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A MAP BETWEEN MODULI SPACES OF CONNECTIONS

FRANK LORAY AND VALENTE RAMÍREZ

ABSTRACT. We are interested in studying moduli spaces of rank 2 logarithmic connections on elliptic curves having two poles. To do so, we investigate certain logarithmic rank 2 connections defined on the Riemann sphere and a transformation rule to lift such connections to an elliptic curve. The transformation is as follows: given an elliptic curve C with elliptic quotient $\pi: C \rightarrow \mathbb{P}^1$, and the logarithmic connection (E, ∇) on \mathbb{P}^1 , we may pullback the connection to the elliptic curve to obtain a new connection $(\pi^*E, \pi^*\nabla)$ on C . After suitable birational modifications we bring the connection to a particular normal form. The whole transformation is equivariant with respect to bundle automorphisms and therefore defines a map between the corresponding moduli spaces of connections.

The aim of this paper is to describe the moduli spaces involved and compute explicit expressions for the above map in the case where the target space is the moduli space of rank 2 logarithmic connections on an elliptic curve C with two simple poles and trivial determinant.

1. INTRODUCTION

Let C be a compact complex curve, E a rank 2 holomorphic vector bundle, and $\nabla: E \rightarrow E \otimes \Omega_C^1(D)$ a connection having simple poles at the (reduced) divisor $D = t_1 + \dots + t_n$. At each pole t_i , consider the residue matrix $\text{Res}_{t_i}(\nabla)$ and denote by ν_i^+, ν_i^- its eigenvalues. Fixing the base curve (C, D) , the *spectral data* $\bar{\nu} = (\nu_1^\pm, \dots, \nu_n^\pm)$, the *trace connection* $(\det E, \text{tr } \nabla)$, and introducing weights $\bar{\mu}$ for stability, we may construct the moduli space $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, D)$ of $\bar{\mu}$ -semistable $\bar{\nu}$ -parabolic connections $(E, \nabla, \bar{\ell})$ using Geometric Invariant Theory (GIT) [Nit93, IIS06a]. This moduli space is a separated irreducible quasi-projective variety of dimension $2N$, where $N = 3g - 3 + n$ is the dimension of deformation of the base curve, and g is the genus of C . This variety is moreover endowed with a holomorphic symplectic structure (which is in fact algebraic) [Boa01, Ina13, IIS06a].

Moduli spaces of connections over the Riemann sphere have been extensively studied, in particular as these correspond to spaces of initial conditions for Garnier systems. The elliptic case with one and two poles have been studied in [Lor16] and [FL18], respectively.

Closely related, we have moduli spaces $\text{Bun}^{\bar{\mu}}(C, D)$ of $\bar{\mu}$ -semistable *parabolic bundles*, and a natural map (which we denote Bun) that assigns to a parabolic connection $(E, \nabla, \bar{\ell})$ its underlying parabolic bundle $(E, \bar{\ell})$. This correspondence is a Lagrangian fibration [LS15], and over the set of *simple bundles* it defines an affine \mathbb{C}^N -bundle which is an affine extension of the cotangent bundle of $\text{Bun}^{\bar{\mu}}(C, D)$ [AL97a, AL97b].

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Let (C, T) be an elliptic curve with two marked points, and let ι be the unique elliptic involution that permutes the marked points. Taking the quotient by this involution defines an elliptic covering $\pi: C \rightarrow \mathbb{P}^1$. Via this ramified covering we can pull bundles and connections from \mathbb{P}^1 back to the elliptic curve C . This correspondence defines a map between the corresponding moduli spaces. In this paper we aim to study a particular map

$$\Phi: \text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Con}_{\bar{\nu}'}^{\bar{\mu}'}(C, T),$$

obtained in this way (see [Section 3](#) for details). This transformation was originally introduced in [\[DL15\]](#), and using the associated monodromy representations it was shown to be dominant and generically 2 : 1. The same transformation rule induces also a map between moduli spaces of parabolic bundles (which we denote by the same symbol Φ), making the following diagram commute:

$$\begin{array}{ccc} \text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) & \xrightarrow{\Phi} & \text{Con}_{\bar{\nu}'}^{\bar{\mu}'}(C, T) \\ \text{Bun} \downarrow & & \downarrow \text{Bun} \\ \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) & \xrightarrow{\Phi} & \text{Bun}^{\bar{\mu}'}(C, T) \end{array} \quad (1.1)$$

The moduli spaces $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ and $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ have been explicitly described in [\[LS15\]](#), as well as the fibration Bun between them. The moduli space of parabolic bundles $\text{Bun}^{\bar{\mu}'}(C, T)$ was later studied in [\[Fer16\]](#). Moreover, the latter paper also describes geometrically and in coordinates the map $\Phi: \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}'}(C, T)$.

The objective of this paper is to complete the explicit description of the commutative diagram [\(1.1\)](#) by describing the space $\text{Con}_{\bar{\nu}'}^{\bar{\mu}'}(C, T)$, endowing it with a coordinate system, and computing the map $\Phi: \text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Con}_{\bar{\nu}'}^{\bar{\mu}'}(C, T)$ in such coordinates.

1.1. Summary of the new results. Let $C \subset \mathbb{P}^2$ be an elliptic curve given by the affine equation $y^2 = x(x-1)(x-\lambda)$, and $\pi: C \rightarrow \mathbb{P}^1$ the elliptic quotient $(x, y) \mapsto x$. Let $t \in \mathbb{P}^1$ be a point different from $0, 1, \lambda, \infty$. From now on we fix the divisors $D = 0 + 1 + \lambda + \infty + t$ on \mathbb{P}^1 , and $T = \pi^*(t)$ on C . We refer the reader to [Section 3](#) for the details of the transformation that takes a connection ∇ on (\mathbb{P}^1, D) and returns a connection $\Phi(\nabla)$ on (C, T) . We also define in [Section 3](#) the weights $\bar{\mu}$ and spectral data $\bar{\nu}$ that we will use throughout the present work.

In [Section 6](#) we construct a family of connections over C , denoted \mathcal{U}_C , birationally parametrized by $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \times \mathbb{C}^2$. This family is the image under Φ of the *universal family* for $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ constructed in [\[LS15, Section 5\]](#). The family \mathcal{U}_C is generated by elements $\nabla_0, \Theta_1, \Theta_2$ in such a way that any element $\nabla \in \mathcal{U}_C$ is given by a unique combination

$$\nabla = \nabla_0(u) + \kappa_1 \Theta_1(u) + \kappa_2 \Theta_2(u), \quad u \in \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D), \quad (\kappa_1, \kappa_2) \in \mathbb{C}^2.$$

The natural map into the moduli space $\mathcal{U}_C \dashrightarrow \text{Con}_{\bar{\nu}'}^{\bar{\mu}'}(C, T)$ is a rational dominant map, generically 2:1. Using this family we are able to give an explicit a birational equivalence

$$\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \dashrightarrow \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \times \mathbb{C}^2.$$

This gives a trivialization of the affine \mathbb{C}^2 -bundle $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \rightarrow \text{Bun}^{\bar{\mu}}(C, T)$ over some open and dense subset of $\text{Bun}^{\bar{\mu}}(C, T)$. Furthermore, over this dense set, it identifies the moduli space of *parabolic Higgs bundles* with the cotangent bundle $T^* \text{Bun}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ in a natural way.

Using the isomorphism $\text{Bun}^{\bar{\mu}}(C, T) \cong \mathbb{P}_z^1 \times \mathbb{P}_w^1$ constructed in [Fer16, Section 4.3], we obtain a coordinate system for the moduli space of connections

$$\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \dashrightarrow \mathbb{P}_z^1 \times \mathbb{P}_w^1 \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2.$$

We have explicitly computed the map Φ in these coordinates. The corresponding formulas are given in Section 8. Moreover, we show that the 2-form ω_C defining the symplectic structure of $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is given by

$$\omega_C = dz \wedge d\kappa_1 + dw \wedge d\kappa_2,$$

which coincides, under our identification, with the Liouville 2-form defining the canonical symplectic structure on $T^* \text{Bun}_{\bar{\nu}}^{\bar{\mu}}(C, T)$. Moreover, we verify that the map Φ is a symplectic map.

Unlike $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$, the moduli space $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is singular. We describe the singular locus and describe the local analytic type of such singularities.

Additionally, we define an *apparent map* for connections from the family \mathcal{U}_C . This map is defined as the set of tangencies of the connection with respect to *two fixed subbundles*. The image of this map belongs to $\mathbb{P}^2 \times \mathbb{P}^2$. This map is not well defined on the moduli space, but after symmetrization, ie. after passing to the quotient $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \text{Sym}^2(\mathbb{P}^2)$, we obtain a well defined map which we denote App_C . Note that this is a map between spaces of the same dimension, thus not a Lagrangian fibration. The map is rational, dominant, and the generic fiber consists of exactly 12 points (cf. Theorem 9.1).

Finally, inspired by the results of [LS15], we combine the maps App and Bun to obtain a generically injective map $\text{App}_C \times \text{Bun}: \text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \dashrightarrow \text{Sym}^2(\mathbb{P}^2) \times \mathbb{P}_z^1 \times \mathbb{P}_w^1$, showing that a generic connection is completely determined by its underlying parabolic bundle together with its image under the apparent map.

1.2. Code repository. All the computations mentioned in the present work have been carried out using the computer algebra system SageMath [Sage]. The code is available at the following repository [GitHub].

1.3. Related work. It is well-known that compact Riemann surfaces of genus g with n punctures are hyperelliptic for

$$(g, n) = (2, 0), (1, 2), (1, 1), \text{ and } (1, 0).$$

It has been observed by W. Goldman in [Gol97, Theorem 10.2] that, $\text{SL}_2(\mathbb{C})$ -representations of the fundamental group of these surfaces, with parabolic representation around each puncture, are invariant under the hyperelliptic involution; moreover, they come from the orbifold quotient representations. From the Riemann-Hilbert correspondance, this means that a similar result should hold true for logarithmic connections, providing a dominant map between the corresponding moduli spaces of connections. This has been studied in details in the genus 2 case in [HL19]. The genus 1 case has been considered much earlier in [Hit95] (see also [LvdPU08]). For the genus 1 case with one puncture, the same results

also revealed to be true with arbitrary local monodromy at the puncture, which has been studied in [Lor16].

The case studied here, 2 punctures on genus 1 curves, was first considered in [DL15] for representations. There it was proved that the result of Goldman extends as follows: Consider the unique elliptic involution permuting the two punctures; then any $\mathrm{SL}_2(\mathbb{C})$ -representation whose image is Zariski dense, and whose boundary components have image into the same conjugacy class, is invariant under the involution and comes from a representation of the orbifold quotient. The goal of the present paper was to provide the similar property for logarithmic connections, and therefore complete the whole picture for hyperelliptic curves. We note that similar constructions also hold within the class of connections on the 4-punctured sphere (see [MV13]).

The present work relies strongly on several results from [LS15, Fer16], which we discuss in Section 5.

Finally, we remark the following for the 2-punctured elliptic curve case. Let E be a rank 2 vector bundle over the elliptic curve C of degree d . By tensoring E with a line bundle L , we can change the degree to any desired value as long as it has the same parity as d . Therefore, the study of moduli spaces of rank 2 connections falls into two cases: odd degree and even degree. Usually the determinant of the bundle is fixed to be either \mathcal{O}_C in the even case (as in the present paper), or $\mathcal{O}_C(\mathbf{w}_\infty)$, where $\mathbf{w}_\infty \in \mathbb{C}$ is the identity element for the group structure of C . The moduli space of connections on C with two poles and fixed determinant $\mathcal{O}_C(\mathbf{w}_\infty)$ has already been described in detail in [FL18], together with its symplectic structure and apparent map. As pointed out in [Fer16], it is possible to pass from the moduli space in the even degree case to that in the odd degree case. This is done by one elementary transformation followed by a twist by a rank 1 connection of degree zero. However, the transformation is not canonical, and this passage makes explicit computations hard to obtain.

1.4. A note about notation. We are going to deal with a lot of objects that are defined over the elliptic curve C , and analogous objects defined over \mathbb{P}^1 . In order to avoid confusion, we will try to use bold typography for objects in C that have a counterpart in \mathbb{P}^1 (eg. ∇ and $\bar{\nabla}$). An exception is the use of $\bar{\mu}, \bar{\nu}$ to denote weight vectors and spectral data. This notation is explained in Definition 3.1.

Throughout this work we will use Φ to denote the transformation described in Section 3, which takes an object (a parabolic bundle, connection, or Higgs bundle) defined over (\mathbb{P}^1, D) , and returns an analogous object defined over (C, T) . Abusing notation, we use the same symbol to denote the induced maps between moduli spaces (cf. Remark 3.1). In a similar fashion, we use the symbol τ to denote the geometric transformation discussed in Remark 5.3, which acts on objects defined over (\mathbb{P}^1, D) . We use the same symbol to denote the involutions that are induced on the corresponding moduli spaces. A closely related transformation τ is defined in Section 7, which acts on the family \mathcal{U}_C but has trivial action on $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$.

Finally, we remark that we write \mathbb{P}_z^1 whenever we want to make explicit the fact that the space \mathbb{P}^1 is endowed with an affine coordinate $z \in \mathbb{C}$. This will allow us to distinguish different occurrences of \mathbb{P}^1 . Similar for affine spaces such as $\mathbb{C}_{(c_1, c_2)}^2$.

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2. GENERAL ASPECTS ABOUT MODULI SPACES OF CONNECTIONS

Let X be a smooth projective complex curve and $D = t_1 + \dots + t_n$ a reduced divisor. A *quasi-parabolic bundle* of rank 2 on (X, D) is a pair $(E, \bar{\ell})$, where E is a holomorphic vector bundle of rank 2 over X , and $\bar{\ell} = \{\ell_1, \dots, \ell_2\}$ a collection of rank 1 subspaces $\ell_i \subset E|_i$. A *parabolic bundle* is a quasi-parabolic bundle endowed with a vector of weights $\bar{\mu} = (\mu_1, \dots, \mu_n)$, where $\mu_i \in [0, 1]$. We will usually omit the vector $\bar{\mu}$ in the notation and denote a parabolic bundle simply by $(E, \bar{\ell})$.

