



HAL
open science

PERCOLATION OF THE EXCURSION SETS OF PLANAR SYMMETRIC SHOT NOISE FIELDS

Raphaël Lachièze-Rey, Stephen Muirhead

► **To cite this version:**

Raphaël Lachièze-Rey, Stephen Muirhead. PERCOLATION OF THE EXCURSION SETS OF PLANAR SYMMETRIC SHOT NOISE FIELDS. 2019. hal-02337951v1

HAL Id: hal-02337951

<https://hal.science/hal-02337951v1>

Preprint submitted on 29 Oct 2019 (v1), last revised 24 Jan 2022 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PERCOLATION OF THE EXCURSION SETS OF PLANAR SYMMETRIC SHOT NOISE FIELDS

RAPHAEL LACHIEZE-REY¹ AND STEPHEN MUIRHEAD²

ABSTRACT. We prove the existence of a sharp phase transition in the global connectivity of the excursion sets of planar symmetric shot noise fields, with the zero level critical. Our results hold for a wide class of mark distributions, including the Gaussian, Uniform, and Rademacher cases. Our main assumption on the shot noise kernel is that it is positive, symmetric, and has sufficient tail decay (depending on the mark distribution); for example, for Gaussian and Uniform marks we require polynomial decay with exponent at least three, whereas for Rademacher marks we require super-polynomial decay.

Keywords: percolation, excursions, shot noise fields, sharp phase transition

1. INTRODUCTION

Let $g(x) \in L^1(\mathbb{R}^2)$ be a continuous function, and let μ be a distribution on \mathbb{R} with finite mean. The *planar shot noise field* with *kernel* g and *mark distribution* μ is defined as

$$(1.1) \quad f(x) = \sum_{i \in \mathcal{P}} Y_i^\mu g(x - i)$$

where \mathcal{P} is a Poisson point process on \mathbb{R}^2 with unit intensity, and $\{Y_i^\mu\}$ are i.i.d. random variables drawn from the distribution μ ; the sum in (1.1) is well-defined almost surely by the integrability of g and μ . For example, one could take $g(x) = (1 + |x|)^{-\alpha}$, $\alpha > 2$, and the mark μ to be a normal distribution (‘Gaussian shot noise’), a uniform distribution (‘Uniform shot noise’), or the Rademacher distribution $1/2(\delta_{-1} + \delta_1)$ (‘Rademacher shot noise’). If μ is a symmetric distribution then we say that f is a *planar symmetric shot noise field*.

Equation (1.1) is the Euclidean separable form of a general abstract class of infinitely divisible fields obtained by convoluting a kernel over a Poisson random measure. Shot noise fields were introduced by Campbell [Cam09] to model thermionic noise, and since then have been used, under several different names, in diverse fields such as image analysis [BD16b, BD16a] and telecommunications networks [BB10, BB15]. In the latter case, the points of the underlying Poisson process can be seen as emitters of an electromagnetic signal, while the field itself represents the total signal at every location of the space; see [BB10] for a detailed mathematical study of theory and applications of such models. At high frequency, i.e. when the density of points is high compared to the scale of the kernel g , shot noise fields are also a good approximation of Gaussian fields with the same covariance structure.

In this paper we are interested in the global connectivity of the (upper-)excursion sets

$$\mathcal{E}_\ell = \{x \in \mathbb{R}^2 : f(x) \geq -\ell\}, \quad \ell \in \mathbb{R},$$

of planar symmetric shot noise fields. In the analysis of telecommunications networks, the percolation properties of \mathcal{E}_ℓ are of high importance in determining the global connectivity of the network [BB15]. More generally, the geometric properties of \mathcal{E}_ℓ have been the focus of many other studies [BD16a, LR19, BST12].

By analogy with other planar percolation models (e.g. Bernoulli edge percolation on the square lattice [Har60, Kes80], level set percolation of planar stationary Gaussian fields

¹UNIVERSITÉ PARIS DESCARTES / UNIVERSITÉ DE PARIS

²SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY UNIVERSITY OF LONDON

E-mail addresses: raphael.lachieze-rey@parisdescartes.fr, s.muirhead@qmul.ac.uk.

Date: October 29, 2019.

The second author would like to thank Michael McAuley and Hugo Vanneville for helpful discussions.

[Ale96, BG17, RV19a, MV19]), it is natural to expect that the global connectivity of the excursion sets of a planar symmetric shot noise field undergoes a sharp phase transition at the zero level, from a sub-critical phase in which all the excursion set components are bounded, to a super-critical phase in which the excursion set has a unique unbounded connected component. Our main result establishes the existence of such a phase transition at the zero level under general conditions on the kernel g and mark distribution μ .

Let us introduce the necessary assumptions on the model; in fact, we will distinguish two alternative conditions, which trade off greater generality of the mark distribution for more restrictions on the kernel. Recall that, for a continuous random variable, the *Mills ratio* is defined as the ratio of the survival function to the density. Similarly, for a symmetric mark distribution μ with an absolutely continuous component $\mu_{a.c.}$, we define

$$c_{\text{Mills}}(\mu) = \operatorname{ess\,sup}_{x \geq 0} \frac{\bar{F}_\mu(x)}{f_{\mu_{a.c.}}(x)}$$

where $\bar{F}_\mu(x) = \int_{s \geq x} d\mu(s)$, $f_{\mu_{a.c.}}(x)$ is the density function of the absolutely continuous component of μ , and where we use the conventions $1/0 = \infty$ and $0/0 = 0$. In particular, these conventions ensure that $c_{\text{Mills}}(\mu)$ is finite whenever μ has an absolutely continuous component with density bounded away from zero on its support. Moreover, for a set $D \subseteq \mathbb{R}$, we define

$$c_{\text{Mills}}(\mu; D) = \operatorname{ess\,sup}_{x \geq 0, x \in D} \frac{\bar{F}_\mu(x)}{f_{\mu_{a.c.}}(x)}$$

clearly $c_{\text{Mills}}(\mu; D) \leq c_{\text{Mills}}(\mu)$ for all $D \subseteq \mathbb{R}$.

Our two alternative conditions are as follows:

Condition 1.1 ('Bounded Mills ratio' case).

- (Mark) The mark distribution μ is symmetric and has bounded Mills ratio, i.e., there exists a $c > 0$ such that $c_{\text{Mills}}(\mu) < c$.
- (Kernel) The kernel g is C^2 -smooth and symmetric with respect to reflection in the x -axis and rotation by $\pi/2$. Moreover, there exist $\alpha > 3$ and $c > 0$ such that, for every multi-index k such that $|k| \leq 2$,

$$|\partial^k g(x)| < c(1 + |x|)^{-\alpha - |k|}.$$

Condition 1.2 ('Bounded Mills ratio in the tails' case).

- (Mark) The mark distribution μ is symmetric and has bounded Mills ratio in the tails, i.e., there exists a $c > 0$ and a compact set $D \subset \mathbb{R}$ such that $c_{\text{Mills}}(\mu; D^c) < c$.
- (Kernel) The kernel g is C^2 -smooth and symmetric with respect to reflection in the x -axis and rotation by $\pi/2$. Moreover, for every multi-index k such that $|k| \leq 2$, as $|x| \rightarrow \infty$,

$$\frac{\log |\partial^k g(x)|}{\log |x|} \rightarrow -\infty.$$

Remark 1.3. Examples of mark distributions satisfying Condition 1.1 are centred normal distributions, centred uniform distributions, or more generally, any symmetric distribution whose density function is C^1 -smooth and log-concave on its support (see [BB05, Corollary 2]). Examples of mark distributions satisfying Condition 1.2 include all of the above, and also any symmetric distribution with compact support (in particular, the Radermacher distribution), and any symmetric distribution whose tail has a C^1 -smooth density that is asymptotically equivalent to a log-concave function.

Notice that, although the assumption on the mark distribution in Condition 1.1 is stronger than that in Condition 1.2, the assumption on the kernel decay is much weaker, requiring only polynomial decay with exponent three rather than super-polynomial decay.

Conditions 1.1 and 1.2 imply certain regularity properties of the shot noise field f . In particular, since g is C^2 -smooth, f can be viewed as an (almost surely) absolutely convergent series in the Sobolev space $W^{2,1}(\mathbb{R}^2)$, which ensures that the sample paths of f are almost surely almost everywhere twice differentiable. Then, since $x \mapsto \sup_{y: |y-x| \leq 1} |\partial^k g(y)|$ is integrable for $|k| \leq 2$, f is actually C^2 -smooth [BB09, Proposition 2.2.3].

On the other hand, these conditions do not on their own guarantee that f and its derivatives have bounded density. We shall impose this as an additional condition:

Condition 1.4 (Bounded density). *The random vector $(f(0), \nabla f(0))$ has a continuous bounded density.*

While the above condition will be sufficient in the case of bounded Mills ratio (Condition 1.1), in the case that the Mills ratio is only bounded in the tails (Condition 1.2) we shall require a slightly stronger version.

For $\lambda > 0$, define $f^{(\lambda)}$ to be the shot noise field obtained by replacing the kernel g in the definition of f by the kernel $g(\lambda \cdot)$ (this is equivalent in distribution, up to a spatial rescaling by $x \mapsto x/\lambda$, to modifying the unit intensity Poisson point process \mathcal{P} that defines f to have intensity λ).

Condition 1.5 (Bounded density, strong version). *There exists a $\lambda \in (0, 1)$ such that the random vector $(f^{(\lambda)}(0), \nabla f^{(\lambda)}(0))$ has a continuous bounded density.*

Remark 1.6. We give examples of shot noise fields satisfying Conditions 1.4 and 1.5 in Section A.2; an obvious necessary condition is that g has unbounded support (see Remark 1.9).

Let us make some comments regarding the relationship between Conditions 1.4 and 1.5. By the Riemann-Lebesgue lemma, the density of $(f^{(\lambda)}(0), \nabla f^{(\lambda)}(0))$ being continuous and bounded is implied by the integrability of its characteristic function

$$\varphi^{(\lambda)}(u, v) := \mathbb{E}(\exp(i(u f^{(\lambda)}(0) + \langle v, \nabla f^{(\lambda)}(0) \rangle))), \quad u \in \mathbb{R}, v \in \mathbb{R}^2.$$

In fact, since the mark distribution μ is symmetric and so $\varphi^{(\lambda)}$ takes values in $[0, 1]$, these are actually equivalent. Moreover, by the Poisson structure of the shot noise, the characteristic function can be expressed as

$$\varphi^{(\lambda)}(u, v) = \exp\left(\lambda \int_{\mathbb{R} \times \mathbb{R}^2} (\exp(im[ug(x) + \langle v, \nabla g(x) \rangle]) - 1) \mu(dm) dx\right) = \varphi^{(1)}(u, v)^\lambda,$$

which implies that if $\varphi^{(\lambda_0)}$ is integrable for some $\lambda_0 > 0$, then it is integrable for all $\lambda \geq \lambda_0$. Hence, if Condition 1.5 is satisfied for some $\lambda_0 \in (0, 1)$, the density of $(f^{(\lambda)}(0), \nabla f^{(\lambda)}(0))$ is uniformly bounded for all $\lambda \geq \lambda_0$, and in particular Condition 1.4 is also satisfied.

