathrm{Camb}(n)$ *the set of all Cambrian trees.*

Throughout the paper, we represent Cambrian trees as follows (see Figure 2 (left)):

(i) each vertex $v \in [n]$ appears at abscissa $v$;

(ii) the edges are oriented form bottom to top;

(iii) negative vertices (with one parent and two children) are represented by $\ominus$, while positive vertices (with one child and two parents) are represented by $\oplus$;

(iv) we sometimes draw a vertical red wall below the negative vertices and above the positive vertices to mark the separation between the left and right subtrees of each vertex.

**Remark 2** (Spines of triangulations)**.** Cambrian trees can be seen as spines (*i.e.* oriented and labeled dual trees) of triangulations of labeled polygons. More precisely, consider an $(n+2)$-gon $\mathrm{P}^\varepsilon$ with vertices labeled by $0, \dots, n+1$ from left to right, and where vertex $i$ is located above the diagonal $[0, n+1]$ if $\varepsilon_i = +$ and below it if $\varepsilon_i = -$. We associate with a triangulation $\sigma$ of $\mathrm{P}^\varepsilon$ its dual tree having a node labeled by $j$ for each triangle $ijk$ of $\sigma$ where $i < j < k$, and an edge between any two adjacent triangles oriented from the triangle below to the triangle above their common diagonal. See Figure 2 and refer to [LP13] for details. Throughout the paper, we denote by $\mathrm{T}^*$ the triangulation of $\mathrm{P}^\varepsilon$ dual to the $\varepsilon$-Cambrian tree T, and we use this interpretation to provide the reader with some geometric intuition of definitions and results of this paper. The first consequence of this geometric interpretation concerns the enumeration of Cambrian trees.

**Proposition 3** ([LP13, IO13])**.** *For any signature* $\varepsilon \in \pm^n$*, the number of* $\varepsilon$*-Cambrian trees is the Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$*. Therefore,* $|\mathrm{Camb}(n)| = 2^n C_n$*. See* [OEI10, A151374].

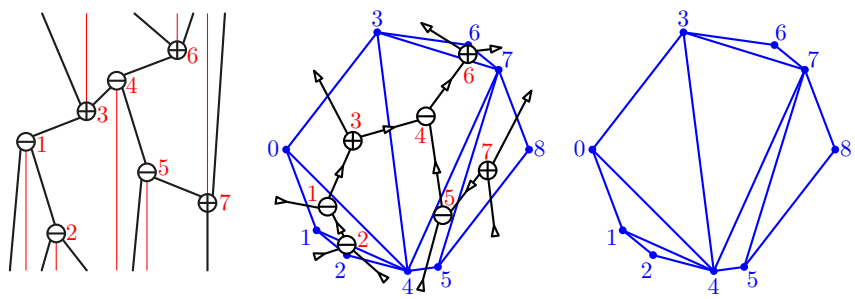



FIGURE 2. Cambrian trees (left) and triangulations (right) are dual to each other (middle).

1.1.2. **Cambrian correspondence.** We represent graphically a permutation $\tau \in \mathfrak{S}_n$ by the $(n \times n)$-table, with rows labeled by positions from bottom to top and columns labeled by values from left to right, and with a dot in row $i$ and column $\tau(i)$ for all $i \in [n]$. (This unusual choice of orientation is necessary to fit later with the existing constructions of [LR98, HNT05].)

A *signed permutation* is a permutation table where each dot receives a $+$ or $-$ sign, see Figure 3 (top left). We could also think of a permutation where the positions or the values receive a sign, but it will be useful later to switch the signature from positions to values. The *p-signature* (resp. *v-signature*) of a signed permutation $\tau$ is the sequence $\varepsilon_p(\tau)$ (resp. $\varepsilon_v(\tau)$) of signs of $\tau$ ordered by positions from bottom to top (resp. by values from left to right). For a signature $\varepsilon \in \pm^n$, we denote by $\mathfrak{S}_\varepsilon$ (resp. by $\mathfrak{S}^\varepsilon$) the set of signed permutations $\tau$ with p-signature $\varepsilon_p(\tau) = \varepsilon$ (resp. with v-signature $\varepsilon_v(\tau) = \varepsilon$). Finally, we denote the set of all signed permutations by

$$\mathfrak{S}_\pm := \bigsqcup_{\substack{n \in \mathbb{N} \\ \varepsilon \in \pm^n}} \mathfrak{S}_\varepsilon = \bigsqcup_{\substack{n \in \mathbb{N} \\ \varepsilon \in \pm^n}} \mathfrak{S}^\varepsilon.$$

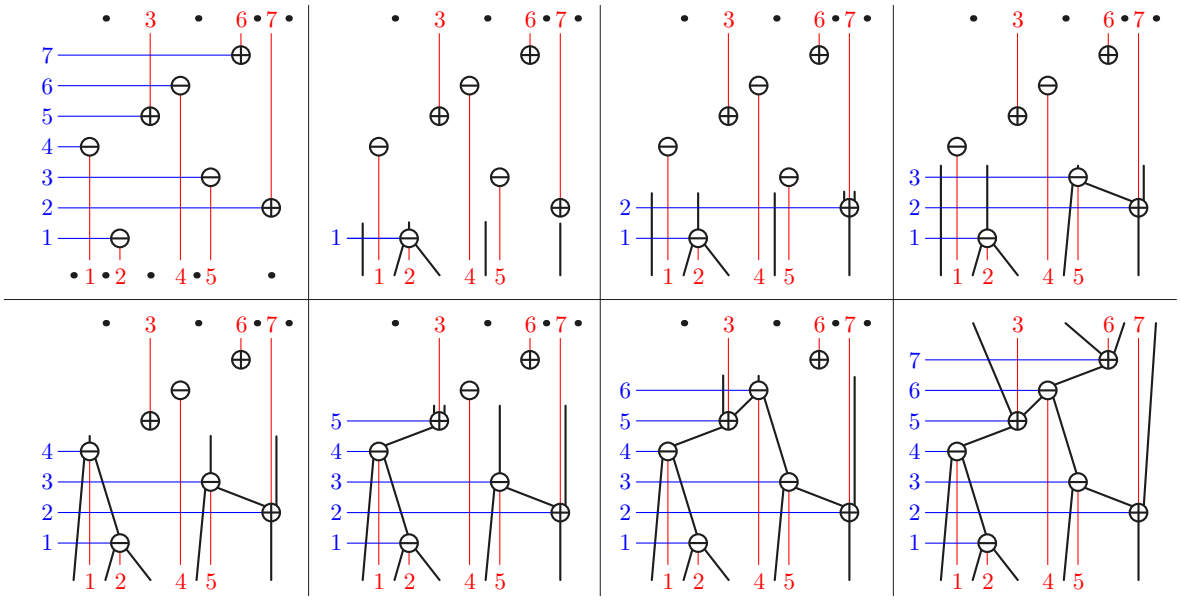


FIGURE 3. The insertion algorithm on the signed permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$.

In concrete examples, we underline negative positions/values while we overline positive positions/values: for example, we write $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$ for the signed permutation represented on the top left corner of Figure 3, where $\tau = [2, 7, 5, 1, 3, 4, 6]$, $\varepsilon_p = -+--+-+$ and $\varepsilon_v = --+--++$.

Following [LP13], we now present an algorithm to construct an $\varepsilon$-Cambrian tree $\mathbf{ct}(\tau)$ from a signed permutation $\tau \in \mathfrak{S}^\varepsilon$. Figure 3 illustrates this algorithm on the permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$. As a preprocessing, we represent the table of $\tau$ (with signed dots in positions $(\tau(i), i)$ for $i \in [n]$) and draw a vertical wall below the negative vertices and above the positive vertices. We then sweep the table from bottom to top (thus reading the permutation $\tau$ from left to right) as follows. The procedure starts with an incoming strand between any two consecutive negative values. A negative dot $\ominus$ connects the two strands immediately to its left and immediately to its right to form a unique outgoing strand. A positive dot $\oplus$ separates the only visible strand (not hidden by a wall) into two outgoing strands. The procedure finishes with an outgoing strand between any two consecutive positive values. See Figure 3. We denote by $\mathbf{ct}(\tau)$ the resulting oriented graph.

**Proposition 4** ([LP13])**.** *The map* $\mathbf{ct}$ *is a surjection from the signed permutations* $\mathfrak{S}^\varepsilon$ *to the Cambrian trees* $\mathsf{Camb}(\varepsilon)$*. The fiber* $\mathbf{ct}^{-1}(\mathrm{T}) := \{\tau \in \mathfrak{S}^\varepsilon \mid \mathbf{ct}(\tau) = \mathrm{T}\}$ *of a Cambrian tree* $\mathrm{T} \in \mathsf{Camb}(\varepsilon)$ *is its set* $\mathcal{L}(\mathrm{T})$ *of linear extensions.*

**Remark 5** (Cambrian correspondence on triangulations)**.** The map $\mathbf{ct}$ was previously described on the triangulations of the polygon $\mathrm{P}^\varepsilon$ in [Rea06] (see also the references therein). Namely, the triangulation $\mathbf{ct}(\tau)^*$ is the union of the paths $\pi_0, \dots, \pi_n$ where $\pi_i$ is the path between vertices 0 and $n+1$ of $\mathrm{P}^\varepsilon$ passing through the vertices in the symmetric difference $\varepsilon^{-1}(-) \triangle \tau([i])$.

1.1.3. **Cambrian congruence.** Following the sylvester congruence in [HNT05], we now characterize by a congruence relation the signed permutations $\tau \in \mathfrak{S}^\varepsilon$ which have the same image $\mathbf{ct}(\tau)$. This Cambrian congruence goes back to the original definition of N. Reading [Rea06].

**Definition 6** ([Rea06])**.** *For a signature* $\varepsilon \in \pm^n$*, the* $\varepsilon$*-Cambrian congruence is the equivalence relation on* $\mathfrak{S}^\varepsilon$ *defined as the transitive closure of the rewriting rules*

$$UacVbW \equiv_\varepsilon UcaVbW \quad \text{if } a < b < c \text{ and } \varepsilon_b = -,$$
$$UbVacW \equiv_\varepsilon UbVcaW \quad \text{if } a < b < c \text{ and } \varepsilon_b = +,$$

*where* $a, b, c$ *are elements of* $[n]$ *while* $U, V, W$ *are words on* $[n]$*. The Cambrian congruence is the equivalence relation on all signed permutations* $\mathfrak{S}_\pm$ *given by all* $\varepsilon$*-Cambrian congruences:*

$$\text{for } \tau \in \mathfrak{S}^\varepsilon, \tau \in \mathfrak{S}^{\varepsilon'} \qquad \tau \equiv \tau' \iff \varepsilon = \varepsilon' \text{ and } \tau \equiv_\varepsilon \tau'.$$

We now focus on the equivalence classes of the Cambrian congruence.

**Proposition 7.** *The $\varepsilon$-Cambrian congruence classes are precisely the fibers of the map* $\mathbf{ct}$*:*

$$\text{for } \tau, \tau' \in \mathfrak{S}^\varepsilon \qquad \tau \equiv_\varepsilon \tau' \iff \mathbf{ct}(\tau) = \mathbf{ct}(\tau').$$

*Proof.* It boils down to observe that two consecutive vertices $a, c$ in a linear extension $\tau$ of a $\varepsilon$-Cambrian tree T can be switched while preserving a linear extension $\tau'$ of T precisely when they belong to distinct subtrees of a vertex $b$ of T. It follows that the vertices $a, c$ lie on either sides of $b$ so that we have $a < b < c$. If $\varepsilon_b = -$, then $a, c$ appear before $b$ and $\tau = UacVbW$ can be switched to $\tau' = UcaVbW$, while if $\varepsilon_b = +$, then $a, c$ appear after $b$ and $\tau = UbVacW$ can be switched to $\tau' = UbVcaW$. □

Remember that the *(right) weak order* on $\mathfrak{S}^\varepsilon$ is defined as the inclusion order of (right) inversions, where a *(right) inversion* of $\tau \in \mathfrak{S}^\varepsilon$ is a pair of values $i < j$ such that $\tau^{-1}(i) > \tau^{-1}(j)$. In this definition, the signs on $\tau$ do not matter (they are seen as decorations, not as actual signs that would change the order on the values). In this paper, we always work with the right weak order, that we simply call weak order for brevity. The following statement is due to N. Reading [Rea06].

**Proposition 8** ([Rea06])**.** *All $\varepsilon$-Cambrian classes are intervals of the weak order on $\mathfrak{S}^\varepsilon$, whose minimal (resp. maximal) elements avoid the patterns $\bar{b}$-ca and ca-$\underline{b}$ (resp. $\bar{b}$-ac and ac-$\underline{b}$).*

1.1.4. **Rotations and Cambrian lattices.** We now present rotations in Cambrian trees, a local operation which transforms a $\varepsilon$-Cambrian tree into another $\varepsilon$-Cambrian tree where a single oriented cut differs (see Proposition 10).

**Definition 9.** *Let $i \to j$ be an edge in a Cambrian tree* T*, with $i < j$. Let $L$ denote the left subtree of $i$ and $B$ denote the remaining incoming subtree of $i$, and similarly, let $R$ denote the right subtree of $j$ and $A$ denote the remaining outgoing subtree of $j$. Let* T′ *be the oriented tree obtained from* T *just reversing the orientation of $i \to j$ and attaching the subtrees $L$ and $A$ to $i$ and the subtrees $B$ and $R$ to $j$. The transformation from* T *to* T′ *is called* rotation *of the edge $i \to j$. See Figure 4.*

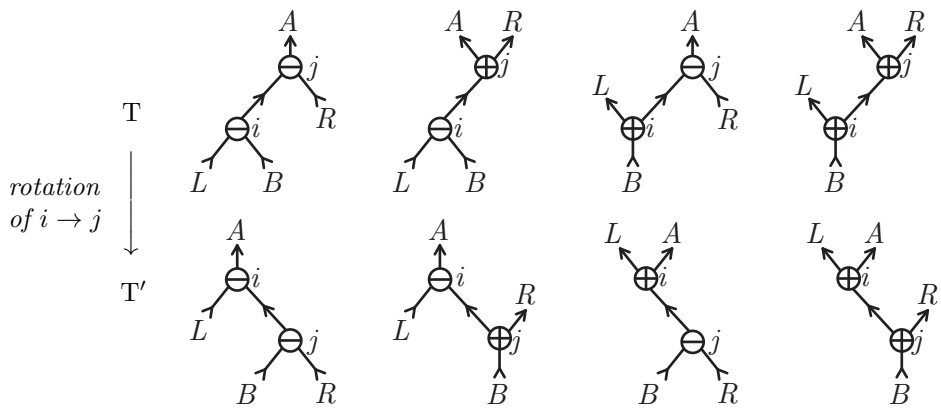



FIGURE 4. Rotations in Cambrian trees: the tree T (top) is transformed into the tree T′ (bottom) by rotation of the edge $i \to j$. The four cases correspond to the possible signs of $i$ and $j$.

The following proposition states that rotations are compatible with Cambrian trees and their edge cuts. An *edge cut* in a Cambrian tree T is the ordered partition $(X \parallel Y)$ of the vertices of T into the set $X$ of vertices in the source set and the set $Y$ of vertices in the target set of an oriented edge of T.

