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► **To cite this version:**

Loic Le Gratiet. Reformulation of the GCQC3 rule for three correlated excitation components with different spectra shapes. 2019. <hal-02341647>

HAL Id: hal-02341647

<https://hal.science/hal-02341647v1>

Preprint submitted on 4 Nov 2019

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Reformulation of the GCQC3 rule for three correlated excitation components with different spectra shapes

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1. Abstract

The response spectrum method is commonly used to design multi-degree-of-freedom structures subject to random vibrations. However, the common direction combination rules (SRSS, Newmark 30-100-100, Newmark 40-100-100) consider uncorrelated excitation components. Theoretically, it is always possible to rotate the directions in order to obtain principal directions along which the excitation components are not correlated. This problem is handled by the combinations CQC3 and GCQC3. However, these rules need simplifications to be applicable in an industrial framework. For specific applications, these simplifications are not consistent (e.g. for substructures anchored to a structure subject to plane crash). This article addresses the problem of reformulating the GCQC3 rule when vibrations along each direction are correlated, spectra have different shapes and without any simplifications. This new formulation avoids a lot of calculations and makes the GCQC3 method industrially applicable in its most general form. In particular, it integrates the instantaneous correlations between excitation components. Furthermore, this formulation is propagated to the elliptical response envelope – also called interaction ellipsoid – method.

Keywords: response spectrum method, GCQC3, CQC3, directional correlation, direction combination, interaction hyper-ellipsoid

2. Introduction

The response spectrum method (RSM) is commonly used to design multi-degree-of-freedom structures subject to random vibrations. This method works with linear structures and it uses a modal superposition procedure where each eigenmode corresponds to a single-degree-of-freedom oscillator. Then, the peak responses of modal oscillators under each excitation component are evaluated and combined. The combinations give a probable upper bound for the temporal response of the structure [4,5,6]. Both modal and direction combinations have to be performed. A well-known rule for the modal combination is the Complete Quadratic Combination (CQC) [4,5]. It allows for taking into account the correlation between the eigenmodes. For the direction combinations, the SRSS, the 30% [18] and the 40% [15] rules are the most common. They all assume that the excitation components are uncorrelated. However, this assumption is frequently not verified in practical applications [16,2,7,12]. In that case, the excitation components can be rotated to the principal orthogonal directions along which their correlations are null [16,3,6]. Then, a question of interest for seismic design is to establish the most critical orientation of the principal axes. Since it is often convenient to work with the structure axes, many works have investigated the generalization of the CQC rule when the principal components are rotated with respect to the structure axes [1, 8, 10, 11, 13, 14, 1]. In that framework, the GCQC3 rule has been developed [8, 14]. However, this method in its most general case could be difficult to handle for practical applications. For that reason, many works deal with this problem by suggesting simplifications. For example, [8] suggests a simplification by imposing a restriction on the maximum inclination or intensity of a principal seismic component. Furthermore, most of them consider rotations only for horizontal directions [17, 26, 19, 11, 13, 9]. In particular, in the work [17] the horizontal spectra are supposed to be identical and the solicitation components are uncorrelated. Then, a rule where the spectra have the same shape (i.e. they are equal up to a multiplicative constant) is presented in [26]. The

correlation are taken into account in [19] where the spectra are supposed to have the same shape. A general rule with different spectra shapes and considering the correlations is given in [11]. Finally, the equations of [11] are reused in [13] with identical spectra shapes, the resulting formulation is called CQC3. This last work has been reformulating in [9] in order to avoid the evaluation of the critical orientation which implies lot of calculations.

The works cited previously essentially focus on structures under seismic solicitations. This article concerns the specific application of a substructure, such as equipment, which is anchored to a main structure subject to a vibrational solicitation. This solicitation could be seismic or induced by a plane crash. This application implies that the solicitations at the anchorages have different spectra shapes along the three directions and can present significant correlations between all directions (especially for vibrations induced by plane crash). Therefore, the suggested simplification of the GCQC3 rule [8, 14] are not appropriate. Another particularity of this application is that the solicitations are determined with a time-history numerical model of the main structure. This point allows for reformulating the GCQC3 rule without performing simplifications by introducing the instantaneous correlations between the pairs of directions. Indeed, in practical application, it is straightforward to obtain these correlations by post-processing. The main result of this work is the establishment of an alternative formulation of the GCQC3 rule depending only on these instantaneous correlations. As a consequence, the resulting rule is quite easy to apply which is mandatory in the nuclear industry. For example, for sizing at plane crash, several thousand tests based on RSM have to be performed on substructures anchored to the main structures.

Finally, the suggested reformulation allows to easily adapt the elliptical response envelopes formulae. The reader is referred to [6,20,21,22,23] for more detail about this approach. It allows to work with signed simultaneous responses for the design. More precisely, under the assumptions considered for the response spectrum method, the responses of interest belong to an interaction hyper-ellipsoid which has a closed-form expression. Then, by bounding the hyper-ellipsoid with a polytope, one can deduce concomitant responses which enfold with high probability all possible responses. As for the Square Root of Sum of Squares (SRSS), the 40% and the 30% rules, the elliptical envelopes make the assumption of uncorrelated excitation components (by using a rotation of directions if necessary). This article generalizes this method by introducing the instantaneous correlations between the components.

