# A Statistical Approach to the Rupture of Brittle Materials

François Hild, Didier Marquis

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ABSTRACT. – Since initial defects are the main cause of the failure of structures made of brittle materials, the scope of this paper is to take these defects into account in the evaluation of the failure probability of a structure. Using the weakest link theory assumption and modeling the failure probability of a single link, a general relationship between the statistical distribution of defects and the failure probability is obtained. After determining the size of the critical defect, some correlations between the defect distribution and the Weibull's parameters are deduced from either Linear Fracture Mechanics or Continuum Damage Mechanics. The effect of the statistical distribution of defects within the material is computed and studied with respect to the distribution itself, the loaded volume of the structure and the heterogeneity of the stress field.

## 1. Introduction

A common feature of all brittle materials (e. g. engineering ceramics, concrete, glass) is the scatter of the failure stress. This is due to scatter in initial defects within the material that become critical for a given load level. These defects are often randomly distributed within the material and induce some specific experimental effects. The strength of such materials is not only material dependent but depends on the structure size and on the loading pattern. The present approach attempts to relate the statistical defect distribution to the failure probability of structures of different sizes and subjected to different types of loading.

The expression for the failure probability of a structure made of brittle material is generally deduced from both the weakest link theory and the independence of the events assumption (*see* [Weibull, 1951]; [Freudenthal, 1968]; [Batdorf, 1977]; [Evans, 1978]). In Section 2 of this paper, using these assumptions and modeling the failure probability of a single link containing only one defect, a relation between the failure probability of the whole structure and the defect distribution is deduced. Section 3 is related to different modelings of the rupture through either Linear Fracture Mechanics or Continuum Damage Mechanics, allowing a relationship between the defect distribution parameters and Weibull's coefficients. Finally, Section 4 is devoted to the computational study of the influence of the defect distribution, the specimen size and the type of loading on the failure of structures made of brittle materials.

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## 2. Correlation between the defect distribution within the structure and the failure probability of the structure

Using the weakest link assumption and the hypothesis of the independence of events ([W, 1951]; [F, 1968]), the cumulative failure probability $P_F$ of a structure $\Omega$ can be expressed as:

$$\ln(1-P_F)=\frac{1}{V_0}\int_{\Omega}\ln(1-P_{F0})\,dV \tag{1}$$

where $P_{F0}$ is the cumulative failure probability of a single link $\Omega_0$ of volume $V_0$. Besides, we suppose that $\Omega_0$ contains only one defect. That is to say $V_0$ is greater than the volume of the largest defect. In the following we assume that the volume $V_0$ is related to the mean distance between two defects $\bar{d}$: $V_0=[\bar{d}]^3$. Hence $V_0$ is a material parameter *i.e.*, physically related to the defect distribution.

$P_{F0}$ being close to 0, relation (1) can be rewritten as:

$$P_F \# 1-\exp\left[-\frac{1}{V_0}\int_{\Omega}P_{F0}\,dV\right] \tag{2}$$

This last expression is generally used for the study of the rupture of brittle materials (e.g. [W, 1951]) but since $P_{F0}$ varies within the range (0, 1), expression (2) is only an approximation to equation (1).

The cumulative failure probability $P_{F0}$ is given by the probability of finding a critical defect within a volume $\Omega_0$ of $\Omega$. The probability of finding a critical defect refers to the defect distribution characterized by a probability density function $f$. The function $f$ may depend upon several parameters. This dependence is linked with the modeling of the defects: first, a defect size, characterized by $a$; second, a defect direction characterized by a unit normal $\mathbf{n}$; third, if necessary, other parameters, characterized by $\omega$, which may be either a single parameter or a set of parameters. Besides, for a given load level, the set of defects $\mathscr{D}$ split into two subsets. The first one $\mathscr{D}_c$ is related to the defects that are critical (*i.e.* they lead to the failure of a link, and therefore of the whole structure). The second one $\mathscr{D}_{nc}$ refers to the defects that are not critical. The higher the load level, the larger $\mathscr{D}_c$ becomes with respect to $\mathscr{D}_{nc}$. The determination of the critical defects depends on the mechanical modeling of the defect. Some particular expressions can be found (*see* [B, 1977]; [Jayatilaca & Trustrum, 1977]; [E, 1978]). A general expression for $P_{F0}$ can be written as:

$$P_{F0}=\int_{\mathscr{D}_c} f(a,\mathbf{n},\omega)\,da\,d\mathbf{n}\,d\omega \tag{3}$$