**Proposition 10** ([LP13])**.** *The result* T′ *of the rotation of an edge $i \to j$ in a $\varepsilon$-Cambrian tree* T *is a $\varepsilon$-Cambrian tree. Moreover,* T′ *is the unique $\varepsilon$-Cambrian tree with the same edge cuts as* T*, except the cut defined by the edge $i \to j$.*

**Remark 11** (Rotations and flips)**.** Rotating an edge $e$ in a $\varepsilon$-Cambrian tree T corresponds to flipping the dual diagonal $e^*$ of the dual triangulation $\mathrm{T}^*$ of the polygon $\mathrm{P}^\varepsilon$. See [LP13, Lemma 13].

Define the *increasing rotation graph* on $\mathrm{Camb}(\varepsilon)$ to be the graph whose vertices are the $\varepsilon$-Cambrian trees and whose arcs are increasing rotations $\mathrm{T} \to \mathrm{T}'$, *i.e.* where the edge $i \to j$ in T is reversed to the edge $i \leftarrow j$ in $\mathrm{T}'$ for $i < j$. See Figure 5 for an illustration. The following statement, adapted from N. Reading's work [Rea06], asserts that this graph is acyclic, that its transitive closure defines a lattice, and that this lattice is closely related to the weak order.

**Proposition 12** ([Rea06])**.** *The transitive closure of the increasing rotation graph on* $\mathrm{Camb}(\varepsilon)$ *is a lattice, called* $\varepsilon$-Cambrian lattice*. The map* $\mathbf{ct} : \mathfrak{S}^\varepsilon \to \mathrm{Camb}(\varepsilon)$ *defines a lattice homomorphism from the weak order on* $\mathfrak{S}^\varepsilon$ *to the* $\varepsilon$*-Cambrian lattice on* $\mathrm{Camb}(\varepsilon)$*.*

**Example 13.** When $\varepsilon = (-)^n$, the Cambrian lattice is the classical Tamari lattice [MHPS12]. It can be defined equivalently by left-to-right rotations in planar binary trees, by slope increasing flips in triangulations of $\mathrm{P}^{(-)^n}$, or as the quotient of the weak order by the sylvester congruence.

1.1.5. **Canopy.** The canopy of a binary tree was already used by J.-L. Loday in [LR98, Lod04] but the name was coined by X. Viennot [Vie07]. It was then extended to Cambrian trees (or spines) in [LP13] to define a surjection from the associahedron $\mathsf{Asso}(\varepsilon)$ to the parallelepiped $\mathsf{Para}(n)$ generated by the simple roots. The main observation is that the vertices $i$ and $i+1$ are always comparable in a Cambrian tree (otherwise, they would be in distinct subtrees of a vertex $j$ which should then lie between $i$ and $i+1$).

**Definition 14.** *The* canopy *of a Cambrian tree* T *is the sequence* $\mathbf{can}(\mathrm{T}) \in \pm^{n-1}$ *defined by* $\mathbf{can}(\mathrm{T})_i = -$ *if* $i$ *is above* $i+1$ *in* T *and* $\mathbf{can}(\mathrm{T})_i = +$ *if* $i$ *is below* $i+1$ *in* T*.*

For example, the canopy of the Cambrian tree of Figure 2 (left) is $-++-+-$. The canopy of T behaves nicely with the linear extensions of T and with the Cambrian lattice. To state this, we define for a permutation $\tau \in \mathfrak{S}^\varepsilon$ the sequence $\mathbf{rec}(\tau) \in \pm^{n-1}$, where $\mathbf{rec}(\mathrm{T})_i = -$ if $\tau^{-1}(i) > \tau^{-1}(i+1)$ and $\mathbf{rec}(\mathrm{T})_i = +$ otherwise. In other words, $\mathbf{rec}(\tau)$ records the *recoils* of the permutation $\tau$, *i.e.* the *descents* of the inverse permutation of $\tau$.

**Proposition 15.** *The maps* $\mathbf{ct}, \mathbf{can}$*, and* $\mathbf{rec}$ *define the following commutative diagram of lattice homomorphisms:*

$$\begin{tikzcd} \mathfrak{S}^\varepsilon \arrow[rr, two heads, "\mathbf{rec}"] \arrow[dr, two heads, "\mathbf{ct}"'] & & \pm^{n-1} \\ & \mathrm{Camb}(\varepsilon) \arrow[ur, two heads, "\mathbf{can}"'] & \end{tikzcd}$$

1.1.6. **Geometric realizations.** We close this section with geometric interpretations of the Cambrian trees, Cambrian classes, Cambrian correspondence, and Cambrian lattices. We denote by $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$ and by $\mathbb{H}$ the hyperplane of $\mathbb{R}^n$ orthogonal to $\sum e_i$. Define the *incidence cone* $\mathrm{C}(\mathrm{T})$ and the *braid cone* $\mathrm{C}^\diamond(\mathrm{T})$ of a directed tree T as

$$\mathrm{C}(\mathrm{T}) := \mathrm{cone}\,\{e_i - e_j \mid \text{for all } i \to j \text{ in } \mathrm{T}\} \quad \text{and} \quad \mathrm{C}^\diamond(\mathrm{T}) := \{\mathbf{x} \in \mathbb{H} \mid x_i \le x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}.$$

These two cones lie in the space $\mathbb{H}$ and are polar to each other. For a permutation $\tau \in \mathfrak{S}_n$, we denote by $\mathrm{C}(\tau)$ and $\mathrm{C}^\diamond(\tau)$ the incidence and braid cone of the chain $\tau(1) \to \cdots \to \tau(n)$. Finally, for a sign vector $\chi \in \pm^{n-1}$, we denote by $\mathrm{C}(\chi)$ and $\mathrm{C}^\diamond(\chi)$ the incidence and braid cone of the oriented path $1 - \cdots - n$, where $i \to i+1$ if $\chi_i = +$ and $i \leftarrow i+1$ if $\chi_i = -$.

These cones (together with all their faces) form complete simplicial fans in $\mathbb{H}$:

(i) the cones $\mathrm{C}^\diamond(\tau)$, for all permutations $\tau \in \mathfrak{S}_n$, form the *braid fan*, which is the normal fan of the *permutahedron* $\mathsf{Perm}(n) := \mathrm{conv}\left\{\sum_{i\in[n]} \tau(i) e_i \mid \tau \in \mathfrak{S}_n\right\}$;

(ii) the cones $\mathrm{C}^\diamond(\mathrm{T})$, for all $\varepsilon$-Cambrian trees T, form the *$\varepsilon$-Cambrian fan*, which is the normal fan of the *$\varepsilon$-associahedron* $\mathsf{Asso}(\varepsilon)$ of C. Hohlweg and C. Lange [HL07] (see also [LP13]);

(iii) the cones $\mathrm{C}^\diamond(\chi)$, for all sign vectors $\chi \in \pm^{n-1}$, form the *boolean fan*, which is the normal fan of the *parallelepiped* $\mathsf{Para}(n) := \left\{\mathbf{x} \in \mathbb{H} \mid i(2n+1-i) \le 2\sum_{j\le i} x_j \le i(i+1) \text{ for all } i \in [n]\right\}$.

In fact, $\mathsf{Asso}(\varepsilon)$ is obtained by deleting certain inequalities in the facet description of $\mathsf{Perm}(n)$, and similarly, $\mathsf{Para}(n)$ is obtained by deleting facets of $\mathsf{Asso}(\varepsilon)$. In particular, we have the geometric inclusions $\mathsf{Perm}(n) \subset \mathsf{Asso}(\varepsilon) \subset \mathsf{Para}(n)$. See Figure 6 for 3-dimensional examples.

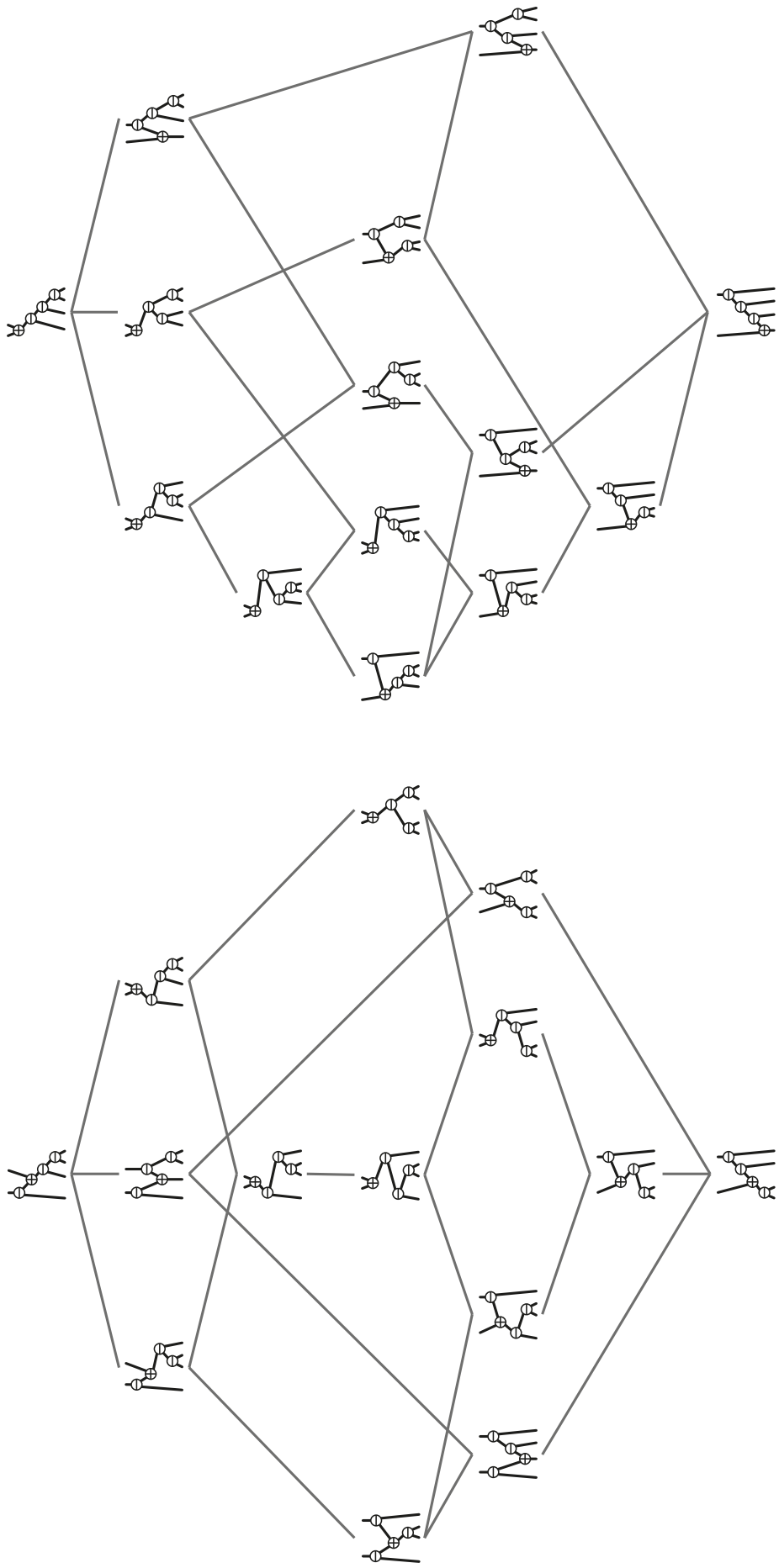


FIGURE 5. The $\varepsilon$-Cambrian lattices for $\varepsilon = -+--$ (left) and $\varepsilon = +---$ (right).

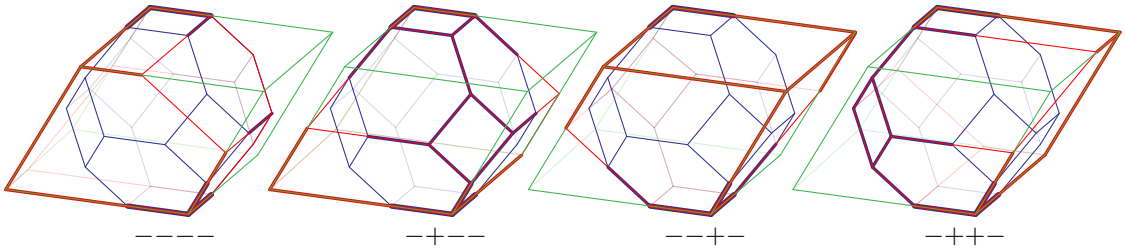



FIGURE 6. The polytope inclusion $\mathsf{Perm}(4) \subset \mathsf{Asso}(\varepsilon) \subset \mathsf{Para}(4)$ for different signatures $\varepsilon \in \pm^4$. The permutahedron $\mathsf{Perm}(4)$ is represented in red, the associahedron $\mathsf{Asso}(\varepsilon)$ in blue, and the parallelepiped $\mathsf{Para}(4)$ in green.

The incidence and braid cones also characterize the maps **ct**, **can**, and **rec** as follows

$$\begin{aligned} \mathrm{T} = \mathbf{ct}(\tau) &\iff \mathrm{C}(\mathrm{T}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\mathrm{T}) \supseteq \mathrm{C}^\diamond(\tau), \\ \chi = \mathbf{can}(\mathrm{T}) &\iff \mathrm{C}(\chi) \subseteq \mathrm{C}(\mathrm{T}) \iff \mathrm{C}^\diamond(\chi) \supseteq \mathrm{C}^\diamond(\mathrm{T}), \\ \chi = \mathbf{rec}(\tau) &\iff \mathrm{C}(\chi) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\chi) \supseteq \mathrm{C}^\diamond(\tau). \end{aligned}$$

In particular, Cambrian classes are formed by all permutations whose braid cone belong to the same Cambrian cone. Finally, the 1-skeleta of the permutahedron $\mathsf{Perm}(n)$, associahedron $\mathsf{Asso}(\varepsilon)$ and parallelepiped $\mathsf{Para}(n)$, oriented in the direction $(n, \dots, 1) - (1, \dots, n) = \sum_{i \in [n]} (n + 1 - 2i)\, e_i$ are the Hasse diagrams of the weak order, the Cambrian lattice and the boolean lattice respectively. These geometric properties originally motivated the definition of Cambrian trees in [LP13].

## 1.2. Cambrian Hopf Algebra

In this section, we introduce the Cambrian Hopf algebra $\mathsf{Camb}$ as a subalgebra of the Hopf algebra $\mathsf{FQSym}_\pm$ on signed permutations, and the dual Cambrian algebra $\mathsf{Camb}^*$ as a quotient algebra of the dual Hopf algebra $\mathsf{FQSym}_\pm^*$. We describe both the product and coproduct in these algebras in terms of combinatorial operations on Cambrian trees. These results extend the approach of F. Hivert, J.-C. Novelli and J.-Y. Thibon [HNT05] to construct the algebra of J.-L. Loday and M. Ronco on binary trees [LR98] as a subalgebra of the algebra of C. Malvenuto and C. Reutenauer on permutations [MR95].

We immediately mention that a different generalization was studied by N. Reading in [Rea05]. His idea was to construct a subalgebra of C. Malvenuto and C. Reutenauer's algebra $\mathsf{FQSym}$ using equivalent classes of a congruence relation defined as the union $\bigcup_{n \in \mathbb{N}} \equiv_{\varepsilon_n}$ of $\varepsilon_n$-Cambrian relation for one fixed signature $\varepsilon_n \in \pm^n$ for each $n \in \mathbb{N}$. In order to obtain a valid Hopf algebra, the choice of $(\varepsilon_n)_{n \in \mathbb{N}}$ has to satisfy certain compatibility relations: N. Reading characterizes the "translational" (resp. "insertional") families $\equiv_n$ of lattice congruences on $\mathfrak{S}_n$ for which the sums over the elements of the congruence classes of $(\equiv_n)_{n \in \mathbb{N}}$ form the basis of a subalgebra (resp. subcoalgebra) of $\mathsf{FQSym}$. These conditions make the choice of $(\varepsilon_n)_{n \in \mathbb{N}}$ rather constrained. In contrast, by constructing a subalgebra of $\mathsf{FQSym}_\pm$ rather than $\mathsf{FQSym}$, we consider simultaneously all Cambrian relations for all signatures.