Note that the above argument actually shows that the maximum of the density of $(f^{(\lambda)}(0), \nabla f^{(\lambda)}(0))$, which is proportional to $\|\varphi^{(\lambda)}\|_{L^1}$, is non-increasing in λ . Another way to see this is by viewing $f^{(\lambda)}$, $\lambda < 1$, as (a spatial rescaling of) the shot noise generated by a λ -thinning of the unit intensity Poisson point process \mathcal{P} (since the maximum density of a random variable is non-increasing under summation with a second independent random variable). We shall use this observation later (see the proof of Lemma 2.2).

We are now ready to state our main result, establishing the existence of a phase transition in the connectivity of the excursion sets at the level $\ell = 0$:

Theorem 1.7. *Suppose that a planar symmetric shot noise satisfies either Conditions 1.1 and 1.4 or Conditions 1.2 and 1.5. Then:*

- (1) *If $\ell \leq 0$, the excursion set \mathcal{E}_ℓ contains only bounded connected components almost surely.*
- (2) *If $\ell > 0$, the excursion set \mathcal{E}_ℓ contains a unique infinite connected component almost surely.*

An immediate consequence is that, for all $\ell \in \mathbb{R}$, the ℓ -level lines (i.e. the connected components of $\{x \in \mathbb{R}^2 : f(x) = \ell\}$) are almost surely bounded.

We furthermore have the following quantitative estimates for the connectivity of excursion sets at the zero level (more formal statements are given in Section 3):

- (1) *For all $\rho > 0$ there are $0 < c_- \leq c_+ < \infty$ such that, for all $R \geq 1$,*

$$c_- \leq \mathbb{P}(\mathcal{E}_0 \text{ crosses horizontally the rectangle } [0, R] \times [0, \rho R]) \leq c_+.$$
- (2) *There are $c, c_{\text{Arm}} > 0$ such that, for all $1 \leq r \leq R$,*

$$\mathbb{P}(\mathcal{E}_0 \text{ connects } \partial B(0, r) \text{ to } \partial B(0, R)) \leq c \left(\frac{r}{R}\right)^{c_{\text{Arm}}}.$$

Remark 1.8. Although we provide a self-contained proof, the fact that \mathcal{E}_ℓ has only bounded connected components at levels $\ell \leq 0$ could, with a bit of work, be deduced from general results of Alexander [Ale96]. The fact that \mathcal{E}_ℓ has unbounded connected components at sufficiently high levels $\ell \gg 1$ can be deduced using the arguments of [MS83, BM17] in the slightly different setting of non-negative shot noise fields (our results are for symmetric shot noise fields). The main novelty of Theorem 1.7 is (i) the statement that \mathcal{E}_ℓ has an infinite connected component for every $\ell > 0$, and (ii) the quantitative estimates on \mathcal{E}_0 .

Remark 1.9. As we mentioned, Conditions 1.4 and 1.5 require that g have unbounded support. In fact, it is known that Theorem 1.7, and more precisely its statements about \mathcal{E}_0 , can fail in general if this assumption is removed. Indeed, the theory of Poisson Boolean percolation (see [BR06, Chapter 8] or [MR96, Chapter 3]) demonstrates that \mathcal{E}_0 contains an unbounded connected component almost surely whenever the support of g is contained within a ball of sufficiently small radius. On the other hand, it is plausible that \mathcal{E}_0 does not contain an unbounded connected component as soon as the support of g contains a sufficiently large compact set.

Remark 1.10. Although Condition 1.2 states that the kernel g decays super-polynomially, our proof of Theorem 1.7 in the case of Conditions 1.2 and 1.5 actually only requires polynomial decay with some very large exponent $\alpha > 0$ (see Remark 5.3). Rather than attempt to quantify this exponent, we have stated the condition as super-polynomial decay for simplicity.

Remark 1.11. Let us give some examples of shot noise fields to which Theorem 1.7 applies. First, we could take μ as either a centred Gaussian or uniform distribution, and

$$g(x) = (1 + |x|)^{-\alpha}, \quad \alpha > 3,$$

since Conditions 1.1 and 1.4 apply to such fields. Second, we could take μ as the Rademacher distribution, and

$$g(x) = \exp(-(1 + |x|^2)^{\alpha/2}), \quad \alpha \in (0, 1),$$

since Conditions 1.2 and 1.5 apply to such fields. See Section A.2 for the proof that Conditions 1.4 and 1.5 are satisfied in these cases.

The conclusion of Theorem 1.7 is a qualitative description of the phase transition, and in fact our proof gives detailed information on the quantitative *sharpness* of the transition, analogous to in Bernoulli percolation [Gri99, BR06]. In particular, in the super-critical phase $\ell > 0$ we prove that the excursion set \mathcal{E}_ℓ crosses large domains outside an event of probability that is exponentially small in the scale of the domain (see Theorem 4.1). Finally, as well as the ‘box-crossing’ estimates and polynomial ‘one-arm’ decay we prove at the zero level (see Theorems 3.9 and 3.10), our techniques can also be used to establish that the ‘near-critical window’ (see, e.g., [MV19, Theorem 1.15]) is polynomially small in the scale, as it is for Bernoulli percolation, although we have chosen not to state this result precisely.

Our proof of Theorem 1.7 is largely inspired by [MV19] (which proved an analogous result for planar Gaussian fields, following [BG17, RV19a]). Nevertheless, shot noise fields marginals are essentially accessible through their characteristic function, and in general no expression is available for the density. As a result, a lot of Gaussian techniques fail in the Poisson shot noise set up, and there are some notable differences in the shot noise case compared to the Gaussian case. Most significantly, our emphasis on the role played by the bounded Mills ratio is entirely novel (although, of course, the Gaussian distribution has bounded Mills ratio), and deducing the phase transition at this level of generality requires more care (see Section 4). Our extension of the proof to cover marks with bounded Mills ratio in the tails is also new (see Section 5).

2. PRELIMINARY RESULTS

In this section we collect preliminary results on the shot noise field f , and various perturbations of this field.

2.1. Exponential decay of the marks. We first note that our assumption on the mark distribution μ implies that it has exponentially decaying tails. This will be useful for deriving large deviation bounds.

Lemma 2.1. *Suppose there exists $c > 0$ and a compact $D \subset \mathbb{R}$ such that $c_{\text{Mills}}(\mu; D^c) < c$. Then there exists a $c_1 > 0$ such that*

$$\bar{F}_\mu(x) < c_1 e^{-cx}, \quad x \geq 0.$$

Proof. If μ is compactly supported the conclusion of the lemma is immediate. So suppose that $\bar{F}_\mu(x) > 0$ for all $x \geq 0$. The assumption of the lemma implies the existence of $x_0, c > 0$ such that

$$(2.1) \quad \frac{f_{\mu_{a.c.}}(x)}{\bar{F}_\mu(x)} \geq c, \quad x \geq x_0.$$

Since $\log \bar{F}_\mu(x)$ is non-increasing, and since the left-hand side of (2.1) equals $-(\log \bar{F}_\mu(x))'$ whenever this derivative is defined, for $x \geq x_0$,

$$\log \bar{F}_\mu(x) \leq \log \bar{F}_\mu(x_0) - \int_{x_0}^x \frac{f_{\mu_{a.c.}}(s)}{\bar{F}_\mu(s)} ds \leq \log \bar{F}_\mu(x_0) - c(x - x_0).$$

Hence, for $x \geq x_0$,

$$\bar{F}_\mu(x) \leq \bar{F}_\mu(x_0) e^{-cx_0} e^{-cx},$$

which implies the result. \square

2.2. Perturbations of the shot noise field. We next introduce various perturbations of the shot noise field f based on (i) smoothing of the mark distribution, (ii) truncation of the kernel, (iii) spatial discretisation, and (iv) adding a constant to the mark.

2.2.1. Smoothing of the mark distribution. In the case of marks with bounded Mills ratio in the tails (Condition 1.2), a certain ‘smoothing’ operation on the mark distribution will enable us to approximate f with a version having marks with globally bounded Mills ratio.

Lemma 2.2. *Suppose Condition 1.2 holds. Then there exists $\eta_0, c > 0$ such that, for all $\eta \in (0, \eta_0]$, there exists a symmetric distribution μ_η such that*

- (1) $c_{\text{Mills}}(\mu_\eta) \leq 1/\eta$;
- (2) $d_{TV}(\mu, \mu_\eta) \leq c\eta$;
- (3) $\bar{F}_{\mu_\eta}(x) \leq ce^{-x/c}$, for all $x \geq 0$.

Suppose in addition that Condition 1.5 holds. Then η_0, c and $(\mu_\eta)_{\eta \in (0, \eta_0]}$ can be chosen so that, in addition,

- (4) *The random vector $(f_\eta(0), \nabla f_\eta(0))$ has density bounded by c , where*

$$(2.2) \quad f_\eta(x) = \sum_{y \in \mathcal{P}} Y_i^{\mu_\eta} g(x - y)$$

and $\{Y_i^{\mu_\eta}\}$ are i.i.d. random variables drawn from μ_η .

Proof. Since Condition 1.2 holds, we can find $c_1, c_2 > 0$ such that $c_{\text{Mills}}(\mu; [-c_1, c_1]^c) < c_2$. We temporarily set

$$\eta_0 = \min\{1/c_1, 1/c_2\},$$

and, for each $\eta \in (0, \eta_0]$, define the symmetric probability measure

$$\mu_\eta(dx) = \mu(dx)(1 - c_1\eta) + (\eta/2)\mathbb{1}_{[-c_1, c_1]}dx,$$

i.e. we first rescale μ by the constant factor $(1 - c_1\eta) \in [0, 1)$, and then we reassign the excess probability mass $c_1\eta$ evenly within $[-c_1, c_1]$.

