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Version of Record: https://www.sciencedirect.com/science/article/pii/S0005109819302754
Manuscript_4778a71921486bbe28d4e6e46fa35568

Observer design for a class of disturbed nonlinear systems with time-varying delayed outputs using mixed time-continuous and sampled measurements

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Abstract

A state observer is proposed for a class of uniformly observable systems involving bounded disturbances. The system outputs are delayed with a known (time-varying) delay and their measurements are continuously available over some time horizons and sampled over other ones. The rational behind the proposed observer design consists in the consideration of an appropriate corrective term provided by a linear Delay Differential Equation (DDE), which accounts for the form under which the output measurements are available as well as for the presence of delay, through a well defined piece-wise continuous function depending on the sampling instants and on the output (time-varying) delay. Of particular interest, the proposed observer has the same dimension as the system and is of the high gain variety in the case where the magnitude of the output (time-varying) delay is relatively small. The arbitrarily long delays are handled thanks to a suitable observer structure composed of cascaded subsystems each one of which has the same dimension as the system. More specifically, the cascade observer is first designed in the constant long delay case before being extended to the time-varying long delay case using a suitable procedure. The convergence analysis is performed thanks to a comprehensive Lyapunov approach under a well-defined condition on the maximum value of the sampling partition diameter. The effectiveness of the proposed observer is emphasized through illustrative simulation results.

Key words: Nonlinear systems, Output delay, Sampled outputs, Time-varying delay, High gain observer, Cascade observer.

1 Introduction

Over the last two decades, an intensive research activity has been devoted to investigate the state estimation problem for linear and nonlinear systems involving delayed outputs. The underlying observer design problem has been comprehensively examined in [9] for linear systems and many state observers have been proposed for nonlinear systems using various approaches (see [10] and references therein for small output delays and [6, 7, 12] for arbitrarily long output delays). A special attention has been paid to the state observer design of systems subject to unknown disturbances and delayed outputs which is still a challenging problem (See for instance [3] and [4]). An unknown input observer has been proposed in [4] using an appropriate LMI-based approach. The observer error was shown to be bounded up to the feasibility of a LMI set. A cascade observer has been proposed in [3] for a class of uniformly observable nonlinear systems assuming that the output measurements are continuously available. This observer has been subsequently redesigned to deal with the output measurement sampling process. Of fundamental interest, it has been shown that the ultimate bound of the asymptotic estimation error is a non increasing function of the cascade length.

The aforementioned results have been obtained by assuming that the delayed output measurements are exclusively available under a continuous form or through an appropriate sampling process. A redesign of the proposed observers is hence always required to account for the nature under which the (delayed) output measurements are available. Moreover, most of available results dealing with output delays have been devoted to the constant delay case. In [2], a class of uniformly observable systems with long time-varying delays has been considered. The authors considered a non uniform partition of the time delays and proposed a cascade observer constituted by m chained dynamical systems which are designed such that the head of the cascade provides an estimate of the actual state. However, in order to account for the delays time variations, the structure of the corrective term involved in each system is not fixed as in the constant delay case but it is appropriately updated according to the instantaneous values of the delays. In [5], the authors proposed a solution that handles the time-

varying delays in a similar way as the constant ones. The idea is to introduce a further delay on the available delayed output in such a way that the resulting overall delay is constant and it is equal to the maximum of the original time-varying output delay. In [2] and [5], the delayed output measurements are assumed to be available in a continuous way; the time delay was assumed to be continuous with respect to time in [2] while it was assumed in [5] to be differentiable with respect to time with a first time derivative strictly lower than 1.

This paper is devoted to further developments of the observer design for a class of MIMO disturbed uniformly observable systems with delayed outputs, similar to that considered in [4]. Three main features are worth to be emphasized with respect to the class considered in [4]. Firstly, the delayed output measurements which were assumed to be available exclusively under one form, i.e. continuously in time or only under samples, may be available in the actual context under a mixed form, i.e. continuously in time over some intervals of time and only under samples over other ones. Secondly, the observer design is particularly alleviated in the case of arbitrarily small output delays. Indeed, one will show that it is possible to handle the observer design in the case of relatively small delays using a high gain observer having the same dimension as the system unlike in [4] where the cascade observer is required whatever the output delay value is. Thirdly, the output delay may be time-varying unlike in [4] where it is assumed to be constant. Of a fundamental interest, the time-varying delay problem is turned into a constant delay problem thanks to a suitable procedure.

The paper is organized as follows. Section 2 is devoted to the introduction of the class of systems that will be considered for the observer design throughout the paper together with a technical lemma that will be used to perform the convergence analysis of the proposed observers. The scenario describing the output measurements availability is detailed in section 3. Section 4 is devoted to the observer design in the case of an output delay, possibly time-varying, with a relatively small magnitude. A particular emphasis is put on the recovery of the delay-free output case within the proposed design framework. The long constant delay case is addressed in section 5 thanks to a suitable cascade structure. Section 6 is dedicated to the case of long time-varying delays. More specifically, an appropriate procedure is used to reformulate the time-varying delay problem as a constant long delay one, allowing thereby to perform the observer design using the approach pursued in section 5. The effectiveness of the proposed observers is highlighted through an illustrative academic example in section 7. Some concluding remarks are given in section 8 to further precise the main motivations of the paper and the related open problems.

Throughout the paper, $\mathbb{R}^+$, I_n and 0_n will respectively denote the set of non negative reals, the n-dimensional identity and zero matrices, and $\|\cdot\|$ denotes the euclidian norm.

2 Problem formulation and preliminaries

Consider the class of multivariable nonlinear systems that are diffeomorphic to the following block triangular form

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y_a(t) = Cx(t - \bar{d}(t)) = x^{(1)}(t - \bar{d}(t)) \end{cases} \tag{1}$$

with

$$x = \left(x^{(1)T} \ \dots \ x^{(q)T} \right)^T \in \mathbb{R}^n, x^{(k)} \in \mathbb{R}^p, k = 1, \dots, q,$$
$$\varphi(u, x) = \left(\varphi^{(1)T}(u, x^{(1)}) \ \dots \ \varphi^{(q)T}(u, x) \right)^T,$$
$$A = \begin{pmatrix} 0_{(q-1)p,p} & I_{(q-1)p} \\ 0_{p,p} & 0_{p,(q-1)p} \end{pmatrix}, \ B = \left(0_p \ \dots \ 0_p \ I_p \right)^T,$$
$$C = \left(I_p \ 0_p \ \dots \ 0_p \right) \tag{2}$$

where $x^{(k)} \in \mathbb{R}^p$ are the state variables blocks, each nonlinear function $\varphi^{(k)}(u, x) \in \mathbb{R}^p$ is triangular with respect to x, for $k = 1, \dots, q$, $u(t) \in U$ a compact subset of $\mathbb{R}^m$ denotes the system input and $y_a \in \mathbb{R}^p$ denotes the available output; $\varepsilon : \mathbb{R}^+ \mapsto \mathbb{R}^p$ is an unknown disturbance. It is worth noticing that the available output y_a is expressed as a function of $\bar{d}$ which denotes a piecewise time continuous function which accounts for the form of the output measurements (continuous in time or a sequence of samples) with the eventual presence of a (time-varying) delay. The specification of this signal will be given later.

The objective consists in designing a state observer which performs a continuous time estimation of the full state of system (1) using the available output measurements. The observer design requires the following assumptions.

A1.The state $x(t)$ and the control $u(t)$ are bounded, i.e. $x(t) \in \Omega$ and $u(t) \in U$ where $\Omega \subset \mathbb{R}^n$ and $U \subset \mathbb{R}$ are compact sets.

A2. The functions $\varphi^{(i)}$, $i = 1, \dots, q$, are Lipschitz with respect to x uniformly in u, i.e.

$$\exists L_\varphi > 0; \ \forall u \in U; \ \forall (x, \bar{x}) \in \Omega \times \Omega,$$
$$\|\varphi^{(i)}(u, x) - \varphi^{(i)}(u, \bar{x})\| \leq L_\varphi \|x - \bar{x}\|. \tag{3}$$

A3. The unknown disturbance ε is an essentially bounded function, i.e.

$$\exists \delta_\varepsilon > 0 \ \text{ such that } \ \|\varepsilon\|_\infty \triangleq ess \sup_{t \geq 0} \|\varepsilon(t)\| \leq \delta_\varepsilon. \tag{4}$$

More specifically, one will use the following notation

- Δ_θ is the (block) diagonal matrix given by
$$\Delta_\theta = diag\left(I_p, 1/\theta I_p, \dots, 1/\theta^{q-1} I_p\right) \ \text{ with } \ \theta > 0, \tag{5}$$
- K is a $n \times p$ matrix defined as follows
$$K = \left(k_1 I_p \ \dots \ k_q I_p \right)^T \in \mathbb{R}^{n \times p} \ \text{ such that}$$
$$\tilde{A} \triangleq A - KC \ \text{ is a Hurwitz matrix}, \tag{6}$$

where A and C are given in (2) and $k_i > 0$, $i = 1, \dots, q$ are real numbers. Notice that since the matrix $\tilde{A} \triangleq A - KC$ is Hurwitz, there exist a $n \times n$ Symmetric Positive

Definite (SPD) matrix P and a positive real μ such that

$$P\tilde{A} + \tilde{A}^T P \leq -2\mu I_n \tag{7}$$

- λ_M (resp. λ_m) is the maximum (resp. minimum) eigenvalue of the SPD matrix P given by (7) and $\sigma = \sqrt{\lambda_M/\lambda_m}$ is its conditioning number.