### 1.2.1. Signed shuffle and convolution products.

For $n, n' \in \mathbb{N}$, let

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n')\}$$

denote the set of permutations of $\mathfrak{S}_{n+n'}$ with at most one descent, at position $n$. The *shifted concatenation* $\tau\bar{\tau}'$, the *shifted shuffle product* $\tau \,\underline{\sqcup\!\sqcup}\, \tau'$, and the *convolution product* $\tau \star \tau'$ of two (unsigned) permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ are classically defined by

$$\tau\bar{\tau}' := [\tau(1), \dots, \tau(n), \tau'(1) + n, \dots, \tau'(n') + n] \in \mathfrak{S}_{n+n'},$$
$$\tau \,\underline{\sqcup\!\sqcup}\, \tau' := \left\{(\tau\bar{\tau}') \circ \pi^{-1} \mid \pi \in \mathfrak{S}^{(n,n')}\right\} \qquad \text{and} \qquad \tau \star \tau' := \left\{\pi \circ (\tau\bar{\tau}') \mid \pi \in \mathfrak{S}^{(n,n')}\right\}.$$

As shown by J.-C. Novelli and J.-Y. Thibon in [NT10], these definitions extend to signed permutations as follows. The *signed shifted shuffle product* $\tau \,\underline{\sqcup\!\sqcup}\, \tau'$ is defined as the shifted product of the permutations where signs travel with their values, while the *signed convolution product* $\tau \star \tau'$

is defined as the convolution product of the permutations where signs stay at their positions. For example,

$$\bar{1}\underline{2} \,\bar{\sqcup\!\sqcup}\, \underline{23\bar{1}} = \{\bar{1}\underline{245\bar{3}}, \bar{1}\underline{425\bar{3}}, \bar{1}\underline{452\bar{3}}, \bar{1}\underline{45\bar{3}2}, \underline{4\bar{1}25\bar{3}}, \underline{4\bar{1}52\bar{3}}, \underline{4\bar{1}5\bar{3}2}, \underline{45\bar{1}2\bar{3}}, \underline{45\bar{1}\bar{3}2}, \underline{45\bar{3}\bar{1}2}\},$$

$$\bar{1}\underline{2} \star \underline{23\bar{1}} = \{\bar{1}\underline{245\bar{3}}, \bar{1}\underline{345\bar{2}}, \bar{1}\underline{435\bar{2}}, \bar{1}\underline{534\bar{2}}, \bar{2}\underline{345\bar{1}}, \bar{2}\underline{435\bar{1}}, \bar{2}\underline{534\bar{1}}, \bar{3}\underline{425\bar{1}}, \bar{3}\underline{524\bar{1}}, \bar{4}\underline{523\bar{1}}\}.$$

1.2.2. **Subalgebra of $\mathsf{FQSym}_\pm$.** We denote by $\mathsf{FQSym}_\pm$ the Hopf algebra with basis $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}_\pm}$ and whose product and coproduct are defined by

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma\in\tau\,\bar{\sqcup\!\sqcup}\,\tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle\mathbb{F}_\sigma = \sum_{\sigma\in\tau\star\tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}.$$

It naturally extends to signed permutations the Hopf algebra $\mathsf{FQSym}$ on permutations defined by C. Malvenuto and C. Reutenauer [MR95].

We denote by $\mathsf{Camb}$ the vector subspace of $\mathsf{FQSym}_\pm$ generated by the elements

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau\in\mathfrak{S}_\pm \\ \mathbf{ct}(\tau)=\mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau\in\mathcal{L}(\mathrm{T})} \mathbb{F}_\tau,$$

for all Cambrian trees T (recall that $\{\tau \in \mathfrak{S}_\pm \mid \mathbf{ct}(\tau) = \mathrm{T}\} = \mathcal{L}(\mathrm{T})$ is the set of linear extensions of T). The following result is a straightforward adaptation of a similar result [HNT05]. We skip the proof, whose details can be found in [CP14].

**Theorem 16.** $\mathsf{Camb}$ *is a Hopf subalgebra of* $\mathsf{FQSym}_\pm$.

In the remaining of this section, we provide direct descriptions of the product and coproduct of $\mathbb{P}$-basis elements of $\mathsf{Camb}$ in terms of combinatorial operations on Cambrian trees.

PRODUCT The product in the Cambrian algebra can be described in terms of intervals in Cambrian lattices. Given two Cambrian trees $\mathrm{T}, \mathrm{T}'$, we denote by $\mathrm{T} \nearrow \bar{\mathrm{T}}'$ the tree obtained by grafting the rightmost outgoing leaf of T on the leftmost incoming leaf of $\mathrm{T}'$ and shifting all labels of $\mathrm{T}'$. See Figure 7 (left). Note that the resulting tree is $\varepsilon\varepsilon'$-Cambrian, where $\varepsilon\varepsilon'$ is the concatenation of the signatures $\varepsilon = \varepsilon(\mathrm{T})$ and $\varepsilon' = \varepsilon(\mathrm{T}')$. We define similarly $\mathrm{T} \nwarrow \bar{\mathrm{T}}'$.

**Proposition 17.** *For any Cambrian trees* $\mathrm{T}, \mathrm{T}'$*, the product* $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}$ *is given by*

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}},$$

*where* S *runs over the interval between* $\mathrm{T} \nearrow \bar{\mathrm{T}}'$ *and* $\mathrm{T} \nwarrow \bar{\mathrm{T}}'$ *in the* $\varepsilon(\mathrm{T})\varepsilon(\mathrm{T}')$*-Cambrian lattice.*

*Proof.* By Proposition 8, the linear extensions $\mathcal{L}(\mathrm{T})$ of any Cambrian tree T form an interval of the weak order. Moreover, the shuffle product of two intervals of the weak order is an interval of the weak order. Therefore, the product $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}$ is a sum of $\mathbb{P}_{\mathrm{S}}$ where S runs over an interval of the Cambrian lattice. It remains to characterize the minimal and maximal elements of this interval. Let $\mu_{\mathrm{T}}$ and $\omega_{\mathrm{T}}$ denote respectively the smallest and the greatest linear extension of T in weak order. The product $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}$ is the sum of $\mathbb{P}_{\mathrm{S}}$ over the interval $[\mu_{\mathrm{T}}, \omega_{\mathrm{T}}] \,\bar{\sqcup\!\sqcup}\, [\mu_{\mathrm{T}'}, \omega_{\mathrm{T}'}] = [\mu_{\mathrm{T}}\bar{\mu}_{\mathrm{T}'}, \bar{\omega}_{\mathrm{T}'}\omega_{\mathrm{T}}]$,

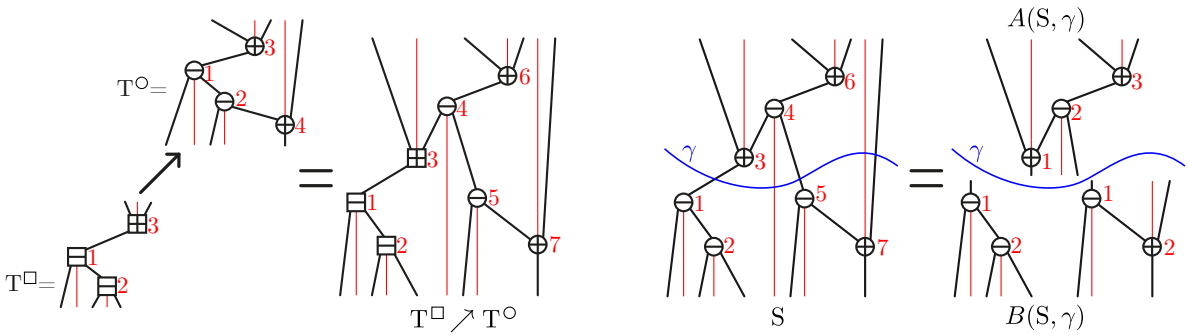



FIGURE 7. Grafting Cambrian trees (left) and cutting a Cambrian tree (right).

where $^{-}$ denotes as usual the shifting operator on permutations. The result thus follows from the fact that

$$\mathbf{ct}(\mu_{\mathrm{T}}\bar{\mu}_{\mathrm{T}'}) = \mathrm{T} \nearrow \bar{\mathrm{T}}' \qquad \text{and} \qquad \mathbf{ct}(\bar{\omega}_{\mathrm{T}'}\omega_{\mathrm{T}}) = \mathrm{T} \nwarrow \bar{\mathrm{T}}'. \qquad \square$$

Coproduct The coproduct in the Cambrian algebra can also be described in combinatorial terms. Define a *cut* of a Cambrian tree S to be a set $\gamma$ of edges such that any geodesic vertical path in S from a down leaf to an up leaf contains precisely one edge of $\gamma$. Such a cut separates the tree S into two forests, one above $\gamma$ and one below $\gamma$, denoted $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$, respectively. See Figure 7 (right).

**Proposition 18.** *For any Cambrian tree* S*, the coproduct* $\triangle\mathbb{P}_{\mathrm{S}}$ *is given by*

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \Big( \prod_{\mathrm{T}\in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \Big) \otimes \Big( \prod_{\mathrm{T}'\in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \Big),$$

*where* $\gamma$ *runs over all cuts of* S*.*

*Proof.* Let $\sigma$ be a linear extension of S and $\tau, \tau' \in \mathfrak{S}_{\pm}$ such that $\sigma \in \tau \star \tau'$. The tables of $\tau$ and $\tau'$ respectively appear in the bottom and top rows of the table of $\sigma$. We can therefore associate a cut of S to each element which appears in the coproduct $\triangle\mathbb{P}_{\mathrm{S}}$. Reciprocally, given a cut $\gamma$ of S, we are interested in the linear extensions of S where all indices below $\gamma$ appear before all indices above $\gamma$. These linear extensions are precisely the permutations formed by a linear extension of $B(\mathrm{S},\gamma)$ followed by a linear extension of $A(\mathrm{S},\gamma)$. But the linear extensions of a forest are obtained by shuffling the linear extensions of its connected components. The result immediately follows since the product $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'}$ precisely involves the shuffle of the linear extensions of T with the linear extensions of T′. $\square$

Matriochka algebras Observe that the commutative diagram of Proposition 15 ensures that the Cambrian algebra is sandwiched between the signed permutation algebra and the recoils algebra, defined as the Hopf subalgebra of $\mathsf{FQSym}_{\pm}$ generated by the elements

$$\mathbb{X}_{\chi} := \sum_{\substack{\tau\in\mathfrak{S}_{\pm} \\ \mathbf{rec}(\tau)=\chi}} \mathbb{F}_{\tau} = \sum_{\substack{\mathrm{T}\in\mathrm{Camb} \\ \mathbf{can}(\mathrm{T})=\chi}} \mathbb{P}_{\mathrm{T}},$$

for all sign vectors $\chi \in \pm^{n-1}$.

Multiplicative bases For a Cambrian tree T, define $\mathbb{E}^{\mathrm{T}} := \sum_{\mathrm{T}\le\mathrm{T}'} \mathbb{P}_{\mathrm{T}'}$ and $\mathbb{H}^{\mathrm{T}} := \sum_{\mathrm{T}'\le\mathrm{T}} \mathbb{P}_{\mathrm{T}'}$. The sets $(\mathbb{E}^{\mathrm{T}})_{\mathrm{T}\in\mathrm{Camb}}$ and $(\mathbb{H}^{\mathrm{T}})_{\mathrm{T}\in\mathrm{Camb}}$ are then multiplicative bases of $\mathsf{Camb}$:

$$\mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'} = \mathbb{E}^{\mathrm{T}\nearrow\bar{\mathrm{T}}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} \cdot \mathbb{H}^{\mathrm{T}'} = \mathbb{H}^{\mathrm{T}\nwarrow\bar{\mathrm{T}}'}.$$

A Cambrian tree S is called *$\mathbb{E}$-indecomposable* if $\mathbb{E}^{\mathrm{S}} = \mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'}$ implies $\mathrm{T} = \mathrm{S}$ or $\mathrm{T}' = \mathrm{S}$. As for J.-L. Loday and M. Ronco's algebra, it turns out that

- $\mathsf{Camb}$ is freely generated by the elements $\mathbb{E}^{\mathrm{S}}$ for all $\mathbb{E}$-indecomposable Cambrian trees S,
- for any $\varepsilon \in \pm^n$, there are $C_{n-1}$ $\mathbb{E}$-indecomposable $\varepsilon$-Cambrian trees,
- the set of $\mathbb{E}$-indecomposable elements is a principal upper ideal of the $\varepsilon$-Cambrian lattice.

We omit here the proofs of these results since they are much more involved than in J.-L. Loday and M. Ronco's algebra. Details can be found in [CP14].

1.2.3. **Quotient algebra of $\mathsf{FQSym}^*_{\pm}$.** We switch to the dual Hopf algebra $\mathsf{FQSym}^*_{\pm}$ with basis $(\mathbb{G}_{\tau})_{\tau\in\mathfrak{S}_{\pm}}$ and whose product and coproduct are defined by

$$\mathbb{G}_{\tau} \cdot \mathbb{G}_{\tau'} = \sum_{\sigma\in\tau\star\tau'} \mathbb{G}_{\sigma} \qquad \text{and} \qquad \triangle\mathbb{G}_{\sigma} = \sum_{\sigma\in\tau\,\underline{\sqcup}\,\tau'} \mathbb{G}_{\tau} \otimes \mathbb{G}_{\tau'}.$$

The following statement is automatic from Theorem 16.

**Theorem 19.** *The graded dual* $\mathsf{Camb}^*$ *of the Cambrian algebra is isomorphic to the image of* $\mathsf{FQSym}^*_\pm$ *under the canonical projection*

$$\pi : \mathbb{C}\langle A\rangle \longrightarrow \mathbb{C}\langle A\rangle/\equiv,$$

*where* $\equiv$ *denotes the Cambrian congruence. The dual basis* $\mathbb{Q}_{\mathrm{T}}$ *of* $\mathbb{P}_{\mathrm{T}}$ *is expressed as* $\mathbb{Q}_{\mathrm{T}} = \pi(\mathbb{G}_\tau)$*, where* $\tau$ *is any linear extension of* $\mathrm{T}$*.*

Similarly as in the previous section, we can describe combinatorially the product and coproduct of $\mathbb{Q}$-basis elements of $\mathsf{Camb}^*$ in terms of operations on Cambrian trees.

PRODUCT Call *gaps* the $n+1$ positions between two consecutive integers of $[n]$, including the position before 1 and the position after $n$. A gap $\gamma$ defines a *geodesic vertical path* $\lambda(\mathrm{T},\gamma)$ in a Cambrian tree $\mathrm{T}$ from the bottom leaf which lies in the same interval of consecutive negative labels as $\gamma$ to the top leaf which lies in the same interval of consecutive positive labels as $\gamma$. See Figure 9. A multiset $\Gamma$ of gaps therefore defines a *lamination* $\lambda(\mathrm{T},\Gamma)$ of $\mathrm{T}$, *i.e.* a multiset of pairwise non-crossing geodesic vertical paths in $\mathrm{T}$ from down leaves to up leaves. When cut along the paths of a lamination, the Cambrian tree $\mathrm{T}$ splits into a forest.

