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# RESIDUES ON AFFINE GRASSMANIANS

M. FLORENCE AND P. GILLE

ABSTRACT. Given a linear group  $G$  over a field  $k$ , we define a notion of index and residue of an element  $g \in G(k((t)))$ . The index  $r(g)$  is a rational number and the residue a group homomorphism  $\text{res}(g) : \mathbb{G}_a$  or  $\mathbb{G}_m \rightarrow G$ . This provides an alternative proof of Gabber's theorem stating that  $G$  has no subgroups isomorphic to  $\mathbb{G}_a$  or  $\mathbb{G}_m$  iff  $G(k[[t]]) = G(k((t)))$ . In the case of a reductive group, we offer an explicit connection with the theory of affine grassmannians.

Keywords: Group schemes, residues, affine grassmannians.

MSC 2000: 14M15, 20G35

## 1. INTRODUCTION

Let  $k$  be a field and let  $G$  be a linear algebraic  $k$ -group. Our goal is to associate in a quite elementary way to each element  $g \in G(k((t))) \setminus G(k[[t]])$  an index  $r(g) \in \mathbb{Q}_{\geq 0}$  and a non-trivial homomorphism  $\text{res}(g) : \mathbb{G}_a \rightarrow G$  (resp. a non-trivial homomorphism  $\text{res}(g) : \mathbb{G}_m \rightarrow G$ ) if  $r(g) > 0$  (resp.  $r(g) = 0$ ). If  $k$  is of characteristic zero (resp.  $p > 0$ ), we show that the index  $r(g)$  is integral (resp. belongs to  $\mathbb{Z}_{(p)}$ ), see Corollary 3.5.

This construction works actually over any ring  $A$  with a closed subgroup scheme of  $\text{SL}_N$ . This permits to recover results by Gabber [O] on the characterization of wound  $k$ -groups (i.e.  $G$  does not contain  $\mathbb{G}_a$  or  $\mathbb{G}_m$ ) and also provides an extension in the group scheme setting. This extends also to points  $x \in X(k((t))) \setminus X(k[[t]])$  for  $X$  a  $G$ -torsor; this is a key ingredient in the proof of the following result.

**Theorem 1.1.** *(see Theorem 5.5) Let  $X$  be a  $G$ -torsor such that  $X(k((t))) \neq \emptyset$ . Then  $X(k) \neq \emptyset$ .*

For reductive groups, this statement is due to Bruhat-Tits (see [Gi, I.1.2.1]). The generalization of that statement over a ring is known for  $\text{GL}_n$  and for tori according to recent results by Bouthier-Česnavičius [B-C, 2.1.17, 3.1.7]; we generalize it as well for wound closed subgroup schemes of  $\text{SL}_N$  (Cor. 4.2) and for  $G$  commutative under further assumptions (Th. 4.3). It is an open question beyond those cases.

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Already over a field it is an open question whether the statement does generalize to homogeneous spaces; this is the case in characteristic 0 according to results by the first author [F].

If  $G$  is split reductive, we relate our construction of index and residue to the affine grassmannians  $\mathcal{Q}_G$ . The index provides a refinement of the stratification of the affine grassmannian  $\mathcal{Q}_G$  of  $G$  (§9.1). In particular we show that an element  $g \in G(k((t))) \setminus G(k[[t]])$  has index 0 if and only if  $g = g_1\mu(t)g_2$  where  $g_1 \in G(k)$ ,  $g_2 \in G(k[[t]])$  for some homomorphism  $\mu : \mathbb{G}_m \rightarrow G$ , see Proposition 9.9.

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## 2. INDICES

**2.1. Notations and conventions.** If  $r \in \mathbb{Q}^\times$ , the notation  $r = m/n$  means that  $(m, n) = 1$  with  $n \geq 1$ . This extends to  $0 = 0/1$ .

For each ring  $A$  (commutative, unital), we denote by  $A^u = A[u]$  the ring of  $A$ -polynomials in the indeterminate  $u$ . We denote by  $A[[t]]$  the ring of power series and define  $A((t)) = A[[t]][x]/(1 - tx)$ . For each non-negative integer  $n \geq 1$ , we define  $A[[t^{1/n}]] = A[[t]][y]/(y^n - t)$  and  $A((t^{1/n})) = A[[t]][x, y]/(y^n - t, 1 - xy)$ . We have natural maps  $A[[t^{1/n}]] \rightarrow A[[t^{1/mn}]]$  and  $A((t^{1/n})) \rightarrow A((t^{1/mn}))$  for  $m \geq 1$ .

If  $r = m/n \in \mathbb{Q}_{\geq 0}$ , we put  $A_r = A^u[[t^{1/n}]]$  and  $\mathcal{A}_r = A^u((t^{1/n}))$ . We have a specialization homomorphism  $j : A_r \rightarrow A^u$ .

For each  $r = m/n \in \mathbb{Q}_{\geq 0}$ , the assignment  $t \rightarrow t(1 + ut^r)$  defines ring homomorphisms  $\sigma_r : A^u[[t]] \rightarrow A_r$  and  $A^u((t)) \rightarrow \mathcal{A}_r$ .

We come now to analogues  $A^{u,+} = A[u][z]/(1 - (1 + u)z)$  where we invert  $1 + u$ . We have the variants  $A_r^+ = A^{u,+}[[t^{1/n}]]$ ,  $\mathcal{A}_r^+ = A^{u,+}((t^{1/n}))$ ,  $\sigma_r$  and  $j^+ : A_r^+ \rightarrow A^{u,+}$ .

**2.2. The ramification index.** Let  $A$  be a ring and let  $G$  be an affine  $A$ -group scheme equipped with a closed embedding  $\rho : G \rightarrow \mathrm{SL}_{N,A}$ .

**Proposition 2.1.** *Let  $g \in G(A((t))) \setminus G(A[[t]])$ .*

(1) *The set*

$$\Sigma(g) = \left\{ r \in \mathbb{Q}_{>0} \mid g^{-1}\sigma_r(g) \in G(A_r) \right\}$$

*is non-empty and let  $r(g)$  be its lower bound in  $\mathbb{R}$ . Then  $r(g) \in \mathbb{Q}_{\geq 0}$  and  $\Sigma(g) = \mathbb{Q}_{>0} \cap [r(g), +\infty[$ .*

(2) *Assume that  $r(g) > 0$ . Then  $j(g^{-1}\sigma_r(g))$  belongs to  $G(A^u) \setminus G(A)$ .*

(3) *Assume that  $r(g) = 0$ . Then  $g^{-1}\sigma_0(g) \in G(A^{u,+}[[t]])$  and  $j(g^{-1}\sigma_0(g))$  belongs to  $G(A^{u,+}) \setminus G(A)$ .*

(4) *Assume that  $r(g) = m/n > 0$ . Let  $\sigma : A[[t]] \rightarrow A_{r(g)}$  be a homomorphism such that  $\sigma(t) = t(1 + ut^r + P_2(u)t^{m_2/n} + \dots)$  with  $m < m_2 < m_3 < \dots < \dots$ . Then  $g^{-1}\sigma(g)$  belongs to  $G(A_{r(g)})$  and  $j(g^{-1}\sigma(g)) = j(g^{-1}\sigma_r(g))$ .*

*Proof.* (1) Clearly the statement reduces to the case of  $\mathrm{SL}_N$ . Our assumption implies that  $g = t^{-d}\underline{g}$  with  $d \geq 1$  and  $\underline{g} \in \mathrm{M}_N(A[[t]]) \setminus t\mathrm{M}_N(A[[t]])$ . It follows that  $\det(\underline{g}) = t^{Nd}$ . We have

$$(2.1) \quad g^{-1}\sigma_r(g) = \frac{t^d}{t^d(1+ut^r)^d} \underline{g}^{-1}\sigma_r(\underline{g}) = (1+ut^r)^{-d} \underline{g}^{-1}\sigma_r(\underline{g}).$$

We write  $\underline{g} = (P_{i,j})_{i,j=1,\dots,N}$  with  $P_{i,j} \in A[[t]]$  and denote by  $\Delta_{i,j} \in A[[t]]$  the minor of index  $(i,j)$  of  $\underline{g}$ . We have  $\underline{g}^{-1} = \left(t^{-Nd} \Delta_{i,j}\right)_{i,j=1,\dots,N}$  so that the  $(i,j)$ -coefficient  $C_{i,j,r}$  of  $\underline{g}^{-1}\sigma_r(\underline{g})$  is

$$(2.2) \quad C_{i,j,r} = t^{-Nd} \sum_{k=1}^N \Delta_{i,k}(t) P_{k,j}(t(t+ut^r)) \in A_r.$$

When  $u = 0$ ,  $C_{i,j,r}$  specializes on  $\delta_{i,j}$  so that

$$(2.3) \quad C_{i,j,r} = \delta_{i,j} + t^{-Nd} \sum_{k=1}^N \Delta_{i,k}(t) (P_{k,j}(t(1+ut^r)) - P_{k,j}(t)).$$

We consider the identity

$$(2.4) \quad \sum_{k=1}^N \Delta_{i,k}(t) (P_{k,j}(t(1+\epsilon)) - P_{k,j}(t)) = \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^a \epsilon^b$$

with  $c_{i,j}^{a,b} \in A$ . Taking  $\epsilon = ut^r$ , we get

$$(2.5) \quad C_{i,j,r} = \delta_{i,j} + t^{-Nd} \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^{a+rb} u^b$$

We consider the sets  $\mathrm{supp}(i,j) = \{(a,b) \mid c_{i,j}^{a,b} \neq 0\}$  and  $\mathrm{supp}(g) = \bigcup_{(i,j)} \mathrm{supp}(i,j)$ .

**Claim 2.2.**  $\mathrm{supp}(g) \neq \emptyset$ .

If  $\mathrm{supp}(g) = \emptyset$ , then  $\underline{g} = \sigma_r(\underline{g})$  and all coefficients of  $\underline{g}$  belong to  $A$  which contradicts the fact  $\det(\underline{g}) = t^{Nd}$ . The Claim is established and enables us to define the function

$$f_g(r) = \mathrm{Inf} \left\{ -Nd + a + rb \mid (a,b) \in \mathrm{supp}(g) \right\}.$$

We have  $f_g(r) \geq r + f_g(0)$  so that  $f$  admits positive values and  $\Sigma(g)$  is not empty. Since  $1+ut^r \in A_r^\times$ , the set  $\Sigma(g)$  consists in the positive rational numbers  $r$  such that  $\underline{g}^{-1}\sigma_r(\underline{g})$  belongs to  $\mathrm{GL}_N(A_r)$ . We get that

$$\Sigma(g) = \{r \in \mathbb{Q}_{>0} \mid f_g(r) \geq 0\}.$$

By definition of  $r(g)$ , we have

$$r(g) = \text{Inf}\{r \in \mathbb{Q}_{>0} \mid f_g(r) \geq 0\} \in \mathbb{R}_{\geq 0}.$$

If  $f_g(0) \geq 0$ , then  $r(g) = 0$ . If  $f_g(0) < 0$ , then there exists  $a, b$  such that  $-Nd + a + r(g)b = 0$  whence  $r(g) \in \mathbb{Q}_{>0}$ . In both cases, we have  $\Sigma(g) = \mathbb{Q}_{>0} \cap [r(g), +\infty[$ .

(2) Along the proof of (1), we have seen that then there exists  $a, b$  such that  $-Nd + a + rb = 0$ . Formula (2.5) shows that  $j(g^{-1}\sigma_r(g)) \notin \text{SL}_N(A)$ .

(3) Once again, it is enough to consider the case of  $\text{SL}_{N,A}$ . If  $r(g) = 0$ , we have  $a - Nd \geq 0$  for each  $a$  occurring in formula (2.5). The point is that the computation of (1) works also for  $r = 0$ . It follows that

$$\underline{g}^{-1}\sigma_0(\underline{g}) - I_N \in uM_N(A_0) \subset M_N(A_0)$$

so that  $\underline{g}^{-1}\sigma_0(\underline{g}) \in M_N(A_0)$ . Taking into account the identity (2.1) for  $r = 0$ , we get

$$\underline{g}^{-1}\sigma_0(\underline{g}) = \lambda^{-d}\underline{g}^{-1}\sigma_0(\underline{g}) = \lambda^{-d}[I_N + uM_0 + tM_1 + \dots] \in M_N(A_0^+).$$

with  $M_0 \in M_N(A)$ . Since  $\underline{g}^{-1}\sigma_0(\underline{g})$  has determinant one, we conclude that  $\underline{g}^{-1}\sigma_0(\underline{g}) \in \text{SL}_N(A_r^+)$ . We have  $j(\underline{g}^{-1}\sigma_0(\underline{g})) = \lambda^{-d}(I_N + uM_0)$ . Assume that  $j(\underline{g}^{-1}\sigma_0(\underline{g})) \in \text{SL}_N(A)$ . Since  $d \geq 0$ , we get that  $d = 1$  and  $M_0 = I_N$ . The formula above reads

$$\underline{g}^{-1}\sigma_0(\underline{g}) = I_n + \lambda^{-1}[tM_1 + t^2M_2 + \dots] \in M_N(A_0^+).$$

with  $M_i \in M_N(A_0^+)$ . Since  $\underline{g}^{-1}\sigma_0(\underline{g}) \in \text{SL}_N(A[\lambda, \lambda^{-1}][[t]])$ , we can take  $\lambda = t$  so that  $\underline{g}^{-1}\sigma_0(\underline{g}) \in M_N(A[[t]])$ . It follows that  $\underline{g}(t^2) \in \underline{g}M_N(A[[t]]) \subseteq \frac{1}{t}M_N(A[[t]])$  which contradicts the fact that  $t^2\underline{g}(t^2) \in M_N(A[[t]]) \setminus tM_N(A[[t]])$ . We conclude that  $j(\underline{g}^{-1}\sigma_0(\underline{g})) \notin \text{SL}_N(A)$ .

(4) We write  $r = r(g) > 0$  for short and continue to work with  $\text{SL}_{N,A}$ . We denote by  $(\tilde{C}_{i,j})$  the entries of  $\underline{g}^{-1}\sigma(\underline{g})$ . Taking  $\epsilon = ut^r + P_2(u)t^{m_2/r} + \dots$ , we have

$$\begin{aligned} \tilde{C}_{i,j} &= \delta_{i,j} + t^{-Nd} \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^a (ut^r + P_2(u)t^{m_2/n} + \dots)^b \\ &= C_{i,j,r} + t^{-Nd} \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^a (u^b t^{br} + bu^{b-1} P_2(u) t^{\frac{(b-1)m+m_2}{n}} + \text{upper terms}). \end{aligned}$$

For each  $(a, b) \in \text{supp}(i, j)$ , we have  $-Nd + a + br \geq 0$ , so that  $-Nd + a + \frac{(b-1)m+m_2}{n} > 0$ . Since  $\underline{g}^{-1}\sigma_r(\underline{g})$  belongs to  $G(A_r)$ , it follows that  $\underline{g}^{-1}\sigma(\underline{g})$  belongs to  $G(A_r)$  and so do  $\underline{g}^{-1}\sigma(\underline{g})$ . Furthermore the above computation shows that  $\underline{g}^{-1}\sigma(\underline{g}) = \underline{g}^{-1}\sigma_r(\underline{g})$  in  $M_N(A_r)$  modulo  $t^{\frac{m_2-m}{n}}$ . Thus  $j(\underline{g}^{-1}\sigma(\underline{g})) = j(\underline{g}^{-1}\sigma_r(\underline{g}))$ .  $\square$

**Remark 2.3.** If  $A$  is a field, by inspection of the proof, we see that

$$r(g) = \text{Inf}\left\{r \in \mathbb{Q}_{>0} \mid \underline{g}^{-1}\sigma_r(\underline{g}) \in G(A(u)[[t]])\right\}.$$

For later use, we record the following consequence of the proof of Proposition 5.3.

