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► **To cite this version:**

Luigi Santocanale. THE INVOLUTIVE QUANTALOID OF COMPLETELY DISTRIBUTIVE LATTICES. 2019. hal-02342655v1

**HAL Id: hal-02342655**

**<https://hal.science/hal-02342655v1>**

Preprint submitted on 1 Nov 2019 (v1), last revised 6 Apr 2020 (v3)

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# THE INVOLUTIVE QUANTALOID OF COMPLETELY DISTRIBUTIVE LATTICES

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ABSTRACT. Let  $L$  be a complete lattice and let  $\mathcal{Q}(L)$  be the unital quantale of join-continuous endo-functions of  $L$ . We prove the following result:  $\mathcal{Q}(L)$  is an involutive (that is, non-commutative cyclic  $\star$ -autonomous) quantale if and only if  $L$  is a completely distributive lattice. If this is the case, then the dual tensor operation corresponds, via Raney's transforms, to composition in the (dual) quantale of meet-continuous endo-functions of  $L$ .

Let  $\mathbf{Latt}_\vee$  be the category of sup-lattices and join-continuous functions and let  $\mathbf{Latt}_\vee^{\text{cd}}$  be the full subcategory of  $\mathbf{Latt}_\vee$  whose objects are the completely distributive lattices. We argue that (i)  $\mathbf{Latt}_\vee^{\text{cd}}$  is itself an involutive quantaloid, and therefore it is the largest full-subcategory of  $\mathbf{Latt}_\vee$  with this property; (ii)  $\mathbf{Latt}_\vee^{\text{cd}}$  is closed under the monoidal operations of  $\mathbf{Latt}_\vee$ , and therefore it is star-autonomous. Consequently, if  $\mathcal{Q}(L)$  is involutive, then it is completely distributive as well.

## 1. INTRODUCTION

Let  $C$  be a finite chain or the unit interval of the reals. In a series of recent works [4, 17, 16] we argued that the unital quantale structure of  $\mathcal{Q}(C)$ , the set of join-continuous functions from  $C$  to itself, plays a fundamental role to solve more complex combinatorial and geometrical problems arising in Computer Science. In [4, 17] we formulated an order theoretic approach to the problem of constructing discrete approximations of curves in higher dimensional unit cubes. On the side of combinatorics, the results in [16] yield bijective proofs for counting idempotent monotone endo-functions of a finite chain [6, 9] and a new algebraic interpretation of well-known combinatorial identities [2].

The quantales  $\mathcal{Q}(C)$ ,  $C$  a finite chain or  $[0, 1] \subseteq \mathbb{R}$ , are *involutive*—or, using another possible naming, *non-commutative cyclic  $\star$ -autonomous*. The involution was indeed used in the mentioned works, yet it was not clear to what extent it was necessary. Also, it was left open whether there are other chains  $C$  such that  $\mathcal{Q}(C)$  is involutive. At its inception, the aim of this reserach was to answer this question. Also, letting  $\mathcal{D}(P)$  be the perfect distributive lattice of downsets of a poset  $P$ ,  $\mathcal{Q}(\mathcal{D}(P))$  is involutive. In this case,  $\mathcal{Q}(\mathcal{D}(P))$  is isomorphic the residuated lattice of weakening relations on  $P$ , known to be involutive, see [15, 10, 7]. Relying

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Partially supported by the “LIA LYSM AMU CNRS ECM INdAM”, by the “LIA LIRCO”, and by the ANR project TICAMORE.

on our earlier and more recent works on involutive quantale structures, we have been able to find the common generalization responsible for a quantale  $\mathcal{Q}(L)$  to be involutive. Let us recall that chains and perfect distributive lattices are *completely distributive*. We prove in this paper the following statements:

**Theorem.** *The quantale  $\mathcal{Q}(L)$  of join-continuous endo-functions of  $L$  is involutive if and only if  $L$  is a completely distributive lattice.*

We also observe that if  $\mathcal{Q}(L)$  is involutive, then it is involutive in a unique way. If  $\mathcal{Q}(L)$  is involutive, then the dual quantale structure of  $\mathcal{Q}(L)$  arises from  $\mathcal{Q}(L^\partial)$ , the quantale of meet-continuous endo-functions of  $L$ , via Raney’s transforms (to be studied in Section 5).

**Theorem.** *A complete lattice is a chain if and only if the inclusion  $0 \leq 1$  (in the language of involutive residuated lattices) holds in  $\mathcal{Q}(L)$ , i.e. if and only if  $\mathcal{Q}(L)$  satisfies the mix law. A completely distributive lattice has no completely join-prime elements if and only if the inclusion  $1 \leq 0$  holds in  $\mathcal{Q}(L)$ .*

We observe that the local involutive quantale structures on each completely distributive lattice fit together in a uniform way. A quantaloid is a category whose homsets are complete lattices and for which composition distributes on both sides with suprema. As a quantale can be considered as a one-object quantaloid, the notion of involutive quantale naturally lifts to the multi-object context—so an involutive quantale is a one-object involutive quantaloid. The following statement, proved in this paper, makes precise the intuition that the local involutive quantaloid structures are uniform:

**Theorem.** *The full subcategory of the category of complete lattices and join-continuous functions whose objects are the completely distributive lattices is an involutive quantaloid.*

Finally, the tools we develop also yield the following result:

**Theorem.** *The quantaloid of completely distributive lattices is closed under the monoidal operations inherited from the category of complete lattices and join-continuous functions. Therefore, it is a  $\star$ -autonomous category.*

From the above statement it is also inferred that  $\mathcal{Q}(L)$  is completely distributive whenever it is an involutive quantale.

These results give an important clarification of the algebra used in our previous works [4, 17, 16] and open new perspectives and research directions on the algebra of chains and of completely distributive lattices. In particular, the first theorem yields a new characterization of completely distributive lattices that adds up to the existing ones, see e.g. [11, 18]. On the side of logic, it is worth observing that enforcing a linear negation (the involution, the star) on the most typical models of intuitionistic non-commutative linear logic also enforces a classical behaviour—that is, distributivity—of the additive logical connectors. Besides the philosophical questions about logic, the above theorems pinpoint an important obstacle in finding Cayley style representation theorems for involutive residuated lattices or a generalization of Holland’s theorem [5] from lattice-ordered groups to involutive residuated lattices: if a residuated lattice embedding of  $\mathcal{Q}$  into some involutive residuated lattice of the form  $\mathcal{Q}(L)$  exists, then  $\mathcal{Q}$  is distributive.

