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Syed Ali Ajwad[a,*], Tomas Ménard[b], Emmanuel Moulay[c], Michael Defoort[d], Patrick Coirault[a]

[a]LIAS (EA 6315), Université de Poitiers, 2 rue Pierre Brousse, 86073 Poitiers Cedex 9, France.
[b]LAC (EA 7478), Université de Caen, 6 Bd du Marchal Juin, 14032 Caen Cedex, France.
[c]XLIM (UMR CNRS 7252), Université de Poitiers, 11 bd Marie et Pierre Curie, 86073 Poitiers Cedex 9, France.
[d]LAMIH, UMR CNRS 8201, Polytechnic University of Hauts-de-France, Valenciennes, 59313 France.

Abstract

This paper deals with the leader-following consensus problem of multi-agent systems with the consideration that each agent can only transmit its position state to the neighbors at irregular discrete sampling times. In the proposed algorithm, a continuous-discrete time observer is designed for the continuous estimation of both position and velocity from the discrete position information of the neighbors. These estimated states are then used for designing a continuous control law which solves the leader-following consensus problem. Moreover, the dynamics of the leader is not fixed and can be controlled through an external input. The stability analysis has been carried out by employing the Lyapunov approach which provides sufficient conditions to tune the parameters according to the maximum allowable sampling period. The developed algorithm has been simulated and then tested on an actual multi-robot system consisting of three differential drive wheeled robots. Both simulation and hardware results validate the effectiveness of the control algorithm.

Keywords: Second-order multi-agent systems, Continuous-discrete time observer, Leader-following consensus, Nonuniform and asynchronous sampling

1. Introduction

Development in technology has enabled advanced operations for autonomous robotic vehicles. The effectiveness of these robots can be further improved by utilizing them in a cooperative manner to accomplish much more complex tasks which cannot be realized by solo operation. Cooperative team work is not only limited to robotics but it also finds applications in the field of sensor networks [1], interferometers [2], electric power systems [3], distributed computation [4], synchronization [5] etc.

Cooperative control of multi-agent systems (MAS) is classified into two categories, namely, centralized control and distributed control. In centralized control, all the agents communicate with a central control unit. On the other hand, distributed control scheme does not have any common control unit. All the agents make decisions by exchanging their information with neighbors. Study of distributed control of MAS has gained much popularity in the research community during the last decade because of its advantages over centralized control such as efficiency, scalability, flexibility, robustness and adaptability [6].

Consensus is considered as a fundamental problem in distributed control in which all agents are required to reach an agreement on some value of interest [7]. For instance, in multi-robotics systems, it may require all robots to either reach a common position or follow the trajectory of a leader. The former case is defined as leaderless consensus while the latter is called leader-following consensus [8]. The topic of leader-following consensus has received significant attention in the past years [9, 10, 11, 12, 13, 14].

*Corresponding author

Email addresses: syed.ali.ajwad@univ-poitiers.fr (Syed Ali Ajwad), tomas.menard@unicaen.fr (Tomas Ménard), emmanuel.moulay@univ-poitiers.fr (Emmanuel Moulay), michael.defoort@uphf.fr (Michael Defoort), patrick.coirault@univ-poitiers.fr (Patrick Coirault)

Various physical systems like robotic vehicles can be represented as linearized double integrator dynamic systems where position and velocity are the information states [15, 16]. Though these dynamics are continuous in nature, information exchange among the agents is in discrete time because of the digital means of communication [17].

Several control techniques, to achieve consensus in double integrator systems with sampled data, have been proposed by the research community. For instance, Pan and Qiao proposed a discrete time consensus algorithm for double integrator MAS in [18] where communication delay was considered and conditions for consensus under such delays were derived using linear matrix inequalities. Several sufficient and necessary conditions on controller gain, communication topology and sampling time were proposed in [19, 20] for the convergence of sampled-data coordination protocols for agents with double integrator dynamics. In [21], average consensus was achieved asymptotically for a dynamic network provided that there always exists a directed spanning tree. Integral sliding mode control based consensus algorithm was proposed in [22] for both leaderless and leader tracking of second-order MAS with disturbance. The authors in [23] investigated a discrete-time consensus algorithm for second-order MAS. It was shown that agents achieve consensus in the presence of switching communication topology with non-uniform time delays and sampled data. Eichler and Werner discussed the optimization of convergence speed for only fixed communication topology [24]. A leader-following protocol for second-order MAS in a sampling setting with Markovian switching topology has been presented in [25]. Wu *et al.* derived mean square consensus tracking with a virtual leader in [26]. In this paper, the tracking error not only depends on the sampling period and delay but also on the velocity and acceleration of the virtual leader. The problem of intermittent communication was addressed by Liu *et al.* in [27]. The authors proposed a consensus algorithm based on persistent-hold techniques for first order systems.

However, the above mentioned techniques require information of both position and velocity states of the neighbors. In a practical scenario, the agents usually have limited on-board measurement resources that makes it difficult to measure their velocity along with their position. That is why, it is more affordable to measure and transmit only position information. To cope with this problem, available position information of the neighbors were used to estimate the velocity for consensus tracking problem in [28]. Chen and Li have proposed an observer to estimate the velocity via sampled data [29]. Second-order consensus was achieved by Yu *et al.* by using only current and previous sampled position data [30]. It was shown that consensus can only be reached for specific sampling time intervals. Some necessary and sufficient conditions for sampling time period were established in [31] for position based second-order consensus algorithms.

In the above cited literature, position information is supposed to be transmitted at regular constant intervals. However in practice, asynchronous and aperiodic transmission of data is inevitable. Furthermore, in many cases, asynchronous and aperiodic transmission is desired in order to avoid unnecessary consumption of the communication resources. For example, in event-triggered based consensus, local information is exchanged only when a specific event occurs. In [32], Dimarogonas *et al.* proposed an event-triggering based distributed control for MAS. The next triggering time is calculated when a particular event takes place. An event-triggering based algorithm with a state-dependant triggering function was proposed in [33]. An observer based event-triggering consensus protocol was proposed in [34, 35, 36] where the control law is updated using the estimated states. One drawback of such event-triggering schemes is that it sometimes requires a continuous monitoring of the full state of the agents. Furthermore, it needs pre-knowledge of the event-triggering function or mechanism which dictates the sampling times for the data transmission.

In the current article, it is aimed at designing a leader-following consensus algorithm for second-order MAS which guarantees that each agent tracks the leader when only position information is transmitted with asynchronous and irregular sampling times. The proposed scheme is based on continuous-discrete time observers. Such observers estimate both the available and unavailable states in continuous time from information received in discrete time. Continuous-discrete time observers have been studied widely in recent years, for example in [37, 38, 39]. A leaderless consensus protocol for second-order MAS with asynchronous sampling and partial measurements has been introduced in [40]. A continuous-discrete time observer has been proposed to reconstruct the position and velocity in continuous time from the available discrete position data. In the current paper, using the basic idea of [40], a novel output-feedback controller is proposed to solve the problem of leader-following consensus problem. The leader is not assumed to be fixed. Its dynamics can be controlled by an external input that influences the stability of the whole system. Here, the novelty of this article lies in the development of a leader-following consensus protocol with an active leader in the presence of the following constraints: (i) an agent only transmits its position state to its neighbors (nor its velocity nor its input), (ii) the agents transmit their data with irregular sampling times, (iii) the sampling instants for each agent are totally independent which means that no synchronization is required for the transmission, (iv) the leader can communicate its position to only a small portion of the followers, (v) the communication topology among agents is directed. The designed controller ensures the practical consensus of MAS for the general case with an ultimate bound that can be made as small as desired by appropriately tuning the parameters. It is also shown that exponential consensus is obtained if the leader input is equal to zero. It is important to mention that the results of [40] cannot be directly applied to the leader-following case due to the presence of aperiodic and asynchronous sampling period and the availability of position state only. Compared to [40], the addition of an active leader changes the dynamics of overall system, resulting in new terms not only in the tracking error dynamics but also in the observer error dynamics. Consequently, the stability analysis can no longer be carried out with the same Lyapunov function as used in [40]. Therefore, in this paper, a new stability proof is derived based on new Lyapunov functions. At last, another important contribution in this paper compared to [40] is the experimental validation of the developed algorithm. Indeed, the algorithm is tested on a group of mobile robots using Robot Operating System (ROS) for linear movement. Both simulation and experiment results have validated the efficacy of this algorithm.

The remaining of the paper is organized as follows. Section 2 describes preliminary knowledge along with the problem formulation. Main results are presented in Section 3. Simulation and experimental results are discussed in Section 4 and 5 respectively while Section 6 concludes the article.

2. Preliminary knowledge and problem formulation

2.1. Notations

In this paper, the set of $n \times n$ real matrices is denoted $\mathbb{R}^{n\times n}$. $I_n \in \mathbb{R}^{n\times n}$ is the $n-$dimensional identity matrix. For any symmetric matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent minimum and maximum eigenvalue of A respectively. The symbol $\triangleq$ means equal by definition while $\otimes$ represents the Kronecker product. $\mathcal{D}_i^N$ denotes the $N \times N$ matrix with all entries equal to zero except the i^{th} diagonal entry which is 1. $\|.\|_2$ and $\|.\|_F$ represent the Euclidean and Frobenius norms respectively. If nothing is specified, then $\|.\|$ denotes the Euclidean norm. $\text{diag}(b_1,\ldots,b_q)$, with $b_i \in \mathbb{R}^{m\times m}$, $i = 1,\ldots,q$, $q, m \in \mathbb{N}$, is the diagonal by block matrix having $b_1,\ldots,b_q$ on its diagonal. $\mathbf{1}_N \in \mathbb{R}^N$ represents the vector with all entries equal to 1.

2.2. Preliminary knowledge

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a nonempty finite set of nodes and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is a set of edges where an edge is an ordered pair of distinct nodes. For an edge (i, j), node j, called child node, can receive information from node i which is the parent node and i is a neighbor of j. A graph has a directed spanning tree if each node has one parent node except for one node, called the root, which has a directed path to all other nodes in the graph. The adjacency matrix $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ with N nodes is defined by $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N\times N}$ is defined as $l_{ii} = \sum_{j\neq i} a_{ij}$, $l_{ij} = -a_{ij}$ for $i \neq j$. $\mathcal{L}$ satisfies $l_{ii} \leqslant 0$, $i \neq j$ and $\sum_{j=1}^{N} l_{ij} = 0$, $i = 1\ldots N$.

Lemma 1. *[41] Let $M = (m_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ be a non-singular M-matrix, such that $m_{ij} \leq 0$, for all $i \neq j$, $i, j = 1,\ldots,n$ and all its eigenvalues have positive real parts. Then, there exists a diagonal matrix $\Omega = diag(\omega_1,\ldots,\omega_n) > 0$ such that $\Omega M + M^T\Omega > 0$.*

2.3. Problem formulation

Consider a group of N followers labeled from 1 to N and one leader labeled 0. The followers have the following second-order dynamics:

$$\begin{cases} \dot{r}_i(t) = v_i(t), \quad i = 1,\ldots,N \\ \dot{v}_i(t) = u_i(t) \end{cases} \tag{1}$$

where $r_i, v_i, u_i \in \mathbb{R}^m$ represent respectively the position, velocity and control input of the i-th agent ($m \in \mathbb{N}$). The dynamics of the leader is given by:

$$\begin{cases} \dot{r}_0(t) = v_0(t) \\ \dot{v}_0(t) = u_0(t) \end{cases} \tag{2}$$

where $r_0, v_0, u_0 \in \mathbb{R}^m$ represent respectively the position, velocity and control input of the leader. It is assumed that the control input u_0 is bounded, that is, there exists $\delta_0 \geq 0$ such that $\|u_0(t)\| \leq \delta_0$ for all $t \geq 0$.