**Lemma 2.4.** *The assumptions are those of Proposition 5.3 and we put  $r = r(g) = \frac{m}{n}$ . Let  $M$  be an integer satisfying  $M \geq m + 1$ .*

(1) *If  $r > 0$  we have*

$$g^{-1}\sigma_s(g) \in \ker\left(G(A^u[[t^{1/n}]]) \rightarrow G(A^u[t^{1/n}]/t^{\frac{M}{n}})\right)$$

*for all  $s = \frac{u}{n}$  with  $u \geq m + M$ .*

(2) *If  $r = 0$  we have*

$$g^{-1}\sigma_s(g) \in \ker\left(G(A^u[[t]]) \rightarrow G(A^u[t]/t^M)\right)$$

*for each integer  $s \geq M$ .*

*Proof.* (1) The  $(i, j)$ -coefficient of  $g^{-1}\sigma_s(g)$  reads

$$(2.6) \quad D_{i,j,s} = (1 + ut^s)^{-d} \left( \delta_{i,j} + t^{-Nd} \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^{a+sb} u^b \right).$$

We write  $s - r = \frac{v}{n}$  with  $v \geq M$  and get

$$(2.7) \quad D_{i,j,s} = (1 + ut^s)^{-d} \left( \delta_{i,j} + \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^{-Nd+a+rb} t^{\frac{vb}{n}} u^b \right).$$

For each non zero  $c^{a,b}$ , we have  $-Nd + a + rb \geq 0$  so that  $D_{i,j,s} - \delta_{i,j} \in t^{\frac{v}{n}} A^u[[t^{\frac{1}{n}}]] \subseteq t^{\frac{M}{n}} A^u[[t^{\frac{1}{n}}]]$ .

(2) For an integer  $s \geq M$ , the  $(i, j)$ -coefficient of  $g^{-1}\sigma_s(g)$  reads

$$(2.8) \quad D_{i,j,s} = (1 + ut^s)^{-d} \left( \delta_{i,j} + \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^{-Nd+a} t^{bs} u^b \right).$$

For each non zero  $c^{a,b}$ , we have  $-Nd + a \geq 0$  so that  $D_{i,j,s} - \delta_{i,j} \in t^s A^u[[t]] \subseteq t^M A^u[[t]]$ .  $\square$

**Definition 2.5.** *Let  $g \in G(A((t)))$ . If  $g \notin G(A[[t]])$ , we define the ramification index  $r(g)$  as in Proposition 2.1. If  $g \in G(A[[t]])$ , we define  $r(g) = -1$ .*

It is straightforward to check that the index does not depend of the choice of the representation  $\rho$ .

**Lemma 2.6.** *We have*

$$\left\{ g \in G(A((t))) \mid r(g) \leq 0 \right\} = \left\{ g \in G(A((t))) \mid g^{-1}g(\lambda t) \in G(A[\lambda, \lambda^{-1}][[t]]) \right\}.$$

*Proof.* Let  $g \in G(A((t)))$ . If  $g \in G(A[[t]])$ , it is obvious that  $g$  belongs to the right handside. If  $g \notin G(A[[t]])$  and  $r(g) \leq 0$ , we have  $r(g) = 0$  and  $g$  belongs to the right handside according to Proposition 2.1.(3).

Conversely we assume that  $g$  belongs to the right handside, that is  $g^{-1}g(\lambda t) \in G(A[\lambda, \lambda^{-1}][[t]])$ . We are given  $r = m/n > 0$ . Since  $1 + ut^r$  is invertible in  $A^u[[t^{1/n}]]$ , we can make  $\lambda = 1 + ut^r$  so that  $g^{-1}\sigma_r(g) \in G(A^u[[t]])$ . Since it holds for each rational  $r > 0$ , we get that  $r(g) = 0$  by definition of the index.  $\square$

**Lemma 2.7.** (1) *The function  $g \rightarrow r(g)$  is right  $G(A[[t]])$ -invariant (resp. left  $G(A)$ -invariant) and is insensible to any injective base change  $A \hookrightarrow A'$ .*

(2) *Let  $\phi : A \rightarrow B$  be a morphism of rings. Then  $r(g_B) \leq r(g)$ .*

(3) *Let  $f : G \rightarrow H$  be a homomorphism between affine  $A$ -group schemes of finite type.*

(i) *We have  $r(f(g)) \leq r(g)$ .*

(ii) *If  $A$  is integral and  $f$  is proper, we have  $r(f(g)) = r(g)$ .*

(4) *Let  $G_1, G_2$  be affine  $A$ -group schemes of finite type and consider the  $A$ -group scheme  $G_1 \times_A G_2$ . For  $g_i \in G_i(A((t)))$  we have  $r(g_1, g_2) \leq \inf(r(g_1), r(g_2))$ .*

(5) *Let  $d$  be a non-negative integer and consider the map  $\phi_d : A((t)) \rightarrow A((T))$  defined by  $\phi_d(t) = T^d$ . We consider the map  $\phi_{d,*} : G(A((t))) \rightarrow G(A((T)))$ .*

(i) *If  $d$  is not a zero divisor in  $A$ , we have  $r(\phi_{d,*}(g)) = dr(g)$ .*

(ii) *If  $A$  is of characteristic  $p > 0$  and  $d = p^e$ , we have  $r(\phi_d(g)) = r(g)$ .*

*Proof.* (1), (2) and (3).(i) readily follows of the definition of  $r(g)$ .

(3)(ii): We can replace  $A$  by its fraction field by (1), allowing us to assume that  $A$  is a field. Since  $f : G \rightarrow H$  is assumed to be proper, the valuative criterion of properness yields  $f^{-1}(H(A(u)[[t^r]]) \cap G(A(u)((t^r))) = G(A(u)[[t^r]])$  for each  $r \in \mathbb{Q}_{>0}$ . Then Remark 2.3 shows that  $r(f(g)) = r(g)$ .

(4) follows from the definition of the index.

(5) We are given  $r = m/n \in \mathbb{Q}_{>0}$ . In case (i), the change  $T \mapsto T(1 + uT^r)$  induces  $t = T^d \mapsto T^d(1 + uT^r)^d = t(1 + udt^{\frac{m}{nd}} + \dots)$ . According to Proposition 2.1.(4), we get that  $r(g) = \frac{r(\phi_{d,*}(g))}{d}$ .

In case (ii), we have  $t = T^{p^e} \mapsto T^{p^e}(1 + uT^r)^{p^e} = t(1 + u^{p^e}t^{m/n} + \dots)$ . We get then  $r(\phi_{d,*}(g)) = r(g)$ .  $\square$

**Corollary 2.8.** *Assume that  $A$  is integral and of characteristic  $p > 0$ . Let  $e$  be a non-negative integer and let  $F_e : G \rightarrow G^{(e)}$  be the  $e$ -iterated Frobenius morphism [D-G, II.7.1.4]. Then for each  $g \in G(A((t)))$ , we have  $r(F_e(g)) = r(g)$ .*

*Proof.* Since  $F_e$  is proper, this follows of Lemma 2.7.(3).(ii).  $\square$

### 3. THE RESIDUE

Let  $A, G$  and  $g \in G(A((t)))$  as in Proposition 2.1. If  $r(g) > 0$ , we define the residue  $\text{res}(g)$  as the image of  $g^{-1}\sigma_{r(g)}(g)$  by the homomorphism

$j_* : G(A_r) \rightarrow G(A^u) = G(A[u])$ . We see it as an  $A$ -map  $\text{res}(g) : \mathbb{G}_{a,A} = \text{Spec}(A[u]) \rightarrow \underline{G}$ .

If  $r(g) = 0$ , we define the residue  $\text{res}(g)$  as the image of  $g^{-1}\sigma_0(g)$  by the homomorphism  $j_* : G(A_0^+) \rightarrow G(A^{u,+}) = G(A[u, \frac{1}{1+u}])$ . Putting  $\lambda = 1 + u$ , we have  $A^{u,+} = A[\lambda, \lambda^{-1}]$  so that we see the residue as an  $A$ -map  $\text{res}(g) : \mathbb{G}_{m,A} = \text{Spec}(A[\lambda, \lambda^{-1}]) \rightarrow G$ .

If  $r(g) = -1$ , i.e.  $g \in G(A[[t]])$ , we put  $\text{res}(g) = 1 \in G(A^u)$ . Again this does not depend of the choice of a representation.

**Examples 3.1.** (1) If  $G = \mathbb{G}_{m,A}$  and  $g = \frac{1}{t^d}$ , we have

$$g^{-1}\sigma_0(g) = (1+u)^d = \lambda^d.$$

In this case we have  $r(g) = 0$  and  $\text{res}(g)(\lambda) = \lambda^d$ .

(2) If  $G = \mathbb{G}_{a,A}$  and  $g = \frac{1}{t^d}$  with  $d \in A^\times$ , we have

$$g^{-1}\sigma_r(g) = \frac{-1}{t^d} + \frac{1}{t^d(1+ut^r)^d} = \frac{-dut^r + \dots}{t^d}$$

In this case we have  $r(g) = d$  and  $\text{res}(g)(u) = -du$ .

(3) If  $G = \mathbb{G}_{a,A}$  and  $g = \frac{1}{t^p}$  with  $A$  of characteristic  $p > 0$  we have

$$g^{-1}\sigma_r(g) = \frac{-1}{t^p} + \frac{1}{t^p(1+ut^r)^p} = \frac{(-u)^p t^{rp} + \dots}{t^p}$$

In this case we have  $r(g) = 1$  and  $\text{res}(g)(u) = (-u)^p$ .

**Example 3.2.** We consider the case  $G = \text{GL}_2$ , and the element  $g = \begin{pmatrix} t^a & P(t) \\ 0 & t^d \end{pmatrix}$  with  $P(t) \in A[t]$  and  $a, d \in \mathbb{Z}$ . Putting  $\epsilon = ut^r$ , we have

$$g^{-1}\sigma_r(g) = t^{-a-d} \begin{pmatrix} t^d & -P(t) \\ 0 & t^a \end{pmatrix} \begin{pmatrix} t^a(1+\epsilon)^a & P(t(1+\epsilon)) \\ 0 & t^d(1+\epsilon)^d \end{pmatrix} = \begin{pmatrix} (1+\epsilon)^a & f \\ 0 & (1+\epsilon)^d \end{pmatrix}$$

with

$$f = t^{-a-d} \left( t^d P(t(1+\epsilon)) - t^d (1+\epsilon)^d P(t) \right) = t^{-a} \left( P(t(1+\epsilon)) - (1+\epsilon)^d P(t) \right).$$

(a) We take  $a, d \geq 1$ ,  $P(t) = 1$  and assume that  $d$  is invertible in  $A$ . In this case,  $P(t(1+\epsilon)) - (1+\epsilon)^d P(t) = 1 - (1+ut^r)^d$  so that

$$g^{-1}\sigma_r(g) = \begin{pmatrix} (1+ut^r)^a & -dt^{-a+r}u + \dots \\ 0 & (1+ut^r)^d \end{pmatrix}$$

It follows that  $r(g) = a$  and that  $\text{res}(g) = \begin{pmatrix} 1 & -du \\ 0 & 1 \end{pmatrix}$ .



(b) Assume that  $A$  is an  $\mathbb{F}_p$ -algebra and take  $d = p^s$  ( $s \geq 1$ ) and  $P(t) = t^{p^{ms}}$  with  $m \geq 2$ . Then

$$f = t^{-a} \left( t^{p^{ms}} (1 + \epsilon^{p^{ms}}) - (1 + \epsilon^{p^s}) t^{p^{ms}} \right) = t^{-a} \left( t^{p^{ms}} (ut^r)^{p^{ms}} - t^{p^{ms}} (ut^r)^{p^s} \right) = -t^{-a+p^{ms}+rp^s} (u)^{p^s} + \dots$$

If  $a > p^{ms}$ , we have  $-a + p^{ms} + rp^s = 0$  so that  $r = \frac{a}{p^s} + p^{m(s-1)}$ . In particular  $r$  can belong in  $\mathbb{Z}[\frac{1}{p}] \setminus \mathbb{Z}$ .

(c) For the multiplicative indeterminate  $\lambda$ , we compute also

$$g^{-1}g(\lambda t) = t^{-a-d} \begin{pmatrix} t^d & -P(t) \\ 0 & t^a \end{pmatrix} \begin{pmatrix} \lambda^a t^a & P(\lambda t) \\ 0 & \lambda^d t^d \end{pmatrix} = \begin{pmatrix} \lambda^a & f \\ 0 & \lambda^d \end{pmatrix}$$

with

$$f = t^{-a-d} \left( t^d P(\lambda t) - \lambda^d t^d P(t) \right) = t^{-a} \left( P(\lambda t) - \lambda^d P(t) \right).$$

If  $a \leq -1$ , we have  $r(g) = 0$  and  $\text{res}(g) = \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^d \end{pmatrix}$ . Furthermore for  $g' = g \text{res}(g)(t^{-1})$ , we have

$$g'^{-1}g'(\lambda t) = \begin{pmatrix} t^a & 0 \\ 0 & t^d \end{pmatrix} \begin{pmatrix} \lambda^a & f \\ 0 & \lambda^d \end{pmatrix} \begin{pmatrix} \lambda^{-a} t^{-a} & 0 \\ 0 & \lambda^{-d} t^{-d} \end{pmatrix} = \begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix}$$

with  $f_1 = t^{a-d} \lambda^{-d} f = \lambda^{-d} t^{-d} \left( P(\lambda t) - \lambda^d P(t) \right)$ . For  $a = -1$ ,  $d = 1$  and  $P(t) = 1$ , we see that  $g'^{-1}g'(\lambda t)$  does not belong in  $\text{GL}_2(A[\lambda, \lambda^{-1}][[t]])$ .

Similarly for  $g'' = \text{res}(g)(t^{-1})g$ , we have

$$g''^{-1}g''(\lambda t) = t^{-a-d} \begin{pmatrix} t^d & -P(t) \\ 0 & t^a \end{pmatrix} \begin{pmatrix} \lambda^{-a} & 0 \\ 0 & \lambda^{-d} \end{pmatrix} \begin{pmatrix} \lambda^a t^a & P(\lambda t) \\ 0 & \lambda^d t^d \end{pmatrix} = \begin{pmatrix} 1 & f_2 \\ 0 & 1 \end{pmatrix}$$

with  $f_2 = t^{-a} \left[ \lambda^{-a} P(\lambda t) - P(t) \right]$ . So for  $a \leq -1$ , we see that  $g''^{-1}g''(\lambda t)$  belongs to  $\text{GL}_2(A[[t]])$ .

**Lemma 3.3.** *Let  $g \in G(A((t)))$ .*

(1) *Let  $g_1 \in G(A)$  and  $g_2 \in G(A[[t]])$ . Then  $\text{res}(g_1 g g_2) = \text{res}(g)^{\overline{g_2}}$  where  $\overline{g_2}$  stands for the specialization of  $g_2$  in  $G(A)$ . In particular we have  $\text{res}({}^{g_1}g) = {}^{g_1}\text{res}(g)$ .*

(2) *Let  $A \rightarrow A'$  be a base change such that  $r(g) = r(g_{A'((t))})$  (it holds for example when  $A$  injects in  $A'$ ). Then  $\text{res}(g_{A'}) = \text{res}(g)_{A'}$ .*

(3) *Assume that  $A$  is integral and let  $f : G \rightarrow H$  be a proper homomorphism between affine  $A$ -group schemes of finite type. We have  $\text{res}(f(g)) = \text{res}(g)$ .*

(4) *Let  $d$  be a non-negative integer and consider the map  $\phi_d : A((t)) \rightarrow A((T))$  defined by  $\phi_d(t) = T^d$ . We consider the map  $\phi_{d,*} : G(A((t))) \rightarrow G(A((T)))$ .*

(i) *If  $d$  is not a zero divisor in  $A$ , we have  $\text{res}(\phi_{d,*}(g))(u) = \text{res}(g)(du)$  if  $r > 0$ , or  $\text{res}(\phi_{d,*}(g))(\lambda) = \text{res}(g)(\lambda^d)$  if  $r = 0$ .*

(ii) If  $A$  is of characteristic  $p > 0$  and  $d = p^e$ , we have  $\text{res}(\phi_d(g)) = \text{res}(g)(u^{p^e})$ .