The paper is organised as follows. We provide in Section 2 definitions and elementary results. In Section 3 we introduce the notion of an involutive quantaloid (we shall identify an involutive quantale with a one-object involutive quantaloid). We prove in Section 4 that if a quantale of the form  $\mathcal{Q}(L)$  is involutive, then it has just one cyclic dualizing element. That is, there can be at most one involutive quantale structure extending the structure of  $\mathcal{Q}(L)$ . Moreover, we prove in this section that if  $\mathcal{Q}(L)$  is involutive, then  $L$  is a completely distributive lattice. The uniqueness of the involutive structure is intimately related to the fact—analyzed at the end of Section 4—that the only central elements of  $\mathcal{Q}(L)$  are the identity and the constant function with value the bottom of  $L$ . We introduce in Section 5 Raney’s transforms and their elementary properties. Raney’s transforms are the main tool used to prove, in Section 6, that completely distributive lattices form an involutive quantaloid. In Section 7 we prove that the full subcategory of  $\mathbf{Latt}_\vee$ —the category of complete lattices and join-continuous functions—whose objects are the completely distributive lattices is closed under the monoidal operations of  $\mathbf{Latt}_\vee$ . Consequently, it is a  $\star$ -autonomous category. In particular, this implies that  $\mathcal{Q}(L)$  is a completely distributive lattice if  $\mathcal{Q}(L)$  involutive, that is, if  $L$  is completely distributive.

## 2. DEFINITIONS AND ELEMENTARY RESULTS

Complete lattices and the category  $\mathbf{Latt}_\vee$ . A *complete lattice* is a poset  $L$  such that  $X \subseteq L$  has a supremum  $\bigvee X$ . A map  $f : L \rightarrow M$  is join-continuous if  $f(\bigvee X) = \bigvee f(X)$ , for each subset  $X \subseteq L$ . We shall denote by  $\mathbf{Latt}_\vee$  the category whose objects are the complete lattices and whose morphisms are the join-continuous maps.

For a poset  $P$ ,  $P^\partial$  denotes the poset with the same elements of  $P$  but with the reverse ordering:  $x \leq_{P^\partial} y$  iff  $y \leq_P x$ . In a complete lattice, the set  $\bigvee\{y \mid y \leq x, \text{ for each } x \in X\}$  is the infimum of  $X$ . Therefore, if  $L$  is complete, then  $L^\partial$  is also a complete lattice. Moreover, if  $L, M$  are complete lattices and  $f : L \rightarrow M$  is join-continuous, then the map  $\rho(f) : M \rightarrow L$ , defined by  $\rho(f)(y) := \bigvee\{x \in L \mid f(x) \leq y\}$ , preserves infima and therefore it belongs to the homset  $\mathbf{Latt}_\vee(M^\partial, L^\partial)$ . The map  $\rho(f)$  is the *right adjoint* of  $f$ , meaning that, for each  $x \in L$  and  $y \in M$ ,  $f(x) \leq y$  if and only if  $x \leq \rho(f)(y)$ . For  $g : M \rightarrow L$  meet-continuous, its *left adjoint*  $\ell(g) : L \rightarrow M$  is defined similarly, and satisfies  $\ell(g)(x) \leq y$  if and only if  $x \leq g(y)$ , for each  $x \in L$  and  $y \in M$ . Consequently,  $\ell(\rho(f)) = f$  and  $\rho(\ell(g)) = g$ . Indeed, by defining with  $f^\partial := \rho(f)$ ,  $(\cdot)^\partial : \mathbf{Latt}_\vee \rightarrow \mathbf{Latt}_\vee^{op}$  is a (contravariant) functor and a category isomorphism.

Let  $\{f_i \mid i \in I\}$  be a family of join-continuous functions from  $L$  to  $M$ . The function  $\bigvee_{i \in I} f_i$ , defined by

$$\left(\bigvee_{i \in I} f_i\right)(x) := \bigvee_{i \in I} f_i(x)$$

is a join-continuous map from  $L$  to  $M$ . Therefore the homset  $\mathbf{Latt}_\vee(L, M)$ , with the pointwise ordering, is a complete lattice, where suprema are computed by the above formula. The same formula shows that the inclusion of  $\mathbf{Latt}_\vee(L, M)$  into  $M^L$ , the set of all functions from  $L$  to  $M$ , is join-continuous. It follows that, for every  $f : L \rightarrow M$ , there is a (uniquely determined) greatest join-continuous function

$\text{int}(f) \in \text{Latt}_{\vee}(L, M)$  such that  $\text{int}(f) \leq f$ . Observe also that, by monotonicity of composition,  $\text{int}(g) \circ \text{int}(f) \leq g \circ f$  and therefore  $\text{int}(g) \circ \text{int}(f) \leq \text{int}(g \circ f)$ . Quantales and involutive quantales. A quantale is a complete lattice  $\mathcal{Q}$  coming with a semigroup operation  $\circ$  that distributes with arbitrary sups. That is, we have  $(\bigvee X) \circ (\bigvee Y) = \bigvee_{x \in X, y \in Y} x \circ y$ , for each  $X, Y \subseteq \mathcal{Q}$ . A quantale is unital if the semigroup operation has a unit. As we shall always consider unital quantales, we shall use the wording quantale as a synonym of unital quantale. In a quantale  $\mathcal{Q}$ , left and right divisions are defined as follows:

$$x \backslash y := \bigvee \{ z \in \mathcal{Q} \mid x \circ z \leq y \}, \quad y / x := \bigvee \{ z \in \mathcal{Q} \mid z \circ x \leq y \}.$$

Clearly, we have the following adjointness relations:  $x \circ y \leq z$  iff  $y \leq x \backslash z$  iff  $x \leq z / y$ . Let us recall that a quantale  $\mathcal{Q}$  is a residuated lattice, as defined for example in [3].

A standard example of quantale is  $\mathcal{Q}(L)$ , the set of join-continuous endo-functions of a complete lattice  $L$ . In this case, the semigroup operation is function composition; otherwise said,  $\mathcal{Q}(L)$  is the homset  $\text{Latt}_{\vee}(L, L)$ . We shall consider special elements of  $\mathcal{Q}(L)$  and of  $\mathcal{Q}(L^{op})$ . For  $x \in L$ , let  $c_x, \gamma_x, a_x, \alpha_x : L \rightarrow L$  be defined as follows:

$$(1) \quad \begin{aligned} c_x(t) &:= \begin{cases} x, & t = \top, \\ \perp, & t = \perp, \end{cases} & \gamma_x(t) &:= \begin{cases} \top & t = \top, \\ x, & t = \perp, \end{cases} \\ a_x(t) &:= \begin{cases} \top, & t \not\leq x, \\ \perp, & t \leq x, \end{cases} & \alpha_x(t) &:= \begin{cases} \top, & x \leq t, \\ \perp, & x \not\leq t. \end{cases} \end{aligned}$$

Clearly,  $c_x, a_x \in \mathcal{Q}(L)$ , while  $\gamma_x, \alpha_x \in \mathcal{Q}(L^{op})$ . Moreover, we have  $\rho(c_x) = \alpha_x$  and  $\rho(a_x) = \gamma_x$ .