The first three points of the lemma are immediate from the construction of μ_η . More precisely, we have

$$c_{\text{Mills}}(\mu_\eta; [-c_1, c_1]^c) = c_{\text{Mills}}(\mu; [-c_1, c_1]^c) < c_2,$$

and

$$c_{\text{Mills}}(\mu_\eta; [-c_1, c_1]) \leq \frac{\bar{F}_\eta(0)}{\eta/2} = 1/\eta,$$

which, since $\eta \leq \eta_0 \leq 1/c_2$, implies that $c_{\text{Mills}}(\mu_\eta) \leq \max\{c_2, 1/\eta\} = 1/\eta$. Moreover, since μ_η has been constructed from μ by moving $c_1\eta$ amount of probability mass,

$$d_{\text{TV}}(\mu, \mu_\eta) \leq c_1\eta.$$

Finally, $\bar{F}_{\mu_\eta}(x) \leq \bar{F}_\mu(x)$ for all $x > c_1$, and so the uniform exponential decay of \bar{F}_{μ_η} follows from the exponential decay of \bar{F}_μ (established in Lemma 2.1).

To prove the last point of the lemma, we use the fact that if $d_{\text{TV}}(\mu, \mu_\eta) \leq c_1\eta \leq 1$, then f_η can be constructed from f by first thinning the Poisson point process used to define f to have intensity $1 - c_1\eta$, and then adding an independent shot noise field. Recall that the thinned version of the shot noise field is a spatial rescaling of $f^{(1-c_1\eta)}$, and moreover adding an independent field acts as a convolution on the density. Therefore, by Condition 1.5 it follows that $(f_\eta(0), \nabla f_\eta(0))$ has a density that is uniformly bounded for all sufficiently small η . Redefining η_0 to be sufficiently small, we have proved the lemma. \square

For the remainder of the paper we shall define $(f_\eta)_{\eta>0}$ in the following way: in the case that Condition 1.2 holds, we fix $\eta_0 > 0$ and $(\mu_\eta)_{\eta \in (0, \eta_0]}$ satisfying Lemma 2.2 and define $(f_\eta)_{\eta \in (0, \eta_0]}$ as in (2.2), with $f_\eta = f_{\eta_0}$ for all $\eta > \eta_0$; in the case that Condition 1.1 holds, we simply define $f_\eta = f$ for all $\eta > 0$.

By the construction of f_η , it is not hard to see that f_η converges to f in the C^0 -topology on compact sets (we will not formally prove this since we do not require it), so it is natural to include f in the set $(f_\eta)_{\eta \geq 0}$ by defining $f_0 = f$.

2.2.2. Truncation of the kernel. Fix $\chi : \mathbb{R}^2 \rightarrow [0, 1]$ to be a smooth approximation of the indicator function of $B(1)$, where $B(r)$ denotes the closed ball of radius r centred at the origin. More precisely, we define χ to take the value 1 on $B(1/2)$, the value 0 on $B(1)^c$, and to be radially non-increasing with uniformly bounded derivatives. For $r \in \mathbb{R}_+ \cup \{\infty\}$, define the truncated kernel

$$g_r(x) = \begin{cases} g(x)\chi(r|x|), & r > 0, \\ g(x), & r = \infty, \end{cases}$$

and define the r -truncated shot noise field

$$f_r = \sum_{i \in \mathcal{P}} Y_i^\mu g_r(x - i).$$

Similarly, for $\eta \geq 0$ let $f_{\eta,r}$ be defined analogously with Y_i^μ replaced by $Y_i^{\mu_\eta}$.

Note that we have formally included f_η in the set $(f_{\eta,r})_{r>0}$ by identifying f_η with the limiting case $r = \infty$. This is natural, since $f_{\eta,r}$ converges to f_η as $r \rightarrow \infty$ in the C^0 -topology on compact sets (we quantify the speed of convergence in Lemma 2.3 below).

2.2.3. Spatial discretisation. For $\varepsilon > 0$, define the ε -discretised shot noise field

$$f^\varepsilon(x) = \sum_{i \in \varepsilon\mathbb{Z}^2} B_i Y_i^\mu g(x - i)$$

where $\{B_i\}$ are i.i.d. Bernoulli random variables with parameter ε^2 , and $\{Y_i^\mu\}$ are, as before, i.i.d. random variables drawn from μ . For $\eta \geq 0$ and $r > 0$ define $f_{\eta,r}^\varepsilon$ analogously by replacing Y_i^μ with $Y_i^{\mu_\eta}$ and g with g_r .

Again, it is not hard to see that $f_{\eta,r}^\varepsilon$ converges to $f_{\eta,r}$ as $\varepsilon \rightarrow 0$ in the C^0 -topology on compact sets (we verify this in Lemma 2.4 below). Hence it is natural to include $f_{\eta,r}$ in the set $(f_{\eta,r}^\varepsilon)_{\varepsilon \geq 0}$ by identifying $f_{\eta,r}$ with the limiting case $f_{\eta,r}^0$.

2.2.4. Adding a constant to the mark. For $\varepsilon > 0$ and $h = (h_i)_{i \in \varepsilon\mathbb{Z}^2}$, we construct $f^{\varepsilon,h}$ from f^ε by adding the constant h_i to the mark distribution Y_i^μ at each site $i \in \mathbb{Z}^2$, i.e.

$$f^{\varepsilon,h} = \sum_{i \in \varepsilon\mathbb{Z}^2} B_i (Y_i^\mu + h_i) g(x - i),$$

with $f_{\eta,r}^{\varepsilon,h}$ defined analogously. When $h \in \mathbb{R}$, we understand this as setting $h_i \equiv h$, and in this case we also define

$$f_{\eta,r}^{0,h} = \sum_{i \in \mathcal{P}} B_i(Y_i^{\mu_\eta} + h)g_r(x - i).$$

2.3. Analysis of the perturbations. We first give bounds on the distance between the (smoothed) shot noise field f_η and its perturbation $f_{\eta,r}^{\varepsilon,h}$. We begin by analysing the truncation.

Lemma 2.3. *Suppose Condition 1.1 holds, or Condition 1.2 holds and $\alpha > 0$ is an arbitrary constant. Then there exist $c_1, c_2 > 0$ such that, for all $\eta \geq 0$ and $r, s, t \geq 2$,*

$$\mathbb{P}(\|f_\eta - f_{\eta,r}\|_{B(s),\infty} > c_1 t^2 r^{1-\alpha} (\log r)^2) < c_1 s^2 e^{-c_2 t}.$$

Proof. The random field $f_\eta - f_{\eta,r}$ is a symmetric shot noise field with kernel

$$h(x) = g(x) - g_r(x) = g(x)(1 - \chi(r|x|)) \leq g(x)\mathbf{1}_{|x| \geq r/2}$$

and mark distribution μ_η . Define the auxiliary kernel

$$j(x) = \sup_{y \in [-1/2, 1/2]^2} |(1 + |\log(|x+y|)|)^2 \|\nabla h(x+y)\|.$$

According to equation (A.5) of Proposition A.1 (since Lemmas 2.1 and 2.2 guarantee the uniform exponential decay of the mark distribution μ_η , we may take $\gamma = 1$ in this proposition), for all $r, s, t \geq 2$,

$$\mathbb{P}\left(\|f_\eta - f_{\eta,r}\|_{B(s)} \geq 2t(\sqrt{2t}\|h\|_{L^2} + \frac{t}{3}\|h\|_\infty + \|j\|_{L^1} + \sqrt{2t}\|j\|_{L^2} + \frac{t}{3}\|j\|_\infty)\right) \leq c_1 s^2 e^{-c_2 t}$$

for some $c_1, c_2 > 0$. By Condition 1.1 (or Condition 1.2 with $\alpha > 0$ arbitrary), and since the derivatives of $\chi(r|x|)$ are uniformly bounded for $r \geq 1$, there is a $c_3 > 0$ such that

$$h(x) \leq c_3 |x|^{-\alpha} \mathbf{1}_{|x| \geq r/2}$$

and

$$j(x) \leq c_3 |x|^{-\alpha-1} (\log |x|)^2 \mathbf{1}_{|x| \geq r/2}.$$

Hence there exists a $c_4 > 0$ such that

$$\|h\|_{L^2} \leq c_4 r^{1-\alpha}, \quad \|j\|_{L^1} \leq c_4 r^{1-\alpha} (\log r)^2,$$

$$\|j\|_{L^2} \leq c_4 r^{-\alpha} (\log r)^2 \quad \text{and} \quad \|j\|_\infty \leq c_4 r^{-1-\alpha} (\log r)^2,$$

and so the bound reduces to

$$(2.3) \quad \mathbb{P}(\|f_\eta - f_{\eta,r}\|_{B(s)} \geq c_5 t^2 r^{1-\alpha} (\log r)^2) \leq c_1 s^2 e^{-c_2 t}$$

for some $c_5 > 0$, which gives the result. \square

For the discretisation, we verify that f and f_ε can be coupled so that they converge in probability in the C^0 -topology on compact sets. Note that this coupling naturally induces a coupling of $f_{\eta,r}^{0,h}$ and $f_{\eta,r}^{\varepsilon,h}$ for all $\eta \geq 0$, $r > 0$ and $h \in \mathbb{R}$.

Lemma 2.4. *Suppose that either Condition 1.1 or Condition 1.2 holds. Fix a compact set $D \subset \mathbb{R}^2$ and a compact interval $I \subset \mathbb{R}$. Then there exists a sequence $g_\varepsilon \rightarrow 0$ of positive numbers and a coupling of (f, f^ε) such that, for every $\eta \geq 0$, $r > 0$ and $h \in I$,*

$$\mathbb{P}(\|f_{\eta,r}^{0,h} - f_{\eta,r}^{\varepsilon,h}\|_{D,\infty} > g_\varepsilon) < g_\varepsilon$$

eventually as $\varepsilon \rightarrow 0$. In particular, for every $\eta \geq 0$, $r > 0$ and $h \in \mathbb{R}$

$$f_{\eta,r}^{\varepsilon,h} \Rightarrow f_{\eta,r}^{0,h}$$

in law in the C^0 -topology on compact sets.

