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# THE INVOLUTIVE QUANTALOID OF COMPLETELY DISTRIBUTIVE LATTICES

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ABSTRACT. Let  $L$  be a complete lattice and let  $\mathcal{Q}(L)$  be the unital quantale of join-continuous endo-functions of  $L$ . We prove that  $\mathcal{Q}(L)$  has at most two cyclic elements, and that if it has a non-trivial cyclic element, then  $L$  is completely distributive and  $\mathcal{Q}(L)$  is involutive (that is, non-commutative cyclic  $\star$ -autonomous). If this is the case, then the dual tensor operation corresponds, via Raney's transforms, to composition in the (dual) quantale of meet-continuous endo-functions of  $L$ .

Let  $\mathbf{Latt}_\vee$  be the category of sup-lattices and join-continuous functions and let  $\mathbf{Latt}_\vee^{\text{cd}}$  be the full subcategory of  $\mathbf{Latt}_\vee$  whose objects are the completely distributive lattices. We argue that  $\mathbf{Latt}_\vee^{\text{cd}}$  is itself an involutive quantaloid, thus it is the largest full-subcategory of  $\mathbf{Latt}_\vee$  with this property. Since  $\mathbf{Latt}_\vee^{\text{cd}}$  is closed under the monoidal operations of  $\mathbf{Latt}_\vee$ , we also argue that if  $\mathcal{Q}(L)$  is involutive, then  $\mathcal{Q}(L)$  is completely distributive as well; consequently, any lattice embedding into an involutive quantale of the form  $\mathcal{Q}(L)$  has, as its domain, a distributive lattice.

## 1. INTRODUCTION

Let  $C$  be a finite chain or the unit interval of the reals. In a series of recent works [6, 22, 21] we argued that the unital quantale structure of  $\mathcal{Q}(C)$ , the set of join-continuous functions from  $C$  to itself, plays a fundamental role to solve more complex combinatorial and geometrical problems arising in Computer Science. In [6, 22] we formulated an order theoretic approach to the problem of constructing discrete approximations of curves in higher dimensional unit cubes. On the side of combinatorics, the results in [21] yield bijective proofs of counting results (that is, bijections, through which these results can easily be established) for idempotent monotone endo-functions of a finite chain [9, 12] and a new algebraic interpretation of well-known combinatorial identities [3].

The quantales  $\mathcal{Q}(C)$ ,  $C$  a finite chain or  $[0, 1] \subseteq \mathbb{R}$ , are *involutive*—or, using another possible naming, *non-commutative cyclic  $\star$ -autonomous*. The involution is used in the mentioned works, even if is not clear to what extent it is necessary. It was left open in these works whether there are other complete chains  $C$  such that  $\mathcal{Q}(C)$  is involutive and, at its inception, the aim of this research was to answer this

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question. Let us use  $\mathcal{Q}(L)$  for the unital quantale of join-continuous endo-functions of a complete lattice  $L$ . Recalling that involutive quantale structures on a given quantale are determined by the cyclic dualizing elements and that complete chains are completely distributive, the following statement from the monograph [4] shows that  $\mathcal{Q}(C)$  is involutive for each complete chain  $C$ .

**Proposition 2.6.18 in [4].** *Let  $L$  be a complete lattice and let  $o_L$  be the join-continuous self-mapping on  $L$  defined by  $o_L(x) := \bigvee_{x \not\leq z} z$ , for  $x \in L$ . Then the following assertions are equivalent: (i)  $o_L$  is a dualizing element of the quantale  $\mathcal{Q}(L)$ , (ii)  $L$  is completely distributive.*

This proposition also covers another important example studied in the literature. Let  $\mathcal{D}(P)$  be the perfect completely distributive lattice of downsets of a poset  $P$ . According to the proposition,  $\mathcal{Q}(\mathcal{D}(P))$  is involutive, a fact that can also be inferred via the isomorphism with the residuated lattice of weakening relations on  $P$ , known to be involutive, see [19, 13, 10].

We strengthen here the above statement in many ways. Firstly, we observe that the quantale  $\mathcal{Q}(L)$  has at most two cyclic elements and that cyclicity of  $o_L$  is almost sufficient for  $\mathcal{Q}(L)$  to be involutive:

**Theorem.** *If  $c \in \mathcal{Q}(L)$  is cyclic, then either  $c$  is the top element of  $\mathcal{Q}(L)$  or  $c = o_L$ . Moreover, if  $o_L$  is cyclic and not equal to the top element of  $\mathcal{Q}(L)$ , then  $L$  is completely distributive (and therefore  $\mathcal{Q}(L)$  is involutive, as from Proposition 2.6.18 of [4]).*

An important consequence of the previous statement is that the quantale  $\mathcal{Q}(L)$  can be made into an involutive quantale in a unique way:

**Theorem.** *If the quantale  $\mathcal{Q}(L)$  is involutive, then its dualizing cyclic element is the join-continuous function  $o_L$ .*

In the direction from  $L$  to  $\mathcal{Q}(L)$ , we observe that the local involutive quantale structures on each completely distributive lattice fit together in a uniform way. A quantaloid is a category whose homsets are complete lattices and for which composition distributes on both sides with suprema. As a quantale can be considered as a one-object quantaloid, the notion of involutive quantale naturally lifts to the multi-object context—so an involutive quantale is a one-object involutive quantaloid. *Involutive quantaloids* are indeed the Girard quantaloids introduced in [19]. The following statement, proved in this paper, makes precise the intuition that the local involutive quantale structures are uniform:

**Theorem.** *The full subcategory of the category of complete lattices and join-continuous functions whose objects are the completely distributive lattices is an involutive quantaloid.*

The tools used in this paper rely on and emphasize Raney’s characterization of completely distributive lattices [15, 16]. A main remark that we develop is that if  $\mathcal{Q}(L)$  is involutive, then the dual quantale structure of  $\mathcal{Q}(L)$  arises from  $\mathcal{Q}(L^\partial)$ , the quantale of meet-continuous endo-functions of  $L$ , via Raney’s transforms (to be studied in Section 5).

Overall, this set of results yields an important clarification of the algebra used in our previous works [6, 22, 21] and, more importantly, new characterizations of

completely distributive lattices adding up to the existing ones, see e.g. [15, 16, 11, 7, 23]. These characterizations strongly rely on the algebra of quantales and residuated lattices thus on relation algebra, in a wider sense.