Consider two Cambrian trees $\mathrm{T}$ and $\mathrm{T}'$ on $[n]$ and $[n']$ respectively. For any shuffle $s$ of their signatures $\varepsilon$ and $\varepsilon'$, consider the multiset $\Gamma$ of gaps of $[n]$ given by the positions of the negative signs of $\varepsilon'$ in $s$ and the multiset $\Gamma'$ of gaps of $[n']$ given by the positions of the positive signs of $\varepsilon$ in $s$. We denote by $\mathrm{T}\,{}_s\!\backslash\mathrm{T}'$ the Cambrian tree obtained by connecting the up leaves of the forest defined by the lamination $\lambda(\mathrm{T},\Gamma)$ to the down leaves of the forest defined by the lamination $\lambda(\mathrm{T}',\Gamma')$.

**Example 20.** Consider the Cambrian trees $\mathrm{T}^{\bigcirc}$ and $\mathrm{T}^{\square}$ of Figure 8. To distinguish signs in $\mathrm{T}^{\bigcirc}$ and $\mathrm{T}^{\square}$, we circle the signs in $\varepsilon(\mathrm{T}^{\bigcirc}) = \ominus\ominus\oplus$ and square the signs in $\varepsilon(\mathrm{T}^{\square}) = \boxminus\boxminus\boxplus\boxminus$. Consider now an arbitrary shuffle $s = \boxminus\ominus\ominus\boxminus\boxplus\oplus\boxminus$ of these two signatures. The resulting laminations of $\mathrm{T}^{\bigcirc}$ and $\mathrm{T}^{\square}$, as well as the Cambrian tree $\mathrm{T}^{\bigcirc}\,{}_s\!\backslash\mathrm{T}^{\square}$ are represented in Figure 8.

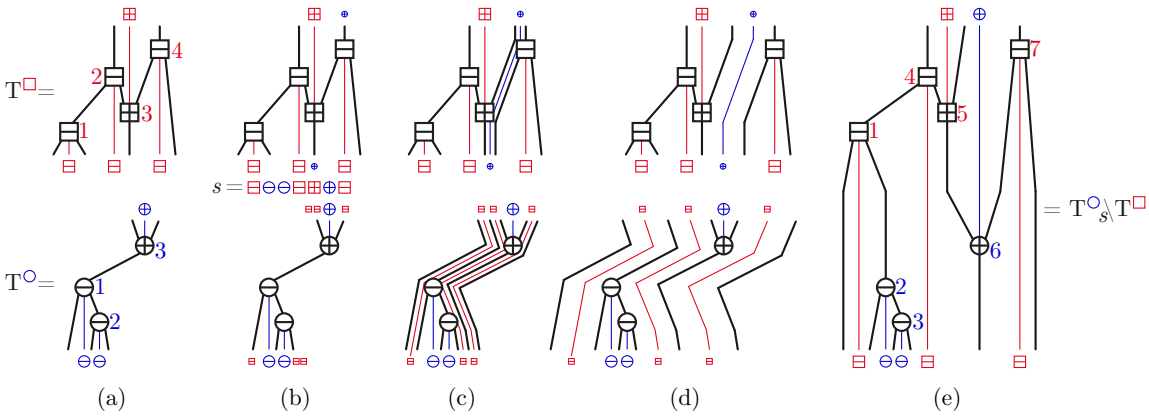



FIGURE 8. (a) The two Cambrian trees $\mathrm{T}^{\bigcirc}$ and $\mathrm{T}^{\square}$. (b) Given the shuffle $s = \boxminus\ominus\ominus\boxminus\boxplus\oplus\boxminus$, the positions of the $\boxminus$ are reported in $\mathrm{T}^{\bigcirc}$ and the positions of the $\oplus$ are reported in $\mathrm{T}^{\square}$. (c) The corresponding laminations. (d) The trees are split according to the laminations. (e) The resulting Cambrian tree $\mathrm{T}^{\bigcirc}\,{}_s\!\backslash\mathrm{T}^{\square}$.

**Proposition 21.** *For any Cambrian trees* $\mathrm{T},\mathrm{T}'$*, the product* $\mathbb{Q}_{\mathrm{T}}\cdot\mathbb{Q}_{\mathrm{T}'}$ *is given by*

$$\mathbb{Q}_{\mathrm{T}}\cdot\mathbb{Q}_{\mathrm{T}'} = \sum_s \mathbb{Q}_{\mathrm{T}\,{}_s\!\backslash\mathrm{T}'},$$

*where* $s$ *runs over all shuffles of the signatures of* $\mathrm{T}$ *and* $\mathrm{T}'$*.*

*Proof.* Let $\tau$ and $\tau'$ be linear extensions of $\mathrm{T}$ and $\mathrm{T}'$ respectively, let $\sigma \in \tau\star\tau'$ and let $\mathrm{S} = \mathbf{ct}(\sigma)$. The convolution $\tau\star\tau'$ shuffles the columns of the tables of $\tau$ and $\tau'$ while preserving the order of their rows. According to the description of the insertion algorithm $\mathbf{ct}$, the tree $\mathrm{S}$ thus consists in $\mathrm{T}$ below and $\mathrm{T}'$ above, except that the vertical walls falling from the negative nodes of $\mathrm{T}'$ split $\mathrm{T}$ and similarly the vertical walls rising from the positive nodes of $\mathrm{T}$ split $\mathrm{T}'$. This corresponds to the description of $\mathrm{T}\,{}_s\!\backslash\mathrm{T}'$, where $s$ is the shuffle of the signatures of $\mathrm{T}$ and $\mathrm{T}'$ given by $\sigma$. $\square$

$\boxed{\text{Coproduct}}$ To describe the coproduct of $\mathbb{Q}$-basis elements of $\mathsf{Camb}^*$, we also use gaps and vertical paths in Cambrian trees. Namely, for a gap $\gamma$, we denote by $L(\mathrm{S},\gamma)$ and $R(\mathrm{S},\gamma)$ the left and right Cambrian subtrees of S when split along the path $\lambda(\mathrm{S},\gamma)$. See Figure 9.

**Proposition 22.** *For any Cambrian tree* S*, the coproduct* $\triangle\mathbb{Q}_{\mathrm{S}}$ *is given by*

$$\triangle\mathbb{Q}_{\mathrm{S}} = \sum_{\gamma} \mathbb{Q}_{L(\mathrm{S},\gamma)} \otimes \mathbb{Q}_{R(\mathrm{S},\gamma)},$$

*where* $\gamma$ *runs over all gaps between vertices of* S.

*Proof.* Let $\sigma$ be a linear extension of S and $\tau, \tau' \in \mathfrak{S}_{\pm}$ such that $\sigma \in \tau \,\bar{\sqcup}\, \tau'$. As discussed in Section 1.2.1, $\tau$ and $\tau'$ respectively appear on the left and right columns of $\sigma$. Let $\gamma$ denote the vertical gap separating $\tau$ from $\tau'$. Applying the insertion algorithm to $\tau$ and $\tau'$ separately then yields the trees $L(\mathrm{S},\gamma)$ and $R(\mathrm{S},\gamma)$. The description follows. □

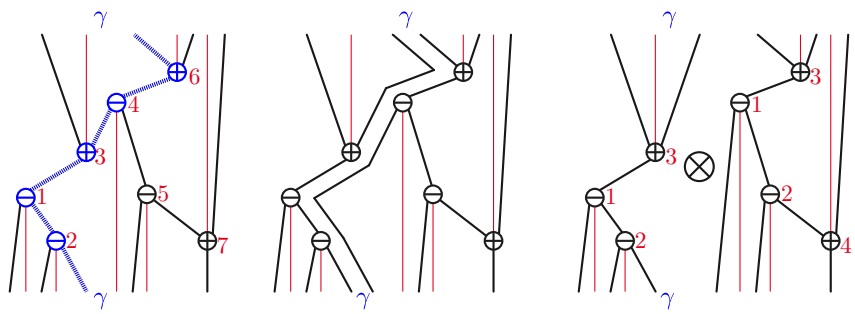



Figure 9. A gap $\gamma$ (left) defines a vertical cut (middle) which splits the Cambrian tree (right).

# Part 2. The Baxter-Cambrian Hopf Algebra

## 2.1. Twin Cambrian trees

We now consider twin Cambrian trees and the resulting Baxter-Cambrian algebra. It provides a straightforward generalization to the Cambrian setting of the work of S. Law and N. Reading on rectangulations [LR12] and S. Giraudo on twin binary trees [Gir12]. The bases of these algebras are counted by the Baxter numbers. In Section 2.1.5 we provide references for the various Baxter families and their bijective correspondences, and we discuss the Cambrian counterpart of these numbers. Definitions and combinatorial properties of twin Cambrian trees are given in this section, while the algebraic aspects are treated in the next section.

2.1.1. **Twin Cambrian trees.** We consider the following pairs of Cambrian trees (see Figure 10).

**Definition 23.** *Two* $\varepsilon$*-Cambrian trees* $\mathrm{T}_\circ, \mathrm{T}_\bullet$ *are* twin *if the union* $\mathrm{T}_\circ \mathbin{\rotatebox[origin=c]{-90}{$\leftrightarrows$}} \mathrm{T}_\bullet$ *of* $\mathrm{T}_\circ$ *with the reverse of* $\mathrm{T}_\bullet$ *(reversing the orientations of all edges) is acyclic.*

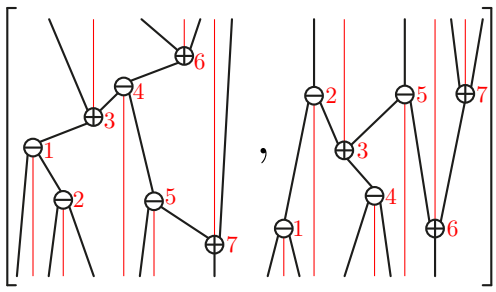



Figure 10. A pair of twin Cambrian trees.

If $\mathrm{T}_\circ, \mathrm{T}_\bullet$ are twin $\varepsilon$-Cambrian trees, they necessarily have opposite canopy (see Section 1.1.5), meaning that $\mathbf{can}(\mathrm{T}_\circ)_i = -\mathbf{can}(\mathrm{T}_\bullet)_i$ for all $i \in [n-1]$. The reciprocal statement for the constant signature $(-)^n$ is proved by S. Giraudo in [Gir12].

**Proposition 24** ([Gir12])**.** *Two binary trees are twin if and only if they have opposite canopy.*

We conjecture that this statement holds for general signatures. Consider two $\varepsilon$-Cambrian trees $\mathrm{T}_\circ, \mathrm{T}_\bullet$ with opposite canopies. It is easy to show that $\mathrm{T}_\circ \between \mathrm{T}_\bullet$ cannot have trivial cycles, meaning that $\mathrm{T}_\circ$ and $\mathrm{T}_\bullet$ cannot both have a path from $i$ to $j$ for $i \neq j$. To prove that $\mathrm{T}_\circ \between \mathrm{T}_\bullet$ has no cycles at all, a good method is to design an algorithm to extract a linear extension of $\mathrm{T}_\circ \between \mathrm{T}_\bullet$. This approach was used in [Gir12] for the signature $(-)^n$. In this situation, it is clear that the root of $\mathrm{T}_\bullet$ is minimal in $\mathrm{T}_\circ$ (by the canopy assumption), and we therefore pick it as the first value of a linear extension of $\mathrm{T}_\circ \between \mathrm{T}_\bullet$. The remaining of the linear extension is constructed inductively. In the general situation, it turns out that not all maximums in $\mathrm{T}_\bullet$ are minimums in $\mathrm{T}_\circ$ (and reciprocally). It is thus not clear how to choose the first value of a linear extension of $\mathrm{T}_\circ \between \mathrm{T}_\bullet$.

2.1.2. **Baxter-Cambrian correspondence.** We send a permutation $\tau \in \mathfrak{S}^\varepsilon$ of $\mathfrak{S}^\varepsilon$ to a pair of twin $\varepsilon$-Cambrian trees by inserting $\tau = \tau_1 \cdots \tau_n$ and its *mirror* $\overleftarrow{\tau} = \tau_n \cdots \tau_1 \in \mathfrak{S}^\varepsilon$ with the map $\mathbf{ct}$ from Section 1.1.2.

**Proposition 25.** *The map $\mathbf{ct}^{\between}$ defined by $\mathbf{ct}^{\between}(\tau) := \big[\mathbf{ct}(\tau), \mathbf{ct}(\overleftarrow{\tau})\big]$ is a surjection from signed permutations to pairs of twin Cambrian trees.*

*Proof.* By definition, $\tau$ is a linear extension of $\mathbf{ct}(\tau)$ and of the reverse of $\mathbf{ct}(\overleftarrow{\tau})$, so that $\mathbf{ct}(\tau) \between \mathbf{ct}(\overleftarrow{\tau})$ is indeed acyclic. Moreover, the fiber $(\mathbf{ct}^{\between})^{-1}([\mathrm{T}_\circ, \mathrm{T}_\bullet])$ of a pair of twin $\varepsilon$-Cambrian trees $\mathrm{T}_\circ, \mathrm{T}_\bullet$ is the set $\mathcal{L}(\mathrm{T}_\circ \between \mathrm{T}_\bullet)$ of linear extensions of the graph $\mathrm{T}_\circ \between \mathrm{T}_\bullet$. This set is non-empty since $\mathrm{T}_\circ \between \mathrm{T}_\bullet$ is acyclic by definition of twin Cambrian trees. □

2.1.3. **Baxter-Cambrian congruence.** We now characterize by a congruence relation the signed permutations $\tau \in \mathfrak{S}^\varepsilon$ which have the same image $\mathbf{ct}^{\between}(\tau)$.