A *logarithmic connection* on X with poles at D is a pair (E, ∇) , where E is a holomorphic vector bundle over X , and $\nabla: E \rightarrow E \otimes \Omega_X^1(D)$ is a \mathbb{C} -linear map satisfying Leibniz' rule. The eigenvalues of the residue $\text{Res}_{t_i}(\nabla)$, ν_i^+, ν_i^- are called the *local exponents*, and the collection $\bar{\nu} = (\nu_1^\pm, \dots, \nu_n^\pm)$ is the *spectral data*. We have the following equality known as *Fuchs' relation*:

$$\sum_{i=1}^n (\nu_i^+ + \nu_i^-) + \deg E = 0.$$

Definition 2.1. Let $\bar{\nu}$ be fixed spectral data and $\bar{\mu}$ a fixed vector of weights. A $\bar{\nu}$ -*parabolic connection* of rank 2 on (X, D) is a triple $(E, \nabla, \bar{\ell})$ where $(E, \bar{\ell})$ is a rank 2 parabolic bundle and (E, ∇) is a logarithmic connection with poles on D , such that at each subspace ℓ_i the residue $\text{Res}_{t_i}(\nabla)$ acts by multiplication by ν_i^+ .

We remark that the difference between two connections is an \mathcal{O}_X -linear operator (known as a *Higgs field*), and the space of connections with a fixed parabolic structure a (finite dimensional) affine space.

Definition 2.2. A *parabolic Higgs bundle* of rank 2 on (X, D) is a triple $(E, \Theta, \bar{\ell})$, where $(E, \bar{\ell})$ is a parabolic vector bundle, $\Theta: E \rightarrow E \otimes \Omega_X^1(D)$ is a \mathcal{O}_X -linear map, and such that for each $t_i \in D$ the residue $\text{Res}_{t_i}(\Theta)$ is nilpotent with null space given by ℓ_i .

We now introduce the notion of $\bar{\mu}$ -semistability.

Definition 2.3. Let $(E, \bar{\ell})$ be a rank 2 parabolic bundle and $\bar{\mu} = (\mu_1, \dots, \mu_n) \in [0, 1]^n$ its weight vector. We define the $\bar{\mu}$ -*parabolic degree* of a line subbundle $L \subset E$ as

$$\deg E - 2 \deg L + \sum_{\ell_i \not\subset L} \mu_i - \sum_{\ell_i \subset L} \mu_i.$$

The parabolic bundle $(E, \bar{\ell})$ is said to be $\bar{\mu}$ -*semistable* ($\bar{\mu}$ -*stable*) if the parabolic degree is non-negative (resp. positive) for every subline bundle L .

In order to define moduli spaces it is convenient to fix the determinant bundle $\det(E)$, and the trace $\text{tr}(\nabla)$ in the case of connections. These choices will not appear explicitly in the notation, but we always assume these objects have been defined and fixed. Thus we denote by $\text{Bun}^{\bar{\mu}}(X, D)$ the moduli space of $\bar{\mu}$ -semistable parabolic bundles modulo s -equivalence, where all bundles are assumed to have determinant equal to some fixed line bundle. The moduli space does not depend on the choice of the prescribed determinant, as we can freely change it by twisting all connections by a given line bundle.

Remark 2.1. A parabolic connection is said to be $\bar{\mu}$ -semistable if every subbundle *invariant by the connection* has non-negative parabolic degree. It is possible for a connection to be $\bar{\mu}$ -semistable while the underlying parabolic bundle is not. Usually, parabolic bundles

admitting a connection ∇ with generic exponents $\bar{\nu}$ are indecomposable and $\bar{\mu}$ -semistable for a suitable choice of weights $\bar{\mu}$ [LS15, Section 3]. Thus semistability is restored by modifying the weights. However, because of the particular definition of the transformation Φ in Section 3, we need to use very specific weights so that the map preserves the notion of semistability [Fer16, Section 6]. Thus, when extending Φ to the moduli space of connections, we cannot allow ourselves to vary the weights and we are forced to replace the usual notion of semistability of connections by semistability of the underlying parabolic bundle.

In virtue of the above remark, the following definition might not be standard.

Definition 2.4. In this paper $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(X, D)$ denotes the moduli space of $\bar{\nu}$ -parabolic connections on (X, D) , where the determinant bundle and trace connection equal some fixed pair (L, ξ) , and such that the underlying parabolic bundle is $\bar{\mu}$ -semistable. Similarly, we denote by $\text{Higgs}^{\bar{\mu}}(X, D)$ the moduli space of parabolic Higgs bundles with $\bar{\mu}$ -semistable underlying bundle. Note that in this way we consider an open subset of the usual moduli spaces of connections (as defined in [IIS06a, IIS06b, Ina13]).

From now on, unless otherwise specified, connections, Higgs fields and bundles are assumed to have trivial determinant and zero trace.

3. THE PULLBACK MAP

Let $C \subset \mathbb{P}^2$ be an elliptic curve such that in some fixed affine chart it is given by the equation

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

This curve is endowed with the *elliptic involution* $(x, y) \mapsto (x, -y)$. With respect to the group structure of C , this involution is precisely $\mathbf{p} \mapsto -\mathbf{p}$. The quotient of C under this involution gives rise to the *elliptic quotient* $\pi: C \rightarrow \mathbb{P}^1$. This is a 2 : 1 cover ramified over the 2-torsion points $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_\lambda, \mathbf{w}_\infty$, which are the points on C that satisfy $x = 0, 1, \lambda, \infty$, respectively.

Let us choose a point $t \in \mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$, and let $\pi^{-1}(t) = \{\mathbf{t}_1, \mathbf{t}_2\}$. We define the following divisors of \mathbb{P}^1 :

$$W = 0 + 1 + \lambda + \infty, \quad T = t, \quad D = W + T.$$

We define analogous divisors for C :

$$W = \mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_\lambda + \mathbf{w}_\infty, \quad T = \mathbf{t}_1 + \mathbf{t}_2, \quad D = W + T.$$

We are going to abuse notation and denote them with the same letters. It should be clear from the context whether we are talking about a divisor on \mathbb{P}^1 or on C .

Now, let us fix the spectral data and weights to use throughout the text. We remark that for the most part we will work with \mathfrak{sl}_2 -connections. Therefore the spectral data will always satisfy $\nu_i^- = -\nu_i^+$.

Definition 3.1. Let ν any complex number such that $2\nu \notin \mathbb{Z}$, and choose μ a real number $0 < \mu < 1$. When working with parabolic bundles over (C, T) , we define the spectral data $\bar{\nu} = (\pm\nu, \pm\nu)$ and the weight vector $\bar{\mu} = (\mu, \mu)$. If working with bundles over (\mathbb{P}^1, D) , we will use the same notation $\bar{\nu}, \bar{\mu}$ to denote the vectors $\bar{\nu} = (\pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\nu)$ and $\bar{\mu} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mu)$.

We now follow the construction presented in [Fer16, Section 6]. Below we make explicit the case of connections, but remark that this simultaneously defines transformations Φ acting on parabolic bundles and Φ acting on connections.

Let ∇ be a connection on the trivial bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, having simple poles over the divisor D , with spectral data given by $\bar{\nu} = (\pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\nu)$. Assume that over W these are *apparent singularities* (ie. after suitable birational transformations these points are no longer poles of the connection). The following series of transformations defines the map Φ :

- (1) Pullback ∇ to C . This gives a connection $\pi^*\nabla$ on $\pi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_C \oplus \mathcal{O}_C$ with poles on D . Locally, the connection near $\mathbf{t}_1, \mathbf{t}_2$ looks like ∇ around t . This is not the case around the ramification points \mathbf{w}_k , but we know this construction multiplies the residual eigenvalues by a factor of two. Therefore the spectral data is given by $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\nu, \pm\nu)$.
- (2) Perform a positive elementary transformation for each pole of the divisor W . This gives a new connection on some bundle E of degree 4. The spectral data over the points \mathbf{t}_i is unchanged, and the new spectral data at the \mathbf{w}_k is $\nu_k^+ = -\frac{1}{2}, \nu_k^- = -\frac{1}{2}$ (not an \mathfrak{sl}_2 -connection).
- (3) Tensor with the rank 1 connection $(\mathcal{O}_C(-2\mathbf{w}_\infty), \xi)$, where ξ is a fixed connection with simple poles on W and residue $\frac{1}{2}$ at each of them (no poles on T). By design, the bundle $E' = E \otimes \mathcal{O}_C(-2\mathbf{w}_\infty)$ has trivial determinant and the residual eigenvalues at \mathbf{w}_k are all zero. Generically the bundle E' is of the form $E' = \mathbf{L} \oplus \mathbf{L}^{-1}$, where \mathbf{L} is a rank 1 bundle of degree zero. The assumption that the poles of ∇ over W were apparent singularities implies that the new connection is in fact holomorphic at each point in W .
- (4) Since the final connection is holomorphic at W , we may forget these points from the divisor of poles and consider it as a connection defined on (C, T) with spectral data $\bar{\nu} = (\pm\nu, \pm\nu)$.

We denote the last connection by $\Phi(\nabla)$.

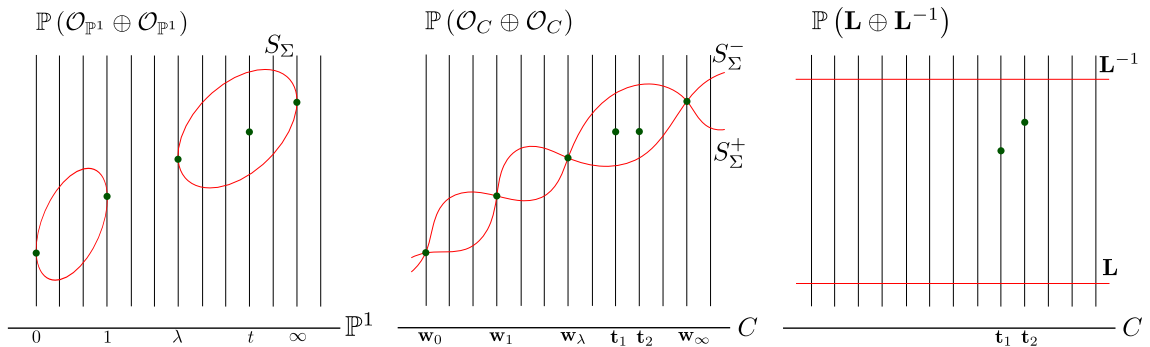


FIGURE 1. Steps of the transformation Φ . The canonical sections corresponding to $\mathbf{L}, \mathbf{L}^{-1}$ in $\mathbb{P}(\mathbf{L} \oplus \mathbf{L}^{-1})$ come from a *multisection* S_Σ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$. This is explained in Remark 7.2.

Remark 3.1. The transformation Φ preserves $\bar{\mu}$ -stability for the underlying parabolic bundles, as long as we use the weight vectors defined in Definition 3.1 (cf. [Fer16, Section 6]).

Therefore, the correspondence $\nabla \mapsto \Phi(\nabla)$ induces a map between moduli spaces

$$\Phi: \text{Con}_{\bar{\mu}}^{\bar{\mu}}(\mathbb{P}^1, D) \longrightarrow \text{Con}_{\bar{\mu}}^{\bar{\mu}}(C, T).$$

We will use the same notation for the geometric transformation defined in this section and the induced map between moduli spaces.

4. GENERICITY ASSUMPTIONS

In this section we will briefly describe the geometry of the moduli spaces of parabolic bundles we work with. We will explain which families of bundles are particularly special, and define a *generic bundle* to be one not belonging to these families. We begin with bundles over \mathbb{P}^1 .

Let $\lambda, t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be different points, and let D be the divisor $D = 0 + 1 + \lambda + \infty + t$. We are interested in $\bar{\mu}$ -semistable parabolic bundles of degree zero over the marked curve (\mathbb{P}^1, D) . For the weights defined in [Definition 3.1](#), all parabolic bundles are in fact $\bar{\mu}$ -stable and have a trivial underlying bundle. Moreover, the moduli space $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ is isomorphic to a Del Pezzo surface of degree 4, which we denote \mathcal{S} [[Fer16](#), Proposition 6.1]. The surface \mathcal{S} is a smooth projective surface that is obtained by blowing-up 5 particular points $D_i \in \mathbb{P}^2$. It is well-known that this surface \mathcal{S} has exactly 16 rational curves of self-intersection -1 . Namely, the five exceptional divisors from the blow-up E_i , the strict transform of the conic Π passing through the five points, and the strict transform of the 10 lines $L_{i,j}$ passing through every possible pair (D_i, D_j) .

As shown in [[LS15](#)], the *coarse moduli space* of indecomposable parabolic bundles is a non-separated variety obtained by gluing together a finite number of spaces $\text{Bun}^{\bar{\mu}'}(\mathbb{P}^1, D)$ for suitable choices of weight vectors $\bar{\mu}'$. As the weights vary, the bundles in the special families $\{\Pi, E_i, L_{i,j}\}$ may become unstable, and new bundles that were previously unstable are now semistable. However, the bundles represented in $\mathcal{S} \setminus \{\Pi, E_i, L_{i,j}\}$ are always stable and thus common to every chart. This motivates the following definition.

Definition 4.1. We will say that a parabolic bundle is *generic* in $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \cong \mathcal{S}$ if it lies outside the union of the 16 (-1) -curves $\{\Pi, E_i, L_{i,j}\}$. A parabolic connection in $\text{Con}_{\bar{\mu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ will be called *generic* if the underlying parabolic bundle is itself generic. We denote by $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$ and $\text{Con}_{\bar{\mu}}^{\bar{\mu}}(\mathbb{P}^1, D)^0$ the open subsets of generic bundles and connections, respectively.

By blowing down 4 out of the 16 (-1) -curves in the surface \mathcal{S} , we arrive to $\mathbb{P}^1 \times \mathbb{P}^1$. This latter space can be endowed with a pair of affine coordinates (u_λ, u_t) , as will be explained later in [Remark 5.1](#). For the most part of this article we will use $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ as a birational model for $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$.