*Proof.* We write  $r = r(g) = m/n$ .

(1) Since  $\sigma_r(g_1) = g_1$ , we have  $(g_1 g g_2)^{-1} \sigma_r(g_1 g g_2) = g_2^{-1} (g^{-1} \sigma_r(g)) \sigma_r(g_2) = g_2^{-1} (g^{-1} \sigma_r(g)) \sigma_r(g_2)$ . When we specialize at  $t = 0$ , we get  $\text{res}(g_1 g g_2) = \text{res}(g)^{g_2}$ . Assertions (2) and (3) follow of Lemma 2.7.

(4) We continue the proof of Lemma 2.7.(5). We have four cases to verify.

*Case (i),  $r > 0$ .* We have  $r(\phi_{d,*}(g)) = dr(g) = \frac{dm}{n}$ . The change  $T \mapsto T(1 + uT^r)$  induces  $t = T^d \mapsto T^d(1 + uT^r)^d = \tau(t) = t(1 + d u t^{\frac{m}{nd}} + \dots)$ . It follows that

$$\phi_{d,*}(g^{-1} \tau(g)) = \phi_{d,*}(g)^{-1} \sigma_{T, \frac{m}{nd}}(\phi_{d,*}(g)) \in G(A^u[[T]])$$

Proposition 2.1.(4) yields that  $j(g^{-1} \tau(g)) = j(g^{-1} \sigma_{t,r}(g)) = d \text{res}(g) \in G(A^u)$ .

*Case (i),  $r = 0$ .* The change  $T \mapsto \lambda T$  induces  $t = T^d \mapsto \lambda^d t$ . It follows that

$$\phi_{d,*}(g^{-1} g(\lambda t)) = \phi_{d,*}(g)^{-1} g(\lambda^d T) \text{res}(g)(\lambda^d) (1 + \epsilon) \in G(A[\lambda^{\pm 1}[[T]])$$

with  $j(1 + \epsilon) = 1$ . We conclude that  $\text{res}(\phi_{d,*}(g))(\lambda) = \text{res}(g)(\lambda^d)$ .

*Case (ii),  $r > 0$ .* We have  $r(\phi_{d,*}(g)) = r(g)$  and consider the base change  $t = T^{p^e} \mapsto T^{p^e} (1 + uT^r)^{p^e} = \tau'(t) = t(1 + u^{p^e} t^{m/n} + \dots)$ . It follows that

$$(3.1) \quad \phi_{d,*}(g^{-1} \tau'(g)) = \phi_{d,*}(g)^{-1} \sigma_{T,r}(\phi_{d,*}(g)) \in G(A^u[[T]]).$$

We put  $v = u^{p^e}$  and consider  $\sigma_r^v : A^v[[t]] \rightarrow A^v[[t^{1/n}]]$ . By using Proposition 2.1.(4) and the functoriality of the construction  $A^v \rightarrow A^u$ , we have  $j(g^{-1} \tau'(g)) = j(g^{-1} \sigma_r^v(g)) = \text{res}(g)(v) = \text{res}(g)(u^{p^e}) \in G(A^u)$ . By specializing formula (3.1) at  $T = 0$ , we get  $\text{res}(\phi_{d,*}(g))(u) = \text{res}(g)(u^{p^e})$ .

*Case (ii),  $r = 0$ .* It is similar. □

**Theorem 3.4.** (1) If  $r(g) > 0$ , then  $\text{res}(g)$  is non-trivial homomorphism  $\mathbb{G}_{a,A} \rightarrow G$ .  
(2) If  $r(g) = 0$ , then  $\text{res}(g)$  is a non-trivial homomorphism  $\mathbb{G}_{m,A} \rightarrow G$ .

We provide an alternative proof for (1) and (2) in Remark 8.3.1.

*Proof.* We can continue to work with  $\text{SL}_N$ . We write  $r = r(g) = m/n$ .

(1) We assume firstly that  $n$  is invertible in  $A$ . By developing the serie  $(1 + ut^r)^{1/n}$  in  $A_r$ , we can extend  $\sigma_r : A^u[[t]] \rightarrow A_r$  to  $\tilde{\sigma}_r : A_r \rightarrow A_r$ . The trick is to use the rings  $A^{v_1, v_2} = A[v_1, v_2]$ ,  $A^{v_1, v_2}[[t]]$  and  $A^{v_1, v_2}[[t^{1/n}]]$  and to define morphisms  $\tau_i : A^{v_1, v_2}[[t]] \rightarrow A^{v_1, v_2}[[t^{1/n}]]$  ( $i = 1, 2, 3$ ) by  $t \mapsto t(1 + v_1 t^r)$ ,  $t(1 + v_2 t^r)$ ,  $t(1 + (v_1 + v_2) t^r)$  respectively. These morphisms extend to morphisms  $\tilde{\tau}_i : A^{v_1, v_2}[[t^{1/n}]] \rightarrow A^{v_1, v_2}[[t^{1/n}]]$  for  $i = 1, 2, 3$ . We have the cocycle relation

$$(3.2) \quad g^{-1}(\tilde{\tau}_1 \tilde{\tau}_2)(g) = g^{-1} \tilde{\tau}_1(g) \tilde{\tau}_1(g^{-1} \tilde{\tau}_2(g))$$

inside  $G(A^{v_1, v_2}((t^{1/n})))$ . By using functoriality properties (Lemmas 2.7.(1) and 3.3.(2)) we have  $g^{-1} \tilde{\tau}_i(g) \in G(A^{v_1, v_2}[[t^{1/n}]])$  with specialization  $\text{res}(g)(v_i)$ . It follows that

$$(3.3) \quad g^{-1}(\tilde{\tau}_1 \tilde{\tau}_2)(g) = \text{res}(g)(v_1) \text{res}(g)(v_2)$$

inside  $G(A^{v_1, v_2}[[t^{1/n}]])$  modulo the kernel of  $G(A^{v_1, v_2}[[t^{1/n}]]) \rightarrow G(A^{v_1, v_2})$ . On the other hand  $\tilde{\tau}_1 \tilde{\tau}_2(t) = t(1 + (v_1 + v_2)t^r + \text{upper terms})$ . Proposition 2.1.(3) applied to the ring  $A[v_1]$  and  $u = v_1 + v_2$  shows that  $g^{-1}(\tilde{\tau}_1 \tilde{\tau}_2)(g) = \text{res}(g)(v_1 + v_2)$  inside  $G(A^{v_1, v_2}[[t^{1/n}]])$  modulo the kernel of  $G(A^{v_1, v_2}[[t^{1/n}]]) \rightarrow G(A^{v_1, v_2})$ . We conclude that  $\text{res}(g)(v_1 + v_2) = \text{res}(g)(v_1) \times \text{res}(g)(v_2)$ . Thus  $\text{res}(g)$  is an  $A$ -group homomorphism.

We explain now the refinement to the case  $n = qn'$  when  $A$  is of characteristic  $p > 0$  and  $(n', p) = 1$  and  $q = p^e$ . We consider  $A^u[[t]] \xrightarrow{\sigma_r} A^u[[t^{1/n}]] \rightarrow A[u^{1/q}][[t^{1/nq}]]$  to  $\tilde{\sigma}_r : A^u[[t^{1/n}]] \rightarrow A[u^{1/q}][[t^{1/nq}]]$  by mapping  $t^{1/n}$  to the series  $(1 + u^{1/q}t^{r/q})^{1/n'}$ .

We extend then similarly the morphisms  $A^{v_1^q, v_2^q}[[t]] \xrightarrow{\tau_i} A^{v_1^q, v_2^q}[[t^{1/n}]] \rightarrow A^{v_1, v_2}[[t^{1/nq}]]$  ( $i = 1, 2, 3$ ) defined by  $t \mapsto t(1 + v_1^q t^r)$ ,  $t(1 + v_2^q t^r)$ ,  $t(1 + (v_1^q + v_2^q)t^r)$  in  $\tilde{\tau}_i : A^{v_1^q, v_2^q}[[t^{1/n}]] \rightarrow A^{v_1, v_2}[[t^{1/nq}]]$  for  $i = 1, 2, 3$ . The cocycle condition reads then

$$g^{-1}(\tilde{\tau}_1 \tilde{\tau}_2)(g) = g^{-1} \tau_1(g) \tilde{\tau}_1(g^{-1} \tau_2(g)).$$

The same method yields  $\text{res}(g)(v_1^q + v_2^q) = \text{res}(g)(v_1^q) \times \text{res}(g)(v_2^q)$ . Thus  $\text{res}(g)$  is an  $A$ -group morphism.

(2) We have seen that  $\text{res}(g) \in G(A[\lambda^{\pm 1}]) \setminus G(A)$  in Lemma 3.3.(4). We consider the ring  $A[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$  and the automorphisms  $\rho_i$  of  $A[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}][[t]]$  defined by  $\rho_i(t) = \lambda_1 t$ ,  $\lambda_2 t$ , and  $\lambda_1 \lambda_2 t$ . Since  $\rho_3 = \rho_2 \circ \rho_1$ , the cocycle relation  $g^{-1}(\rho_1 \rho_2)(g) = g^{-1} \rho_1(g) \rho_1(g^{-1} \rho_2(g))$  yields  $\text{res}(g)(\lambda_1 \lambda_2) = \text{res}(g)(\lambda_1) \text{res}(g)(\lambda_2)$  as desired.  $\square$

This provides some control on the indices in the integral case.

**Corollary 3.5.** *Assume that  $A$  is integral. Let  $d \geq 1$  be an integer such that  $t^d g \in M_N(A[[t]])$ .*

(1) *If  $A$  contains  $\mathbb{Z}$ , then  $r(g) \in \mathbb{Z}$  and  $r(g) \leq Nd$ .*

(2) *If  $A$  is a  $\mathbb{F}_p$ -algebra for a prime  $p$ , then  $r(g) \in \mathbb{Z}[\frac{1}{p}]$  and there exists  $s \geq 0$  such that  $p^s r(g) \in \mathbb{Z}$  and  $p^s r(g) \leq Nd$ .*

*Proof.* (1) and (2). If  $r(g) = 0$  the statements are clear so that we can assume that  $r(g) > 0$ . We are allowed to replace  $A$  by its fraction field  $F$  according to Lemma 3.3.(2). We use now the decomposition  $\text{SL}_N(F((t))) = B_N(F((t))) \text{SL}_N(F[[t]])$  where  $B_N$  stands for the  $F$ -subgroup of upper triangular matrices [Br-T, 4.4.3]. Lemma 3.3.(1) permits to assume that  $g \in B_N(F((t)))$ . Coming back in the proof of Proposition 2.1, we consider the coefficients of  $g^{-1} \sigma_r(g)$

$$(3.4) \quad D_{i,j,r} = (1 + ut^r)^{-d} \left( \delta_{i,j} + t^{-Nd} \sum_{a \geq 0, b \geq 1} c_{i,j}^{a,b} t^{a+rb} u^b \right).$$

We have  $D_{i,j,r} = 0$  if  $j < i$ . We consider the non-empty set

$$\Upsilon(g) = \left\{ (i, j, a, b) \mid -Nd + a + rb = 0 \text{ and } c_{i,j}^{a,b} \neq 0 \right\}.$$

It follows that the  $(i, j)$ -entry of  $\text{res}(g) \in B_N(k[u])$  is

$$(3.5) \quad c_{i,j} = \delta_{i,j} + \sum_{(i,j,a,b) \in \Upsilon(g)} c_{i,j}^{a,b} u^b$$

It follows that  $c_{i,i} = 1$  for each  $i = 1, \dots, N$ , that is  $\text{res}(g) \in U_N(k[u])$  where  $U_N$  stands for the unipotent radical of  $B_N$ . Let  $(i, j, a, b) \in \Upsilon(g)$  such that  $i+j$  is minimal. Since  $\text{res}(g)$  is a group homomorphism, it follows that  $u \mapsto c_{i,j}^{a,b} u^b$  is a group homomorphism.

*Case of characteristic zero.* In this case we have  $b = 1$ . The equation  $-Nd + a + rb = 0$  yields that  $r \in \mathbb{Z}$  and that  $r = Nd - a \leq Nd$ .

*Case of characteristic  $p$ .* It follows that  $b$  is a  $p$ -power, i.e.  $b = p^s$ . Thus  $r \in \mathbb{Z}[\frac{1}{p}]$  and  $p^s r = Nd - a \leq Nd$ .  $\square$

**Remark 3.6.** In the case of an  $A$ -ring  $B$ , we prove later than  $r(g) \leq Nd$  for  $g \in G(B((t)))$  (see Lemma 9.1.(1)).

**3.1. Left index, left residue.** Given  $g \in G(A((t)))$ , we define the left index and left residue by  $r^L(g) = r(g^{-1})$  and  $\text{res}^L(g) = \text{res}(g^{-1})$ . If  $G$  is commutative, we have  $r^L(g) = r(g)$  and  $\text{res}^L(g) = -\text{res}(g)$ . This breaks in the non-commutative case.

**Example 3.7.** In  $\text{GL}_2(k((t)))$ , we take  $g = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$ . We have seen that  $r(g) = 1$ .

We have  $g^{-1} = \begin{pmatrix} t^{-1} & -t^{-2} \\ 0 & t^{-1} \end{pmatrix}$  and

$$g \sigma_r(g^{-1}) = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix} \begin{pmatrix} t^{-1}(1 - ut^r)^{-1} & -t^{-2}(1 + ut^r)^{-2} \\ 0 & t^{-1}(1 - ut^r)^{-1} \end{pmatrix} = \begin{pmatrix} (1 - ut^r)^{-1} & 0 \\ 0 & (1 + ut^r)^{-1} \end{pmatrix}$$

so that  $r(g^{-1}) = 0$ .

**3.2. Wound  $A$ -groups.**

**Corollary 3.8.** *The following conditions are equivalent:*

- (i)  $G(A[[t]]) = G(A((t)))$ ;
- (ii)  $G(A) = G(A[\lambda, \lambda^{-1}])$ ;
- (iii)  $\text{Hom}_{A-gr}(\mathbb{G}_a, G) = 0$  and  $\text{Hom}_{A-gr}(\mathbb{G}_m, G) = 1$ .

*Proof.* (i)  $\implies$  (ii). We have  $G(A[[t]]) = G(A((t)))$  and similarly  $G(A[[t^{-1}]]) = G(A((t^{-1})))$ . It follows that  $G(\mathbb{P}_A^1) = G(A[\lambda^{\pm 1}])$ . Since  $G$  is affine of finite type over  $A$ , we conclude that  $G(A) = G(A[\lambda^{\pm 1}])$ .

(ii)  $\implies$  (iii). Let  $h : \mathbb{G}_a \rightarrow G$  be an  $A$ -homomorphism. Our assumption provides an element  $g \in G(A)$  such that  $h(u) = g$ . Since  $h(0) = 1$ , we conclude that  $h$  is trivial. Similarly we have  $\mathrm{Hom}_{A\text{-gr}}(\mathbb{G}_m, G) = 1$ .

(iii)  $\implies$  (i). This follows of Theorem 3.4.  $\square$

**Remark 3.9.** If  $A = F$  is a field, Corollary 3.8.(1) was known in the reductive case as the Bruhat-Tits-Rousseau's theorem [P] and the general case is a consequence of Gabber's compactifications, this will be discussed in §6.

**Definition 3.10.** *If the affine  $A$ -group scheme  $G$  of finite type satisfies the conditions of Corollary 3.8, we say that  $G$  is  $A$ -wound ( $A$ -ployé in French).*

Our definition extends over any ring the case of wound algebraic groups over a field [C, 5.1].