This definition ensures that $P_{F0}$ lies between 0 (*i.e.*, $\mathscr{D}_c=\varnothing$: no defect is critical) and 1 (*i.e.*, $\mathscr{D}_c=\mathscr{D}$: all the defects are critical). It may be seen that $\mathscr{D}_c$ depends upon both the defects within the material and upon the load level, which is related to the mechanical modeling.

## 3. A mesomechanical study of the link failure

This section deals with the mechanical modeling of a defect within the link and the corresponding cumulative failure probability. The local behavior of a single link is described by an internal variable $\delta$ expressing its degradation. Using the maximal dissipation principle ([Nguyen, 1984, 1987]), the evolution law of $\delta$ is given by:

$$(\Delta - \Delta^*)\,\dot{\delta} \geqq 0 \quad \text{for every } \Delta^* \text{ belonging to } \mathscr{C} \tag{4}$$

where $\mathscr{C}$ is the admissible force domain and $\Delta$ the thermodynamical force associated with $\delta$. Relation (4) gives for a given loading the critical degradation $\delta_c$ under which the degradation does not evolve. If the degradation is greater than the critical degradation, its rate of evolution is infinite and leads to a brittle failure. This allows us to find, for a given loading and a given load level, the set of critical defects $\mathscr{D}_c$. Within this framework, a preexisting or initial defect is modelled by an initial value of the degradation $\delta_0$.

### 3.1. General uniaxial cases

The probability density function $f$ is here assumed to be only dependent on the size of the defect $a$. Then the cumulative failure probability $P_{F0}$ is the probability of finding a defect size larger than the critical defect size $a_C$. The critical defect size is the lower bound of the set $\mathscr{D}_c$ defined above. Therefore, using expression (3), the cumulative failure probability of a single link is written as:

$$P_{F0} = \int_{a_C}^{\infty} f(a)\,da \tag{5}$$

The critical defect size is linked with an uniaxial stress $\sigma$ by a general fracture criterion:

$$\sigma = g(a_C) \tag{6}$$

where $a_c$ is the first critical defect size (then $\mathscr{D}_c = \{a / a \in [a_c, \infty[\}$, $\mathscr{D}_{nc} = \{a / a \in [0, a_c[\}$) and $g$ is a positive strictly decreasing $C^1$ function defined between 0 and 1. Function $g$ will be specified later in this paper. Using expressions (1) and (5), the failure probability is given by:

$$P_F = 1 - \exp\left[\frac{1}{V_0}\int_{\Omega} \ln\left\{1 - \int_{g^{-1}(\sigma)}^{\infty} f(a)\,da\right\} dV\right] \tag{7}$$

Expression (7) gives a relationship between the defect distribution and the failure probability for the structure $P_F$. The critical defect size ($a_C = g^{-1}(\sigma)$) is obtained through a mesomechanical analysis.

### 3.2. Study of unidimensional cases

With some additional assumptions, the general expression (7) leads to the Weibull law. When the maximum defect size is not bounded a two parameter Weibull law is derived.

On the other hand, if the maximum defect size is bounded ($a < a_M < \infty$) a three parameter Weibull law is deduced.