Let us now move on to parabolic bundles over (C, T) . As shown in [[Fer16](#), Theorem A], the moduli space $\text{Bun}^{\bar{\mu}}(C, T)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. With respect to the coordinate system $(z, w) \in \mathbb{P}_z^1 \times \mathbb{P}_w^1$ introduced in the latter paper, the map $\Phi: \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}}(C, T)$ transforms the special (-1) -curves of \mathcal{S} to either horizontal or vertical lines defined by $z = 0, 1, \lambda, \infty$, and $w = 0, 1, \lambda, \infty$. Our definition of generic bundle will exclude these special lines, together with two more curves that we describe below.

Definition 4.2. We denote the ramification locus of Φ by $\Sigma \subset \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$, and by $\Sigma = \Phi(\Sigma) \subset \text{Bun}^{\bar{\mu}}(C, T)$ the branch locus.

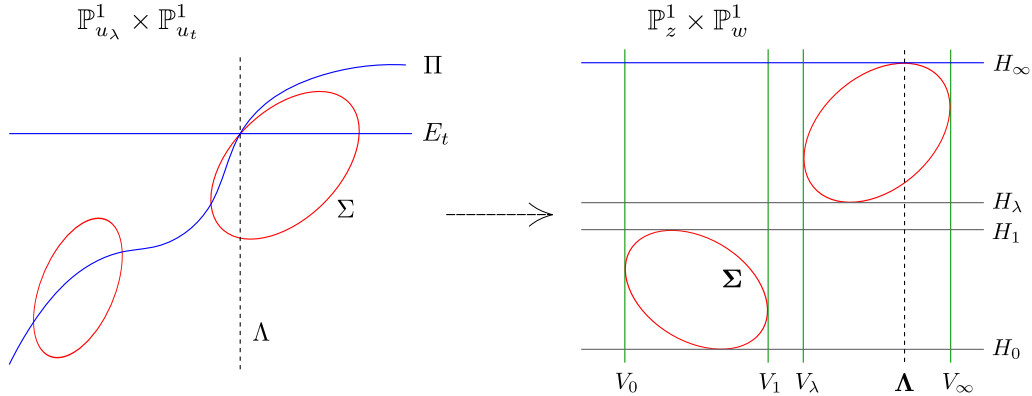


FIGURE 2. The non-generic curves on $\mathbb{P}_z^1 \times \mathbb{P}_w^1$, and (some of) their counterparts in $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$.

The curves Σ and Σ are smooth curves isomorphic to the elliptic curve C . Moreover, according to [Fer16, Section 6.4], Σ coincides with the strictly $\bar{\mu}$ -semistable locus of $\text{Bun}^{\bar{\mu}}(C, T)$. This is a curve of bidegree $(2, 2)$ in $\mathbb{P}_z^1 \times \mathbb{P}_w^1$, and it is isomorphic to the elliptic curve C itself. We remark that the ramification locus Σ represents bundles that are $\bar{\mu}$ -stable. These only become semistable once pulled-back to the elliptic curve.

Finally, for technical reasons, we need to exclude the vertical line $\Lambda = \{z = t\}$. This corresponds to another vertical line $\Lambda \subset \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$. See Figure 2.

Definition 4.3. We will say that a parabolic bundle is *generic* in $\text{Bun}^{\bar{\mu}}(C, T) \cong \mathbb{P}_z^1 \times \mathbb{P}_w^1$ if it lies outside the following loci:

- The union of the 8 lines $z = 0, 1, \lambda, \infty$, and $w = 0, 1, \lambda, \infty$,
- The strictly $\bar{\mu}$ -semistable locus Σ ,
- The vertical line Λ defined by $z = t$.

A parabolic connection in $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ will be called *generic* if the underlying parabolic bundle is itself generic. We denote by $\text{Bun}^{\bar{\mu}}(C, T)^0$ and $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)^0$ the open subsets of generic bundles and connections, respectively.

5. RECAP OF PREVIOUSLY KNOWN RESULTS

In this section we will further recall several facts from [LS15, Fer16] in order to make our results precise and to put them into context. We restrict ourselves to the cases that are relevant to us. We refer the reader to the original papers cited for a detailed treatment and for more general cases. Some basic definitions and results about parabolic bundles and connections can be found on Section 2.

5.1. Moduli spaces of parabolic bundles. In the previous section we have discussed that the moduli space $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ is isomorphic to a Del Pezzo surface, which we continue to denote \mathcal{S} . Below we note two birational models we can use to describe such a surface in coordinates.

Remark 5.1. There are two coordinate systems we can use to describe the set $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$. The first one is based on the fact that \mathcal{S} is defined as the blow-up of \mathbb{P}^2 at five points, so there is a canonical birational map $\mathbb{P}^2 \dashrightarrow \mathcal{S}$. Since $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$ excludes (together with other curves) the exceptional divisors of the blow-up, this map defines a one-to-one map between an open subset of \mathbb{P}^2 and $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$. Fixing homogeneous coordinates $[b_0 : b_1 : b_2]$, the space \mathbb{P}_b^2 defines a coordinate system for $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$. The second alternative relies on the fact that every element of $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$ has the trivial bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ as underlying bundle [Fer16, Proposition 6.1]. Let us introduce an affine coordinate ζ on the fibers of the projectivized bundle. In the generic case and after a fractional linear transformation, we may assume that the parabolic structure over the points $0, 1, \infty$ is given by $\zeta = 0, 1, \infty$, respectively. Under this situation, any parabolic bundle is completely determined by $u_\lambda, u_t \in \mathbb{P}^1$, the parabolic structures over t and λ . This assignment defines a birational map $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0 \dashrightarrow \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$.

Now, let $C \subset \mathbb{P}^2$ be an elliptic curve such that in some fixed affine chart it is given by the equation

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

As described in [Fer16], a parabolic bundle on \mathbb{P}^1 can be lifted to C using the elliptic covering $\pi: C \rightarrow \mathbb{P}^1$. After a series of transformations we obtain a parabolic bundle on C with parabolic structure supported over the divisor $T = \pi^*(t)$. This defines a map

$$\Phi: \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}}(C, T)$$

between moduli spaces (see Section 3 for details). The map Φ is a $2 : 1$ smooth covering, which ramifies over a smooth divisor. The domain space is the Del Pezzo surface \mathcal{S} discussed above, and the target space is proved to be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ in [Fer16, Theorem A]. With respect to the coordinate system (z, w) used in [Fer16, Section 4.3] for the latter space, and using the coordinate chart \mathbb{P}_b^2 in Remark 5.1, the map Φ is explicitly given by

$$[b_0 : b_1 : b_2] \mapsto \left(\frac{b_1 t - b_2}{b_0 t - b_1}, -b_1 \frac{b_0 \lambda - b_1 \lambda - b_1 + b_2}{b_1^2 - b_0 b_2} \right) \in \mathbb{P}_z^1 \times \mathbb{P}_w^1.$$

Definition 5.1. We define $\tau \in \text{Aut}(\mathcal{S})$ as the involution of $\mathcal{S} \cong \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ which permutes the two sheets of the map Φ and fixes every point in the ramification divisor.

The above involution is a lift of a de Jonquières automorphism of \mathbb{P}_b^2 (a birational automorphism of degree 3 that preserves a pencil of lines through a point, and a pencil of conics through four other points). The curve $\Sigma \subset \mathcal{S}$ in Definition 4.2 is precisely the curve of fixed points of τ . A detailed description of this involution can be found in [Fer16, Section 6].

Remark 5.2. The moduli space $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ is endowed with an involution τ , in such a way that the quotient of $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ by the action of τ is precisely $\text{Bun}^{\bar{\mu}}(C, D)$. Because of this, the involution τ will play a crucial role in the present work.

Remark 5.3. Given a parabolic bundle of degree zero over (\mathbb{P}^1, D) , we may perform the following birational modifications: an elementary transformation at each of the parabolics over the divisor $W = 0 + 1 + \lambda + \infty$, and a twist by the bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$. The resulting parabolic bundle is again of degree zero, and so this defines an automorphism on the space $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$. It is shown in [Fer16, Proposition 6.5] that this automorphism is precisely τ . Note that this construction naturally extends to parabolic connections if we

twist by a suitable fixed rank 1 connection $(\mathcal{O}_{\mathbb{P}^1}(-2), \xi)$ in the last step. Following the same convention we have adopted for the transformation Φ (cf. [Remark 3.1](#)), we will use the symbol τ to denote both the geometric transformation defined above and the induced self-map on the moduli space $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$.

5.2. Moduli spaces of connections over \mathbb{P}^1 . Recall that the space $\text{Con}_{\nabla}^{\bar{\mu}}(\mathbb{P}^1, D)$ carries a natural symplectic structure in such a way that the map $\text{Bun}: \text{Con}_{\nabla}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ is a Lagrangian fibration. In [\[LS15\]](#) it is shown that the so-called *apparent map* defines a dual Lagrangian fibration. Given a connection ∇ on a bundle E and a rank 1 subbundle $L \subset E$, the apparent map is defined by the zero divisor of the composite map

$$L \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega_{\mathbb{P}^1}^1(D) \rightarrow (E/L) \otimes \Omega_{\mathbb{P}^1}^1(D).$$

Note that the apparent map is defined geometrically as the set of points of tangency between the Riccati foliation defined by ∇ on $\mathbb{P}(E)$ and the section induced by the subbundle L .

For a generic connection of degree -1 , the underlying bundle is $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. This bundle has a unique trivial subbundle $L = \mathcal{O}_{\mathbb{P}^1}$, which provides a canonical choice for the apparent map. In this case we obtain a rational map

$$\text{App}: \text{Con}_{\nabla}^{\bar{\mu}}(\mathbb{P}^1, D) \dashrightarrow |\mathcal{O}_{\mathbb{P}^1}(n-3)| \cong \mathbb{P}^{n-3},$$

where n denotes the number of singularities (in our particular case $n = 5$). For generic connections of degree zero the underlying bundle is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, and we may perform an elementary transformation to replace it by $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. After this, we may proceed as above. This extends the definition of the apparent map to bundles of degree zero.

The Lagrangian fibrations provide a description of the geometric structure of the space of connections. Indeed, over the space of generic connections (and under a simple assumption on the residual eigenvalues) the morphism

$$\text{App} \times \text{Bun}: \text{Con}_{\nabla}^{\bar{\mu}}(\mathbb{P}^1, D) \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \tag{5.1}$$

defines an open embedding [\[LS15, Theorem 4.2\]](#) (moreover, a suitable compactification of the space of generic bundles makes the above map an isomorphism).

5.3. A universal family of connections. Another result that we try to imitate is the construction of a *universal family* for $\text{Con}_{\nabla}^{\bar{\mu}}(\mathbb{P}^1, D)$. By this we mean a family of connections, which we denote \mathcal{U} , that represent every (generic) isomorphism class in $\text{Con}_{\nabla}^{\bar{\mu}}(\mathbb{P}^1, D)$. This is done in such a way that the correspondence is one-to-one (ie. no two elements of the family are isomorphic).

As pointed out in [Remark 5.1](#), a generic parabolic bundle $(E, \bar{\ell})$ of degree zero and polar divisor $D = 0 + 1 + \lambda + \infty + t$ has $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ as underlying bundle, and we may assume the parabolic structure is given by $\bar{\ell} = (0, 1, u_{\lambda}, \infty, u_t)$, for some $u_{\lambda}, u_t \in \mathbb{P}^1$.

In [\[LS15, Section 5.1\]](#), the authors provide explicitly a connection $\nabla_0(u_{\lambda}, u_t)$ and two parabolic Higgs bundles $\Theta_1(u_{\lambda}, u_t), \Theta_2(u_{\lambda}, u_t)$, in such a way that any connection on a generic parabolic bundle (defined by the parameters u_{λ}, u_t) is written uniquely as

$$\nabla = \nabla_0(u_{\lambda}, u_t) + c_1 \Theta_1(u_{\lambda}, u_t) + c_2 \Theta_2(u_{\lambda}, u_t), \tag{5.2}$$

for some $(c_1, c_2) \in \mathbb{C}^2$. Note that the above description defines a birational map

$$\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \dashrightarrow \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \times \mathbb{C}_{(c_1, c_2)}^2.$$

This provides an alternative description of the geometry of the moduli space of connections. Moreover, this map provides, over the space of generic bundles, a trivialization of the affine \mathbb{C}^2 -bundle $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \mathrm{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$. In these coordinates, the symplectic structure is given by the 2-form

$$\omega = dc_1 \wedge du_t + dc_2 \wedge du_\lambda.$$

Remark 5.4. In general, the moduli space of parabolic Higgs bundles is naturally identified with the total space of the cotangent bundle to the moduli space of parabolic bundles [AL97a, AL97b]. As explained in [LS15, Section 5.1], under this correspondence we have $\Theta_1 \mapsto du_t$, $\Theta_2 \mapsto du_\lambda$. Note that the 1-form ω given above corresponds with the canonical symplectic form on the cotangent bundle $T^* \mathrm{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$.

6. STATEMENT OF THE MAIN RESULTS

We begin with the map $\Phi: \mathrm{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \mathrm{Bun}^{\bar{\mu}}(C, T)$ between moduli spaces of parabolic bundles. Our main objective is to describe the map Φ obtained by extending Φ to the moduli spaces of connection. We know little about the space $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$, except it is an affine \mathbb{C}^2 -bundle over the space $\mathrm{Bun}^{\bar{\mu}}(C, T)$. However, as explained in Section 1, we know beforehand that the extended map Φ is dominant and generically $2 : 1$ [DL15]. Therefore our strategy is to understand $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ as a quotient of $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ by the involution τ that permutes the two sheets of this double cover (this, at least, in some Zariski open subset).

6.1. A “double” universal family of connections. In this and in the forthcoming sections we will use extensively the transformation τ and the involution it induces on the moduli spaces. The reader may consult Definition 5.1 and the discussion around Remark 5.3.

Note that we can use the transformation Φ described in Section 3 to pull the universal family of connections \mathcal{U} on (5.2) from \mathbb{P}^1 to C . This is a one-to-one correspondence that yields a family of connections on C with poles over T , which we denote \mathcal{U}_C . This family of connections represents almost every generic class in the moduli space $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$, but each class has two representatives in \mathcal{U}_C . Indeed, if ∇ is a connection on \mathbb{P}^1 , then $\Phi(\nabla)$ and $\Phi(\tau\nabla)$ are two different connections on C yet they are isomorphic. Hence representing the same equivalence class in the moduli space. Because of this, the natural correspondence $\mathcal{U}_C \rightarrow \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is a generically $2 : 1$ dominant map.