An ideal goal of future research is to characterize the equational theory of the involutive residuated lattices of the form  $\mathcal{Q}(L)$ . For the moment being, we observe that the units of the involutive quantale  $\mathcal{Q}(L)$ ,  $L$  a completely distributive lattice, may be used to characterize properties of  $L$ :

**Theorem.** *A complete lattice is a chain if and only if the inclusion  $0 \leq 1$  (in the language of involutive residuated lattices) holds in  $\mathcal{Q}(L)$ , i.e. if and only if  $\mathcal{Q}(L)$  satisfies the mix law. A completely distributive lattice has no completely join-prime elements if and only if the inclusion  $1 \leq 0$  holds in  $\mathcal{Q}(L)$ .*

It is known that the full subcategory of the category of complete lattices and join-continuous functions whose objects are the completely distributive lattices, the involutive quantaloid of completely distributive lattice, is closed under the monoidal operations inherited from the super category, see e.g. [20, 7, 4]. In particular, this quantaloid is itself a  $\star$ -autonomous category. For the sake of studying the equational theory of the  $\mathcal{Q}(L)$ , this fact and the previous results jointly yield the following remarkable consequence:

**Corollary.** *If  $\mathcal{Q}(L)$  is an involutive quantale, then it is completely distributive.*

On the side of logic, it is worth observing that enforcing a linear negation (the involution, the star) on the most typical models of intuitionistic non-commutative linear logic also enforces a classical behaviour, distributivity, of the additive logical connectors. Apart from the philosophical questions about logic, the above corollary pinpoints an important obstacle in finding Cayley style representation theorems for involutive residuated lattices or a generalization of Holland's theorem [8] from lattice-ordered groups to involutive residuated lattices:

**Corollary.** *If a residuated lattice embedding of  $\mathcal{Q}$  into some involutive residuated lattice of the form  $\mathcal{Q}(L)$  exists, then  $\mathcal{Q}$  is distributive.*

Finally, we observe that these mathematical results pinpoint the importance and the naturalness of considering a linear logic based on a distributive setting. This algebraic setting has already many established facets and applications. Among them, let us mention bunched implication logic, for which our last theorem provides non-standard pointless models. Let us also mention the usage of this algebra in pointfree topology: here embeddability problems for quantales dual to topological groupoids, problems analogous to the ones we are raising, have already been investigated in depth, see e.g. [14].

The paper is organised as follows. In Section 2 we provide definitions and elementary results. In Section 3 we introduce the notion of an involutive quantaloid (we shall identify an involutive quantale with a one-object involutive quantaloid). We prove in Section 4 that if a quantale of the form  $\mathcal{Q}(L)$  is involutive, then it has just one cyclic dualizing element. That is, there can be at most one involutive quantale structure extending the structure of  $\mathcal{Q}(L)$ . Moreover, we prove in this section that if  $\mathcal{Q}(L)$  has a non-trivial cyclic element, then  $L$  is a completely distributive lattice. The uniqueness of the involutive structure is intimately related to the fact—analyzed at the end of Section 4—that the only central elements of  $\mathcal{Q}(L)$  are

the identity and the constant function mapping to the bottom of  $L$ . In Section 5 we introduce Raney's transforms and their elementary properties. Raney's transforms are the main tool used to prove, in Section 6, that completely distributive lattices form an involutive quantaloid. In Section 7 we develop some considerations on the equational theories of the lattices  $\mathcal{Q}(L)$  among which, the use of the multiplicative units of  $\mathcal{Q}(L)$  to characterize properties of  $L$  and the fact that  $\mathcal{Q}(L)$  is completely distributive whenever it is  $\mathcal{Q}(L)$  involutive.

## 2. DEFINITIONS AND ELEMENTARY RESULTS

Complete lattices and the category  $\mathbf{Latt}_\vee$ . A *complete lattice* is a poset  $L$  such that each  $X \subseteq L$  has a supremum  $\bigvee X$ . A map  $f : L \rightarrow M$  is *join-continuous* if  $f(\bigvee X) = \bigvee f(X)$ , for each subset  $X \subseteq L$ . We shall denote by  $\mathbf{Latt}_\vee$  the category whose objects are the complete lattices and whose morphisms are the join-continuous maps.

For a poset  $P$ ,  $P^\partial$  denotes the poset with the same elements of  $P$  but with the reverse ordering:  $x \leq_{P^\partial} y$  iff  $y \leq_P x$ . In a complete lattice, the set  $\bigvee\{y \mid y \leq x, \text{ for each } x \in X\}$  is the infimum of  $X$ . Therefore, if  $L$  is complete, then  $L^\partial$  is also a complete lattice. Moreover, if  $L, M$  are complete lattices and  $f : L \rightarrow M$  is join-continuous, then the map  $\rho(f) : M \rightarrow L$ , defined by  $\rho(f)(y) := \bigvee\{x \in L \mid f(x) \leq y\}$ , preserves infima and therefore it belongs to the homset  $\mathbf{Latt}_\vee(M^\partial, L^\partial)$ . The map  $\rho(f)$  is the *right adjoint* of  $f$ , meaning that, for each  $x \in L$  and  $y \in M$ ,  $f(x) \leq y$  if and only if  $x \leq \rho(f)(y)$ . For  $g : M \rightarrow L$  meet-continuous, its *left adjoint*  $\ell(g) : L \rightarrow M$  is defined similarly, and satisfies  $\ell(g)(x) \leq y$  if and only if  $x \leq g(y)$ , for each  $x \in L$  and  $y \in M$ . Consequently,  $\ell(\rho(f)) = f$  and  $\rho(\ell(g)) = g$ . Indeed, by defining with  $f^\partial := \rho(f)$ ,  $(\cdot)^\partial : \mathbf{Latt}_\vee \rightarrow \mathbf{Latt}_\vee^{op}$  is a (contravariant) functor and a category isomorphism.

Let  $\{f_i \mid i \in I\}$  be a family of join-continuous functions from  $L$  to  $M$ . The function  $\bigvee_{i \in I} f_i$ , defined by  $(\bigvee_{i \in I} f_i)(x) := \bigvee_{i \in I} f_i(x)$ , is a join-continuous map from  $L$  to  $M$ . Therefore the homset  $\mathbf{Latt}_\vee(L, M)$ , with the pointwise ordering, is a complete lattice, where suprema are computed by the above formula. The same formula shows that the inclusion of  $\mathbf{Latt}_\vee(L, M)$  into  $M^L$ , the set of all functions from  $L$  to  $M$ , is join-continuous. It follows that, for every  $f : L \rightarrow M$ , there is a (uniquely determined) greatest join-continuous function  $h \in \mathbf{Latt}_\vee(L, M)$  such that  $h \leq f$ ; in the following we shall use  $\mathbf{int}(f)$  to denote such  $h$ . Observe also that, by monotonicity of composition,  $\mathbf{int}(g) \circ \mathbf{int}(f) \leq g \circ f$  and therefore  $\mathbf{int}(g) \circ \mathbf{int}(f) \leq \mathbf{int}(g \circ f)$ .

Quantales and involutive quantales. A *quantale* is a complete lattice  $\mathcal{Q}$  coming with a semigroup operation  $\circ$  that distributes with arbitrary sups. That is, we have  $(\bigvee X) \circ (\bigvee Y) = \bigvee_{x \in X, y \in Y} x \circ y$ , for each  $X, Y \subseteq \mathcal{Q}$ . A quantale is unital if the semigroup operation has a unit. As we shall always consider unital quantales, we shall use the wording quantale as a synonym of unital quantale. In a quantale  $\mathcal{Q}$ , left and right residuals are defined as follows:  $x \backslash y := \bigvee\{z \in \mathcal{Q} \mid x \circ z \leq y\}$  and  $y / x := \bigvee\{z \in \mathcal{Q} \mid z \circ x \leq y\}$ . Clearly, we have the following adjointness relations:  $x \circ y \leq z$  iff  $y \leq x \backslash z$  iff  $x \leq z / y$ . Let us recall that a quantale  $\mathcal{Q}$  is a *residuated lattice*, that is an algebra on the signature  $\wedge, \vee, 1, \circ, \backslash, /$ , satisfying a finite identities, see e.g. [5, §2.2].