**Definition 26.** *For a signature $\varepsilon \in \pm^n$, the $\varepsilon$-Baxter-Cambrian congruence is the equivalence relation on $\mathfrak{S}^\varepsilon$ defined as the transitive closure of the rewriting rules*

$$\begin{aligned} UbVadWcX &\equiv_\varepsilon^{\between} UbVdaWcX \quad \text{if } a < \{b,c\} < d \text{ and } \varepsilon_b = \varepsilon_c, \\ UbVcWadX &\equiv_\varepsilon^{\between} UbVcWdaX \quad \text{if } a < \{b,c\} < d \text{ and } \varepsilon_b \neq \varepsilon_c, \\ UadVbWcX &\equiv_\varepsilon^{\between} UdaVbWcX \quad \text{if } a < \{b,c\} < d \text{ and } \varepsilon_b \neq \varepsilon_c, \end{aligned}$$

*where $a, b, c, d$ are elements of $[n]$ while $U, V, W, X$ are words on $[n]$. The Baxter-Cambrian congruence is the equivalence relation on all signed permutations $\mathfrak{S}_\pm$ given by all $\varepsilon$-Baxter-Cambrian congruences:*

$$\text{for } \tau \in \mathfrak{S}^\varepsilon, \tau' \in \mathfrak{S}^{\varepsilon'} \qquad \tau \equiv^{\between} \tau' \iff \varepsilon = \varepsilon' \text{ and } \tau \equiv_\varepsilon^{\between} \tau'.$$

**Proposition 27.** *The $\varepsilon$-Cambrian congruence classes are precisely the fibers of the map $\mathbf{ct}^{\between}$:*

$$\text{for } \tau, \tau' \in \mathfrak{S}^\varepsilon \qquad \tau \equiv_\varepsilon^{\between} \tau' \iff \mathbf{ct}^{\between}(\tau) = \mathbf{ct}^{\between}(\tau').$$

*Proof.* The proof of this proposition consists essentially in seeing that $\mathbf{ct}^{\between}(\tau) = \mathbf{ct}^{\between}(\tau')$ if and only if $\tau \equiv_\varepsilon \tau'$ and $\overleftarrow{\tau} \equiv_\varepsilon \overleftarrow{\tau'}$ (by definition of $\mathbf{ct}^{\between}$). The definition of the $\varepsilon$-Baxter-Cambrian equivalence $\equiv_\varepsilon^{\between}$ is exactly the translation of this observation in terms of rewriting rules. □

**Proposition 28.** *The $\varepsilon$-Baxter-Cambrian class indexed by a pair $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ of twin $\varepsilon$-Cambrian trees is the intersection of the $\varepsilon$-Cambrian class indexed by $\mathrm{T}_\circ$ with the $(-\varepsilon)$-Cambrian class indexed by the reverse of $\mathrm{T}_\bullet$.*

*Proof.* The $\varepsilon$-Baxter-Cambrian class indexed by $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ is the set of linear extensions of $\mathrm{T}_\circ \between \mathrm{T}_\bullet$, *i.e.* of permutations which are both linear extensions of $\mathrm{T}_\circ$ and linear extensions of the reverse of $\mathrm{T}_\bullet$. The former form the $\varepsilon$-Cambrian class indexed by $\mathrm{T}_\circ$ while the latter form the $(-\varepsilon)$-Cambrian class indexed by the reverse of $\mathrm{T}_\bullet$. □

2.1.4. **Rotations and Baxter-Cambrian lattices.** We now present the rotation operation on pairs of twin $\varepsilon$-Cambrian trees.

**Definition 29.** *Let $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ be a pair of $\varepsilon$-Cambrian trees and $i \to j$ be an edge of $\mathrm{T}_\circ \Updownarrow \mathrm{T}_\bullet$. We say that the edge $i \to j$ is* rotatable *if and only if one of the following holds:*

- *$i \to j$ is an edge in $\mathrm{T}_\circ$ and $j \to i$ is an edge in $\mathrm{T}_\bullet$,*
- *$i \to j$ is an edge in $\mathrm{T}_\circ$ while $i$ and $j$ are incomparable in $\mathrm{T}_\bullet$,*
- *$i$ and $j$ are incomparable in $\mathrm{T}_\circ$ while $j \to i$ is an edge in $\mathrm{T}_\bullet$.*

*If $i \to j$ is rotatable in $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$, its* rotation *transforms $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ to the pair of trees $[\mathrm{T}'_\circ, \mathrm{T}'_\bullet]$, where*

- *$\mathrm{T}'_\circ$ is obtained by rotation of $i \to j$ in $\mathrm{T}_\circ$ if possible and $\mathrm{T}'_\circ = \mathrm{T}_\circ$ otherwise, and*
- *$\mathrm{T}'_\bullet$ is obtained by rotation of $j \to i$ in $\mathrm{T}_\bullet$ if possible and $\mathrm{T}'_\bullet = \mathrm{T}_\bullet$ otherwise.*

**Proposition 30.** *Rotating a rotatable edge $i \to j$ in a pair $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ of twin $\varepsilon$-Cambrian trees yields a pair $[\mathrm{T}'_\circ, \mathrm{T}'_\bullet]$ of twin $\varepsilon$-Cambrian trees.*

*Proof.* By Proposition 10, the trees $\mathrm{T}_\circ, \mathrm{T}_\bullet$ are $\varepsilon$-Cambrian trees. To see that they are twins, observe that switching $i$ and $j$ in a linear extension of $\mathrm{T}_\circ \Updownarrow \mathrm{T}_\bullet$ yields a linear extension of $\mathrm{T}'_\circ \Updownarrow \mathrm{T}'_\bullet$. □

**Remark 31** (Number of rotatable edges)**.** Note that a pair $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ of $\varepsilon$-Cambrian trees has always at least $n-1$ rotatable edges. This will be immediate from the considerations of Section 2.1.6.

Consider the *increasing rotation graph* whose vertices are pairs of twin $\varepsilon$-Cambrian trees and whose arcs are increasing rotations $[\mathrm{T}_\circ, \mathrm{T}_\bullet] \to [\mathrm{T}'_\circ, \mathrm{T}'_\bullet]$, *i.e.* for which $i < j$ in Definition 29. This graph is illustrated on Figure 11 for the signatures $\varepsilon = -+--$ and $\varepsilon = +---$.

**Proposition 32.** *For any cover relation $\tau < \tau'$ in the weak order on $\mathfrak{S}^\varepsilon$, either $\mathbf{ct}^{\Updownarrow}(\tau) = \mathbf{ct}^{\Updownarrow}(\tau')$ or $\mathbf{ct}^{\Updownarrow}(\tau) \to \mathbf{ct}^{\Updownarrow}(\tau')$ in the increasing rotation graph.*

*Proof.* Let $i, j \in [n]$ be such that $\tau'$ is obtained from $\tau$ by switching two consecutive values $ij$ to $ji$. If $i$ and $j$ are incomparable in $\mathbf{ct}(\tau)$, then $\mathbf{ct}(\tau) = \mathbf{ct}(\tau')$. Otherwise, there is an edge $i \to j$ in $\mathbf{ct}(\tau)$, and $\mathbf{ct}(\tau')$ is obtained by rotating $i \to j$ in $\mathbf{ct}(\tau)$. The same discussion is valid for the trees $\mathbf{ct}(\overleftarrow{\tau})$ and $\mathbf{ct}(\overleftarrow{\tau'})$ and edge $j \to i$. The result immediately follows. □

It follows that the increasing rotation graph on pairs of twin $\varepsilon$-Cambrian trees is acyclic and we call *$\varepsilon$-Baxter-Cambrian poset* its transitive closure. In other words, the previous statement says that the map $\mathbf{ct}^{\Updownarrow}$ defines a poset homomorphism from the weak order on $\mathfrak{S}^\varepsilon$ to the $\varepsilon$-Baxter-Cambrian poset. The following statement extends the results of N. Reading [Rea06] on Cambrian lattices and S. Law and N. Reading [LR12] on the lattice of diagonal rectangulations.

**Proposition 33.** *The $\varepsilon$-Baxter-Cambrian poset is a lattice quotient of the weak order on $\mathfrak{S}^\varepsilon$.*

*Proof.* By Proposition 28, the $\varepsilon$-Baxter-Cambrian congruence is the intersection of two Cambrian congruences. The statement follows since the Cambrian congruences are lattice congruences of the weak order [Rea06] and an intersection of lattice congruences is a lattice congruence. □

**Remark 34** (Extremal elements and pattern avoidance)**.** Since the Baxter-Cambrian classes are generated by rewriting rules, we immediately obtain that the minimal elements of the Baxter-Cambrian classes are precisely the signed permutations avoiding the patterns:

$$\underline{b}\text{-}da\text{-}\underline{c},\ \overline{b}\text{-}da\text{-}\overline{c},\ \underline{c}\text{-}da\text{-}\underline{b},\ \overline{c}\text{-}da\text{-}\overline{b},\ \underline{b}\text{-}\overline{c}\text{-}da,\ \overline{b}\text{-}\underline{c}\text{-}da,\ \underline{c}\text{-}\overline{b}\text{-}da,\ \overline{c}\text{-}\underline{b}\text{-}da,\ da\text{-}\underline{b}\text{-}\overline{c},\ da\text{-}\overline{b}\text{-}\underline{c},\ da\text{-}\underline{c}\text{-}\overline{b},\ da\text{-}\overline{c}\text{-}\underline{b}.$$

Similarly, the maximal elements of the Baxter-Cambrian classes are precisely the signed permutations avoiding the patterns:

$$(\star)\quad \underline{b}\text{-}ad\text{-}\underline{c},\ \overline{b}\text{-}ad\text{-}\overline{c},\ \underline{c}\text{-}ad\text{-}\underline{b},\ \overline{c}\text{-}ad\text{-}\overline{b},\ \underline{b}\text{-}\overline{c}\text{-}ad,\ \overline{b}\text{-}\underline{c}\text{-}ad,\ \underline{c}\text{-}\overline{b}\text{-}ad,\ \overline{c}\text{-}\underline{b}\text{-}ad,\ ad\text{-}\underline{b}\text{-}\overline{c},\ ad\text{-}\overline{b}\text{-}\underline{c},\ ad\text{-}\underline{c}\text{-}\overline{b},\ ad\text{-}\overline{c}\text{-}\underline{b}.$$

2.1.5. **Baxter-Cambrian numbers.** In contrast to the number of $\varepsilon$-Cambrian trees, the number of pairs of twin $\varepsilon$-Cambrian trees depends on the signature $\varepsilon$. For example, there are 22 pairs of twin $(----)$-Cambrian trees and only 20 pairs of twin $(-+--)$-Cambrian trees. See Figures 11 and 12.

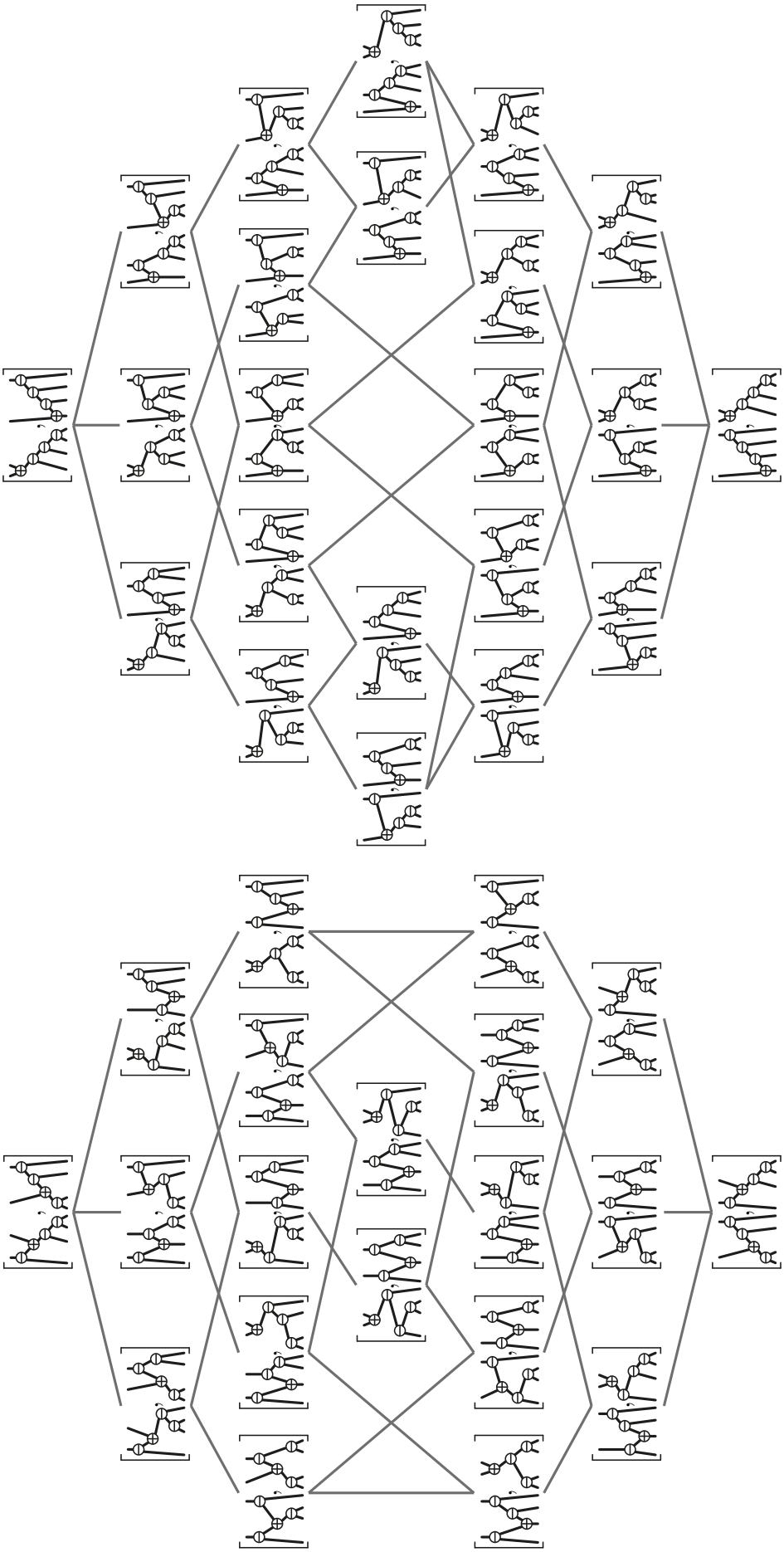


FIGURE 11. The $\varepsilon$-Baxter-Cambrian lattices for $\varepsilon = -+--$ (left) and $\varepsilon = +---$ (right).

For a signature $\varepsilon$, we define the *$\varepsilon$-Baxter-Cambrian number* $B_\varepsilon$ to be the number of pairs of twin $\varepsilon$-Cambrian trees. We immediately observe that $B_\varepsilon$ is preserved when we change the first and last sign of $\varepsilon$, inverse simultaneously all signs of $\varepsilon$, or reverse the signature $\varepsilon$:

$$B_\varepsilon = B_{\chi_0(\varepsilon)} = B_{\chi_n(\varepsilon)} = B_{-\varepsilon} = B_{\overleftarrow{\varepsilon}},$$

where $\chi_0$ and $\chi_n$ change the first and last sign, $(-\varepsilon)_i = -\varepsilon_i$ and $(\overleftarrow{\varepsilon})_i = \varepsilon_{n+1-i}$.

In the following statements, we provide an inductive formula to compute all $\varepsilon$-Baxter-Cambrian numbers, using a two-parameters refinement. The pairs of twin $\varepsilon$-Cambrian trees are in bijection with the weak order maximal permutations of $\varepsilon$-Baxter-Cambrian classes. These permutations are precisely the permutations avoiding the patterns $(\star)$ in Remark 34. We consider the generating tree $\mathcal{T}_\varepsilon^{\Downarrow}$ for these permutations. This tree has $n$ levels, and the nodes at level $m$ are labeled by permutations of $[m]$ whose values are signed by the restriction of $\varepsilon$ to $[m]$ and avoiding the patterns $(\star)$. The parent of a permutation in $\mathcal{T}_\varepsilon^{\Downarrow}$ is obtained by deleting its maximal value. See Figure 12.

We consider the possible positions of $m+1$ in the children of a permutation $\tau$ at level $m$ in this generating tree $\mathcal{T}_\varepsilon^{\Downarrow}$. Index by $\{0,\dots,m\}$ from left to right the gaps before the first letter, between two consecutive letters, and after the last letter of $\tau$. *Free gaps* are those where placing $m+1$ does not create a pattern of $(\star)$. Free gaps are marked with a blue dot in Figure 12. It is important to observe that gap 0 as well as the gaps immediately after $m-1$ and $m$ are always free, no matter $\tau$ or the signature $\varepsilon$.

Define the *free-gap-type* of $\tau$ to be the pair $(\ell, r)$ where $\ell$ (resp. $r$) denote the number of free gaps on the left (resp. right) of $m$ in $\tau$. For a signature $\varepsilon$, let $B_\varepsilon(\ell, r)$ denote the number of free-gap-type $(\ell, r)$ weak order maximal permutations of $\varepsilon$-Baxter-Cambrian classes. These refined Baxter-Cambrian numbers enables us to write inductive equations.