Completely distributive lattices. A complete lattice  $L$  is said to be *completely distributive* if, for each pair of families  $\pi : J \rightarrow I$  and  $x : J \rightarrow L$ , the following equality holds

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_j = \bigvee_{\psi} \bigwedge_{i \in I} x_{\psi(i)},$$

where  $J_i = \pi^{-1}(i)$ , for each  $i \in I$ , and the meet on the right is over all sections  $\psi$  of  $\pi$ , that is, those functions such that  $\pi \circ \psi = id_I$ . Let us recall that the notion of a completely distributive lattice is auto-dual, meaning that a complete lattice  $L$  is completely distributive iff  $L^{op}$  is such. For each complete lattice  $L$ , define

$$(2) \quad o_L(x) := \bigvee \{ t \mid x \not\leq t \}, \quad \omega_L(y) := \bigwedge \{ t \mid t \not\leq y \}.$$

It is easy to see that  $o_L \in \mathcal{Q}(L)$  and that  $\rho(o_L) = \omega_L$ . The following statement appears in [12, Theorem 4]:

**Theorem 1.** *A lattice is completely distributive if and only if any of the following equivalent conditions hold:*

$$(3) \quad \bigvee_{x \not\leq t} \omega_L(t) = x, \quad \bigwedge_{t \not\leq y} o_L(t) = y.$$

### 3. INVOLUTIVE QUANTALOIDS

We recall that a *quantaloid*, see e.g. [18], is a category  $\mathcal{Q}$  enriched over the category of sup-lattices. This means that, for each pair of objects  $L, M$  of  $\mathcal{Q}$ , the homset  $\mathcal{Q}(L, M)$  is a complete lattice and that composition distributes over

suprema in both variables,  $(\bigvee_{i \in I} g_i) \circ (\bigvee_{j \in J} f_j) = \bigvee_{i \in I, j \in J} f_i \circ g_j$ . A quantale, see e.g. [13], might be seen as a one-object quantaloid. The category  $\text{Latt}_{\vee}$  is itself a quantaloid.

We define now involutive quantaloids and state elementary facts.

**Definition 1.** An *involutive quantaloid*<sup>1</sup> is a quantaloid  $\mathcal{Q}$  coming with operations

$$(\cdot)^{\star L, M} : \mathcal{Q}(L, M) \longrightarrow \mathcal{Q}(M, L), \quad L, M \text{ objects of } \mathcal{Q},$$

satisfying the following conditions:

- (1)  $(f^{\star L, M})^{\star M, L} = f$ , for each  $f \in \mathcal{Q}(L, M)$ ,
- (2) for each  $f, g \in \mathcal{Q}(L, M)$ ,

$$f \leq g \quad \text{iff} \quad f \circ g^{\star L, M} \leq 0_M \quad \text{iff} \quad g^{\star L, M} \circ f \leq 0_L,$$

$$\text{where } 0_M := (id_M)^{\star M, M} \text{ and } 0_L := (id_L)^{\star L, L}.$$

The definition mimics in a multisorted setting the definition of an involutive quantale (and that of an involutive residuated lattice). Indeed, we shall consider an involutive quantale as an involutive quantaloid with just one object.

If the superscripts in  $(\cdot)^{\star L, M}$  might be inferred from the context, then we shall avoid writing them.

For a category  $\mathcal{C}$  enriched over posets, we use  $\mathcal{C}^{co}$  for the category with same objects and homsets, but for which the order is reversed.

**Lemma 1.** *In an involutive quantaloid, the operations  $\star$  are order reversing. Thus,  $\star$  is the arrow part of a functor  $\mathcal{Q} \longrightarrow (\mathcal{Q}^{op})^{co}$  which is the identity on objects.*

*Proof.* We have  $f \leq g$  iff  $g^{\star} \circ f^{\star\star} = g^{\star} \circ f \leq 0$  iff  $g^{\star} \leq f^{\star}$ . □

**Lemma 2.** *In an involutive quantaloid, if any of the inequalities below holds, then so do the other two:*

$$\begin{array}{ccc} L \xrightarrow{h} N & N \xrightarrow{g^{\star}} M & M \xrightarrow{f^{\star}} L \\ \begin{array}{c} \searrow f \\ \downarrow \vee \\ \nearrow g \end{array} & \begin{array}{c} \searrow h^{\star} \\ \downarrow \vee \\ \nearrow f \end{array} & \begin{array}{c} \searrow g \\ \downarrow \vee \\ \nearrow h^{\star} \end{array} \\ & M & L & N \end{array}$$

*Proof.* Suppose  $g \circ f \leq h$ , then  $g \circ (f \circ h^{\star}) = (g \circ f) \circ h^{\star} \leq 0_N$ , and so  $f \circ h^{\star} \leq g^{\star}$ .

Using the same argument, from  $f \circ h^{\star} \leq g^{\star}$ , it follows  $h^{\star} \circ g \leq f^{\star}$ . □

Let us recall that in any quantaloid residuals (i.e. division operators) exist being defined as follows: for  $f : L \longrightarrow M$ ,  $g : M \longrightarrow N$ , and  $h : L \longrightarrow N$ ,

$$g \backslash h : L \longrightarrow M := \bigvee \{ k \mid g \circ k \leq h \}, \quad h / f : M \longrightarrow N := \bigvee \{ k \mid k \circ f \leq h \},$$

so, the usual adjointness relations hold:

$$g \circ f \leq h \quad \text{iff} \quad f \leq g \backslash h \quad \text{iff} \quad g \leq h / f.$$

**Lemma 3.** *In an involutive quantaloid, for  $f : L \longrightarrow M$ ,  $g : M \longrightarrow N$ , and  $h : L \longrightarrow N$ , we have the following equalities:*

$$g \backslash h = (h^{\star L, N} \circ g)^{\star M, L}, \quad h / f = (f \circ h^{\star L, N})^{\star N, M}.$$

*In particular (for  $L = N$  and  $h = 0_L$ ) we have  $g \backslash 0_L = g^{\star M, L}$  and  $0_L / f = f^{\star L, M}$ .*

<sup>1</sup>Another possible naming for the same concept is *non-commutative, cyclic, star-autonomous quantaloid*. However, for the sake of conciseness, we prefer the wording involutive quantaloid. Also, in [14], involutive quantaloids are called *Girard quantaloids*.

*Proof.* We use uniqueness of adjoints and the previous lemmas. We have  $g \leq h/f$  iff  $g \circ f \leq h$  iff  $f \circ h^* \leq g^*$  iff  $g = g^{**} \leq (f \circ h^*)^*$ .

Similarly,  $f \leq g \backslash h$  iff  $g \circ f \leq h$  iff  $h^* \circ g \leq f^*$  iff  $f = f^{**} \leq (h^* \circ g)^*$ .  $\square$

#### 4. CYCLIC DUALIZING ELEMENTS OF $\mathcal{Q}(L)$ AND COMPLETE DISTRIBUTIVITY OF $L$

We prove in this section that if a quantale of the form  $\mathcal{Q}(L)$  is involutive, then  $id_L^*$  equals  $o_L$  defined in equation (2). From this it follows that there is at most one involutive quantale structure on  $\mathcal{Q}(L)$  extending the quantale structure. At the end of this section, we also argue that if  $\mathcal{Q}(L)$  is involutive, then  $L$  is a completely distributive lattice. To this end, let us firstly recall the following standard definitions:

**Definition 2.** Let  $\mathcal{Q}$  be a quantale. An element  $\alpha \in \mathcal{Q}$  is said to be

- *cyclic* if  $f \backslash \alpha = \alpha / f$ , for each  $f \in \mathcal{Q}$ ,
- *dualizing* if  $\alpha / f \backslash \alpha = f$ , for each  $f \in \mathcal{Q}$ .