The communication connection between the followers is described by graph $\mathcal{G}$ and the corresponding adjacency and Laplacian matrices are $\mathcal{A}$ and $\mathcal{L}$, respectively. The position of the leader is considered to be transmitted to a small portion of the followers. Let the diagonal matrix $\mathcal{B} = diag(b_1, b_2, \ldots, b_N)$ be the interconnection relationship between the leader and followers, where $b_i = 1$ if the information of the leader is accessible by the i^{th} follower, otherwise $b_i = 0$. The communication graph including the followers and the leader is denoted $\tilde{\mathcal{G}}$. Let us define the following matrix

$$\mathcal{H} = \mathcal{L} + \mathcal{B}$$

Lemma 2. *[42] Matrix $\mathcal{H}$ is a nonsingular M-matrix if and only if the pinning joint communication topology $\tilde{\mathcal{G}}$ has a directed spanning tree.*

Definition 1. *The leader-following practical exponential consensus is achieved if*

$$\sum_{i=1}^{N} \|e_i(t)\| \leq \alpha e^{-\beta t} + \gamma, \quad \forall t \geq 0$$

where $e_i = x_i - x_0$ with $x_i = \left[r_i^T, v_i^T\right]^T$, $x_0 = \left[r_0^T, v_0^T\right]^T$, $\alpha, \beta > 0$ and γ is a positive constant.

It is considered that each agent transmits only its position r_j to its neighbor i at times $t_k^{i,j}$ with $k \in \mathbb{N}$, but not its velocity v_j nor its input u_j. The sampling instants $t_k^{i,j}$ are supposed to verify $0 = t_0^{i,j} < t_1^{i,j} < \cdots < t_k^{i,j} < \ldots$. Furthermore, one assumes that there exist constants $\tau_m, \tau_M > 0$, called respectively the minimum sampling period and the maximum sampling period, such that $\tau_m < t_{k+1}^{i,j} - t_k^{i,j} < \tau_M$, for all $k \in \mathbb{N}$ and $i = 1, \ldots, N$, $j = 0, \ldots, N$. Figure 1 shows an example of sampling instants for data transmission between the neighbors. The problem under investigation is to design a distributed control law to solve the leader-following consensus problem under these communication constraints.

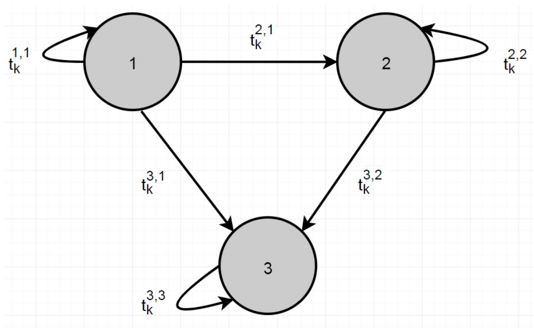


Figure 1: Example of sampling instants for data transmission under a directed graph

3. Observer based leader-following consensus

In this paper, the basic idea is to use classical continuous linear consensus controller for double integrator MAS with discrete position measurements only. In a classical consensus protocol, it is assumed that all states of the neighbors are available in continuous time. Whereas, the current article considers that only position data is transmitted to the neighbors at irregular and asynchronous time instants. The idea is to estimate the position and velocity states in continuous time from the available discrete position data by using a continuous-discrete time observer (as discussed in [40] for the leaderless consensus problem). The proposed observer-based consensus protocol is given for $t \geq 0$ by

$$\begin{aligned} u_i(t) &= -\bar{c}\lambda^2 \sum_{j=1}^{N} a_{ij} \left[\hat{r}_{i,i}(t) - \hat{r}_{i,j}(t)\right] - \bar{c}2\lambda \sum_{j=1}^{N} a_{ij} \left[\hat{v}_{i,i}(t) - \hat{v}_{i,j}(t)\right] \\ &\quad -\bar{c}\lambda^2 b_i \left[\hat{r}_{i,i}(t) - \hat{r}_{i,0}(t)\right] - \bar{c}2\lambda b_i \left[\hat{v}_{i,i}(t) - \hat{v}_{i,0}(t)\right], \quad i = 1, \dots, N \end{aligned} \tag{3}$$

where $\hat{r}_{i,j}$ and $\hat{v}_{i,j}$ are the estimated position and velocity of agent j by agent i. Their dynamics are given by

$$\dot{\hat{r}}_{i,j}(t) = \hat{v}_{i,j}(t) - 2\theta e^{-2\theta(t - t_k^{i,j})} \left(\hat{r}_{i,j}(t_k^{i,j}) - r_j(t_k^{i,j})\right) \tag{4}$$

$$\dot{\hat{v}}_{i,j}(t) = -\theta^2 e^{-2\theta(t - t_k^{i,j})} \left(\hat{r}_{i,j}(t_k^{i,j}) - r_j(t_k^{i,j})\right) \tag{5}$$

for $i = 1, \dots, N$, $j = 0, 1, \dots, N$ and $t \in \left[t_k^{i,j}, t_{k+1}^{i,j}\right)$, $k \in \mathbb{N}$, where $\bar{c} > 0$ is the coupling strength, a_{ij} is the (i,j)-th entry of the adjacency matrix $\mathcal{A}$ of the directed graph $\mathcal{G}$ while θ and $\lambda > 0$ are the observer and controller tuning parameter respectively. The initial conditions $\hat{r}_{i,j}(0), \hat{v}_{i,j}(0) \in \mathbb{R}^m$ of the observers can be chosen arbitrarily. Before stating the main result, note that if $\tilde{\mathcal{G}}$ has a directed spanning tree, then according to Lemma 1, there exists a diagonal matrix $\Omega = \mathrm{diag}(\omega_1, \dots, \omega_N)$ such that $\mathcal{H}^T\Omega + \Omega\mathcal{H} > 0$. Furthermore, the following notations will be used hereafter.

$$\omega_{\max} = \max\{\omega_1, \dots, \omega_N\}, \tag{6}$$

$$\omega_{\min} = \min\{\omega_1, \dots, \omega_N\}, \tag{7}$$

$$\rho = \lambda_{\min}(\mathcal{H}^T\Omega + \Omega\mathcal{H}). \tag{8}$$

$$h_{\max} = \max_{i,j} |\mathcal{H}_{ij}| \tag{9}$$

Theorem 1. *Consider the MAS* (1)-(2) *with the consensus protocol* (3)-(5) *and assume that the communication topology $\tilde{\mathcal{G}}$ contains a directed spanning tree. If the control parameters $\theta, \lambda \geq 1$ and $\bar{c} > 0$ satisfy the following*

$$\theta < \frac{\bar{\varrho}}{\tau_M} \tag{10}$$

$$\bar{c} \geq \frac{\omega_{\max}}{\rho} \tag{11}$$

$$\lambda < \epsilon^*\theta \tag{12}$$

where $\bar{\varrho}$ is a positive constant, $\epsilon^* \in (0,1)$, ω_{max} and ρ are given by (6) *and* (8)*, respectively, then the leader-following practical consensus problem is solved in the sense of Definition 1.*

Remark 1. *The conditions provided in Theorem 1 to achieve the leader-following consensus are only sufficient since they are obtained through Lyapunov based stability analysis which may lead to conservative bounds. Despite the limitation of Lyapunov based approaches, it provides some useful hints about the choice of the gains. For instance, it is clear from inequality* (10) *that the maximum sampling period directly influences the choice of θ. If the maximum sampling time is high, then θ must be chosen small enough and vice versa. It should be noted that θ and λ dictate the convergence speed of the observer and controller, respectively. In order to guarantee closed-loop stability, the controller dynamics must be slower than the observer dynamics which is represented by the fact $\epsilon^* < 1$. Therefore, for large maximum sampling period, the system convergence rate will be slow.*

Before the proof of Theorem 1, we need some important results which are presented below.

Lemma 3. *Let $v_1(t)$ and $v_2(t)$ be real valued functions verifying*

$$\frac{d}{dt}\left(v_1^2(t)+v_2^2(t)\right) \leq -av_1^2(t) - bv_2^2(t) + c\int_{t-\delta}^{t} v_2^2(s)ds + k, \tag{13}$$

for all $t \geq 0$, where $a, b, c, \delta > 0$ and $k \geq 0$. There exists $\varrho > 0$, independent of a, b, c, k, and $\bar{\alpha} \geq 0$ such that if $\delta < \varrho \min\left(\frac{b}{c}, \frac{1}{\sigma}\right)$, then $v_1(t)$ and $v_2(t)$ verify the following inequality

$$v_1^2(t) + v_2^2(t) \leq \bar{\alpha}e^{-\sigma t} + \frac{k}{\sigma}, \quad \forall t \geq 0 \tag{14}$$

where σ is given by

$$\sigma = \frac{1}{2}\min(a, b) \tag{15}$$

PROOF. Let $v = \min\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$, $\xi = 2\frac{c\delta}{b}$ and $\kappa = 1 - \xi$. Since $\delta \in \left(0, \varrho\min\left(\frac{b}{c}, \frac{1}{\sigma}\right)\right)$, one has

$$0 < 2\frac{c\delta}{b} < 2\frac{c}{b}\varrho\min\left(\frac{b}{c}, \frac{1}{\sigma}\right) \leq 2\varrho \quad \Rightarrow \quad 1 - 2\varrho < \underbrace{1-\xi}_{=\kappa} < 1 \tag{16}$$

$$0 < v\kappa\delta < v\delta < v\varrho\min\left(\frac{b}{c}, \frac{1}{\sigma}\right) \leq \sqrt{2}\varrho \tag{17}$$

Then $\varrho > 0$ can be chosen, independently of a, b, c, k such that for all $\delta \in \left(0, \varrho\min\left(\frac{b}{c}, \frac{1}{\sigma}\right)\right)$

$$\kappa \in \left(\frac{1}{\sqrt{2}}, 1\right) \tag{18}$$

$$e^{v\kappa\delta} \leq 1 + 2v\kappa\delta \tag{19}$$

Consider the following Lyapunov function

$$W(v_t) = v_1^2(t) + v_2^2(t) + c\int_0^{\delta}\int_{t-s}^{t} e^{v\kappa(\mu-t+s)}v_2^2(\mu)d\mu ds \tag{20}$$

where $v_t(s) = [v_1(t+s), v_2(t+s)]^T$, $s \in [-\delta, 0]$. One has

$$\dot{W}(v_t) = \frac{d}{dt}\left(v_1^2(t)+v_2^2(t)\right) + c\int_0^{\delta}\frac{d}{dt}\int_{t-s}^{t} e^{v\kappa(\mu-t+s)}v_2^2(\mu)d\mu ds.$$