**Proposition 35.** *Consider two signatures $\varepsilon \in \pm^n$ and $\varepsilon' \in \pm^{n-1}$, where $\varepsilon'$ is obtained by deleting the last sign of $\varepsilon$. Then*

$$B_\varepsilon(\ell, r) = \begin{cases} \sum_{\ell' \geq \ell} B_{\varepsilon'}(\ell', r-1) \;+\; \sum_{r' \geq r} B_{\varepsilon'}(\ell-1, r') & \text{if } \varepsilon_{n-1} = \varepsilon_n, \quad (=) \\ \delta_{\ell=1} \cdot \delta_{r\geq 2} \cdot \sum_{\substack{\ell' \geq r-1 \\ r' \geq 1}} B_{\varepsilon'}(\ell', r') \;+\; \delta_{\ell \geq 2} \cdot \delta_{r=1} \cdot \sum_{\substack{\ell' \geq 1 \\ r' \geq \ell-1}} B_{\varepsilon'}(\ell', r') & \text{if } \varepsilon_{n-1} \neq \varepsilon_n, \quad (\neq) \end{cases}$$

*where $\delta$ denote the Kronecker $\delta$ (defined by $\delta_X = 1$ if $X$ is satisfied and $0$ otherwise).*

*Proof.* Assume first that $\varepsilon_{n-1} = \varepsilon_n$. Consider two permutations $\tau$ and $\tau'$ at level $n$ and $n-1$ in $\mathcal{T}_\varepsilon^{\Downarrow}$ such that $\tau'$ is obtained by deleting $n$ in $\tau$. Denote by $\alpha$ and $\beta$ the gaps immediately after $n-1$ and $n$ in $\tau$, by $\alpha'$ the gap immediately after $n-1$ in $\tau'$, and by $\beta'$ the gap in $\tau'$ where we insert $n$ to get $\tau$. Then, besides gaps 0, $\alpha$ and $\beta$, the free gaps of $\tau$ are precisely the free gaps of $\tau'$ not located between gaps $\alpha'$ and $\beta'$. Indeed,

- inserting $d := n+1$ just after a value $a$ located between $b := n-1$ and $c := n$ in $\tau$ would create a pattern *b-ad-c* or *c-ad-b* with $\varepsilon_b = \varepsilon_c$;
- conversely, consider a gap $\gamma$ of $\tau$ not located between $\alpha$ and $\beta$. If inserting $n+1$ at $\gamma$ in $\tau$ creates a forbidden pattern of $(\star)$ with $c = n$, then inserting $n$ at $\gamma$ in $\tau'$ would also create the same forbidden pattern of $(\star)$ with $c = n-1$. Therefore, all free gaps not located between gaps $\alpha'$ and $\beta'$ remain free.

Let $(\ell, r)$ denote the free-gap-type of $\tau$ and $(\ell', r')$ denote the free-gap-type of $\tau'$. We obtain that

- $\ell' \geq \ell$ and $r' = r-1$ if $n$ is inserted on the left of $n-1$;
- $\ell' = \ell - 1$ and $r' \geq r$ if $n$ is inserted on the right of $n-1$.

The formula follows immediately when $\varepsilon_{n-1} = \varepsilon_n$.

Assume now that $\varepsilon_{n-1} = -\varepsilon_n$, and keep the same notations as before. Using similar arguments, we observe that besides gaps 0, $\alpha$ and $\beta$, the free gaps of $\tau$ are precisely the free gaps of $\tau'$ located between gaps $\alpha'$ and $\beta'$. Therefore, we obtain that

- $\ell = 1$, $r \geq 2$, and $\ell' \geq r-1$ if $n$ is inserted on the left of $n-1$;
- $\ell \geq 2$, $r = 1$, and $r' \geq \ell - 1$ if $n$ is inserted on the right of $n-1$.

The formula follows for $\varepsilon_{n-1} = -\varepsilon_n$. □

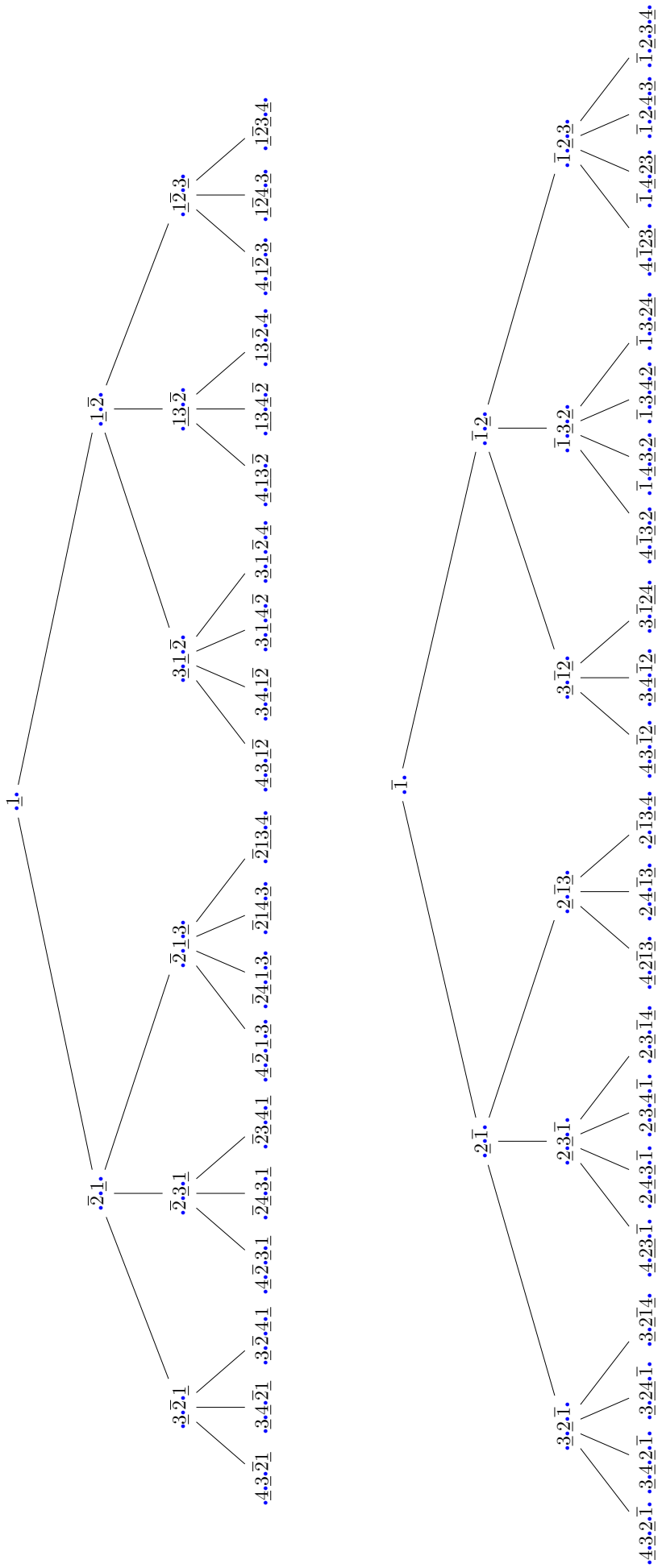



FIGURE 12. The generating trees $\mathcal{T}_\varepsilon^{\P}$ for the signatures $\varepsilon = -+--$ (top) and $\varepsilon = +---$ (bottom). Free gaps are marked with a blue dot.

Before applying these formulas to obtain bounds on $B_\varepsilon$ for arbitrary signatures $\varepsilon$, let us consider two special signatures: the constant and the alternating signature.

ALTERNATING SIGNATURE Since it is the easiest, we start with the *alternating signature* $(+-)^{\frac{n}{2}}$ (where we define $(+-)^{\frac{n}{2}}$ to be $(+-)^m+$ when $n = 2m+1$ is odd).

**Proposition 36.** *The Baxter-Cambrian numbers for alternating signatures are central binomial coefficients (see [OEI10, A000984]):*

$$B_{(+-)^{\frac{n}{2}}} = \binom{2n-2}{n-1}.$$

*Proof.* We prove by induction on $n$ that the refined Baxter-Cambrian numbers are

$$B_{(+-)^{\frac{n}{2}}}(\ell, r) = \delta_{\ell=1} \cdot \delta_{r\geq 2} \cdot \binom{2n-2-r}{n-r} + \delta_{\ell\geq 2} \cdot \delta_{r=1} \cdot \binom{2n-2-\ell}{n-\ell}.$$

This is true for $n = 2$ since $B_{+-}(1, 2) = 1$ (counting the permutation 21) and $B_{+-}(2, 1) = 1$ (counting the permutation 12). Assume now that it is true for some $n \in \mathbb{N}$. Then Equation $(\neq)$ of Proposition 35 shows that

$$\begin{aligned} B_{(+-)^{\frac{n+1}{2}}}(\ell, r) &= \delta_{\ell=1} \cdot \delta_{r\geq 2} \cdot \sum_{\ell'\geq r-1} \binom{2n-2-\ell'}{n-\ell'} + \delta_{\ell\geq 2} \cdot \delta_{r=1} \cdot \sum_{r'\geq \ell-1} \binom{2n-2-r'}{n-r'} \\ &= \delta_{\ell=1} \cdot \delta_{r\geq 2} \cdot \binom{2n-r}{n+1-r} + \delta_{\ell\geq 2} \cdot \delta_{r=1} \cdot \binom{2n-\ell}{n+1-\ell}, \end{aligned}$$

since a sum of binomial coefficients along a diagonal $\sum_{i=0}^{p} \binom{q+i}{i}$ simplifies to the binomial coefficient $\binom{q+p+1}{p}$ by multiple applications of Pascal's rule. Finally, we conclude observing that

$$B_{(+-)^{\frac{n}{2}}} = \sum_{\ell,r\in[n]} B_{(+-)^{\frac{n}{2}}}(\ell, r) = 2\sum_{u\geq 2} \binom{2n-2-u}{n-u} = 2\binom{2n-3}{n-2} = \binom{2n-2}{n-1}. \qquad \square$$

CONSTANT SIGNATURE We now consider the *constant signature* $(+)^n$. The number $B_{(+)^n}$ is the classical *Baxter number* (see [OEI10, A001181]) defined by

$$B_{(+)^n} = B_n = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

These numbers have been extensively studied, see in particular [CGHK78, Mal79, DG96, DG98, YCCG03, FFNO11, BBMF11, LR12, Gir12]. The Baxter number $B_n$ counts several families:

- Baxter permutations of $[n]$, *i.e.* permutations avoiding the patterns *b-da-c* and *c-ad-b*,
- weak order maximal (resp. minimal) permutations of Baxter congruence classes on $\mathfrak{S}_n$, *i.e.* permutations avoiding the patterns *b-ad-c* and *c-ad-b* (resp. *b-da-c* and *c-da-b*),
- pairs of twin binary trees on $n$ nodes,
- diagonal rectangulations of an $n \times n$ grid,
- plane bipolar orientations with $n$ edges,
- non-crossing triples of path with $k-1$ north steps and $n-k$ east steps, for all $k \in [n]$,
- etc.

Bijections between all these *Baxter families* are discussed in [DG96, DG98, FFNO11, BBMF11].

**Remark 37** (Two proofs of the summation formula)**.** There are essentially two ways to obtain the above summation formula for Baxter numbers: it was first proved analytically in [CGHK78], and then bijectively in [Vie81, DG98, FFNO11]. Let us shortly comment on these two techniques and discuss the limits of their extension to the Baxter-Cambrian setting.

(i) The *bijective proofs* in [DG98, FFNO11] transform pairs of binary trees to triples of non-crossing paths, and then use the Gessel-Viennot determinant lemma [GV85] to get the summation formula. The middle path of these triples is given by the canopy of the twin binary trees, while the other two paths are given by the structure of the trees. We are not yet able to adapt this technique to provide summation formulas for all Baxter-Cambrian numbers.

(ii) The *analytic proof* in [CGHK78] is based on Equation (=) of Proposition 35 and can be partially adapted to arbitrary signatures as follows. Define the *extension* of a signature $\varepsilon \in \pm^n$ by a signature $\delta \in \pm^m$ to be the signature $\varepsilon \lhd \delta \in \pm^{n+m}$ such that $(\varepsilon \lhd \delta)_i = \varepsilon_i$ for $i \in [n]$ and $(\varepsilon \lhd \delta)_{n+j} = \delta_j \cdot (\varepsilon \lhd \delta)_{n+j-1}$ for $j \in m$. For example, $++- \lhd +--+ = ++--+--$. Then for any $\varepsilon \in \pm^n$ and $\delta \in \pm^m$, we have

$$B_{\varepsilon \lhd \delta} = \sum_{\ell, r \geq 1} X_\delta(\ell, r)\, B_\varepsilon(\ell, r),$$

where the coefficients $X_\delta(\ell, r)$ are obtained inductively from the formulas of Proposition 35. Namely, for any $\ell, r \geq 1$, we have $X_\varnothing(\ell, r) = 1$ and

$$X_{(+\delta)}(\ell, r) = \sum_{1 \leq \ell' \leq \ell} X_\delta(\ell', r+1) \;+\; \sum_{1 \leq r' \leq r} X_\delta(\ell+1, r'),$$
$$X_{(-\delta)}(\ell, r) = \sum_{2 \leq \ell' \leq r+1} X_\delta(\ell', 1) \;+\; \sum_{2 \leq r' \leq \ell+1} X_\delta(1, r').$$

These equations translate on the generating function $\mathfrak{X}_\delta(u, v) := \sum_{\ell, r \geq 1} X_\delta(\ell, r) u^{\ell-1} v^{r-1}$ to the formulas $\mathfrak{X}_\varnothing(u, v) = \frac{1}{(1-u)(1-v)}$ and

$$\mathfrak{X}_{(+\delta)}(u, v) = \frac{\mathfrak{X}_\delta(u, v) - \mathfrak{X}_\delta(u, 0)}{(1-u)v} + \frac{\mathfrak{X}_\delta(u, v) - \mathfrak{X}_\delta(0, v)}{u(1-v)},$$
$$\mathfrak{X}_{(-\delta)}(u, v) = \frac{\mathfrak{X}_\delta(v, 0) - \mathfrak{X}_\delta(0, 0)}{(1-u)(1-v)v} + \frac{\mathfrak{X}_\delta(0, u) - \mathfrak{X}_\delta(0, 0)}{u(1-u)(1-v)}.$$

Note that the $u/v$-symmetry of $\mathfrak{X}_\delta(u, v)$ is reflected in a symmetry on these inductive equations. We can thus write this generating function $\mathfrak{X}_\delta(u, v)$ as

$$\mathfrak{X}_\delta(u, v) = \sum_{\substack{i, j \geq 0 \\ k \in [|\delta|+1]}} Y_\delta^{i,j,k} \frac{(-u)^i\,(-v)^j}{(1-u)^{|\delta|+2-k}(1-v)^k},$$

where the non-vanishing coefficients $Y_\delta^{i,j,k}$ are computed inductively by $Y_\varnothing^{0,0,1} = 1$ and

$$Y_{(+\delta)}^{i,j,k} = \binom{k}{j+1} Y_\delta^{i,0,k} - Y_\delta^{i,j+1,k} + \binom{|\delta|+3-k}{i+1} Y_\delta^{0,j,k-1} - Y_\delta^{i+1,j,k-1},$$
$$Y_{(-\delta)}^{i,j,k} = \binom{k-1}{j}\left[\binom{|\delta|+2-k}{i+1} Y_\delta^{0,0,k} - Y_\delta^{i+1,0,k}\right] + \binom{|\delta|+2-k}{i}\left[\binom{k-1}{j+1} Y_\delta^{0,0,k-1} - Y_\delta^{0,j+1,k-1}\right].$$

We used that $Y_\delta^{i,j,k} = Y_\delta^{j,i,|\delta|+2-k}$ to simplify the second equation. Note that this decomposition of $\mathfrak{X}_\delta$ is not unique and the inductive equations on $Y_\delta^{i,j,k}$ follow from a particular choice of such a decomposition.