It is well-known that involutive quantale structures on a quantale  $\mathcal{Q}$  are in bijection with cyclic dualizing elements of  $\mathcal{Q}$ . Let us also recall that, for an involutive quantaloid  $\mathcal{Q}$  and an object  $L$  of  $\mathcal{Q}$ ,  $0_L := (id_L)^{*,L,L}$  is both a cyclic and a dualizing element of the quantale  $\mathcal{Q}(L, L)$ .

An important first observation, stated in the next lemma, is that residuals of the form  $g \backslash h$  in  $\mathbf{Latt}_V$  can be constructed by means of the operations  $\mathbf{int}(\cdot)$  (greatest join-continuous map below a given one) and  $\rho(\cdot)$  (taking the right adjoint of a join-continuous map).

**Lemma 4.** For each  $g \in \mathbf{Latt}_V(M, N)$ ,  $h \in \mathbf{Latt}_V(L, N)$ , we have

$$g \backslash h = \mathbf{int}(\rho(g) \circ h).$$

*Proof.* Indeed, for each  $f \in \mathbf{Latt}_V(L, M)$ , we have  $f \leq g \backslash h$  iff  $g \circ f \leq h$ , iff  $g(f(x)) \leq h(x)$ , for each  $x \in L$ , iff  $f(x) \leq \rho(g)(h(x))$ , for each  $x \in L$ , iff  $f \leq \rho(g) \circ h$ , iff  $f \leq \mathbf{int}(\rho(g) \circ h)$ .  $\square$

For the next lemma, recall that the join-continuous map  $o_L$  has been defined in (2) and that the maps  $c_t$  and  $a_t$  have been defined in (1).

**Lemma 5.** We have  $o_L = \bigvee_{t \in L} c_t \circ a_t$ .

*Proof.* Observe that  $c_t(a_t(x)) = \perp$ , if  $x \leq t$ , and  $c_t(a_t(x)) = t$ , if  $x \not\leq t$ . Therefore

$$\square \quad \left( \bigvee_{t \in L} c_t \circ a_t \right)(x) = \bigvee_{t \in L} (c_t(a_t(x))) = \bigvee_{\substack{t \in L, \\ c_t(a_t(x)) \neq \perp}} c_t(a_t(x)) = \bigvee_{\substack{t \in L, \\ x \not\leq t}} t = o_L(x).$$

**Lemma 6.** For each  $x \in L$ ,  $\mathbf{int}(\alpha_x) = a_{o_L(x)}$ .

*Proof.* Let us observe that  $a_{o(x)} \leq \alpha_x$ . This amounts to verifying that if  $\alpha_x(t) = \perp$ , then  $a_{o(x)}(t) = \perp$ . Now,  $\alpha_x(t) = \perp$  iff  $x \not\leq t$ , and so  $t \leq o(x)$ , thus  $a_{o(x)}(t) = \perp$ . Next, let us suppose that  $f : L \rightarrow L$  is join-continuous and below  $\alpha_x$ . Thus, if  $\alpha_x(t) = \perp$ , that is, if  $x \not\leq t$ , then  $f(t) = \perp$ . Then  $f(o(x)) = f(\bigvee_{x \not\leq t} t) = \bigvee_{x \not\leq t} f(t) = \perp$ . By monotonicity of  $f$ , if  $t \leq o(x)$ , then  $f(t) = \perp$ , showing that  $f \leq a_{o(x)}$ .  $\square$

**Lemma 7.** *If  $L$  is not trivial, then  $c_\top$  is not a dualizing element of  $\mathcal{Q}(L)$ .*

*Proof.* Observe that  $c_\top$  is the greatest element of  $\mathcal{Q}(L)$  and, for this reason,  $f \setminus c_\top = c_\top / f = c_\top$ , for each  $f \in \mathcal{Q}(L)$ . If  $c_\top$  is dualizing, then  $c_\perp = (c_\top / c_\perp) \setminus c_\top = c_\top$ . Considering that the mapping from sending  $x \in L$  to  $c_x \in \mathcal{Q}(L)$  is an embedding, this shows that  $\perp = \top$  in  $L$ .  $\square$

**Theorem 2.** *If  $\mathcal{Q}(L)$  is an involutive quantale, then  $id_L^* = o_L$ . That is, a cyclic dualizing element is unique and necessarily equal to  $o_L$ .*

*Proof.* Let  $h = id_L^*$ , so  $h$  is cyclic and dualizing. We firstly prove that  $o_L \leq h$ . Consider that, for each  $x \in L$ ,  $a_x \circ c_x = c_\perp \leq h$ . Thus, since  $g \circ f \leq h$  if and only if  $f \circ g \leq h$ , we also have  $c_x \circ a_x \leq h$ . Since this relation holds for each  $x \in L$ , then, using Lemma 5, the relation  $o_L = \bigvee_{x \in L} c_x \circ a_x \leq h$  holds.

We argue now that  $h = o_L$ . Let  $x \in L$  and consider that  $c_x^* \circ c_x \leq h$ . Now,  $c_x^* = c_x \setminus h = \mathbf{int}(\rho(c_x) \circ h) = \mathbf{int}(\alpha_x \circ h)$  and therefore, using Lemma 6,

$$a_{o_L(x)} \circ h \circ c_x = \mathbf{int}(\alpha_x) \circ \mathbf{int}(h) \circ c_x \leq \mathbf{int}(\alpha_x \circ h) \circ c_x = c_x^* \circ c_x \leq h.$$

If  $t \neq \perp$ , then, by evaluating the above inequality at  $t$ , we get  $a_{o_L(x)}(h(x)) \leq h(t)$ . Since  $a_{o_L(x)}(h(x))$  takes values  $\perp$  and  $\top$ , this means that  $a_{o_L(x)}(h(x)) = \top$  implies  $\top \leq h(t)$ , for all  $t \neq \perp$ . That is, if  $h(x) \not\leq o_L(x)$ , then  $h(t) = \top$ , for all  $t \neq \perp$  and  $x \in L$ . Otherwise stated, if  $h \not\leq o_L$  (or  $h \neq o_L$ , given the inclusion  $o_l \leq h$  has already been proved), then  $h = c_\top$ . By Lemma 7, the equality  $h = c_\top$  does not hold, unless  $L$  is trivial. Thus, if  $L$  is not trivial, then  $h \leq o_L$ , and if  $L$  is trivial, then  $h = o_L$  since  $\mathcal{Q}(L)$  has just one element.  $\square$

**Theorem 3.** *If  $\mathcal{Q}(L)$  is an involutive quantaloid, then  $x = \bigwedge_{t \not\leq x} o_L(t)$ , for each  $x \in L$ . Consequently,  $L$  is a completely distributive lattice.*

*Proof.* If  $L$  is a one-element lattice, then the above equation trivially holds. Thus we shall suppose that  $L$  has at least two elements. In this case, assuming  $\mathcal{Q}(L)$  an involutive quantaloid, then  $o_L$  is dualizing and  $o_L \neq c_\top$  by Lemma 7.

Since  $o_L$  is cyclic, then, for each  $y, x \in L$ , the two conditions  $c_y \circ a_x \leq o_L$  and  $a_x \circ c_y \leq o_L$  are equivalent.