It is important to note that (the image of) the original basis $\nabla_0(u_\lambda, u_t)$, $\Theta_1(u_\lambda, u_t)$, $\Theta_2(u_\lambda, u_t)$ in (5.2) is not the most suitable for describing the family \mathcal{U}_C . Indeed, these are not equivariant with respect to the involution τ , which is a key player in the description of $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ (cf. Remark 5.2). Let us precise the above claim. The action τ is defined by the same series of transformations for parabolic bundles, connections and Higgs bundles. Given a parabolic bundle defined by $u = (u_\lambda, u_t)$, it is not true in general that $\tau\nabla_0(u) = \nabla_0(\tau u)$ nor $\tau\Theta_1(u) = \Theta_1(\tau u)$ (incidentally, Θ_2 is always equivariant). We now seek for a basis that is equivariant. This will descend well as a basis to the quotient $\mathcal{U}/\tau \cong \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$.

Remark 6.1. The connection $\tau\nabla_0(u)$ is a parabolic connection whose underlying parabolic structure is given by τu . Since \mathcal{U} is a universal family, the latter connection can be expressed uniquely as $\nabla_0(\tau u) + c_1\Theta_1(\tau u) + c_2\Theta_2(\tau u)$, for some coefficients c_1, c_2 . However, the claim is that the coefficients c_1, c_2 are not both zero. Equivalently, we could state that $\tau\nabla_0(\tau u) \neq \nabla_0(u)$, even though both left hand side and right hand side are connections with parabolic structure given by $\tau(\tau u) = u$.

Note that because $\tau\nabla_0(\tau u)$ is a connection with underlying parabolic structure u , the mean $\frac{1}{2}(\nabla_0(u) + \tau\nabla_0(\tau u))$ also has u as its parabolic structure. Moreover, this average is equivariant with respect to τ .

Definition 6.1. We define the following elements of \mathcal{U} :

$$\nabla_0^\tau(u) = \frac{1}{2}(\nabla_0(u) + \tau\nabla_0(\tau u)), \quad \Theta_i^\tau(u) = \frac{1}{2}(\Theta_i(u) + \tau\Theta_i(\tau u)), \quad i = 1, 2,$$

and of \mathcal{U}_C :

$$\nabla_0^\tau(u) = \Phi(\nabla_0^\tau(u)), \quad \text{and} \quad \Theta_i^\tau(u) = \Phi\Theta_i^\tau(u), \quad i = 1, 2,$$

and we call them the *equivariant bases* for \mathcal{U} and \mathcal{U}_C , respectively.

By construction, all the above are equivariant with respect to τ . The underlying parabolic structure of $\nabla_0^\tau(u)$ is precisely u , and the parabolic structure of $\nabla_0(u)$ is $\Phi(u)$.

By combining Θ_1^τ and Θ_2^τ in a suitable manner, we arrive to a new equivariant basis $\nabla_0^\tau, \Theta_z, \Theta_w$, defined in [Definition 8.1](#). We shall refer to this as the *canonical basis* (it comes from the canonical identification of the moduli space of parabolic Higgs bundles and the cotangent bundle of $\text{Bun}^\mu(C, T)$, cf. [Corollary 6.1](#)). This is the basis we will use to describe the family \mathcal{U}_C (alas it is only well defined for generic connections). Being a basis, each generic element of \mathcal{U}_C can be written uniquely as a linear combination $\nabla_0^\tau(u) + \kappa_1\Theta_z(u) + \kappa_2\Theta_w(u)$. The change of basis $(c_1, c_2) \mapsto (\kappa_1, \kappa_2)$ is given by an affine transformation that depends rationally on u .

Note that even though the connections on \mathcal{U}_C are defined over C , they are partly parametrized by the moduli space of parabolic bundles over \mathbb{P}^1 . More precisely, we have that $\mathcal{U}_C \cong \text{Bun}^\mu(\mathbb{P}^1, D) \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2$.

The family \mathcal{U}_C is a double cover of $\text{Con}_D^\mu(C, T)$. Let us denote by τ the transformation that permutes the fibers, which corresponds to the action of τ on \mathcal{U} . It follows that

$$\text{Con}_D^\mu(C, T) \cong \mathcal{U}_C/\tau. \tag{6.1}$$

This is analogous to the description of $\text{Bun}^\mu(C, T)$ in [Remark 5.2](#). However, we remark that, unlike the case of parabolic bundles, the fixed-point set of τ has codimension bigger than one, making the space $\text{Con}_D^\mu(C, T)$ singular. This is discussed in [Theorem 6.3](#) below.

Because of equivariance of the basis $\nabla_0^\tau, \Theta_z, \Theta_w$, we have that τ acts only on the first factor of $\text{Bun}^\mu(\mathbb{P}^1, D) \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2$. Namely,

$$\tau(\nabla_0^\tau(u) + \kappa_1\Theta_z(u) + \kappa_2\Theta_w(u)) = \nabla_0^\tau(\tau u) + \kappa_1\Theta_z(\tau u) + \kappa_2\Theta_w(\tau u),$$

and so, in coordinates, $\tau: (u, \kappa_1, \kappa_2) \mapsto (\tau u, \kappa_1, \kappa_2)$. We conclude that

$$\mathcal{U}_C/\tau \cong \text{Bun}^\mu(\mathbb{P}^1, D)/\tau \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2. \tag{6.2}$$

From the description of the moduli space of parabolic bundles in [Fer16], we have that

$$\mathrm{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)/\tau \cong \mathrm{Bun}^{\bar{\mu}}(C, T) \cong \mathbb{P}_z^1 \times \mathbb{P}_w^1. \quad (6.3)$$

Finally, combining (6.1)–(6.3) we obtain

$$\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \cong \mathbb{P}_z^1 \times \mathbb{P}_w^1 \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2. \quad (6.4)$$

Note that in particular this gives a local trivialization of the affine \mathbb{C}^2 -bundle Bun over some open and dense set.

We recall the reader that $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)^0$ denotes the moduli space of connections whose underlying parabolic bundle is *generic*, as specified in Definition 4.3 (see also Definition 4.1 for connections on \mathbb{P}^1).

Theorem 6.1. *The family of connections ∇_0^τ in Definition 6.1 defines a global section $\nabla_0^\tau: \mathrm{Bun}^{\bar{\mu}}(C, T)^0 \rightarrow \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)^0$, thus identifying the affine bundle $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)^0 \rightarrow \mathrm{Bun}^{\bar{\mu}}(C, T)^0$ to the vector bundle $\mathrm{Higgs}^{\bar{\mu}}(C, T)^0 \rightarrow \mathrm{Bun}^{\bar{\mu}}(C, T)^0$. Moreover, the section is Lagrangian, making the above identification symplectic with respect to the natural symplectic structures on the moduli spaces of connections and Higgs bundles. Finally, the latter vector bundle is analytically trivial, thus*

$$\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)^0 \cong \mathrm{Bun}^{\bar{\mu}}(C, T)^0 \times \mathbb{C}^2.$$

Theorem 6.2. *Let $\Sigma, \Lambda \subset \mathrm{Bun}^{\bar{\mu}}(C, T)$ be the strictly-semistable locus and the exceptional line introduced in Definition 4.3, and let $\Sigma, \Lambda \subset \mathrm{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ be their preimages under Φ . The morphism $\Phi: \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is well defined and holomorphic on $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)^0 \setminus \mathrm{Bun}^{-1}(\Sigma \cup \Lambda)$. When restricted to this set, the map Φ is an unramified double cover. Explicit expressions for this map in terms of the coordinates defined by (6.4) are given in Section 8.*

Under the canonical identification between $T^*\mathrm{Bun}^{\bar{\mu}}(C, T)$ and $\mathrm{Higgs}^{\bar{\mu}}(C, T)$, we have the correspondence $dz \mapsto \Theta_z, dw \mapsto \Theta_w$. Therefore, the symplectic structure of $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is given by the 2-form

$$\omega_C = d\kappa_1 \wedge dz + d\kappa_2 \wedge dw.$$

The explicit formulas found in the previous theorem allow us to conclude the following.

Corollary 6.1. *The map $\Phi: \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is symplectic.*

The map Φ is globally well defined as a morphism of quasi-projective varieties, but our coordinate systems cannot be used to describe this map. In fact, the coordinate system fails to describe the space $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ precisely because the latter it is singular over Σ . Recall that our description of the moduli space $\mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is based on the fact that such space can be realized as the quotient of the family \mathcal{U}_C by the involution τ .

Theorem 6.3. *The set of fixed points $\mathrm{Fix}(\tau)$ is a codimension 2 subvariety of \mathcal{U}_C . In fact, it defines a subbundle of rank 1 on the affine \mathbb{C}^2 bundle $\mathrm{Bun}|_{\Sigma}$. A connection $(\nabla, \mathbf{E}, \{\ell_1, \ell_2\}) \in \mathrm{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ belongs to $\mathrm{Fix}(\tau)$ if and only if it decomposes as a direct sum*

$$(\nabla, \mathbf{E}, \bar{\ell}) = (\zeta, \mathbf{L}, \ell) \oplus \iota^*(\zeta, \mathbf{L}, \ell),$$

where ζ is a rank 1 connection with a single pole at either \mathbf{t}_1 or \mathbf{t}_2 , \mathbf{L} is a line bundle of degree zero, and $\iota: C \rightarrow C$ is the elliptic involution.

We remark that the fact that the fixed-point set of τ has codimension bigger than one causes the space $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ to have singularities at $\Phi(\text{Fix}(\tau))$. The next theorem gives the local description of the singular set.

Theorem 6.4. *Around a generic point of the singular locus $\Phi(\text{Fix}(\tau))$, the moduli space $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is locally isomorphic to the hypersurface in $\mathbb{C}_{\bar{x}}^5$ given by the equation $x_1^2 = x_2x_3$. Thus, locally, the singularities look like the product of a quadratic conic singularity and a bidisk.*

6.2. The apparent map in the elliptic case. Consider the projectivization of the trivial bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$. In [LS15] the apparent map is defined with respect to the constant horizontal section σ_{∞} , defined by $\zeta = \infty$ with respect to an affine coordinate ζ on the fiber. After applying the transformation τ , the section σ_{∞} becomes a section of self-intersection $+2$, which we denote σ_{τ} . In general, the tangencies of a connection with σ_{∞} , and with σ_{τ} occur at different points. This means that the apparent map is not invariant under τ . If we want to use Φ to push the concept of the apparent map from $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ to $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$, we need to redefine the apparent map in such a way that it becomes invariant under τ . To do this, we consider both the tangency loci of the connection with σ_{∞} and with σ_{τ} simultaneously. This defines an element of $\mathbb{P}^2 \times \mathbb{P}^2$. The action of τ permutes these factors, so we need to pass to the symmetric product $\text{Sym}^2(\mathbb{P}^2)$.

Definition 6.2. We define the *apparent map* as the unique map $\text{App}_C: \text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \rightarrow \text{Sym}^2(\mathbb{P}^2)$ that completes the following commutative diagram:

$$\begin{array}{ccc} \text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) & \xrightarrow{\text{App} \times (\text{App} \circ \tau)} & \mathbb{P}^2 \times \mathbb{P}^2 \\ \Phi \downarrow & & \downarrow \text{Sym} \\ \text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) & \xrightarrow{\exists! \text{App}_C} & \text{Sym}^2(\mathbb{P}^2) \end{array}$$

Note that, unlike the case for \mathbb{P}^1 , the above map is defined between spaces of the same dimension. Therefore, the map cannot be Lagrangian. It is a generically finite map, but the correspondence is not one-to-one.

In section [Section 9](#) we discuss some properties of the map App_C and the closely related map

$$\text{App}_C \times \text{Bun}: \text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \dashrightarrow \text{Sym}^2(\mathbb{P}^2) \times \mathbb{P}_z^1 \times \mathbb{P}_w^1,$$

and we show that the latter is generically injective.

7. GEOMETRIC DESCRIPTION OF A GENERIC PARABOLIC BUNDLE

In this section we want to describe some geometric aspects of parabolic bundles in $\text{Bun}_{\bar{\nu}}^{\bar{\mu}}(C, T)^0$.

We begin with a generic parabolic connection $(E, \nabla, \bar{\ell})$ of degree zero and polar divisor $D = 0 + 1 + \lambda + \infty + t$ over \mathbb{P}^1 . As we have pointed out before, in the generic case we can assume $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Introducing an affine coordinate ζ on the fibers of $\mathbb{P}(E)$, the parabolic structure is given by $\bar{\ell} = (0, 1, u_{\lambda}, \infty, u_t)$, for some $u_{\lambda}, u_t \in \mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$. We denote by σ_{∞} the constant horizontal section $\zeta = \infty$. This section plays a central role, since it is used to define the apparent map and the universal family \mathcal{U} (cf. [Section 5.2](#)

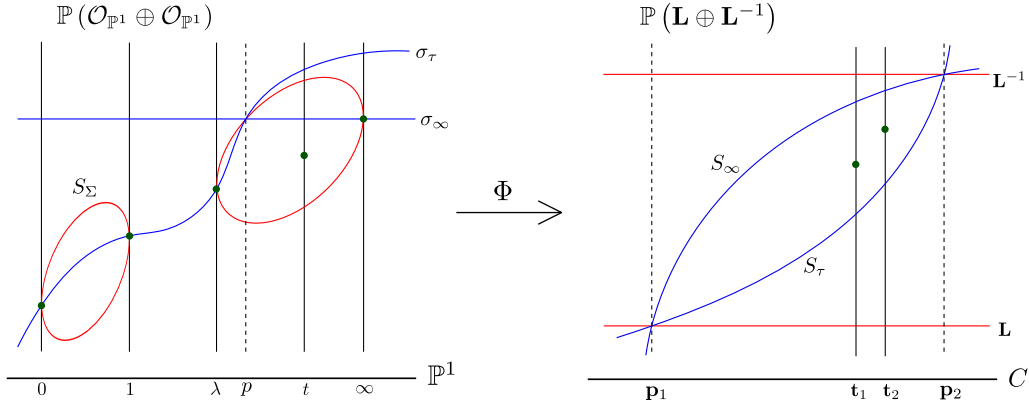


FIGURE 3. The sections σ_∞ and σ_τ under Φ .

and Section 5.3). The birational involution τ defined in Remark 5.3 preserves the trivial bundle E and transforms σ_∞ into a section σ_τ of self-intersection $+2$. This section passes through the parabolic points over $0, 1, \lambda$ but not ∞ . The sections σ_∞ and σ_τ intersect (transversally) over a unique point $x = p$.