Applying Leibniz integration leads to

$$\begin{aligned}
\dot{W}(v_t) &= \frac{d}{dt}\left(v_1^2(t)+v_2^2(t)\right) - v\kappa c\int_0^{\delta}\int_{t-s}^{t} e^{v\kappa(\mu-t+s)}v_2^2(\mu)d\mu ds + c\int_0^{\delta} e^{v\kappa s}v_2^2(t) - v_2^2(t-s)ds \\
&\leq -av_1^2(t) - bv_2^2(t) + c\int_{t-\delta}^{t} v_2^2(s)ds + k - v\kappa c\int_0^{\delta}\int_{t-s}^{t} e^{v\kappa(\mu-t+s)}v_2^2(\mu)d\mu ds \\
&\quad + c\int_0^{\delta} e^{v\kappa s}v_2^2(t)ds - c\int_0^{\delta} v_2^2(t-s)ds \\
&\leq -av_1^2(t) - bv_2^2(t) + k + c\left(\frac{e^{v\kappa\delta}-1}{v\kappa}\right)v_2^2(t) - v\kappa\left(W(v_t) - v_1^2(t) - v_2^2(t)\right)
\end{aligned}$$

Since $\frac{e^{v\kappa\delta}-1}{v\kappa} \leq 2\delta$ and given the definition of v, the following inequalities are achieved

$$\begin{aligned}
\dot{W}(v_t) + v\kappa W(v_t) &\leq (-a + v\kappa)\, v_1^2(t) + (-b + 2c\delta + v\kappa) v_2^2(t) + k \\
\dot{W}(v_t) + v\kappa W(v_t) &\leq -a\left(1 - \frac{1}{\sqrt{2}}\right) v_1^2(t) - b\left(1 - (1-\kappa) - \frac{\kappa}{\sqrt{2}}\right) v_2^2(t) + k \\
\dot{W}(v_t) + v\kappa W(v_t) &\leq -a\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) v_1^2(t) - b\kappa\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) v_2^2(t) + k \\
\dot{W}(v_t) &\leq -v\kappa W(v_t) + k \\
\dot{W}(v_t) &\leq -\sigma W(v_t) + k
\end{aligned}$$

In order to get an over-valuation of W, one uses the comparison Lemma 2.5 p85 [43]. The solution of the following ODE

$$\dot{z}(t) = -\sigma z(t) + k$$

is given by

$$z(t) = \left(z(0) - \frac{k}{\sigma}\right) e^{-\sigma t} + \frac{k}{\sigma}$$

Taking $\bar{\alpha} = (W(v_0) - \frac{k}{\sigma})$ ends the proof. □

Remark 2. *It is worth mentioning that the results of Lemma 2 of [40], which deal with the exponential stability problem, cannot be directly applied to the current case where the problem of practical stability is considered. In the case of practical stability, the error dynamics does not converge exactly toward zero but stays in a ball of arbitrary small radius. Therefore, it is important to estimate the radius of this convergence ball and its dependency on the control parameters. Lemma 2 of [40] does not provide such information. Therefore, Lemma 3 is needed in order to get an insight on the effect of tuning parameters on the final error bounds.*

3.1. *Proof of Theorem 1*

The proof of Theorem 1 is divided into several steps. In the first step, system (1) along with the control law (3) and the observer (4)-(5) are re-written in a more compact form. Equations for tracking and observer errors are also derived in this step. New coordinates for high-gain design are introduced in the second step, while, in step 3, variables of the state of tracking and observer errors are combined in new variables. Candidate Lyapunov functions are introduced in step 4 and inequalities involving their derivatives are derived. Then in step 5, it is shown that Lemma 3 can be applied if the conditions given in Theorem 1 are satisfied. Finally in the last step, an inequality involving the tracking error in original coordinates is obtained.

Step 1. The dynamics of the agents can be written as

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i \quad i = 0, \ldots, N \\ r_i = Cx_i \end{cases} \tag{21}$$

where $A = \begin{pmatrix} 0_m & I_m \\ 0_m & 0_m \end{pmatrix}$, $B = \begin{pmatrix} 0_m \\ I_m \end{pmatrix}$ and $C = \begin{pmatrix} I_m & 0_m \end{pmatrix}$. Denoting $\hat{x}_{i,j} = (\hat{r}_{i,j}^T, \hat{v}_{i,j}^T)^T$, system (4)-(5) can be written as

$$\dot{\hat{x}}_{i,j}(t) = A\hat{x}_{i,j}(t) - \theta\Delta_\theta^{-1} K_o e^{-2\theta(t-\kappa_{i,j}(t))}(\hat{r}_{i,j}(\kappa_{i,j}(t)) - r_j(\kappa_{i,j}(t))), \quad i = 1 \ldots N, j = 0 \ldots N$$

where $\kappa_{i,j}(t) = max\left\{t_k^{i,j} \mid t_k^{i,j} \leq t, k \in \mathbb{N}\right\}$ represents the last instant, among all the sampling times represented by $t_k^{i,j}$, when the measurements of agent j have been received by agent i, $\Delta_\theta = \begin{pmatrix} I_m & 0_m \\ 0_m & \frac{1}{\theta} I_m \end{pmatrix}$, $K_o = P^{-1}C^T =$

$\begin{bmatrix}2I_m & I_m\end{bmatrix}^T$, with P the symmetric positive definite matrix solution of the equation $P + A^TP + PA = C^TC$ (see [44] for more details).

Defining the estimation error $\tilde{x}_{i,j} = \hat{x}_{i,j} - x_j$ for $j = 0,\ldots,N$ and tracking error $e_i = x_i - x_0$ for $i = 1,\ldots,N$ gives

$$\begin{aligned}\dot{\tilde{x}}_{i,j}(t) &= A\tilde{x}_{i,j}(t) - \theta\Delta_\theta^{-1}K_o e^{-2\theta(t-\kappa_{i,j}(t))}\left(C\hat{x}_{i,j}(\kappa_{i,j}(t)) - Cx_j(\kappa_{i,j}(t))\right) - Bu_j(t)\\ &= (A - \theta\Delta_\theta^{-1}K_oC)\tilde{x}_{i,j}(t) - \theta\Delta_\theta^{-1}K_oz_{i,j}(t) - Bu_j(t)\end{aligned}$$

where $z_{i,j}(t) = \left[e^{-2\theta(t-\kappa_{i,j}(t))}C\tilde{x}_{i,j}(\kappa_{i,j}(t)) - C\tilde{x}_{i,j}(t)\right]$ and

$$\begin{aligned}\dot{e}_i(t) &= Ae_i(t) + Bu_i(t) - Bu_0(t)\\ u_i &= -\bar{c}K_c\Gamma_\lambda\sum_{k=1}^{N}\mathcal{H}_{ik}e_k - \bar{c}K_c\Gamma_\lambda\sum_{k=1}^{N}\mathcal{H}_{ik}\tilde{x}_{i,k} + b_i\bar{c}K_c\Gamma_\lambda\tilde{x}_{i,0}\end{aligned}$$

for $i = 1,\ldots,N$ and $K_c = B^TQ = \begin{pmatrix}I_m & 2I_m\end{pmatrix}$ with Q the symmetric positive definite matrix solution of $Q + QA + A^TQ = QBB^TQ$ and $\Gamma_\lambda = \begin{pmatrix}\lambda^2I_m & 0_m\\ 0_m & \lambda I_m\end{pmatrix}$ (see [45] for more details).

Step 2. Consider the coordinates for high-gain design $\bar{e}_i = \Gamma_\lambda e_i$ and $\bar{x}_{i,j} = \Delta_\theta\tilde{x}_{i,j}$ using the equalities, $\Delta_\theta A\Delta_\theta^{-1} = \theta A$, $C\Delta_\theta^{-1} = C$, $\Gamma_\lambda A\Gamma_\lambda^{-1} = \lambda A$, $\Gamma_\lambda B = \lambda B$, $\Delta_\theta B = \frac{1}{\theta}B$ and $B^T\Delta_\theta^{-1} = \theta B^T$ yields

$$\begin{aligned}\dot{\bar{e}}_i(t) &= \lambda A\bar{e}_i(t) + \lambda Bu_i(t) - \lambda Bu_0(t)\\ \dot{\bar{x}}_{i,j}(t) &= \theta(A - K_oC)\bar{x}_{i,j}(t) - \theta K_oz_{i,j}(t) - \frac{1}{\theta}Bu_j(t)\\ u_i &= -\bar{c}K_c\sum_{k=1}^{N}\mathcal{H}_{ik}\bar{e}_k - \bar{c}K_c\Gamma_\lambda\Delta_\theta^{-1}\sum_{k=1}^{N}\mathcal{H}_{ik}\bar{x}_{i,k} + b_i\bar{c}K_c\Gamma_\lambda\Delta_\theta^{-1}\bar{x}_{i,0}\end{aligned}$$

Step 3. Denoting $\eta^c = [\bar{e}_1^T\ldots\bar{e}_N^T]^T$, $\eta_i^o = [(\bar{x}_{i,1})^T\ldots(\bar{x}_{i,N})^T]^T$, $i = 1\ldots N$ and $\eta_0^o = [(\bar{x}_{1,0})^T\ldots(\bar{x}_{N,0})^T]$, the tracking error can be written in compact form as

$$\begin{aligned}\dot{\eta}^c &= \lambda[I_N\otimes A]\eta^c - \bar{c}\lambda[\mathcal{H}\otimes(BK_c)]\eta^c - \bar{c}\lambda\sum_{i=1}^{N}[(\mathcal{D}_i^N\mathcal{H})\otimes(BK_c\Gamma_\lambda\Delta_\theta^{-1})]\eta_i^o\\ &\quad +\bar{c}\lambda[I_N\otimes(BK_c\Gamma_\lambda\Delta_\theta^{-1})][\mathcal{B}\otimes I_{2m}]\eta_0^o - \lambda[\mathbf{1}_N\otimes B]u_0\end{aligned}$$

Step 4. Let us consider the following Lyapunov functions

$$\bar{V}_c(\eta^c) = (\eta^c)^T[\Omega\otimes Q]\eta^c \tag{22}$$

$$V_o(\bar{x}_{i,j}) = (\bar{x}_{i,j})^TP(\bar{x}_{i,j}) \tag{23}$$

$$\bar{V}_o(\eta^o) = \sum_{i=1}^{N}\sum_{j=0}^{N}s_{ij}V_o(\bar{x}_{i,j}) \tag{24}$$

where $s_{ij} = 1$ if agent i receives information from agent j and 0 otherwise for $i = 1,\ldots,N$, $j = 0,\ldots,N$ and η^o is the vector containing all the $\bar{x}_{i,j}$ such that $s_{ij} = 1$.