Remark 2.1 *As pointed out in [4], since the system state trajectory lies in a bounded set Ω, one can extend the nonlinearities $\varphi(u,x)$ in such a way that this extension becomes globally Lipschitz on the entire state space $\mathbb{R}^n$. The observer synthesis will then be based on the resulting extended system which coincides with the original system on the domain of interest i.e. Ω. One can refer to [11, 8] and references therein for more details on how to carry globally Lipschitz prolongations in the state observation context.*

Before detailing the scenario describing the output measurements availability, one states the following technical lemma the proof of which is given in Appendix A.

Lemma 2.1 *Consider a differentiable function $v : t \in [t_0 - \delta, +\infty[\mapsto v(t) \in \mathbb{R}^+$ with $t_0, \delta \geq 0$, satisfying the following inequality*

$$\dot{v}(t) \leq -av(t) + b\int_{t-\delta}^{t} v(s)ds + p(t), \quad \forall t \geq t_0, \tag{8}$$

where $a > 0$, $b \geq 0$ and $p(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is an essentially bounded function with $|p|_\infty = ess \sup_{t \geq t_0} p(t) \leq \delta_p$.
If $b\delta/a < 1$, then the function v satisfies for all $t \geq t_0$,

$$v(t) \leq e^{-\epsilon a(t-t_0)}(1 + (1-\epsilon)\delta a) \max_{\nu \in [t_0-\delta, t_0]} v(\nu) + \delta_p/(\epsilon a), \tag{9}$$

where ϵ is such that $0 < \epsilon \leq 1$ and satisfies

$$\frac{b}{a}\frac{(e^{\epsilon a\delta} - 1)}{\epsilon a} \leq 1 - \epsilon. \tag{10}$$

Now, one shall describe more precisely the scenario related to the availability of the output measurements.

3 Availability of the output measurements

The output measurements may be available under a continuous form over some time intervals and only under sampling measurements over other ones. In both cases, these output measurements are available with no delay or affected by a delay which may be time-varying. More precisely, let $N_0 \geq 0$ and $N_i > 0$, $i = 1, 2, \ldots$ be integer numbers and let $0 \leq t_0^{(0)} \leq t_0^{(N_0)} < t_1^{(0)} < t_1^{(N_1)} < \ldots < t_k^{(0)} < t_k^{(N_k)} < t_{k+1}^{(0)} < \ldots$ be real numbers. The output measurements take one of the following forms

1. For all $t \in [t_k^{(0)}, t_k^{(N_k)}[, k \geq 0, n = 0, \ldots, N_k - 1$, the output measurements are available only at some sampling time instants. This means that each interval $[t_k^{(0)}, t_k^{(N_k)}[$ is the union of N_k sub-intervals, i.e. $[t_k^{(0)}, t_k^{(N_k)}[= \bigcup_{n=0}^{N_k-1} [t_k^{(n)}, t_k^{(n+1)}[$ with $t_k^{(0)} < t_k^{(1)} < \ldots < t_k^{(N_k)}$, such that a sample of the output measurements is available at each sampling instant $t_k^{(n)}$, $n = 0, \ldots, N_k - 1$. Furthermore, one shall naturally assume that there exist $0 < \tau_m \leq \tau_M < +\infty$ such that, $\forall k \geq 0, n = 0, \ldots, N_k - 1$,

$$0 < \tau_m \leq \tau_k^{(n)} = t_k^{(n+1)} - t_k^{(n)} \leq \tau_M, \tag{11}$$

2. For all $t \in [t_k^{(N_k)}, t_{k+1}^{(0)}[$, the output measurements are continuously available.

Notice that the time instants $t_0^{(0)}$ and $t_0^{(N_0)}$ are such that $t_0^{(0)} \leq t_0^{(N_0)}$. Hence, in the case where $N_0 = 0$, the first output measurements are continuously available over the time interval $[t_0^{(0)}, t_1^{(0)}[$ and they become available only at sampling time instants over the time interval $[t_1^{(0)}, t_1^{(N_1)}[$. The output measurement at the sampling time instant $t_1^{(0)}$ can be interpreted as being the last output measurement received from a continuous flow output measurements over the interval $[t_0^{(0)}, t_1^{(0)}[$. In the case where $N_0 > 0$, the first output measurements are available under the form of samples at the time instants $t_0^{(0)}, \ldots, t_0^{(N_0)}$. The output measurement at the sampling time instant $t_0^{(N_0)}$ can be interpreted as the last sampled output measurement on the interval $[t_0^{(0)}, t_0^{(N_0)}[$. But, since the output measurements become continuously available at the time instant $t_0^{(N_0)}$, the sampled output measurement at the instant $t_0^{(N_0)}$ can also be interpreted as the initial data received from the output measurements continuous flow over the interval $[t_0^{(N_0)}, t_1^{(0)}[$.
Now, let us denote by $d(t)$ the delay which may affect the output. Such a delay can be expressed as follows

$$\begin{cases} d(t) = d_c(t), \ \forall t \in [t_k^{(N_k)}, t_{k+1}^{(0)}], \ k \geq 0, \\ d(t) = d_c\left(t_k^{(n)}\right), \ \forall t \in [t_k^{(n)}, t_k^{(n+1)}[, k \geq 0, n = 0, \ldots, N_k - 1, \end{cases} \tag{12}$$

where $d_c : [t_0^{(0)}, +\infty[\to \mathbb{R}^+$ is a non negative real-valued function which is assumed to be known, continuous with respect to time and bounded by $d_M > 0$. According to the definition (12), the delay function $d(t)$ is bounded for all $t \geq t_0^{(0)}$, i.e.

$$\exists d_M > 0; \forall t \geq t_0^{(0)}, \ |d(t)| \leq d_M. \tag{13}$$

Notice that, in the case where the output measurements are available with no delay, one obviously has $d(t) \equiv 0$, for all $t \geq t_0^{(0)}$.
To summarize, the (delayed) output measurements are available according to the following scenario.
• For $t \in [t_k^{(N_k)}, t_{k+1}^{(0)}[$, the (delayed) output measurements are continuously available

$$y_a(t) = y(t - d(t)), \ \forall t \in [t_k^{(N_k)}, t_{k+1}^{(0)}[, \ k \geq 0. \tag{14}$$

• For $t \in [t_k^{(n)}, t_k^{(n+1)}[$, the (delayed) output measurements are available only at sampling time instants

$$y_a(t_k^{(n)}) = y\left(t_k^{(n)} - d\left(t_k^{(n)}\right)\right), \ \forall k \geq 0, \ n = 0, \ldots, N_k - 1, \tag{15}$$

$$= y\left(t - (t - t_k^{(n)} + d\left(t_k^{(n)}\right))\right). \tag{16}$$

Now, one shall define the time-varying signal $\bar{d} : [t_0^{(0)}, +\infty[\to \mathbb{R}^+$ given in (1), which allows to express the output measurements whatever is the form under which it is available, i.e. continuous or sampled form, and in both cases, the output delay-free case and the delayed output case. Indeed, one defines $\bar{d}$ as follows

$$\begin{cases} \bar{d}(t) = d(t), \ \forall t \in [t_k^{(N_k)}, t_{k+1}^{(0)}], \ k \geq 0, \\ \bar{d}(t) = t - t_k^{(n)} + d\left(t_k^{(n)}\right), \\ \qquad \forall t \in [t_k^{(n)}, t_k^{(n+1)}[, \ k \geq 0, \ n = 0, \ldots, N_k - 1. \end{cases}$$

It is worth noticing that the function $\bar{d}$ is well defined on $[t_0^{(0)}, +\infty[$ and is continuous on each $[t_k^{(N_k)}, t_{k+1}^{(0)}[$ for $k \geq 0$ as well as on each $[t_k^{(n)}, t_k^{(n+1)}[$ for $k \geq 0$ and $n = 0, \ldots, N_k - 1$. Furthermore, this function is piece-wise continuous on $[t_0^{(0)}, +\infty[$ and is bounded, i.e.

$$\forall t \geq t_0^{(0)}, \ \bar{d}(t) \leq d_M + \tau_M, \tag{17}$$

where τ_M and d_M are given by (11) and (13), respectively. Now, according to equations (14) and (16) which give the expressions of the available output measurements under the continuous and sampled forms, respectively, it is easy to check that these expressions can be merged into the following single expression:

$$\forall t \geq t_0^{(0)}, \ y_a(t) = y(t - \bar{d}(t)) = Cx(t - \bar{d}(t)). \tag{18}$$

In the remaining of the paper, one shall consider the observer design in three situations, namely the relatively small (time-varying) delay, the constant arbitrarily long delay and finally the time varying arbitrarily long delay. It should be emphasized, that the output delay is assumed to be known in all these situations.