## 4. RESIDUES FOR TORSORS

### 4.1. Definition.

**Proposition 4.1.** *Let  $X$  be a  $G$ -torsor over  $A$ . Let  $x \in X(A((t))) \setminus X(A[[t]])$ . For each  $r \in \mathbb{Q}_{\geq 0}$ , we denote by  $g_r(x)$  the unique element of  $G(\mathcal{A}_r)$  such that  $\sigma_r(x) = x \cdot g_r(x)$ .*

(1) *The set*

$$\Sigma(x) = \left\{ r \in \mathbb{Q}_{>0} \mid g_x(r) \in G(A_r) \right\}$$

*is non-empty and let  $r(x)$  be its lower bound in  $\mathbb{R}$ . Then  $r(x) \in \mathbb{Q}_{\geq 0}$  and  $\Sigma(x) = \mathbb{Q}_{>0} \cap [r(x), +\infty[$ .*

(2) *If  $r(x) > 0$ , then  $j(g_{r(x)}(x)) \in G(A^u)$  is a non-trivial homomorphism  $\mathrm{res}(x) : \mathbb{G}_a \rightarrow G$ .*

(3) *If  $r(x) = 0$ , then  $j(g_{r(x)}(x)) \in G(A^{u,+})$  is a non-trivial homomorphism  $\mathrm{res}(x) : \mathbb{G}_m \rightarrow G$ .*

*Proof.* As in the proof of Proposition 2.1, we can work with  $\mathrm{SL}_{N,F}$  where  $F$  is the fraction field of  $A$ . In this case, the torsor is trivial so that the result follows of Proposition 2.1 and of Theorem 3.4.  $\square$

Of course this extension of the residue to torsors satisfy the same functorialities as the residue for group schemes.

#### 4.2. The wound case.

**Corollary 4.2.** *We assume that  $G$  is  $A$ -wound. Let  $X$  be a  $G$ -torsor over  $A$ . Then  $X$  is trivial if and only if  $X \times_A A((t))$  is trivial.*

*Proof.* The direct sense is obvious. We assume that  $X \times_A A((t))$  is trivial, that is  $X(A((t))) \neq \emptyset$ . If  $X(A[[t]]) \neq \emptyset$ , then  $X(A) \neq \emptyset$  and  $X$  is a trivial  $G$ -torsor. We can assume then that  $X(A[[t]]) = \emptyset$ . We pick  $x \in X(A((t)))$  and Proposition 4.1 provides an element  $\text{res}(x)$  which is a non-trivial morphism  $\mathbb{G}_a \rightarrow G$  or a non-trivial morphism  $\mathbb{G}_m \rightarrow G$ . This contradicts our assumption.  $\square$

#### 4.3. The commutative case.

**Theorem 4.3.** *We assume that  $A$  is an integral  $\mathbb{Q}$ -algebra, and that the  $A$ -group scheme  $G$  is commutative. Let  $X$  be a  $G$ -torsor over  $A$ . Then  $X$  is trivial if and only if  $X \times_A A((t))$  is trivial.*

*Proof.* We assume that the  $G$ -torsor  $X \times_A A((t))$  is trivial, that is  $X(A((t))) \neq \emptyset$ . If  $X(A) = \emptyset$ , then the indices of points of  $X(A((t)))$  are all non-negative integers. Let  $r$  be the minimal value of those indices and consider a point  $x \in X(A((t)))$  such that  $r(x) = r$ .

*Additive case:*  $r \geq 1$ . We consider the point  $y = x \cdot \text{res}(x)(-\frac{1}{rt^r})$ . According to Example 3.1.(2), we have

$$(4.1) \quad -\frac{1}{t^r} + \sigma_r\left(\frac{1}{t^r}\right) = ru + r\epsilon$$

with  $\epsilon \in tA^u[[t]]$ . Since  $G$  is commutative, we have

$$\begin{aligned} g_r(y) &= g_r(x) \left(\text{res}(x)\left(-\frac{1}{rt^r}\right)\right)^{-1} \sigma_r\left(\text{res}(x)\left(-\frac{1}{rt^r}\right)\right) \\ &= g_r(x) \text{res}(x)(-ut + \epsilon) \end{aligned}$$

by reporting the above identity (4.1). We have  $g_r(y) \in G(A^u[[t]])$  so that  $r(y) \leq r$ . The reduction mod  $t$  of  $g_r(y)$  is trivial by construction so that Proposition 4.1.(2) shows that  $r(y) < r$ , which contradicts the minimality of  $r$ .

*Multiplicative case:*  $r = 0$ . Similarly we use the homomorphism  $\text{res}(x) : \mathbb{G}_{m,A} \rightarrow G$  for constructing the point  $y = x \cdot \text{res}(x)(t^{-1})^{-1}$ . Since  $G$  is commutative, we have

$$\begin{aligned} g_0(y) &= g_0(x) \text{res}(x)(t^{-1}) \sigma_0\left(\text{res}(x)(t^{-1})\right)^{-1} \\ &= g_0(x) \text{res}(x)(\lambda)^{-1}. \end{aligned}$$

We have  $g_0(y) \in G(A^+[[t]])$  so that  $r(y) \leq 0$ . The reduction mod  $t$  of  $g_r(y)$  is  $\text{res}(x)(\lambda) \text{res}(x)(\lambda^{-1}) = 1$  so that Proposition 4.1.(2) shows that  $r(y) < 0$ . Thus  $y \in X(A[[t]])$  and  $X(A)$  is not empty.  $\square$

## 5. THE CASE OF FIELDS, II

A  $k$ -variety is a separated  $k$ -scheme of finite type. If  $X$  is a  $k$ -variety, we denote by  $X^\dagger$  its largest geometrically reduced  $k$ -subscheme [C-G-P, C.4.1].

A  $k$ -compactification of a  $k$ -variety  $V$  is an open immersion  $j : V \rightarrow V^c$  where  $V^c$  is a proper  $F$ -variety.

We use intensively the notions of the book [C-G-P], for example the different radicals of a smooth algebraic  $k$ -group  $H$ .

$\mathcal{R}_{u,k}(H)$ :  $k$ -unipotent radical, i.e. the largest smooth connected unipotent normal  $F$ -subgroup of  $H$ .

$\mathcal{R}_k(H)$ :  $k$ -radical, i.e. the largest smooth connected solvable normal  $k$ -subgroup of  $H$ .

$\mathcal{R}_{us,k}(H)$ : split unipotent  $k$ -radical, i.e. the largest smooth connected  $k$ -split unipotent normal  $k$ -subgroup of  $H$ .

$\mathcal{R}_{s,k}(H)$ : split  $k$ -radical i.e. the largest smooth connected  $k$ -split solvable normal  $k$ -subgroup of  $H$ .

**5.1. Pseudo-complete varieties.** We say that a  $k$ -variety is *pseudo-complete* if  $X(A) = X(K)$  for each discretely valued  $k$ -ring  $A$  with fraction fields  $K$  and whose residue field is separable over  $k$  [C-G-P, app. C.1.]. To check this property, it is enough to consider the case of a complete discretely valued  $k$ -ring whose residue field is separably closed and separable over  $k$  (*ibid*, C.1.2.). In particular,  $X$  is pseudo-complete if and only if  $X_{k_s}$  is pseudo-complete.

We say that  $X$  is  $k$ -pseudo-complete if  $X(A) = X(F)$  for each discretely valued  $F$ -ring  $A$  of fraction field  $F$  and of residue field  $k$ . Similarly, it is enough to consider the case of  $O = k[[t]]$  whose fraction field is denoted by  $K = k((t))$ .

There is related a notion of strong pseudo-completeness characterized by the following result.

**Theorem 5.1.** (*Moret-Bailly, unpublished*) *Let  $X$  be a  $k$ -variety. The following assertions are equivalent:*

- (i) *The  $k$ -variety  $X$  is strongly pseudo-complete, that is for each valuation  $k$ -ring  $A$  whose residue field is separable over  $k$ , we have  $X(A) = X(\text{Frac}(A))$ .*
- (ii)  *$X$  admits a  $k$ -compactification  $X^c$  such that  $X(k_s) = X^c(k_s)$ .*

The next lemma was used in a preliminary version, we kept it since it can be useful.

**Lemma 5.2.** *Let  $H$  be a smooth  $k$ -algebraic group. We assume that  $H$  is pseudo-complete. Let  $f : Y \rightarrow X$  be an  $H$ -torsor where  $Y, X$  are  $k$ -varieties. If  $X$  is  $k$ -pseudo-complete (resp. pseudo-complete), then  $Y$  is  $k$ -pseudo-complete (resp. pseudo-complete).*

*Proof.* We are given  $y \in Y(K)$  where  $K = k((t))$ . There exists a point  $x \in X(O)$  such that  $\pi(y) = x_K$ . Let  $x_0$  be the image of  $x$  by the map  $X(O) \rightarrow X(F)$  and

consider the fiber  $Z = f^{-1}(x)$ . We observe that  $Z$  is  $H$ -torsor which is smooth since  $H$  is smooth. It follows that  $Z_{k_s} \cong H_{k_s}$  so that  $Z$  is pseudo-complete. In the other hand, the fiber  $\mathfrak{Z} = f^{-1}(x)$  is a  $H_O$ -torsor. The map  $H^1(O, H) \rightarrow H^1(k, H)$  is bijective [SGA3, XXIV.8.1]; it follows that the  $H_O$ -torsor  $\mathfrak{Z}$  is isomorphic to  $Z \times_k O$ . We have  $Z(O) = Z(K)$  whence  $\mathfrak{Z}(O) = \mathfrak{Z}(K)$ . But  $y \in \mathfrak{Z}(K)$ , so we conclude that  $y \in Y(O)$ .  $\square$

## 5.2. Affine $k$ -groups.

**Proposition 5.3.** *Let  $G$  be an affine algebraic  $k$ -group. Let  $P$  be a minimal pseudo-parabolic  $k$ -subgroup of  $(G^\dagger)^0$ . We put  $R = \mathcal{R}_{s,k}(G)$ .*

- (1) *The quotient  $Q = P/R$  is  $k$ -wound.*
- (2) *For each  $G$ -torsor  $E$  over  $k$  trivialized over  $k_s$ ,  $E/R$  is  $k$ -pseudo-complete and  $E(k) \neq \emptyset$  if and only if  $(E/R)(K) \neq \emptyset$ .*
- (3) *Let  $E$  be a  $G$ -torsor. If  $E(K) \neq \emptyset$ , then  $E(k) \neq \emptyset$ .*

**Remarks 5.4.** (a) In the perfect field case, statement (1) follows of a compactification result due to Borel-Tits [B-T, th. 8.2].

(b) Statement (3) was known for groups of multiplicative type [CTS2, th. 4.1].

*Proof.* (1) We have  $\mathcal{R}_{s,k}(P) \subseteq \mathcal{R}_k(P)$  and  $\mathcal{R}_k(P)/\mathcal{R}_{s,k}(P)$  is a smooth connected solvable  $k$ -group [C, th. 5.4] which fits in the exact sequence  $1 \rightarrow \mathcal{R}_k(P)/\mathcal{R}_{s,k}(P) \rightarrow P/\mathcal{R}_{s,k}(P) \rightarrow P/\mathcal{R}_k(P) \rightarrow 1$ . If  $P/\mathcal{R}_{s,k}(P)$  admits a split central  $k$ -torus  $S$ , then its pre-image in  $P$  is normal and solvable  $k$ -split. By maximality of  $\mathcal{R}_{s,k}(P)$ , it follows that  $S = 1$ .

If  $P/\mathcal{R}_{s,k}(P)$  admits a  $k$ -subgroup  $U$  isomorphic to  $\mathbb{G}_a$ ,  $U$  cannot be a  $k$ -subgroup of the wound  $k$ -group  $\mathcal{R}_k(P)/\mathcal{R}_{s,k}(P)$ . It follows that the morphism  $U \rightarrow P/\mathcal{R}_k(P)$  is non trivial. Since any non-trivial quotient of  $\mathbb{G}_a$  is isomorphic to  $\mathbb{G}_a$  [D-G, IV.2.1.1], it follows that  $P/\mathcal{R}_k(P)$  admits a  $k$ -subgroup  $V$  isomorphic to  $\mathbb{G}_a$ . We use now that the quotient  $P/\mathcal{R}_k(P)$  is pseudo-reductive. This implies that  $V \subseteq \mathcal{R}_{us,k}(Q')$  where  $Q'$  is a pseudo-parabolic  $k$ -subgroup of  $P/\mathcal{R}_k(P)$  [C-G-P, C.3.8]. The pre-image  $P'$  of  $P$  in  $Q'$  is a (proper) pseudo-parabolic  $k$ -subgroup of  $P$  (*ibid*, 2.2.10), and in the same time a pseudo-parabolic  $k$ -subgroup of  $(G^\dagger)^0$  (*ibid*, 3.5.5). This leads to a contradiction with the minimality of  $P$ . Thus  $Q = P/\mathcal{R}_{s,k}(P)$  is  $k$ -wound.

(2) *The smooth case.* Let  $E$  be a  $G$ -torsor over  $k$  and consider the quotient  $X = E/R$ . We want to show that  $X(O) = X(K)$ . We consider the morphism  $f : X \rightarrow Y = E/P$  which is a  $Q = P/R$ -torsor. Since  $E(k_s) \neq \emptyset$ , we have  $(E/P)_{k_s} \cong (G/P)_{k_s}$ . We recall that  $(G/P)_{k_s}$  is pseudo-complete [C-G-P, C.1.6], so that  $(E/P)_{k_s}$  is pseudo-complete. It follows that  $E/P$  is pseudo-complete; in particular we have  $Y(O) = Y(K)$ .

We are given  $x \in X(K)$ . Since  $Y(O) = Y(K)$ , there exists a unique point  $y \in Y(O)$  such that  $f(x) = y$ . We denote by  $y_0$  its image in  $Y(k)$ . The fiber  $f : X \rightarrow Y = E/P$  at  $y_0$  defines a  $Q$ -torsor  $L = f^{-1}(x_0)$ . On the other hand, we consider the  $P_O$ -torsor  $\mathfrak{L} = f^{-1}(y)$ ; we have  $\mathfrak{L} \times_O F = L$ . Since  $P$  is smooth, the  $Q_O$ -torsor  $\mathfrak{L}$  is isomorphic



to the  $Q_O$ -torsor  $L \times_k O$ . By construction, we have  $L(K) \cong \mathfrak{L}(K) \neq \emptyset$ . Corollary 3.8.(2).(b) shows that  $L(k) \neq \emptyset$ . Since  $k$ -group  $Q$  is wound, we have  $\mathfrak{L}(O) = \mathfrak{L}(K)$ , and this enables us to conclude that  $x \in X(O)$ .

*General case.* Since  $E(k_s) \neq \emptyset$ ,  $E^\dagger$  is a  $G^\dagger$ -torsor which satisfies  $E^\dagger \wedge^{G^\dagger} G = E$ . We consider the morphism  $G/R \rightarrow G/G^\dagger$  and its variant  $q : E/R \rightarrow E/G^\dagger$ . We have  $(E/G^\dagger)(k) = (E/G^\dagger)(O) = (E/G^\dagger)(K) = \{\bullet\}$ . The pre-image of this point is the  $k$ -scheme  $E^\sharp/R$  (actually  $X^\sharp$ ). It follows that  $(E^\sharp/R)(K) = (E/R)(K)$ . Appealing to the smooth case, we get  $(E^\sharp/R)(O) = (E^\sharp/R)(K)$ . Thus  $(E/R)(O) = (E/R)(K)$ .

For the second property, we assume that  $X(k) \neq \emptyset$ . The morphism  $E \rightarrow E/R = X$  is a  $R$ -torsor with  $R$  solvable and  $k$ -split. Since  $H^1(k, R) = 1$  (by dévissage from the cases of  $\mathbb{G}_m$  and  $\mathbb{G}_a$ ), we conclude that  $E(k) \neq \emptyset$ .