At that stage, F. Chung, R. Graham, V. Hoggatt, and M. Kleiman [CGHK78], guess and check that the first equation is always satisfied by

$$Y_{(+)^{n-1}}^{i,j,k} = \frac{\binom{n+1}{k}\binom{n+1}{k+i+1}\binom{n+1}{k-j-1}\left[\binom{k+i-2}{i}\binom{n+j-k-1}{j} - \binom{k+i-2}{i-1}\binom{n+j-k-1}{j-1}\right]}{\binom{n+1}{1}\binom{n+1}{2}}$$

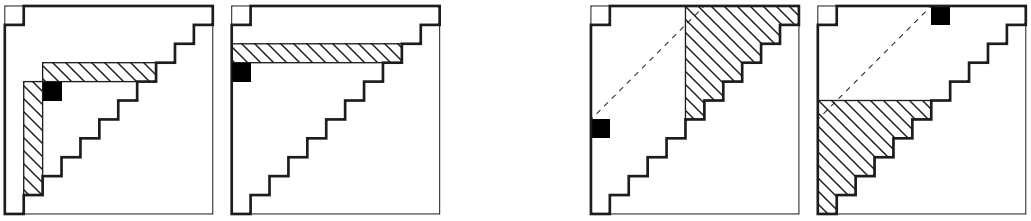
from which they derive immediately that

$$B_{(+)^n} = B_{+\lhd(+)^{n-1}} = \sum_{\ell, r \geq 1} X_{(+)^{n-1}}(\ell, r)\, B_+(\ell, r) = X_{(+)^{n-1}}(1, 1) = \mathfrak{X}_{(+)^{n-1}}(0, 0)$$
$$= \sum_{k \in [n]} Y_{(+)^{n-1}}^{0,0,k} = \binom{n+1}{1}^{-1}\binom{n+1}{2}^{-1} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

Unfortunately, we have not been able to guess closed formulas for the coefficients $Y_\delta^{i,j,k}$ for arbitrary $\delta \in \pm^n$.

ARBITRARY SIGNATURES We now come back to an arbitrary signature $\varepsilon$. We were not able to derive summation formulas for arbitrary signatures using the techniques presented in Remark 37 above. However, we use here the inductive formulas of Proposition 35 to bound the Baxter-Cambrian number $B_\varepsilon$ for an arbitrary signature $\varepsilon$.

For this, we consider the matrix $\mathbf{B}_\varepsilon := \big(B_\varepsilon(\ell, r)\big)_{\ell,r\in[n]}$. The inductive formulas of Proposition 35 provide an efficient inductive algorithm to compute this matrix $\mathbf{B}_\varepsilon$ and thus the $\varepsilon$-Baxter-Cambrian number $B_\varepsilon = \sum_{\ell,r\in[n]} B_\varepsilon(\ell, r)$. Namely, if $\varepsilon$ is obtained by adding a sign at the end of $\varepsilon'$, then each entry of $\mathbf{B}_\varepsilon$ is the sum of entries of $\mathbf{B}_{\varepsilon'}$ in a region depending on whether $\varepsilon_n = \varepsilon_{n-1}$. These regions are sketched in Figure 13.



$\varepsilon_n = \varepsilon_{n-1}$ $\qquad$ $\varepsilon_n = -\varepsilon_{n-1}$

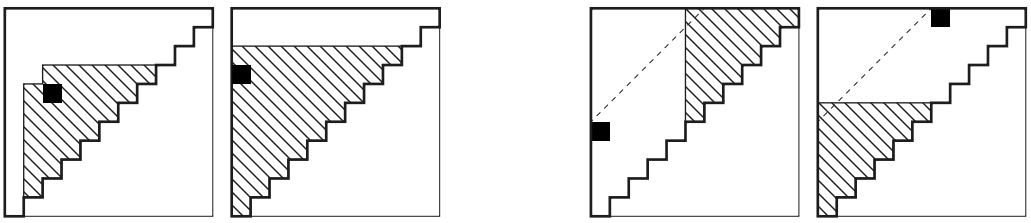
FIGURE 13. Inductive computation of $\mathbf{B}_\varepsilon$: the black entry of $\mathbf{B}_\varepsilon$ is the sum of the entries of $\mathbf{B}_{\varepsilon'}$ over the shaded region. Entries outside the upper triangular region always vanish. When $\varepsilon_n = -\varepsilon_{n-1}$, the only non-vanishing entries of $\mathbf{B}_\varepsilon$ are in the first row or in the first column.

We observe that the transformations of Figure 13 are symmetric with respect to the diagonal of the matrix. Since $\mathbf{B}_{\varepsilon_1\varepsilon_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is symmetric, and $\mathbf{B}_\varepsilon$ is obtained from $\mathbf{B}_{\varepsilon_1\varepsilon_2}$ by successive applications of these symmetric transformations, we obtain that $\mathbf{B}_\varepsilon$ is always symmetric. Although this fact may seem natural to the reader, it is not at all immediate as there is an asymmetry on the three forced free gaps: for example gap 0 is always free.

For a matrix $M := (m_{i,j})$, we consider the matrix $M^{\mathrm{SE}} := \big(m^{\mathrm{SE}}_{i,j}\big)$ where

$$m^{\mathrm{SE}}_{i,j} := \sum_{p\ge i,\ q\ge j} m_{p,q}$$

is the sum of all entries located south-east of $(i, j)$ (in matrix notation). Observe that $(\mathbf{B}_\varepsilon)^{\mathrm{SE}}_{1,1}$ is the sum of all entries of $\mathbf{B}_\varepsilon$, and thus equals the $\varepsilon$-Baxter-Cambrian number $B_\varepsilon$. Using Figure 13, we obtain a similar rule to compute the entries of $B^{\mathrm{SE}}_\varepsilon$ as sums of entries of $B^{\mathrm{SE}}_{\varepsilon'}$ when $\varepsilon$ is obtained by adding a sign at the end of $\varepsilon'$. This rule is presented in Figure 14.



$\varepsilon_n = \varepsilon_{n-1}$ $\qquad$ $\varepsilon_n = -\varepsilon_{n-1}$

FIGURE 14. Inductive computation of $\mathbf{B}^{\mathrm{SE}}_\varepsilon$: the black entry of $\mathbf{B}^{\mathrm{SE}}_\varepsilon$ is the sum of the entries of $\mathbf{B}^{\mathrm{SE}}_{\varepsilon'}$ over the shaded region. Entries outside the triangular shape always vanish. When $\varepsilon_n = -\varepsilon_{n-1}$, the only non-vanishing entries of $\mathbf{B}^{\mathrm{SE}}_\varepsilon$ are in the first row or in the first column.

This matrix interpretation of the formulas of Proposition 35 provides us with tools to bound the Baxter-Cambrian numbers. For a signature $\varepsilon$, we denote by $\mathsf{switch}(\varepsilon)$ the set of gaps where $\varepsilon$ switches sign.

**Proposition 38.** *For any two signatures $\varepsilon, \tilde\varepsilon \in \pm^n$, if $\mathsf{switch}(\varepsilon) \subset \mathsf{switch}(\tilde\varepsilon)$ then $B_\varepsilon > B_{\tilde\varepsilon}$.*

*Proof.* For two matrices $M := (m_{i,j})$ and $\tilde{M} := (\tilde{m}_{i,j})$, we write $M \succcurlyeq \tilde{M}$ when $m_{i,j} \geq \tilde{m}_{i,j}$ for all indices $i, j$ (entrywise comparison), and we write $M \succ \tilde{M}$ when $M \succcurlyeq \tilde{M}$ and $M \neq \tilde{M}$. Consider four signatures $\varepsilon, \tilde{\varepsilon} \in \pm^n$ and $\varepsilon', \tilde{\varepsilon}' \in \pm^{n-1}$ such that $\varepsilon'$ (resp. $\tilde{\varepsilon}'$) is obtained by deleting the last sign of $\varepsilon$ (resp. $\tilde{\varepsilon}$). From Figure 14, and using the fact that $\mathbf{B}_\varepsilon$ is symmetric, we obtain that:

- if $\varepsilon_n = \varepsilon_{n-1}$ while $\tilde{\varepsilon}_n = -\tilde{\varepsilon}_{n-1}$, then $\mathbf{B}^{\mathrm{SE}}_{\varepsilon'} \succcurlyeq \mathbf{B}^{\mathrm{SE}}_{\tilde{\varepsilon}'}$ implies $\mathbf{B}^{\mathrm{SE}}_{\varepsilon} \succ \mathbf{B}^{\mathrm{SE}}_{\tilde{\varepsilon}}$.
- if either both $\varepsilon_n = \varepsilon_{n-1}$ and $\tilde{\varepsilon}_n = \tilde{\varepsilon}_{n-1}$, or both $\varepsilon_n = -\varepsilon_{n-1}$ and $\tilde{\varepsilon}_n = -\tilde{\varepsilon}_{n-1}$, then $\mathbf{B}^{\mathrm{SE}}_{\varepsilon'} \succ \mathbf{B}^{\mathrm{SE}}_{\tilde{\varepsilon}'}$ implies $\mathbf{B}^{\mathrm{SE}}_{\varepsilon} \succ \mathbf{B}^{\mathrm{SE}}_{\tilde{\varepsilon}}$.

By repeated applications of these observations, we therefore obtain that $\mathsf{switch}(\varepsilon) \subset \mathsf{switch}(\tilde{\varepsilon})$ implies $\mathbf{B}^{\mathrm{SE}}_{\varepsilon} \succ \mathbf{B}^{\mathrm{SE}}_{\tilde{\varepsilon}}$, and thus $B_\varepsilon > B_{\tilde{\varepsilon}}$. □

**Corollary 39.** *Among all signatures of $\pm^n$, the constant signature maximizes the Baxter-Cambrian number, while the alternating signature minimizes it: for all $\varepsilon \in \pm^n$,*

$$\binom{2n-2}{n-1} = B_{(+-)^{\frac{n}{2}}} \leq B_\varepsilon \leq B_{(+)^n} = \binom{n+1}{1}^{-1}\binom{n+1}{2}^{-1}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k-1}.$$

**Remark 40.** The proof of Proposition 38 may seem unnecessarily intricate. Observe however that the situation is rather subtle:

- If $\mathsf{switch}(\varepsilon) \not\subseteq \mathsf{switch}(\tilde{\varepsilon})$, we may have $B_\varepsilon < B_{\tilde{\varepsilon}}$ even if $|\mathsf{switch}(\varepsilon)| < |\mathsf{switch}(\tilde{\varepsilon})|$. The smallest example is given by $B_{+++-++---} = 18376 < 18544 = B_{++-+++-++}$.
- We may have $\mathbf{B}^{\mathrm{SE}}_{\varepsilon} \succcurlyeq \mathbf{B}^{\mathrm{SE}}_{\tilde{\varepsilon}}$ but $\mathbf{B}_\varepsilon \not\succcurlyeq \mathbf{B}_{\tilde{\varepsilon}}$.

2.1.6. **Geometric realizations.** Using similar tools as in Section 1.1.6 and following [LR12], we present geometric realizations for pairs of twin Cambrian trees, for the Baxter-Cambrian lattice, and for the map $\mathbf{ct}^{\mathbb{1}}$. For a partial order $\prec$ on $[n]$, we still define its *incidence cone* $\mathrm{C}(\prec)$ and its *braid cone* $\mathrm{C}^\diamond(\prec)$ as

$$\mathrm{C}(\prec) := \mathrm{cone}\,\{e_i - e_j \mid \text{for all } i \prec j\} \quad \text{and} \quad \mathrm{C}^\diamond(\prec) := \{\mathbf{x} \in \mathbb{H} \mid x_i \leq x_j \text{ for all } i \prec j\}.$$

The cones $\mathrm{C}(\mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet)$ for all pairs $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ of twin $\varepsilon$-Cambrian trees form (together with all their faces) a complete polyhedral fan that we call the *$\varepsilon$-Baxter-Cambrian fan*. It is the common refinement of the $\varepsilon$- and $(-\varepsilon)$-Cambrian fans. It is therefore the normal fan of the Minkowski sum of the associahedra $\mathsf{Asso}(\varepsilon)$ and $\mathsf{Asso}(-\varepsilon)$. We call this polytope *Baxter-Cambrian associahedron* and denote it by $\mathsf{BaxAsso}(\varepsilon)$. Note that $\mathsf{BaxAsso}(\varepsilon)$ is clearly centrally symmetric (since $\mathsf{Asso}(\varepsilon) = -\mathsf{Asso}(-\varepsilon)$) but not necessarily simple. Examples are illustrated on Figure 15. The graph of $\mathsf{BaxAsso}(\varepsilon)$, oriented in the direction $(n, \dots, 1) - (1, \dots, n) = \sum_{i \in [n]} (n + 1 - 2i)\, e_i$, is the Hasse diagram of the $\varepsilon$-Baxter-Cambrian lattice. Finally, the map $\mathbf{ct}^{\mathbb{1}}$ can be read geometrically as

$$[\mathrm{T}_\circ, \mathrm{T}_\bullet] = \mathbf{ct}^{\mathbb{1}}(\tau) \iff \mathrm{C}(\mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet) \supseteq \mathrm{C}^\diamond(\tau).$$

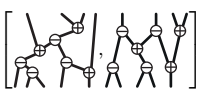
## 2.2. Baxter-Cambrian Hopf Algebra

In this section, we define the Baxter-Cambrian Hopf algebra $\mathsf{BaxCamb}$, extending simultaneously the Cambrian Hopf algebra and the Baxter Hopf algebra studied by S. Law and N. Reading [LR12] and S. Giraudo [Gir12]. We present again the construction of $\mathsf{BaxCamb}$ as a subalgebra of $\mathsf{FQSym}_\pm$ and that of its dual $\mathsf{BaxCamb}^*$ as a quotient of $\mathsf{FQSym}^*_\pm$.