Condition  $c_y \circ a_x \leq o_L$  states that, for each  $t \in L$ ,  $t \not\leq x$  implies  $y \leq o_L(t)$ ; that is  $y \leq \bigwedge_{t \not\leq x} o_L(t)$ . Condition  $a_x \circ c_y \leq o_L$  states that, for each  $t \neq \perp$ , if  $y \not\leq x$  then  $o_L(t) = \top$ . This condition is equivalent to  $y \not\leq x$  implies  $o_L = c_\top$  or, equivalently, to  $o_L \neq c_\top$  implies  $y \leq x$ . We have argued that  $o_L \neq c_\top$  holds, and therefore the second condition is equivalent to  $y \leq x$ . Thus,  $y \leq x$  iff  $y \leq \bigwedge_{t \not\leq x} o_L(t)$ , for each  $x, y \in L$  and then the equality  $x = \bigwedge_{t \not\leq x} o_L(t)$  follows, for each  $x \in L$ . The last sentence of the theorem follows from Theorem 1.  $\square$

4.0.1. *The center of  $\mathcal{Q}(L)$ .* Uniqueness of an involutive quantale structure extending the quantale structure of  $\mathcal{Q}(L)$  can also be achieved through the observation that the unique central elements of  $\mathcal{Q}(L)$  are  $id_L$  and  $c_\perp$ . We are thankful to Claudia Muresan for her help investigating the center of  $\mathcal{Q}(L)$ .

**Definition 3.** We say that an element  $\beta$  of a quantale  $\mathcal{Q}$  is

- *central* if  $\beta \circ x = x \circ \beta$ , for each  $x \in \mathcal{Q}$ ,
- *codualizing* if  $x = \beta \setminus (\beta \circ x)$ , for each  $x \in \mathcal{Q}$ .

**Lemma 8.** *If  $\mathcal{Q}$  is an involutive quantale, then  $\alpha \in \mathcal{Q}$  is cyclic if and only if  $\alpha^*$  is central and it is dualizing if and only if  $\alpha^*$  is codualizing.*

*Proof.* Since  $x \setminus \alpha = (\alpha^* \circ x)^*$ ,  $\alpha/x = (x \circ \alpha^*)^*$ , and  $(\cdot)^*$  is invertible, the equality  $x \setminus \alpha = \alpha/x$  holds if and only if the equality  $\alpha^* \circ x = x \circ \alpha^*$  holds.

Now  $\alpha$  is dualizing if and only if, for each  $x \in \mathcal{Q}$ ,  $x = \alpha/(x \setminus \alpha) = \alpha^* \setminus (x \setminus \alpha)^* = \alpha^* \setminus (\alpha^* \circ x)$ .  $\square$

**Proposition 1.** *The only central elements of  $\mathcal{Q}(L)$  are  $id_L$  and  $c_\perp$ .*

*Proof.* Clearly,  $id_L$  and  $c_\perp$  are central, so we shall be concerned to prove that they are the only ones with this property. To this end, for  $x_0 \in L$ , define

$$\nu_{x_0}(t) := \begin{cases} \perp, & t \leq x_0 \\ t, & \text{otherwise.} \end{cases}$$

Notice that if  $x_0 = \perp$ , then  $\nu_{x_0} = id_L$ , while if  $x_0 = \top$ , then  $\nu_{x_0} = c_\perp$ . We firstly claim that if  $\beta$  is central in  $\mathcal{Q}(L)$ , then  $\beta = \nu_{x_0}$ , for some  $x_0 \in L$ . Suppose  $\beta$  is central. For each  $x \in L$ , we have  $c_x(x) = x$  and therefore

$$\beta(x) = (\beta \circ c_x)(x) = c_x(\beta(x)).$$

If  $\beta(x) \neq \perp$ , then, evaluating the rightmost expression, we obtain  $\beta(x) = x$ . Let  $x_0 := \bigvee \{y \mid \beta(y) = \perp\}$ , so  $\beta(x_0) = \perp$ . If  $t \leq x_0$ , then  $\beta(t) \leq \beta(x_0) = \perp$  and, otherwise,  $\beta(t) \neq \perp$  and so  $\beta(t) = c_t(\beta(t)) = t$ . Therefore,  $\beta = \nu_{x_0}$ .

Next, we claim that if  $x_0 \notin \{\perp, \top\}$ , then  $\nu_{x_0}$  is not central. Observe that

$$\nu_{x_0}(f(x)) = \begin{cases} \perp, & f(x) \leq x_0, \\ f(x), & \text{otherwise,} \end{cases} \quad f(\nu_{x_0}(x)) = \begin{cases} \perp, & x \leq x_0, \\ f(x), & \text{otherwise.} \end{cases}$$

It follows that if  $\nu_{x_0} \circ f = f \circ \nu_{x_0}$ , then  $f(x_0) \leq x_0$ . Indeed, if  $f(x_0) \not\leq x_0$ , then  $f(x_0) \neq \perp$ ,  $\nu_{x_0}(f(x_0)) = f(x_0) \neq \perp$ , and  $f(\nu_{x_0}(x_0)) = \perp$ . Now, if  $x_0 \notin \{\perp, \top\}$ , then  $c_\top$  is such that  $x_0 < \top = c_\top(x_0)$ , and therefore  $\nu_{x_0} \circ c_\top \neq c_\top \circ \nu_{x_0}$ .  $\square$

It is possible now to argue that, for a complete lattice  $L$ , there exists at most one extension of  $\mathcal{Q}(L)$  to an involutive quantale as follows. Let  $(\cdot)^*$  be a fixed involutive quantale structure. If  $\alpha$  is cyclic and dualizing, then  $\beta = \alpha^*$  is central and codualizing and, by the previous proposition,  $\beta \in \{c_\perp, id_L\}$ . Since  $\beta$  is codualizing, then it is an injective function: if  $\beta(x) = \beta(y)$ , then  $\beta \circ c_x = \beta \circ c_y$  and  $c_x = \beta \setminus (\beta \circ c_x) = \beta \setminus (\beta \circ c_y) = c_y$ ; since the mapping sending  $t$  to  $c_t$  is an embedding, we obtain  $x = y$ . Thus  $\beta \neq c_\perp$  (if  $L$  is not trivial) and in any case  $\beta = id_L$ . It follows that  $\alpha = id_L^*$ .

## 5. RANEY'S TRANSFORMS

Let  $L, M$  be two complete lattices. For  $f : L \longrightarrow M$ , define

$$f^\vee(x) := \bigvee_{x \not\leq t} f(t), \quad f^\wedge(x) := \bigwedge_{t \not\leq x} f(t), \quad \text{for each } x \in L.$$

We call  $f^\vee$  and  $f^\wedge$  the Raney's transforms of  $f$ .