A standard example of quantale is  $\mathcal{Q}(L)$ , the set of join-continuous endo-functions of a complete lattice  $L$ . In this case, the semigroup operation is function composition; otherwise said,  $\mathcal{Q}(L)$  is the homset  $\mathbf{Latt}_V(L, L)$ . We shall consider special elements of  $\mathcal{Q}(L)$  and of  $\mathcal{Q}(L^\partial)$ . For  $x \in L$ , let  $c_x, a_x, \alpha_x : L \longrightarrow L$  be defined as follows:

$$c_x(t) := \begin{cases} x, & t \neq \perp, \\ \perp, & t = \perp, \end{cases} \quad a_x(t) := \begin{cases} \top, & t \not\leq x, \\ \perp, & t \leq x, \end{cases} \quad \alpha_x(t) := \begin{cases} \top, & x \leq t, \\ \perp, & x \not\leq t. \end{cases} \quad (1)$$

Clearly,  $c_x, a_x \in \mathcal{Q}(L)$  while  $\alpha_x \in \mathcal{Q}(L^\partial)$ . Moreover, we have  $\rho(c_x) = \alpha_x$ . Completely distributive lattices. A complete lattice  $L$  is said to be *completely distributive* if, for each pair of families  $\pi : J \longrightarrow I$  and  $x : J \longrightarrow L$ , the following equality holds

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_j = \bigvee_{\psi} \bigwedge_{i \in I} x_{\psi(i)},$$

where  $J_i = \pi^{-1}(i)$ , for each  $i \in I$ , and the meet on the right is over all sections  $\psi$  of  $\pi$ , that is, those functions such that  $\pi \circ \psi = id_I$ . Let us recall that the notion of a completely distributive lattice is auto-dual, meaning that a complete lattice  $L$  is completely distributive iff  $L^\partial$  is such. For each complete lattice  $L$ , define

$$o_L(x) := \bigvee \{ t \mid x \not\leq t \}, \quad \omega_L(y) := \bigwedge \{ t \mid t \not\leq y \}. \quad (2)$$

It is easy to see that  $o_L \in \mathcal{Q}(L)$  and that  $\rho(o_L) = \omega_L$ . The following statement appears in [16, Theorem 4]:

**Theorem 1** (Raney). *A lattice is completely distributive if and only if any of the following equivalent conditions hold:*

$$\bigvee_{x \not\leq t} \omega_L(t) = x, \quad \bigwedge_{t \not\leq y} o_L(t) = y. \quad (3)$$

### 3. INVOLUTIVE QUANTALOIDS

The purpose of this section is to define *involutive quantaloids* which, not surprisingly, turn out to be the *Girard quantaloids* of [19]. Let us mention that, following [1, 2] and [6, 22], another possible naming for the same concept is *non-commutative, cyclic, star-autonomous quantaloid*. For the sake of conciseness, we prefer the wording involutive quantaloid.

We recall that a *quantaloid*, see e.g. [23], is a category  $\mathcal{Q}$  enriched over the category of sup-lattices. This means that, for each pair of objects  $L, M$  of  $\mathcal{Q}$ , the homset  $\mathcal{Q}(L, M)$  is a complete lattice and that composition distributes over suprema in both variables,  $(\bigvee_{i \in I} g_i) \circ (\bigvee_{j \in J} f_j) = \bigvee_{i \in I, j \in J} f_i \circ g_j$ . A quantale, see e.g. [18], might be seen as a one-object quantaloid. The category  $\mathbf{Latt}_V$  is itself a quantaloid. The definition below mimics, in a multisorted setting, a possible definition of involutive quantale or of involutive residuated lattice. For the possible equivalent definitions of these notions, see e.g. [2] or [5, §3.3].

**Definition 1.** An *involutive quantaloid* is a quantaloid  $\mathcal{Q}$  coming with operations

$$(\cdot)^{\star_{L,M}} : \mathcal{Q}(L, M) \longrightarrow \mathcal{Q}(M, L), \quad L, M \text{ objects of } \mathcal{Q},$$

satisfying the following conditions:

$$(1) \quad (f^{\star_{L,M}})^{\star_{M,L}} = f, \text{ for each } f \in \mathcal{Q}(L, M),$$

(2) for each  $f, g \in \mathcal{Q}(L, M)$ ,

$$f \leq g \quad \text{iff} \quad f \circ g^{*L,M} \leq 0_M \quad \text{iff} \quad g^{*L,M} \circ f \leq 0_L,$$

where  $0_M := (id_M)^{*M,M}$  and  $0_L := (id_L)^{*L,L}$ .

An *involutive quantale* is a one-object involutive quantaloid.

The superscripts  $L$  and  $M$  in  $(\cdot)^{*L,M}$  shall be omitted if they are clear from the context. We state next elementary facts without proofs, the reader shall have no difficulty providing them. For a category  $\mathcal{C}$  enriched over posets, we use  $\mathcal{C}^{co}$  for the category with same objects and homsets, but for which the order is reversed.

**Lemma 1.** *In an involutive quantaloid  $\mathcal{Q}$ , if any of the inequalities below holds, then so do the other two:*

$$\begin{array}{ccc} L & \xrightarrow{h} & N \\ & \searrow f \quad \vee \quad \nearrow g & \\ & M & \end{array} \quad \begin{array}{ccc} N & \xrightarrow{g^*} & M \\ & \searrow h^* \quad \vee \quad \nearrow f & \\ & L & \end{array} \quad \begin{array}{ccc} M & \xrightarrow{f^*} & L \\ & \searrow g \quad \vee \quad \nearrow h^* & \\ & N & \end{array}$$

In particular, the operations  $\star$  are order reversing, so  $\star$  is the arrow part of a functor  $\mathcal{Q} \longrightarrow (\mathcal{Q}^{op})^{co}$  which is the identity on objects.

Let us recall that in any quantaloid residuals exist being defined as follows: for  $f : L \longrightarrow M$ ,  $g : M \longrightarrow N$ , and  $h : L \longrightarrow N$ ,

$$g \backslash h : L \longrightarrow M := \bigvee \{ k \mid g \circ k \leq h \}, \quad h / f : M \longrightarrow N := \bigvee \{ k \mid k \circ f \leq h \},$$

so, the usual adjointness relations hold:  $g \circ f \leq h$  iff  $f \leq g \backslash h$  iff  $g \leq h / f$ .