Now we apply the transformation Φ to obtain a parabolic connection on (C, T) with trivial determinant. This time the underlying vector bundle \mathbf{E} will not be trivial, and it depends on the parabolic structure of $(E, \bar{\ell})$. Generically, it is of the form $\mathbf{E} = \mathbf{L} \oplus \mathbf{L}^{-1}$, where \mathbf{L} is a line bundle of degree zero. As such, there exists a unique point $\mathbf{p}_1 \in C$ such that $\mathbf{L} = \mathcal{O}_C(\mathbf{p}_1 - \mathbf{w}_\infty)$. The other summand is given by $\mathbf{L}^{-1} = \mathcal{O}_C(\mathbf{p}_2 - \mathbf{w}_\infty)$, in such a way that $\mathbf{p}_1, \mathbf{p}_2$ is a pair of points in involution. Moreover, these points project to the point p defined at the end of the last paragraph (ie. $\pi^{-1}(p) = \{\mathbf{p}_1, \mathbf{p}_2\}$). The bundle \mathbf{E} contains two sections S_∞, S_τ , which are the images of $\sigma_\infty, \sigma_\tau$, respectively. These sections are exchanged by an automorphism of the bundle \mathbf{E} , which we denote τ , making the following diagram commute.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & \mathbf{E} \\ \tau \downarrow & & \downarrow \tau \\ E & \xrightarrow{\Phi} & \mathbf{E} \end{array}$$

Remark 7.1. Note that the involution τ , which acts on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$, has four points of indeterminacy: one at each of the parabolic points above $0, 1, \lambda, \infty$. Thus τ is merely birational and not a bundle automorphism. The transformation Φ pulls the bundle to the elliptic curve, and then blows-up the former points of indeterminacy. As a consequence, the involution τ acting on $\mathbb{P}(\mathbf{E})$ does not have points of indeterminacy and is thus a holomorphic bundle automorphism. This is the reason why a pair of connections $\nabla, \tau\nabla$ represent different classes in $\text{Con}_\nu^{\bar{\mu}}(\mathbb{P}^1, D)$, while $\Phi(\nabla)$ and $\Phi(\tau\nabla)$ represent the same point in $\text{Con}_\nu^{\bar{\mu}}(C, T)$.

Remark 7.2. The subbundles $\mathbf{L}, \mathbf{L}^{-1}$ do not come from subbundles of E in \mathbb{P}^1 . Rather, there exists a curve S_Σ in $\mathbb{P}(E) \cong \mathbb{P}_x^1 \times \mathbb{P}_\zeta^1$ of bidegree $(2, 2)$ projecting down to \mathbb{P}_x^1 as a double cover ramified at the points $x = 0, 1, \lambda, \infty$ (hence isomorphic to the elliptic curve C itself). In fact, this curve is defined by the fixed points of the transformation τ . Once pulled back to C via π , this curve splits into two different sections S_Σ^+, S_Σ^- of $\mathbb{P}(\pi^*(E))$ that intersect over the points $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_\lambda, \mathbf{w}_\infty$. After the elementary transformations dictated by Φ , the sections no longer intersect in $\mathbb{P}(\mathbf{E})$. The action of Φ on S_Σ can be seen in Figure 1, Section 3.

The curve S_Σ passes through the parabolic points above $0, 1, \lambda, \infty$. In general, it does not pass through the parabolic above t . When it does, the parabolic structure is unchanged by τ . Therefore the fixed points of $\tau \in \text{Birat}(\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1)$ are precisely those for which $(t, u_t) \in S_\Sigma$. Moreover, if the curve S_Σ passes through the parabolic point above t in \mathbb{P}^1 , then, after performing the transformation Φ , the parabolics over T are in either \mathbf{L} or \mathbf{L}^{-1} . The elliptic involution permutes the summands $\mathbf{L} \oplus \mathbf{L}^{-1}$. Since the parabolic bundles that come from \mathbb{P}^1 via Φ are invariant under the elliptic involution, we conclude that each direct summand contains exactly one parabolic point. In such case we have a direct sum decomposition $(\mathbf{E}, \{\ell_1, \ell_2\}) = (\mathbf{L}, \{\ell_1\}) \oplus (\mathbf{L}^{-1}, \{\ell_2\})$. Thus we conclude that the generic elements of Σ are precisely the decomposable parabolic bundles.

8. COMPUTATIONS IN COORDINATES

Consider the universal family \mathcal{U} of connections on \mathbb{P}^1 defined by (5.2). As explained in Section 6.1, the universal family \mathcal{U} , and so also \mathcal{U}_C , is birationally parametrized by $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \times \mathbb{C}_{(c_1, c_2)}^2$. Using the canonical basis introduced in Definition 8.1, which is equivariant with respect to τ , instead of the original basis defined in [LS15] allows us to identify

$$\mathcal{U}_C/\tau \cong \text{Bun}^{\bar{\mu}}(C, T) \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2.$$

We have the following diagram:

$$\begin{array}{ccccccc} \text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D) & \longrightarrow & \mathcal{U} & \longrightarrow & \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \times \mathbb{C}_{(c_1, c_2)}^2 & \longrightarrow & \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \times \mathbb{C}_{(c_1, c_2)}^2 \\ \Phi \downarrow & & \downarrow & & \downarrow & & \downarrow \Phi \\ \text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) & \longrightarrow & \mathcal{U}_C/\tau & \longrightarrow & \text{Bun}^{\bar{\mu}}(C, T) \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2 & \longrightarrow & \mathbb{P}_z^1 \times \mathbb{P}_w^1 \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2. \end{array}$$

All the horizontal arrows are birational isomorphisms. In order to describe the left-most vertical map Φ between moduli spaces, we will compute explicitly the right-most arrow Φ in the given coordinates.

Let us split the map Φ as follows: $\Phi = (\Phi, \Psi)$, where Φ is the map between moduli spaces of parabolic bundles, and Ψ determines the change of basis $(c_1, c_2) \mapsto (\kappa_1, \kappa_2)$. The map $\Phi: \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}}(C, T)$ has already been explicitly described both in coordinates and in geometric terms in [Fer16].

Proposition 8.1 ([Fer16]). *The map $\Phi: \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \rightarrow \mathbb{P}_z^1 \times \mathbb{P}_w^1$ is given by $(u_\lambda, u_t) \mapsto (z, w)$, where*

$$z = \frac{\lambda(u_\lambda - 1)}{u_\lambda - \lambda}, \quad w = \frac{\lambda u_t(\lambda u_t - t u_\lambda + t - \lambda - u_t + u_\lambda)}{t \lambda u_t - t \lambda u_\lambda - t u_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + t u_\lambda} \quad (8.1)$$

8.1. The canonical basis. As explained in Section 6.1, in order to have good coordinates on the quotient \mathcal{U}/τ we need to replace the basis $(\nabla_0, \Theta_1, \Theta_2)$ by one that is equivariant with respect to τ . In principle, any equivariant basis would do. However, there is a canonical identification between the spaces $\text{Higgs}^{\bar{\mu}}(C, T)$ and the contangent bundle $T^*(\mathbb{P}_z^1 \times \mathbb{P}_w^1)$, so we seek for the parabolic Higgs bundles Θ_z, Θ_w that correspond to dz, dw under this identification. In fact, we know that the elements Θ_1, Θ_2 in the original basis correspond to du_t, du_λ under the analogous identification.

Consider the pullback of the forms dz, dw under Φ .

$$\Phi^*dz = \frac{\partial z}{\partial u_t} du_t + \frac{\partial z}{\partial u_\lambda} du_\lambda, \quad \Phi^*dw = \frac{\partial w}{\partial u_t} du_t + \frac{\partial w}{\partial u_\lambda} du_\lambda.$$

We use these formulas to introduce the following parabolic Higgs bundles.

Definition 8.1. We define the following family of parabolic Higgs bundles, which are parametrized by (u_λ, u_t) :

$$\Theta_z = \frac{\partial z}{\partial u_t} \Theta_1 + \frac{\partial z}{\partial u_\lambda} \Theta_2, \quad \Theta_w = \frac{\partial w}{\partial u_t} \Theta_1 + \frac{\partial w}{\partial u_\lambda} \Theta_2.$$

We denote their images under Φ by Θ_z and Θ_w . We will refer to the triples $(\nabla_0^\tau, \Theta_z, \Theta_w)$ and $(\nabla_0^\tau, \Theta_z, \Theta_w)$ as the *canonical bases* for \mathcal{U} and \mathcal{U}_C , respectively.

Proposition 8.2. *The parabolic Higgs bundles defined above are equivariant with respect to τ , i.e. $\tau\Theta_i(u) = \Theta_i(\tau u)$, for $i = z, w$, and $u \in \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$. Moreover, with respect to the equivariant basis in [Definition 6.1](#), they can be expressed as*

$$\Theta_z = \frac{(z - \lambda)^2}{\lambda(1 - \lambda)} \Theta_2^\tau, \quad \Theta_w = \frac{2(z - \lambda)}{z - t} \Theta_1^\tau + \frac{(wt - w\lambda - t\lambda + \lambda)(z - \lambda)}{(z - t)(\lambda - 1)\lambda} \Theta_2^\tau, \quad (8.2)$$

where z, w are the functions given in [\(8.1\)](#).

As explained in the proof of [Corollary 6.1](#), given in [Section 8.5](#), the elements Θ_z, Θ_w correspond to dz, dw under the canonical identification of the moduli space of parabolic Higgs bundles and the cotangent space to the moduli space of parabolic bundles. Given the nature of the canonical basis, it is of course expected that Θ_z, Θ_w should be equivariant respect to τ . A rigorous proof follows from the formulas in [\(8.2\)](#), since they are combinations of equivariant elements with τ -invariant coefficients. We delay the proof of such formulas until we have explicit expressions for the base change going from the original basis to the equivariant one.

8.2. The involution τ and exceptional curves in coordinates.

Proposition 8.3. *The involution $\tau \in \text{Birat}(\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1)$ is given by $(u_\lambda, u_t) \mapsto (u_\lambda, \bar{u}_t)$, where \bar{u}_t is given by*

$$\bar{u}_t = \frac{tu_\lambda(\lambda u_t - tu_\lambda + t - \lambda - u_t + u_\lambda)}{t\lambda u_t - t\lambda u_\lambda - tu_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + tu_\lambda}. \quad (8.3)$$

Proof. Let $(z, w) = \Phi(u_\lambda, u_t)$. We know that (z, w) has another preimage under Φ , which by definition is $\tau(u_\lambda, u_t)$. Therefore, we need to solve $\Phi(u'_\lambda, u'_t) = (z, w)$ for u'_λ, u'_t . From [\(8.1\)](#) we can see that z is uniquely determined by u_λ , and that fixing the value for w imposes a quadratic condition on u_t . Solving the equation we recover [\(8.3\)](#). \square

Remark 8.1. The polynomial

$$P_\Pi = t\lambda u_t - t\lambda u_\lambda - tu_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + tu_\lambda, \quad (8.4)$$

which appears as the denominator of [\(8.3\)](#) defines a rational curve in $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$, which corresponds to the rational curve $\Pi \subset \mathcal{S}$ introduced in [Section 4](#) (and corresponds to the conic $b_1^2 - b_0 b_2$ in the birational model \mathbb{P}_b^2 discussed in [Remark 5.1](#)). The involution τ

permutes this curve with the line $u_t = \infty$, thus the former appears as a pole of \bar{u}_t . These rational curves correspond to the two (-1) -curves in the Del Pezzo surface \mathcal{S} that are mapped by Φ to the horizontal line $w = \infty$. Therefore, the polynomial Π also appears in the denominator of the formula for w in (8.1). Indeed, comparing (8.1) to (8.3), we can see that $w = \frac{\lambda}{tu_\lambda} u_t \bar{u}_t$.

The last equation above shows that we can write the product $u_t \bar{u}_t$ in terms of w . We can further express u_λ in terms on z . Moreover, we can also write $u_t + \bar{u}_t$ in terms of z, w . We have the following:

$$u_t + \bar{u}_t = \frac{zw + (z-w)t - \lambda}{z - \lambda}, \quad u_t \bar{u}_t = \frac{wt(z-1)}{z - \lambda}.$$

We conclude that any rational function on u_λ, u_t that is invariant under τ can be expressed as a rational function on z, w .

Remark 8.2. In the chart $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ the curve Σ which was defined as the ramification locus of Φ is defined by the zeros of the following polynomial:

$$P_\Sigma = t\lambda u_t^2 - 2t\lambda u_t u_\lambda - tu_t^2 u_\lambda + \lambda u_t^2 u_\lambda + t^2 u_\lambda^2 - \lambda u_t^2 - t^2 u_\lambda + t\lambda u_\lambda + 2tu_t u_\lambda - tu_\lambda^2. \quad (8.5)$$

Remark 8.3. We have defined $\mathbf{\Lambda} \subset \mathbb{P}_z^1 \times \mathbb{P}_w^1$ to be the vertical line given by $z = t$, and $\Lambda = \Phi^{-1}(\mathbf{\Lambda}) \subset \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$. In our chart $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$, the curve Λ is defined by the vertical line $u_\lambda = \lambda(1-t)/(\lambda-t)$, or equivalently, by the zeros of the polynomial

$$P_\Lambda = \lambda u_\lambda - tu_\lambda + \lambda t - \lambda. \quad (8.6)$$

The special curves discussed in the above remarks can be seen in [Figure 2, Section 4](#).