3. Response to three-dimensional excitations

3.1 Temporal response of a linear structures

The equations of relative motion for a linear multi-degree-of-freedom structure subject to a three-dimensional excitation \mathcal{S} can be written as:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = -\mathbf{M}\mathbf{O}\mathbf{A},$$

where \mathbf{M} , \mathbf{C} and \mathbf{K} respectively are the mass, damping and stiffness matrices, \mathbf{U} is the vector of nodal displacements, $\mathbf{A} = (A_x(t) \ A_y(t) \ A_z(t))^T$ is the vector of accelerations exciting the structure and $\mathbf{O} = (\mathbf{O}_x \ \mathbf{O}_y \ \mathbf{O}_z)^T$ is the matrix coupling the degrees of freedom of the structure to the excitation motion. Let us introduce the matrix $\boldsymbol{\phi}$ containing the eigenmodes of the eigenvalue equation $(\mathbf{K} - \mathbf{M}\omega^2)\boldsymbol{\phi} = 0$ and let us suppose that $(\boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi})^{-1} \boldsymbol{\phi}^T \mathbf{K} \boldsymbol{\phi}$ is diagonal. Then, using the transformation $\mathbf{U} = \boldsymbol{\phi}\mathbf{q}$, the i th uncoupled eigenmode motion equation is:

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q_i = -\mathbf{p}_i\mathbf{A},$$

where $\mathbf{p}_i = \boldsymbol{\phi}_i^T \mathbf{M} \mathbf{O} / (\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i)$ is the vector of participation factors for mode i , $\boldsymbol{\phi}_i$ is the i th column of $\boldsymbol{\phi}$, ω_i is its natural pulsation of mode i and ξ_i is its damping ratio.

Consider a response of interest $R(t)$ that can be written as a linear combination of the nodal displacements, i.e. it has the form $R(t) = \mathbf{d}^T \mathbf{U}$. For a linear structure, responses can be forces, strains, stresses, nodal forces or loads. Then, using a mode superposition approach, $R(t)$ can be written as:

$$R(t) = \sum_{k=x,y,z} \sum_i^n S_{i,k}(t) R_{i,k},$$

where n is the number of eigenmodes, $R_{i,k} = (\boldsymbol{\phi}_i^T \mathbf{M} \mathbf{O}_k / (\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i)) \mathbf{d}^T \boldsymbol{\phi}_i$ is the modal response weighted by its participation factor and $S_{i,k}(t)$ is the response of an 1dof oscillator of pulsation ω_i and damping ratio ξ_i to the excitation $A_k(t)$.

3.2 Response Power Spectral Density for correlated excitation components

Similarly to [4,5] the excitation A is supposed to be a zero-mean stationary random process. Nevertheless, the components of A are not anymore considered as independent. Therefore, one cannot consider them separately as in [4,5]. According to the multivariate Bochner's Theorem [24], the Power Spectral Density (PSD) of a multivariate stationary random process has the following form:

$$G_A(\omega) = \begin{pmatrix} G_{A_{xx}}(\omega) & G_{A_{xy}}(\omega) & G_{A_{xz}}(\omega) \\ G_{A_{xy}}(\omega) & G_{A_{yy}}(\omega) & G_{A_{yz}}(\omega) \\ G_{A_{xz}}(\omega) & G_{A_{yz}}(\omega) & G_{A_{zz}}(\omega) \end{pmatrix},$$

where $\forall \omega, G_A(\omega)$ is nonnegative definite. In practice, this property needs to be verified a posteriori. Then, by the means of linearity, the power spectral density of the response is:

$$G_R(\omega) = \sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i=1}^n \sum_{j=1}^n G_{A_{kl}}(\omega) R_{i,k} R_{j,l} H_i(\omega) H_j^*(\omega),$$

where $*$ stands for the complex conjugate and $H_i(\omega)$ is the complex frequency response function of a 1dof oscillator with damping ratio ξ_i and pulsation ω_i :

$$H_i(\omega) = \frac{1}{\omega_i^2 - \omega^2 + 2i\xi_i\omega_i\omega}.$$

It can be emphasized that $G_R(\omega)$ is a natural generalization of the single excitation component case presented in [4,5]. The m -order spectral moment of the response is:

$$\lambda_m = \int_0^\infty \omega^m G_R(\omega) d\omega = \sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i=1}^n \sum_{j=1}^n R_{i,k} R_{j,l} \lambda_{m,ij,kl},$$

where:

$$\lambda_{m,ij,kl} = \text{Re} \left[\int_0^\infty \omega^m G_{A_{kl}}(\omega) H_i(\omega) H_j^*(\omega) d\omega \right].$$

The term $\lambda_{0,ij,kl}$ represents the covariance between $S_{i,k}(t)$ and $S_{j,l}(t)$ and $\lambda_{2,ij,kl}$ represents the covariance between $\dot{S}_{i,k}(t)$ and $\dot{S}_{j,l}(t)$ [4,5]. Therefore, the correlation between $S_{i,k}(t)$ and $S_{j,l}(t)$ writes:

$$\rho_{ij,kl} = \frac{\lambda_{0,ij,kl}}{\sqrt{\lambda_{0,ii,kk}\lambda_{0,jj,ll}}}$$