(3) Let  $E$  be a  $G$ -torsor such that  $E(K) \neq \emptyset$ . Since  $K$  is separable over  $F$ , we have  $E^\sharp(K) = E(K)$  whence  $E^\sharp \neq \emptyset$ . But  $E^\sharp$  is geometrically reduced and is then generically smooth [St, 32.25.7]. It follows that  $E^\sharp(k_s) \neq \emptyset$  (*ibid*, 32.25.6). A fortiori we have  $E(k_s) \neq \emptyset$  and the statement becomes a straightforward consequence of (2).  $\square$

**5.3. Torsors over fields.** Let  $G$  be an  $k$ -algebraic group. We continue with the notations  $K = k((t))$  and  $O = k[[t]]$ . Our purpose is to discuss the kernels of mappings  $a_G : H^1(O, G) \rightarrow H^1(K, G)$ ,  $b_G : H^1(k, G) \rightarrow H^1(K, G)$ ,  $c_G : H^1(O, G) \rightarrow H^1(k, G)$ .

If  $G$  is smooth, the specialisation  $c_G : H^1(O, G) \rightarrow H^1(k, G)$  is bijective according to Hensel's lemma [SGA3, XXIV.8.1].

**Theorem 5.5.** *The map  $b_G : H^1(k, G) \rightarrow H^1(K, G)$  is injective.*

**Lemma 5.6.** *Let  $1 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 1$  be an exact sequence of  $k$ -algebraic groups such that  $G_3(O) = G_3(K)$ .*

(1) *We assume that  $G_3$  is smooth. If  $\ker(b_{G_1}) \subseteq \ker(c_{G_1})$  and  $\ker(b_{G_3}) \subseteq \ker(c_{G_3})$ , then  $\ker(b_{G_2}) \subseteq \ker(c_{G_2})$ .*

(2) *If  $\ker(b_{G_1}) = 1$  and  $\ker(b_{G_3}) = 1$ , then  $\ker(b_{G_2}) = 1$ .*

*Proof.* (1) It is a classical “dévissage”, see for example the proof of [CTO, Th. 2.1]. The above sequence gives rise to the exact commutative diagram of pointed sets

$$\begin{array}{ccccccc}
G_3(k) & \xrightarrow{\varphi_k} & H^1(k, G_1) & \xrightarrow{\alpha_k} & H^1(k, G_2) & \xrightarrow{\beta_k} & H^1(k, G_3) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
G_3(O) & \xrightarrow{\varphi_O} & H^1(O, G_1) & \xrightarrow{\alpha_O} & H^1(O, G_2) & \xrightarrow{\beta_O} & H^1(O, G_3) \\
\downarrow \cong & & a_{G_1} \downarrow & & a_{G_2} \downarrow & & a_{G_3} \downarrow \\
G_3(K) & \xrightarrow{\varphi_K} & H^1(K, G_1) & \xrightarrow{\alpha_K} & H^1(K, G_2) & \xrightarrow{\beta_K} & H^1(K, G_3).
\end{array}$$

We are given an element  $\gamma_2 \in \ker(H^1(O, G_2) \rightarrow H^1(K, G_2))$ . Since  $a_{G_3}$  has trivial kernel (our assumption), there exists a class  $\gamma_1 \in H^1(O, G_1)$  which maps on  $\gamma_2$  and such that  $\gamma_{1,K}$  belongs to the image of the characteristic map  $\varphi_K$ .

We use now the right action of  $G_3(O)$  on  $H^1(O, G_1)$  and the fibers of  $\alpha_O$  are the orbits for that action [Gd, III.3.3.3]. The same fact holds for  $G_3(K)$  and  $\alpha_K$ . It follows that there exists  $g \in G_3(K)$  such that  $\gamma_{1,K} \cdot g_3 = 1 \in H^1(K, G_1)$ . On the other hand, we have  $g_3 \in G_3(O)$  hence  $\gamma_{1,O} \cdot g_3 \in \ker(H^1(O, G_1) \rightarrow H^1(K, G_1))$ . We can assume that  $\gamma_1 \in \ker(a_{G_1})$ . One of our assumption is that  $\ker(a_{G_1}) \subseteq \ker(c_{G_1})$ . Thus  $c_{G_2}(\gamma_2) = c_{G_2}(\alpha(\gamma_1)) = \alpha_k(c_{G_1}(\gamma_1)) = 1 \in H^1(k, G_2)$ .

The second fact is of the same vein.  $\square$

We can proceed to the proof of Theorem 5.5.

*Proof.* In the affine case, this is Proposition 5.3.(3). We assume  $G$  connected. According to Chevalley [SGA3, VI<sub>B</sub>.12.5.(5)], there is an exact sequence  $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$  where  $G/H$  is an abelian variety and  $H$  is affine. By application of the valuative criterion of properness to  $A$ -torsors, we have that  $\ker(b_A) = 1$ . On the other hand, we have  $\ker(b_H) = 1$ , so that Lemma 5.6.(2) shows that  $\ker(b_G) = 1$ . To reach the non-connected case is of the same vein by using the exact sequence  $1 \rightarrow G \rightarrow G \rightarrow G/G^0 \rightarrow 1$ .  $\square$

## 6. LINK WITH GABBER'S COMPACTIFICATIONS

**6.1. The statements.** The following two results have been announced by Gabber [O]. Let  $k$  be a field. If  $G$  is an algebraic  $k$ -group, we denote by  $L'G$  its largest smooth affine connected  $k$ -subgroup, by  $L_{\bar{k}}G$  the largest largest smooth affine connected subgroup of  $G_{\bar{k}}$  and by  $\bar{L}G$  the smallest  $k$ -subgroup such that  $\bar{L}G \supset L_{\bar{k}}G$ .

We say that  $G$  satisfies the property  $(*)$  iff all tori of  $G_{\bar{k}}$  are in  $(G^\dagger)_{\bar{k}}$ .

**Theorem 6.1.** *Let  $G$  be a  $k$ -group of finite type and  $P$  a pseudo-parabolic subgroup of  $L'G$ . Then  $G/P$  has an equivariant projective compactification, compatible with  $(G^\dagger)^0$ ,  $L'G$ ,  $\bar{L}G$ , with a  $G$ -linearized line bundle relatively ample for  $G/P \rightarrow G/\bar{L}G$  such that the boundary has no separable point and if  $(*)$  holds there is no  $k_s$ -orbit contained set-theoretically in the boundary.*

**Theorem 6.2.** *Let  $G$  be a  $k$ -group of finite type. Then  $G$  admits a projective compactification  $G \hookrightarrow G^c$  with a left action of  $G$ , right action of  $G^\dagger$ , an invariant ample effective divisor with support  $G^c \setminus G$  if  $G$  is affine, such that*

- (1) *For every separable extension  $K$  of  $k$  the following are equivalent:*
  - (i)  $G_K$  has a subgroup isomorphic to  $\mathbb{G}_a$  or  $\mathbb{G}_m$ ;
  - (ii)  $\overline{G}(K) \neq G^c(K)$ ;
  - (iii)  $\overline{L'G}(K) \neq L'G(K)$ ;
  - (iv) *There is an  $K$ -orbit for the left action of  $G$  on  $G^c$  admitting a separable point and contained in  $G^c \setminus G$ .*

(2) For every separably closed separable extension  $K$  of  $k$ ,  $\overline{G}(K) = G^\dagger(K) = G(K)\overline{(L'G)}(K)$ , and if  $(*)$  holds every  $K$ -orbit of the action of  $G$  on  $G^c$  has a  $K$ -point.

**6.2. Alternative proof of Corollary 3.8.(1) in the field case.** Theorem 6.2 provides an alternative proof of that statement. The only relevant implication is (iii)  $\implies$  (i). We are given an affine algebraic  $k$ -group  $G$  such that  $\mathrm{Hom}_k(\mathbb{G}_m, G) = 1$  and  $\mathrm{Hom}_k(\mathbb{G}_a, G) = 0$ . This implies that  $G$  has no  $k$ -subgroup isomorphic to  $\mathbb{G}_a$  or  $\mathbb{G}_m$ . Let  $G^c$  be a projective compactification of  $G$  provided by Theorem 6.2. Then  $G(k) = G^c(k)$  so that  $G$  is a  $k$ -pseudo-complete variety according to Theorem 5.1.(1). Thus  $G(k[[t]]) = G(k((t)))$  as desired.

Similarly one can prove Corollary 3.8.2.b.

### 6.3. Refinement of Theorem 3.4.

**Theorem 6.3.** *Let  $G$  be an affine algebraic  $k$ -group which satisfies the property  $(*)$ . Then  $\ker(a_G) = 1$ .*

*Proof.* We consider the exact sequence of fppf  $k$ -sheaves

$$1 \rightarrow G^\dagger \rightarrow G \rightarrow G/G^\dagger \rightarrow 1.$$

We put  $Q = G/G^\dagger$ ,  $\mathfrak{G} = G \times_k O$ ,  $\mathfrak{Q} = Q \times_k O$  and  $\mathfrak{Q}^c = Q \times_k O$ . According to Gabber [GGMB, th. 5.2],  $Q$  admits a  $G$ -equivariant  $Q^c$  such that the boundary  $\partial Q = Q^c \setminus Q$  has no  $G$ -orbit over any separable field extension of  $k$ . Furthermore  $Q^c$  carries a  $G$ -linearized line bundle.

Let  $\mathfrak{X}$  be a  $\mathfrak{G}$ -torsor such that  $\mathfrak{X}_K$  is trivial, that is  $\mathfrak{X}(K) \neq \emptyset$ . We consider the contracted products  $\mathfrak{Z} = \mathfrak{X} \wedge^G Q$  and  $\mathfrak{Z}^c = \mathfrak{G} \wedge^G Q^c$  (which are representable according to Lemma 10.1) over  $O$ . We put  $X = \mathfrak{X}_k$ ,  $Z = \mathfrak{Z}_k$  and  $Z^c = \mathfrak{Z}_k^c$ .

**Claim 6.4.**  $Z(k) = Z^c(k)$ .

The argument is similar with that of the proof of [GGMB, Lemme 6.1]. We consider the  $k$ -group scheme  $G' = \mathrm{Aut}_G(X)$ , it acts (on the left) on the morphism  $X \rightarrow Z$ . Let  $z \in Z^c(k)$ . According to [GGMB, 2.3.2], the  $k$ -orbit  $T$  of  $Z$  under  $G'$  corresponds canonically to a  $k$ -orbit  $T_0$  of  $G$  on  $Q^c$ . By assumption, we have  $T_0 \subset Q$ , so that  $T \subset Z$ . In particular  $z \in Z(k)$  and the Claim is proven.

We have compatible isomorphisms  $\mathfrak{Z}_K^c \xrightarrow{\sim} Q^c \times_k K$  and  $\mathfrak{Z}_K \xrightarrow{\sim} Q \times_k K$ . Since  $\{\bullet\} = Q(K) = Q^c(K)$ , we have that  $\{z_0\} = \mathfrak{Z}(K) = \mathfrak{Z}^c(K)$ . The Claim implies that the specialization of  $z_0$  in  $Z^c(k)$  belongs to  $Z(k)$ . It follows that  $z_0 \in Z(O)$  so that  $\{z_0\} = \mathfrak{Z}(O)$ . Hence the  $\mathfrak{G}$ -torsor  $\mathfrak{X}$  admits a reduction  $\mathfrak{F}$  to the subgroup  $G^\dagger \times_k O$ . The exact commutative diagram

$$\begin{array}{ccccc} \{\bullet\} = Q(O) & \longrightarrow & H^1(O, G^\dagger) & \longrightarrow & H^1(O, G) \\ \downarrow & & \downarrow a_{G^\dagger} & & \downarrow a_G \\ \{\bullet\} = Q(K) & \longrightarrow & H^1(K, G^\dagger) & \longrightarrow & H^1(K, G) \end{array}$$

enables us to conclude that  $\mathfrak{X}$  is a trivial  $\mathfrak{G}$ -torsor. □

## 7. APPLICATIONS AND EXAMPLES

**7.1. Applications.** We shall apply the main result to nice torsors.

**Corollary 7.1.** *Let  $G$  be an algebraic  $k$ -group.*

(1) *We assume that  $G$  acts (on the left) on a  $k$ -variety  $X$ . Let  $x, x' \in X(k)$ . Then  $x$  and  $x'$  are  $G(k)$ -conjugated if and only if  $x_K$  and  $x'_K$  are  $G(K)$ -conjugated.*

(2) *Let  $H, H'$  be  $k$ -subgroups of  $G$ . Then  $H$  and  $H'$  are  $G(k)$ -conjugated if and only if  $H_K$  and  $H'_K$  are  $G(K)$ -conjugated.*

*Proof.* (1) We assume that  $x, x'$  are  $G(K)$ -conjugated. We consider the transporter  $E = \{g \in G \mid g.x = x'\}$ . Since  $E(K) \neq \emptyset$ ,  $E$  is non-empty and is a torsor under the stabilizer  $G_x = \{g \in G \mid g.x = x\}$ . Theorem 3.4 yields that  $E(k) \neq \emptyset$ . Thus  $x$  and  $x'$  are  $G(k)$ -conjugated.

(2) We assume that  $H$  and  $H'$  are  $G(K)$ -conjugated. We denote by  $N = N_G(H)$  the normalizer of  $H$  in  $G$ . We consider the strict transporter  $T$  of  $H$  to  $H'$  as defined in [D-G, VI<sub>B</sub>.6.2.4]. Since  $T(K)$  is non-empty,  $T$  is a  $N$ -torsor. Theorem 3.4 yields that  $T(k) \neq \emptyset$ . Thus  $H$  and  $H'$  are  $G(k)$ -conjugated. □

Another useful statement is the following.

**Corollary 7.2.** *Let  $X$  be a  $k$ -variety (resp. an algebraic  $k$ -group, etc..) whose automorphism group is representable by an algebraic  $k$ -group. Let  $X'$  be a  $k$ -form of  $X$ . Then  $X$  and  $X'$  are  $k$ -isomorphic if and only if  $X_K$  and  $X'_K$  are  $K$ -isomorphic.*

*Proof.* We apply Theorem 3.4 to the  $\text{Aut}(X)$ -torsor  $\text{Isom}(X, X')$ . □

**7.2. Examples of  $k$ -groups such that  $\ker(a_G) \neq 1$ .** We assume that  $k$  is imperfect of characteristic  $p > 0$  and we pick an element  $a \in k \setminus k^p$ .

(a) We denote by  $G_0 = \text{Aut}(\alpha_p)$  the  $k$ -group (affine, algebraic) of automorphisms of the finite  $k$ -scheme  $\alpha_p$ . The pointed set  $H^1(k, G_0)$  classifies the  $k$ -forms of the  $k$ -algebra  $A_0 = k[x]/x^p$  [D-G, III.5.1.10]. The  $k$ -algebra  $A = k[x]/(x^p - a)$  is a  $k$ -form of  $A_0$  and we denote by  $G$  its automorphism group;  $G$  is a  $k$ -form of  $G_0$ .

We consider now the  $O$ -algebra  $B = O[x]/(x^p - at^p)$ , which is an  $O$ -form of  $A \times_k O$ . This gives rise to a class  $[B] \in H^1(O, G)$ . Since  $A \otimes_k K \xrightarrow{\sim} B \otimes_O K$ ,  $[B]$  has trivial image in  $H^1(K, G)$ . On the other hand,  $a$  is not a  $p$ -power in  $B$  so that  $A \otimes_k O$  and  $B$  are not  $O$ -isomorphic. Thus  $[B] \neq 1 \in H^1(O, G)$  and  $\ker(a_G) \neq 1$ .