Proof. Fix $c > 0$ and define

$$\tilde{f}_{\eta,r}^{0,h} = \sum_{i \in \mathcal{P} \cap [-c,c]^2} (Y_i^{\mu_\eta} + h)g_r(x-i) \quad \text{and} \quad \tilde{f}_{\eta,r}^{\varepsilon,h} = \sum_{i \in \varepsilon\mathbb{Z}^2 \cap [-c,c]^2} B_i(Y_i^{\mu_\eta} + h)g_r(x-i)$$

to be the contributions to $f_{\eta,r}$ (resp. $f_{\eta,r}^\varepsilon$) from the Poisson points (resp. Bernoulli points) inside $[-c, c]^2$. Since $g(x)$ decreases at most like an integrable power of $\|x\|$ (Conditions 1.1 and 1.2), the discrepancies

$$\|f_{\eta,r} - \tilde{f}_{\eta,r}\|_{D,\infty} \quad \text{and} \quad \|f_{\eta,r}^\varepsilon - \tilde{f}_{\eta,r}^\varepsilon\|_{D,\infty}$$

can be made arbitrarily small (uniformly in $\eta \geq 0$, $r > 0$ and $h \in I$) with arbitrarily high probability by taking $c > 0$ large enough. Hence it is sufficient to show that

$$\|\tilde{f}_{\eta,r}^{0,h} - \tilde{f}_{\eta,r}^{\varepsilon,h}\|_{D,\infty} \rightarrow 0$$

in probability, uniformly over $\eta \geq 0$, $r > 0$ and $h \in I$. Notice that, due to the continuity of g , the fields $\tilde{f}_{\eta,r}^{\varepsilon,h}|_D$, $\tilde{f}_{\eta,r}^{0,h}|_D$, $\tilde{f}_{\eta,r}^{\varepsilon,h}|_D$ and $\tilde{f}_{\eta,r}^{0,h}|_D$ can all be viewed as continuous functionals (in the C_0 -topology) of the respective point sets $P = \{(i, Y_i^{\mu_\eta}) : i \in \mathcal{P} \cap [-c, c]^2\}$ and $P^\varepsilon = \{(i, Y_i^{\mu_\eta}) : i \in \varepsilon\mathbb{Z}^2 \cap [-c, c]^2, B_i = 1\}$. Note also that, by the standard binomial approximation of a Poisson point process, the point process P^ε converges vaguely to P as $\varepsilon \rightarrow 0$. Since the support of a point process on a compact set is continuous in the topology induced by vague convergence [Res07, Proposition 3.3], the result follows. \square

We next state basic regularity properties for the fields $(f_\eta)_{\eta \geq 0}$:

Lemma 2.5 (Regularity of level lines). *Suppose that either Conditions 1.1 and 1.4 or Conditions 1.2 and 1.5 hold. Then for every $\eta \geq 0$ and $\ell \in \mathbb{R}$, the level set $\mathcal{L}_\ell = \{x : f_\eta(x) = \ell\}$ consists of a collection of simple closed curves. Moreover, for every line-segment L , the intersections of \mathcal{L}_ℓ with L are transverse almost surely.*

Proof. Since f_η is almost surely C^2 -smooth and the density of $(f_\eta(0), \nabla f_\eta(0))$ is bounded, Bulinskaya's lemma [AT07, Lemma 11.2.10] states that f_η almost surely has no critical points at level ℓ , and hence the level sets are simple closed curves by the implicit function theorem. Similarly, $f_\eta|_L$ almost surely has no critical points at level ℓ , which ensures that \mathcal{L}_ℓ intersects L transversally almost surely. \square

Finally we give an estimate on the number of critical points inside a narrow window of critical values:

Proposition 2.6 (Critical points in a narrow window). *Suppose that either Condition 1.1 and 1.4 or Conditions 1.2 and 1.5 hold. Then there exist $c, \varepsilon_0 > 0$ such that, for every $\eta \geq 0$, $\ell \in \mathbb{R}$, $\varepsilon < \varepsilon_0$ and $s \geq 1$,*

$$\mathbb{P}(\exists x \in B(s) : \nabla f_\eta(x) = 0, |f_\eta(x) - \ell| < \varepsilon) < cs^2\varepsilon|\log \varepsilon|^2,$$

and moreover, for every direction $v \in \mathbb{S}^1$,

$$\mathbb{P}(\exists x \in sL_v : \partial_v f_\eta(x) = 0, |f_\eta(x) - \ell| < \varepsilon) < cs\varepsilon|\log \varepsilon|,$$

where L_v denotes a unit line segment in direction v .

Proof. For each multi-index k such that $|k| \leq 2$, the random field $\partial^k f_\eta$ is a symmetric shot noise field with kernel $\partial^k g$ and mark distribution μ_η . Hence, applying equation (A.3) of Proposition A.1 (since Lemmas 2.1 and 2.2 guarantee the uniform exponential decay of the mark distribution μ_η , we may take $\gamma = 1$ in this proposition), there exist constants $c_1, c_2 > 0$ such that, for all $\eta \geq 0$, $|k| \leq 2$ and $t \geq 1$,

$$(2.4) \quad \mathbb{P}(\|\partial^k f_\eta\|_{[0,1]^2,\infty} \geq c_1 t^2) \leq c_1 e^{-c_2 t}.$$

Now define $t_\varepsilon = |\log \varepsilon|^{1/2}$, and let Ω_ε be the event that

$$\max_{|k|=1,2} \|\partial^k f_\eta\|_{[0,1]^2,\infty} \leq t_\varepsilon^2$$

which, by (2.4), has probability greater than $1 - c_1 \exp(-c_3 t_\varepsilon)$ for some $c_3 > 0$ and all sufficiently small $\varepsilon > 0$. Choose an integer $n \geq t_\varepsilon^2/\varepsilon$, and tile $B(s)$ with $\lceil s \rceil^2 n^2$ squares

$S_1, \dots, S_{\lceil s \rceil^2 n^2}$ of side-length $1/n$. Let x_k be the center of the square C_k . Then we have, for some $c_4, c_5 > 0$,

$$\begin{aligned} & \mathbb{P}(\exists x \in B(s) : \nabla f_\eta(x) = 0, |f_\eta(x) - \ell| \leq \varepsilon) \\ & \leq \sum_{k=1}^{\lceil s \rceil^2 n^2} \mathbb{P}(\Omega_\varepsilon, \exists x \in S_k : \nabla f_\eta(x) = 0, |f_\eta(x) - \ell| \leq \varepsilon) + \mathbb{P}(\Omega_\varepsilon^c) \\ & \leq \sum_{k=1}^{\lceil s \rceil^2 n^2} \mathbb{P}(|\nabla f_\eta(x_k)| \leq t_\varepsilon^2/n, |f_\eta(x_k) - \ell| \leq \varepsilon + t_\varepsilon^2/n) + \mathbb{P}(\Omega_\varepsilon^c) \\ & \leq c_4 s^2 n^2 \times (t_\varepsilon^2/n)^2 \times (\varepsilon + t_\varepsilon^2/n) + \mathbb{P}(\Omega_\varepsilon^c) \\ & \leq 2c_4 s^2 \varepsilon t_\varepsilon^4 + c_1 \exp(-c_3 t_\varepsilon) \leq c_5 s^2 \varepsilon t_\varepsilon^4, \end{aligned}$$

with the third inequality following from the uniform bound on the density of $(f_\eta, \nabla f_\eta)$, and the final inequality holding for all sufficiently small $\varepsilon > 0$. The proof of the first statement is complete.

The proof of the second statement is analogous, except we choose an $n \geq t_\varepsilon/\varepsilon$ and tile sL_ν with $\lceil s \rceil n$ line-segments of length $1/n$. \square

3. CRITICALITY OF THE ZERO LEVEL

In this section we study the ‘critical’ properties of the excursion set \mathcal{E}_0 at the zero level. In particular, we establish (i) positive association for crossing events, (ii) quasi-independence for crossing events, (iii) the ‘box-crossing’ estimates, and finally (iv) the absence of percolation (i.e. the first statement of Theorem 1.7). Since many of the arguments are standard in percolation theory, we emphasise only on the aspects that differ in the shot noise case.

Throughout this section we shall suppose that either Conditions 1.1 and 1.4 or Conditions 1.2 and 1.5 hold; in the latter case we fix $\alpha > 0$ to be an arbitrary constant. Note that only the first two perturbations from Section 2.2 will play a role in this section; i.e. we will only consider the fields $f_{\eta,r}$.

3.1. Crossing events. We begin by introducing ‘crossing events’ for rectangles and annuli. Let g be a continuous planar function let $\ell \in \mathbb{R}$ be a level. For $\rho_1, \rho_2 > 0$, define $R[\rho_1, \rho_2] = [0, \rho_1] \times [0, \rho_2]$, and let $\{g \in \text{Cross}_\ell(\rho_1, \rho_2)\}$ denote the event that there exists a connected component of $\{g \geq -\ell\} \cap R[\rho_1, \rho_2]$ that intersects both the ‘left’ and ‘right’ sides of R , i.e. intersects both $\{0\} \times [0, \rho_2]$ and $\{\rho_1\} \times [0, \rho_2]$. Moreover, let $\{g \in \text{Cross}_\ell^*(\rho_1, \rho_2)\}$ denote the event that there exists a connected component of $\{g \leq -\ell\} \cap R[\rho_1, \rho_2]$ that intersects both the ‘top’ and ‘bottom’ sides of $R[\rho_1, \rho_2]$.

Similarly, for $0 < \rho_1 < \rho_2$ define $A[\rho_1, \rho_2] = \{[-\rho_2, \rho_2]^2 \setminus [-\rho_1, \rho_1]^2\}$, and let $\{g \in \text{Arm}_\ell(\rho_1, \rho_2)\}$ denote the event that there exists a connected component of $\{g \geq -\ell\} \cap A[\rho_1, \rho_2]$ that intersects both $\partial[-\rho_1, \rho_1]^2$ and $\partial[-\rho_2, \rho_2]^2$.

Collectively we shall refer to the events Cross_ℓ , Cross_ℓ^* and Arm_ℓ as ‘crossing events’. To each crossing event we will associate its *level* ℓ , and also its *supporting domain*, being the compact domain $D \subset \mathbb{R}^2$ on which the event is defined (i.e. either the rectangle $R[\rho_1, \rho_2]$ or the annulus $A[\rho_1, \rho_2]$). Notice that the crossing events Cross_ℓ and Arm_ℓ are increasing in both the function g and the level ℓ (i.e. $\{g \in \text{Cross}_\ell\}$ implies that $\{g' \in \text{Cross}_{\ell'}\}$ for any $g' \geq g$ and $\ell' \geq \ell$), whereas the event Cross_ℓ^* is decreasing in both the field and the level.

Let us state some basic properties of these events:

Lemma 3.1. *For each $\eta \geq 0$, the crossing events for f_η are almost surely continuity events in the C^0 -topology on their supporting domains. Moreover,*

$$\{f_\eta \in \text{Cross}_\ell(\rho_1, \rho_2)\} \quad \text{and} \quad \{f_\eta \in \text{Cross}_\ell^*(\rho_1, \rho_2)\}$$

form a partition of the probability space up to a null set.