Then, the mean square of the response $R(t)$ can be written as follows:

$$\lambda_0 = \sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i=1}^n \sum_{j=1}^n R_{i,k} R_{j,l} \rho_{ij,kl} \sqrt{\lambda_{0,ii,kk}\lambda_{0,jj,ll}}$$

According to [5] the mean of the peak response over a duration τ is given by:

$$\bar{F}^{max} = p\sqrt{\lambda_0},$$

where p is a peak factor depending on the duration τ , the mean zero-crossing rate and a shape factor for the response PSD [5]. An estimate value of p is given in Equation (16) in [5]. Similarly to the response $R(t)$, the mean value of the maximum absolute response of a 1dof oscillator of pulsation ω_i and damping ratio ξ_i to the excitation $A_k(t)$ writes:

$$\bar{S}_{i,k} = E \left[\max_t |S_{i,k}(t)| \right] = p_{i,k} \sqrt{\lambda_{0,ii,kk}},$$

where $p_{i,k}$ is given in Equation (16) in [5] with the mean zero-crossing rate $v_{i,k} = \sqrt{\frac{\lambda_{2,ii,kk}}{\lambda_{0,ii,kk}}} / \pi$ and

the response PSD shape factor $\delta_{i,k} = \sqrt{1 - \frac{\lambda_{1,ii,kk}^2}{\lambda_{2,ii,kk}\lambda_{0,ii,kk}}}$. Using the expression of the expected maximum value $\bar{S}_{i,k}$, the mean of the peak response can be expressed as follows:

$$\bar{F}^{max} = \sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i=1}^n \sum_{j=1}^n \frac{p^2}{p_{i,k} p_{j,l}} R_{i,k} R_{j,l} \rho_{ij,kl} \bar{S}_{i,k} \bar{S}_{j,l}}$$

To obtain an estimate of \bar{F}^{max} , the expected maximum value of $\bar{S}_{i,k}$ is substituted by its observed maximal value $\max_t |S_{i,k}(t)|$. Moreover, it is shown in [5] that the ratio $p/p_{i,k}$ is around unity. Their argument is based on the slight dependency of the peak factor to the shape factor and on the fact that the response mean zero-crossing rate is a weighted root-mean-square of the mean zero-crossing rates $v_{i,k}$. From these points, the mean of the peak response value can be approximated by:

$$F^{max} = \sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i=1}^n \sum_{j=1}^n F_{i,k} F_{j,l} \rho_{ij,kl}}$$

where $F_{i,k}$ is the peak modal response :

$$F_{i,k} = \max_t |S_{i,k}(t)| \times R_{i,k}$$

We emphasize that the approximation $p/p_{i,k} \approx 1$ can be avoided by using Equations (14)-(18), (21) and (22) in [5].

4. Generalized CQC rule for correlated excitation components

4.1 Three-components stationary random excitation

A general closed-form expression is given in Section 3 for the response PSD of a structure enduring a multivariate stationary random excitation. By analogy with [4,5], we suppose a

particular probabilistic form for \mathbf{A} . In particular, the following expression is considered for its PSD:

$$G_{\mathbf{A}}(\omega) = G(\omega) \begin{pmatrix} \gamma_x^2 & \gamma_x \gamma_y \tilde{\rho}_{x,y} & \gamma_x \gamma_z \tilde{\rho}_{x,z} \\ \gamma_x \gamma_y \tilde{\rho}_{x,y} & \gamma_y^2 & \gamma_y \gamma_z \tilde{\rho}_{y,z} \\ \gamma_x \gamma_z \tilde{\rho}_{x,z} & \gamma_y \gamma_z \tilde{\rho}_{y,z} & \gamma_z^2 \end{pmatrix},$$

where $G(\omega)$ is the PSD a stationary random process, $\tilde{\rho}_{k,l}$ is the instantaneous correlation between $A_k(t)$ and $A_l(t)$ and γ_k^2 is the second-order moment of $A_k(t)$. From a practical point of view, γ_k^2 is related to the magnitude of the excitation component k and $\tilde{\rho}_{k,l}$ has to be estimated statistically (instantaneous correlation means that no lag is introduced between the excitation components). The nonnegativity of $G_{\mathbf{A}}(\omega)$ can be verified by evaluating the determinant of the right-hand term matrix. The response DSP is:

$$G_{\mathbf{R}}(\omega) = \sum_{k=x,y,z} \sum_{l=x,y,z} \tilde{\rho}_{k,l} \gamma_k \gamma_l \sum_{i=1}^n \sum_{j=1}^n G(\omega) R_{i,k} R_{j,l} H_i(\omega) H_j^*(\omega),$$

and the covariance between $S_{i,k}(t)$ and $S_{j,l}(t)$ writes:

$$\lambda_{0,ij,kl} = \gamma_k \gamma_l \tilde{\rho}_{k,l} \text{Re} \left[\int_0^{\infty} G(\omega) H_i(\omega) H_j^*(\omega) d\omega \right].$$