Step 4.1. The derivative of the Lyapunov function (22) can be written as

$$\begin{aligned}\dot{\bar{V}}_c(\eta^c) &= \lambda(\eta^c)^T[\Omega\otimes(A^TQ + QA)]\eta^c - \bar{c}\lambda(\eta^c)^T[(\mathcal{H}^T\Omega)\otimes((BK_c)^TQ)]\eta^c\\ &\quad -\bar{c}\lambda(\eta^c)^T[(\Omega\mathcal{H})\otimes(QBK_c)]\eta^c + 2\bar{c}\lambda(\eta^c)^T[\Omega\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})][\mathcal{B}\otimes I_{2m}]\eta_0^o\\ &\quad -2\bar{c}\lambda\sum_{i=1}^{N}(\eta^c)^T[(\Omega\mathcal{D}_i^N\mathcal{H})\otimes(QBK^c\Gamma_\lambda\Delta_\theta^{-1})]\eta_i^o - 2\lambda(\eta^c)^T[(\Omega\mathbf{1}_N)\otimes(QB)]u_0\end{aligned}$$

One has the following inequalities, whose derivations are detailed in Appendix B:

$$\begin{aligned}
&2\bar{c}\lambda(\eta^c)^T[\Omega\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})][\mathcal{B}\otimes I_{2m}]\eta_0^o \\
&\quad -2\bar{c}\lambda\sum_{i=1}^{N}(\eta^c)^T[(\Omega\mathcal{D}_i^N\mathcal{H})\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})]\eta_i^o \leq 2k_1\lambda\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\bar{V}_c(\eta^c)}\sqrt{\bar{V}_o(\eta^o)} \qquad (25)
\end{aligned}$$

$$-2\lambda(\eta^c)^T[(\Omega\mathbf{1}_N)\otimes(QB)]u_0 \leq 2\lambda\bar{k}_2\delta_0\sqrt{V_c(\eta^c)}. \qquad (26)$$

with

$$\begin{aligned}
k_1 &= \bar{c}\sqrt{N+1}\sqrt{\omega_{\max}}\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\max\{1,\|\mathcal{H}\|\}\,\|K_c\| && (27)\\
\bar{k}_2 &= \sqrt{N}\sqrt{\omega_{\max}}\sqrt{\lambda_{\max}(Q)} && (28)
\end{aligned}$$

furthermore, if $\bar{c} \geq \omega_{\max}/\rho$, then we have

$$\lambda(\eta^c)^T[\Omega\otimes(AQ^T+QA)]\eta^c-\bar{c}\lambda(\eta^c)^T[(\mathcal{H}^T\Omega)\otimes((BK_c)^TQ)]\eta^c-\bar{c}\lambda(\eta^c)^T[(\Omega\mathcal{H})\otimes(QBK_c)]\eta^c \leq -\lambda\bar{V}_c(\eta^c) \quad (29)$$

These inequalities lead to

$$\dot{\bar{V}}_c(\eta^c) \leq -\lambda\bar{V}_c(\eta^c) + 2k_1\lambda\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\bar{V}_c(\eta^c)}\sqrt{\bar{V}_o(\eta^o)} + 2\bar{k}_2\lambda\delta_0\sqrt{\bar{V}_c(\eta^c)}$$

Step 4.2. For i, j such that $s_{ij} = 1$, the derivative of (23) is given by

$$\begin{aligned}
\dot{V}_o(\bar{x}_{i,j}) &= \theta(\bar{x}_{i,j})^T[((A-K_oC)^TP + P(A-K_oC))](\bar{x}_{i,j})\\
&\quad -2\theta(\bar{x}_{i,j})^TPK_oz_{i,j} - \frac{2}{\theta}(\bar{x}_{i,j})^TPBu_j
\end{aligned}$$

One has the following inequalities, whose derivations are detailed in Appendix B:

$$-2\theta(\bar{x}_{i,j}(t))^TPK_oz_{i,j}(t) \leq 2\theta^2\bar{k}_3\sqrt{V_o(\bar{x}_{i,j}(t))}\int_{t-\tau_M}^{t}\sqrt{V_o(\bar{x}_{i,j}(s))}ds \qquad (30)$$

$$\begin{aligned}
-\frac{2}{\theta}(\bar{x}_{i,j})^TPBu_j &\leq 2\frac{\bar{k}_4}{\theta}\sqrt{V_o(\bar{x}_{i,j})}\sqrt{\bar{V}_c(\eta^c)} && (31)\\
&\quad + 2\frac{\bar{k}_5}{\theta}\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{V_o(\bar{x}_{i,j})}\sum_{k=0}^{N}s_{jk}\sqrt{V_0(\bar{x}_{j,k})}\\
-\frac{2}{\theta}(\bar{x}_{i,0})^TPBu_0 &\leq 2\frac{\bar{k}_6}{\theta}\delta_0\sqrt{V_0(\bar{x}_{i,0})} && (32)
\end{aligned}$$

where

$$\begin{aligned}
\bar{k}_3 &= \frac{\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(P)}}\|K_o\|\\
\bar{k}_4 &= \frac{\bar{c}\|K_c\|h_{\max}\sqrt{N}\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(Q)}\sqrt{\omega_{\min}}}\\
\bar{k}_5 &= \frac{\bar{c}\|K_c\|h_{\max}\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(P)}}\\
\bar{k}_6 &= \sqrt{\lambda_{\max}(P)}
\end{aligned}$$

Using inequalities (30), (31) and (32) together with the definition of P leads to

$$\begin{aligned}
\dot{V}_o(\bar{x}_{i,j}) &\leq -\theta V_o(\bar{x}_{i,j}) + 2\theta^2 \bar{k}_3 \sqrt{V_o(\bar{x}_{i,j})} \int_{t-\tau_M}^{t} \sqrt{V_o(\bar{x}_{i,j}(s))} ds + 2\frac{\bar{k}_4}{\theta} \sqrt{V_o(\bar{x}_{i,j})} \sqrt{\bar{V}_c(\eta^c)} \\
&\quad + 2\frac{\bar{k}_5}{\theta} \|\Gamma_\lambda \Delta_\theta^{-1}\| \sqrt{V_o(\bar{x}_{i,j})} \sum_{k=0}^{N} s_{jk} \sqrt{V_o(\bar{x}_{j,k})} \\
&\leq -\theta V_o(\bar{x}_{i,j}) + 2\theta^2 \bar{k}_3 \sqrt{V_o(\bar{x}_{i,j})} \int_{t-\tau_M}^{t} \sqrt{\bar{V}_o(\eta^o(s))} ds + 2\frac{\bar{k}_4}{\theta} \sqrt{V_o(\bar{x}_{i,j})} \sqrt{\bar{V}_c(\eta^c)} \\
&\quad + 2\frac{\bar{k}_5}{\theta} \|\Gamma_\lambda \Delta_\theta^{-1}\| \sqrt{V_o(\bar{x}_{i,j})} \sqrt{N+1} \sqrt{\bar{V}_o(\eta^o)}
\end{aligned}$$

for $j = 1, \ldots, N$, while

$$\dot{V}_o(\bar{x}_{i,0}) \leq -\theta V_o(\bar{x}_{i,0}) + 2\theta^2 \bar{k}_3 \sqrt{V_o(\bar{x}_{i,j})} \int_{t-\tau_M}^{t} \sqrt{\bar{V}_o(\eta^o(s))} ds + 2\frac{\bar{k}_6}{\theta} \delta_0 \sqrt{V_o(\bar{x}_{i,0})}$$

Step 4.3. Letting $\lambda = \epsilon\theta$ with $\epsilon \in (0,1)$, then $\|\Gamma_\lambda \Delta_\theta^{-1}\| = \lambda\theta$ and using the above inequalities, one gets

$$\dot{\bar{V}}_o(\eta^o) \leq -\theta \bar{V}_o(\eta^o) + 2\theta^2 k_3 \sqrt{\bar{V}_o(\eta^o)} \int_{t-\tau_M}^{t} \sqrt{\bar{V}_o(\eta^o(s))} ds + 2\lambda k_5 \bar{V}_o(\eta^o) + 2\frac{k_4}{\theta} \sqrt{\bar{V}_o(\eta^o)} \sqrt{\bar{V}_c(\eta^c)} + 2\frac{k_6}{\theta} \delta_0 \sqrt{\bar{V}_o(\eta^o)}$$

where

$$k_3 = \bar{k}_3 \sqrt{N} \sqrt{N+1} \tag{33}$$
$$k_4 = \bar{k}_4 N \tag{34}$$
$$k_5 = \bar{k}_5 N \sqrt{N+1} \tag{35}$$
$$k_6 = \bar{k}_6 \sqrt{N} \tag{36}$$

Moreover, we obtain

$$\frac{d}{dt}\left(\sqrt{\bar{V}_c(\eta^c)}\right) = \frac{1}{2\sqrt{\bar{V}_c(\eta^c)}} \dot{\bar{V}}_c(\eta^c) \leq -\frac{\lambda}{2} \sqrt{\bar{V}_c(\eta^c)} + k_1 \lambda^2 \theta \sqrt{\bar{V}_o(\eta^o)} + \bar{k}_2 \lambda \delta_0$$

and similarly

$$\frac{d}{dt}\left(\sqrt{\bar{V}_o(\eta^o)}\right) \leq -\frac{\theta}{2} \sqrt{\bar{V}_o(\eta^o)} + k_3 \theta^2 \int_{t-\tau_M}^{t} \sqrt{\bar{V}_o(\eta^o(s))} ds + \frac{k_4}{\theta} \sqrt{\bar{V}_c(\eta^c)} + k_5 \lambda \sqrt{\bar{V}_o(\eta^o)} + \frac{k_6}{\theta} \delta_0.$$

Step 5. We have

$$\begin{aligned}
\frac{d}{dt}\left(\sqrt{\bar{V}_c(\eta^c)} + \epsilon^{\frac{3}{2}} \theta^2 \sqrt{\bar{V}_o(\eta^o)}\right) &\leq -\frac{\epsilon\theta}{4}\left(1 - 4k_4 \epsilon^{\frac{1}{2}}\right) \sqrt{\bar{V}_c(\eta^c)} - \frac{\epsilon^{\frac{3}{2}} \theta^3}{4}\left(1 - 4k_1 \epsilon^{\frac{1}{2}} - 4k_5 \epsilon\right) \sqrt{\bar{V}_o(\eta^o)} \\
&\quad - \frac{\epsilon\theta}{4} \sqrt{\bar{V}_c(\eta^c)} - \frac{\epsilon^{\frac{3}{2}} \theta^3}{4} \sqrt{\bar{V}_o(\eta^o)} + k_3 \epsilon^{\frac{3}{2}} \theta^4 \int_{t-\tau_M}^{t} \sqrt{\bar{V}_o(\eta^o(s))} ds \\
&\quad + \bar{k}_2 \epsilon\theta\delta_0 + k_6 \epsilon^{\frac{3}{2}} \theta \delta_0
\end{aligned}$$