Proof. The first property is a consequence of the regularity of the level lines in Lemma 2.5. Indeed, outside a null set (namely, the event that the regularity properties in Lemma 2.5 do not hold) the crossing events $\text{Cross}_\ell(\rho_1, \rho_2)$ and $f_\eta \in \text{Cross}_\ell^*(\rho_1, \rho_2)$ can each be expressed as a countable disjoint union of sets that are open in the C^0 -topology. Moreover, these open sets are contained in exactly one of $\text{Cross}_\ell(\rho_1, \rho_2)$ or $\text{Cross}_\ell^*(\rho_1, \rho_2)$, yielding the second statement. \square

Remark 3.2. Note that Lemma 3.1 fails for the truncated fields $f_{\eta,r}$; for such fields there is a positive probability that the zero-level set $\{x \in \mathbb{R}^2 : f_{\eta,r} = 0\}$ covers $R[\rho_1, \rho_2]$, which means that the crossing events are discontinuous in the C^0 -topology, and moreover,

$$\mathbb{P}[f_{\eta,r} \in \text{Cross}_\ell(\rho_1, \rho_2)] + \mathbb{P}[f_{\eta,r} \in \text{Cross}_\ell^*(\rho_1, \rho_2)] > 1.$$

We observe a simple consequence of the ‘self-duality’ of the zero level:

Proposition 3.3 (Square crossings). *Let $\rho > 0$ and $\eta \geq 0$. Then*

$$\mathbb{P}[f_\eta \in \text{Cross}_0(\rho, \rho)] = 1/2.$$

Proof. By Lemma 3.1,

$$\{f_\eta \in \text{Cross}_0(\rho, \rho)\} \quad \text{and} \quad \{f_\eta \in \text{Cross}_0^*(\rho, \rho)\}$$

form a partition of the probability space. On the other hand, by the symmetry of the mark μ and the symmetry of the kernel g under reflection in the line $\{y = x\}$, the events $\text{Cross}_0(\rho, \rho)$ and $\text{Cross}_0^*(\rho, \rho)$ are of equal probability, and so this probability is $1/2$. \square

3.2. Positive associations. We next verify the crucial ‘positive-association’ property for crossing events (which is the only reason we insist that $g \geq 0$ in Conditions 1.1 and 1.2):

Proposition 3.4 (Positive associations). *Let $\eta \geq 0$, and let E_1 and E_2 be two crossing events that are either both increasing or both decreasing. Then*

$$\mathbb{P}[\{f_\eta \in E_1\} \cap \{f_\eta \in E_2\}] \geq \mathbb{P}[f_\eta \in E_1]\mathbb{P}[f_\eta \in E_2].$$

Proof. Without loss of generality suppose that E_1 and E_2 are both increasing. Recall the definition of the ε -discretised field f_η^ε , and notice that the crossing events

$$\{f_\eta^\varepsilon \in E_1\} \quad \text{and} \quad \{f_\eta^\varepsilon \in E_2\}$$

can be viewed as increasing functions on the (countable) product spaces that generate f_η^ε . Hence by the classical Harris/FKG inequality for product spaces (see [Gri99, Section 2.2]),

$$\mathbb{P}[\{f_\eta^\varepsilon \in E_1\} \cap \{f_\eta^\varepsilon \in E_2\}] \geq \mathbb{P}[f_\eta^\varepsilon \in E_1]\mathbb{P}[f_\eta^\varepsilon \in E_2].$$

Since $f_\eta^\varepsilon \rightarrow f_\eta$ in the C^0 -topology on compact sets (Lemma 2.4), and since crossing events are continuity events for the limit f_η in this topology (Lemma 3.1), sending $\varepsilon \rightarrow 0$ yields the result. \square

Remark 3.5. We do not claim positive associations for the truncated field $f_{\eta,r}$ because it does not follow from our proof (see Remark 3.2), and we do not need it.

3.3. Quasi-independence. We next show that crossing events that are supported on well-separated domains are approximately independent.

As a preliminary, we have a general comparison result that bounds the effect of truncation on the probability of crossing events. Recall that crossing events are either increasing or decreasing; we call a collection of crossing events *monotonic* if they are either all increasing or all decreasing.

Proposition 3.6. *There exist $c_0, c_1, c_2 > 0$ such that, for every $\eta \geq 0$, $r, R, t \geq c_0$, every compact set $D \subset \mathbb{R}^2$ of diameter at most R , and every monotonic collection $\{A_i\}_{i \leq n}$ of crossing events whose supports are contained in D ,*

$$(3.1) \quad |\mathbb{P}(f_\eta \in A) - \mathbb{P}(f_{\eta,r} \in A)| < c_1 n R^2 t^2 r^{1-\alpha} (\log r)^2 ((\log r)^2 + (\log t)^2) + c_1 R^2 e^{-c_2 t},$$

where $A = \cap_{i \leq n} A_i$.

Proof. For each crossing event A_i , let ℓ_i and $S_i \subset D$ denote its associated level and support respectively. Recall that each S_i has a piece-wise linear boundary (with one or two connected components, depending on the crossing event). For a $c > 0$, define the event Ω_c^1 that, for each S_i :

- There is no critical point $x \in S_i$ of f_η such that $|f_\eta(x) - \ell_i| < c$;
- For each boundary line-segment $L \subset \partial S_i$, $f_\eta|_L$ has no critical point $x \in L$ such that $|f_\eta(x) - \ell_i| < c$;
- None of the corners of ∂S_i have $|f_\eta(x) - \ell_i| < c$.

By the Morse lemma (see, e.g., [Han02, Theorem 7]), on the event Ω_c^1 the level set $\{f_\eta = \ell\}$ inside S_i (considered as a set stratified by its boundary line-segments and corners) has the same topology for all levels $\ell \in [\ell_i - c, \ell_i + c]$. In particular, the events

$$\{f_\eta \in A\} \cap \Omega_c^1, \quad \{f_\eta + c \in A\} \cap \Omega_c^1 \quad \text{and} \quad \{f_\eta - c \in A\} \cap \Omega_c^1$$

all agree up to a null set. Define now the event Ω_c^2 that

$$\|f_\eta - f_{\eta,r}\|_{\infty,D} \leq c.$$

By the monotonicity of the collection of crossing events,

$$\{f_\eta \in A\} \cap \Omega_c^1 \cap \Omega_c^2 \quad \text{and} \quad \{f_{\eta,r} \in A\} \cap \Omega_c^1 \cap \Omega_c^2$$

also agree up to a null set. To finish the proof, recall that by Lemma 2.3, there exist $c_1, c_2 > 0$ such that $\mathbb{P}(\Omega_c^2) > 1 - c_1 R^2 e^{-c_2 t}$ for the choice

$$(3.2) \quad c = c_1 t^2 r^{1-\alpha} (\log r)^2.$$

Moreover, by Proposition 2.6 and the union bound, there is a $c_3 > 0$ such that, for all small enough $c > 0$,

$$\mathbb{P}(\Omega_c^1) > 1 - c_3 n R^2 c |\log c|^2.$$

Hence, setting c as (3.2), there is a $c_4 > 0$ such that

$$\mathbb{P}(\Omega_c^1 \cap \Omega_c^2) > 1 - c_4 n R^2 t^2 r^{1-\alpha} (\log r)^2 ((\log r)^2 + (\log t)^2) - c_1 R^2 e^{-c_2 t}$$

which gives the result. \square

Our main quasi-independence result is a corollary of the previous proposition.

Theorem 3.7 (Quasi-independence). *There exist $c_0, c_1, c_2 > 0$ such that, for every $\eta \geq 0$ and $r, R, t \geq c_0$, every pair of compact sets $D_1 \subset \mathbb{R}^2$ (resp. D_2) of diameter at most R and such that $r = \text{dist}(D_1, D_2)$, and every pair of monotonic collections of n_1 (resp. n_2) crossing events $(E_i^1)_{i \leq n_1}$ (resp. $(E_i^2)_{i \leq n_2}$) that are supported on D_1 (resp. D_2),*

$$(3.3) \quad \left| \mathbb{P}(f_\eta \in E^1 \cap E^2) - \mathbb{P}(f_\eta \in E^1) \mathbb{P}(f_\eta \in E^2) \right| \\ < c_1 \max\{n_1, n_2\} R^2 t^2 r^{1-\alpha} (\log r)^2 ((\log r)^2 + (\log t)^2) + c_1 R^2 e^{-c_2 t}$$

where $E^j = \cap_{i \leq n_j} E_i^j$ for $j \in \{1, 2\}$.

Proof. This follows immediately from Proposition 3.6 by replacing f_η with its truncated version $f_{\eta,r}$, and noticing that the events $\{f_{\eta,r} \in E_1\}$ and $\{f_{\eta,r} \in E_2\}$ are independent (see the proof of [MV19, Theorem 4.2] for details). \square

Remark 3.8. Equation (3.3) (with the setting $t = (\log R)^2$) implies that two crossing events that are supported on domains of diameter at most R and separated by a distance greater than $r > c_1 R$ are approximately independent in the sense that, as $R \rightarrow \infty$,

$$(3.4) \quad \left| \mathbb{P}(f_\eta \in E_1 \cap E_2) - \mathbb{P}(f_\eta \in E_1) \mathbb{P}(f_\eta \in E_2) \right| < c_2 R^{3-\alpha} (\log R)^6 \rightarrow 0.$$

In fact, this is precisely the origin of the assumption $\alpha > 3$ in Condition 1.1.

3.3.1. *Box-crossing estimates.* We next establish the ‘box-crossing’ estimates (also known as the Russo-Seymour-Welsh estimates), which state that the probability of crossing events for rectangles are bound away from zero and one uniformly in the scale.

Theorem 3.9 (Box-crossing estimates). *For each $\eta \geq 0$ and $\rho_1, \rho_2 > 0$,*

$$\inf_{R \geq 1} \mathbb{P}(f_\eta \in \text{Cross}_0(R\rho_1, R\rho_2)) > 0 \quad \text{and} \quad \sup_{R \geq 1} \mathbb{P}(f_\eta \in \text{Cross}_0(R\rho_1, R\rho_2)) < 1.$$

Proof. In [Tas16], Tassion gave general conditions for a translation invariant colouring of the plane to satisfy the box-crossing estimates. In our setting, Tassion’s argument is valid since the following four conditions are satisfied for the fields f_η (see [RV19b, Section 4] and [BG17, Section 4.2] for details):

- (1) The square-crossing property in Proposition 3.3;
- (2) The positive association in Proposition 3.4;
- (3) The symmetry guaranteed by Conditions 1.1 and 1.2; and
- (4) The quasi-independence guaranteed by Theorem 3.7. □

3.3.2. *Absence of percolation at the zero level.* Finally, we deduce that the connected components of \mathcal{E}_ℓ are finite for all $\ell \leq 0$, completing the proof of the first statement of Theorem 1.7.