4.2 Generalized CQC rule (GCQC3)

In the expression of $G_{\mathbf{A}}(\omega)$, we consider a unique PSD $G(\omega)$ for all margins. It means that the excitation components have the same probabilistic form. As in [4,5], we consider a white noise excitation. It corresponds to the following DSP:

$$G(\omega) = G_0,$$

where G_0 is a constant. Since the magnitude of the DSP is already taken into account through the coefficient γ_k , we set $G_0 = 1$. In this framework, the covariance between $S_{i,k}(t)$ and $S_{j,l}(t)$ has the following expression [4,5]:

$$\lambda_{0,ij,kl} = \gamma_k \gamma_l \tilde{\rho}_{k,l} \frac{2\pi}{(\omega_i^2 - \omega_j^2)^2 + 4\xi_i \xi_j (\omega_i^2 + \omega_j^2) + 4(\xi_i^2 + \xi_j^2) \omega_i^2 \omega_j^2} (\xi_i \omega_i + \xi_j \omega_j),$$

and their correlation is:

$$\rho_{ij,kl} = \tilde{\rho}_{k,l} \frac{8\sqrt{\xi_j \xi_i \omega_i \omega_j} (\xi_i \omega_i + \xi_j \omega_j) \omega_i \omega_j}{(\omega_i^2 - \omega_j^2)^2 + 4\xi_i \xi_j (\omega_i^2 + \omega_j^2) + 4(\xi_i^2 + \xi_j^2) \omega_i^2 \omega_j^2} = \tilde{\rho}_{k,l} \rho_{ij}.$$

According to Subsection 3.2, the mean of the peak response value is:

$$F^{max} = \sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i,j=1}^n \rho_{i,j} \tilde{\rho}_{k,l} F_{i,k} F_{j,l}},$$

For the single component case, we retrieve the classical CQC given in [4,5] (i.e. $F^{max} = \sqrt{\sum_{i,j=1}^n \rho_{i,j} F_{i,k} F_{j,k}}$). The expression of F^{max} provides a generalization of the CQC rule – denoted GCQC3 – when excitation components are correlated.

4.3 Discussions about the GCQC3 rule

As presented in Subsection 4.2, one can obtain the peak response value of a quantity of interest by considering the following combination rule:

$$\sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i,j=1}^n \rho_{i,j} \tilde{\rho}_{k,l} F_{i,k} F_{j,l}}.$$

Contrary to the SRSS, the Newmark 30-100-100 and the Newmark 40-100-100 rules, it requires to perform the modal and the direction combinations simultaneously. However, the peak modal responses $(F_{i,k})_{\substack{k=x,y,z \\ i=1,\dots,n}}$ are those already evaluated when using classical modal and direction combinations. Therefore, this generalized rule only requires in addition the evaluation of the correlation coefficients $\tilde{\rho}_{k,l}$. Moreover, the independent case corresponds to $\tilde{\rho}_{k,l} = \delta_{k=l}$ where δ is the delta of Kroenecker. This assumption leads to the following rule:

$$\sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i,j=1}^n \rho_{i,j} \delta_{k=l} F_{i,k} F_{j,l}} = \sqrt{\sum_{k=x,y,z} \sum_{i,j=1}^n \rho_{i,j} F_{i,k} F_{j,l}},$$

which is exactly the classical SRSS rule. Introducing the notation:

$$F^{k,l} = \sum_{i,j=1}^n \rho_{i,j} F_{i,k} F_{j,l},$$

we can explicit the GCQC3:

$$(F^{max})^2 = \sum_{k=x,y,z} \sum_{l=x,y,z} \tilde{\rho}_{k,l} F^{k,l} = \sum_{k=x,y,z} F^{k,k} + 2\tilde{\rho}_{x,y} F^{x,y} + 2\tilde{\rho}_{y,z} F^{y,z} + 2\tilde{\rho}_{x,z} F^{x,z},$$

where $F^{k,k}$ is the square of the classical direction response deduced from a CQC rule. Finally, if only the classical CQC responses are available, one can deduce an upper bound to F^{max} . Indeed, the Cauchy-Schwarz inequality implies:

$$\sum_{k=x,y,z} \sum_{l=x,y,z} |\tilde{\rho}_{k,l} F^{k,l}| \leq \sum_{k=x,y,z} \sum_{l=x,y,z} |\tilde{\rho}_{k,l}| \sqrt{\sum_{i,j=1}^n \rho_{i,j} F_{i,k} F_{j,k}} \sqrt{\sum_{i,j=1}^n \rho_{i,j} F_{i,l} F_{j,l}}.$$

Then, using the triangle inequality the following inequality holds:

$$F^{max} \leq \sqrt{F^{x,x} + F^{y,y} + F^{z,z} + 2|\tilde{\rho}_{x,y}| \sqrt{F^{x,x}} \sqrt{F^{y,y}} + 2|\tilde{\rho}_{y,z}| \sqrt{F^{y,y}} \sqrt{F^{z,z}} + 2|\tilde{\rho}_{x,z}| \sqrt{F^{x,x}} \sqrt{F^{z,z}}}.$$

As a consequence, when the instantaneous correlations are unknown, an upper bound for the response can be provided.