By choosing $\epsilon < \epsilon^*$ where $\epsilon^* = \min\left\{1, \frac{1}{(4k_4)^2}, \frac{1}{(8k_1)^2}, \frac{1}{8k_5}\right\}$, we obtain

$$\frac{d}{dt}\left(\sqrt{\bar{V}_c(\eta^c)} + \epsilon^{\frac{3}{2}} \theta^2 \sqrt{\bar{V}_o(\eta^o)}\right) \leq -\frac{\epsilon\theta}{4} \sqrt{\bar{V}_c(\eta^c)} - \frac{\epsilon^{\frac{3}{2}} \theta^3}{4} \sqrt{\bar{V}_o(\eta^o)} + k_3 \epsilon^{\frac{3}{2}} \theta^4 \int_{t-\tau_M}^{t} \sqrt{\bar{V}_o(\eta^o(s))} ds + k_2 \epsilon\theta\delta_0$$

where

$$k_2 = \max\{\bar{k}_2, k_6\} \tag{37}$$

Applying Lemma 3 with

$$\begin{aligned} a &= \frac{\epsilon\theta}{4} && (38)\\ b &= \frac{\epsilon^{\frac{3}{2}}\theta^3}{4} && (39)\\ c &= k_3\epsilon^{\frac{3}{2}}\theta^4 && (40)\\ k &= k_2\epsilon\theta\delta_0 && (41) \end{aligned}$$

one obtains the existence of $\varrho > 0$ and $\bar{\alpha} > 0$ such that if

$$\tau_M < \varrho \min\left(\frac{b}{c}, \frac{1}{\sigma}\right) \tag{42}$$

with $\sigma = \frac{1}{2}\min(a, b) = \frac{\lambda}{8}$ since $\lambda \geq 1$, then

$$\sqrt{\bar{V}_c(\eta^c)} + \epsilon^{\frac{3}{2}}\theta^2\sqrt{\bar{V}_o(\eta^o)} \leq \bar{\alpha}e^{-\sigma t} + \frac{k_2\epsilon\theta\delta_0}{\sigma}$$

Since

$$\min\left(\frac{b}{c}, \frac{1}{\sigma}\right) = \min\left(\frac{1}{4\theta k_3}, \frac{8}{\lambda}\right) \geq \min\left(\frac{1}{4k_3}, 8\right)\min\left(\frac{1}{\theta}, \frac{1}{\lambda}\right) \geq \frac{1}{4k_3\theta} \tag{43}$$

then, if θ verifies

$$\tau_M < \frac{\bar{\varrho}}{\theta} \tag{44}$$

with $\bar{\varrho} = \frac{\varrho}{4k_3}$, the following inequality holds true

$$\sqrt{\bar{V}_c(\eta^c)} + \epsilon^{\frac{3}{2}}\theta^2\sqrt{\bar{V}_o(\eta^o)} \leq \bar{\alpha}e^{-\frac{\lambda}{8}t} + 8k_2\delta_0 \tag{45}$$

Step 6. One now comes back to the original coordinates of both observer and tracking errors. Since $\lambda \geq 1$ and $\theta \geq 1$, one has

$$\sqrt{\bar{V}_c(\eta^c)} \geq \lambda l_1 \sum_{i=1}^{N} \|e_i\| \tag{46}$$

$$\sqrt{\bar{V}_o(\eta^o)} \geq \frac{l_2}{\theta}\sum_{i=1}^{N}\sum_{j=0}^{N} \|\tilde{x}_{i,j}\| \tag{47}$$

where

$$\begin{aligned} l_1 &= \frac{\sqrt{\lambda_{\min}(Q)}\sqrt{\omega_{\min}}}{\sqrt{N}} && (48)\\ l_2 &= \frac{\sqrt{\lambda_{\min}(P)}}{\sqrt{N}\sqrt{N+1}} && (49) \end{aligned}$$

The derivation of (46) and (47) is provided in Appendix B. Using these inequalities, one obtains

$$\sqrt{\bar{V}_c(\eta^c)} + \epsilon^{\frac{3}{2}}\theta^2\sqrt{\bar{V}_o(\eta^o)} \geq \lambda l_1 \sum_{i=1}^{N} \|e_i\| + \epsilon^{\frac{3}{2}}\theta l_2 \sum_{i=1}^{N}\sum_{j=0}^{N} \|\tilde{x}_{i,j}\|$$

Over-valuation of the tracking error e_i is given by

$$\sqrt{\bar{V}_c(\eta^c)} + \epsilon^{\frac{3}{2}}\theta^2\sqrt{\bar{V}_o(\eta^o)} \quad \geq \quad \lambda l_1 \sum_{i=1}^{N} \|e_i\|$$

by using inequality (45), it gives

$$\sum_{i=1}^{N} \|e_i\| \quad \leq \quad \alpha e^{-\frac{\lambda}{8}t} + \frac{\beta\delta_0}{\lambda} \tag{50}$$

with

$$\alpha = \frac{\bar{\alpha}}{\lambda l_1} \tag{51}$$

$$\beta = \frac{8k_2}{l_1} \tag{52}$$

This ends the proof. □

Remark 3. *It is clear from inequality* (50) *that the error will gradually converge and will enter in a ball centered at the origin. It means that practical consensus is achieved. The radius of the convergence ball of the tracking error is directly proportional to the bound of the leader input/acceleration* δ_0. *The size of this ball can also be reduced by increasing the controller gain* λ *(while keeping in mind that the controller dynamics must remain slower than the observer dynamics in order to guarantee stability of the closed-loop system). Furthermore, if the leader is moving with constant velocity i.e.* $u_0 = 0$, *then the MAS achieves exponential consensus.*

Remark 4. *One can remark that the conditions on the control parameters given in Theorem 1 require some global information like* $\mathcal{H}$ *and* N. *Hence, each agent must have some global knowledge about the communication topology similarly to many existing works on consensus. Nevertheless, the tuning parameters* θ, λ *and* $\bar{c}$ *are chosen beforehand and then they remain constant for all* $t \geq 0$ *as only fixed communication topology is considered. Once the gains are set, only local information is needed to achieve leader-following consensus.*

Remark 5. *In [40], the problem of leaderless consensus has been discussed whereas the current paper deals with the problem of leader-following consensus. It is important to mention that the results of [40] cannot be directly applied to the leader-following case. Indeed, the addition of an active leader is not straightforward because of aperiodic sampling periods and availability of partial data only. Both the tracking and estimation error dynamics are different from [40] due to the presence of leader dynamics. Therefore, the stability analysis is different from the one presented in [40].*

4. Simulation results

Consider a multi-agent system with four followers, labeled from 1 to 4 and a leader denoted as 0. The communication topology of the system is shown in Figure 2.

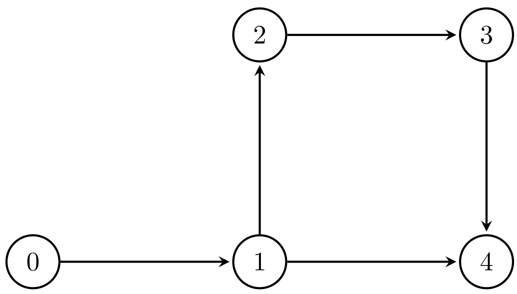
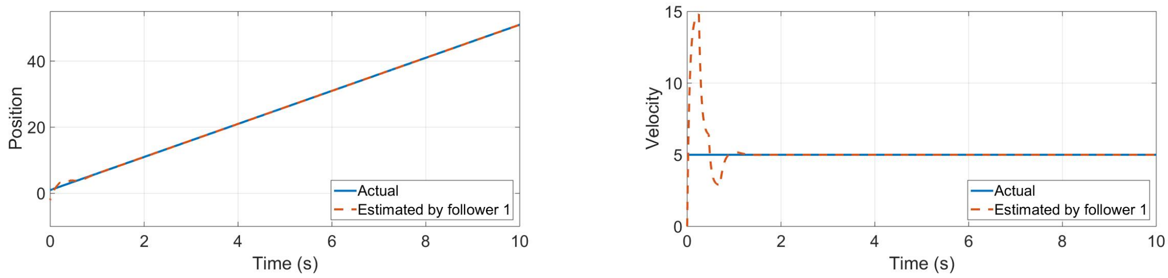


Figure 2: Communication topology.

The adjacency and pinning matrices corresponding to this topology are

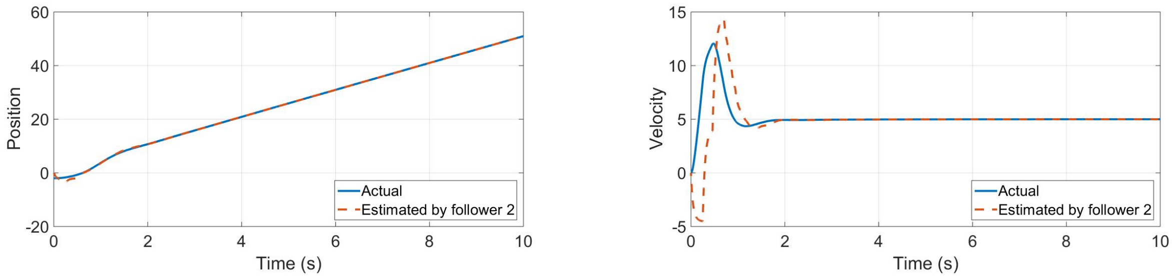
$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Based on Theorem 1 and Remark 1, the gains for simulation purpose are chosen as $\bar{c} = 1$, $\lambda = 1.0$ and $\theta = 10$. Figure 3 depicts the estimation of both position and velocity of the leader by follower 1. Similarly, the state estimation of follower 1 by follower 2 is also shown in Figure 4. It can be seen that the observer estimates both position and velocity quite efficiently. The time intervals between transmission of position state information from follower 1 to follower 2 is presented in Figure 5. Figures 6 and 7 show the leader-following consensus results for step and ramp position trajectories of the leader respectively. In case of non-zero leader input, only practical stability is achieved which is shown in Figure 8. It is worth noting that only position information is transmitted through the communication network, given in Figure 2, at nonuniform and asynchronous sampling instants. Velocities and inputs are completely unknown to the neighbors. Despite these constraints, the system achieved leader-following consensus.



(a) position (b) velocity

Figure 3: Estimation of the leader's states by follower 1.



(a) position (b) velocity

Figure 4: Estimation of follower 1's states by follower 2.