Theorem 3.10 (One-arm decay). *There exist $c, c_{\text{Arm}} > 0$ such that, for every $\eta \geq 0$ and $1 \leq r \leq R$,*

$$\mathbb{P}(f_\eta \in \text{Arm}_0(r, R)) < c \left(\frac{r}{R} \right)^{c_{\text{Arm}}}.$$

Hence almost surely \mathcal{E}_0 has bounded connected components.

Proof. This follows from the positive association in Proposition 3.4, the quasi-independence in Theorem 3.7, and the box-crossing estimates in Theorem 3.9 (see the proof of [RV19b, Proposition 4.5] for details). Note that the constant c_{Arm} depends on the mark μ and the kernel g , but can be chosen uniformly over a collection of kernels g_i whose derivatives $|\partial^k g_i|$, $|k| \leq 2$, decay more and more rapidly. □

Remark 3.11. As shown in [BG17, Section 4.2], given the quasi-independence in Theorem 3.7, the properties established for the excursion set \mathcal{E}_0 in Theorems 3.9 and 3.10 hold equally for the zero level set $\{x \in \mathbb{R}^2 : f_\eta(x) = 0\}$.

4. THE PHASE TRANSITION IN THE CASE OF BOUNDED MILLS RATIO

We now study the phase transition that occurs when the level passes through zero. The main result will be the following description of the phase transition for crossings of $2R \times R$ rectangles:

Theorem 4.1. *For every $\ell > 0$ there exists a $c > 0$ such that, for every $R \geq 1$,*

$$\mathbb{P}(f \in \text{Cross}_\ell(2R, R)) \geq 1 - e^{-cR}.$$

Theorem 1.7 can be deduced from Theorem 4.1 (along with the previously stated Theorems 3.9 and 3.10) in a straightforward way:

Proof of Theorem 1.7. The boundedness of excursion sets components at levels $\ell \leq 0$ follows from the fact that, by Theorem 3.10, $\mathbb{P}(f \in \text{Arm}_0(1, R)) \rightarrow 0$ as $R \rightarrow \infty$. More precisely, suppose that \mathcal{E}_0 had an unbounded component with positive probability. Then trivial regularity considerations imply that it would have a positive Lebesgue measure, and hence by stationarity 0 would be in this component with positive probability, which we just proved was impossible.

The existence of a unique unbounded excursion sets component in the case $\ell > 0$ is a consequence of Theorem 4.1 by standard gluing arguments. Indeed, by Theorem 4.1 we have

$$\sum_{k \geq 1} (1 - \mathbb{P}(\text{Cross}_\ell(2^{k+1}, 2^k))) < \infty,$$

and so, by the Borel-Cantelli lemma, there exists a $k_0 \geq 1$ such that

$$\{f \in \text{Cross}_\ell(2^{k+1}, 2^k)\}$$

occurs for each $k \geq k_0$. Arranging the $2^{k+1} \times 2^k$ rectangles (and rotated versions) so that the resulting crossings overlap, this implies the existence of a unique unbounded connected component in $\{f \geq -\ell\}$.

The fact that ℓ -level sets are almost surely bounded is an immediate consequence of the above, since an unbounded ℓ -level line would lie at the interface between unbounded components of \mathcal{E}_ℓ and \mathcal{E}_ℓ^c , and by the above (and the symmetric $f \stackrel{d}{=} -f$) at least one of these sets has only bounded components almost surely.

The remainder of the claims in Theorem 1.7 are proven in Theorems 3.9 and 3.10. \square

In this section we will prove Theorem 4.1 in the case of bounded Mills ratio (Conditions 1.1 and 1.4); the case of bounded Mills ratio in the tails (Condition 1.2) will be the focus of the next section. In particular, in this section we set the smoothing parameter η to zero.

4.1. A differential inequality. The first step is to establish a differential inequality, with respect to the level ℓ , for the probability of $\{f_r^\epsilon \in \text{Cross}_\ell(2R, R)\}$. Our method makes use of the OSSS inequality (described below).

In fact, instead of differentiating with respect to the level ℓ , it will initially be more natural to differentiate with respect to a change in the mean of the mark distribution, i.e. the variables $h = (h_i)$ in the definition of the field $f_r^{\epsilon, h}$ (see Section 2.2). The differential inequality is the following:

Proposition 4.2. *There exists a $h_0 > 0$ such that, for all $r \geq 1$, $R > 4r$, $\ell \in \mathbb{R}$, $\epsilon > 0$ and $h \in [0, h_0]$,*

$$\begin{aligned} & -\frac{\partial}{\partial h} \log(1 - \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))) \\ & \geq (c_{\text{Mills}}^{-1}/4) \times \frac{\mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))}{\inf_{\bar{r} \in (2r, R/2)} (2\bar{r}/R + \mathbb{P}(f_{\bar{r}}^\epsilon \in \text{Arm}_{-\ell}(2r, \bar{r}))}. \end{aligned}$$

Before proving Proposition 4.2, let us recall the OSSS inequality. Let Λ be a finite set, and let $(E^\Lambda, \mathcal{E}^{\otimes \Lambda}, \mu^{\otimes \Lambda})$ be a product probability space. Given an event $A \in \mathcal{E}^{\otimes \Lambda}$ and coordinate $i \in \Lambda$, the *influence* $I_i^\mu(A)$ of the i^{th} coordinate on A is defined as the probability that resampling the i^{th} coordinate modifies $\mathbb{1}_A$, i.e.,

$$I_i^\mu(A) = \mathbb{P}(\mathbb{1}_A(\omega) \neq \mathbb{1}_A(\tilde{\omega})),$$

where $\omega \stackrel{d}{=} \mu^{\otimes \Lambda}$ and where $\tilde{\omega} = \omega$ except that ω_i is resampled independently.

Now, let \mathcal{A} be a *random algorithm* that determines A , i.e. a procedure that reveals step-by-step the coordinates of ω and stops once the value of $\mathbb{1}_A(\omega)$ is known (the algorithm should also be adapted, i.e. the next coordinate to be revealed should be measurable with respect to the coordinates already revealed and auxiliary randomness). The *revelment* $\delta_i^\mu(\mathcal{A})$ of the i^{th} coordinate for the algorithm \mathcal{A} is the probability that ω_i is revealed by the algorithm.

The OSSS inequality gives an upper bound on the variance of an event A (in a product space) in terms of the influences and revealments of a random algorithm determining A :

Theorem 4.3 (OSSS inequality (see [OSSS05] or [MV19, Theorem 2.1])). *For every $A \in \mathcal{E}^{\otimes \Lambda}$ and random algorithm \mathcal{A} determining A ,*

$$\text{Var}_\mu(\mathbb{1}_A(\omega)) \leq \sum_{i \in \Lambda} \delta_i^\mu(\mathcal{A}) I_i^\mu(A).$$

Let us now describe how we apply the OSSS inequality. Recall the field $f_r^{\epsilon, h}$, and notice that the event $\{f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)\}$ is measurable with respect to the (finite-dimensional) product space indexed by $\Lambda = \epsilon\mathbb{Z}^2 \cap [-2R-r, 2R+r]^2$, where each coordinate

consists of the pair (B_i, Y_i^μ) , and where B_i is a Bernoulli random variable with parameter ϵ^2 and Y_i^μ is an independent random variable drawn from the mark distribution μ .

The algorithm \mathcal{A} that we use to determine the crossing event $\{f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)\}$ is the same as that used in [MV19] (and inspired by [AB18, BKS99, SS10]). More precisely, we first fix a random horizontal line $L = \{y = \mathcal{U}\}$ where \mathcal{U} is uniform in $[0, R]$. Then we reveal all the coordinates $i \in \Lambda$ that are within a distance $2r$ from the line L , which determines the values of the field up to a distance at least r from L . Finally, we iteratively reveal all coordinates that are within a distance $2r$ from a component of $\{f_r^{\epsilon, h} \leq -\ell\}$ that intersects L , terminating with value 0 if a connected component is found to intersect both the top and bottom sides of the $2R \times R$ rectangle, and terminating with value 1 if no such component is found.

The key properties of this algorithm are that it:

- almost surely terminates with value $\mathbb{1}_{\{f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)\}}$; and
- reveals a coordinate $i \in \Lambda$ if and only if there exists a connected component of $\{f_r^{\epsilon, h} \leq -\ell\}$ that connects the ball $B_i(2r)$ to the line L .

Let us give a simple upper bound for the revealments $\delta_i(\mathcal{A})$ of this algorithm. Suppose that $h \geq 0$. By symmetry, and since arm events are increasing, $\delta_i(\mathcal{A})$ is bound above by the probability that a connected component of $\{f_r^\epsilon \geq \ell\}$ connects $B_i(2r)$ to L . Since for any $\bar{r} < R/2$ the distance between i and L is larger than \bar{r} with probability greater than $1 - 2\bar{r}/R$, by averaging over L we have

$$\delta_i(\mathcal{A}) \leq 2\bar{r}/R + \mathbb{P}(f_r^{\epsilon, h} \in \text{Arm}_{-\ell}(2r, \bar{r})),$$

valid for any $\bar{r} \in (2r, R/2)$.

Now suppose that $R > 4r$ and let I_i be the influence of coordinate $i \in \Lambda$ on the event $\{f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)\}$, that is, the probability that resampling the pair (B_i, Y_i^μ) modifies (the indicator function of) the event. Then applying the OSSS equality to the algorithm described above and rearranging,

$$(4.1) \quad \frac{\sum_{i \in \Lambda} I_i}{1 - \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))} \geq \frac{\mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))}{\inf_{\bar{r} \in (2r, R/2)} (2\bar{r}/R + \mathbb{P}(f_r^\epsilon \in \text{Arm}_{-\ell}(2r, \bar{r})))}.$$

We can now complete the proof of the differential inequality:

Proof of Proposition 4.2. First, we fix $h_0 > 0$ sufficiently small so that

$$\frac{\mathbb{P}[Y_i^\mu \geq h_0]}{\mathbb{P}[Y_i^\mu \leq h_0]} > 1/2;$$

such an h_0 exists since μ is symmetric and not supported on $\{0\}$. It is sufficient to prove that, for each $i \in \Lambda$ and $h \in [0, h_0]$,

$$(4.2) \quad \frac{\partial}{\partial h_i} \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)) \geq (c_{\text{Mills}}^{-1}/4) I_i,$$

where I_i is defined above (4.1), since then we have

$$\begin{aligned} -\frac{\partial}{\partial h} \log(1 - \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))) &\geq \frac{\sum_i \frac{\partial}{\partial h_i} \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))}{1 - \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))} \\ &\geq (c_{\text{Mills}}^{-1}/4) \frac{\sum_i I_i}{1 - \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R))} \end{aligned}$$

and so (4.1) gives the result.