5. GCQC3 method for elliptical response envelopes

As for the SRSS, Newmark 30-100-100 and Newmark 40-100-100 rules, the GCQC3 is valid for a single response [25]. As state in [25], to deal with multiple response, SRSS, 30%, 40% and GCQC3 rules can be performed for each response and then all combinations of maximum responses have to be considered. This can lead to strong overestimates. The elliptical response envelopes method can be used to avoid them. It allows to work with concomitant responses by considering their cross-correlation [6]. Like the classical rules, it considers uncorrelated excitation components by arguing that there are principal directions leading to un-correlation. However, it could be difficult in practice to work with rotated directions. To address this issue, the elliptical response envelopes is re-written by integrating excitation component correlations.

First of all, let us consider a multiple-response $\mathbf{R}(t) = (R_1(t) \cdots R_p(t))^T$ with p components and $R_m(t) = \mathbf{d}_m^T \mathbf{U}$. It can be written [6]:

$$\mathbf{R}(t) = \sum_{k=x,y,z} \sum_i^n \alpha_{i,k}(t) \mathbf{F}_{i,k},$$

where $-1 \leq \alpha_{i,k}(t) \leq 1$,

$$\mathbf{F}_{i,k} = \begin{pmatrix} F_{1,i,k} = \max_t |S_{i,k}(t)| \times R_{1,i,k} \\ \vdots \\ F_{p,i,k} = \max_t |S_{i,k}(t)| \times R_{p,i,k} \end{pmatrix},$$

and $R_{m,i,k} = (\boldsymbol{\phi}_i^T \mathbf{M} \mathbf{O}_k / (\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i)) \mathbf{d}_m^T \boldsymbol{\phi}_i$. The multiple-response can be written in the following form:

$$\mathbf{R}(t) = \mathbf{F}^T \boldsymbol{\alpha}(t),$$

with $\mathbf{F} = (\mathbf{F}_1 \cdots \mathbf{F}_p)$,

$$\mathbf{F}_m = \left([F_{m,i,x}]_{i=1,\dots,n}^T \quad [F_{m,i,y}]_{i=1,\dots,n}^T \quad [F_{m,i,z}]_{i=1,\dots,n}^T \right)^T,$$

and

$$\boldsymbol{\alpha}(t) = \left([\alpha_{i,x}(t)]_{i=1,\dots,n}^T \quad [\alpha_{i,y}(t)]_{i=1,\dots,n}^T \quad [\alpha_{i,z}(t)]_{i=1,\dots,n}^T \right)^T.$$

For $p = 1$, this is equivalent to the equation of $R(t)$ in Subsection 3.1. From the GCQC3 method, we have the flowing inequality:

$$|R_m(t)| = |\mathbf{F}_m^T \boldsymbol{\alpha}(t)| \leq F_m^{\max} = \sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i,j=1}^n \rho_{i,j} \tilde{\rho}_{k,l} F_{m,i,k} F_{m,j,l}} = \sqrt{\mathbf{F}_m^T \mathbf{H} \mathbf{F}_m},$$

Where:

$$\mathbf{H} = \begin{pmatrix} [\rho_{i,j}]_{i,j=1,\dots,n} & [\tilde{\rho}_{x,y} \rho_{i,j}]_{i,j=1,\dots,n} & [\tilde{\rho}_{x,z} \rho_{i,j}]_{i,j=1,\dots,n} \\ [\tilde{\rho}_{x,y} \rho_{i,j}]_{i,j=1,\dots,n} & [\rho_{i,j}]_{i,j=1,\dots,n} & [\tilde{\rho}_{y,z} \rho_{i,j}]_{i,j=1,\dots,n} \\ [\tilde{\rho}_{x,z} \rho_{i,j}]_{i,j=1,\dots,n} & [\tilde{\rho}_{y,z} \rho_{i,j}]_{i,j=1,\dots,n} & [\rho_{i,j}]_{i,j=1,\dots,n} \end{pmatrix}.$$

As demonstrated in [6], $\boldsymbol{\alpha}(t)$ must fulfills the following inequality:

$$\alpha(t)^T \mathbf{H}^{-1} \alpha(t) \leq 1,$$

which means that $\alpha(t)$ belong to an hyper-ellipsoid of dimension $3n$. Therefore, $\alpha(t)$ verifies:

$$\begin{cases} \mathbf{R}(t) = \mathbf{F}^T \alpha(t) \\ \alpha(t)^T \mathbf{H}^{-1} \alpha(t) \leq 1 \end{cases}$$

To obtain a unique value for $\alpha(t)$, the one minimizing the quadratic form $\alpha(t)^T \mathbf{H}^{-1} \alpha(t)$ is considered [6]. It leads to the following closed-form expression:

$$\alpha_{min}(t) = \mathbf{H} \mathbf{F} (\mathbf{F}^T \mathbf{H} \mathbf{F})^{-1} \mathbf{R}(t).$$

Integrating the value of $\alpha_{min}(t)$ into $\alpha(t)^T \mathbf{H}^{-1} \alpha(t) \leq 1$, we obtain the inequality:

$$\mathbf{R}^T(t) (\mathbf{F}^T \mathbf{H} \mathbf{F})^{-1} \mathbf{R}(t) \leq 1,$$

which implies that the multiple response $\mathbf{R}(t)$ belongs to an hyper-ellipsoid of dimension p . It is so-called the elliptical response envelopes. From it, we can deduce a set of concomitant responses. In practice, they are deduced from the vertices of a polytope bounding the elliptical response envelopes.