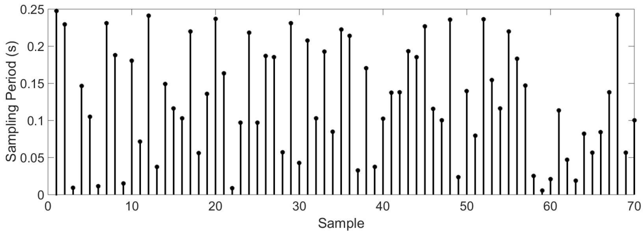
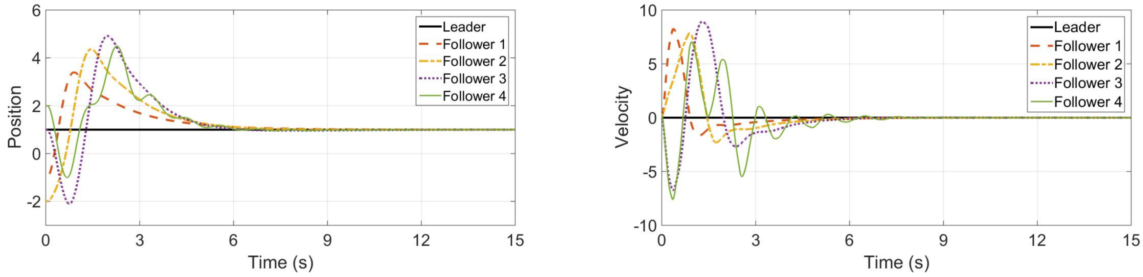


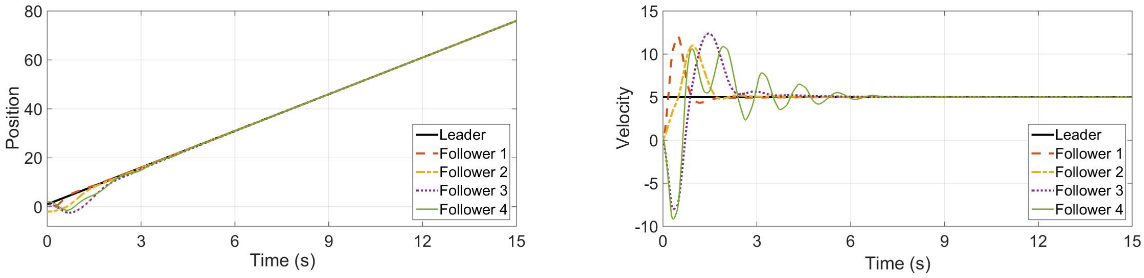
Figure 5: Sampling period for transmission of position information from follower 1 to follower 2.



(a) position

(b) velocity

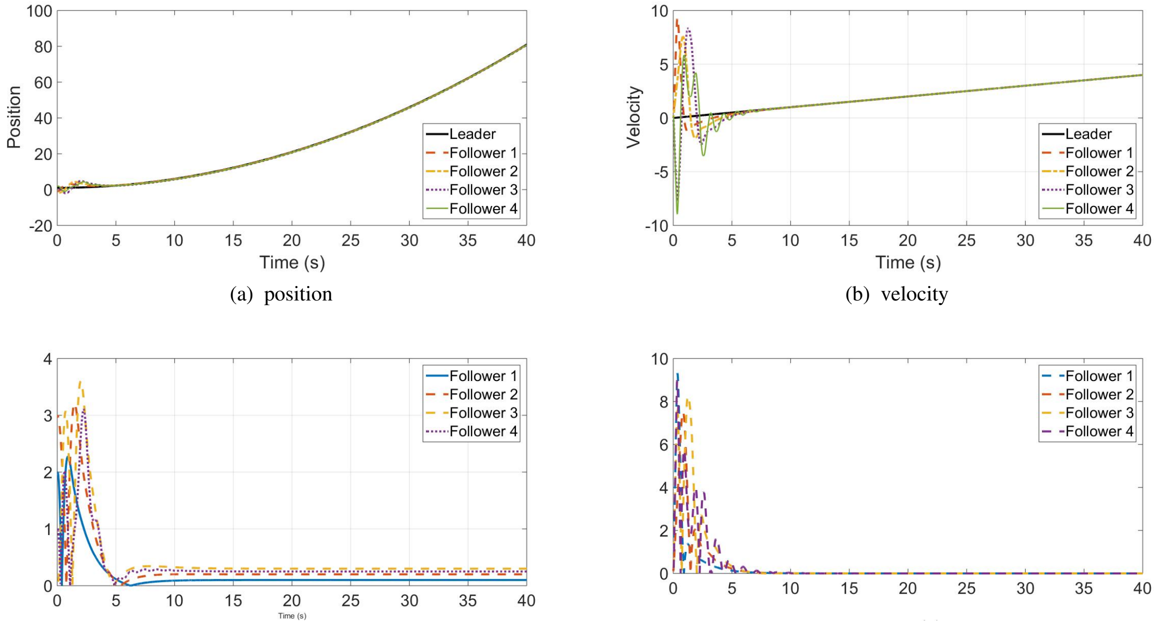
Figure 6: Consensus tracking when the leader trajectory is a step function.



(a) position

(b) velocity

Figure 7: Consensus tracking when the leader trajectory is a ramp function.



(a) position

(b) velocity

(c) position error $|r_0 - r_i|$

(d) velocity error $|v_0 - v_i|$

Figure 8: Consensus tracking when $u_0 = 0.1$.

5. Experimental results

The proposed algorithm has been implemented on a group of wheeled mobile robots. The group consists of 3 Mini-Lab robots by Enova ROBOTICS. Table 1 illustrates the specifications of Mini-Lab robot. It is equipped with various sensors including a camera for depth images and an odometer for position measurements. The open source control architecture of the robot allows the implementation of the developed algorithm using Robot Operating System (ROS). ROS is a multi-lingual framework where modules can be written in various languages like C++, Python and Lisp. Each process in ROS is separately designed and programmed and is referred to as a node. Nodes can be run on different computers since ROS provides a distributed computation environment. Moreover, data among the nodes can be shared by passing messages over a topic. A node can publish or subscribe to multiple topics at a time. A ROS master manages the naming and registration of the nodes in an overall ROS system and tracks the publishers and subscribers to the topics. All nodes can communicate to the ROS master for registration and then they can receive information of other registered nodes. In the current experimental setup, the robots are connected over a wireless network and each robot has its own ROS master. The ROS package *rosmaster_fkie* has been used to establish and manage a ROS multi-master network that consists of three robots. The proposed consensus tracking algorithm has been simulated for Mini-Lab using ROS Gazebo simulator prior to the actual implementation. ENOVA ROBOTICS has provided URDF file of Mini-Lab which gives 3D visualization of the robot along with the kinematic and dynamic properties.

In the following experiment, robot motion is restricted to $1-$D space (i.e. a robot is allowed to move only in a straight line), Hence, the nonholonomic constraints can be ignored and robots can be modelled as double integrator systems with position and velocity as states. Therefore, the proposed algorithm is implemented with the consideration that robots are moving only in the $x-$axis with some constant offset in their $y-$positions. The consensus is supposed to be achieved for the $x-$axis positions and the $x-$axis velocities only. The experimental setup is shown in Figure 9 while the communication topology for the experiments is illustrated in Figure 10. Keeping in mind the hardware limitation of the robots, the controller gains have been chosen in such a way that the maximum speed limit is never crossed. Experimental results of the leader-following consensus, for both step and ramp position trajectories of the leader, are shown in Figure 11. It is clear from these results that MAS achieved the leader-following consensus efficiently in spite of the presence of uncertainties and disturbances inherent for a real application. An example of the time period between information transmission from one robot to the other is depicted in Figure 12. This nonuniform transmission of data may be caused by multiple reasons including computation time and measurement delays.

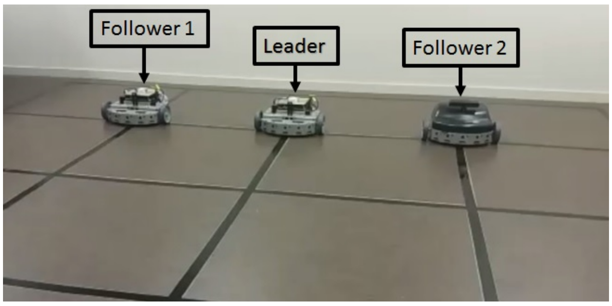


Figure 9: Experimental Setup.

Mechanical	Dimensions (W $\times$ L $\times$ H)	409 $\times$ 364 $\times$ 231 mm
	Weight	11.5 Kg
	Load Capacity	3 Kg
	Max Speed	1.5 m/s
	Max Slope Angle	10 degree
Electronics	Processor	Atom N2800
	Sensor Interface	Ardino
	Depth Camera	Asus Xtion Live Pro
	Sensors	Ultrasonic ($\times$5)
		Infrared ($\times$5)
Communication	Wireless	IEEE 802.11b/g/n-
	Extension with	USB, Ethernet
Power	Battery	12V
	Autonomy	4h
	On-board Voltages	5V/12V

Table 1: Specifications of Mini-Lab robot

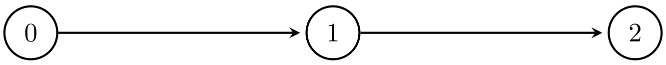
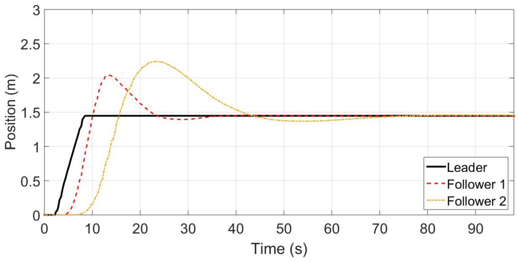
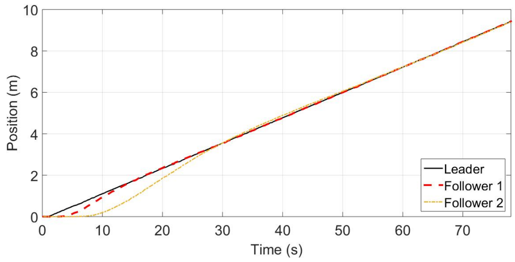


Figure 10: Communication topology for hardware experiments



(a)



(b)

Figure 11: Experimental results for leader-following consensus(a) step (b) ramp

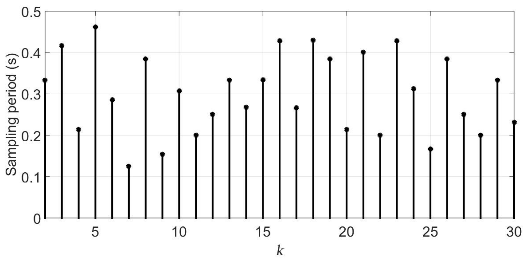


Figure 12: Sampling period for transmission from follower 1 to follower 2

6. Conclusion

In this article, an algorithm has been proposed to deal with the communication constraints in the leader-following consensus problem. These communication constraints include nonuniform and asynchronous sampling periods and availability of only the position state. An observer has been designed to estimate the states of the leader and followers through discrete position information of neighbors which is available at irregular time intervals. The effectiveness of the proposed law has been demonstrated through both simulation and implementation on a fleet of wheeled mobile robots. The design of consensus algorithms to further cope with the nonlinear model of the robot is planned as a future work.

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Appendix A.