**Lemma 9.** *For any  $f : L \longrightarrow M$ , define*

$$(4) \quad g_f(y) := \bigwedge \{z \mid f(z) \not\leq y\}.$$

*Then  $g_f$  is right adjoint to  $f^\vee$  and therefore  $f^\vee$  is join-continuous. Dually,  $f^\wedge$  is meet-continuous.*

*Proof.* Indeed, we have  $f^\vee(x) \leq y$  if and only if, for all  $z \in L$ ,  $x \not\leq z$  implies  $f(z) \leq y$ , and this is logically (classically) equivalent to stating that, for all  $z \in L$ ,  $f(z) \not\leq y$  implies  $x \leq z$ , that is,  $x \leq g_f(y)$ .  $\square$

We call the operation  $(\cdot)^\vee$  Raney's transform for the following reason. For  $\theta \subseteq L \times M$  an arbitrary relation, Raney [12] defined (up to some dualities)

$$r_\theta(x) := \bigwedge \{ y \in M \mid \forall (t, v). (t, v) \in \theta \text{ implies } x \leq t \text{ or } v \leq y \}.$$

Recall that a left adjoint  $\ell : L \rightarrow M$  can be expressed from its right adjoint  $\rho : M \rightarrow L$  by the formula  $\ell(x) = \bigwedge \{ y \mid x \leq \rho(y) \}$ . Using this expression with  $\ell = f^\vee$  and  $\rho = g_f$  defined in (4), we obtain

$$f^\vee(x) = \bigwedge \{ y \in M \mid \forall t. f(t) \not\leq y \text{ implies } x \leq t \}.$$

Clearly, if we let  $\theta$  be the graph of  $f$ , defined by  $(t, v) \in \theta$  if and only if  $f(t) = v$ , then we obtain that  $f^\vee = r_\theta$ .

**Lemma 10.** *The transform  $(\cdot)^\vee$  has the following properties:*

- (1) if  $f \leq g : L \rightarrow M$ , then  $f^\vee \leq g^\vee$ ,
- (2) if  $g : L \rightarrow M$  and  $f : M \rightarrow N$  is monotone, then  $(f \circ g)^\vee \leq f \circ (g^\vee)$ ,
- (3) if  $g : L \rightarrow M$  and  $f : M \rightarrow N$  is join-continuous, then  $(f \circ g)^\vee = f \circ (g^\vee)$ .

*Proof.* 1. If  $t \not\leq x$ , then  $f(t) \leq g(t)$  for each  $t \in L$  and therefore  $\bigvee_{t \not\leq x} f(t) \leq \bigvee_{t \not\leq x} g(t)$ . 2. The relation  $\bigvee_{x \not\leq t} f(g(t)) \leq f(\bigvee_{x \not\leq t} f(g(t)))$  is an immediate consequence of monotonicity of  $f$ . 3. If  $f$  is join-continuous, then clearly  $f(\bigvee_{x \not\leq t} g(t)) = \bigvee_{x \not\leq t} f(g(t))$ .  $\square$

**Lemma 11.** *If  $L$  and  $M$  are arbitrary complete lattices and  $f : L \rightarrow M$  is join-continuous, then*

$$(5) \quad \ell(f^\wedge) = \rho(f)^\vee : M \rightarrow L.$$

*Proof.* We show that, for  $x \in M$  and  $y \in L$ ,  $\rho(f)^\vee(x) \leq y$  if and only if  $x \leq f^\wedge(y)$ . The condition  $\rho(f)^\vee(x) \leq y$  amounts to

$$(6) \quad x \not\leq t \text{ implies } \rho(f)(t) \leq y, \quad \text{for all } t \in M,$$

while the condition  $x \leq f^\wedge(y)$  amounts to

$$u \not\leq y \text{ implies } x \leq f(u), \quad \text{for all } u \in L,$$

or, equivalently, to

$$(7) \quad x \not\leq f(u) \text{ implies } u \leq y, \quad \text{for all } u \in L.$$

Let us show that (6) implies (7). If  $x \not\leq f(u)$ , then, by (6),  $\rho(f)(f(u)) \leq y$ , and then  $u \leq y$ , since  $u \leq \rho(f)(f(u))$ . Conversely, let us assume (7) and argue for (6). If  $x \not\leq t$ , then, considering that  $f(\rho(f)(t)) \leq t$ ,  $x \not\leq f(\rho(f)(t))$  as well. Therefore, using (7),  $\rho(f)(t) \leq y$ .  $\square$

## 6. THE INVOLUTIVE QUANTALOID OF COMPLETELY DISTRIBUTIVE LATTICES

We prove now that  $\mathbf{Latt}_V^{\text{cd}}$ , the full subcategory of  $\mathbf{Latt}_V$  whose objects are the completely distributive lattices, is an involutive quantaloid. By the results of Section 4, this is also the largest full subcategory of  $\mathbf{Latt}_V$  with this property.

Recall from Theorem 1 that a complete lattice is completely distributive if and only if  $\omega_L^\vee = \text{id}_L$  (or, equivalently,  $o_L^\wedge = \text{id}_L$ ).

**Lemma 12.** *If  $L$  is a completely distributive lattice and  $f : L \longrightarrow M$  is monotone, then  $\text{int}(f) = (f \circ \omega_L)^\vee$  and  $f^\vee = \text{int}(f \circ o_L)$ .*

*Proof.* By monotonicity of  $f$ , we have  $(f \circ \omega_L)^\vee \leq f \circ (\omega_L^\vee) = f$ . Suppose that  $g$  is join-continuous and  $g \leq f$ . Then  $g = g \circ (\omega_L^\vee) = (g \circ \omega_L)^\vee \leq (f \circ \omega_L)^\vee$ . To see that  $f^\vee = \text{int}(f \circ o_L)$ , observe that  $f^\vee = (f \circ \text{id})^\vee \leq f \circ (\text{id}_L^\vee) = f \circ o_L$ , and therefore  $f^\vee \leq \text{int}(f \circ o_L)$ . On the other hand,  $\text{int}(f \circ o_L) = (f \circ o_L \circ \omega_L)^\vee \leq f^\vee$ , using the counit of the adjunction,  $o_L \circ \omega_L \leq \text{id}_L$ .  $\square$

The interior operator so defined is quite peculiar, since for  $g : L \longrightarrow M$  monotone and  $f : M \longrightarrow N$  join-continuous, we have

$$\text{int}(f \circ g) = (f \circ g \circ \omega_L)^\vee = f \circ (g \circ \omega_L)^\vee = f \circ \text{int}(g).$$

In general, if  $L$  is not a completely distributive lattice, then we would have, above, only an inequality, since  $\text{int}(f \circ g) \geq \text{int}(f) \circ \text{int}(g) = f \circ \text{int}(g)$ .