6. GCQC3 with rigid mode

To take into account the missing mass due to the projection $\mathbf{U} = \boldsymbol{\phi} \mathbf{q}$, an additional mode called "rigid mode" or "pseudo-mode" is often considered. It assumes that the dynamic amplification of the modes with frequency larger than the one of the Zero-Period Acceleration ξ_{ZPA} is negligible. Then, the response becomes:

$$\mathbf{R}(t) = \sum_{k=x,y,z} A_k(t) \mathbf{R}_{n+1,k} + \sum_i^n S_{i,k}(t) \mathbf{R}_{i,k} = \sum_i^{n+1} S_{i,k}(t) \mathbf{R}_{i,k},$$

where $\mathbf{R}_{n+1,k} = \mathbf{K}^{-1} \mathbf{M} \boldsymbol{\phi}_{R,k}$, $\boldsymbol{\phi}_{R,k} = \mathbf{O}_k - \sum_{i=1}^n \mathbf{R}_{i,k}$ is the rigid mode and $S_{n+1,k}(t) = A_k(t)$ is the ground acceleration for direction k . Then, considering the following correlations:

$$\text{cor} \left(S_{n+1,k}(t), S_{j,k}(t) \right) = \delta_{j=n+1},$$

$$\text{cor} \left(S_{n+1,k}(t), S_{n+1,l}(t) \right) = \tilde{\rho}_{k,l},$$

and using the notation $\rho_{n+1,j} = \delta_{j=n+1}$, taking into account the rigid mode is straightforward :

$$F^{max} = \sqrt{\sum_{k=x,y,z} \sum_{l=x,y,z} \sum_{i,j=1}^{n+1} \rho_{i,j} \tilde{\rho}_{k,l} F_{i,k} F_{j,l}}.$$

7. Example

Let us consider an embedded structure composed of three beams and four nodes. The structure and its eigenmodes are illustrated in Figure 1. The node N3 in Figure 1 has a punctual mass of 10kg and the masses of the others nodes and beams are null. The structure is embedded at node N0, the beam lengths equal 1m and they have square sections with length 3cm and thickness 5mm. The Young's modulus of the beams is 200GPa and their Poisson coefficient is 0.3. Furthermore, the degrees of freedom along the Y axis are blocked as well as the rotations around the X and Z axes. The structure has two modes presented in Figure 1 that take into account the whole mass.

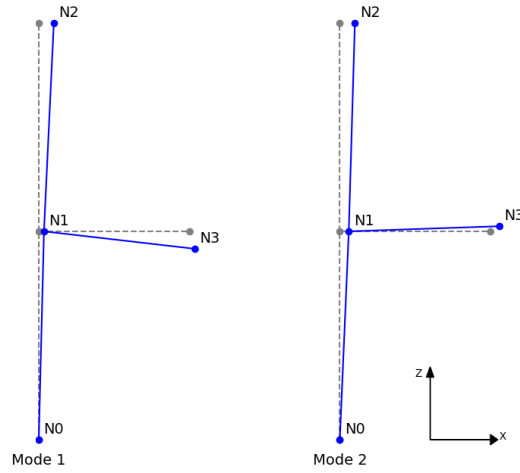


Figure 1. Eigenmodes of the structure.

The frequencies, the participation factors and the modal masses are presented in Table 1. For the participation factors, the eigenmodes are normalized such that the largest translation value equals one. Moreover, the modal damping is the same for the two modes and equals $\xi = 5\%$.

Table 1. Modal frequencies, participation factors and masses.

	Mode 1	Mode 2
Frequency (Hz)	4.22	14.74
Participation factor along X	4.23e-1	1.43
Participation factor along Z	-1.02	5.93
Modal masse along X (kg)	1.46	8.54
Modal masse along Z (kg)	8.54	1.46

The excitations along the X and Z directions have the following form:

$$A(t) = \Gamma(t)Y(t),$$

where $\Gamma(t)$ is the Gamma modulation function defined by:

$$\Gamma(t) = \alpha t^{\beta-1} \exp(\gamma t),$$

and $Y(t)$ is a zero-mean stationary Gaussian process with an exponential covariance function:

$$k(h) = \sigma^2 \exp\left(-\frac{|h|}{\theta}\right).$$

The coefficients of the excitation model are given in Table 2.

Table 2. Model coefficients for the excitation.

α	1/6
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β	4
γ	0.6
σ^2	1
θ	2

The correlation between the excitation components is driven by the correlation between the Gaussian processes $Y(t)$ for each direction X and Z. Examples of excitations for instantaneous correlations of 0.8 and -0.6 are presented in Figure 2.