4 Small (time-varying) delay case

Assume that the considered class of systems exhibits a relatively small (time-varying) delay, then the state estimation problem can be handled by the following observer

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - G(t), \ \textit{where } \forall t \geq t_0^{(0)}, \\ G(t) = \theta \Delta_\theta^{-1} K \left(g(t) - g(t - \bar{d}(t)) + C\hat{x}(t - \bar{d}(t)) - y_a(t) \right), \\ \dot{g}(t) = -\theta k_1 \left(g(t) - g(t - \bar{d}(t)) + C\hat{x}(t - \bar{d}(t)) - y_a(t) \right), \\ g(s) = \psi(s), \ \forall s \in [t_0^{(0)} - d_M - \tau_M, t_0^{(0)}], \end{cases} \tag{19}$$

where $\hat{x} \in \mathbb{R}^n$ is the state estimate, K is a $n \times p$ matrix satisfying the property (6), Δ_θ is the (block) diagonal matrix given by (5) where θ constitutes a design parameter, y_a is the available measured output given by (18), $\bar{d}$ is given by (17), $\psi(s)$ is any arbitrary continuous function on $[t_0^{(0)} - d_M - \tau_M, t_0^{(0)}]$, d_M is the upper bound of the output (time-varying) delay given by (13) and finally τ_M is the upper bound of the sampling partition diameter defined in (11).
The following theorem, the proof of which is given in Appendix B, provides the main properties of the observer while specifying the notion of small (time-varying) delay.

Theorem 4.1 *Consider system (1) subject to Assumption* **A1** *to* **A3** *together with system (19). Then,* $\exists \theta_0 > 0$, $\forall \theta \geq \theta_0$, $\forall u \in U$, *if the upper bound of the sampling partition diameter* τ_M *and the upper bound of the time delay* d_M *respectively given by (11) and (13) are such that*

$$(\tau_M + d_M) < \chi_\theta, \tag{20}$$

then for every $\hat{x}(0) \in \mathbb{R}^n$, *one has*

$$\|\tilde{x}(t)\| \leq \bar{\rho}_0 e^{-\bar{\epsilon} a_\theta (t - t_0^{(0)})} \sup_{s \in [t_0^{(0)} - \tau_M - d_M, t_0^{(0)}]} \|\tilde{x}(s)\| + \bar{M}_0 \delta_\varepsilon, \ \forall t \geq t_0^{(0)}, \ \textit{where} \tag{21}$$

$$\begin{cases} \chi_\theta = \dfrac{\mu}{2(L_\varphi + \theta)\sigma \|K\| \lambda_M}, \ a_\theta = \dfrac{\mu\theta}{2\lambda_M}, \ \bar{M}_0 = \dfrac{\sigma}{\bar{\epsilon} a_\theta}, \\ \bar{\rho}_0 = \sigma \theta^{q-1} \left(1 + (1 - \bar{\epsilon}) a_\theta (\tau_M + d_M)\right), \ \bar{\epsilon} \ \textit{ is such that} \\ 0 < \bar{\epsilon} \leq 1 \ \textit{with } (e^{\bar{\epsilon} a_\theta (\tau_M + d_M)} - 1)/(\bar{\epsilon} a_\theta) \leq (1 - \bar{\epsilon})\chi_\theta, \end{cases} \tag{22}$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$ *with* x *being the unknown trajectory of (1) associated to the input* u, $\hat{x}$ *is any trajectory of system (19) associated to* (u, y_a), $\theta_0 = max(1, 2L_\varphi \sqrt{n} \lambda_M / \mu)$, L_φ *and* μ *are defined in (3) and (7), respectively and finally* λ_M *is the largest eigenvalue of the matrix SPD matrix* P *defined in (7).*

Remark 4.1 *From (22), one can easily check that the ultimate bound of the asymptotic observation error* $\bar{M}_0 \delta_\varepsilon$ *as given by (21) can be made small enough by choosing values of* θ *sufficiently high (while satisfying condition (20)).*

Notice that the upper bound of the output delay, d_M, has to satisfy condition (20), i.e. the magnitude of the output delay has to be relatively small. It should be emphasized that observer (19) can be used in the delay-free case, i.e. $d_M = 0$. In what follows, one shall show that in the absence of output delay, observer (19) can be interpreted as a generalization of two high gain observers designed in [1] for system (1) in the case where the output measurements are available either exclusively under a continuous form for the first observer or exclusively under a sampled form for the second one. Indeed, let us express the gain G in the output delay-free case, i.e. $d(t) \equiv 0$ and let us show that the underlying expressions meet those given in [1].

Let $t \in [t_k^{(N_k)}, t_{k+1}^{(0)}[, k \geq 0$. Then, $\bar{d}(t) \equiv 0$ and $y_a(t) \equiv y(t)$. It is easy to check that the expression of $G(t)$ given by (19) becomes $G(t) = \theta \Delta_\theta^{-1} K \left(C\hat{x}(t) - y(t)\right)$. The so derived expression of G is identical to that corresponding to the high gain observer proposed in [1] in the case where the output measurements are available exclusively under a continuous form.
Now, let $t \in [t_k^{(n)}, t_k^{(n+1)}[, k \geq 0, n = 0, \ldots, N_k - 1$. Then, $\bar{d}(t) = t - t_k^{(n)}$ and $y_a(t) \equiv y(t_k^{(n)})$. The dynamics of $g(t)$ given by (19) specializes as follows

$$\dot{g}(t) = -\theta k_1 \left(g(t) - g(t_k^{(n)}) + C\hat{x}(t_k^{(n)}) - y(t_k^{(n)}) \right). \tag{23}$$

The above first order ODE (23) can be easily solved and one can check that $g(t) - g(t_k^{(n)}) - C\hat{x}(t_k^{(n)}) - y(t_k^{(n)}) =$

$-e^{-\theta k_1(t-t_k^{(n)})}(C\hat{x}(t_k^{(n)}) - y(t_k^{(n)}))$, allowing thereby to rewrite the expression of $G(t)$ given by (19) as follows

$$G(t) = \theta\Delta_\theta^{-1} K e^{-\theta k_1(t-t_k^{(n)})}(C\hat{x}(t_k^{(n)}) - y(t_k^{(n)})). \quad (24)$$

Again, the expression of G given by (24) is identical to that involved in the continuous-discrete time high observer proposed in [1] in the case where the output measurements are available exclusively under samples.

5 Constant long delay case

Assume that the delay is constant and arbitrarily long so that condition (20) is no longer satisfied, i.e. $\tau_M + d \geq \chi_\theta$ where χ_θ is given by (22). Then, the observer (19) cannot be used to deal with the state estimation problem under consideration. The latter can be handled using a cascade observer similar to those proposed in [4]. More specifically, suppose that $\tau_M < \chi_\theta$ and let d_0 be a non negative real such that $\tau_M + d_0 < \chi_\theta$; one defines the function: $\bar{d}_0 : [t_0^{(0)}, +\infty[\to \mathbb{R}^+$ as follows

$$\begin{cases} \bar{d}_0(t) = d_0, \ \forall t \in [t_k^{(N_k)}, t_{k+1}^{(0)}[, \ k \geq 0, \\ \bar{d}_0(t) = t - t_k^{(n)} + d_0, \ \forall t \in [t_k^{(n)}, t_k^{(n+1)}[, \\ \qquad k \geq 0, n = 0, \ldots, N_k - 1. \end{cases} \quad (25)$$