**Lemma 2.** *In an involutive quantaloid, for  $f : L \longrightarrow M$ ,  $g : M \longrightarrow N$ , and  $h : L \longrightarrow N$ , we have the following equalities:*

$$g \backslash h = (h^{*L,N} \circ g)^{*M,L}, \quad h / f = (f \circ h^{*L,N})^{*N,M}.$$

In particular (for  $L = N$  and  $h = 0_L$ ) we have  $g \backslash 0_L = g^{*M,L}$  and  $0_L / f = f^{*L,M}$ .

Let us argue that our definition coincides with the definition of a Girard quantaloid given in [19]. It is readily seen that, given an involutive quantaloid  $\mathcal{Q}$ , the collection  $\{0_L = id_L^* \mid L \text{ an object of } \mathcal{Q}\}$  is a cyclic dualizing family in the sense of [19]. Conversely, given such a family and  $f : L \longrightarrow M$ , we can define  $f^* := f \backslash 0_M$  and this definition yields an involutive quantaloid structure as defined here. This definition also sets a bijective correspondence between the two kind of structures.

#### 4. CYCLIC ELEMENTS OF $\mathcal{Q}(L)$

We prove in this section that if a quantale of the form  $\mathcal{Q}(L)$  is involutive, then  $id_L^*$  equals  $o_L$  defined in equation (2). From this it follows that there is at most one involutive quantale structure on  $\mathcal{Q}(L)$  extending the quantale structure. Moreover, we also prove that if  $o_L$  is cyclic and distinct from  $c_\top$ , then  $L$  is completely distributive. To this end, let us firstly recall the following standard definitions:

**Definition 2.** Let  $\mathcal{Q}$  be a quantale. An element  $\alpha \in \mathcal{Q}$  is said to be

- *cyclic* if  $f \backslash \alpha = \alpha / f$ , for each  $f \in \mathcal{Q}$ ,
- *dualizing* if  $(\alpha / f) \backslash \alpha = \alpha / (f \backslash \alpha) = f$ , for each  $f \in \mathcal{Q}$ .

We already mentioned that involutive quantale structures on a quantale  $\mathcal{Q}$  are in bijection with cyclic dualizing elements of  $\mathcal{Q}$ . Let us also recall that, for an involutive quantaloid  $\mathcal{Q}$  and an object  $L$  of  $\mathcal{Q}$ ,  $0_L := (id_L)^{\star_{L,L}}$  is both a cyclic and a dualizing element of the quantale  $\mathcal{Q}(L, L)$ .

An important first observation, stated in the next lemma, is that residuals of the form  $g \backslash h$  in  $\mathbf{Latt}_V$  can be constructed by means of the operations  $\mathbf{int}(\cdot)$  (greatest join-continuous map below a given one) and  $\rho(\cdot)$  (taking the right adjoint of a join-continuous map).

**Lemma 3.** *For each  $g \in \mathbf{Latt}_V(M, N)$ ,  $h \in \mathbf{Latt}_V(L, N)$ , we have*

$$g \backslash h = \mathbf{int}(\rho(g) \circ h).$$

*Proof.* Indeed, for each  $f \in \mathbf{Latt}_V(L, M)$ , we have  $f \leq g \backslash h$  iff  $g \circ f \leq h$ , iff  $g(f(x)) \leq h(x)$ , for each  $x \in L$ , iff  $f(x) \leq \rho(g)(h(x))$ , for each  $x \in L$ , iff  $f \leq \rho(g) \circ h$ , iff  $f \leq \mathbf{int}(\rho(g) \circ h)$ .  $\square$

For the next lemma, recall that the join-continuous map  $o_L$  has been defined in (2) and that the maps  $c_t$  and  $a_t$  have been defined in (1).

**Lemma 4.** *We have  $o_L = \bigvee_{t \in L} c_t \circ a_t$ .*

*Proof.* Observe that  $c_t(a_t(x)) = \perp$ , if  $x \leq t$ , and  $c_t(a_t(x)) = t$ , if  $x \not\leq t$ . Therefore

$$\left(\bigvee_{t \in L} c_t \circ a_t\right)(x) = \bigvee_{t \in L} (c_t(a_t(x))) = \bigvee_{\substack{t \in L, \\ c_t(a_t(x)) \neq \perp}} c_t(a_t(x)) = \bigvee_{\substack{t \in L, \\ x \not\leq t}} t = o_L(x). \quad \square$$

**Lemma 5.** *For each  $x \in L$ ,  $\mathbf{int}(\alpha_x) = a_{o_L(x)}$ .*

*Proof.* Let us observe that  $a_{o(x)} \leq \alpha_x$ . This amounts to verifying that if  $\alpha_x(t) = \perp$ , then  $a_{o(x)}(t) = \perp$ . Now,  $\alpha_x(t) = \perp$  iff  $x \not\leq t$ , and so  $t \leq o(x)$ , thus  $a_{o(x)}(t) = \perp$ . Next, let us suppose that  $f : L \rightarrow L$  is join-continuous and below  $\alpha_x$ . Thus, if  $\alpha_x(t) = \perp$ , that is, if  $x \not\leq t$ , then  $f(t) = \perp$ . Then  $f(o(x)) = f(\bigvee_{x \not\leq t} t) = \bigvee_{x \not\leq t} f(t) = \perp$ . By monotonicity of  $f$ , if  $t \leq o(x)$ , then  $f(t) = \perp$ , showing that  $f \leq a_{o(x)}$ .  $\square$

**Theorem 2.** *For each complete lattice  $L$ , the quantale  $\mathcal{Q}(L)$  has at most two cyclic elements, among  $c_\top$  and  $o_L$ .*

*Proof.* Now, let  $h \in \mathcal{Q}(L)$  be cyclic. First we prove that  $o_L \leq h$ . Consider that, for each  $x \in L$ ,  $a_x \circ c_x = c_\perp \leq h$ . Thus, since  $g \circ f \leq h$  if and only if  $f \circ g \leq h$ , we also have  $c_x \circ a_x \leq h$ . Since this relation holds for each  $x \in L$ , then, using Lemma 4, the relation  $o_L = \bigvee_{x \in L} c_x \circ a_x \leq h$  holds.

We argue now that if  $h \neq c_\top$ , then  $h \leq o_L$  and therefore  $h = o_L$ . Let  $x \in L$  and consider that  $c_x \circ c_x \backslash h \leq h$ . By cyclicity, we also have  $c_x \backslash h \circ c_x \leq h$ .