We shall establish (4.2) conditional on $(B_j)_{j \in \Lambda \setminus \{i\}}$ and $(Y_j^\mu)_{j \in \Lambda \setminus \{i\}}$ (note that if μ has an atom the derivative may be infinite for some values of the conditioning, but that only helps us).

First, let us bound the influence I_i from above. Since the event $\text{Cross}_\ell(2R, R)$ is increasing, there exists a threshold $\omega \in [-\infty, \infty]$, measurable with respect to the conditioning on $(B_j)_{j \in \Lambda \setminus \{i\}}$ and $(Y_j^\mu)_{j \in \Lambda \setminus \{i\}}$, such that $f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)$ if $B_i(Y_i^\mu + h_i) > \omega$ and

$f_r^{\epsilon, h} \notin \text{Cross}_\ell(2R, R)$ if $B_i(Y_i^\mu + h_i) < \omega$. Hence the influence I_i is at most the probability that $B_i(Y_i^\mu + h_i) \geq \omega$ but $B_i(Y_i^\mu + h_i) \leq \omega$ after resampling (B_i, Y_i^μ) , or vice versa.

If $\omega > 0$, this event implies that $B_i = 1, Y_i^\mu + h_i \geq \omega$, and after resampling either $B_i = 1, Y_i^\mu - h_i \leq \omega$ or $B_i = 0$ (or vice versa), and hence the probability of this event is at most

$$2\epsilon^2 \mathbb{P}(Y_i^\mu \geq \omega - h_i) (\epsilon^2 \mathbb{P}(Y_i \leq \omega - h_i) + (1 - \epsilon^2)) \leq 2\epsilon^2 \mathbb{P}(Y_i^\mu \geq \omega - h_i).$$

If $\omega \leq 0$, this event implies that either $B_i = 1, Y_i^\mu - h_i \geq \omega$ or $B_i = 0$, and after resampling $B_i = 1, Y_i^\mu - h_i \leq \omega$ (or vice versa), and hence the probability of this event is at most

$$2\epsilon^2 \mathbb{P}(Y_i^\mu < \omega - h_i) (\epsilon^2 \mathbb{P}(Y_i \geq \omega - h_i) + (1 - \epsilon^2)) \leq 2\epsilon^2 \mathbb{P}(Y_i^\mu \leq \omega - h_i).$$

Putting this together, we have shown that

$$(4.3) \quad I_i \leq 2\epsilon^2 \begin{cases} \mathbb{P}(Y_i^\mu \geq \omega - h_i) & \text{if } \omega > 0 \\ \mathbb{P}(Y_i^\mu \leq \omega - h_i) & \text{if } \omega \leq 0 \end{cases}$$

Turning to the derivative, we see that

$$\frac{\partial}{\partial h_i} \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)) \geq \epsilon^2 f_{\mu_{a.c.}}(\omega - h_i).$$

If $\omega - h_i \geq 0$, then by the definition of the Mill's ratio

$$f_{\mu_{a.c.}}(\omega - h_i) \geq c_{\text{Mills}}^{-1} \mathbb{P}(Y_i^\mu \geq \omega - h_i).$$

Similarly, if $\omega - h_i \leq 0$ then by symmetry

$$f_{\mu_{a.c.}}(\omega - h_i) \geq c_{\text{Mills}}^{-1} \mathbb{P}(Y_i^\mu \leq \omega - h_i).$$

Altogether we have shown that

$$(4.4) \quad \frac{\partial}{\partial h_i} \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)) \geq c_{\text{Mills}}^{-1} \epsilon^2 \begin{cases} \mathbb{P}(Y_i^\mu \geq \omega - h_i) & \text{if } \omega \geq h_i \\ \mathbb{P}(Y_i^\mu \leq \omega - h_i) & \text{if } \omega \leq h_i \end{cases}.$$

Combining (4.3) and (4.4), we see that

$$\frac{\partial}{\partial h_i} \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)) \geq (c_{\text{Mills}}^{-1}/2) \epsilon^2 I_i$$

whenever $\omega \leq 0$ or $\omega \geq h_i$. On the other hand, if $\omega \in [0, h_i]$ then we still have

$$\frac{\partial}{\partial h_i} \mathbb{P}(f_r^{\epsilon, h} \in \text{Cross}_\ell(2R, R)) \geq (c_{\text{Mills}}^{-1}/2) I_i \times \frac{\mathbb{P}(Y_i^\mu \leq \omega - h_i)}{\mathbb{P}(Y_i^\mu \geq \omega - h_i)}.$$

Moreover, since this implies $h_i - \omega \leq h_i \leq h_0$, we have

$$\frac{\mathbb{P}(Y_i^\mu \leq \omega - h_i)}{\mathbb{P}(Y_i^\mu \geq \omega - h_i)} = \frac{\mathbb{P}(Y_i^\mu \geq h_i - \omega)}{\mathbb{P}(Y_i^\mu \leq h_i - \omega)} \geq \frac{\mathbb{P}(Y_i^\mu \geq h_0)}{\mathbb{P}(Y_i^\mu \leq h_0)} > 1/2$$

by the symmetry of μ and our definition of h_0 . Hence we have established (4.2) in all cases, which completes the proof. \square

4.2. A first description of the phase transition. We next use the differential inequality in Proposition 4.2 to establish a ‘qualitative’ description of the phase transition. The main result of this section is the following:

Theorem 4.4. *For every $\ell > 0$, as $R \rightarrow \infty$,*

$$\mathbb{P}(f \in \text{Cross}_\ell(2R, R)) \rightarrow 1.$$

Let us fix $\ell > 0$. We also define positive exponents

$$(4.5) \quad \gamma \in (0, 1), \quad \zeta = \frac{\gamma c_{\text{Arm}} + 1}{c_{\text{Arm}} + 1} \in (\gamma, 1) \quad \text{and} \quad \xi \in (0, 1 - \zeta)$$

where $c_{\text{Arm}} > 0$ is the constant appearing in Theorem 3.10. For the remainder of the section we fix the scales

$$r = r_R = R^\gamma \rightarrow \infty \quad \text{and} \quad h = h_R = R^{-\xi} \rightarrow 0.$$

We first state an auxiliary result that shows that, if $\varepsilon = \varepsilon_R \rightarrow 0$ decays sufficiently quickly, the fields f , f_R^ε and $f_r^{\varepsilon,h}$ are all close with overwhelming probability.

Lemma 4.5. *There exists a scale $\varepsilon = \varepsilon_R \rightarrow 0$, constants $c_1, c_2 > 0$, and a coupling of (f, f^ε) such that, for all $R \geq 1$,*

$$(4.6) \quad \mathbb{P}(\|f - f_r^\varepsilon\|_{B(2R),\infty} \geq \ell/2) < c_1 e^{-c_2(\log R)^2}$$

and

$$(4.7) \quad \mathbb{P}(\|f - f_r^{\varepsilon,h}\|_{B(2R),\infty} \geq \ell/2) < c_1 e^{-c_2(\log R)^2}.$$

Proof. By Lemma 2.4, one can find a scale $\varepsilon_R \rightarrow 0$ and a coupling of (f, f^ε) such that

$$\|f_r^{0,h} - f_r^{\varepsilon,h}\|_{B(2R),\infty} \geq \ell/2$$

has probability converging to zero as rapidly as desired. Moreover, by Lemma 2.3 ($t = R^\gamma$ for some $\gamma' \in (0, (\alpha - 1)\gamma/2)$), there are $c_1, c_2 > 0$ such that

$$\mathbb{P}(\|f - f_r\|_{B(2R),\infty} \geq c_1 R^{2\gamma'} r^{1-\alpha} (\log r)^2) < c_1 R^2 e^{-c_2 R^{\gamma'}}.$$

Since

$$R^{2\gamma'} r^{1-\alpha} (\log r)^2 = R^{2\gamma' - \gamma(\alpha-1)} (\log R^\gamma)^2 \rightarrow 0$$

by the definition of γ' , we deduce that $\|f - f_r\|_{B(2R),\infty}$ is also larger than $\ell/2$ with probability decaying faster than any polynomial. Finally, applying equation (A.3) of Proposition A.1 (and taking $\gamma = \infty$) there are $c_1, c_2 > 0$ such that

$$\mathbb{P}(\|f_r - f_r^{0,h}\|_{B(2R),\infty} > h(\log R)^2) \leq c_1 R^2 e^{-c_2 \log(R)^2}.$$

Since $h_r = R^{-\xi}$ for $\xi > 0$, this shows that $\|f_r - f_r^{0,h}\|_{B(2R),\infty}$ is also larger than $\ell/2$ with probability decaying faster than any polynomial. Putting this together gives the result. \square

We can now prove Theorem 4.4:

Proof. Define $\varepsilon_R \rightarrow 0$ as in Lemma 4.5. We first make use of the simple fact that if f and g are fields such that $\mathbb{P}[\|f - g\| \geq t] < s$, and A is an increasing event, then

$$(4.8) \quad \mathbb{P}(f + t \in A) \geq \mathbb{P}(g \in A) - s.$$

In particular, by (4.6) and (4.7) this implies the existence of $c_1, c_2 > 0$ such that, for sufficiently large R ,

$$\mathbb{P}(f_r^\varepsilon \in \text{Arm}_{-\ell/2}(2r, \bar{r})) \leq \mathbb{P}(f \in \text{Arm}_0(2r, \bar{r})) + c_1 e^{-c_2(\log R)^2}$$

and

$$\mathbb{P}(f_r^{\varepsilon,h} \in \text{Cross}_{\ell/2}(2R, R)) \geq \mathbb{P}(f \in \text{Cross}_0(2R, R)) - c_1 e^{-c_2(\log R)^2}.$$

Setting $\bar{r} = R^\zeta \in (2r, R/2)$ (where $\zeta > 0$ is defined in (4.5)), by Theorems 3.9 and 3.10 we deduce the existence of a $c > 0$ such that

$$\mathbb{P}(f_r^\varepsilon \in \text{Arm}_{-\ell/2}(2r, \bar{r})) \leq \mathbb{P}(f \in \text{Arm}_0(2r, \bar{r})) + c_1 e^{-c_2(\log R)^2} \leq c_3 R^{-(\zeta-\gamma)c_{\text{Arm}}},$$

and

$$(4.9) \quad \mathbb{P}(f_r^{\varepsilon,h} \in \text{Cross}_{\ell/2}(2R, R)) \geq \mathbb{P}(f \in \text{Cross}_0(2R, R)) - c_1 e^{-c_2(\log R)^2} \geq c_4,$$

eventually as $R \rightarrow \infty$, for some $c_3, c_4 > 0$. Applying Proposition 4.2 we have, for some $c_5, c_6 > 0$,

$$\begin{aligned} & - \frac{\partial}{\partial h} \log(1 - \mathbb{P}(f_r^{\varepsilon,h} \in \text{Cross}_{\ell/2}(2R, R))) \\ & \geq c_5 c_{\text{Mills}}^{-1} \times \frac{\mathbb{P}(f_r^{\varepsilon,h} \in \text{Cross}_{\ell/2}(2R, R))}{2\bar{r}/R + \mathbb{P}(f_r^\varepsilon \in \text{Arm}_{-\ell}(2r, \bar{r}))} \\ & \geq c_6 c_{\text{Mills}}^{-1} R^{1-\zeta}, \end{aligned}$$

where we used the fact $1 - \zeta = (\zeta - \gamma)c_{\text{Arm}}$ by definition. Integrating both sides of this inequality from 0 to h , and using (4.9) to bound the evaluation at $h = 0$, we have

$$\mathbb{P}(f_r^{\varepsilon,h} \in \text{Cross}_{\ell/2}(2R, R)) \geq 1 - (1 - c_4) e^{-c_6 c_{\text{Mills}}^{-1} R^{1-\zeta} h}.$$

By our choice of $h = h_R$, the latter quantity converges to 1, and so applying (4.7) and (4.8) one more time gives the result. \square

4.3. Sharpness of the phase transition. The proof of Theorem 4.1 follows from Theorem 4.4 from a classical bootstrapping argument that is borrowed from [MV19].