It is easy to check that the function $\bar{d}_0$ is piece-wise continuous on $[t_0^{(0)}, +\infty[$ and bounded by $d_0 + \tau_M$ where τ_M is given by (11).
Now, one introduces the following notations that have already been used in cascade observers [6, 7, 5], $x_j(t) = x\left(t - \underline{d} + \frac{j}{m}\underline{d}\right)$, $u_j(t) = u\left(t - \underline{d} + \frac{j}{m}\underline{d}\right)$, $j = 0, \ldots, m, t \geq -j\frac{d}{m}$, where m is a positive integer that shall be specified more precisely later and

$$\underline{d} = d - d_0. \quad (26)$$

It should be emphasized that the rational behind the cascade observer design is based upon the following properties

$$x_j\left(t - \frac{\underline{d}}{m}\right) = x_{j-1}(t), \ u_j\left(t - \frac{\underline{d}}{m}\right) = u_{j-1}(t), \ j = 1, \ldots, m. \quad (27)$$

The considered state estimation problem can be handled by the cascade observer described by the following cascaded dynamical subsystems

$$\begin{cases} \dot{\hat{x}}_0(t) = A\hat{x}_0(t) + \varphi(u_0(t), \hat{x}_0(t)) - G_0(t), \\ G_0(t) = \theta\Delta_\theta^{-1} K \left(g(t) - g(t - \bar{d}_0(t)) \right. \\ \qquad \left. + C\hat{x}_0(t - \bar{d}_0(t)) - y_a(t)\right), \\ \dot{g}(t) = -k_1\theta \left(g(t) - g(t - \bar{d}_0(t)) \right. \\ \qquad \left. + C\hat{x}_0(t - \bar{d}_0(t)) - y_a(t)\right), \end{cases}$$

and for $j = 1, \ldots, m$, (28)

$$\begin{cases} \dot{\hat{x}}_j(t) = A\hat{x}_j(t) + \varphi(u_j(t), \hat{x}_j(t)) - G_j(t), \\ G_j(t) = e^{\bar{A}\frac{d}{m}} \left(G_{j-1}(t) + (A - \bar{A})\left(\hat{x}_j\left(t - \frac{d}{m}\right) - \hat{x}_{j-1}(t)\right)\right. \\ \left. + \varphi\left(u_{j-1}(t), \hat{x}_j\left(t - \frac{d}{m}\right)\right) - \varphi(u_{j-1}(t), \hat{x}_{j-1}(t))\right), \end{cases}$$

where $\hat{x}_j \in \mathbb{R}^n$ is the estimate of x_j for $j = 0, \ldots, m$, K is a $n \times p$ matrix satisfying the property (6), Δ_θ is the (block) diagonal matrix given by (5), y_a is the available measured output given by (18) where $\bar{d}$ is given by (17), $\bar{d}_0$ is defined as in (25) and finally $\bar{A} \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix which constitutes a design parameter for the cascade observer (28). Now, since the matrix $\bar{A}$ is Hurwitz, it satisfies the following property which is of fundamental interest for the observer convergence analysis [4, 5]

$$\begin{aligned} &\exists \beta \geq 1 \text{ such that } \|e^{\bar{A}t}\| \leq \beta e^{-\bar{a}t} \ \forall t \geq 0 \\ &\text{with } \bar{a} = \min_{i\in\{1,\ldots,n\}} |\Re(\lambda_i(\bar{A}))|, \end{aligned} \quad (29)$$

where $\lambda_i(\bar{A})$, $i = 1, \ldots, n$ are the n eigenvalues of $\bar{A}$ (with negative real parts).
The above observer delay differential equations are initialized as follows

$$\hat{x}_0(t_0^{(0)}) = \hat{x}\left(t_0^{(0)} - \underline{d}\right), \ g(s) = \psi(s), \ \forall s \in [t_0^{(0)} - d_0 - \tau_M, t_0^{(0)}],$$

$$\hat{x}_j(s) = \hat{x}\left(s - \underline{d} + \frac{j}{m}\underline{d}\right), \ \forall s \in \left[t_0^{(0)} - \frac{\underline{d}}{m}, t_0^{(0)}\right], j = 1, \ldots, m,$$

where $\hat{x}(s), s \in [t_0^{(0)} - \underline{d}, t_0^{(0)}]$ is any *a priori* selected estimate of the state vector and $\psi(s), s \in [t_0^{(0)} - d_0 - \tau_M, t_0^{(0)}]$ is any arbitrary continuous function.
Observer (28) has a cascade structure with $m+1$ chained subsystems. The head of the cascade the state of which is denoted $\hat{x}_0$ is a high gain observer and it provides an estimate of the system delayed state $x(t - d + d_0)$. Such an estimate is indeed a prediction of the delayed state $x(t-d)$ with a prediction horizon equal to d_0. Each of the remaining subsystems in the cascade, the state of which is denoted $\hat{x}_j$, predicts the state of the preceding subsystem with a prediction horizon equal to $\frac{d}{m} = \frac{d-d_0}{m}$ in such a way that the state of the last subsystem is an estimate of the system actual state $x(t)$. Notice that an interesting property of observer (28) lies in the fact that only the expression of G_0 depends on the form under which the output measurements are available. The remaining subsystems in the cascade, i.e. subsystems with states $\hat{x}_j, j = 1, \ldots, m$, are by no means of the high gain variety and they keep the same structure over time whatever is the form under which the output measurements are available. Notice that two observers have been proposed in [4] for system (1) and each one is devoted to the case where the (delayed) output measurements are available exclusively under one form. It is worth to be noticed that in the case where $d_0 = 0$, observer (28) coincides with each of these observers and the subsystem located at the head of the cascade provides an estimate of the system delayed state $x(t - d)$.
The properties of the cascade observer are similar to those of the cascade observers proposed in [4]. These properties are summarized in the following theorem which proof is quite similar to those of Theorems 3.1 and 4.1 in [4].

Theorem 5.1 *Consider system (1) subject to Assumption* **A1** *to* **A3** *together with system (28). Then,* $\exists\theta_0 > 0$*;* $\forall\theta \geq \theta_0$*;* $\forall u \in U$*, if*

- *the upper bound of the sampling partition diameter* τ_M *and* d_0 *satisfy* $\tau_M + d_0 < \chi(\theta)$ *where* $\chi(\theta)$ *is given by (22),*

- *the matrix $\bar{A}$ is chosen such that $\bar{a} \leq a_\theta$ where $\bar{a}$ and a_θ are respectively given by (29) and (22),*
- *the number m is chosen such that*

$$\xi(d-d_0)/m < 1, \text{ with } \xi \triangleq \beta(L_\varphi + \|\bar{A} - A\|), \tag{30}$$

where L_φ is the Lipschitz constant of φ, A is the block anti-shift matrix given by (2) and β is given by (29),

then, one has for $j = 1, \ldots, m$,

$$\|\tilde{x}_j(t)\| \triangleq \|\hat{x}_j(t) - x_j(t)\| \leq \bar{\rho}_j e^{-\bar{a}t} + \bar{M}_j \delta_\varepsilon, \ t \geq t_0^{(0)}, \text{ with} \tag{31}$$

$$\left\{ \begin{array}{rcl} \bar{\rho}_j & = & \beta\chi_m^j \bar{\rho}_0 \|\tilde{x}_0(0)\| + 1/\left(1 - \xi(d-d_0)/m\right) \\ & & \left(\xi \displaystyle\int_{-(d-d_0)/m}^{0} \|\tilde{x}_j(s)\| ds + \beta \sum_{k=0}^{j-1} \chi_m^k \|r_{j-k}(0)\|\right), \\ \bar{M}_j & = & \beta\chi_m^j \bar{M}_0 + \left((\beta(d-d_0)/m)\,/\,(1 - \xi(d-d_0)/m\right) \displaystyle\sum_{i=0}^{j-1} \chi_m^i, \end{array} \right.$$

where $\bar{\rho}_0$ and $\bar{M}_0$ are given by (22), respectively, and δ_ε is the upper essential bound of $\|\varepsilon\|$ defined in Assumption **A3**. *The r_k's, $k = 1, \ldots, j$, are time-varying functions which are governed by the following stable ODE, $\dot{r}_k(t) = \bar{A} r_k(t), \forall t \geq 0$. Finally, the expression of χ_m is given by*

$$\chi_m = e^{-\bar{a}\frac{d-d_0}{m}} / \left(1 - \xi(d-d_0)/m\right). \tag{32}$$

Remark 5.1 *Let $\bar{M}_m \delta_\varepsilon$ be the ultimate bound of the error between the actual state $x(t)$ of system (1) and the state of the last subsystem $\hat{x}_m(t)$ in the cascade observer (28) as given by (31) for $j = m$, i.e. $\limsup_{t \to \infty} \|\hat{x}_m(t) - x(t)\| \leq \bar{M}_m \delta_\varepsilon$. Then, it has been shown in ([4], Proposition 3.1) that the sequence $(\bar{M}_m)_{m \in I\!N^\star}$ is non increasing with*

$$\lim_{m \to \infty} \bar{M}_m = \beta \bar{M}_0 e^{(\xi - \bar{a})d} + \beta \left(e^{(\xi - \bar{a})d} - 1\right) / (\xi - \bar{a}), \tag{33}$$

where $\bar{M}_0$ is given by (22), $\beta, \bar{a}$ are given by (29) and ξ by (30).