Now,  $c_x \backslash h = \mathbf{int}(\rho(c_x) \circ h) = \mathbf{int}(\alpha_x \circ h)$  and therefore, using Lemma 5,

$$a_{o_L(x)} \circ h \circ c_x = \mathbf{int}(\alpha_x) \circ \mathbf{int}(h) \circ c_x \leq \mathbf{int}(\alpha_x \circ h) \circ c_x = c_x \backslash h \circ c_x \leq h.$$

If  $t \neq \perp$ , then, by evaluating the above inequality at  $t$ , we get  $a_{o_L(x)}(h(x)) \leq h(t)$ . Since  $a_{o_L(x)}(h(x))$  takes values  $\perp$  and  $\top$ , this means that  $a_{o_L(x)}(h(x)) = \top$  implies  $\top \leq h(t)$ , for all  $t \neq \perp$ . That is, if  $h(x) \not\leq o_L(x)$ , then  $h(t) = \top$ , for all  $t \neq \perp$  and  $x \in L$ . Otherwise stated, if  $h \not\leq o_L$ , then  $h = c_\top$ .  $\square$



Let us recall that a nucleus on a quantale  $\mathcal{Q}$  is a closure operator  $j$  such that  $j(g) \circ j(f) \leq f(g \circ f)$ . Nuclei are sort of congruences in the category of quantales while quotients into some involutive quantale bijectively correspond to nuclei  $j$  of the form  $j(f) = (f \setminus 0) \setminus 0$  where  $0$  is cyclic [17, Theorem 1]. Thus, the above theorem exhibits the quantales  $\mathcal{Q}(L)$  as sort of simple w.r.t. involutive quantales.

**Lemma 6.** *If  $L$  is not trivial, then  $c_\top$  is not a dualizing element of  $\mathcal{Q}(L)$ .*

*Proof.* Observe that  $c_\top$  is the greatest element of  $\mathcal{Q}(L)$  and, for this reason,  $f \setminus c_\top = c_\top / f = c_\top$ , for each  $f \in \mathcal{Q}(L)$ . If  $c_\top$  is dualizing, then  $c_\perp = (c_\top / c_\perp) \setminus c_\top = c_\top$ . Considering that the mapping from sending  $x \in L$  to  $c_x \in \mathcal{Q}(L)$  is an embedding, this shows that  $\perp = \top$  in  $L$ .  $\square$

**Corollary 1.** *If  $h \in \mathcal{Q}(L)$  is a cyclic and dualizing element, then  $h = o_L$ . That is, if  $\mathcal{Q}(L)$  is an involutive quantale, then  $id_L^* = o_L$ .*

*Proof.* If  $L$  is trivial, then so is  $\mathcal{Q}(L)$ , and  $h = c_\perp = o_L$ . If  $L$  is not trivial, then, by Theorem 2,  $h \in \{o_L, c_\top\}$  and, by Lemma 6,  $h \neq c_\top$ .  $\square$

With respect to Theorem 2, we notice that  $c_\top$ , being the top element of  $\mathcal{Q}(L)$ , is always cyclic. It is therefore pertinent to ask when  $o_L$  is cyclic. Of course, this is the case if  $o_L = c_\top$ .

**Theorem 3.** *If  $o_L$  is a cyclic element of  $\mathcal{Q}(L)$  and  $o_L \neq c_\top$ , then  $x = \bigwedge_{t \not\leq x} o_L(t)$ , for each  $x \in L$ . Consequently,  $L$  is a completely distributive lattice.*

*Proof.* Since  $o_L$  is cyclic, then, for each  $x, y \in L$ , the two conditions (a)  $c_y \circ a_x \leq o_L$  and (b)  $a_x \circ c_y \leq o_L$  are equivalent.

Condition (a) states that, for each  $t \in L$ ,  $t \not\leq x$  implies  $y \leq o_L(t)$ ; that is  $y \leq \bigwedge_{t \not\leq x} o_L(t)$ . Condition (b) states that, for each  $t \neq \perp$ , if  $y \not\leq x$  then  $o_L(t) = \top$ . This condition is equivalent to  $y \not\leq x$  implies  $o_L = c_\top$  or, equivalently, to  $o_L \neq c_\top$  implies  $y \leq x$ . Thus we have that, if  $o_L$  is cyclic and  $o_L \neq c_\top$ , then (c) for each  $x, y \in L$ ,  $y \leq x$  iff  $y \leq \bigwedge_{t \not\leq x} o_L(t)$ . Now, condition (c) is easily recognized to be equivalent to the equality  $x = \bigwedge_{t \not\leq x} o_L(t)$ , holding for each  $x \in L$ . From the latter identity, complete distributivity of  $L$  follows using Raney's characterization of complete distributivity, Theorem 1.  $\square$

In this way we also obtain a refinement of one side of the equivalence stated in Proposition 2.6.18 of [4], where we do not need to refer to the cyclic dualizing element.

**Corollary 2.** *If  $\mathcal{Q}(L)$  is an involutive quantale, then  $L$  is a completely distributive lattice.*

*Proof.* If  $\mathcal{Q}(L)$  is an involutive quantale, then its dualizing cyclic element is, necessarily,  $o_L$ . In particular,  $o_L$  is cyclic and distinct from  $c_\top$ . By Theorem 3,  $L$  is completely distributive.  $\square$

We shall see that  $o_L$  is also dualizing if  $L$  is completely distributive. A remarkable fact arising from these considerations is that, on the class of pointed residuated lattices  $\langle \mathcal{Q}(L), p \rangle$  (where  $p \in \mathcal{Q}(L)$  is the point), the universal sentence  $p \neq \top \ \& \ \forall x. x \setminus p = p / x$  implies distributivity as well as the linear double negation principle,  $x = (x \setminus p) \setminus p$ .

4.0.1. *The center of  $\mathcal{Q}(L)$ .* Uniqueness of an involutive quantale structure extending the quantale structure of  $\mathcal{Q}(L)$  can also be achieved through the observation that the unique central elements of  $\mathcal{Q}(L)$  are  $id_L$  and  $c_\perp$ . We are thankful to Claudia Muresan for her help with investigating the center of  $\mathcal{Q}(L)$ .

**Definition 3.** We say that an element  $\beta$  of a quantale  $\mathcal{Q}$  is

- *central* if  $\beta \circ x = x \circ \beta$ , for each  $x \in \mathcal{Q}$ ,
- *codualizing* if  $x = \beta \backslash (\beta \circ x)$ , for each  $x \in \mathcal{Q}$ .

**Lemma 7.** *If  $\mathcal{Q}$  is an involutive quantale, then  $\alpha \in \mathcal{Q}$  is cyclic if and only if  $\alpha^*$  is central and it is dualizing if and only if  $\alpha^*$  is codualizing.*

*Proof.* Since  $x \backslash \alpha = (\alpha^* \circ x)^*$ ,  $\alpha / x = (x \circ \alpha^*)^*$ , and  $(\cdot)^*$  is invertible, the equality  $x \backslash \alpha = \alpha / x$  holds if and only if the equality  $\alpha^* \circ x = x \circ \alpha^*$  holds.