(b) We shall construct an example of dimension 1 which arises from [GGMB, §7.1]. We consider the action of the  $k$ -group  $G := \mathbb{G}_a \rtimes \mathbb{G}_m$  (semi-direct product for the standard action of  $\mathbb{G}_m$  on  $\mathbb{G}_a$ ) on the affine line  $\mathbf{A}_k^1$  defined by

$$(x, y).z = x^p + y^p z \quad (x \in \mathbb{G}_a, y \in \mathbb{G}_m, z \in \mathbf{A}^1).$$

We observe that  $\mathbf{A}_k^1$  is a homogeneous  $K$ -space (on the left) under  $G$ ; also the stabilizer of 0 is the closed  $k$ -subgroup  $\alpha_p \rtimes_k \mathbb{G}_m$ . For an element  $t_0 \in O$ , we denote by  $\mathfrak{G}_{t_0}$  the stabilizer of  $t_0$ , i.e.

$$\mathfrak{G}_{t_0} = \left\{ (x, y) \in G \times_k O \mid x^p + y^p t_0 = t_0 \right\}.$$

We are given  $t_1 \in O$ , the strict transporter from  $t_0$  to  $t_1$  is the  $(\mathfrak{G}_{t_0}, \mathfrak{G}_{t_1})$ -bitorsor

$$\mathfrak{E}_{t_0, t_1} = \left\{ (x, y) \in G \times_k O \mid x^p + y^p t_0 = t_1 \right\}.$$

We consider the special cases  $t_0 = a$  and  $t_1 = at^p$  which is taylor made for having  $\mathfrak{E}_{t_0, t_1}(K) \neq \emptyset$ . Since  $a \notin k^p$ , one has  $\mathfrak{E}_{t_0, t_1}(k) = \emptyset$  and a fortiori  $\mathfrak{E}_{t_0, t_1}(O) = \emptyset$ .

The point is that the  $O$ -scheme  $\mathfrak{G}_a$  arises from the  $k$ -scheme  $H = \left\{ (x, y) \in G \mid x^p + y^p a = a \right\}$ . We conclude that the map  $a_H : H^1(O, H) \rightarrow H^1(K, H)$  has a non trivial kernel. Furthermore we have  $\ker(a_H) \not\subset \ker(c_H)$ .

**Remark 7.3.** Theorem 6.3 shows that the  $k$ -group  $G$  of Example (a) (resp.  $H$  of Example (b)) does not satisfy property (\*). This fact can be checked directly.

(a) The  $k$ -group  $G_0$  contains  $\mathbb{G}_m$  so that  $G_{\bar{k}}$  contains  $\mathbb{G}_{m, \bar{k}}$ . On the other hand, we have  $G(k_s) = \text{Aut}(k_s[x]/(x^p - a) = 1$  so that  $G^\dagger = 1$ .

(b) We have  $H(k_s) = 1$  so that  $H^\dagger = 1$ . On the other hand, we have  $H_{\bar{k}} \cong (\alpha_p \rtimes \mathbb{G}_m)_{\bar{k}}$  which contains  $\mathbb{G}_{m, \bar{k}}$ .

## 8. A MORE ADVANCED VIEWPOINT

In this section, our goal is to give an abstract exposition of the group structure on automorphisms of  $A[[t]]$  used in the second section.

### 8.1. Automorphisms of Laurent series and pro-group schemes. .

For each  $w \geq 0$ , we consider the affine  $\mathbb{Z}$ -group scheme  $J_w$  of automorphisms of the sequence of rings

$$\mathbb{Z}[t]/t^{w+1} \rightarrow \mathbb{Z}[t]/t^w \rightarrow \cdots \rightarrow \mathbb{Z}$$

For each ring  $Z$ ,  $J_w(R)$  consists of elements  $f_i \in \text{Aut}_{R\text{-ring}}(R[t]/t^i)$  for  $i = 0, \dots, w+1$  such that  $f_i = f_{i+1} \bmod t$  for  $i = 0, \dots, w$ . We have transition maps  $\pi^{w, w-1} : J_w \rightarrow J_{w-1}$  for each  $w \geq 1$ . The projective limit in the sense of [EGAIV, §8] is denoted by  $J = \varprojlim J_w$ . This is an affine  $\mathbb{Z}$ -group scheme whose coordinate ring is  $\mathbb{Z}[J] = \varinjlim \mathbb{Z}[J_w]$ . We have projections  $j_w : J \rightarrow J_w$  and we put  $J^w = \ker(j_w)$ . Since  $J_0$  is trivial, we have  $J^0 = J$ . The following statement is straightforward.

**Lemma 8.1.** *Let  $w \geq 1$  and let  $A$  be a ring.*

- (1) *An element of  $J_w(A)$  is given by  $t \mapsto a_1 t + a_2 t^2 + \cdots + a_w t^w$  with  $a_1 \in A^\times$  and  $a_2, \dots, a_n \in A$ .*
- (2) *We have an exact sequence  $1 \rightarrow \mathbb{G}_a \rightarrow J_{w+1} \rightarrow J_w \rightarrow 1$ .*

In particular, we have  $J_1 = \mathbb{G}_m$  and the map  $j^1 : J \rightarrow J_1 = \mathbb{G}_m$  is split by mapping a scalar  $\lambda \in R^\times$  to  $f_i = \times \lambda$ . It follows that  $J = J^1 \rtimes \mathbb{G}_m$  where  $J^1$  is a pro-unipotent  $\mathbb{Z}$ -group scheme.

For a ring  $R$ , we have  $R[[t]] = \varprojlim R[t]/t^{w+1}$  so that  $J(R) = \varprojlim J_w(R)$  acts on  $R[[t]]$ . We name  $J(R)$  the group of continuous automorphisms of  $R[[t]]$ . An important thing is that  $J(R)$  acts also on  $A[[t]]$  for each  $R$ -algebra  $A$ .

**8.2. Torsors and cocycles for Hochschild cohomology.** We are given a ring  $A$ , an  $A$ -group scheme  $G$  equipped with a closed immersion  $G \hookrightarrow \mathrm{SL}_N$ , a  $G$ -torsor  $X$  and a point  $x \in X(A((t)))$ . Of course an important case is  $G$  itself. For each  $A$ -ring  $B$  and each  $\sigma \in J(B)$  we write  $\sigma(x_B) = x_B \cdot z_\sigma(x)$  with  $z_\sigma(x) \in G(B((t)))$ .

For each  $w \geq 0$  we denote by  $G_w = \prod_{A[t]/t^{w+1}/A} G_{A[t]/t^{w+1}}$  the Weil restriction of  $G$  with respect to the finite  $A$ -algebra  $A[t]/t^{w(x)+1}$ .

We denote by  $L^+G$  the  $A$ -functor in groups  $B \mapsto G(B[[t]])$ ; there is a natural map  $L^+G \rightarrow G_w$  for each  $w \geq 0$ . Similarly we denote by  $LG$  the  $A$ -functor in groups  $B \mapsto G(B((t)))$ .

**Lemma 8.2.** *There exists a smallest integer  $w(x)$  such that the restriction of  $z(x)$  to  $J^w$  factorizes through  $L^+G$ .*

*Proof.* We have to prove that there exists  $w \in \mathbb{N}$  such that  $z_\sigma(x) \in G(B[[t]])$  for all  $R$ -algebras  $B$  and all  $\sigma \in J(B)$ . Without loss of generality, we can replace  $A$  by a faithfully flat extension so we can assume that  $X$  is a trivial  $G$ -torsor and  $x = g \in G(A((t)))$ . We can replace then  $G$  by  $\mathrm{SL}_N$  and use the setting of the proof of Proposition 2.1. We write  $g = t^{-d} \underline{g}$  with  $d \geq 0$  and  $\underline{g} \in \mathrm{M}_N(A[[t]]) \setminus t\mathrm{M}_N(A[[t]])$ . It follows that  $\det(\underline{g}) = t^{Nd}$ . For an  $A$ -algebra  $B$  and  $\sigma \in J(B)$ , we have

$$(8.1) \quad g_{B[[t]]}^{-1} \sigma(g_{B[[t]])} = \left( \frac{\sigma(t)}{t} \right)^d \underline{g}_{B[[t]]}^{-1} \sigma(\underline{g}_{B[[t]])}.$$

We write  $\underline{g} = (P_{i,j})_{i,j=1,\dots,N}$  with  $P_{i,j} \in A[[t]]$  and denote by  $\Delta_{i,j} \in A[[t]]$  the minor of index  $(i,j)$  of  $\underline{g}$ . We have  $\underline{g}^{-1} = \left( t^{-Nd} \Delta_{i,j} \right)_{i,j=1,\dots,N}$  so that the  $(i,j)$ -coefficient  $D_{i,j,\sigma}$  of  $g_{B[[t]]}^{-1} \sigma(g_{B[[t]])}$  is

$$(8.2) \quad D_{i,j,\sigma} = \left( \frac{\sigma(t)}{t} \right)^d t^{-Nd} \sum_{k=1}^N \Delta_{i,k}(t) P_{k,j}(\sigma(t)) \in B((t)).$$

When  $\sigma = 1$ ,  $C_{i,j,\sigma}$  specializes on  $\delta_{i,j}$  so that

$$(8.3) \quad D_{i,j,\sigma} = \delta_{i,j} + \left( \frac{\sigma(t)}{t} \right)^d t^{-Nd} \sum_{k=1}^N \Delta_{i,k}(t) (P_{k,j}(\sigma(t)) - P_{k,j}(t)).$$

It follows that there exists a uniform integer  $w \geq 0$  such that  $g_{B[[t]]}^{-1} \sigma(g_{B[[t]])}$  belongs to  $G(B[[t]])$  for  $\sigma \in J^w(B)$ .  $\square$

Our construction defines then a 1-cocycle  $z(x) : J^{w(x)} \rightarrow L^+G$  for the Hochschild cohomology as defined by Demarche [D, §2.1] for the  $A$ -functor in groups  $J^{w(x)}$  and  $LG$ . This induces an 1-cocycle for the Hochschild cohomology

$$\text{Res}(x) : J^{w(x)} \rightarrow G_{w(x)}.$$

Since  $J^{w(x)}$  acts trivially on  $G_{w(x)}$ , the map  $\text{Res}(x)$  is actually a homomorphism of  $A$ -groups. This defines classes  $\gamma(x) = [z(x)] \in H_{\text{coc}}^1(J^{w(x)}, LG)$  and  $[\text{Res}(x)] \in H_{\text{coc}}^1(J^{w(x)}, G_{w(x)})$ .

**Lemma 8.3.** *Let  $A'$  be a flat cover of  $A$ . Then  $w(x_{A'((t))}) = w(x)$  and  $\text{Res}(x_{A'((t))}) = \text{Res}(x)$ .*

*Proof.* We have the obvious relation  $w' := w(x_{A'((t))}) \leq w(x) =: w$  (which holds in general). We observe that  $w = 0$  implies that  $w' = 0$  as well. We can assume  $w \geq 1$ . By definition of  $w$ , there exists a ring extension  $B$  of  $A$  and  $\sigma \in J^{w-1}(B)$  such that  $z_\sigma(x) \in G(B((t))) \setminus G(B[[t]])$ . We consider  $B' = A' \otimes_A B$ , this is a flat cover of  $B$  so that  $B$  injects in  $B'$ . It follows that  $B[[t]] = B'[[t]] \cap B((t))$ . Since  $G$  is affine, we get that  $(z_\sigma(x))_{B'}$  belongs to  $G(B'((t))) \setminus G(B'[[t]])$ . Thus  $w' > w - 1$  and  $w' = w$ .  $\square$

**8.3. Comparison with the elementary construction.** For each integer  $r \geq 1$ , we consider the map of  $\mathbb{Z}$ -functors<sup>1</sup>  $\phi^r : \mathbb{G}_a \rightarrow J$ , which associates to the coordinate  $u$  the element of  $J(\mathbb{Z}[u])$  defined by  $t \mapsto t(1 + ut^r) = t + ut^{r+1}$ . We observe that  $\phi$  factorizes through  $J^{r+1}$ .

**Lemma 8.4.** *The composite  $\mathbb{G}_a \xrightarrow{\phi^r} J^r \rightarrow J^r/J^{r+1}$  is a  $\mathbb{Z}$ -group isomorphism.*

*Proof.* We take two parameters  $u_1, u_2$  and see that  $\phi^r(u_1) \circ \phi^r(u_2) \in J(\mathbb{Z}[u_1, u_2])$  maps  $t$  to

$$(t + u_2 t^{r+1}) + u_1 (t + u_2 t^{r+1})^{r+1} = t + (u_1 + u_2) t^r + (r+1) u_1 u_2 t^{r+2} + \dots$$

$\square$

The following statement is then obvious.

**Lemma 8.5.** *For each integer  $w \geq 1$  and each ring  $B$ , we have*

$$J^w(B) = \langle \phi^r(B) \rangle_{r \geq w}.$$

We consider firstly the case when the index of  $x$  is integral.

**Lemma 8.6.** *We assume that  $r(x) \in \mathbb{N}$ .*

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<sup>1</sup>It is not a group functor.

(1) If  $r(x) \geq 1$ , then  $w(x) = r(x) + 1$  and the following diagrams

$$\begin{array}{ccc} J^{w(x)} & \xrightarrow{\text{Res}(x)} & G_{w(x)} \\ \uparrow \phi^{r(x)} & & \downarrow \\ \mathbb{G}_a & \xrightarrow{\text{res}(x)} & G \end{array} \quad \begin{array}{ccc} J^{w(x)} & \xrightarrow{\text{Res}(x)} & G_{w(x)} \\ \downarrow & & \downarrow \\ J^{w(x)}/J^{w(x)+1} = \mathbb{G}_a & \xrightarrow{\text{res}(x)} & G \end{array}$$

commute. Furthermore  $\text{Res}(x)$  is trivial on  $J^{2w(x)+1}$ .

(2) If  $r(x) = 0$ , then  $w(x) = 0$  or  $1$  and the following diagrams

$$\begin{array}{ccc} J & \xrightarrow{\text{Res}(x)} & G \\ \uparrow & \nearrow \text{res}(x) & \\ \mathbb{G}_m & & \end{array} \quad \begin{array}{ccc} J & \xrightarrow{\text{Res}(x)} & G \\ \downarrow & \nearrow \text{res}(x) & \\ \mathbb{G}_m & & \end{array}$$

commute.

*Proof.* Once again we can localize for the flat topology and assume that the  $G$ -torsor is trivial.

(1) Since the maps  $\phi^r$  factorize by  $L^+G$  for all  $r \geq r(x)$ , it follows that  $w(x) \leq r(x)$  according to the identity of Lemma 8.5. On the other hand,  $\phi^{r(x)-1}$  does not factorize by  $L_G^+$ , so that  $w(x) > r(x) - 1$ . We conclude that  $w(x) = r(x)$ . The commutativity of the left handside diagram is clear. For the right handside diagram we apply lemma 2.4.(1) to  $r(g)$  and  $M = 1$ . For  $s \geq r(x) + 1$ , it follows that  $g^{-1} \sigma_s(g) \in \ker \left( G(A^u[[t]]) \rightarrow G(A^u[t]/t) \right)$  so that

$$\phi^s(A) \subseteq \ker \left( J^{w(x)}(A) \xrightarrow{\text{Res}} G(A[t]/t^{w(x)+1}) \rightarrow G(A) \right)$$

Lemma 8.5 yields that

$$J^{w(x)+1}(A) \subseteq \ker \left( J^{w(x)}(A) \xrightarrow{\text{Res}} G(A[t]/t^{w(x)+1}) \rightarrow G(A) \right).$$

Since we have the same property for each  $A$ -algebra  $B$ , the right handside compatibility is established. The last assertion follows from Lemma 2.4.(1) applied to  $M = r(x) + 1$ .

(2) This is similar with (1) and the last assertion follows from Lemma 2.4.(2).  $\square$

**Remarks 8.3.1.** (a) Since  $\text{Res}$  is a group homomorphism, Lemma 8.6 shows that  $\text{res}$  is a group homomorphism. We have then an alternative proof of Theorem 3.4.(1) and (2) when the index is integral. However we still need Proposition 2.1 for showing that  $\text{res}(x)$  (and a fortiori  $\text{Res}(x)$ ) are not trivial.

(b) In case (1), for  $v = 0, \dots, w(x)$ , the same argument shows more generally that  $\text{Res} : J^{w(x)} \rightarrow G_{w(x)} \rightarrow G_v$  factorizes through  $J^{w(x)}/J^{w(x)+v+1}$ .