Remark 5.2 *The expression of the limit of $\bar{M}_m$ given by (33) is constituted by two additive terms. The first term can be made as small as desired by choosing values of θ high enough as pointed out in Remark 4.1. The second term is fixed and corresponds to some limit when m is chosen sufficiently high. An appropriate advise consists in specifying the cascade length m according to the condition (30) with an appropriate trade-off between the observer complexity (due to high values of m) and an admissible behavior of the estimation error (see Remarks 3.2 and 3.3 in [4]).*

6 Time-varying long delay case

Assume that the time delay is still arbitrarily large as before but now it is time-varying. As in the constant case, this delay is assumed to be known. A similar problem has been already addressed in [4] where the output measurements are assumed to be continuously available and the time delay is differentiable with $\dot{d}(t) < 1, \ \forall t \geq 0$. The rational behind the proposed solution consists in reformulating the problem with a time-varying delay as an appropriate problem with a constant delay that can be naturally handled using the cascade observer given in the above section. The involved reformulation allows to recover the essential property of the cascade observer design, namely the identity $x_j\left(t - \frac{d}{m}\right) = x_{j-1}(t)$, which is no longer satisfied in the case of a time-varying delay. Such a recovery requirement has been achieved by introducing an additional time-varying delay on the available delayed output measurements in such a way that the resulting overall delay is constant and is equal to the maximum of the original time-varying output delay. It is worth noticing that the extension of the approach proposed in [5], where the (delayed) output measurements are assumed to be available exclusively under a continuous form, to the case under study is by no means obvious. It requires a significant reconsideration that shall be detailed hereafter using the time-varying delay definition given by the expression (12).

The output measurement samples are available at the time instants $t_k^{(n)}$, $k \geq 0$, $0 \leq n \leq N_k$ and each of these samples corresponds to the output measurements at the time instant $t_k^{(n)} - d(t_k^{(n)})$. Hence, one shall naturally assume that $\left(t_k^{(n)} - d(t_k^{(n)})\right)_{0 \leq n \leq N_k - 1}$ is an increasing sequence. Notice that the time instant $\left(t_k^{(N_k)} - d(t_k^{(N_k)})\right)$ corresponds to the starting point of an interval along which the output measurements will be continuously received; this is why this time instant has not been considered in the above sequence. Similarly, one also naturally assumes that the function $t \mapsto t - d(t)$ is increasing over the time intervals $[t_k^{(N_k)}, t_{k+1}^{(0)}[$, $k \geq 0$, throughout which the output measurements are continuously available. Otherwise said, one assumes that the following conditions hold true.

- $\forall t \in [t_k^{(N_k)}, t_{k+1}^{(0)}[, \ k \geq 0,$ the function $t \mapsto t - d(t)$ is increasing.
- For a fixed $k \geq 0$,

$$(i) \text{ the sequence } \left(t_k^{(n)} - d(t_k^{(n)})\right)_{0 \leq n \leq N_k - 1} \text{ is increasing,} \tag{34}$$

$$(ii) \ t_k^{(N_k - 1)} - d(t_k^{(N_k - 1)}) \leq t_k^{(N_k)} - d(t_k^{(N_k)}). \tag{35}$$

Furthermore, one naturally assumes that the delay function $d(t)$ is bounded, i.e.

$$\exists d_M > 0; \forall t \geq 0 : \ 0 \leq d(t) \leq d_M. \tag{36}$$

Notice that since the function $t - d(t)$ is increasing on $[t_k^{(N_k)}, t_{k+1}^{(0)}[$, then according to Lebesgue's Theorem for the differentiability of monotone functions, this function is differentiable almost everywhere on $[t_k^{(N_k)}, t_{k+1}^{(0)}[$. To simplify, one shall assume that $t - d(t)$ is differentiable everywhere on each interval $]t_k^{(N_k)}, t_{k+1}^{(0)}[$, i.e. as long as the (delayed) output measurements are available under a continuous form, the underlying time delay is assumed to be differentiable. It should be emphasized that this assumption was not required in the case of small delays. Now, let $\left(\nu_k^{(n)}\right)_{0 \leq n \leq N_k}$ be the sequence of time instants defined as follows

$$\nu_k^{(n)} = t_k^{(n)} - d(t_k^{(n)}) + d_M, \ k \geq 0, 0 \leq n \leq N_k. \tag{37}$$

For $0 \leq n \leq N_k - 1$, set $\tau_k^{(n)} = \nu_k^{(n+1)} - \nu_k^{(n)}$. The variable $\tau_k^{(n)}$ is the sampling period taking into account the time-

varying delay $d(t)$ and according to (34) and (35), one has $\tau_k^{(n)} > 0$ for all $k \geq 0,\ n = 0, \ldots, N_k - 2$, and $\tau_k^{(N_k-1)} \geq 0$. One shall denote by $\bar{\tau}_M$ the maximum value of $\tau_k^{(n)}$, i.e.

$$\bar{\tau}_M = \max_{0 \leq n \leq N_k - 1, k \geq 0} \tau_k^n. \tag{38}$$

The rational behind the observer design in the time-varying long delay case is based on the use of a delayed version of the available delayed output measurements in such a way that the overall delay in the so obtained delayed version is constant. Now, one shall explain how to define and compute the additional delay. Indeed, let us define the function $\zeta : \left[\nu_0^{(0)}, +\infty\right[\longrightarrow \mathbb{R}^+$ as follows

$$\begin{cases} \bullet\ \zeta(t) = t + d(\zeta(t)) - d_M,\ \forall t \in \left[\nu_k^{(N_k)}, \nu_{k+1}^{(0)}\right[, \\ \quad with\ \zeta(\nu_k^{(N_k)}) = t_k^{(N_k)}, k \geq 0, \\ \bullet\ \zeta(t) = t_k^{(n)},\ \forall t \in \left[\nu_k^{(n)}, \nu_k^{(n+1)}\right[, k \geq 0, n = 0, \ldots, N_k - 1. \end{cases} \tag{39}$$

The function $\zeta(t)$ satisfies some properties which are summarized in the following proposition the proof of which is given in the Appendix C.

Proposition 6.1 *The function ζ defined by (39) is such that*

1. *(i)* $\forall t \geq \nu_k^{(0)},\ \ \zeta(t) \leq t$,
   (ii) $\forall t \in \left[\nu_k^{(N_k)}, \nu_{k+1}^{(0)}\right[,\ \zeta(t) - d(\zeta(t)) = t - d_M \in \left[t_k^{(N_k)} - d(t_k^{(N_k)}), t_{k+1}^{(0)} - d(t_{k+1}^{(0)})\right[$,
   (iii) $\forall t \in \left[\nu_k^{(n)}, \nu_k^{(n+1)}\right[, k \geq 0, 0 \leq n \leq N_k - 1$, $\zeta(t) - d(\zeta(t)) = t_k^{(n)} - d(t_k^{(n)}) = \nu_k^{(n)} - d_M$.
2. $\forall t \in \left]\nu_k^{(N_k)}, \nu_{k+1}^{(0)}\right[,\ \zeta(t) \in \left]t_k^{(N_k)}, t_{k+1}^{(0)}\right[$ *and $\zeta(t)$ is differentiable at t with*
   $\dot{\zeta}(t) = 1/\left(1 - \dot{d}(\zeta(t))\right),\ \zeta(\nu_k^{(N_k)}) = t_k^{(N_k)}$. (40)

Let us evaluate the available output $y_a(t)$ at $\zeta(t)$. Notice that the obtained signal, i.e. $y_a(\zeta(t))$, is a delayed version of $y_a(t)$ since $\zeta(t) \leq t$ according to 1.(i) of Proposition 6.1. Besides, according to (18), $y_a(\zeta(t)) = y(\zeta(t) - \bar{d}(\zeta(t)))$ where $\bar{d}(t)$ is defined as in (17).