Now  $\alpha$  is dualizing if and only if, for each  $x \in \mathcal{Q}$ ,  $x = \alpha / (x \backslash \alpha) = \alpha^* \backslash (x \backslash \alpha)^* = \alpha^* \backslash (\alpha^* \circ x)$ .  $\square$

**Proposition 1.** *The only central elements of  $\mathcal{Q}(L)$  are  $id_L$  and  $c_\perp$ .*

*Proof.* Clearly,  $id_L$  and  $c_\perp$  are central, so we shall be concerned to prove that they are the only ones with this property. To this end, for  $x_0 \in L$ , define

$$\nu_{x_0}(t) := \begin{cases} \perp, & t \leq x_0 \\ t, & \text{otherwise.} \end{cases}$$

Notice that if  $x_0 = \perp$ , then  $\nu_{x_0} = id_L$ , while if  $x_0 = \top$ , then  $\nu_{x_0} = c_\perp$ . We firstly claim that if  $\beta$  is central in  $\mathcal{Q}(L)$ , then  $\beta = \nu_{x_0}$ , for some  $x_0 \in L$ . Suppose  $\beta$  is central. For each  $x \in L$ , we have  $c_x(x) = x$  and therefore

$$\beta(x) = (\beta \circ c_x)(x) = c_x(\beta(x)).$$

If  $\beta(x) \neq \perp$ , then, evaluating the rightmost expression, we obtain  $\beta(x) = x$ . Let  $x_0 := \bigvee \{y \mid \beta(y) = \perp\}$ , so  $\beta(x_0) = \perp$ . If  $t \leq x_0$ , then  $\beta(t) \leq \beta(x_0) = \perp$  and, otherwise,  $\beta(t) \neq \perp$  and so  $\beta(t) = c_t(\beta(t)) = t$ . Therefore,  $\beta = \nu_{x_0}$ .

Next, we claim that if  $x_0 \notin \{\perp, \top\}$ , then  $\nu_{x_0}$  is not central. Observe that

$$\nu_{x_0}(f(x)) = \begin{cases} \perp, & f(x) \leq x_0, \\ f(x), & \text{otherwise,} \end{cases} \quad f(\nu_{x_0}(x)) = \begin{cases} \perp, & x \leq x_0, \\ f(x), & \text{otherwise.} \end{cases}$$

It follows that if  $\nu_{x_0} \circ f = f \circ \nu_{x_0}$ , then  $f(x_0) \leq x_0$ . Indeed, if  $f(x_0) \not\leq x_0$ , then  $f(x_0) \neq \perp$ ,  $\nu_{x_0}(f(x_0)) = f(x_0) \neq \perp$ , and  $f(\nu_{x_0}(x_0)) = \perp$ . Now, if  $x_0 \notin \{\perp, \top\}$ , then  $c_\top$  is such that  $x_0 < \top = c_\top(x_0)$ , and therefore  $\nu_{x_0} \circ c_\top \neq c_\top \circ \nu_{x_0}$ .  $\square$

It is now possible to argue that, for a complete lattice  $L$ , there exists at most one extension of  $\mathcal{Q}(L)$  to an involutive quantale as follows. Suppose that  $\mathcal{Q}(L)$  is involutive, so let  $(\cdot)^*$  be a fixed involutive quantale structure. We shall argue that  $id_L^*$  is the unique cyclic and dualizing element of  $\mathcal{Q}(L)$ . If  $\alpha$  is an arbitrary cyclic and dualizing element of  $\mathcal{Q}(L)$ , then  $\beta := \alpha^*$  is central and codualizing and  $\beta \in \{c_\perp, id_L\}$  using Proposition 1. Since  $\beta$  is codualizing, then it is an injective function: if  $\beta(x) = \beta(y)$ , then  $\beta \circ c_x = \beta \circ c_y$  and  $c_x = \beta \backslash (\beta \circ c_x) = \beta \backslash (\beta \circ c_y) = c_y$ ; since the mapping sending  $t$  to  $c_t$  is an embedding, we obtain  $x = y$ . Thus, if  $L$  is not trivial,  $\beta \neq c_\perp$  (since  $c_\perp$  is constant). Whether or not  $L$  is trivial, we derive  $\beta = id_L$ . It follows that  $\alpha = \alpha^{**} = \beta^* = id_L^*$ .

## 5. RANEY'S TRANSFORMS

Let  $L, M$  be two complete lattices. For  $f : L \longrightarrow M$ , define

$$f^\vee(x) := \bigvee_{x \not\leq t} f(t), \quad f^\wedge(x) := \bigwedge_{t \not\leq x} f(t), \quad \text{for each } x \in L.$$

We call  $f^\vee$  and  $f^\wedge$  the *Raney's transforms* of  $f$ . Notice that  $f$  is not required to be monotone in order to define  $f^\vee$  or  $f^\wedge$  which, on the other hand, are easily seen to be monotone; these functions are even join and meet-continuous, respectively, as argued in the next lemma.

**Lemma 8.** *For any  $f : L \longrightarrow M$ , define*

$$g_f(y) := \bigwedge \{ z \mid f(z) \not\leq y \}. \quad (4)$$

*Then  $g_f$  is right adjoint to  $f^\vee$  and therefore  $f^\vee$  is join-continuous. Dually,  $f^\wedge$  is meet-continuous.*

We call the operation  $(\cdot)^\vee$  Raney's transform for the following reason. For  $\theta \subseteq L \times M$  an arbitrary relation, Raney [16] defined (up to some dualities)

$$r_\theta(x) := \bigwedge \{ y \in M \mid \forall (t, v). (t, v) \in \theta \text{ implies } x \leq t \text{ or } v \leq y \}. \quad (5)$$

Recall that a left adjoint  $\ell : L \longrightarrow M$  can be expressed from its right adjoint  $\rho : M \longrightarrow L$  by the formula  $\ell(x) = \bigwedge \{ y \mid x \leq \rho(y) \}$ . Using this expression with  $\ell = f^\vee$  and  $\rho = g_f$  defined in (4), we obtain

$$f^\vee(x) = \bigwedge \{ y \in M \mid \forall t. f(t) \not\leq y \text{ implies } x \leq t \}. \quad (6)$$

Clearly, if in (5) we let  $\theta$  be the graph of  $f$ , defined by  $(t, v) \in \theta$  if and only if  $f(t) = v$ , then we obtain equality between the right-hand sides of (5) and (6), and so  $f^\vee = r_\theta$ .

We list next the few properties we need to know about these transforms.

**Lemma 9.** *The transform  $(\cdot)^\vee$  has the following properties:*

- (1) *if  $f \leq g : L \longrightarrow M$ , then  $f^\vee \leq g^\vee$ ,*
- (2) *if  $g : L \longrightarrow M$  and  $f : M \longrightarrow N$  is monotone, then  $(f \circ g)^\vee \leq f \circ (g^\vee)$ ,*
- (3) *if  $g : L \longrightarrow M$  and  $f : M \longrightarrow N$  is join-continuous, then  $(f \circ g)^\vee = f \circ (g^\vee)$ ,*
- (4) *if  $f : L \longrightarrow M$  is join-continuous (with  $L$  and  $M$  complete), then*

$$\ell(f^\wedge) = \rho(f)^\vee : M \longrightarrow L. \quad (7)$$

The proof of these properties does not present difficulties, possibly apart for the last item, for which we refer the reader to [7, Proposition 4.6 (b.iii)].