2.2.1. **Subalgebra of $\mathsf{FQSym}_\pm$.** We denote by $\mathsf{BaxCamb}$ the vector subspace of $\mathsf{FQSym}_\pm$ generated by the elements

$$\mathbb{P}_{[\mathrm{T}_\circ, \mathrm{T}_\bullet]} := \sum_{\substack{\tau \in \mathfrak{S}_\pm \\ \mathbf{ct}^{\mathbb{1}}(\tau) = [\mathrm{T}_\circ, \mathrm{T}_\bullet]}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet)} \mathbb{F}_\tau,$$

for all pairs of twin Cambrian trees $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ (recall that $\{\tau \in \mathfrak{S}_\pm \mid \mathbf{ct}^{\mathbb{1}}(\tau) = \mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet\} = \mathcal{L}(\mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet)$ is the set of linear extensions of $\mathrm{T}_\circ \mathbin{\rotatebox{90}{$\leq$}} \mathrm{T}_\bullet$). For example, for the pair of twin Cambrian trees of Figure 10 (left), we have

$\mathbb{P}$  $= \mathbb{F}_{\underline{2}1\overline{7}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}1\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{5}1\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}1\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{2}\underline{5}1\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{5}\underline{2}1\overline{3}\underline{4}\overline{6}}.$

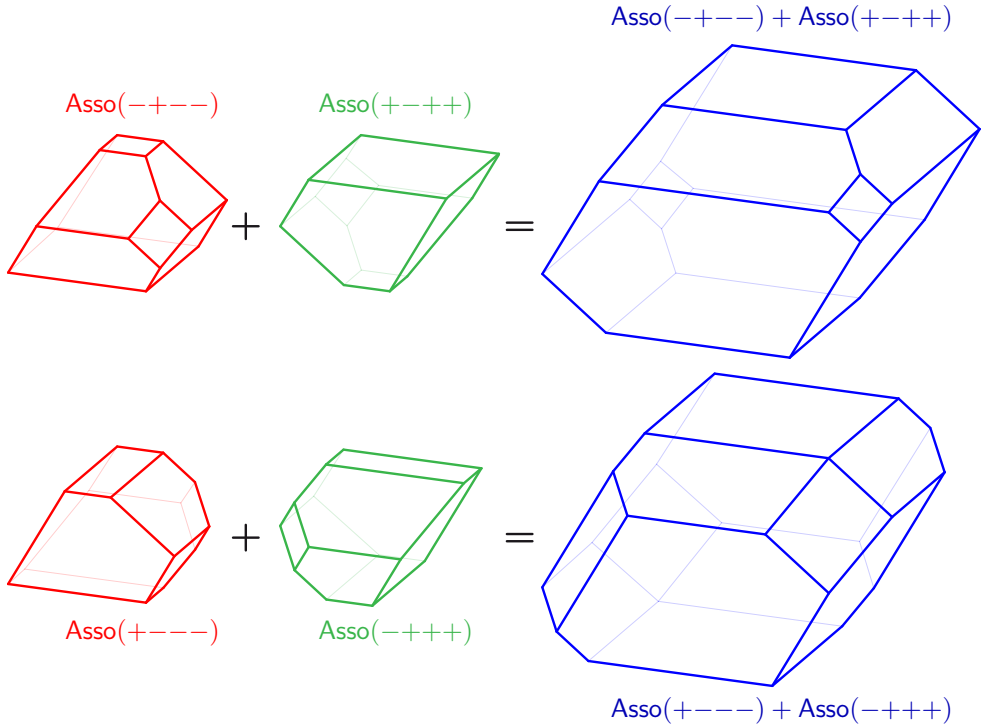



FIGURE 15. The Minkowski sum (blue, right) of the associahedra $\mathsf{Asso}(\varepsilon)$ (red, left) and $\mathsf{Asso}(-\varepsilon)$ (green, middle) gives a realization of the $\varepsilon$-Baxter-Cambrian lattice. Illustrated with the signatures $\varepsilon = -+--$ (top) and $\varepsilon = +---$ (bottom) whose $\varepsilon$-Baxter-Cambrian lattice appear in Figure 11.

The following statement is similar to Theorem 16.

**Theorem 41.** $\mathsf{BaxCamb}$ *is a Hopf subalgebra of* $\mathsf{FQSym}_{\pm}$.

As for the Cambrian algebra, we can describe combinatorially the product and coproduct of $\mathbb{P}$-basis elements of $\mathsf{BaxCamb}$ in terms of operations on pairs of twin Cambrian trees.

PRODUCT The product in the Baxter-Cambrian algebra $\mathsf{BaxCamb}$ can be described in terms of intervals in Baxter-Cambrian lattices.

**Proposition 42.** *For any two pairs* $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ *and* $[\mathrm{T}'_\circ, \mathrm{T}'_\bullet]$ *of twin Cambrian trees, the product* $\mathbb{P}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]} \cdot \mathbb{P}_{[\mathrm{T}'_\circ,\mathrm{T}'_\bullet]}$ *is given by*

$$\mathbb{P}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]} \cdot \mathbb{P}_{[\mathrm{T}'_\circ,\mathrm{T}'_\bullet]} = \sum_{[\mathrm{S}_\circ,\mathrm{S}_\bullet]} \mathbb{P}_{[\mathrm{S}_\circ,\mathrm{S}_\bullet]},$$

*where* $[\mathrm{S}_\circ, \mathrm{S}_\bullet]$ *runs over the interval between* $\big[\mathrm{T}_\circ \nearrow \bar{\mathrm{T}}'_\circ, \mathrm{T}_\bullet \nwarrow \bar{\mathrm{T}}'_\bullet\big]$ *and* $\big[\mathrm{T}_\circ \nwarrow \bar{\mathrm{T}}'_\circ, \mathrm{T}_\bullet \nearrow \bar{\mathrm{T}}'_\bullet\big]$ *in the* $\varepsilon(\mathrm{T}_\circ)\varepsilon(\mathrm{T}'_\circ)$*-Baxter-Cambrian lattice.*

*Proof.* The result relies on the fact that the $\varepsilon$-Baxter-Cambrian classes are intervals of the weak order on $\mathfrak{S}^\varepsilon$, and that the shuffle product of two intervals of the weak order is again an interval of the weak order. See the similar proof of Proposition 17. □

COPRODUCT A *cut* of a pair of twin Cambrian trees $[\mathrm{S}_\circ, \mathrm{S}_\bullet]$ is a pair $\gamma = [\gamma_\circ, \gamma_\bullet]$ where $\gamma_\circ$ is a cut of $\mathrm{S}_\circ$ and $\gamma_\bullet$ is a cut of $\mathrm{S}_\bullet$ such that the labels of $\mathrm{S}_\circ$ below $\gamma_\circ$ coincide with the labels of $\mathrm{S}_\bullet$ above $\gamma_\bullet$. Equivalently, it can be seen as a lower set of $\mathrm{T}_\circ \rotatebox{90}{$\leftrightarrows$} \mathrm{T}_\bullet$. An example is illustrated in Figure 16.

We denote by $AB([\mathrm{S}_\circ, \mathrm{S}_\bullet], [\gamma_\circ, \gamma_\bullet])$ the set of pairs $[A_\circ, B_\bullet]$, where $A_\circ$ appears in the product $\prod_{\mathrm{T} \in A(\mathrm{S}_\circ)} \mathbb{P}_\mathrm{T}$ while $B_\bullet$ appears in the product $\prod_{\mathrm{T} \in B(\mathrm{S}_\circ)} \mathbb{P}_\mathrm{T}$, and $A_\circ$ and $B_\bullet$ are twin Cambrian trees. We define $BA([\mathrm{S}_\circ, \mathrm{S}_\bullet], [\gamma_\circ, \gamma_\bullet])$ similarly exchanging the role of $A$ and $B$. We obtain the following description of the coproduct in the Baxter-Cambrian algebra $\mathsf{BaxCamb}$.

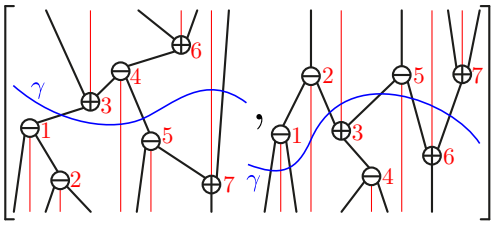



FIGURE 16. A cut $\gamma$ of a pair of twin Cambrian trees.

**Proposition 43.** *For any pair of twin Cambrian trees* $[\mathrm{S}_\circ, \mathrm{S}_\bullet]$*, the coproduct* $\triangle\mathbb{P}_{[\mathrm{S}_\circ,\mathrm{S}_\bullet]}$ *is given by*

$$\triangle\mathbb{P}_{[\mathrm{S}_\circ,\mathrm{S}_\bullet]} = \sum_{\gamma} \Big( \sum_{[B_\circ,A_\bullet]} \mathbb{P}_{[B_\circ,A_\bullet]} \Big) \otimes \Big( \sum_{[A_\circ,B_\bullet]} \mathbb{P}_{[A_\circ,B_\bullet]} \Big),$$

*where* $\gamma$ *runs over all cuts of* $[\mathrm{S}_\circ, \mathrm{S}_\bullet]$*,* $[B_\circ, A_\bullet]$ *runs over* $BA([\mathrm{S}_\circ, \mathrm{S}_\bullet], [\gamma_\circ, \gamma_\bullet])$ *and* $[A_\circ, B_\bullet]$ *runs over* $AB([\mathrm{S}_\circ, \mathrm{S}_\bullet], [\gamma_\circ, \gamma_\bullet])$*.*

*Proof.* The proof is similar to that of Proposition 18. The difficulty here is to describe the linear extensions of the union of the forest $A(\mathrm{S}_\circ, \gamma_\circ)$ with the opposite of the forest $B(\mathrm{S}_\bullet, \gamma_\bullet)$. This difficulty is hidden in the definition of $AB([\mathrm{S}_\circ, \mathrm{S}_\bullet], [\gamma_\circ, \gamma_\bullet])$. □

2.2.2. **Quotient algebra of $\mathsf{FQSym}^*_\pm$.** As for the Cambrian algebra, the following result is automatic from Theorem 41.

**Theorem 44.** *The graded dual* $\mathsf{BaxCamb}^*$ *of the Baxter-Cambrian algebra is isomorphic to the image of* $\mathsf{FQSym}^*_\pm$ *under the canonical projection*

$$\pi : \mathbb{C}\langle A\rangle \longrightarrow \mathbb{C}\langle A\rangle / \equiv^{\mathbb{1}},$$

*where* $\equiv^{\mathbb{1}}$ *denotes the Baxter-Cambrian congruence. The dual basis* $\mathbb{Q}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]}$ *of* $\mathbb{P}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]}$ *is expressed as* $\mathbb{Q}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]} = \pi(\mathbb{G}_\tau)$*, where* $\tau$ *is any linear extension of* $\mathrm{T}_\circ \mathbb{1} \mathrm{T}_\bullet$*.*

We now describe the product and coproduct in $\mathsf{BaxCamb}^*$ by combinatorial operations on pairs of twin Cambrian trees. We use the definitions and notations introduced in Section 1.2.3.

PRODUCT The product in $\mathsf{BaxCamb}^*$ can be described using gaps and laminations similarly to Proposition 21. An example is illustrated on Figure 17. For two Cambrian trees T and T′ and a shuffle $s$ of the signatures $\varepsilon(\mathrm{T})$ and $\varepsilon(\mathrm{T}')$, we still denote by $\mathrm{T}\,{}_s\!\backslash\mathrm{T}'$ the tree described in Section 1.2.3.

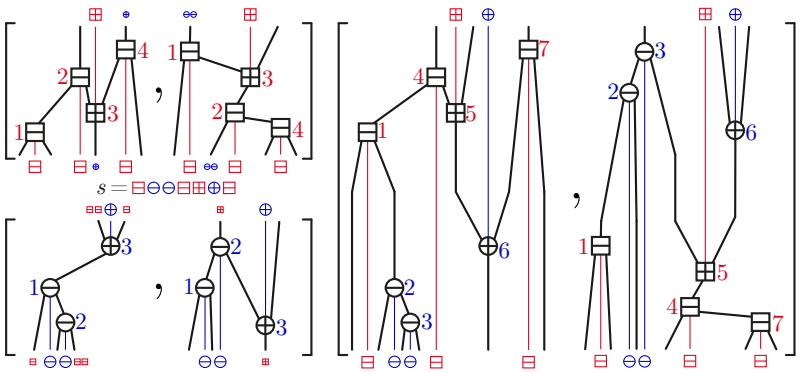



FIGURE 17. Two pairs of twin Cambrian trees $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ and $[\mathrm{T}'_\circ, \mathrm{T}'_\bullet]$ (left), and a pair of twin Cambrian tree which appear in the product $\mathbb{Q}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]} \cdot \mathbb{Q}_{[\mathrm{T}'_\circ,\mathrm{T}'_\bullet]}$ (right).

**Proposition 45.** *For any two pairs of twin Cambrian trees* $[\mathrm{T}_\circ, \mathrm{T}_\bullet]$ *and* $[\mathrm{T}'_\circ, \mathrm{T}'_\bullet]$*, the product* $\mathbb{Q}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]} \cdot \mathbb{Q}_{[\mathrm{T}'_\circ,\mathrm{T}'_\bullet]}$ *is given by*

$$\mathbb{Q}_{[\mathrm{T}_\circ,\mathrm{T}_\bullet]} \cdot \mathbb{Q}_{[\mathrm{T}'_\circ,\mathrm{T}'_\bullet]} = \sum_{s} \mathbb{Q}_{[\mathrm{T}_\circ\,{}_s\!\backslash\mathrm{T}'_\circ,\mathrm{T}'_\bullet\,{}_s\!\backslash\mathrm{T}_\bullet]},$$

*where* $s$ *runs over all shuffles of the signatures* $\varepsilon(\mathrm{T}_\circ) = \varepsilon(\mathrm{T}_\bullet)$ *and* $\varepsilon(\mathrm{T}'_\circ) = \varepsilon(\mathrm{T}'_\bullet)$*.*

*Proof.* The proof follows the same lines as that of Proposition 21. The only difference is that if $\tau \in \mathcal{L}(\mathrm{T}_\circ \between \mathrm{T}_\bullet)$, $\tau' \in \mathcal{L}(\mathrm{T}'_\circ \between \mathrm{T}'_\bullet)$, and $\sigma \in \tau \star \tau'$, then $\mathrm{T}_\circ = \mathbf{ct}(\tau)$ appears below $\mathrm{T}'_\circ = \mathbf{ct}(\tau')$ in $\mathbf{ct}(\sigma)$ since $\sigma$ is inserted from left to right in $\mathbf{ct}(\sigma)$, while $\mathrm{T}_\bullet = \mathbf{ct}(\overleftarrow{\tau})$ appears above $\mathrm{T}'_\bullet = \mathbf{ct}(\overleftarrow{\tau'})$ in $\mathbf{ct}(\overleftarrow{\sigma})$ since $\sigma$ is inserted from right to left in $\mathbf{ct}(\overleftarrow{\sigma})$. □

$\boxed{\text{Coproduct}}$ The coproduct in $\mathsf{BaxCamb}^*$ can be described combinatorially as in Proposition 22. For a Cambrian tree S and a gap $\gamma$ between two consecutive vertices of S, we still denote by $L(\mathrm{S}, \gamma)$ and $R(\mathrm{S}, \gamma)$ the left and right Cambrian subtrees of S when split along the path $\lambda(\mathrm{S}, \gamma)$.

**Proposition 46.** *For any pair of twin Cambrian trees* $[\mathrm{S}_\circ, \mathrm{S}_\bullet]$*, the coproduct* $\triangle\mathbb{Q}_{[\mathrm{S}_\circ,\mathrm{S}_\bullet]}$ *is given by*

$$\triangle\mathbb{Q}_{[\mathrm{S}_\circ,\mathrm{S}_\bullet]} = \sum_\gamma \mathbb{Q}_{[L(\mathrm{S}_\circ,\gamma),L(\mathrm{S}_\bullet,\gamma)]} \otimes \mathbb{Q}_{[R(\mathrm{S}_\circ,\gamma),R(\mathrm{S}_\bullet,\gamma)]},$$

*where* $\gamma$ *runs over all gaps between consecutive positions in* $[n]$*.*

*Proof.* The proof is identical to that of Proposition 22. □

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