Lemma 4. *We have the following results:*

a) $\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2$ *for all* $A \in \mathbb{R}^{n\times n}$*;*

b) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$ *for all* $x \in \mathbb{R}^n$*;*

c) $\sum_{i=1}^n \sqrt{x_i} \leq \sqrt{n}\sqrt{\sum_{i=1}^n x_i}$ *for all* $x_i \in \mathbb{R}^+$ *with* $i = 1, \ldots, n$*;*

d) $\sum_{i=1}^n (x_i)^2 \leq \left(\sum_{i=1}^n x_i\right)^2$ *for all* $x_i \in \mathbb{R}^+$ *with* $i = 1, \ldots, n$*;*

e) let μ_i, $i = 1, \ldots, m$ *and* ν_j, $j = 1, \ldots, n$ *be respectively the eigenvalues of* $A \in \mathbb{R}^{m\times m}$ *and* $B \in \mathbb{R}^{n\times n}$*, then the eigenvalues of* $A \otimes B$ *are* $\mu_i\nu_j$ *with* $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$*;*

f) $\|A \otimes B\|_2 = \|A\|_2\,\|B\|_2$ *for all* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{m\times m}$*;*

g) let $A, B \in \mathbb{R}^{n\times n}$ *be positive definite symmetric matrices, then*

$$\begin{aligned} \lambda_{\max}(A \otimes B) &\leq \lambda_{\max}(A)\lambda_{\max}(B) \\ \lambda_{\min}(A \otimes B) &\geq \lambda_{\min}(A)\lambda_{\min}(B) \end{aligned}$$

h) for any symmetric definite positive matrix $M \in \mathbb{R}^{n\times n}$ *and* $x, y \in \mathbb{R}^n$, $x^TMy \leq \sqrt{x^TMx}\sqrt{y^TMy}$ *(Cauchy-Schwarz inequality);*

i) for any symmetric matrix $M \in \mathbb{R}^{n\times n}$ *and* $x \in \mathbb{R}^n$, $\lambda_{\min}(M)x^Tx \leq x^TMx \leq \lambda_{\max}(M)x^Tx$ *(Rayleigh inequality);*

j) let $(u^i)_{1\leq i\leq n}$ *a basis of* $\mathbb{R}^n$ *and* $(v^j)_{1\leq j\leq m}$ *a basis of* $\mathbb{R}^m$*, then* $(u^i \otimes v^j)_{1\leq i\leq n, 1\leq j\leq m}$ *is a basis of* $\mathbb{R}^{nm}$*;*

k) let $A \in \mathbb{R}^{n\times n}$ *be a symmetric definite positive matrix and* $B \in \mathbb{R}^{\times m}$ *be a symmetric semi-definite matrix, then the following inequality hold*

$$\lambda_{\min}(A)I_n \otimes B \leq A \otimes B \leq \lambda_{\max}(A)I_n \otimes B.$$

PROOF. a) See [46, p.304], section 5.6.23

b) See [46, p.304], section 5.6.23

c) Consider $X = \begin{pmatrix} \sqrt{x_1} \\ \vdots \\ \sqrt{x_n} \end{pmatrix}$, then we have

$$\begin{aligned} \|X\|_1 &= \sum_{i=1}^n \sqrt{x_i} \\ \|X\|_2 &= \sqrt{\sum_{i=1}^n (\sqrt{x_i})^2)} = \sqrt{\sum_{i=1}^n x_i} \end{aligned}$$

Applying Lemma 4-b) gives the result.

d) The proof is straightforward and then not reported here.

e) See [47, p.27], property *IX*.

f) Denoting $\rho(A)$ the spectral radius of matrix A, one has

$$\begin{aligned}\|A\otimes B\|_2^2 &= \rho\left((A\otimes B)^T(A\otimes B)\right)\\ &= \rho((A^TA)\otimes(B^TB))\\ &= \rho(A^TA)\rho(B^TB)) \quad \text{(by applying Lemma 4-e)}\\ &= \|A\|_2^2\,\|B\|_2^2\end{aligned}$$

g) The two inequalities are obtained directly from Lemma 4-e).

h) See [46, p.15], subsection 0.6.3.

i) See [46, p.234], Theorem 4.2.2.

j) From property *X* in [47, p. 27], one has $det(A\otimes B) = (det(A))^m(det(B))^n$ for any matrices $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{m\times m}$. Then taking $A=(u^1,\dots,u^n)$ and $B=(v^1,\dots,v^m)$, one has $det(A\otimes B) = det([u^1\otimes v^1,\dots,u^1\otimes v^m, u^2\otimes v^1,\dots,u^n\otimes v^m]) = (det(A))^n(det(B))^m \neq 0$ since (u^i) and (v^j) are basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

k) Let x be a non zero vector $\mathbb{R}^{mn}$. Let (u^i) (resp. (v^j)) be a basis of orthogonal eigenvectors of A (resp. B) of $\mathbb{R}^n$ (resp. $\mathbb{R}^m$), that is $Au^i=\mu^iu^i$, with μ^i an eigenvalue of A (resp. $Bv^j=\lambda^jv^j$, with λ^j an eigenvalue of B). Since, according to point *j)*, $(u^i\otimes v^j)_{1\leq i,1\leq j\leq m}$ is a basis of $\mathbb{R}^{mn}$, there exist reals α_{ij} such that

$$x=\sum_{i=1}^n\sum_{j=1}^m\alpha_{ij}u^i\otimes v^j.$$

One has

$$\begin{aligned}x^T(A\otimes B)x &= \sum_{i_1,i_2=1}^n\sum_{j_1,j_2=1}^m\alpha_{i_1j_1}\alpha_{i_2j_2}(u^{i_1}\otimes v^{j_1})^T(A\otimes B)(u^{i_2}\otimes v^{j_2})\\ &= \sum_{i_1,i_2=1}^n\sum_{j_1,j_2=1}^m\alpha_{i_1j_1}\alpha_{i_2j_2}((u^{i_1})^TAu^{i_2})\otimes((v^{j_1})^TBu^{j_2})\\ &= \sum_{i=1}^n\sum_{j=1}^m\alpha_{ij}^2((u^i)^TAu^i)\otimes((v^j)^TBu^j) \text{ since } (u^i) \text{ and } (v^j) \text{ are orthogonal basis,}\\ &= \sum_{i=1}^n\sum_{j=1}^m\alpha_{ij}^2\mu^i\lambda^j((u^i)^Tu^i)\otimes((v^j)^Tu^j)\end{aligned}$$

Then, since B is semidefinite positive symmetric, one has $\lambda^j\geq 0$, $j=1,\dots,m$. Furthermore $((u^i)^Tu^i)\otimes((v^j)^Tv^j) = ((u^i)^Tu^i)((v^j)^Tv^j)\geq 0$, then

$$\begin{aligned}\sum_{i=1}^n\sum_{j=1}^m\alpha_{ij}^2\mu^i\lambda^j((u^i)^Tu^i)\otimes((v^j)^Tv^j) &\leq \max_i\{\mu^i\}\sum_{i=1}^n\sum_{j=1}^m\alpha_{ij}^2\lambda^j((u^i)^Tu^i)\otimes((v^j)^Tv^j)\\ &= \max_i\{\mu^i\}\sum_{i=1}^n\sum_{j=1}^m\alpha_{ij}^2((u^i)^Tu^i)\otimes((v^j)^TBv^j)\\ &= \max_i\{\mu^i\}x^T(I_n\otimes B)x\end{aligned}$$

The same can be done for the other inequality. $\square$

Appendix B.

The detailed derivations of the inequalities, introduced in the proof of Theorem 1, are given in this Appendix.

a) Derivation of inequality (25)
Inequality (25) is derived in two steps. An over-valuation of the two terms in the left hand-side of (25) are first obtained and then combined in order to obtain (25).
For the first term, using Lemma 4-h)-i)-g) yields

$$\begin{aligned} 2\bar{c}\lambda(\eta^c)^T[\Omega\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})][\mathcal{B}\otimes I_{2m}]\eta_0^o &= 2\bar{c}\lambda(\eta^c)^T[(\Omega\otimes Q)(I_N\otimes BK_c\Gamma_\lambda\Delta_\theta^{-1})][\mathcal{B}\otimes I_{2m}]\eta_0^o \\ &\leq 2\bar{c}\lambda\sqrt{\bar{V}_c(\eta^c)}\sqrt{\omega_{\max}\lambda_{\max}(Q)}\|I_N\otimes BK_c\Gamma_\lambda\Delta_\theta^{-1}\|_2\|[\mathcal{B}\otimes I_{2m}]\eta_0^o\|_2 \end{aligned} \quad \text{(B.1)}$$

Moreover

$$\begin{aligned} \|[\mathcal{B}\otimes I_{2m}]\eta_0^o\|_2^2 &= \sum_{i=1}^N b_i\|\bar{x}_{i,0}\|_2^2 \\ &\leq \sum_{i=1}^N \frac{b_i}{\lambda_{\min}(P)}\left((\bar{x}_{i,0})^T P\bar{x}_{i,0}\right) \quad \text{by using Lemma 4-i)} \\ &\leq \frac{1}{\lambda_{\min}(P)}\sum_{i=1}^N s_{i0}V_o(\bar{x}_{i,0}) \end{aligned} \quad \text{(B.2)}$$

By Lemma 4-f) and since $\|I_N\|_2 = 1$, we obtain

$$\begin{aligned} &2\bar{c}\lambda(\eta^c)^T[\Omega\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})][\mathcal{B}\otimes I_{2m}]\eta_0^o \\ &\leq 2\bar{c}\lambda\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\omega_{\max}}\frac{\sqrt{\lambda_{\max}(Q)}}{\sqrt{\lambda_{\min}(P)}}\|BK_c\|\sqrt{\bar{V}_c(\eta^c)}\sqrt{\sum_{i=0}^N s_{i0}V_o(\bar{x}_{i0})} \\ &\leq 2\bar{c}\lambda\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\omega_{\max}}\frac{\sqrt{\lambda_{\max}(Q)}}{\sqrt{\lambda_{\min}(P)}}\|K_c\|\sqrt{\bar{V}_c(\eta^c)}\sqrt{\sum_{i=0}^N s_{i0}V_o(\bar{x}_{i0})} \quad \text{since } \|B\| = 1 \end{aligned} \quad \text{(B.3)}$$

For the second term, we have

$$\begin{aligned} &-2\bar{c}\lambda\sum_{i=1}^N(\eta^c)^T[(\Omega\mathcal{D}_i^N\mathcal{H})\otimes(QBK^c\Gamma_\lambda\Delta_\theta^{-1})]\eta_i^o \\ = \;&-2\bar{c}\lambda\sum_{i=1}^N(\eta^c)^T(\Omega\otimes Q)[(\mathcal{D}_i^N\mathcal{H})\otimes(BK_c\Gamma_\lambda\Delta_\theta^{-1})]\eta_i^o \\ \leq \;& 2\bar{c}\lambda\sum_{i=1}^N\sqrt{\bar{V}_c(\eta^c)}\sqrt{\lambda_{\max}(\Omega\otimes Q)}\|(\mathcal{D}_i^N\mathcal{H})\otimes(BK_c\Gamma_\lambda\Delta_\theta^{-1})\|_2\sqrt{\sum_{j=1}^N s_{ij}\|\bar{x}_{i,j}\|} \quad \text{by Lemma 4-h) and i)} \\ \leq \;& 2\bar{c}\lambda\sqrt{\omega_{\max}\lambda_{\max}(Q)}\|\mathcal{H}\|\,\|K_c\|\,\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\bar{V}_c(\eta^c)}\sum_{i=1}^N\sqrt{\sum_{j=1}^N s_{ij}\|\bar{x}_{i,j}\|} \\ &\qquad \text{by using Lemma 4-e) and } \|\mathcal{D}_i^N\|_2 = 1,\ \|B\|_2 = 1 \\ \leq \;& 2\bar{c}\lambda\sqrt{\omega_{\max}}\frac{\sqrt{\lambda_{\max}(Q)}}{\sqrt{\lambda_{\min}(P)}}\|\mathcal{H}\|\,\|K_c\|\,\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\bar{V}_c(\eta^c)}\sum_{i=1}^N\sqrt{\sum_{j=1}^N s_{ij}V(\bar{x}_{i,j})} \quad \text{by using Lemma 4-i)} \end{aligned} \quad \text{(B.4)}$$