6.  $\mathbf{Latt}_\vee^{\text{cd}}$  IS AN INVOLUTIVE QUANTALOID

We prove now that  $\mathbf{Latt}_\vee^{\text{cd}}$ , the full subcategory of  $\mathbf{Latt}_\vee$  whose objects are the completely distributive lattices, is an involutive quantaloid. By the results of Section 4, this is also the largest full subcategory of  $\mathbf{Latt}_\vee$  with this property.

Recall from Theorem 1 that a complete lattice is completely distributive if and only if  $\omega_L^\vee = id_L$  (or, equivalently,  $\omega_L^\wedge = id_L$ ).

**Lemma 10.** *If  $L$  is a completely distributive lattice and  $f : L \longrightarrow M$  is monotone, then  $\mathbf{int}(f) = (f \circ \omega_L)^\vee$  and  $f^\vee = \mathbf{int}(f \circ \omega_L)$ .*

*Proof.* By monotonicity of  $f$ , we have  $(f \circ \omega_L)^\vee \leq f \circ (\omega_L^\vee) = f$ . Suppose that  $g$  is join-continuous and  $g \leq f$ . Then  $g = g \circ (\omega_L^\vee) = (g \circ \omega_L)^\vee \leq (f \circ \omega_L)^\vee$ . To see that  $f^\vee = \text{int}(f \circ o_L)$ , observe that  $f^\vee = (f \circ id)^\vee \leq f \circ (id_L^\vee) = f \circ o_L$ , and therefore  $f^\vee \leq \text{int}(f \circ o_L)$ . On the other hand,  $\text{int}(f \circ o_L) = (f \circ o_L \circ \omega_L)^\vee \leq f^\vee$ , using the counit of the adjunction,  $o_L \circ \omega_L \leq id_L$ .  $\square$

The interior operator so defined is quite peculiar, since for  $g : L \longrightarrow M$  monotone and  $f : M \longrightarrow N$  join-continuous, we have

$$\text{int}(f \circ g) = (f \circ g \circ \omega_L)^\vee = f \circ (g \circ \omega_L)^\vee = f \circ \text{int}(g).$$

In general, if  $L$  is not a completely distributive lattice, then we would have, above, only an inequality, since  $\text{int}(f \circ g) \geq \text{int}(f) \circ \text{int}(g) = f \circ \text{int}(g)$ .

**Lemma 11.** *If  $L$  is a completely distributive lattice and  $f : L \longrightarrow M$  is join-continuous, then  $f = f^{\wedge\vee}$ .*

*Proof.* We firstly show that  $f^{\wedge\vee} \leq f$ . If  $x \not\leq t$ , then  $f^\wedge(t) = \bigwedge_{u \not\leq t} f(u) \leq f(x)$  and therefore  $f^{\wedge\vee}(x) = \bigvee_{x \not\leq t} f^\wedge(t) \leq f(x)$ , for all  $x \in L$ . Let us argue that  $f \leq f^{\wedge\vee}$ :

$$f = f \circ id_L = f \circ (\omega_L^\vee) = (f \circ \omega_L)^\vee \leq f^{\wedge\vee},$$

where we have used the fact, dual to the relation  $f^\vee = \text{int}(f \circ o_L)$  established in Lemma 10, that  $f^\wedge$  is the least meet-continuous function above  $f \circ \omega_L$ , so in particular  $f \circ \omega_L \leq f^\wedge$ .  $\square$

For  $f : L \longrightarrow M$  join-continuous, define  $f^{*L,M} : M \longrightarrow L$  as follows:

$$f^{*L,M} := \rho(f)^\vee = \ell(f^\wedge).$$

Let us remark that the mappings  $(\cdot)^*$  so defined are the maps witnessing that completely distributive lattices are nuclear, see [7, Theorem 4.7]. We leave for future research to establish an exact connection between the notions of involutive quantaloid and of nuclear object in an autonomous category.

**Theorem 4.** *The operations  $(\cdot)^{*L,M}$  so defined yield an involutive quantaloid structure on  $\text{Latt}_V^{\text{cd}}$ .*

*Proof.* Firstly, we verify that  $f^{**} = f$  using Lemmas 9 and 11, and the fact that the join-continuous functions are in bijection with meet-continuous functions via taking adjoints:  $f^{**} = \rho(\rho(f)^\vee)^\vee = \rho(\ell(f^\wedge))^\vee = f^{\wedge\vee} = f$ .

We now verify that  $(\cdot)^*$  satisfies the constraints needed to have an involutive quantaloid. Let us remark that  $id_L^* = \rho(id_L)^\vee = id_L^\vee = o_L$ .

Observe that since  $(\cdot)^*$  is defined by composing an order reversing and an order preserving function, it is order reversing. Since it is an involution, then  $f \leq g$  if and only if  $g^* \leq f^*$ .

Now we assume that  $f : L \longrightarrow M$  and  $h : M \longrightarrow L$  and recall (see Lemma 10) that  $h^{*M,L} = \rho(h)^\vee = \text{int}(\rho(h) \circ o_L) : L \longrightarrow M$ . Therefore,  $h \circ f \leq o_L$  if and only if  $f \leq \rho(h) \circ o_L$ , if and only if  $f \leq \text{int}(\rho(h) \circ o_L) = h^*$ . Therefore, if  $g : L \longrightarrow M$ , then (letting  $h = g^*$ )  $f \leq g$  if and only if  $g^* \circ f \leq o_L$ . Then, also,  $f \leq g$  if and only if  $g^* \leq f^*$  if and only if  $f \circ g^* \leq o_M$ .  $\square$

Putting together Theorems 3 and 4, we obtain the following generalization of Proposition 2.6.18 in [4], where no mention of the choice of the cyclic dualizing element is required.

**Corollary 3.** *The quantale  $\mathcal{Q}(L)$  is involutive if and only if  $L$  is a completely distributive lattice.*

For  $f : L \longrightarrow M$  and  $g : M \longrightarrow N$  (with  $L, M, N$  completely distributive lattices), let us define

$$g \oplus f := (f^* \circ g^*)^* : L \longrightarrow M,$$

and observe that

$$(g \oplus f) = (f^* \circ g^*)^* = \rho((\ell(f^\wedge) \circ \ell(g^\wedge)))^\vee = (g^\wedge \circ f^\wedge)^\vee.$$

That is:

**Proposition 2.** *The dual quantaloid structure arises via Raney's transforms from the composition  $\mathbf{Latt}_\vee(L^\partial, M^\partial) \times \mathbf{Latt}_\vee(M^\partial, N^\partial) \longrightarrow \mathbf{Latt}_\vee(L^\partial, N^\partial)$ .*

## 7. REMARKS ON THE EQUATIONAL THEORY OF THE $\mathcal{Q}(L)$

We develop in this section few considerations concerning the equational theory of the involutive residuated lattices  $\mathcal{Q}(L)$ .