3.2.1. *Bounded maximum value of the defect size*

For this case the threshold stress is the lowest value of the stress, under which the failure probability has a zero value:

$$\sigma_u = g(a_M) \tag{8}$$

For $a$ close to $a_M$, the function $f$ is assumed to be equivalent to $k\,(a_M - a)^\beta$, with $k>0$ and $\beta>0$. Assuming that $a_c$ is close to $a_M$, the cumulative failure probability of a single link $P_{F0}$ is approximatively given by:

$$P_{F0} \# \frac{k\,(a_M - a_c)^{\beta+1}}{\beta+1} \tag{9}$$

Using expression (6), $a_M - a_c$ is fitted by the following expression:

$$a_M - a_c \# - \frac{\sigma - \sigma_u}{g'(a_M)} \tag{10}$$

Finally, $P_{F0}$ is approximatively equal to:

$$P_{F0} \# \left[\frac{\langle \sigma - \sigma_u \rangle}{\sigma_0}\right]^{\beta+1}, \qquad \sigma_0 = -g'(a_M)\left[\frac{\beta+1}{k}\right]^{1/(\beta+1)} \tag{11 a, b}$$

Using relations (2) and (11), we obtain a three parameter Weibull expression where $m$ (the Weibull's parameter) is linked with $\beta$ (a parameter of the defect distribution $f$) by:

$$m = \beta + 1 \tag{12}$$

The $\beta$ parameter gives the trend of the defect distribution for large sizes (*i.e.*, $a$ being close to $a_M$). Relation (12) gives a relationship between a parameter of the defect distribution (*i.e.*, a physical parameter) and Weibull's parameter $m$ (i.e., a mechanical parameter) that can be obtained through an analysis of a set of macroscopical failure tests.

3.2.2. *Unbound maximum value of the defect size*

For this case, we assume that the function $f$ is equivalent to $K a^{-n}$ for large values of $a$. The relationship between the critical defect size and the applied stress $\sigma$ is assumed to be given by:

$$\sigma = g(a_c) = K'[a_c]^{-(1/p)} \tag{13}$$

Then using expressions (5) and (13), the expression of the cumulative failure probability is approximated by a two parameter Weibull law:

$$P_{F0} \# \left[\frac{\sigma}{\sigma_0}\right]^{p(n-1)} \qquad \text{where} \quad \sigma_0 = K'\left[\frac{n-1}{K}\right]^{1/p(n-1)} \tag{14}$$

Then the Weibull's parameter is given by:

$$m = p(n-1) \tag{15}$$

It is worth noting that the case $p=2$ has been studied by Jayatilaca & Trustrum [J & T, 1977].

### 3.3. Applications in unidimensional cases

#### 3.3.1. *Linear Fracture Mechanics*

In Linear Fracture Mechanics, the defects are modelled by initial cracks. The cracks are supposed to be orthogonal to the tensile direction (see *Fig.* 1). Here, $\Delta$ represents the

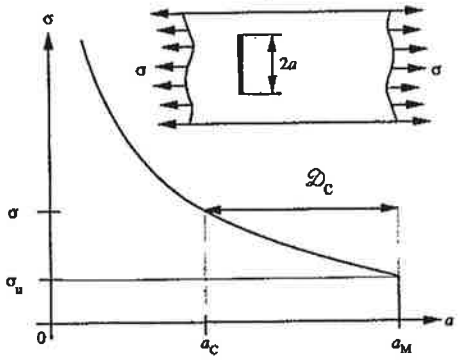


Fig. 1. – Critical crack size in fracture mode I for a given stress.

coplanar strain-energy release rate G [relation (4)]. If the size of a defect is very small in comparison with the size of the structure and the average distance between defects is large, expression (11) and (14) can be written in terms of the critical stress intensity factors $K_{IC}$ of the material and we obtain:

$$\sigma_u = \frac{K_{IC}}{\sqrt{\pi a_M}}, \qquad \sigma_0 = \frac{\sigma_u}{2a_M}\left[\frac{\beta+1}{k}\right]^{1/(\beta+1)} \tag{16 a, b}$$

with bounded maximum defect size

$$\sigma_0 = \frac{K_{IC}}{\sqrt{\pi}}\left[\frac{n-1}{K}\right]^{1/2(n-1)} \tag{17}$$

with unbounded maximum defect size

#### 3.3.2. *Continuum Damage Mechanics*

An alternative to the evolution of the degradation modeling is given by Continuum Damage Mechanics (C.D.M.), C.D.M. refers to the notion of effective stress. Because of the presence of a defect, the resisting area is reduced and the effective stress $\tilde{\sigma}$ (*see* [Kachanov, 1958] is related to the applied stress $\sigma$ by:

$$\tilde{\sigma} = \frac{\sigma}{(1-D)} \tag{18}$$

The so-called damage variable D represents globally the influence of a degradation on the mesomechanical behavior. The constitutive equations for strain behavior and failure behavior for a damaged material are given by using the effective stress instead of the classical Cauchy's stress.