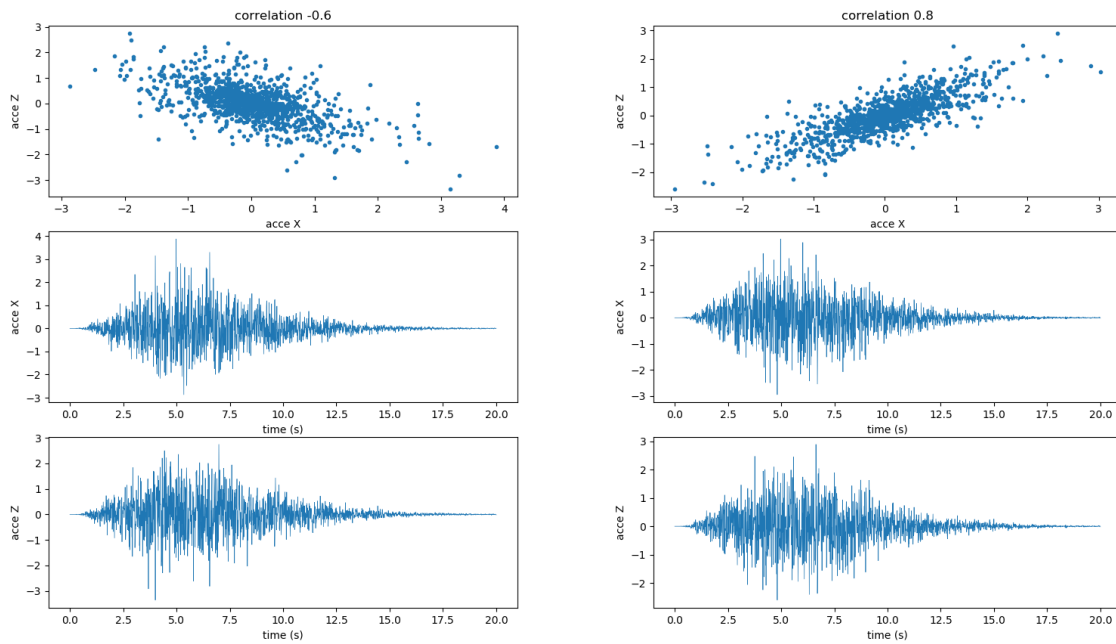


Figure 2. Examples of excitations along the X and Z directions with instantaneous correlations of -0.6 and 0.8.

Then, the displacements obtained with the response spectrum analysis are compared to the ones obtained with a time-history simulation. To perform the comparison, 200 couples of excitations along the X and Z directions have been generated for each correlation $\tilde{\rho}_{x,z}$ between -1 and 1. The mean of the maximum displacements with respect to the solicitation correlation is presented in Figure 3. The results in Figure 3 show that the response spectrum method with the GCQC3 rule fits very well the time-history simulation. This emphasizes the efficiency of this rule for taking into account the correlations between the excitation components. Furthermore, Figure 3 confirms that the classical SRSS rule behaves very badly. Indeed, its relative error can be greater than 50%. It is worth to highlight that the SRSS and the GCQC3 rules are equivalent when the correlation coefficient is zero.

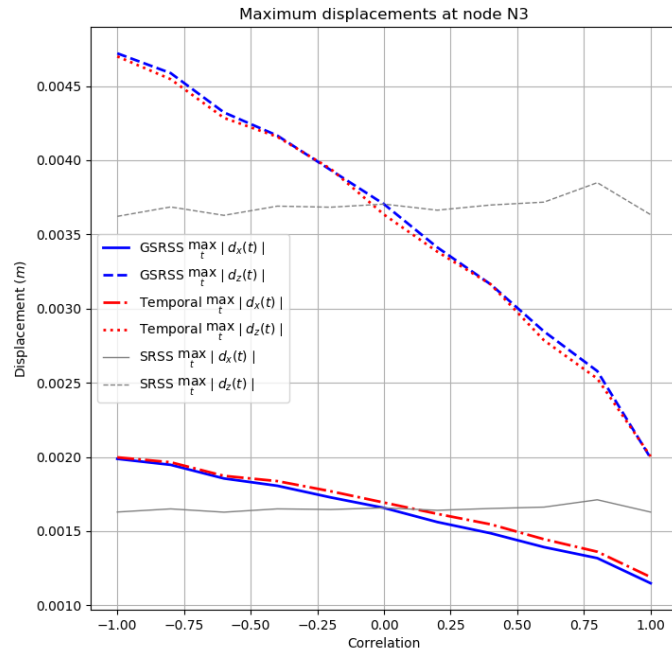


Figure 3. Maximum displacements with respect to the excitation component correlations.

Finally, we present in the series of figures in Figure 4, the results of the elliptical response envelopes method with the GCQC3 rule for displacements along the X and Z directions. The black points in Figure 4 represent the values of the time-history simulations. For each correlation value, the solid lines represent the elliptical envelopes for the 200 excitations and the dashed line is the mean of these envelopes. Figure 4 highlights the good behavior of the GCQC3 rule since the elliptical envelopes fit the cloud of concomitant responses obtained with the time-history analysis.