**Theorem 5.** *A complete lattice  $L$  is a chain if and only if  $\mathcal{Q}(L)$  is an involutive quantale satisfying the mix rule, i.e. the inclusion  $x \circ y \leq x \oplus y$ .*

*Proof.* It is well known that the mix rule is equivalent to the inclusion  $0 \leq 1$ —where  $1$  is the unit for  $\circ$  and  $0$  is the unit for  $\oplus$ . Therefore, an involutive quantale of the form  $\mathcal{Q}(L)$  satisfies the mix rule if and only if  $o_L \leq id_L$ . This relation is easily seen to be equivalent to the statement that if  $x \not\leq t$ , then  $t \leq x$ , so  $L$  is a chain. For the converse, we just need to recall that every chain is a completely distributive lattice.  $\square$

Let us recall that an element  $x$  of a complete lattice  $L$  is *completely join-prime* if, for every  $Y \subseteq L$ , the relation  $x \leq \bigvee Y$  implies  $x \leq y$  for some  $y \in Y$ . It is not difficult to see that  $x$  is completely join-prime if and only if  $x \not\leq o_L(x)$ . Thus we say that a complete lattice is *smooth* if it has no completely join-prime element. For example, the interval  $[0, 1]$  of the reals is a smooth completely distributive lattice. The following statement is an immediate consequence of these considerations.

**Theorem 6.** *A complete lattice  $L$  is smooth if and only if  $id_L \leq o_L$ . Thus, a completely distributive lattice  $L$  is smooth if and only if  $\mathcal{Q}(L)$  satisfies the inclusion  $1 \leq 0$  in the language of involutive residuated lattices.*

These statements generalize the remarks by Galatos and Jipsen, this collection, on the involutive residuated lattice of weakening relations on  $P$ . Let us recall that this involutive residuated lattice is isomorphic to  $\mathcal{Q}(\mathcal{D}(P))$  where  $\mathcal{D}(P)$  is the collection of downsets of  $P$ . Thus, they observe that  $\mathcal{Q}(\mathcal{D}(P))$  satisfies the mix rule if and only if  $P$  is a chain, and that there are no non-trivial posets  $P$  such that  $\mathcal{Q}(\mathcal{D}(P))$  satisfies the inclusion  $1 \leq 0$ . These facts might be seen as consequences Theorems 5 and 6, considering that  $\mathcal{D}(P)$  is a chain if and only if  $P$  is a chain, and that  $\mathcal{D}(P)$  is spatial, meaning that every element of  $\mathcal{D}(P)$  is the join of the completely join-prime elements below it (so,  $\mathcal{D}(P)$  has plenty of completely join-prime elements).

For a family  $\{f_i \in \mathbf{Latt}_V(L, M) \mid i \in I\}$ , let us define  $\bigwedge_{i \in I} f_i$  and  $\bigwedge_{i \in I} f_i$  by

$$\left(\bigwedge_{i \in I} f_i\right)(x) := \bigwedge_{i \in I} (f_i(x)), \quad \left(\bigwedge_{i \in I} f_i\right)(x) := \bigvee_{x \not\leq t} \bigwedge_{i \in I} f_i(\omega_L(t)). \quad (8)$$

Notice that  $\bigwedge_{i \in I} f_i$  need not be join-continuous while  $\bigwedge_{i \in I} f_i$  is join-continuous and  $\bigwedge_{i \in I} f_i = \mathbf{int}(\bigwedge_{i \in I} f_i)$  if  $L$  is completely distributive, see Lemma 10. Under the latter condition,  $\bigwedge_{i \in I} f_i$  is the infimum of  $\{f_i \mid i \in I\}$  within the complete lattice  $\mathbf{Latt}_V(L, M)$ . The explicit description of the infimum given in (8) can be exploited to prove that  $\mathbf{Latt}_V^{\text{cd}}$  is closed under the monoidal operations inherited from  $\mathbf{Latt}_V$ , see e.g. [20, 7, 4], thus it is  $\star$ -autonomous [1]. We expect the formula in (8) also to be useful for computational issues, see Ramirez et al., this collection.

Coming back to the equational theory of the  $\mathcal{Q}(L)$ , an important consequence of  $\mathbf{Latt}_V^{\text{cd}}$  being  $\star$ -autonomous is that  $\mathcal{Q}(L)$  is completely distributive if  $L$  is completely distributive (the converse holds as well). Then, the following obstacle arises towards finding representation theorems for involutive residuated lattices via the  $\mathcal{Q}(L)$ :

**Corollary 4.** *If an involutive residuated lattice  $\mathcal{Q}$  has an embedding into an involutive quantale of the form  $\mathcal{Q}(L)$ , then  $\mathcal{Q}$  is distributive.*

Indeed, if  $\mathcal{Q}(L)$  is an involutive quantale, then  $L$  is a completely distributive lattice and  $\mathcal{Q}(L)$  as well. Thus, if  $\mathcal{Q}$  has a lattice embedding into  $\mathcal{Q}(L)$ , then  $L$  is distributive.

## 8. CONCLUSIONS AND FUTURE STEPS

The research exposed in this paper tackles and solves a natural problem encountered during our investigations of certain quantales built from complete chains [21, 22, 6]. The problem asks to characterize the complete chains whose quantale of join-continuous endomaps is involutive. Every complete chain is a completely distributive lattice and by now we know that every complete chain has this property; in particular, other properties of chains and posets, such as self-duality, are not relevant.

The solution provided, building on [4, Proposition 2.6.18], is as general as possible, in two respects. On the one hand, an exact characterization of all the complete lattices—not just the chains— $L$  for which  $\mathcal{Q}(L)$  is involutive becomes available: these are the completely distributive lattices; improving on [4, Proposition 2.6.18], we argue that the choice of a cyclic dualizing element does not matter. In particular, the characterization covers different kind of involutive quantales known in the literature, those discovered in our investigation of complete chains and those known as the residuated lattices of weakening relations—arising from the relational semantics of distributive linear logic. On the other hand, we show that the involutive quantale structures on completely distributive lattices are uniform, yielding and involutive quantaloid structure on the category of completely distributive lattices and join-continuous functions.

We have drawn several consequences from the observations developed, among them, the fact that if an involutive quantale  $\mathcal{Q}$  can be embedded into an quantale of the form  $\mathcal{Q}(L)$ , then it is distributive. This fact calls for a characterization of the involutive residuated lattices embeddable into some  $\mathcal{Q}(L)$ , a research track that might require to or end up with determining the variety of involutive residuated lattices generated by the  $\mathcal{Q}(L)$ . A second research goal, that we might tackle in

a close future, demands to investigate the algebra developed in connection with the continuous weak order [22] in the wider and abstract setting of completely distributive lattices. Let us recall that in [22] a surprising bijection was established between two kind of objects, the maximal chains in the cube lattice  $[0, 1]^d$  and the families  $\{f_{i,j} \in \mathcal{Q}([0, 1]) \mid 1 \leq i < j \leq d\}$  such that, for  $i < j < k$ ,  $f_{j,k} \circ f_{i,j} \leq f_{i,k} \leq f_{j,k} \oplus f_{i,j}$ . So, are there other surprising bijections if the interval  $[0, 1]$  is replaced by an arbitrary completely distributive lattice, and if we move from the involutive quantale setting to the multisorted setting of involutive quantaloids?

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