## 9. LOOP GROUPS AND AFFINE GRASSMANIANS

Our goal is to relate our construction with the theory of affine grassmannians.

**9.1. Ind-schemes.** We continue with the same setting with  $G \subset \mathrm{SL}_N =: H$  a closed  $A$ -subgroup scheme. The loop groups are the  $A$ -functors

$$B \mapsto L_G^+(B) = G(B[[t]]), \quad B \mapsto L_G(B) = G(B((t))).$$

Both  $A$ -functors are equipped with the rotation action of  $\mathbb{G}_m$ : for each  $A$ -ring  $B$ , each  $b \in B^\times$  and each  $g \in L_G(B) = G(B((t)))$  we put  $\delta(b).g = g(b^{-1}t)$ . The  $A$ -functor  $L^+$  is representable by an  $A$ -scheme and the  $A$ -functor  $L^+$  is representable by an ind  $A$ -scheme. More precisely  $L_H$  is the union of the subfunctors  $L_{H,d}$  given by those matrices  $A$  for which the entries of  $A$  are Laurent series of the form  $\sum_{i \geq -d} a_i t^i$ .

Each  $L_{H,d}$  is representable by an affine  $A$ -scheme and we set  $L_{G,d} = L_{H,d} \cap L_G$ .

For  $n \geq 1$ , we denote by  ${}_n L_G(B) = G(B((t^{1/n})))$  and similarly for  $L_G^+$ . We have a natural morphism  $L_G \rightarrow {}_n L_G$ . Let  $r = m/n \in \mathbb{Q}_{\geq 0}$  and consider the  $A$ -subfunctors of  $L_G$  defined by

$${}^r L_G(B) = \left\{ [g] \in L_G(B) \mid g^{-1} \sigma_r(g) \in G(B^u[[t^{1/n}]]) \right\}.$$

if  $r > 0$  and

$${}^0 L_G(B) = \left\{ [g] \in L_G(B) \mid g^{-1} g(\lambda t) \in G(B[\lambda, \lambda^{-1}][[t^{1/n}]]) \right\}.$$

For  $r > s \geq 0$ , we have  ${}^s L_G \subseteq {}^r L_G$  according to Proposition 2.1 (resp. Lemma 2.6) in the case  $s > 0$  (resp.  $s = 0$ ).

**Lemma 9.1.** (1) For each  $d \in \mathbb{Z}_{\geq 0}$ ,  ${}^r L_{G,d} = L_{G,d} \cap {}^r L_G$  is a closed  $A$ -subscheme of  $L_{G,d}$  and even of  $L_{G,d/N}$ .

(2)  ${}^r L_G$  is a closed ind-subscheme of  $L_G$ .

*Proof.* (1) Once again one can work with  $H = \mathrm{SL}_N$ . *First case:  $r > 0$ :* For each  $d \geq 0$ , we have by definition

$${}^r L_{H,d}(B) = \left\{ g \in L_{H,d}(B) \mid g^{-1} \sigma_r(g) \in \mathrm{SL}_N(B^u[[t^{1/n}]]) = {}_n L_{H,d}(B^u) \right\}.$$

We consider the  $\mathbf{A}_A^1$ -morphism  $\psi_{r,d} : L_{H,d} \times_A \mathbf{A}_A^1 \rightarrow {}_n L_{H,nNd} \times_A \mathbf{A}_A^1$  defined by  $(g, u) \mapsto g^{-1} \sigma_r(g)$ . The  $A$ -functor  ${}^r L_{H,d}$  is the  $A$ -subfunctor of  $L_{H,d}$  consisting in the  $g \in L_{H,d}$  such that the map  $u \mapsto g^{-1} \sigma_r(g) = \psi_{r,d}(g)$  factorizes through the closed  $A$ -subscheme  ${}_n L_{H,0} \times_A \mathbf{A}_A^1$ . Since  $\mathbb{G}_{a,A}$  is free over  $A$  (i.e.  $A[t]$  is a free  $A$ -module), it follows that the  $A$ -functor  ${}^r L_{H,d}$  is representable by a closed  $A$ -subscheme of  $L_{H,d}$  according to [SGA3, VI<sub>B</sub>.6.2.4]. According to Corollary 3.5, for each  $A$ -field  $E$ , we have  ${}^r L_{H,d}(E) \subseteq L_{H,N/d}(E)$  so that  ${}^r L_H \subseteq L_{H,N/d}$ .

*Case  $r = 0$ .* The preceding argument works verbatim with the morphism  $\psi_d : L_{H,d} \times_A \mathbb{G}_{m,A} \rightarrow L_{H,Nd} \times_A \mathbb{G}_{m,A}$ ,  $(g, \lambda) \mapsto g^{-1} g(\lambda t)$ .

(2) This is a straightforward consequence of (1).  $\square$

**Remark 9.2.** The  ${}^rL_G$  have no reason to be  $A$ -schemes and this happens already for  $\mathbb{G}_m$  since  ${}^0L_{\mathbb{G}_m} = L_{\mathbb{G}_m}$ .

By specialization at  $t = 0$ , we get the residue  $A$ -functors

$${}^r\text{res}_G : {}^rL_G \rightarrow \text{Hom}_{gr}(\mathbb{G}_a, G) \quad (r > 0), \quad {}^0\text{res}_G : {}^0L_G \rightarrow \text{Hom}_{gr}(\mathbb{G}_m, G).$$

By definition, the affine grassmannian  $\mathcal{Q}_G$  of  $G$  is the fppf sheafification of the  $A$ -functor

$$B \mapsto \mathcal{F}(B) = G(B((t)))/G(B[[t]]).$$

Let  $r \in \mathbb{Q}_{\geq 0}$  and consider the  $A$ -subfunctor of  $\mathcal{F}$  defined by

$$\mathcal{F}^r(B) = \left\{ [g] \in {}^rL_G(B)/L_G^+(B) \right\}.$$

It is an  $A$ -subfunctor granting to Lemma 2.7.(2) and we denote its fppc sheafification by  ${}^r\mathcal{Q}_G$ . Clearly the map  $L_G \rightarrow \mathcal{Q}_G$  induces an isomorphism  $({}^rL_G/L_G^+)_{fppf} \xrightarrow{\sim} {}^r\mathcal{Q}_G$ . Also the residue  $A$ -functors give rise to the  $A$ -functors

$${}^r\text{res}_G : {}^r\mathcal{Q}_G \rightarrow \overline{\text{Hom}}_{gr}(\mathbb{G}_a, G) \quad (r > 0), \quad {}^0\text{res}_G : {}^0\mathcal{Q}_G \rightarrow \overline{\text{Hom}}_{gr}(\mathbb{G}_m, G);$$

where  $\overline{\text{Hom}}_{gr}(\mathbb{G}_a, G)$  is the fppf quotient  $\text{Hom}_{gr}(\mathbb{G}_a, G)/G$  and similarly for  $\mathbb{G}_m$ .

**Lemma 9.3.**  ${}^0\mathcal{Q}_G$  is the fixed locus for the rotation action of  $\mathbb{G}_m$  on  $\mathcal{Q}_G$ .

*Proof.* The fixed locus  ${}^\sharp\mathcal{Q}_G$  is the  $A$ -subfunctor of  $\mathcal{Q}_G$  defined by

$${}^\sharp\mathcal{Q}_G(B) = \left\{ x \in \mathcal{Q}_G(B) \mid x(ct) = x(t) \quad \forall B\text{-algebra } C \text{ and } \forall c \in C^\times \right\}.$$

Clearly  ${}^0\mathcal{Q}_G$  is an  $A$ -subfunctor of  ${}^\sharp\mathcal{Q}_G$ . Conversely we are given an  $A$ -algebra  $B$  and an element  $x \in {}^\sharp\mathcal{Q}_G(B)$ . To show that  $x$  is fixed is local for the fppf topology so that we may assume that  $x$  lifts to an element  $g \in L_G(B)$ . We take  $C = B[\lambda^{\pm 1}]$  so that  $x(\lambda t) = x(t) \in \mathcal{Q}_G(C)$  It follows that  $g(\lambda t) = g(t)h$  with  $h \in L_G^+(C)$ . hence  $g^{-1}g(\lambda t) \in G(B[\lambda^{\pm 1}] [[t]])$ . In other words  $g$  belongs to  ${}^0L_G(B)$  and  $x \in {}^0\mathcal{Q}_G(B)$ .  $\square$

**9.2. Ind-schemes, II.** Now we assume that the quotient  $\text{SL}_N/G$  is representable by an quasi-affine  $A$ -scheme. In this case, the structure of ind  $A$ -scheme of  $\mathcal{Q}_{\text{SL}_N}$  induces a structure of ind  $A$ -scheme on  $\mathcal{Q}_G$  such that the map  $\mathcal{Q}_G \rightarrow \mathcal{Q}_{\text{SL}_N}$  is a locally closed immersion (which is closed if  $\text{SL}_N/G$  is affine), see [Go, lemma 2.14].

**Proposition 9.4.**  ${}^r\mathcal{Q}_G$  is a closed  $A$ -ind-subscheme of  $\mathcal{Q}_G$ .

*Proof.* It is enough to consider the case of  $\text{SL}_N$ . For each  $d \geq 0$ , the map  $\text{SL}_{N,d} \rightarrow \mathcal{Q}_{H,d}$  is a  $L_H^+$ -torsor (locally trivial for the étale topology). It follows that the quotient sheaf  ${}^r\text{SL}_{N,d}/L_H^+$  is representable by a closed  $A$ -subscheme of the  $A$ -scheme  $\mathcal{Q}_{H,d}$ . Thus  ${}^r\mathcal{Q}_G$  is a closed  $A$ -ind-subscheme of  $\mathcal{Q}_G$ .  $\square$

**9.3. Case of a split reductive  $k$ -group.** We work here over a base field  $k$  and assume that  $G$  is a split reductive  $k$ -group equipped with a Killing couple  $(B, T)$ . We denote by  $\Phi = \Phi(G, T)$  the associated root system and by  $\Delta$  the subset of simple roots with respect to  $B$ . For each  $\alpha \in \Phi$ , we denote by  $U_\alpha$  the image of the root homomorphism  $u_\alpha : \mathbb{G}_a \rightarrow G$  attached to  $\alpha$ . We shall use also the  $k$ -group  $L_G^{\leq 0}$  defined by  $L_G^{\leq 0}(R) = G(R[\frac{1}{t}])$  for each  $k$ -algebra  $R$ .

We consider the cell  $L_G^\mu = L_G^+ t^\mu L_G^+$  where  $\mu$  is a non-negative coweight  $\mu : \mathbb{G}_m \rightarrow T$  and the corresponding affine Schubert cell  $\mathcal{Q}^\mu = (L_G^+ t^\mu L_G^+)/L_G^+$  of  $\mathcal{Q}$ . We know that  $\mathcal{Q}_\mu$  is a smooth  $k$ -variety. Also the map  $s_0 : L_G^+ \rightarrow G$  induces a  $G$ -equivariant map

$$p_\mu : \mathcal{Q}_\mu \rightarrow G/P_G(-\mu)$$

where  $P_G(-\mu)$  stands for the Richardson parabolic subgroup attached to  $\mu$  [NP, §2]. The orbit map  $G \rightarrow \mathcal{Q}_\mu$ ,  $g \mapsto g t^\mu$  induces a closed immersion  $i_\mu : G/P_G(-\mu) \rightarrow \mathcal{Q}_\mu$  which is a section of  $p_\mu$ .

We consider the  $k$ -groups  $L^{\geq \mu}G = t^\mu L_G^+ t^{-\mu}$  and  $L^{< \mu}G = t^\mu L_G^{\leq 0} t^{-\mu}$ . Let  $J = L_G^+ \cap s_0^{-1}(B)$  and define  $J^{\geq \mu} = J \cap L^{\geq \mu}G$  and  $J^\mu = J \cap L^{\leq \mu}G$ .

According to [NP, lemme 2.2], we have an isomorphism  $J^\mu \times J^{\geq \mu} \xrightarrow{\sim} J$  and the map  $J^\mu \rightarrow \mathcal{Q}_\mu$ ,  $g \rightarrow g t^\mu$  is an open immersion; we denote by  $\Omega^\mu$  its image. We choose a total order on  $\Phi_+$ ; furthermore the product in  $L_G^+$  induces an isomorphism (of  $k$ -varieties)

$$j_\mu : \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 1} \prod_{i=0}^{\langle \mu, \alpha \rangle} U_{\alpha, i} \xrightarrow{\sim} J^\mu$$

where  $U_{\alpha, i}$  is the image of  $u_{\alpha, i} : \mathbb{G}_a \rightarrow L_G^+$ ,  $x \mapsto u_\alpha(t^i x)$ .

**Lemma 9.5.** *The following diagram*

$$\begin{array}{ccc} \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 1} \prod_{i=0}^{\langle \mu, \alpha \rangle} U_{\alpha, i} & \xrightarrow{\sim} & J^\mu \hookrightarrow \mathcal{Q}_\mu \\ \downarrow & & \downarrow p_\mu \\ \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 1} \mathbb{G}_a & \xrightarrow{\sim} & U_G(\mu) \hookrightarrow G/P_G(-\mu) \end{array}$$

*commutes and is cartesian where the bottom map is induced by the embedding  $U_G(\mu) \hookrightarrow G$  and the left vertical map is the projection on the factors involving  $i = 0$ .*

Note that  $U_G(\mu) \rightarrow G/P_G(-\mu)$  is an open embedding (Bruhat big cell).

*Proof.* The commutativity is obvious. We have the inclusions  $\Omega_\mu \subseteq p_\mu^{-1}(U_G(\mu) \cdot [1]) \subset \mathcal{Q}_\mu$  and we have to show that  $\Omega_\mu \subseteq p_\mu^{-1}(U_G(\mu) \cdot [1])$ .

Up to extend  $k$  to its algebraic closure, it enough to show that  $p_\mu^{-1}(U_G(\mu).[1])(k) \subseteq \Omega_\mu(k)$ . We are given an element  $g.t^\mu \in p_\mu^{-1}(U_G(\mu).[1])(k)$  with  $g \in L_G^+(k)$ . Since  $g(0).[1]$  belongs in the big Bruhat cell  $V = U_G(\mu).[1]$  of  $G/P_G(-\mu)$ , it follows that  $g.[1] \in V(k[[t]])$ . We get that  $g \in U_G(\mu)(k[[t]]) P_G(-\mu)(k[[t]])$ . We can assume that  $g \in U_G(\mu)(k[[t]]) = L_{U_G(\mu)}^+(k)$ . The decomposition  $J^\mu \times J^{\geq \mu} \xrightarrow{\sim} J$  induces a decomposition  $J^\mu \times (J^{\geq \mu} \cap L_{U_G(\mu)}^+) \xrightarrow{\sim} L_{U_G(\mu)}^+$ . Since  $J^{\geq \mu}$  fixes  $t^\mu$ , we get that  $g.t^\lambda \in \Omega_\mu(k)$ .  $\square$

We get then an an isomorphism

$$J_+^\mu := \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 1} \prod_{i=1}^{\langle \mu, \alpha \rangle} U_{\alpha, i} \quad \xrightarrow{\sim} \quad p_\mu^{-1}([1]) \subset \mathcal{Q}_\mu.$$

We put  $E^\mu = p_\mu^{-1}([1])$ , it is an affine  $k$ -space equipped with a left action of  $P_G(-\mu)$ . We get then a  $P_G(-\mu)$ -equivariant map  $G \times_k E^\mu \rightarrow \mathcal{Q}_\mu$  where  $G$  is equipped with the left action provided by right translations. The next statement is well-known.