8.3. Geometry of the apparent map. Before we move on, let us recall the geometric picture of the universal family \mathcal{U} on \mathbb{P}^1 . As usual, we consider parabolic bundles on (\mathbb{P}^1, D) with trivial underlying bundle. We assume that, with respect to an affine coordinate ζ on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$, the parabolic structure is given by $\bar{\ell} = (0, 1, u_\lambda, \infty, u_t)$ (cf. [Remark 5.1](#)). For each parabolic bundle represented in $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$, we define a connection $\nabla_0(u_\lambda, u_t)$. This connection is characterized as the unique connection compatible with the parabolic structure such that the divisor of the apparent map is given by $\text{App}(\nabla_0) = \lambda + t$. In fact, every connection $\nabla \in \mathcal{U}$ is completely determined by its parabolic structure and its image under the apparent map. Recall that the apparent map is defined by the tangencies of the Riccati foliation with the section $\sigma_\infty = \{\zeta = \infty\}$.

Now, in order to understand the family \mathcal{U}_C we need to understand the action of τ defined in [Remark 5.3](#). For generic bundles, τ acts on the trivial bundle in such a way that it exchanges σ_∞ with the $(+2)$ -section σ_τ .

Proposition 8.4. *Assume (u_λ, u_t) defines a generic parabolic bundle. Then τ transforms σ_∞ into the section $\sigma_\tau: \mathbb{P}^1 \rightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}_x^1 \times \mathbb{P}_\zeta^1$ defined by*

$$\sigma_\tau(x) = \left(x, \frac{u_\lambda(1-\lambda)x}{(u_\lambda - \lambda)x - \lambda(u_\lambda - 1)} \right).$$

Proof. Recall that τ is defined as the transformation obtained by blowing up the parabolic points above the divisor $W = 0+1+\lambda+\infty$, and subsequently a twist by the bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$.

Since σ_∞ is a section of degree zero (constant) passing only through the parabolic above $x = \infty$, we conclude that σ_τ must be a section of self-intersection $+2$ passing through the parabolics above $0, 1, \lambda$ (but not ∞). A simple computation shows that there is a unique such section and it is given by the expression above. \square

Remark 8.4. Let $\nabla \in \mathcal{U}$ be defined by the parabolic structure (u_λ, u_t) and divisor Z for the apparent map. Then $\tau\nabla$ is the unique connection with parabolic structure $\tau(u_\lambda, u_t) = (u_\lambda, \bar{u}_t)$ and whose tangencies with the section σ_τ are exactly given by the divisor Z .

In order to be more explicit, let us denote App_∞ the usual apparent map with respect to the constant section σ_∞ . Now that we have a formula for σ_τ , we can compute the tangencies of this section with a given connection. Let us detail this construction.

Definition 8.2. Given a connection $\nabla \in \mathcal{U}$, define the vector $v(x) = (1, \sigma_\tau(x))^\top$. Let $v_1 = v(x)$ and $v_2 = \nabla v_1$. We define $\text{App}_\tau(\nabla)$ as the numerator of the rational expression $\det(v_1, v_2)$. We call the map $\nabla \mapsto \text{App}_\tau(\nabla)$ the *apparent map* with respect to σ_τ .

Explicit expressions for App_∞ in terms of the variables u_λ, u_t, c_1, c_2 are given in [LS15, Section 6]. We omit those for App_τ here since they are considerably more intricate.

8.4. The base change map. Below we compute the map $\Psi: \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \times \mathbb{C}_{(c_1, c_2)}^2 \rightarrow \mathbb{C}_{(\kappa_1, \kappa_2)}^2$, given by the base change from the original basis into the canonical one. We need to describe in coordinates the base change from the original basis into the equivariant basis. The passage from the equivariant basis to the canonical one is dictated by [Proposition 8.2](#). Note that it is enough to work directly with the universal family \mathcal{U} (and not on \mathcal{U}_C). We begin by describing in (c_1, c_2) -coordinates the action of the involution τ on $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$. To do so we will exploit the idea presented in [Remark 8.4](#).

Let $\nabla \in \mathcal{U}$ have parabolic structure $\text{Bun}(\nabla) = (u_\lambda, u_t)$, and assume that with respect to the original basis for \mathcal{U} it is written as $\nabla = \nabla_0(u_\lambda, u_t) + c_1\Theta_1(u_\lambda, u_t) + c_2\Theta_2(u_\lambda, u_t)$. Following [Remark 8.4](#), we seek for the unique connection ∇' such that

$$\text{Bun}(\nabla') = (u_\lambda, \bar{u}_t), \quad \text{and} \quad \text{App}_\tau(\nabla') = \text{App}_\infty(\nabla).$$

The connection ∇' is precisely the image of ∇ under τ . A straight forward computation allows us to find coefficients c'_1, c'_2 as functions of u_λ, u_t, c_1, c_2 , such that

$$\nabla' = \nabla_0(u_\lambda, \bar{u}_t) + c'_1\Theta_1(u_\lambda, \bar{u}_t) + c'_2\Theta_2(u_\lambda, \bar{u}_t).$$

If we hold the parabolic structure fixed, the coefficients c'_1, c'_2 are affine functions of c_1, c_2 . This means that there exists a 3×3 matrix $\mathbb{T}(u_\lambda, \bar{u}_t)$ such that

$$\begin{pmatrix} 1 \\ c'_1 \\ c'_2 \end{pmatrix} = \mathbb{T}(u_\lambda, \bar{u}_t) \begin{pmatrix} 1 \\ c_1 \\ c_2 \end{pmatrix}.$$

Proposition 8.5. *The matrix $\mathbb{T} = \mathbb{T}(u_\lambda, \bar{u}_t)$ is given as follows:*

$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbb{T}_{10}/\delta & \mathbb{T}_{11}/\delta & 0 \\ \mathbb{T}_{20}/\delta & \mathbb{T}_{21}/\delta & 1 \end{pmatrix},$$

where δ, T_{ij} are functions of u_λ, u_t given by

$$\begin{aligned}\delta &= t(t-1)(t-\lambda)u_\lambda(u_\lambda-1)(u_\lambda-\lambda), \\ T_{10} &= -2\nu(t\lambda u_t - t\lambda u_\lambda - tu_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + tu_\lambda)(t\lambda - tu_\lambda + \lambda u_\lambda - \lambda), \\ T_{20} &= \nu t(t-1)(2\lambda^2 u_t - 2t\lambda u_\lambda + tu_\lambda^2 - \lambda u_\lambda^2 + t\lambda - \lambda^2 - 2\lambda u_t + 2\lambda u_\lambda), \\ T_{11} &= -(t\lambda u_t - t\lambda u_\lambda - tu_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + tu_\lambda)^2, \\ T_{21} &= -t(t-1)(-\lambda^2 u_t^2 + 2t\lambda u_t u_\lambda - t\lambda u_\lambda^2 - tu_t u_\lambda^2 + \lambda u_t u_\lambda^2 - t\lambda u_t \\ &\quad + \lambda^2 u_t + \lambda u_t^2 - 2\lambda u_t u_\lambda + tu_\lambda^2).\end{aligned}$$

Note that the factors that appear in T_{10} are P_Π and P_Λ (defined in [Section 8.2](#)), and that $T_{11} = -P_\Pi^2$.

The above proposition is just the result of the computations mentioned earlier. We remark that the first row of T had to be that way in order to map connections to connections. The last column of T takes this form since the Higgs bundle $\Theta_2(u_\lambda, u_t)$ is equivariant with respect to τ . This in turn is a consequence of the fact that u_λ is unaffected by τ (cf. [Proposition 8.3](#)).

Remark 8.5. The fact that τ is an involution translates to the identity $T(u_\lambda, u_t) = T(u_\lambda, \bar{u}_t)^{-1}$, which can be easily verified from the above expressions.

The equivariant basis is defined by the conditions:

$$\nabla_0^\tau(u) = \frac{1}{2}(\nabla_0(u) + \tau\nabla_0(\tau u)), \quad \Theta_i^\tau(u) = \frac{1}{2}(\Theta_i(u) + \tau\Theta_i(\tau u)), \quad i = 1, 2.$$

From these, we deduced that the matrix $\frac{1}{2}(\text{Id} + T(u_\lambda, \bar{u}_t))$ dictates the base change from the equivariant basis to the original basis (it tells us how $\nabla_0^\tau, \Theta_i^\tau$ are written in terms of ∇_0, Θ_i). The base change that we seek to define Ψ is its inverse. Namely

$$B(u_\lambda, u_t) = 2(\text{Id} + T(u_\lambda, \bar{u}_t))^{-1}.$$

Proposition 8.6. *The explicit expressions for B are as follows:*

$$B = \begin{pmatrix} 1 & 0 & 0 \\ B_{10}/\beta & B_{11}/\gamma\beta & 0 \\ B_{20}/\alpha\beta & B_{21}/\gamma\beta & 1 \end{pmatrix},$$

where the numerators are

$$\begin{aligned}B_{10} &= 2\nu(t\lambda u_t - t\lambda u_\lambda - tu_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + tu_\lambda), \\ B_{20} &= -\nu(-2t\lambda^2 u_t^2 u_\lambda + 3t\lambda u_t^2 u_\lambda^2 + 2t^2 \lambda u_\lambda^3 - 2t\lambda u_t u_\lambda^3 - tu_t^2 u_\lambda^3 + \lambda u_t^2 u_\lambda^3 - t^2 u_\lambda^4 + t\lambda^2 u_t^2 \\ &\quad + 2t\lambda^2 u_t u_\lambda - t\lambda u_t^2 u_\lambda + \lambda^2 u_t^2 u_\lambda - 3t^2 \lambda u_\lambda^2 - 3\lambda u_t^2 u_\lambda^2 + t^2 u_\lambda^3 - t\lambda u_\lambda^3 + 2tu_t u_\lambda^3 \\ &\quad + tu_\lambda^4 - \lambda^2 u_t^2 + t^2 \lambda u_\lambda - t\lambda^2 u_\lambda - 2t\lambda u_t u_\lambda + 2\lambda u_t^2 u_\lambda + 3t\lambda u_\lambda^2 - 2tu_\lambda^3), \\ B_{11} &= 2(t\lambda u_t - t\lambda u_\lambda - tu_t u_\lambda + \lambda u_t u_\lambda - \lambda u_t + tu_\lambda)^2, \\ B_{21} &= -t(t-1)(\lambda^2 u_t^2 - 2t\lambda u_t u_\lambda + t\lambda u_\lambda^2 + tu_t u_\lambda^2 - \lambda u_t u_\lambda^2 + t\lambda u_t - \lambda^2 u_t \\ &\quad - \lambda u_t^2 + 2\lambda u_t u_\lambda - tu_\lambda^2),\end{aligned}$$

and the denominators are given by

$$\begin{aligned}\alpha &= 2u_\lambda(u_\lambda - 1)(u_\lambda - \lambda), \\ \beta &= t\lambda u_t^2 - 2t\lambda u_t u_\lambda - tu_t^2 u_\lambda + \lambda u_t^2 u_\lambda + t^2 u_\lambda^2 - \lambda u_t^2 - t^2 u_\lambda + t\lambda u_\lambda + 2tu_t u_\lambda - tu_\lambda^2, \\ \gamma &= \lambda u_\lambda - tu_\lambda + \lambda t - \lambda.\end{aligned}$$

Comparing with the polynomials introduced in [Section 8.2](#), we have $\beta = P_\Sigma$, and $\gamma = P_\Lambda$. The polynomial P_Π appears again in B_{10} and B_{11} .

Proof of [Proposition 8.2](#). We have three bases to describe the universal family \mathcal{U} , and with them the following base-change matrices:

- (1) B going from the original basis to the equivariant one,
- (2) J going from the canonical basis to the original one,
- (3) C going from the canonical basis to the equivariant one.

The matrix B is given by the above proposition. The matrix J is given by putting together the first column of $B^{-1} = \frac{1}{2}(\text{Id} + \mathbb{T}(u_\lambda, \bar{u}_t))$ (which defines the connection ∇_0^τ), and the (transposed) Jacobian matrix $\frac{\partial(z,w)}{\partial(u_t, u_\lambda)}$, which defines the parabolic Higgs bundles Θ_z, Θ_w . More precisely,

$$J = \begin{pmatrix} 1 & 0 & 0 \\ c_1^0 & \frac{\partial z}{\partial u_t} & \frac{\partial w}{\partial u_t} \\ c_2^0 & \frac{\partial z}{\partial u_\lambda} & \frac{\partial w}{\partial u_\lambda} \end{pmatrix},$$

where $\nabla_0^\tau = \nabla_0 + c_1^0 \Theta_1 + c_2^0 \Theta_2$. The matrix $C = BJ$ is the one we are interested in. Having explicit expressions for both B and J, we can compute the explicit expressions for C. It is straightforward to check that the entries of this matrix are invariant under τ , and in fact can be rewritten in terms of the τ -invariant functions z, w that appear in [\(8.1\)](#). These expressions coincide with the coefficients given in [\(8.2\)](#). In particular, this proves that the parabolic Higgs bundles Θ_z and Θ_w are equivariant with respect to τ , since they are expressed as a linear combination of the τ -equivariant bundles Θ_1^-, Θ_2^- , with τ -invariant coefficients. \square

For future reference, we provide here an explicit expression of the matrix C^{-1} .

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & K_{11} & K_{12} \\ 0 & K_{21} & 0 \end{pmatrix},$$

where

$$K_{11} = \frac{wt - w\lambda - t\lambda + \lambda}{2(z - \lambda)^2}, \quad K_{21} = \frac{z - t}{2(z - \lambda)}, \quad K_{12} = \frac{\lambda(1 - \lambda)}{(z - \lambda)^2}. \quad (8.7)$$

We arrive finally to a complete description of the map Φ in coordinates:

Proposition 8.7. *The map*

$$\Phi: \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \times \mathbb{C}_{(c_1, c_2)}^2 \dashrightarrow \mathbb{P}_w^1 \times \mathbb{P}_z^1 \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2$$

decomposes as $\Phi = (\Phi, \Psi)$ where $\Phi: \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \rightarrow \mathbb{P}_w^1 \times \mathbb{P}_z^1$ is given by (8.1), and $\Psi: \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \times \mathbb{C}_{(c_1, c_2)}^2 \rightarrow \mathbb{C}_{(\kappa_1, \kappa_2)}^2$ is defined by

$$\begin{pmatrix} 1 \\ \kappa_1 \\ \kappa_2 \end{pmatrix} = J(u_\lambda, u_t)^{-1} \begin{pmatrix} 1 \\ c_1 \\ c_2 \end{pmatrix},$$

where $J(u_\lambda, u_t)^{-1}$ is given by the product $J^{-1} = C^{-1}B$. The explicit expression of the matrix B appears in Proposition 8.6, and the entries of C^{-1} are given in (8.7).