**Lemma 13.** *If  $L$  is a completely distributive lattice and  $f : L \longrightarrow M$  is join-continuous, then  $f = f^{\wedge\vee}$ .*

*Proof.* We firstly show that  $f^{\wedge\vee} \leq f$ . If  $x \not\leq t$ , then  $f^\wedge(t) = \bigwedge_{u \not\leq t} f(u) \leq f(x)$  and therefore  $f^{\wedge\vee}(x) = \bigvee_{x \not\leq t} f^\wedge(t) \leq f(x)$ , for all  $x \in L$ . Let us argue that  $f \leq f^{\wedge\vee}$ :

$$f = f \circ \text{id}_L = f \circ (\omega_L^\vee) = (f \circ \omega_L)^\vee \leq f^{\wedge\vee},$$

where we have used the fact, dual to the relation  $f^\vee = \text{int}(f \circ o_L)$  established in Lemma 12, that  $f^\wedge$  is the least meet-continuous function above  $f \circ \omega_L$ , so in particular  $f \circ \omega_L \leq f^\wedge$ .  $\square$

For  $f : L \longrightarrow M$  join-continuous, define  $f^{*L,M} : M \longrightarrow L$  as follows:

$$f^{*L,M} := \rho(f)^\vee.$$

**Theorem 4.** *The operations  $(\cdot)^{*L,M}$  so defined yield an involutive quantaloid structure on  $\mathbf{Latt}_V^{\text{cd}}$ , the full subcategory of  $\mathbf{Latt}_V$  whose objects are the completely distributive lattices.*

*Proof.* Firstly, we verify that  $f^{**} = f$  using Lemmas 11 and 13, and the fact the join-continuous functions are in bijection with meet-continuous functions via taking adjoints:

$$f^{**} = \rho(\rho(f)^\vee)^\vee = \rho(\ell(f^\wedge))^\vee = f^{\wedge\vee} = f.$$

We verify now that  $(\cdot)^*$  satisfies the constraints needed to have an involutive quantaloid. Let us remark that  $\text{id}_L^* = \rho(\text{id}_L)^\vee = \text{id}_L^\vee = o_L$ .

Observe that since  $(\cdot)^*$  is defined by composing an order reversing and an order preserving function, it is order reversing. Since it is an involution, then  $f \leq g$  if and only if  $g^* \leq f^*$ .

Now we assume that  $f : L \longrightarrow M$  and  $h : M \longrightarrow L$  and recall (see Lemma 12) that  $h^{*M,L} = \rho(h)^\vee = \text{int}(\rho(h) \circ o_L) : L \longrightarrow M$ . Therefore,  $h \circ f \leq o_L$  if and only if  $f \leq \rho(h) \circ o_L$ , if and only if  $f \leq \text{int}(\rho(h) \circ o_L) = h^*$ . Therefore, if  $g : L \longrightarrow M$ , then (letting  $h = g^*$ )  $f \leq g$  if and only if  $g^* \circ f \leq o_L$ . Then, also,  $f \leq g$  if and only if  $g^* \leq f^*$  if and only if  $f \circ g^* \leq o_M$ .  $\square$

Putting together Theorems 3 and 4, we obtain:

**Corollary 1.** *The quantale  $\mathcal{Q}(L)$  is involutive if and only if  $L$  is a completely distributive lattice.*

For  $f : L \longrightarrow M$  and  $g : M \longrightarrow N$  (with  $L, M, N$  completely distributive lattices), let us define

$$g \oplus f := (f^* \circ g^*)^* : L \longrightarrow M,$$

and observe that

$$(g \oplus f) = (f^* \circ g^*)^* = \rho((\ell(f^\wedge) \circ \ell(g^\wedge)))^\vee = (g^\wedge \circ f^\wedge)^\vee.$$

That is, the dual quantale structure arise via Raney's transforms from the composition  $\text{Latt}_\vee(L^\partial, M^\partial) \times \text{Latt}_\vee(M^\partial, N^\partial) \longrightarrow \text{Latt}_\vee(L^\partial, N^\partial)$ .

An immediate consequence of Corollary 1 is the following:

**Theorem 5.** *A complete lattice  $L$  is a chain if and only if  $\mathcal{Q}(L)$  is an involutive quantale satisfying the mix rule.*

*Proof.* Let us recall that, in the language of involutive residuated lattices, the mix rule is the inclusion  $x \circ y \leq x \oplus y$ —with  $x \oplus y = (y^* \circ x^*)^*$ . It is well known that this inclusion is equivalent to the inclusion  $0 \leq 1$ —where  $1$  is the unit for  $\circ$  and  $0$  is the unit for  $\oplus$ . Therefore, an involutive quantale of the form  $\mathcal{Q}(L)$  satisfies the mix rule if and only if  $o_L \leq id_L$ . This relation is easily seen to be equivalent to the statement that if  $x \not\leq t$ , then  $t \leq x$ , so  $L$  is a chain. For the converse, we just need to recall that every chain is a completely distributive lattice.  $\square$

Let us recall that an element of a lattice  $L$  is said to be completely join-irreducible (equivalently, completely join-prime, if  $L$  is distributive) if it has a unique lower cover. It is not difficult to see that, if  $L$  is a completely distributive lattice, then  $x \not\leq o_L(x)$  if and only if  $x$  is completely join-prime. Thus we say that a completely distributive lattice is *smooth* if it has no completely join-prime element. For example, the interval  $[0, 1]$  of the reals is a smooth completely distributive lattice. The following statement is an immediate consequence of these considerations.

**Theorem 6.** *A completely distributive lattice  $L$  is smooth if and only if  $id_L \leq o_L$ , that is, if and only if  $\mathcal{Q}(L)$  satisfies the inclusion  $1 \leq 0$  in the language of involutive residuated lattices.*

## 7. FURTHER PROPERTIES OF $\text{Latt}_\vee^{\text{cd}}$

For a family  $\{f_i \in \text{Latt}_\vee(L, M) \mid i \in I\}$  let us define its pointwise meet  $\bigwedge_{i \in I} f_i$  as usual by

$$\left(\bigwedge_{i \in I} f_i\right)(x) := \bigwedge_{i \in I} (f_i(x)), \quad \text{for each } x \in L.$$

Notice that  $\bigwedge_{i \in I} f_i$  need not to be join-continuous; however, its interior is join-continuous and, necessarily, it is the infimum of the family  $\{f_i \mid i \in I\}$  within the

complete lattice  $\mathbf{Latt}_V(L, M)$ . Moreover, Lemma 12 yields for completely distributive lattices an explicit description of the interior  $\mathbf{int}(\bigwedge_{i \in I} f_i)$  that we shall exploit later on. We collect these observations in the following proposition.

**Proposition 2.** *If  $L$  is a completely distributive lattice and  $\{f_i \mid i \in I\}$  is a family as above, then the function  $\mathbb{A}_{i \in I} f_i$  defined by*

$$(8) \quad \left(\mathbb{A}_{i \in I} f_i\right)(x) := \bigvee_{x \not\leq t} \bigwedge_{i \in I} f_i(\omega_L(t)), \quad \text{for each } x \in L,$$

is the infimum of  $\{f_i \mid i \in I\}$  in  $\mathbf{Latt}_V(L, M)$ .

*Proof.* Observe that  $\mathbb{A}_{i \in I} f_i = ((\bigwedge_{i \in I} f_i) \circ \omega_L)^\vee = \mathbf{int}(\bigwedge_{i \in I} f_i)$ , by Lemma 12.