Finally, by using inequalities (B.3) and (B.4), one obtains

$$\begin{aligned}
& 2\bar{c}\lambda(\eta^c)^T[\Omega\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})][b\otimes I_{2m}]\eta_0^o - 2\bar{c}\lambda\sum_{i=1}^{N}(\eta^c)^T[(\Omega\mathcal{D}_i^N\mathcal{H})\otimes(QBK_c\Gamma_\lambda\Delta_\theta^{-1})]\eta_i^o \\
\leq\ & 2\bar{c}\lambda\sqrt{\omega_{\max}}\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\max\{1,\|\mathcal{H}\|\}\,\|K_c\|\,\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{\bar{V}_c(\eta^c)}\sum_{i=0}^{N}\sqrt{V_o(\eta_i^o)} \\
\leq\ & 2\lambda\|\Gamma_\lambda\Delta_\theta^{-1}\|\underbrace{\bar{c}\sqrt{N+1}\sqrt{\omega_{\max}}\sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\max\{1,\|\mathcal{H}\|\}\,\|K_c\|}_{\triangleq k_1}\sqrt{\bar{V}_c(\eta^c)}\sqrt{\bar{V}_o(\eta^o)} \quad \text{by using Lemma 4-c).}
\end{aligned}$$

b) Derivation of inequality (26)
One has

$$\begin{aligned}
-2\lambda(\eta^c)^T[(\Omega\mathbf{1}_N)\otimes(QB)]u_0 &= -2\lambda(\eta^c)^T(\Omega\otimes Q)(\mathbf{1}_N\otimes B)u_0 \\
&\leq 2\underbrace{\sqrt{N}\sqrt{\omega_{\max}}\sqrt{\lambda_{\max}(Q)}}_{\triangleq \bar{k}_2}\lambda\delta_0\sqrt{\bar{V}_c(\eta^c)} \quad \text{by using Lemma 4-g), h) and i)}
\end{aligned}$$

c) Derivation of inequality (29)
Using the equality $(BK^c)^TQ = QBK^c = QBB^TQ$, one gets with point k) of Lemma 4

$$\begin{aligned}
(\mathcal{H}^T\Omega)\otimes((BK_c)^TQ) + (\Omega\mathcal{H})\otimes(QBK_c) &= [\mathcal{H}^T\Omega+\Omega\mathcal{H}]\otimes[QBB^TQ] \\
&\geq \rho(I_N\otimes[QBB^TQ]) \\
&\geq \frac{\rho}{\omega_{\max}}\Omega\otimes[QBB^TQ]
\end{aligned}$$

If $\bar{c}\geq\frac{\omega_{\max}}{\rho}$, one then obtains

$$\lambda(\eta^c)^T[\Omega\otimes(A^TQ+QA)]\eta^c - \bar{c}\lambda(\eta^c)^T[(\mathcal{H}^T\Omega)\otimes((BK_c)^TQ)]\eta^c - \bar{c}\lambda(\eta^c)^T[(\Omega\mathcal{H})\otimes(QBK_c)]\eta^c \leq -\lambda\bar{V}_c(\eta^c)$$

d) Derivation of inequality (30)
One has

$$-2\theta(\bar{x}_{i,j})^TPK_oz_{i,j}(t) \leq 2\theta\sqrt{\lambda_{\max}(P)}\|K_o\|\sqrt{V_o(\bar{x}_{i,j})}\|z_{i,j}(t)\| \quad \text{by using Lemma 4-i)} \qquad \text{(B.5)}$$

Since

$$\dot{z}_{i,j}(t) = -2\theta e^{-2\theta(t-\kappa_{i,j}(t))}C\tilde{x}_j^i(\kappa_{i,j}(t)) - C\dot{\tilde{x}}_{i,j}(t) = -B^T\tilde{x}_{i,j}(t)$$

and $z_{i,j}(\kappa_{i,j}(t))=0$, $\forall t\geq 0$, then we have

$$\begin{aligned}
z_{i,j}(t) &= -\int_{\kappa_{i,j}(t)}^{t}B^T\tilde{x}_{i,j}(s)ds = -\theta\int_{\kappa_{i,j}(t)}^{t}B^T\bar{x}_{i,j}(s)ds \\
\|z_{i,j}\| &= \theta\left\|\int_{\kappa_{i,j}(t)}^{t}B^T\bar{x}_{i,j}(s)ds\right\| \\
&\leq \theta\int_{t-\tau_M}^{t}\|\bar{x}_{i,j}(s)\|ds \quad \text{from the fact that } t-\kappa_{i,j}(t)\leq\tau_M,\ \forall t\geq 0,\ \text{and } \left\|B^T\right\|=1 \\
&\leq \frac{\theta}{\sqrt{\lambda_{\min}(P)}}\int_{t-\tau_M}^{t}\sqrt{((\bar{x}_{i,j}(s))^TP(\bar{x}_{i,j}(s)))}ds \quad \text{by using Lemma 4-i)}
\end{aligned}$$

Therefore, (B.5) becomes

$$-2\theta(\bar{x}_{i,j})^T PK_o z_{i,j}(t) \leq 2\theta^2 \underbrace{\frac{\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(P)}}\|K_o\|}_{\triangleq \bar{k}_3} \sqrt{V_o(\bar{x}_{i,j}(t))} \int_{t-\tau_M}^{t} \sqrt{V_o(\bar{x}_{i,j}(s))}ds$$

e) Derivation of inequality (31)
One has

$$\begin{aligned}
\|u_j\| &\leq \bar{c}\|K_c\|h_{\max}\left(\sum_{k=1}^{N}\|\bar{e}_k\| + \|\Gamma_\lambda\Delta_\theta^{-1}\|\sum_{k=0}^{N}s_{jk}\|\bar{x}_{j,k}\|\right)\\
\|\bar{e}_k\| &\leq \frac{1}{\sqrt{\lambda_{\min}(Q)}}\sqrt{\bar{e}_k^T Q\bar{e}_k} \quad \text{by using lemma 4-i)}\\
\sum_{k=1}^{N}\|\bar{e}_k\| &\leq \frac{\sqrt{N}}{\lambda_{\min}(Q)\sqrt{\omega_{min}}}\sqrt{\bar{V}_c(\eta^c)} \quad \text{by using Lemma 4-c)}\\
\|\bar{x}_{j,k}\| &\leq \frac{1}{\sqrt{\lambda_{\min}(P)}}\sqrt{V_o(\bar{x}_{j,k})} \quad \text{by using Lemma 4-i)}
\end{aligned}$$

So it leads to

$$\begin{aligned}
-\frac{2}{\theta}(\bar{x}_{i,j})^T PBu_j &\leq \frac{2}{\theta}\sqrt{V_o(\bar{x}_{i,j})}\sqrt{\lambda_{\max}(P)}\|u_j\|\\
&\leq \frac{2}{\theta}\underbrace{\frac{\bar{c}\|K_c\|h_{\max}\sqrt{N}\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(Q)}\sqrt{\omega_{\min}}}}_{\triangleq \bar{k}_4}\sqrt{V_o(\bar{x}_{i,j})}\sqrt{\bar{V}_c(\eta^c)}\\
&\quad +\frac{2}{\theta}\underbrace{\frac{\bar{c}\|K_c\|h_{\max}\sqrt{\lambda_{\max}(P)}}{\sqrt{\lambda_{\min}(P)}}}_{\triangleq \bar{k}_5}\|\Gamma_\lambda\Delta_\theta^{-1}\|\sqrt{V_o(\bar{x}_{i,j})}\sum_{k=0}^{N}s_{jk}\sqrt{V_o(\bar{x}_{j,k})}
\end{aligned}$$

f) Derivation of inequality (32)
One has

$$\begin{aligned}
-\frac{2}{\theta}(\bar{x}_{i,0})^T PBu_0 &\leq \frac{2}{\theta}\sqrt{\lambda_{\max}(P)}\sqrt{(\bar{x}_{i,0})^T P(\bar{x}_{i,0})}\sqrt{(Bu_0)^T(Bu_0)} \quad \text{by using Lemma 4-h)}\\
&\leq \frac{2}{\theta}\delta_0\underbrace{\sqrt{\lambda_{\max}(P)}}_{\triangleq \bar{k}_6}\sqrt{V_0(\bar{x}_{i,0})}
\end{aligned}$$

by using Lemma 4-i) and the fact that $\|B\| = 1$ and $\|u_0(t)\| \leq \delta_0$, $\forall t \geq 0$

g) Derivation of inequalities (46) and (47)

One has

$$\begin{aligned}
\lambda\sqrt{\lambda_{\min}(Q)}\|e_i\| &\leq \sqrt{\bar{e}_i^T Q\bar{e}_i} \\
\|e_i\| &\leq \frac{1}{\lambda\sqrt{\lambda_{\min}(Q)}\sqrt{\omega_i}}\sqrt{\omega_i\bar{e}_i^T Q\bar{e}_i} \\
\sum_{i=1}^{N}\|e_i\| &\leq \frac{1}{\lambda}\frac{\sqrt{N}}{\sqrt{\lambda_{\min}(Q)}\sqrt{\omega_{\min}}}\sqrt{\bar{V}_c(\eta^c)} \\
&\quad \text{from the fact that } \sum_{i=1}^{N}\omega_i\left(\bar{e}_i^T Q\bar{e}_i\right) = \bar{V}_c(\eta^c)) \text{ and Lemma 4-c)} \\
\sqrt{\bar{V}_c(\eta^c)} &\geq \lambda\underbrace{\frac{\sqrt{\lambda_{\min}(Q)}\sqrt{\omega_{\min}}}{\sqrt{N}}}_{\triangleq l_1}\sum_{i=1}^{N}\|e_i\| \qquad \text{(B.6)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{\sqrt{\lambda_{\min}(P)}}{\theta}\|\tilde{x}_{i,j}\| &\leq \sqrt{(\bar{x}_{i,j})^T P\bar{x}_{i,j}} \\
\sum_{j=0}^{N}\|\tilde{x}_{i,j}\| &\leq \frac{\theta}{\sqrt{\lambda_{\min}(P)}}\sum_{j=0}^{N}\sqrt{V_o(\bar{x}_{i,j})} \\
\sum_{i=1}^{N}\sum_{j=0}^{N}\|\tilde{x}_{i,j}\| &\leq \theta\frac{\sqrt{N}\sqrt{N+1}}{\sqrt{\lambda_{\min}(P)}}\sqrt{\bar{V}_o(\eta^o)} \quad \text{by using Lemma 4-c)} \\
\sqrt{\bar{V}_o(\eta^o)} &\geq \frac{1}{\theta}\underbrace{\frac{\sqrt{\lambda_{\min}(P)}}{\sqrt{N}\sqrt{N+1}}}_{\triangleq l_2}\sum_{i=1}^{N}\sum_{j=0}^{N}\|\tilde{x}_{i,j}\| \qquad \text{(B.7)}
\end{aligned}$$