For $t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[= [t_k^{(N_k)} + d_M - d(t_k^{(N_k)}), t_{k+1}^{(0)} + d_M - d(t_{k+1}^{(0)})[$ and according to (17), one has $y_a(\zeta(t)) = y(\zeta(t) - d(\zeta(t)))$ and such a signal is always available since on one hand $\zeta(t) - d(\zeta(t)) \in [t_k^{(N_k)} - d(t_k^{(N_k)}), t_{k+1}^{(0)} - d(t_{k+1}^{(0)})[$ according to 1.(ii) of Proposition 6.1, and on the other hand the output is continuously available on $[t_k^{(N_k)} - d(t_k^{(N_k)}), t_{k+1}^{(0)} - d(t_{k+1}^{(0)})[$.
Similarly, for $t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[= [t_k^{(n)} + d_M - d(t_k^{(n)}), t_k^{(n+1)} + d_M - d(t_k^{(n+1)})[$ and according to (17), one has $y_a(\zeta(t)) = y\left(\zeta(t) - (\zeta(t) - t_k^{(n)} + d(t_k^{(n)}))\right) = y(t_k^{(n)} - d(t_k^{(n)}))$ which is available since $t_k^{(n)} - d(t_k^{(n)}) \leq \nu_k^{(n)}$ according to (37).

Hence, the signal $y_a(\zeta(t))$ is available for all $t \geq \nu_0^{(0)}$. Notice that the obtention of $y_a(\zeta(t))$ requires the calculation of $\zeta(t)$. The latter is equal to $t_k^{(n)}$ for $t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[$ and it is the solution of the ODE given by (40) for $t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$, i.e.

$$\begin{cases} \bullet\ \forall t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[, k \geq 0, 0 \leq n \leq N_k - 1,\ \ \zeta(t) = t_k^{(n)}, \\ \bullet\ \forall t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[, k \geq 0, \\ \quad \dot{\zeta}(t) = 1/\left(1 - \dot{d}(\zeta(t))\right),\ \zeta(\nu_k^{(N_k)}) = t_k^{(N_k)}. \end{cases} \tag{41}$$

At this step, one has proved the availability of the signal $y_a(\zeta(t))$. Now, let us put forward a simple expression of $y_a(\zeta(t))$. Indeed, according to 1.(ii) and 1.(iii) of Proposition 6.1, the delayed output $y_a(\zeta(t))$ can be expressed as follows

$$\begin{cases} \bullet\ \forall t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[, y_a(\zeta(t)) = y(\zeta(t) - d(\zeta(t))) = y(t - d_M), \\ \bullet\ \forall t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[, k \geq 0, 0 \leq n \leq N_k - 1, \\ \quad y_a(\zeta(t)) = y(\zeta(t) - d(\zeta(t))) = y\left(\nu_k^{(n)} - d_M\right). \end{cases} \tag{42}$$

According to (42), the delayed output $y_a(\zeta(t))$ is nothing else than a particular delayed version of the actual output, namely the actual output delayed with a constant delay equal to d_M. Hence, an observer similar to that proposed in the constant long delay case can be designed to estimate the system actual state. The equations of the observer are identical to those of the observer proposed in the constant long delay case, i.e. equations of system (28) with $\underline{d} = d_M - d_0$ and $\bar{d}_0 : [\nu_0^{(0)}, +\infty[\to \mathbb{R}^+$ is given by (compare with (17) and (25))

$$\begin{cases} \bar{d}_0(t) = d_0,\ \forall t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[,\ k \geq 0, \\ \bar{d}_0(t) = t - \nu_k^{(n)} + d_0, \\ \qquad \forall t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[, k \geq 0, n = 0, \ldots, N_k - 1. \end{cases} \tag{43}$$

Recall that $\bar{d}_0$ allows an automatic update of the observer corrective term and as in the small delay case, $\bar{d}_0(t)$ is piece-wise continuous on $[\nu_0^{(0)}, +\infty[$ and is bounded by $d_0 + \bar{\tau}_M$ where $\bar{\tau}_M$ is given by (38). Notice that the delayed available output $y_a(t)$ used in the constant delay case has to be replaced by its delayed version $y_a(\zeta(t))$ where $\zeta(t)$ is computed as in (41).

7 Example

The effectiveness of the proposed cascade observer is investigated through the following dynamical system

$$\begin{cases} \dot{x}_1(t) = x_2(t) - 0.1x_1(t) + u_1(t) \\ \dot{x}_2(t) = x_3(t) - 0.01x_2 - 0.5x_2^3(t) + u_2(t) - x_1(t) \\ \dot{x}_3(t) = -0.3x_3(t) + 0.1\ \tanh(x_3(t)) - x_2 + \varepsilon(t) \\ y_a(t) = x_1(t - \bar{d}(t)) \end{cases} \tag{44}$$

It is easy to see that system (44) is under form (1) with $q = 3,\ p = 1,\ x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T \in \mathbb{R}^3$ is the system state and $u = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T \in \mathbb{R}^2$ is the system input. Moreover, one can check that the system states are bounded.
One assumes that the measurements are recovered at regular time intervals $\left[t_k^{(N_k)}, t_{k+1}^{(1)}, \ldots, t_{k+1}^{(N_{k+1})}\right[$ which the length of each is equal to $\tau = \tau_c + (N-1)\tau_m + \tau_M$ where $\tau_c,\ \tau_m$ and τ_M are positive reals, with $N_0 = 0$ (i.e. $t_0^{(0)} = 0$), $N_k = N > 0$ for $k \geq 1$, and

$$t_k^{(N_k)} = k\tau \text{ and } t_{k+1}^{(n)} = k\tau + \tau_c + n\tau_m,\ \ n = 0, \ldots, N-1,\ k \geq 0.$$

The delay $d(t)$ is chosen equal to $d(t) = 3.75 + 0.25\ \sin(\pi t)$ for all $t \geq t_0^{(0)} = 0$ which leads to $d_M = 4s$. The available delayed output measurements can be expressed as $y_a(t) = y(t - \bar{d}(t))$ where $\bar{d}$ is defined as in (17).
A cascade observer with a time-varying delay has been designed and simulated using the following parameters setting and design parameters

$$\tau = 2s,\ \tau_c = 0.5s,\ \tau_m = 0.25s,\ \tau_M = 0.5s,\ N = 6,$$
$$\theta = 5,\ m = 6,\ d_0 = 0.1s,\ K = \begin{pmatrix} 3 & 3 & 1 \end{pmatrix}^T,\ \bar{A} = A - 2I_3. \quad (45)$$

The system inputs have been set to $u_1(t) = 20\ \cos(0.4\pi t)$ and $u_2(t) = 20\ \sin(0.4\pi t)$, and the initial conditions of the system and the cascade observer have been chosen as $x^T(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and $\hat{x}^T(s) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ for $s \in [-d_M + \tau_M, 0]$. The time evolution of the available delayed output is reported in Figure 1 together with the unknown actual output. Two sets of simulation results have been obtained depending on whether the disturbance is considered or not. The first set of results corresponds to the disturbance-free case and the underlying curves are reported in Figure 2 where the estimate of each state of the system is compared to its actual value issued from the system simulation. In the second set of simulation, the disturbance has been chosen as $\varepsilon(t) = 0.1 + 0.5\ \cos(5t)$. The resulting ultimate estimation errors obtained with 3 values of the cascade length m, namely $m = 4, 10, 30$ are reported in Figure 3. The obtained results actually corroborate the fundamental results and more specifically an interesting property that has been pointed out in Remark 5.2, namely the ultimate estimation error is a non increasing function of the cascade observer length. Notice that other values of m greater than 30 have been considered in the presence of the disturbance but no significant decrease of the ultimate bound has been noticed.

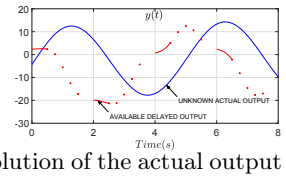

Fig. 1. Time evolution of the actual output with its available delayed measurements

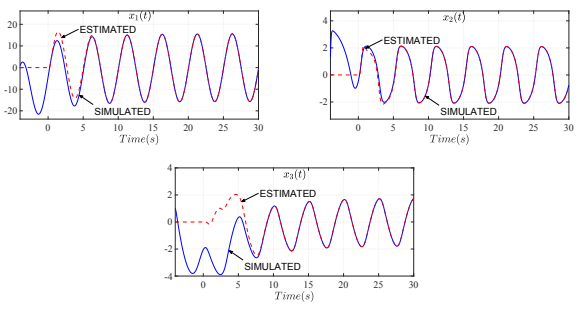

Fig. 2. Estimates of x_1, x_2 and x_3

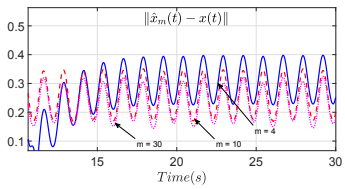

Fig. 3. Norm of the ultimate estimation errors for $m = 4, 10, 30$

8 Conclusion

This paper was mainly motivated by the design of state observers incorporating a suitable ability to account for the nature of the output measurements availability, namely time-varying delayed output measurements that are continuously available over some time horizons and sampled over other ones. Two major assets are worth to be emphasized from a design and fundamental points of view. The first asset concerns the three observer design features. Firstly, the dimension of the observer is equal to that of the system in the case where the (time-varying) output delay is small. Secondly, an appropriate cascade observer is used to handle arbitrarily long delays. Thirdly, the design of the cascade observer has been first proposed in the constant delay case before being extended through a systematic procedure to the time-varying delay one. The second asset is closely related to the nature of the fundamental results. The exponential convergence to zero of the observation error has been first established in the disturbance-free case under a well-defined condition on the maximum value of the sampling partition diameter. Further analysis have been carried out to establish the ultimate convergence of the observation error in the presence of disturbances. A particular emphasis has been put on the fact that the underlying ultimate bound is a non increasing function of the cascade length for arbitrarily long delays.
The proposed design assumes that all the outputs are sampled simultaneously and are affected by the same (time-varying) delay. The extension of the design to multiple sampling rates and multiple output delays is actually under study.