The presence of a defect in a volume element is modelled by a damage value for this element. The defect size is specified by the radius $a$ of the smallest sphere containing the defect. The volume element size is denoted by L (see *Fig.* 2).

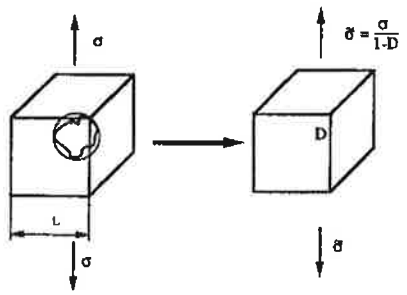


Fig. 2. – Definition of the damage variable in the uniaxial case.

In the following, the relation between the damage value and the defect size is assumed to be:

(19) $$D = h\left(\frac{a}{L}\right)$$

where $h$ is a positive strictly increasing $C^1$ function defined between 0 and 1. It is worth noting that the size L has to be strictly greater than the maximum size. Since the size L is assumed to be bounded, the maximum defect size has to be bounded.

A structure with some initial defects randomly distributed is modelled as a structure containing initial damage randomly distributed. The initial damage value $D_0$ in a volume element is given by:

(20) $$D_0 = h\left(\frac{a}{L}\right)$$

where $a$ is the size of the initial defect within the volume element.

In the following we assume that the defect distribution is dilute. Then the interactions between the stress fields around the defects are low. Therefore the influence of the initial defects on the strain behavior of the material is neglected. On the other hand, the influence of the defects on the failure behavior is given by a criterion related to the effective stress [Lemaitre & Chaboche, 1978]:

(21) $$\tilde{\sigma} = \sigma_M$$

where $\sigma_M$ is the ultimate strength of the virgin material. With the initial defect distribution modelled by an initial damage distribution, the preceding criterion yields :

(22) $$\frac{\sigma}{(1-D_0)} = \sigma_M$$

As mentioned earlier the defect size has to be bounded when using this modeling. We consider the same assumption for the defect distribution. Namely, for $a$ close to $a_M$, the defect distribution $f$ is assumed to be equivalent to $k(a_M - a)^\beta$ with $k>0$ and $\beta>0$. With this modeling, we get the same type of relationship as above mentioned in expression (11) and (12) with:

$$(23\,a,b) \qquad \sigma_u = \sigma_M (1 - D_{0M}); \qquad \sigma_0 = \sigma_M h'\left(\frac{a_M}{L}\right)\left\{\frac{\beta+1}{k}\right\}^{1/(\beta+1)}$$

where $D_{0M}$ is the maximal value of initial damage.

### 3.4. Two dimensional cases

The cracks are considered with variable size (characterized by $a$) and direction (characterized by $\gamma$: see *Fig.* 4): $\mathscr{D} = \{a, \gamma / a \in [0, a_M],\ \gamma \in [0, (\pi/2)]\}$. Approximatively, the failure criterion (see *Fig.* 4) is given by $K_{IC}^2 = K_I^2 + K_{II}^2$ (*see* [Amestoy *et al.*, 1979]) for positive stresses $\sigma$. The cumulative failure probability of a single link $P_{F0}$ takes the form:

$$(24) \qquad P_{F0} = \int_{K_{IC}^2/\pi\sigma^2}^{\infty} \left[ \int_{\text{Arcsin}\,(K_{IC}/\sigma\sqrt{\pi a})}^{\pi/2} f(a,\gamma)\, d\gamma \right] da$$

If we suppose an equiangular distribution of defects, $P_{F0}$ becomes:

$$(25) \qquad P_{F0} = \int_{a_c}^{\infty} \frac{2}{\pi} \text{Arccos}\left[\frac{K_{IC}}{\sigma\sqrt{\pi a}}\right] f(a)\, da \qquad \text{with} \quad a_c = \frac{K_{IC}^2}{\pi\sigma^2}$$

Expression (25) has a similar form as relation (5).