**Lemma 9.6.** *The quotient  $G \wedge^{P_G(-\mu)} E^\mu$  is representable by the smooth  $k$ -variety  $\mathcal{Q}_\mu$ .*

*Proof.* The action  $P_G(-\mu)$  on  $G \times E^\mu$  (and on  $G$ ) is free. We apply Lemma 10.2 to the  $P_G(-\mu)$ -morphism  $G \times_k E^\mu \rightarrow G$  and get that the fppf quotient  $G \wedge^{P_G(-\mu)} E^\mu$  is representable by a  $k$ -scheme; we observe that this  $k$ -scheme is of finite type according to the permanence properties [EGAIV, 2.2.7.1]. Similarly, using [EGAIV, 4.17.7.3], we see that  $G \wedge^{P_G(-\mu)} E^\mu$  is smooth over  $G/P_G(-\mu)$ .

By construction, the map  $G/P_G(-\mu)$ -map  $G \wedge^{P_G(-\mu)} E^\mu \rightarrow \mathcal{Q}_\mu$  is an isomorphism over all geometric fibers over  $G/P_G(-\mu)$ . According to Grothendieck's fiberwise isomorphism criterion [EGAIV, 4.17.9.5], we conclude that  $Z \rightarrow \mathcal{Q}_\mu$  is an isomorphism.  $\square$

**Remarks 9.7.** (a) If  $\mu$  is minuscule, then the map  $p_\mu$  is an isomorphism.

(b) If  $\mu$  is quasi-minuscule (i.e. minimal but not minuscule),  $p_\mu$  is the line bundle of Ngô-Polo [NP, §7].

**Proposition 9.8.** [Z, after 2.1.11] *The morphism  $i_\mu$  induces an isomorphism  $G/P_G(-\mu) \xrightarrow{\sim} \sharp \mathcal{Q}_\mu$  on the fixed point locus for the rotation action.*

*Proof.* Since  $\mathcal{Q}_\mu$  is smooth, the  $k$ -subvariety  $\sharp \mathcal{Q}_\mu$  is smooth as well. It is then enough to check that  $i_\mu$  induces an isomorphism  $(G/P_G(-\mu))(\bar{k}) \xrightarrow{\sim} \sharp \mathcal{Q}_\mu(\bar{k})$ . Let  $q \in \sharp \mathcal{Q}_\mu(\bar{k})$ . Up to conjugate by an element of  $G(k)$  we can assume that  $p_\mu(x) = 1$  so that  $x \in E^\mu(k)$ . But  $\sharp E_\mu(k) = \{p_\mu(0)\}$  so that  $x = i_\mu([1])$ .  $\square$

Taking into account Lemma 9.3 we get an isomorphism  $G/P_G(-\mu) \xrightarrow{\sim} {}^0\mathcal{Q}_\mu$ .

**Proposition 9.9.** *Assume that  $\mu > 0$ . Let  $g \in L_G^\mu(R)$  where  $R$  is a  $k$ -algebra.*

(1) *If  $R$  is semilocal, then  $r(g) = 0$  if and only if  $g \in G(R) t^\mu L_G^+(R)$ .*

(2) Assume that  $r(g) = 0$  and denote by  $x$  the image of  $g$  in  $G/P_G(-\mu)(R) \xrightarrow{\sim} {}^0\mathcal{Q}_\mu(R)$ . It defines a  $P_G(-\mu)$ -torsor  $E(g)$ . Then  $E(g)$  is a trivial  $P_G(-\mu)$ -torsor if and only if  $g \in G(R) t^\mu L_G^+(R)$ .

It follows of (1) that the strata  $\mathcal{Q}_\mu$  which contains  $[g]$  is encoded in  $\text{res}(g)$ .

*Proof.* (1) If  $g \in G(R) t^\mu L_G^+(R)$ , then  $r(g) = r(t^\mu) = 0$ . Conversely we assume that  $r(g) = 0$ . We denote by  $x \in {}^0\mathcal{Q}_\mu(R)$  the image of  $g$ . Since  $G/P_G(-\mu) \xrightarrow{\sim} {}^0\mathcal{Q}_\mu$  and  $R$  is semilocal there exists  $h \in G(R)$  such that  $x = [h.t^\mu]$  [SGA3, XXVI.5.10.(i)]. We conclude that  $g \in G(R) t^\mu L_G^+(R)$ .

(2) If  $g = h t^\mu h' \in G(R) t^\mu L_G^+(R)$ , then  $x = [h] \in (G/P_G(-\mu))(R)$ . We have  $E(g)(R) \neq \emptyset$  so that the  $P_G(-\mu)$ -torsor  $E(g)$  is trivial. Conversely we assume that  $E(g)$  is the trivial  $P_G(-\mu)$ -torsor. Then  $E(g)(R) \neq \emptyset$ , that is there exists  $h \in G(R)$  such that  $x = [h]$ . It follows that  $h t^\mu = g \in L_G(R)/L_G^+(R)$  whence  $g \in G(R) t^\mu L_G^+(R)$ .  $\square$

**Proposition 9.10.** (1) The ind  $k$ -variety  $\sharp\mathcal{Q}$  is a  $k$ -scheme and we have  $\sharp\mathcal{Q} = \bigsqcup_{\mu \geq 0} {}^0\mathcal{Q}_\mu$ .

(2) Let  $R$  be a semi-local connected  $k$ -algebra. We have

$${}^0L_G(R) = \bigsqcup_{\mu \geq 0} G(R) t^\mu L_G^+(R).$$

*Proof.* (1) We write  $\mathcal{Q} = \varinjlim_{\theta \geq 0} \overline{\mathcal{Q}}_\theta$  as an inductive limit of projective varieties. Let  $\theta$  be a non-negative coweight. The  $\mathcal{Q}_\mu$ 's for  $0 \leq \mu \leq \theta$  provide a stratification of  $\overline{\mathcal{Q}}_\theta$ . We consider the map  $\psi_\theta : \bigsqcup_{0 \leq \mu \leq \theta} {}^0\mathcal{Q}_\mu \rightarrow \sharp\overline{\mathcal{Q}}_\theta$  and claim that it is an isomorphism. The

left handside is a projective variety and so is the right handside. Each piece  ${}^0\mathcal{Q}_\mu$  is a closed (smooth) subvariety of  $\overline{\mathcal{Q}}_\theta$ . Also  ${}^0\mathcal{Q}_\mu \cap {}^0\mathcal{Q}_{\mu'} = \emptyset$  for  $\mu' \neq \mu$  since the residue encodes the strata. It follows that  $\psi_\theta$  is a closed immersion. On the other hand, the map  $\bigsqcup_{0 \leq \mu \leq \theta} {}^0\mathcal{Q}_\mu(\overline{k}) \rightarrow \sharp\overline{\mathcal{Q}}_\theta(\overline{k})$  is bijective so that  $\psi_\theta$  is a bijective closed immersion. If

follows that  $\sharp\overline{\mathcal{Q}}_\theta = \bigsqcup_{0 \leq \mu \leq \theta} Z_{\theta,\mu}$  where  $Z_{\theta,\mu}$  is the unique connected component of  $\sharp\overline{\mathcal{Q}}_\theta$

containing the image of  ${}^0\mathcal{Q}_\mu$  for each  $\mu$ . We observe that  $Z_{\theta,\theta}$  is closed in  $\sharp\overline{\mathcal{Q}}_\theta$  and does not intersect the boundary  $\sharp\overline{\mathcal{Q}}_\theta \setminus \mathcal{Q}_\theta$  so that  $\sharp\mathcal{Q}_\theta \subseteq Z_{\theta,\theta} \subset \mathcal{Q}_\theta$ . Thus  $\sharp\mathcal{Q}_\theta = Z_{\theta,\mu}$  by taking the invariants under the rotation action.

We observe that  $Z_{\theta,\mu} = Z_{\theta',\mu}$  for all coweights satisfying  $0 \leq \theta \leq \theta'$  so that  $\sharp\mathcal{Q}_\mu = Z_{\theta,\mu}$  for all coweights satisfying  $0 \leq \mu \leq \theta$ . Thus  $\psi_\theta$  is an isomorphism for all non-negative coweights  $\theta$ .

Passing to the limit on  $\theta$  yields the wished statement.

(2) We combine (1) and Proposition 9.9.  $\square$

Given an element  $g$  of  $L_G^\mu(k)$ , we would like to investigate in a few cases its index and its residue. We write it  $g = g_0 \left( \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 2} u_\alpha \left( \sum_{i=1}^{\langle \mu, \alpha \rangle - 1} t^i x_{\alpha, i} \right) \right) t^\mu$ . Then  $g$  and  $\left( \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 2} u_\alpha \left( \sum_{i=1}^{\langle \mu, \alpha \rangle - 1} t^i x_{\alpha, i} \right) \right) t^\mu$  have same index and conjugated residues, so that we may work with  $g = \left( \prod_{\alpha \in \Phi | \langle \mu, \alpha \rangle \geq 2} u_\alpha \left( \sum_{i=1}^{\langle \mu, \alpha \rangle - 1} t^i x_{\alpha, i} \right) \right) t^\mu$ .

*The minuscule case.* We have  $g = t^\mu$  hence  $r(g) = 0$  and  $\text{res}(g) = t^\mu$ .

*The quasi-minuscule case.* Let  $\gamma$  be the unique root satisfying  $\langle \mu, \gamma \rangle \geq 2$ . We have  $g = u_\gamma(tx) t^\mu$  for some  $x \in k$ . For  $r = m/n$ , it follows that

$$\begin{aligned} g^{-1} \sigma_r(g) &= t^{-\mu} u_\gamma(-tx) u_\gamma(t(1+ut^r)x) t^\mu (\sigma_r(t)/t)^\mu \\ &= t^{-\mu} u_\gamma(ut^{r+1}x) t^{-\mu} (\sigma_r(t)/t)^\mu \\ &= u_\gamma(ut^{r+1-\langle \mu, \gamma \rangle}x) t^{-\mu} (\sigma_r(t)/t)^\mu. \end{aligned}$$

It follows that  $r = \langle \mu, \gamma \rangle - 1$  if  $x \neq 0$ .

*The rank one case.* There exists a unique root  $\alpha$  satisfying  $\langle \mu, \alpha \rangle \geq 1$  and we assume that  $\langle \mu, \alpha \rangle \geq 2$  (since the case  $\langle \mu, \alpha \rangle = 1$  is minuscule). We have  $g = (P(t)) t^\mu$  for some polynomial  $P \in k[t]$  of degree  $\leq \langle \mu, \alpha \rangle - 1$  satisfying  $P(0) = 0$ . For  $r = m/n$ , it follows that

$$g^{-1} \sigma_r(g) = t^{-\mu} u_\alpha(-P(t)) u_\alpha(P(t(1+ut^r))) t^\mu (\sigma_r(t)/t)^\mu = u_\alpha \left( \frac{P(t) - P(t(1+ut^r))}{t^{\langle \mu, \alpha \rangle}} \right) (\sigma_r(t)/t)^\mu.$$

It follows that

$$r \geq r(g) \iff \frac{P(t) - P(t(1+ut^r))}{t^{\langle \mu, \alpha \rangle}} \in k^u[[t^{1/n}]].$$

For example, for  $P(t) = t$ , we find that  $r(g) = \langle \mu, \alpha \rangle$  and that  $\text{res}(g) = u_\alpha$ .

**Lemma 9.11.** *With the preceding notations we have  $r(g) = 0$  if and only if  $P = 0$ .*

Note that this is coherent with Proposition 9.9.

*Proof.* If  $P = 0$ , we have  $g = t^\mu$  so that  $r(g) = 0$ . Conversely, we assume that  $r(g) = 0$  so that  $g^{-1} g(\lambda t) \in G(k[\lambda^{\pm 1}][[t]])$ . The above computation with  $\lambda = 1 + ut^0$  shows that

$$g^{-1} g(\lambda t) = u_\alpha \left( \frac{P(t) - P(\lambda t)}{t^{\langle \mu, \alpha \rangle}} \right) \mu(\lambda) \in G(k[\lambda^{\pm 1}][[t]])$$

Since  $P$  is of degree  $< \langle \mu, \alpha \rangle$ , we conclude that  $P = 0$ .  $\square$

## 10. APPENDIX: DESCENT

**Lemma 10.1.** *Let  $S$  be a scheme. Let  $G$  be a  $S$ -group scheme flat of finite type. We are given a left action of  $G$  on an  $S$ -scheme  $X$  of finite type such that  $X$  admits a  $G$ -linearized line bundle  $\mathcal{L}$  which is relatively ample over  $S$ . Let  $T$  be an  $S$ -scheme and let  $f : E \rightarrow T$  be a  $G_T$ -torsor. Then the contracted product  $E \wedge^G X_T$  over  $T$  is representable by a  $T$ -scheme. Furthermore we have*

- (i) *the line bundle  $\mathcal{M} = E \wedge^G \mathcal{L}$  on  $E \wedge^G X_T$  is relatively ample over  $T$ .*
- (ii) *If a  $S$ -group scheme  $J$  acts (on the left) on  $f : E \rightarrow T$  such that  $T$  admits a  $J$ -linearized line bundle which is relatively ample over  $S$ , then  $E \wedge^G X_T$  admits a  $J$ -linearized line bundle which is relatively ample over  $T$ .*

*Proof.* The first part with property (i) is [B-L-R, §10.2, lemme 6]. For establishing (ii), we are given a  $J$ -linearized line bundle  $\mathcal{N}_0$  on  $T$  which is relatively ample over  $S$ . We consider the mapping  $h : E \wedge^G X_T \rightarrow T$ , we know that there exists a positive integer  $n$  such that the line bundle  $\mathcal{N} = h^*(\mathcal{N}_0)^{\otimes n} \otimes \mathcal{M}$  on  $E \wedge^G X_T \rightarrow T$  is relatively ample over  $T$  [EGAII, 4.6.13.(ii)]. This line bundle is  $J$ -linearized as desired.  $\square$

**Lemma 10.2.** *Let  $S$  be a scheme and let  $G$  be a flat  $S$ -group scheme locally of finite type. We are given a  $G$ -morphism of  $S$ -schemes  $f : X \rightarrow Y$ . We assume that  $G$  acts freely on  $X$  and on  $Y$ , that  $f$  is affine and that the fppf quotient  $Y/G$  is representable by a  $S$ -scheme. Then the fppf quotient  $X/G$  is representable by an  $S$ -scheme.*

*Proof.* Put  $Z := Y/G$ . Assume first that the  $G$ -torsor  $Y \rightarrow Z$  is trivial. Equivalently, the  $G$ -scheme  $Y$  is isomorphic, over  $S$ , to  $G \times_S Z$ . Choosing such an isomorphism allows to consider  $f$  as a  $G$ -morphism  $X \rightarrow G \times_S Z$ . Put  $X_0 := f^{-1}(e \times_S Z)$ ; it is a closed subscheme of  $X$ , which is transverse to the  $G$ -action: the natural  $G$ -morphism  $G \times_S X_0 \rightarrow X$  is an isomorphism. Thus,  $X/G$  is represented by  $X_0$ . For the general case, we proceed by descent. Considering  $f$  as a morphism of  $Z$ -schemes, we may replace  $S$  by  $Z$  (and  $G$  by  $G \times_S Z$ ), and assume for simplicity that  $Z = S$ . Hence,  $Y$  is a  $G$ -torsor over  $S$ . Put  $S' := Y$ . We are going to base-change the situation via the morphism  $S' \rightarrow S$ . Put

$$f' := f \times_S S' : X' := X \times_S S' \rightarrow Y' := Y \times_S S'.$$

Now, the  $G$ -torsor  $Y' \rightarrow S'$  is trivial, so that the quotient  $X' \rightarrow X'/G$  exists (and is a trivial  $G$ -torsor) by the discussion above. Since  $X' \rightarrow Y'$  is affine,  $X'/G \rightarrow Y'/G = S'$  is affine as well. It is equipped with a canonical descent data for the fpqc morphism  $S' \rightarrow S$ . Hence, this data is effective (by fpqc descent for affine schemes), yielding an arrow  $\tilde{X} \rightarrow S$ . Now, using descent for morphisms, the  $S'$ -arrow  $X' \rightarrow X'/G$  descends to an  $S$ -arrow  $X \rightarrow \tilde{X}$ . This is the quotient  $X \rightarrow X/G$  we sought for. This fact can, again, be checked by descent.  $\square$

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