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A Proof of Lemma 2.1

Let us first prove the existence of ϵ given by (10). To this end, notice that since $b\delta/a < 1$ and from the fact that the function $\alpha \mapsto (e^{\alpha\delta}-1)/\alpha$ where $\alpha > 0$ is a real number, is increasing on $[0, +\infty[$, there exists $\bar{\alpha} > 0$ such that, for all $\alpha \in]0, \bar{\alpha}]$, one has [6, 5]

$$(b/a)(e^{\alpha\delta}-1)/\alpha < 1. \tag{A.1}$$

Now, let $0 < \varepsilon_1 \leq 1$ be chosen such that $\varepsilon_1 a \leq \bar{\alpha}$. It is clear that (A.1) is satisfied with $\alpha = \varepsilon_1 a$. Moreover, since all the variables involved in the left side of inequality (A.1) are constant, there exists ε_2 with $0 < \varepsilon_2 \leq 1$ such that

$$(b/a)(e^{\varepsilon_1 a\delta}-1)/(\varepsilon_1 a) \leq 1-\varepsilon_2 < 1. \tag{A.2}$$

Two cases shall be considered depending on whether $\varepsilon_1 \leq \varepsilon_2$ or not.
Case 1: $\varepsilon_1 \leq \varepsilon_2$. In this case, one has $1-\varepsilon_2 \leq 1-\varepsilon_1$ and according to (A.2), one gets $(b/a)(e^{\varepsilon_1 a\delta}-1)/(\varepsilon_1 a) \leq 1-\varepsilon_1$. Hence, inequality (10) is checked with $\epsilon = \varepsilon_1$.
Case 2: $\varepsilon_1 > \varepsilon_2$. In this case, one has

$$\begin{aligned}(b/a)(e^{\varepsilon_2 a\delta}-1)/(\varepsilon_2 a) &< (b/a)(e^{\varepsilon_1 a\delta}-1)/(\varepsilon_1 a)\\ &\leq 1-\varepsilon_2 \text{ according to (A.2).}\end{aligned}$$

Here too, inequality (10) is checked with $\epsilon = \varepsilon_2$.
Consider the following candidate Lyapunov function

$$W(v,t) = v(t) + b\int_0^\delta \int_{t-s}^t e^{\epsilon a(\nu-t+s)} v(\nu)\, d\nu\, ds, \tag{A.3}$$

where ϵ is defined as in (10). One has

$$\dot{W}(v,t) = \dot{v}(t) - \epsilon a\,(W(v,t)-v(t)) + b\frac{e^{\epsilon a\delta}-1}{\epsilon a}v(t) - b\int_{t-\delta}^t v(s)\,ds. \tag{A.4}$$

Using (10), inequality (A.4) leads to

$$\begin{aligned}\dot{W}(v,t) + \epsilon a W(v,t) &\leq \dot{v}(t) + \epsilon a v(t) + (1-\epsilon)av(t)\\ &\quad - b\int_{t-\delta}^t v(s)\,ds \leq p(t).\end{aligned} \tag{A.5}$$

Using the comparison lemma, one gets

$$W(v(t),t) \leq e^{-\epsilon a(t-t_0)} W(v(t_0),t_0) + \delta_p/(\epsilon a),\ \forall t \geq t_0. \tag{A.6}$$

According to (A.3), one has

$$\begin{aligned}W(v(t_0),t_0) &\leq v(t_0) + b\int_0^\delta \int_{t_0-s}^{t_0} e^{\epsilon a(\nu-t_0+s)} d\nu\, ds \max_{\nu\in[t_0-\delta,t_0]} v(\nu)\\ &\leq v(t_0) + b\delta\int_0^\delta e^{\epsilon a s} ds \max_{\nu\in[t_0-\delta,t_0]} v(\nu)\\ &\leq v(t_0) + (1-\epsilon)\delta a \max_{\nu\in[t_0-\delta,t_0]} v(\nu) \text{ according to (10).}\end{aligned} \tag{A.7}$$

From the fact that $v(t) \leq W(v(t),t), \forall t \geq t_0$, the combination of (A.6) and (A.7) leads to (9). This ends the proof of Lemma 2.1.□

B Proof of Theorem 4.1

Let $\tilde{x}(t) = \hat{x}(t) - x(t)$ be the observation error and set $\bar{x} = \theta^{q-1}\Delta_\theta \tilde{x}$ where Δ_θ is given by (5). According to (1) and (19), one has

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + \tilde{\varphi}(u(t),\hat{x}(t),x(t)) - B\varepsilon(t) - G(t), \tag{B.1}$$

where $\tilde{\varphi}(u(t),\hat{x}(t),x(t)) = \varphi(u(t),\hat{x}(t)) - \varphi(u(t),x(t))$. Using (18), the first time derivative of $g(t)$ as well as $G(t)$ given by (19) can be expressed as follows

$$\begin{cases}\dot{g}(t) &= -\theta k_1(g(t) - g(t-\bar{d}(t)) + C\tilde{x}(t-\bar{d}(t)),\\ G(t) &= \theta\Delta_\theta^{-1}K(g(t) - g(t-\bar{d}(t)) + C\tilde{x}(t-\bar{d}(t)).\end{cases} \tag{B.2}$$

Taking into account the facts that $\Delta_\theta A\Delta_\theta^{-1} = \theta A$ and $C\Delta_\theta^{-1} = C$ yields

$$\begin{aligned}\dot{\bar{x}}(t) = \theta A\bar{x}(t) &+ \theta^{q-1}\Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t)) - B\varepsilon(t)\\ &-\theta K(\bar{g}(t) - \bar{g}(t-\bar{d}(t)) + C\bar{x}(t-\bar{d}(t)),\end{aligned} \tag{B.3}$$

where $\bar{g}(t) = \theta^{q-1}g(t)$. Now, according to (B.2), one has

$$\dot{\bar{g}}(t) = -\theta k_1(\bar{g}(t) - \bar{g}(t-\bar{d}(t)) + C\bar{x}(t-\bar{d}(t)). \tag{B.4}$$

Adding and subtracting $\theta KC\bar{x}(t)$ to the right hand side of equation (B.3), one gets

$$\dot{\bar{x}}(t) = \theta\tilde{A}\bar{x}(t) + \theta^{q-1}\Delta_\theta\tilde{\varphi}(u(t),\hat{x}(t),x(t)) - B\varepsilon(t) + \theta Kz(t),$$

where $\tilde{A} = A - KC$ is the Hurwitz matrix defined by (6) and $z(t) = (C\bar{x}(t) - C\bar{x}(t-\bar{d}(t))) - (\bar{g}(t) - \bar{g}(t-\bar{d}(t)))$. The expression of $z(t)$ can be rewritten as follows

$$z(t) = \int_{t-\bar{d}(t)}^t \left(\dot{\bar{x}}^{(1)}(s) - \dot{\bar{g}}(s)\right) ds. \tag{B.5}$$

Moreover, it is easy to check from (B.3) that

$$\begin{aligned}\dot{\bar{x}}^{(1)}(t) = \theta\bar{x}^{(2)} &+ \theta^{q-1}\tilde{\varphi}^{(1)}(u(t),\hat{x}(t),x(t))\\ &-\theta k_1(\bar{g}(t) - \bar{g}(t-\bar{d}(t)) + C\bar{x}(t-\bar{d}(t)).\end{aligned}$$

Using (B.4), the above equation can be rewritten as

$$\dot{\bar{x}}^{(1)}(t) = \theta\bar{x}^{(2)} + \theta^{q-1}\tilde{\varphi}^{(1)}(u(t),\hat{x}(t),x(t)) + \dot{\bar{g}}(t). \tag{B.6}$$