#### 3.4.1. *Bounded maximum value of the defect size*

When $a_c$ is close to $a_M$ ($x$ being close to 1, Arccos ($x$) is about $\sqrt{2[1-x]}$), expression (25) can be approximated by expression (11 $a$) which has the same form as the Weibull's expression. Under these hypotheses, the same threshold stress as that mentioned above [relation (14)] is obtained. It corresponds to the failure stress of the most critical defect, *i.e.* the largest one with its orientation orthogonal to the tensile direction (*Fig.* 3). The introduction of mode II leads to a modification of the relationship between $m$ and $\beta$:

$$(26) \qquad m = \beta + 1 + \frac{1}{2}$$

Since $m$ is generally much larger than 1/2 (for monolithic ceramics, the order of $m$ is 10: *see* [H *et al.*, 1990 *a*]), it appears that the influence of mode II is low. For a given defect size, the most critical orientation corresponds to a mode I loading. As the defect size is

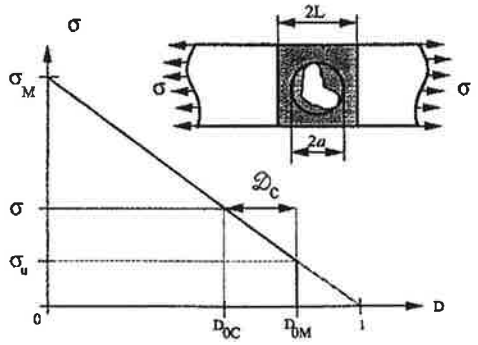


Fig. 3

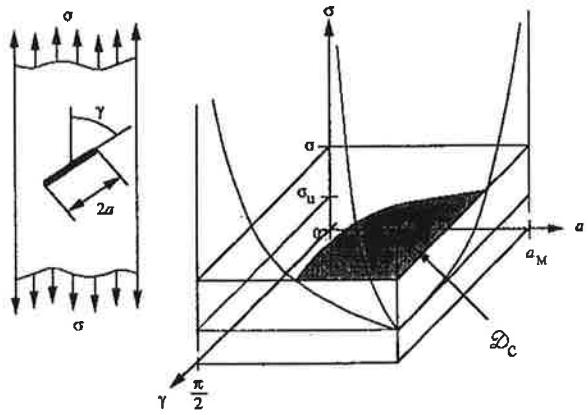


Fig. 4

Fig. 3. – Critical value of damage for a given stress.

Fig. 4. – Critical crack size and orientation in fracture mode I and II for a given stress.

bounded the probability of an orientation of the critical defect close to $\pi/2$ (see *Fig.* 4) is high.

3.4.2. *Unbounded maximum value of the defect size*

This case was studied by Jayatilaca & Trustrum [J & T, 1977]. Relation (25) is approximated by a two-parameter Weibull law. The relation between the power $n$ of the defect distribution ($f(a) \sim K a^{-n}$) and the Weibull's exponent is given by:

$$m = 2n - 2 \tag{27}$$

This relation is the same as the one derived in the uniaxial case (see *Sec.* 3.2.2). As the defect distribution is not bounded, any orientation of the defect can be critical.

3.5. THREE DIMENSIONAL CASES

The same relationship as in Section 3.3.2 between the initial value of damage and defect size is now assumed:

$$D_0 = h\left(\frac{a}{L}\right) \tag{28}$$

It is worth noting that a scalar damage variable is still used. This corresponds to the assumption of isotropic damage. This assumption is relevant to the case of sphere-like defects. As Continuum Damage Mechanics is used in this section, the equivalent failure stress is referring to the equivalent damage stress (*see* [Lemaitre & Chaboche, 1990]). Since the failure is mainly driven by tensile stresses (*see* fratographic observations in [H *et al.*, 1990] the equivalent damage stress is referring to the positive stress tensor $\boldsymbol{\sigma}^+$:

$$(\sigma_r)^* = \sqrt{\frac{2}{3}(1+\nu)(\sigma_{eq}^+)^2 + 3(1-2\nu)(\sigma_H^+)^2} \tag{29}$$

where $\nu$ is the Poisson's ratio and $\sigma_H^+$ and $\sigma_{eq}^+$ are defined as:

$$\sigma_H^+ = J_1(\boldsymbol{\sigma}^+), \qquad \sigma_{eq}^+ = J_2(\boldsymbol{\sigma}^+) \tag{30 a, b}$$

from the positive stress tensor $\boldsymbol{\sigma}^+$:

$$\boldsymbol{\sigma}^+ = \sum_{I=1}^{3} \langle \sigma_I \rangle_+ \mathbf{V}_I \otimes \mathbf{V}_I \tag{31}$$

where $\otimes$ is the tensorial product, $\sigma_I$ are the principal stresses and $\mathbf{V}_I$ the corresponding normalized eigenvectors. For all the computations, the failure criterion was the following:

$$\frac{(\sigma_r)^*}{(1-D_0)} = \sigma_M \tag{32}$$

that can be rewritten as:

$$Y^+ = Y_c \tag{33}$$

where $Y^+$ is the damage energy density release rate referring to the positive tensor, and $Y_c$ is a characteristic of the material.

$D_{0c}$ being close to $D_{0M}$, a relationship between $m$ and $\beta$ and an expression of the threshold stress are derived:

$$m = \beta + 1, \qquad \sigma_u = \sigma_M \left[ 1 - h\left( \frac{a_M}{L} \right) \right] \tag{34 a, b}$$

A scalar damage variable is sufficient to describe the statistical aspect of the failure of brittle materials when Eq. (12), (26) and (34 *a*) are compared. Thus an expression, *a priori* taking into account only the size of the defects, seems *a posteriori* sufficient (*see* [Hild & Marquis, 1990]):

$$P_F = 1 - \exp\left[ \frac{1}{V_0} \int_\Omega \ln\left\{ 1 - \int_{a_c}^{a_M} f(a)\, da \right\} dV \right] \quad \text{with} \quad a_c = L h^{-1}\left[ 1 - \frac{(\sigma_r)^*}{\sigma_M} \right] \tag{35}$$

The correlation between defect distribution parameters ($\beta$ and $a_M$) and Weibull's parameters ($m$, $\sigma_u$ and $\sigma_0$) could be used to get informations on the flaw distribution within a brittle material from a statistical analysis of a set of macroscopic failure tests.

## 4. Application to a beta distribution

The scope of this section is to study the influence of the experimental parameters using expressions (35) and (19). After performing failure tests for different types of loading, and analyzing the failure reasons (*see* [Hild *et al.*, 1990]), it is proposed to characterize the various influences on the failure stresses by three main parameters, viz. the distribution of defects, the size of the structure and the heterogeneity of the stress field.

In the sequel, the defect distribution is modelled by a Beta function (see *Fig.* 5), which satisfies the assumptions of Section 3.1 and takes into account the value of the most critical flaw $a_M$:

$$f(a)=\frac{a_M^{-1-\alpha-\beta}}{B_{\alpha\beta}}a^{\alpha}(a_M-a)^{\beta}\, a\in[0,a_M], \qquad \alpha>-1 \quad \text{et} \quad \beta>0 \tag{36}$$

where $\alpha$ and $\beta$ are the parameters of the Beta function, $B_{\alpha\beta}$ is equal to $B(\alpha+1,\beta+1)$, the function B is the Euler's integral of the first kind.

The failure modeling uses the above described damage model with $h(x)=x$.

### 4.1. Effect of the defect distribution on the failure probability

For different values of the parameters $\alpha$ and $\beta$ of the defect distribution function, the failure probability is studied in pure tension. Using expression (35), the cumulative failure probability is computed (see *Fig.* 6).