We have, therefore,  $\mathbb{A}_{i \in I} f_i = \mathbf{int}(\bigwedge_{i \in I} f_i) \leq \bigwedge_{i \in I} f_i \leq f_i$ , for each  $i \in I$ . Conversely, if  $g \in \mathbf{Latt}_V(L, M)$  and  $g \leq f_i$  for each  $i \in I$ , then  $g \leq \bigwedge_{i \in I} f_i$  and  $g \leq \mathbf{int}(\bigwedge_{i \in I} f_i) = \mathbb{A}_{i \in I} f_i$ .  $\square$

**Theorem 7.** *If  $L$  and  $M$  are completely distributive lattices, then so is the complete lattice  $\mathbf{Latt}_V(L, M)$ .*

*Proof.* Let us recall that the join  $\bigvee f_i$  of a family  $\{f_i \mid i \in I\}$  is computed pointwise. We need to argue that, for  $\pi : J \rightarrow I$  and a family  $\{f_j \mid j \in J\}$ ,

$$\mathbb{A}_{i \in I} \bigvee_{j \in J_i} f_j = \bigvee_{\psi} \mathbb{A}_{i \in I} f_{\psi(i)},$$

where as usual  $J_i = \pi^{-1}(i)$  and the meet on the right ranges over all sections  $\psi$  of  $\pi$ . The inclusion from right to left always holds, so we argue for the inclusion from left to right. The formula given in equation (8) allows to verify this inclusion pointwise, since for  $x \in L$  we have

$$\left(\mathbb{A}_{i \in I} \left(\bigvee_{j \in J_i} f_j\right)\right)(x) = \bigvee_{x \not\leq t} \left(\bigwedge_{i \in I} \left(\bigvee_{j \in J_i} f_j\right)\right) \circ \omega_L(t) = \bigvee_{x \not\leq t} \bigwedge_{i \in I} \bigvee_{j \in J_i} f_j(\omega_L(t)).$$

Thus, it will be enough to verify that if  $u \in L$  is arbitrary such that  $x \not\leq u$ , then

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} f_j(\omega_L(u)) \leq \left(\bigvee_{\psi} \left(\mathbb{A}_{i \in I} f_{\psi(i)}\right)\right)(x).$$

This is achieved as follows:

since  $M$  is completely distributive,

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} f_j(\omega_L(u)) = \bigvee_{\psi} \bigwedge_{i \in I} f_{\psi(i)}(\omega_L(u)),$$

since  $x \not\leq u$ ,

$$\leq \bigvee_{\psi} \bigvee_{x \not\leq t} \bigwedge_{i \in I} f_{\psi(i)}(\omega_L(t)),$$

$\square$

$$= \bigvee_{\psi} \left(\mathbb{A}_{i \in I} f_{\psi(i)}\right)(x) = \left(\bigvee_{\psi} \left(\mathbb{A}_{i \in I} f_{\psi(i)}\right)\right)(x).$$

Let in the following  $\mathbf{Latt}_V^{\text{cd}}$  be the full subcategory of  $\mathbf{Latt}_V$  whose objects are the completely distributive lattices. We refer the reader to [8] for an in-depth study of the category  $\mathbf{Latt}_V$  and to [1] for the notion of a  $\star$ -autonomous category.

**Theorem 8.**  *$\mathbf{Latt}_V^{\text{cd}}$  is closed under the monoidal closed operations of  $\mathbf{Latt}_V$  and, consequently, it is  $\star$ -autonomous.*

*Proof.* Let us recall that  $\mathbf{Latt}_\vee$  is monoidal closed and  $\star$ -autonomous. The unit of the tensor is the two-element Boolean algebra, which we denote by  $2$ . Clearly  $2$  is a completely distributive lattice and moreover it is the dualizing object of  $\mathbf{Latt}_\vee$ . We have argued that the homset  $\mathbf{Latt}_\vee(L, M)$  is a completely distributive lattice, whenever  $L$  and  $M$  are such, and so  $\mathbf{Latt}_\vee^{\text{cd}}$  is closed under exponentiation. A particular case of exponentiation is when  $M = 2$ , in which case we derive once more that  $L^\partial$ , being isomorphic to  $\mathbf{Latt}_\vee(L, 2)$ , is a completely distributive lattice. It follows then that also the tensor product of two completely distributive lattices is completely distributive, since  $L \otimes M = \mathbf{Latt}_\vee(L, M^\partial)^\partial$ . Thus  $\mathbf{Latt}_\vee^{\text{cd}}$  is closed under the operations making  $\mathbf{Latt}_\vee$  into a  $\star$ -autonomous category, and the last statement of the theorem also follows.  $\square$

An important consequence of the remarks developed in this section is the following:

**Corollary 2.** *If an involutive residuated lattice  $\mathcal{Q}$  has an embedding into an involutive quantale of the form  $\mathcal{Q}(L)$ , then  $\mathcal{Q}$  is distributive.*

Indeed, if  $\mathcal{Q}(L)$  is an involutive quantale, then  $L$  is a completely distributive lattice. Then Theorem 7 ensures that  $\mathcal{Q}(L)$  is also a completely distributive lattice, and therefore, if  $\mathcal{Q}$  has a lattice embedding into  $\mathcal{Q}(L)$ , then  $L$  is distributive.

## 8. CONCLUSIONS AND FUTURE STEPS

The researches exposed in this paper solve a natural problem encountered during our investigations of certain quantales built from complete chains [16, 17, 4]. The problem asks to characterize the complete chains whose quantale of join-continuous endomaps is involutive. Indeed, every complete chain is a completely distributive lattice and therefore the results presented in this paper prove that every complete chain has this property; in particular, other properties of chains and posets, such as self-duality, are not relevant.

The solution we provide is as general as possible, in two respects. On the one hand, we have been able to give an exact characterization of all the complete lattices—not just the chains— $L$  for which  $\mathcal{Q}(L)$  is involutive; these are the completely distributive lattices. In particular, the characterization covers different kind of involutive quantales known in the literature, those discovered in our investigation of complete chains and those known as the residuated lattices of weakening relations—arising from the relational semantics of distributive linear logic. On the other hand, we show that the involutive quantale structures on completely distributive lattices are uniform, yielding an involutive quantaloid structure on the category of completely distributive lattices and join-continuous functions.

We have drawn several consequences from the observations developed, among them, the fact that if an involutive quantale  $\mathcal{Q}$  can be embedded into an involutive quantale of the form  $\mathcal{Q}(L)$ , then it is distributive. This fact calls for a characterization of the involutive residuated lattices embeddable into some  $\mathcal{Q}(L)$ , a research track that might require to or end up with determining the variety of involutive residuated lattices generated by the  $\mathcal{Q}(L)$ . A second research goal, that we might tackle in a close future, demands to investigate the algebra developed in connection with the continuous weak order [17] in the wider and abstract setting of completely distributive lattices. Let us recall that in [17] a surprising bijection was established between two kind of objects, the maximal chains in the cube lattice  $[0, 1]^d$  and the

families  $\{f_{i,j} \in \mathcal{Q}([0,1]) \mid 1 \leq i < j \leq d\}$  such that, for  $i < j < k$ ,  $f_{j,k} \circ f_{i,j} \leq f_{i,k} \leq f_{j,k} \oplus f_{i,j}$ . So, are there other surprising bijections if the interval  $[0,1]$  is replaced by an arbitrary completely distributive lattice, and if we move from the involutive quantale setting to the multisorted setting of involutive quantaloids?

Acknowledgment. The author is thankful to Srečko Brlek, Claudia Muresan, and André Joyal for the fruitful discussions these scientists shared with him on this topic during winter 2018.

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