Combining (B.5) and (B.6), one gets

$$z(t) = \int_{t-\bar{d}(t)}^t \left(\theta\bar{x}^{(2)}(s) + \theta^{q-1}\tilde{\varphi}^{(1)}(u(s),\hat{x}(s),x(s))\right) ds. \tag{B.7}$$

Now, let us consider the following candidate quadratic Lyapunov function: $V(\bar{x}) = \bar{x}^T P \bar{x}$ where P is defined as in (7). Proceeding as in [1] and using Assumption **A2**, one can show that for $\theta \geq 1$, one has

$$\begin{aligned}\dot{V}(\bar{x}) &= 2\bar{x}^T P \left(\theta \tilde{A}\bar{x} + \theta^{q-1}\Delta_\theta \tilde{\varphi}(u,\hat{x},x) - B\varepsilon + \theta K z\right) \\ &\leq -(2\mu\theta - 2L_\varphi\sqrt{n}\lambda_M)\|\bar{x}\|^2 + 2\theta\bar{x}^T PKz - 2\bar{x}^T PB\varepsilon. \quad \text{(B.8)}\end{aligned}$$

Choosing θ such that $2\mu\theta - 2L_\varphi\sqrt{n}\lambda_M > \mu\theta$, i.e. $\theta > \theta_0 \triangleq 2L_\varphi\sqrt{n}\lambda_M/\mu$, inequality (B.8) becomes

$$\begin{aligned}\dot{V}(\bar{x}) &\leq -\frac{\mu\theta}{\lambda_M}V(\bar{x}) + 2\theta\|P\bar{x}\|\|K\|\|z(t)\| + 2\|P\bar{x}\|\|\varepsilon(t)\| \\ &\leq -\frac{\mu\theta}{\lambda_M}V(\bar{x}) + 2\sqrt{\lambda_M}\sqrt{V(\bar{x})}\,(\theta\|K\|\|z(t)\| + 2\|\varepsilon(t)\|) \quad \text{(B.9)}\end{aligned}$$

Now, using (17), the variable z given by (B.7) can be bounded as follows for all $t \geq t_0^{(0)}$,

$$\|z(t)\| \leq (\theta + L_\varphi)/\sqrt{\lambda_m}\int_{t-\tau_M-d_M}^{t}\sqrt{V(\bar{x}(s))}ds. \quad \text{(B.10)}$$

Using (B.10), (B.8) leads to

$$\frac{d}{dt}\sqrt{V(\bar{x}(t))} \leq -a_\theta\sqrt{V(\bar{x}(t))} + b_\theta\int_{t-\tau_M-d_M}^{t}\sqrt{V(\bar{x}(s))}ds + p(t),$$

where

$$a_\theta = (\mu\theta)/(2\lambda_M), \quad b_\theta = \theta\sigma\|K\|(L_\varphi + \theta), \quad p(t) = \sqrt{\lambda_M}\|\varepsilon(t)\|. \quad \text{(B.11)}$$

Notice that according to Assumption **A3**, the function $p(t)$ given by (B.11) is essentially bounded by $\sqrt{\lambda_M}\delta_\varepsilon$. Now, if the upper diameter of the sampling partition τ_M as well as the upper bound of the delay d_M satisfy the following condition $(b_\theta(\tau_M + d_M))/a_\theta < 1$, i.e. $\tau_M + d_M < a_\theta/b_\theta \triangleq \chi_\theta = \mu/(2\lambda_M\sigma\|K\|(L_\varphi + \theta))$, then all the conditions of Lemma 2.1 are fulfilled and according to this lemma, one gets

$$\begin{aligned}\sqrt{V(\bar{x}(t))} \leq\ & e^{-\bar{\epsilon}a_\theta(t-t_0^{(0)})}\,(1 + (1-\bar{\epsilon})a_\theta(\tau_M + d_M)) \\ & \max_{\nu\in[t_0^{(0)}-\tau_M-d_M,\,t_0^{(0)}]}\sqrt{V(\bar{x}(\nu))} + \frac{\sqrt{\lambda_M}\delta_\varepsilon}{\bar{\epsilon}a_\theta}, \quad \forall t \geq t_0^{(0)},\end{aligned}$$

where $\bar{\epsilon}$ is such that $0 < \bar{\epsilon} \leq 1$ and $(b_\theta/a_\theta)(e^{\bar{\epsilon}a_\theta(\tau_M+d_M)} - 1)/(\bar{\epsilon}a_\theta) \leq 1 - \bar{\epsilon}$, i.e. $(e^{\bar{\epsilon}a_\theta(\tau_M+d_M)} - 1/(\bar{\epsilon}a_\theta) \leq (1-\bar{\epsilon})\chi_\theta$. Coming back to $\tilde{x}(t)$ and using the fact that $\|\tilde{x}(t)\| \leq \|\bar{x}(t)\| \leq \theta^{q-1}\|\tilde{x}(t)\|$ for all $t \geq t_0^{(0)}$ with $\theta \geq 1$, one gets

$$\|\tilde{x}(t)\| \leq \bar{\rho}_0 e^{-\bar{\epsilon}a_\theta(t-t_0^{(0)})}\max_{\nu\in[t_0^{(0)}-\tau_M-d_M,\,t_0^{(0)}]}\|\tilde{x}(\nu)\| + \bar{M}_0\delta_\varepsilon,$$

where $\bar{\rho}_0$ and $\bar{M}_0$ are given by (22). The proof of Theorem 4.1 is ended. □

C Proof of Proposition 6.1

1. (i) On one hand, for $t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$, one has $\zeta(t) = t - (d_M - d(\zeta(t))) \leq t$, according to the definition of d_M. On the other hand, for $t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[$, one has $\zeta(t) = t_k^{(n)} \leq t_k^{(n)} + d_M - d(t_k^{(n)}) = \nu_k^{(n)} \leq t$.

(ii) Let $t \in [\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$. Since $\zeta(t) - d(\zeta(t)) = t - d_M$, one has $\nu_k^{(N_k)} - d_M \leq \zeta(t) - d(\zeta(t)) < \nu_{k+1}^{(0)} - d_M$. Substituting in this inequality $\nu_k^{(N_k)}$ and $\nu_{k+1}^{(0)}$ by their respective expressions given by (37), one gets

$$t_k^{(N_k)} - d(t_k^{(N_k)}) \leq \zeta(t) - d(\zeta(t)) < t_{k+1}^{(0)} - d(t_{k+1}^{(0)}). \quad \text{(C.1)}$$

(iii) Let $t \in [\nu_k^{(n)}, \nu_k^{(n+1)}[$. According to the definition of ζ given by (39), one has

$$\zeta(t) - d(\zeta(t)) = t_k^{(n)} - d(t_k^{(n)}) = \nu_k^{(n)} - d_M \text{ according to (37).}$$

2. Let $t \in]\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$. One shall first show that $\zeta(t) \in]t_k^{(N_k)}, t_{k+1}^{(0)}[$. Indeed, proceeding as for (C.1), one gets

$$t_k^{(N_k)} - d(t_k^{(N_k)}) < \zeta(t) - d(\zeta(t)) < t_{k+1}^{(0)} - d(t_{k+1}^{(0)}). \quad \text{(C.2)}$$

Suppose that there exists $t^\star \in]\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$ such that $\zeta(t^\star) \geq t_{k+1}^{(0)}$. Then, since $t \mapsto t - d(t)$ is non decreasing for all $t \geq t_0^{(0)}$, one gets $\zeta(t^\star) - d(\zeta(t^\star)) \geq t_{k+1}^{(0)} - d(t_{k+1}^{(0)})$, which is in contradiction with (C.2). Similarly, suppose that there exists $t^\star \in]\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$ such that $\zeta(t^\star) \leq t_k^{(N_k)}$. Then, since $t \mapsto t - d(t)$ is non decreasing for all $t \geq t_0^{(0)}$, one gets $\zeta(t^\star) - d(\zeta(t^\star)) \leq t_k^{(N_k)} - d(t_k^{(N_k)})$, which is again in contradiction with (C.2).
Let us now prove that $\zeta(t)$ is differentiable at t with $\dot{\zeta}(t)$ given as in (40). Indeed, from the fact that $\zeta(t) \in]t_k^{(N_k)}, t_{k+1}^{(0)}[$ and that $d(t)$ is differentiable on $]t_k^{(N_k)}, t_{k+1}^{(0)}[$ with $\dot{d}(t) < 1$ (since $t \mapsto t - d(t)$ is increasing and $d(t)$ is differentiable on this interval), one directly concludes that $\zeta(t)$ is differentiable for all $t \in]\nu_k^{(N_k)}, \nu_{k+1}^{(0)}[$ and equation (40) can be derived by taking the time derivative of both sides of $\zeta(t) = t + d(\zeta(t)) - d_M$. This ends the proof of Proposition 6.1. □