8.5. Final details on the map Φ . We conclude this section by filling in details to establish Theorem 6.1, Theorem 6.2, and Corollary 6.1.

Proof of Theorem 6.1 and Theorem 6.2. The coordinate system $\text{Con}_{\mathbb{P}}^{\bar{\mu}}(C, T) \dashrightarrow \mathbb{P}_z^1 \times \mathbb{P}_w^1 \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2$ and the map Φ are defined in terms of the canonical family $\nabla_0^\tau, \Theta_z, \Theta_w$. Therefore, in order to prove both theorems it is enough to establish that, when the underlying parabolic bundle belongs to $\text{Bun}^{\bar{\mu}}(C, T)^0$, the canonical family $\nabla_0^\tau, \Theta_z, \Theta_w$ is well defined (thus holomorphic), and that the parabolic Higgs bundles Θ_z, Θ_w are linearly independent. We will prove these properties for the canonical family $\nabla_0^\tau, \Theta_z, \Theta_w$. According to Proposition 8.7, we need to focus on the base change $J^{-1} = C^{-1}B$ which converts the original family $\nabla_0, \Theta_t, \Theta_\lambda$ used in [LS15] into the canonical one.

First, let us note the following. The map $\mathcal{S} \dashrightarrow \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ is obtained by contracting four of the sixteen (-1) -curves in \mathcal{S} . Therefore, the space of generic bundles $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$ is in one-to-one correspondence with an open subset of $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ (the complement of the remaining twelve rational curves we did not contract). This means that every generic bundle is represented in $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$. We remark that our genericity assumptions also exclude the lines $u_\lambda = \infty, u_t = \infty$ (these are mapped by Φ to $z = \lambda$ and $w = \infty$, respectively). Therefore the affine coordinates we are using in $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ suffice for all computations.

Next, we recall that the map $\Phi: \mathcal{S} \rightarrow \mathbb{P}_z^1 \times \mathbb{P}_w^1$ is everywhere well defined and holomorphic. Our formula for w in (8.1) has points of indeterminacy precisely at the four points obtained by contracting (-1) -curves in \mathcal{S} , but these points are not in $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$.

Finally, we analyze the poles of the entries of the matrices B and C^{-1} , as well as the zeros of their determinants. Let us start with B . The polynomial α in Proposition 8.6 vanishes at the lines $u_\lambda = 0, u_\lambda = 1, u_\lambda = \lambda$. These are among the rational curves excluded by $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$. They are mapped by Φ to the curves $z = 1, z = 0, z = \infty$, respectively. The polynomial β is exactly the polynomial P_Σ defining the ramification locus Σ in (8.5). Lastly, the polynomial γ coincides with P_Λ which defines the special line Λ in (8.6). Since the matrix is triangular we evidently have $\det B = B_{11}/\gamma\beta$. The denominators we have discussed, and the numerator is precisely $2P_\Pi^2$, where P_Π was given in (8.4). Since this is another rational curve excluded by $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$, we conclude that $\det B$ is never zero for a generic parabolic bundle. From the equations (8.7) it is straightforward that the poles of C^{-1} are given by $z = \lambda$ and the determinant only vanishes at $z = t$. Again, these lines are excluded by our genericity assumptions.

We conclude that over $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0 \setminus (\Sigma \cup \Lambda)$ the canonical family $\nabla_0^\tau, \Theta_z, \Theta_w$ is well defined and Θ_z, Θ_w linearly independent. As a consequence, the canonical family $\nabla_0^\tau, \Theta_z, \Theta_w$ enjoys the same properties over $\text{Bun}^{\bar{\mu}}(C, T)^0$. \square

Remark 8.6. All our constructions are well defined for parabolic bundles in $\text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)^0$, except that the base-change matrix J^{-1} has poles over Σ . This is because the Jacobian matrix $\frac{\partial(z,w)}{\partial(u_t, u_\lambda)}$ drops rank over Σ . Thus, the canonical basis introduced in [Definition 8.1](#) is not a true basis: it spans only a one-dimensional linear space when the underlying parabolic bundle belongs to Σ (see also [Remark 8.9](#)). Finally, this means that the coordinate system $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T) \dashrightarrow \mathbb{P}_z^1 \times \mathbb{P}_w^1 \times \mathbb{C}_{(\kappa_1, \kappa_2)}^2$ cannot be used to describe connections with such underlying parabolic bundles. The equivariant basis also has lower rank over Λ , thus for technical reasons we have avoided such curve, although the final base change J^{-1} doesn't have poles over Λ .

Proof of Corollary 6.1. As explained in [[LS15](#), Section 6], the parabolic Higgs bundles Θ_1, Θ_2 are chosen in such way that, under the natural identification, they correspond to the 1-forms du_t, du_λ in $T^* \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$. Because of this, the symplectic structure of the space $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ is given by the 2-form

$$\omega = dc_1 \wedge du_t + dc_2 \wedge du_\lambda,$$

which gives the canonical symplectic structure on the cotangent bundle.

The map $\Phi: \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}}(C, T)$ induces a map between the spaces of global sections

$$\Phi^*: \Gamma(T^* \text{Bun}^{\bar{\mu}}(C, T)) \longrightarrow \Gamma(T^* \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)). \quad (8.8)$$

We have used this map to find the parabolic Higgs bundles Θ_z, Θ_w that correspond to $\Phi^* dz, \Phi^* dw$, respectively. Since these are equivariant with respect to τ , they descend to the quotient by τ as parabolic Higgs bundles Θ_z, Θ_w , which correspond to dz, dw under the canonical identification. Therefore, as before, the symplectic form on $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ is given by

$$\omega_C = d\kappa_1 \wedge dz + d\kappa_2 \wedge dw.$$

Having explicit expressions for the map Φ , it is straightforward to verify that $\omega = \Phi^* \omega_C$, and so the map Φ is symplectic.

Alternatively, we can use the explicit expressions for the involution τ to verify that ω is invariant under τ , and so it descends to the quotient $\mathcal{U}/\tau = \mathcal{U}_C/\tau$. Therefore, it is possible to express the resulting 2-form in terms of the z, w, κ_1, κ_2 variables. This can be done by a straightforward, albeit tedious, computation. In this way, we recover the 2-form ω_C given above. \square

Remark 8.7. Fiberwise, the map $\Phi: \text{Higgs}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Higgs}^{\bar{\mu}}(C, T)$ coincides with (the inverse of) the map (8.8) on the corresponding fibers. This basically follows from the definition of the canonical Higgs bundles Θ_z, Θ_w . Similarly, it is not hard to show that the map τ acting on $\text{Higgs}^{\bar{\mu}}(\mathbb{P}^1, D)$ coincides with the natural action that $\tau: \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$ induces on the cotangent bundle $T^* \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$.

8.6. The singular locus on the space of connections. We have described $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ as a quotient of $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$ by the involution τ . As announced in [Section 6](#), the set of fixed points of τ is a codimension 2 subvariety, which causes the quotient to be singular at the image of the fixed-point locus. We shall first characterize the fixed-point set in $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(\mathbb{P}^1, D)$, and then describe the singularities of $\text{Con}_{\bar{\nu}}^{\bar{\mu}}(C, T)$ locally.

Proof of Theorem 6.3. We analyze the fixed points of τ in \mathcal{U} . This is equivalent to the fixed points of τ in \mathcal{U}_C . Any such fixed point must be a connection defined over a parabolic bundle that is fixed by τ , namely a parabolic bundle $(u_\lambda, u_t) \in \Sigma$. Those connections that are fixed by τ are the solutions to the linear inhomogeneous system

$$(\mathbb{T}(u_\lambda, \bar{u}_t) - \text{Id}) \begin{pmatrix} 1 \\ c_1 \\ c_2 \end{pmatrix} = 0. \quad (8.9)$$

Because of the particular shape of \mathbb{T} (cf. Proposition 8.5), the last column and the first row of the matrix on the left-hand side of (8.9) are zero. Therefore we are left with two inhomogeneous equations on c_1 only (and so c_2 is free to take any value). Recall that the central entry in $\mathbb{T}(u_\lambda, u_t)$ is given by \mathbb{T}_{11}/δ . We can easily verify that $(\mathbb{T}_{11}/\delta)|_\Sigma \equiv -1$. Indeed, a quick computation shows that

$$\frac{\mathbb{T}_{11}}{\delta} = -1 - \frac{P_\Sigma P_\Lambda}{\delta},$$

where $P_\Pi, P_\Sigma, P_\Lambda$ are defined in (8.4)–(8.6).

Remark 8.8. The above equations shows that $\Sigma \cup \Lambda$ is precisely the locus where the involution τ acts as $\Theta_1 \mapsto -\Theta_1 + k\Theta_2$, where k is a scalar. Because of this, the matrix $\frac{1}{2}(\text{Id} + \mathbb{T}(u_\lambda, \bar{u}_t))$ used to define the equivariant basis in Definition 8.1 drops rank and we are unable to define such basis above these parabolic bundles.

The determinant of the bottom-left 2×2 minor of the matrix in (8.9) vanishes at Σ , and the middle row imposes an equation

$$\mathbb{T}_{10}/\delta - 2c_1 = 0.$$

We conclude that the linear space of solutions is one-dimensional and defined by

$$c_1 = \mathbb{T}_{10}/2\delta = \nu \frac{P_\Pi P_\Lambda}{\delta}. \quad (8.10)$$

In Section 7, parabolic bundles in Σ are characterized as those for which the parabolic direction belongs to the curve $S_\Sigma \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$. A quick analysis shows that connections that moreover satisfy (8.10) are precisely those for which the second eigenvector of its residue over t (corresponding to the eigenvalue $-\nu$) also belongs to the curve S_Σ . Further analysis reveals that in this case the Riccati foliation is indeed tangent to S_Σ . After performing the transformation Φ we recover a connection on a bundle of the form $\mathbf{L} \oplus \mathbf{L}^{-1}$ for which each summand is invariant. Invariance by the elliptic involution (which exchanges \mathbf{L} and \mathbf{L}^{-1}) implies that the connection must be of the form

$$(\nabla, \mathbf{E}, \bar{\ell}) = (\zeta, \mathbf{L}, \ell) \oplus \iota^*(\zeta, \mathbf{L}, \ell).$$

□

Remark 8.9. We have shown that the equivariant basis fails to describe the universal family \mathcal{U} whenever the underlying parabolic bundle belongs to Σ . This is in fact true for *any* equivariant basis, including the canonical one. Indeed, since Σ defines the fixed points of τ , we have that any equivariant connection ∇ , parametrized by $u \in \text{Bun}^{\bar{\mu}}(\mathbb{P}^1, D)$, satisfies

$$\tau\nabla(u) = \nabla(\tau u) = \nabla(u), \quad \text{if } u \in \Sigma.$$

Therefore, if $u \in \Sigma$ equivariance implies invariance under τ . As shown above, for each fixed parabolic bundle $u \in \Sigma$, the space of τ -invariant connections is one-dimensional. Thus any equivariant basis drops rank over Σ .

Proof of Theorem 6.4. Let us choose a generic point $u^0 \in \Sigma$ (by generic we mean that $u_\lambda \neq 0, 1, \lambda, \infty$), and $\nabla(u^0)$ a connection fixed by τ . We will describe the space $\text{Con}_v^{\tilde{\mu}}(C, T)$ around $\Phi(\nabla(u^0))$.

Consider the curve $\Sigma \subset \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$, which is the set of fixed points of the involution τ . Up to a linear factor, the curve is given by hyperelliptic equation

$$P_{\text{II}} = \sqrt{\delta}, \tag{8.11}$$

where P_{II} and δ are given in (8.4) and Proposition 8.5, respectively. We propose the following *local* change of variables

$$(u_\lambda, u_t) \mapsto (u_\lambda, U), \quad \text{where} \quad U = \frac{P_{\text{II}} - \sqrt{\delta}}{P_{\text{II}} + \sqrt{\delta}}.$$

Let us assume that $u^0 \mapsto U^0$, and choose a branch of the square root such that Σ is given by $U = 0$ around the point U^0 . A straightforward computation shows that in these coordinates the involution τ is given by $(u_\lambda, U) \mapsto (u_\lambda, -U)$.

As discussed in Remark 8.8, the matrix T has a an eigenvalue equal to -1 whenever the underlying parabolic bundle belongs to Σ . Therefore, with respect to a suitably chosen (non-equivariant) basis $\tilde{\Theta}_1, \Theta_2$, the involution τ acts as $\tilde{\Theta}_1(u^0) \mapsto -\tilde{\Theta}_1(u^0)$, and Θ_2 is unchanged by τ .

Using the coordinates and the basis above, we arrive to local coordinates $(u_\lambda, U, \tilde{c}_1, c_2)$ in which, locally around $\nabla(u^0)$, the action of τ is given by

$$\tau: (u_\lambda, U, \tilde{c}_1, c_2) \mapsto (u_\lambda, -U, -\tilde{c}_1 + \dots, c_2),$$

where multiple dots denote higher order terms. Indeed, the action is not linear, but by the Bochner linearization theorem, we can find yet another set of local coordinates (y_1, y_2, y_3, y_4) in which the action is indeed linear. Focusing on the coordinates (y_2, y_3) where the action is non-trivial, we have that $\mathbb{C}_{(y_2, y_3)}^2 / \tau$ defines a quadratic conic singularity, isomorphic to the cone in \mathbb{C}^3 given by $x_1^2 = x_2 x_3$. \square

9. THE APPARENT MAP

In order to study the *apparent map* App_C introduced in Definition 6.2, we begin with connections on \mathbb{P}^1 and switch from the birational model $\text{Con}_v^{\tilde{\mu}}(\mathbb{P}^1, D) \cong \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1 \times \mathbb{C}_{(c_1, c_2)}^2$, to the model $\text{Con}_v^{\tilde{\mu}}(\mathbb{P}^1, D) \cong \mathbb{P}_a^2 \times \mathbb{P}_b^2$ defined by the map (5.1). This is studied in detail in [LS15]. The first factor, \mathbb{P}_a^2 defines the image of the apparent map. Indeed, the tangencies of a generic connection $\nabla(a, b)$ with the section σ_∞ are precisely at the roots of the polynomial $\text{App}_\infty(a, b) = a_2 x^2 + a_1 x + a_0$. The second factor, \mathbb{P}_b^2 defines the underlying parabolic bundle. Explicit formulas to go from one coordinate system to the other are given in [LS15, Section 6], and so we omit them here.