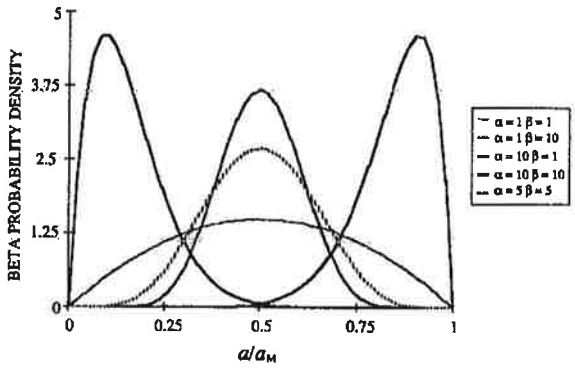


Fig. 5

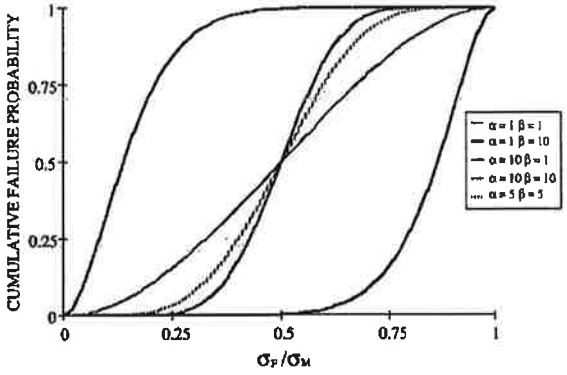


Fig. 6

Fig. 5. – Function Beta for different values of the parameters $\alpha$ and $\beta$.

Fig. 6. – Effect of the defect distribution on the failure probability in tension for a given volume $V=V_0$.

In the studied case ($h(x)=x$), it is possible to relate the two first moments of the failure stress and the two first moment of the defect size.

$$\bar{\sigma}=\sigma_M\left[1-\frac{\bar{a}}{L}\right] \tag{37}$$

$$\tilde{\sigma}=\sigma_M\frac{\tilde{a}}{L} \tag{38}$$

where $\bar{\sigma}$ and $\tilde{\sigma}$ stand for the mean failure stress and the failure stress standard deviation, and $\bar{a}$ and $\tilde{a}$ stand for the mean defect size and the corresponding standard deviation. It is worth noting that when $\alpha=\beta=1$; $\alpha=\beta=5$; $\alpha=\beta=10$ the value of $\bar{a}$ is equal to $0.5\,a_M$ and the mean failure stress is for these cases equal to $0.5\,\sigma_M$.

### 4.2. Effect of the size of a structure on its failure probability

Experimentally, for brittle materials (*see* [Kadlecek & Spetla, 1967] and [Katayama & Hattori, 1982]), the larger the loaded volume, the higher the failure probability. Since expression (1) is used, the volume effect is directly included in Expression (35).

For a given defect distribution ($\alpha=1$, $\beta=10$) and a given stress field (tensile stress field), the mean failure stress decreases and tends asymptotically to $\sigma_u=\sigma_M\ (1-D_{0M})$ when the volume of the structure increases and to $\sigma_u=\sigma_M$ when the volume of the structure tends to zero. Using Eq. (35), Figure 7 shows the results of the computation for five different orders of volumes: the larger the volume, the higher the probability to find a critical defect, the higher the failure probability. When increasing the volume by two orders of magnitude, the mean failure stress is reduced approximately by one third. This result is consistent with experimental observations on silicon nitride specimens [K & H, 1982].

### 4.3. Effect of the stress heterogeneity (H effect) on the failure probability

Experimentally, for a given loaded volume V and a given defect distribution, the more heterogeneous the stress field for a given criterion, the lower the probability to find a critical defect in the most loaded area; thus, the lower the failure probability (*see* [H *et al.*, 1992]). Since the value of the critical defect size depends on the stress field (therefore on its heterogeneity), expression (35) can also give the trend of H effect.