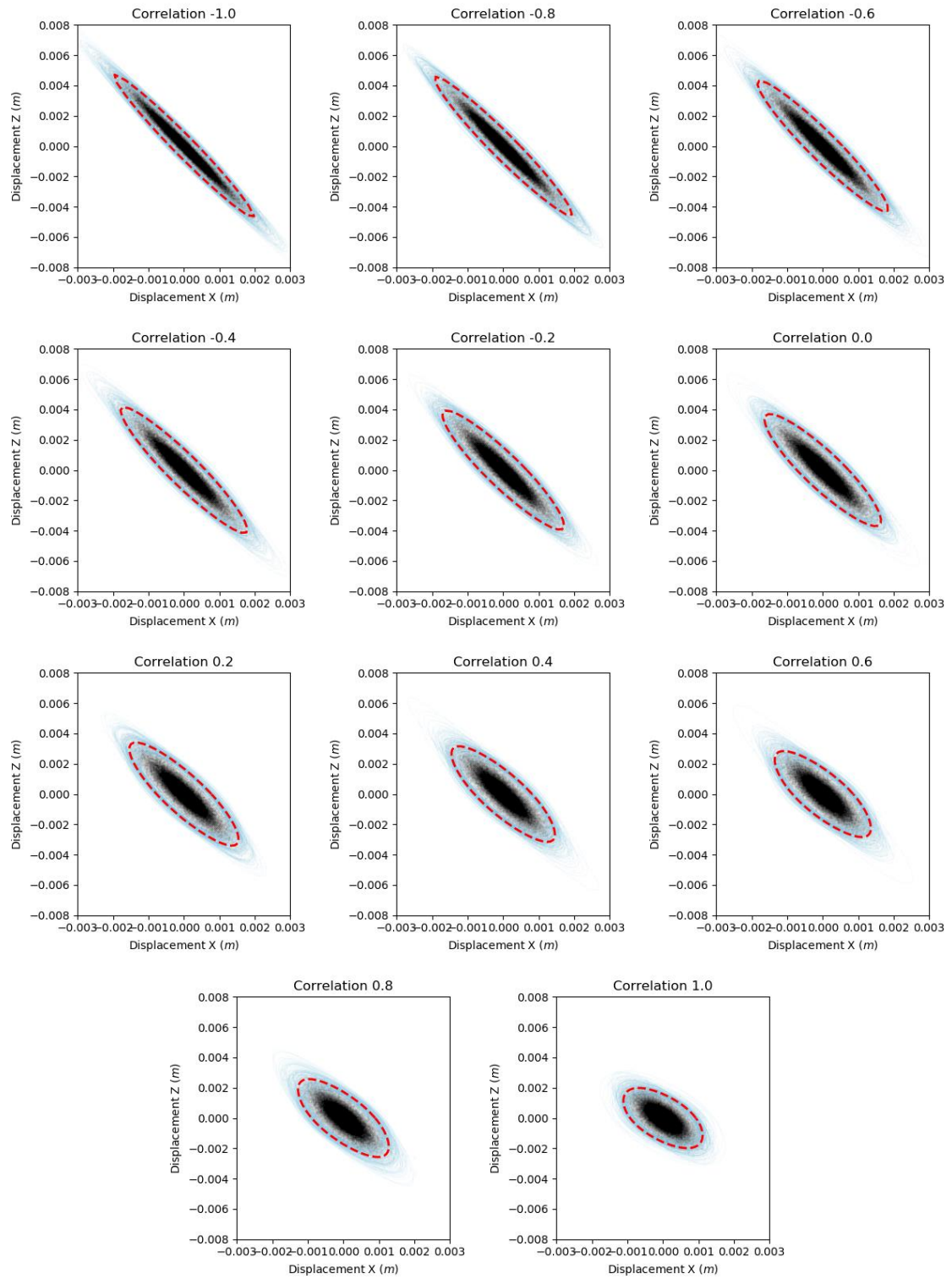


Figure 4. GCQC3 elliptical response envelopes for different component correlations.

8. Conclusion

This article presents a reformulation of the GCQC3 rule when vibrations along each direction are correlated, spectra have different shapes and without any simplifications. Such a rule is necessary in the nuclear industry, for example to size substructures subject to vibrations induced by a plane crash. Furthermore, the reformulation is also required since the general GCQC3 rule without simplifications implies lot of calculations which are difficult to achieve in an industrial context. The obtained GCQC3 rule only requires the evaluation of the instantaneous correlations between excitation components and the peak modal responses. Therefore, its implementation is as easy as the one of the classic rules (SRSS, 40%, 30%). Furthermore, this rule can be adapted for the elliptical response envelopes method which allows to consider concomitant responses. Finally, the efficiency of the GCQC3 rule has been illustrated in a numerical example. It emphasizes that it allows to fit the maximum responses of a time-history analysis even in presence of correlations. Furthermore, a good consistency between the time-history responses and the elliptical response envelopes is also observed.

9. References

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