Recall from Section 4 that the Del Pezzo surface \mathcal{S} can be identified with the blow-up of \mathbb{P}_b^2 at five points, which we call $D_0, D_1, D_\lambda, D_\infty, D_t$. As shown in [Fer16, Section 6], the

involution τ is the lift of the de Jonquières automorphism of \mathbb{P}_b^2 preserving the pencil of lines through D_t and the pencil of conics through $D_0, D_1, D_\lambda, D_\infty$. The following five rational curves in \mathbb{P}_b^2 are important for the upcoming discussion: the conic Π through all five points D_i , and the lines L_{it} passing through the points D_i and D_t , for $i = 0, 1, \lambda, \infty$. These become (-1) -curves in \mathcal{S} , and so they are excluded from $\text{Con}_{\bar{b}}^{\bar{\mu}}(\mathbb{P}^1, D)^0$ (cf. [Definition 4.1](#)). Some of these rational curves have already appeared in previous sections. They correspond to the curves defined by P_Π in [\(8.4\)](#), and the lines $u_\lambda = 0, 1, \infty$. Only the line $L_{\lambda,t}$ is absent from the $\mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ model. In fact, the birational isomorphism $\mathbb{P}_b^2 \dashrightarrow \mathbb{P}_{u_\lambda}^1 \times \mathbb{P}_{u_t}^1$ is obtained by blowing-up D_λ and D_t , and contracting the line $L_{\lambda t}$ through them.

Remark 9.1. As always, the involution τ plays a crucial role in the passage from connections over \mathbb{P}^1 to connections over C . In these coordinates, the involution acts as $\tau: (a, b) \mapsto (s, \bar{b})$. The action on the parabolic bundles $b \mapsto \bar{b}$ is the de Jonquières automorphism of \mathbb{P}_b^2 discussed above. Below we seek to understand the correspondence $(a, b) \mapsto s$. This is given by a matrix M_b , whose entries are parametrized by b . This matrix will be the main object of study in this section. Note that since a is given by the apparent map App_∞ , we have that $s = \text{App}_\infty \circ \tau$.

We now analyze the map $\text{App}_\infty \times (\text{App}_\infty \circ \tau): \text{Con}_{\bar{b}}^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$, which was introduced in [Definition 6.2](#). Recall we have defined $\text{App}_\tau = \text{App}_\infty \circ \tau$. Let us introduce homogeneous coordinates $s = [s_0 : s_1 : s_2]$ on \mathbb{P}^2 so that App_∞ takes values on \mathbb{P}_a^2 , and App_τ takes values on \mathbb{P}_s^2 . Under these coordinate systems the former map is nothing but

$$\text{pr}_1 \times \text{App}_\tau: \mathbb{P}_a^2 \times \mathbb{P}_b^2 \longrightarrow \mathbb{P}_a^2 \times \mathbb{P}_s^2, \quad (9.1)$$

where pr_1 denotes projection onto the first factor \mathbb{P}_a^2 .

We remark that, if we fix b a generic bundle, the map $\text{App}_\tau(_, b): \mathbb{P}_a^2 \rightarrow \mathbb{P}_s^2$ is holomorphic and invertible. Therefore it defines an element of $\text{PGL}(3, \mathbb{C})$. It is a straight forward computation to translate the formula for App_τ mentioned in [Section 8](#) to these new coordinates. Once this is done, a direct inspection yields the following.

Proposition 9.1. *The map $\text{App}_\tau: \mathbb{P}_a^2 \times \mathbb{P}_b^2 \rightarrow \mathbb{P}_s^2$ is defined by a 3×3 matrix M_b via*

$$(a, b) \longmapsto M_b \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

The entries of M_b are homogeneous polynomials on b of degree four. Moreover, its determinant only vanishes along the curves Π and L_{it} , for $i = 0, 1, \lambda, \infty$ (each divisor with multiplicity two).

Remark 9.2. The above proposition implies that the map $\text{App}_\infty \times (\text{App}_\infty \circ \tau)$ is everywhere well defined on $\text{Con}_{\bar{b}}^{\bar{\mu}}(\mathbb{P}^1, D)^0$.

That [\(9.1\)](#) is dominant is straightforward: it is a map between irreducible spaces of the same dimension, and we can check at a generic point that the derivative has maximal rank. The fact that the determinant of M_b factors as a product of rational curves makes it easy to analyze the behavior of this matrix along such divisor (since we can parametrize each branch). Below we describe the kernel of M_b at every point where the determinant vanishes. These are linear subspaces of \mathbb{C}_a^3 which are naturally identified with subsets of $\mathbb{P}_a^2 = \mathbb{P}(\mathbb{C}_a^3)$.

- (1) At a generic point of the conic Π the matrix M_b has rank 2 and its kernel is a fixed point $\tilde{D}_t \in \mathbb{P}_a^2$.
- (2) At a generic point of the line $L_{i,t}$, for $i = 0, 1, \lambda, \infty$, the matrix M_b has rank 1 with a one-dimensional kernel that can be identified with a fixed point $\tilde{D}_i \in \mathbb{P}_a^2$.
- (3) At the point D_i , for $i = 0, 1, \lambda, \infty$, the matrix M_b has rank 1 with a two-dimensional kernel which we identify with the line L_{it} through \tilde{D}_t and \tilde{D}_i in \mathbb{P}_a^2 .
- (4) At the point D_t the matrix M_b vanishes identically.

The above description implies that for a generic choice of $a \in \mathbb{P}_a^2$ (namely distinct from the points \tilde{D}_i and not on any line $\tilde{\Pi}_{it}$), the only way in which $a \in \mathbb{P}_a^2$ could be in the kernel of M_b is if $b = D_t$. In particular, fixing a , the map $\text{App}_\tau(a, _): \mathbb{P}_b^2 \rightarrow \mathbb{P}_s^2$ is a rational map of degree four with a single point of indeterminacy at D_t . A simple analysis shows that the generic fiber of this map consists of 12 points. This has the following consequence.

Theorem 9.1. *The map $\text{App}_C: \text{Con}_\nu^{\bar{\mu}}(C, T) \dashrightarrow \text{Sym}^2(\mathbb{P}^2)$ is a rational dominant map whose generic fiber consists of exactly 12 points. This map is everywhere well defined over the space $\text{Con}_\nu^{\bar{\mu}}(C, T)^0$ of generic connections.*

Proof. This theorem readily follows from the fact that the map $\mathbb{P}_a^2 \times \mathbb{P}_b^2 \rightarrow \mathbb{P}_a^2 \times \mathbb{P}_s^2$ in (9.1) is dominant and generically 12 : 1. Consider the following diagram

$$\begin{array}{ccc}
 \text{Con}_\nu^{\bar{\mu}}(\mathbb{P}^1, D) & \xrightarrow{1:1} & \mathbb{P}_a^2 \times \mathbb{P}_b^2 & \xrightarrow{12:1} & \mathbb{P}_a^2 \times \mathbb{P}_s^2 \\
 \downarrow 2:1 & & & & \downarrow 2:1 \\
 \text{Con}_\nu^{\bar{\mu}}(C, T) & \dashrightarrow & & \dashrightarrow & \text{Sym}^2(\mathbb{P}^2)
 \end{array} \tag{9.2}$$

In principle, the bottom arrow, which represents App_C , *should* be generically 12 : 1. There is one place where we need to be careful: the first two maps on the top row are not surjective. This could decrease the cardinality of the fibers once we trace preimages from right to left.

Take a point $[(a, s)] \in \text{Sym}^2(\mathbb{P}^2)$ in the image of App_C . Then either (a, s) or (s, a) is in the image of the map $\text{App}_\infty \times \text{App}_\tau$. However, the involution τ acts in such a way that $\text{App}_\infty \times \text{App}_\tau(\nabla) = (a, s)$ then $\text{App}_\infty \times \text{App}_\tau(\tau\nabla) = (s, a)$. Hence the image of this map is invariant under permuting the factors in $\mathbb{P}_a^2 \times \mathbb{P}_s^2$. Thus both (a, s) and (s, a) are in the image of such map. From the above discussion, it follows that each of these two points has 12 preimages in $\mathbb{P}_a^2 \times \mathbb{P}_b^2$. It is shown in [LS15, Theorem 1.1] that the image of the map $\text{Con}_\nu^{\bar{\mu}}(\mathbb{P}^1, D) \rightarrow \mathbb{P}_a^2 \times \mathbb{P}_b^2$ coincides with the complement of the incidence variety defined by $a_0b_0 + a_1b_1 + a_2b_2 = 0$. By computing a particular example, we are able to confirm that generically none of the 24 points coming from our original $[(a, s)]$ lie on the incidence variety. Therefore the composition of the horizontal arrows with the vertical arrow on the right gives a map which is invariant under τ and generically 24 : 1. Such map descends to the quotient $\text{Con}_\nu^{\bar{\mu}}(\mathbb{P}^1, D)/\tau \cong \text{Con}_\nu^{\bar{\mu}}(C, T)$. The resulting map, which by definition is App_C , is thus generically 12 : 1. \square

We conclude by showing that the map $\text{App}_C \times \text{Bun}$ is generically injective. This means that, generically, a connection is completely defined by its underlying parabolic bundle and the image of the apparent map App_C . Note however that the domain of this map is four-dimensional while the target space has dimension six.

Theorem 9.2. *The map*

$$\text{App}_C \times \text{Bun}: \text{Con}_V^{\bar{\mu}}(C, T) \dashrightarrow \text{Sym}^2(\mathbb{P}^2) \times \mathbb{P}_z^1 \times \mathbb{P}_w^1$$

is a generically injective.

Proof. As usual, let us start by analyzing the corresponding map for connections on \mathbb{P}^1 . Let $A: \mathbb{P}_a^2 \times \mathbb{P}_b^2 \rightarrow \text{Sym}^2(\mathbb{P}^2)$ be the map obtained by the composition of the last two horizontal arrows in (9.2). Let us denote by $B: \mathbb{P}_a^2 \times \mathbb{P}_b^2 \rightarrow \mathbb{P}_z^1 \times \mathbb{P}_w^1$ the map $B = \Phi \circ \text{Bun}$. Consider the map $A \times B: \mathbb{P}_a^2 \times \mathbb{P}_b^2 \rightarrow \text{Sym}^2(\mathbb{P}^2) \times \mathbb{P}_z^1 \times \mathbb{P}_w^1$. In order to prove the theorem, we are going to show that the generic fibers of $A \times B$ consist of two points that are in involution with respect to τ . Therefore, in the quotient $\text{Con}_V^{\bar{\mu}}(\mathbb{P}^1, D)/\tau \cong \text{Con}_V^{\bar{\mu}}(C, T)$, the induced map App_C will be generically injective.

Let ∇ be a generic connection defined by $(a, b) \in \mathbb{P}_a^2 \times \mathbb{P}_b^2$. Suppose $A(a, b) = [(a, s)] \in \text{Sym}^2(\mathbb{P}^2)$. Let $\Phi(b) = (z, w)$, and denote $\bar{b} = \tau(b)$. We have the following straightforward constraints for a point (a', b') to be on the fiber of $A \times B$ over the point $([(a, s)], (z, w))$. First, b' must equal either b or \bar{b} , since the map Φ is $2 : 1$ and the fiber over (z, w) is precisely $\{b, \bar{b}\}$. For the second, let $s' = \text{App}_\tau(a', b')$, such that $A(a', b') = [(a', s')]$. Since $A(a', b') = A(a, b)$, we must have that either $a' = a$ and $s' = s$, or $a' = s$ and $s' = a$.

At this point, we have shown that the fibers of $A \times B$ consist of at most four points (a', b') that satisfy $a' \in \{a, s\}$, $b' \in \{b, \bar{b}\}$. By design, this map is invariant under τ . Thus, points that are in involution with respect to τ belong to the same fiber. According to Remark 8.4, $\tau(a, b) = (s, \bar{b})$. Therefore both (a, b) and (s, \bar{b}) belong to the fiber. We now need to show that, generically, the points (a, \bar{b}) and (s, b) do not belong to the fiber.

Assume that (a, \bar{b}) belongs to the same fiber as (a, b) , namely, $A(a, \bar{b}) = A(a, b)$. Let us consider the matrix M_b that appears on Proposition 9.1 as an element of $\text{PGL}(3, \mathbb{C})$. Since $A(a, b) = [(a, M_b(a))]$, equality $A(a, \bar{b}) = A(a, b)$ means that $M_b(a) = M_{\bar{b}}(a)$. Note that because τ is an involution, we must have that the composition $M_{\bar{b}}M_b$ is the identity (cf. Remark 8.5). Thus, applying M_b on the left, we have that

$$M_b^2(a) = M_b M_{\bar{b}}(a) = a. \tag{9.3}$$

This implies that a , viewed as a line on \mathbb{C}^3 , is an invariant linear subspace for the matrix M_b^2 . This imposes non-trivial polynomial conditions on the space $\mathbb{P}_a^2 \times \mathbb{P}_b^2$, which are only satisfied in a proper subvariety of $\mathbb{P}_a^2 \times \mathbb{P}_b^2$. The case $A(s, b) = A(a, b)$ is treated in the same way as above (it imposes the same conditions).

We conclude that on a Zariski open subset of $\mathbb{P}_a^2 \times \mathbb{P}_b^2$, the fibers of the map $A \times B$ consists of two points which are permuted by τ . This implies that $\text{App}_C \times \text{Bun}$ is generically injective. \square

Remark 9.3. The Zariski closure of the image of $\text{App}_C \times \text{Bun}$ is a codimension 2 subvariety $X \subset \text{Sym}^2(\mathbb{P}^2) \times \mathbb{P}_z^1 \times \mathbb{P}_w^1$. Unfortunately, we were unable to compute the polynomial equations that define the variety X .

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