To distinguish quantitatively the different loading patterns, a stress heterogeneity factor H has been introduced to characterize the heterogeneity of the stress field:

$$H=\begin{cases}\dfrac{1/V\displaystyle\int_\Omega \sigma_r(M)\,dV}{\underset{\Omega}{\mathrm{Max}}\,\sigma_r(M)} & \text{if } \underset{\Omega}{\mathrm{Max}}\,\sigma_r(M)>0\\ 0 & \text{otherwise}\end{cases} \tag{39}$$

where $\sigma_r$ is an equivalent failure stress. This heterogeneity factor $H(H\in[0,1])$ also represents the first normalized moment of a stress distribution function (*see* [Hild *et al.*, 1990]). The more heterogeneous the stress field, the lower the stress heterogeneity factor. For example, using criterion (29) H is equal to 1 for a tensile loading as the stress is homogeneous whereas H is equal to 0.25 for a pure bending loading as the stress field is linearly heterogenous in the beam thickness and as only the tensile stresses are considered in criterion (29).

Figure 8 shows the computation for tension ($H=1$), torsion of cylinders ($H=0.667$), tension + bending of parallelepipeds ($H=0.5$), pure bending of parallelepipeds ($H=0.25$), three-point bending of parallelepipeds ($H=0.125$) and disc bending ($H=0.086$) by means of expression (35). Expression (35) leads to a good qualitative agreement with experimental results (*see* [H *et al.*, 1992]). But these computations overestimate the influence of the stress heterogeneity on the failure strength. For example experimentally the ratio between the mean failure stresses in disc bending and in three-point bending is equal to 1.1 (*see*

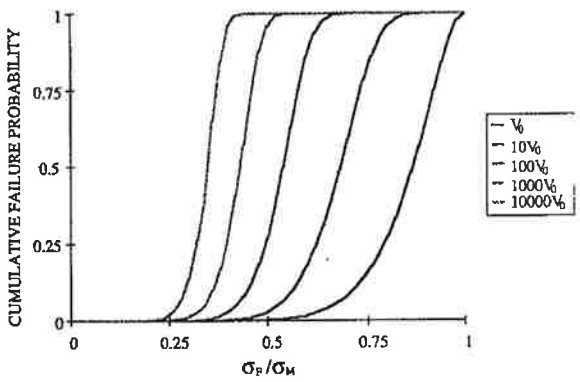


Fig. 7

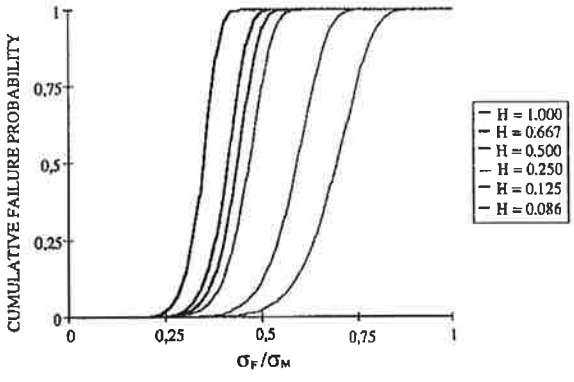


Fig. 8

Fig. 7. – Volume effect in tension for a given defect distribution ($\alpha = 1$, $\beta = 1$).

Fig. 8. – Stress heterogeneity effect for a given volume $V = 10\,000\,V_0$ and a given defect distribution ($\alpha = 1$, $\beta = 10$).

[H. *et al.*, 1992]). On the other hand the computed ratio is 1.2. Reducing the difference between these two ratios could be a good guideline to select the proper defect distribution. This work is in progress.

## 5. Conclusion

Expression (35) gives a simple relationship between the defect distribution and the failure probability of a structure. Therefore the reliability of structures made of brittle material is directly linked to the defects (*i.e.*, their size) distributed within the material. Expression (35) is qualitatively in good agreement with the main experimental effects, and can be used as post-processor of a reliability analysis through F.E.M. calculation for more complex structures and loading (*see* [Hild & Marquis, 1991]). Local fracture models, deduced from both Linear Fracture Mechanics or Continuum Damage Mechanics, define the critical defect size over which a defect leads to the rupture of a structure. Under certain hypotheses, some correlations between the parameters of the defect distribution and Weibull's parameters are derived.

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