We shall need two auxiliary results. The first result is a kind of ‘sprinkled’ version of the quasi-independence statement in Theorem 3.7 that we deduce from Lemma 2.3.

Proposition 4.6. *Let $\gamma' > 0$. There exist $c_1, c_2 > 0$ such that, for every $R \geq 1$, every pair of compact sets $D_1 \subset \mathbb{R}^2$ (resp. D_2) of diameter at most $5R$ and such that $\text{dist}(D_1, D_2) \geq R^{\gamma'}$, and every pair of every decreasing collections of n_1 (resp. n_2) crossing events $(E_i^1)_{i \leq n_1}$ (resp. $(E_i^2)_{i \leq n_2}$) that are supported on D_1 (resp. D_2),*

$$\mathbb{P}(f + c_1 R^{2+\gamma'(1-\alpha)} (\log R)^2 \in E^1 \cap E^2) \leq \mathbb{P}(f \in E^1) \mathbb{P}(f \in E^2) + c_1 e^{-c_2 R},$$

where $E^j = \cap_{i \leq n_j} E_i^j$ for $j \in \{1, 2\}$.

Proof. This follows by applying Lemma 2.3 with the settings $r = R^{\gamma'}$ and $t = R$ (see [MV19, Proposition 6.2] for details, although note that in [MV19, Proposition 6.2] it was possible to get an improved bound by taking $t = \sqrt{R}$ since the relevant error probabilities decayed square-exponentially). \square

The second is a functional inequality which we take from [MV19, Lemma 6.3]:

Lemma 4.7. *Let $\gamma' \in (0, 1)$ and let $(a_R)_{R \geq 0}$ be a positive function such that $a_R \rightarrow 0$ and for which there exist $c_1, c_2, R_0 > 0$ such that, for all $R \geq R_0$,*

$$a_{2R+R^{\gamma'}} \leq c_1 a_R^2 + e^{-c_2 R}.$$

Then there exist $c_3, c_4 > 0$ and a positive sequence $(m_n)_{n \geq 1}$ such that, for all $n \geq 1$,

$$2^n \leq m_n \leq c_3 2^n \quad \text{and} \quad a_{m_n} \leq e^{-c_4 m_n}.$$

Proof. The proof is identical to the proof of Lemma [MV19, Lemma 6.3] after replacing $a_{2R+\sqrt{R}}$ with $a_{2R+R^{\gamma'}}$. Indeed the only properties of the function $f(x) = \sqrt{R}$ used in that proof are that (i) $f(x)/x$ decreases monotonically to 0 as $x \rightarrow \infty$, and (ii) $f(2^n)/2^n$ is summable over n . Since these are satisfied also for $f(x) = x^{\gamma'}$, the proof goes through. \square

We can now complete the proof of Theorem 4.1 (in the bounded Mills ratio case):

Proof of Theorem 4.1 (under Conditions 1.1 and 1.4). Define the exponents

$$\gamma' \in \left(\frac{2}{\alpha - 1}, 1 \right) \quad \text{and} \quad \gamma'' \in (0, (\alpha - 1)\gamma' - 2),$$

which is possible since $\alpha > 3$. Fix $\ell = 0$, and define the increasing sequence of levels $\ell_R = \ell - R^{-\gamma''}$. Note that

$$(4.10) \quad \ell_{2R+R^{\gamma'}} - \ell_R \geq \ell_{2R} - \ell_R = (1 - 2^{-\gamma''}) R^{-\gamma''} \geq c_1 R^{2+\gamma'(1-\alpha)} (\log R)^2$$

eventually for sufficiently large R . Since $\text{Cross}_\ell(2R, R)$ is increasing in ℓ and since $\ell_R < \ell$, defining

$$a_R = \mathbb{P}(f \notin \text{Cross}_{\ell_R}(2R, R)),$$

it is sufficient to prove the existence of a $c_1 > 0$ such that, for sufficiently large R ,

$$(4.11) \quad a_R \leq e^{-c_1 R}.$$

We deduce (4.11) from the following functional inequality for a_R , whose proof is as in [MV19, Lemma 6.3] (with Proposition 4.6 and (4.10) the crucial inputs): There exists a $c_2 > 0$ such that, for sufficiently large $R \geq 1$,

$$(4.12) \quad a_{2R+R^{\alpha'}} \leq 49a_R^2 + e^{-c_2 R}.$$

Recalling that Theorem 4.4 implies that $a_R \rightarrow 0$, an application of Lemma 4.7 then yields the existence of constants $c_3, c_4 > 0$ and a positive subsequence $(m_n)_{n \geq 1}$ such that, for all $n \geq 1$,

$$2^n \leq m_n \leq c_3 2^n \quad \text{and} \quad a_{m_n} \leq e^{-c_4 m_n}.$$

This implies (4.11) for $R \in \{m_n\}_{n \geq 1}$, which can be extended to all $R \geq 0$ by standard gluing arguments. \square

5. THE PHASE TRANSITION IN THE GENERAL CASE

We now turn to the proof of Theorem 4.1 in the general case of bounded Mills ratio in the tails (Conditions 1.2 and 1.5).

Recall the family of smoothed fields f_η introduced in Section 2.2. The first step is to adapt the argument we used to establish Theorem 4.1 in the bounded Mills ratio case to the field f_η for a carefully chosen sequence $\eta = \eta_R \rightarrow 0$.

Recall the exponents $\gamma, \zeta, \xi > 0$ defined in (4.5). For this section we shall redefine these exponents, and introduce a new exponent $\kappa > 0$, as follows:

$$(5.1) \quad \gamma \in \left(0, \frac{c_{\text{Arm}}}{2 + 3c_{\text{Arm}}}\right), \quad \zeta = \frac{\gamma c_{\text{Arm}} + 1}{c_{\text{Arm}} + 1} \in (\gamma, 1), \quad \kappa \in (2\gamma, 1 - \zeta), \quad \eta \in (0, 1 - \zeta - \kappa).$$

It is easy to check that such a choice is possible; indeed the condition

$$(5.2) \quad \gamma < \frac{c_{\text{Arm}}}{2 + 3c_{\text{Arm}}}$$

has been defined so as to be equivalent to

$$2\gamma < 1 - \frac{\gamma c_{\text{Arm}} + 1}{c_{\text{Arm}} + 1} = 1 - \zeta,$$

which ensures that κ is well-defined. Define also the scales $r = r_R = R^\gamma \rightarrow \infty$ and $h = h_R = R^{-\xi} \rightarrow 0$ as before, and introduce the scale $\eta = \eta(R) = R^{-\kappa} \rightarrow 0$.

Theorem 5.1. *For every $\ell > 0$ there exists a $c > 0$ such that, for every $R \geq 1$,*

$$\mathbb{P}(f_\eta \in \text{Cross}_\ell(2R, R)) \geq 1 - e^{-cR}.$$

Proof. It is straightforward to check that the proof of Theorem 4.1 in the case of bounded Mills ratio (in Section 4) goes through replacing f with f_η . Indeed, all of the estimates used are uniform in $\eta \geq 0$, with the single exception of the final step of the proof of Theorem 4.4, in which we need to verify that

$$(5.3) \quad \mathbb{P}(f_{\eta,r}^{\varepsilon,h} \in \text{Cross}_{\ell/2}(2R, R)) \geq 1 - (1 - c_4)e^{-c_6 c_{\text{Mills}}^{-1}(\mu_\eta) R^{1-\zeta} h}$$

tends to 1 as $R \rightarrow \infty$. Since $c_{\text{Mills}}(\mu_\eta) \leq 1/\eta$ by construction (Lemma 2.2), we have

$$c_{\text{Mills}}^{-1}(\mu_\eta) R^{1-\zeta} h \geq R^{-\kappa} R^{1-\zeta} R^{-\xi} \rightarrow \infty,$$

where the convergence holds by the choice of the parameters in (5.1). Hence we verify that the probability in (5.3) converges to 1, as required. \square

The result will now follow by exploiting a natural coupling of f_η and f (or rather their truncations) such that they agree outside a small set.

Lemma 5.2. *Let $c_1 > 0$ be as in Lemma 2.2, and for every $t > 0$ let \mathcal{P}_t be a Poisson point process on \mathbb{R}^2 with intensity t . Then there exists a coupling of $f_{\eta,r}$, f_r and $\mathcal{P}_{c_1\eta}$ such that $f_{\eta,r}$ and f_r are equal outside the set*

$$\bigcup_{i \in \mathcal{P}_{c_1\eta}} (\{i\} + B(r)).$$

Proof. Recall that $d_{\text{TV}}(\mu, \mu_\eta) \leq c_1\eta$ by construction (Lemma 2.2). By the definition of total variation distance, there exists a probability distribution μ' such that that $f_{r,\eta}$ can be constructed from f_r by thinning the Poisson point process \mathcal{P} used to define f_r to have intensity $1 - c_1\eta$, and adding the field

$$\sum_{i \in \mathcal{P}_{\text{thin}}} Y_i^{\mu'} g_r(x - i)$$

where $\mathcal{P}_{\text{thin}}$ denotes the set of thinned points, and Y_i^μ are independent draws from μ' . Since $\mathcal{P}_{\text{thin}} \stackrel{d}{=} \mathcal{P}_{c_1\eta}$ by the thinning property of the Poisson point process, and the support of g_r lies in $B(r)$, we have the result. \square

Let us now finish the proof of Theorem 4.1 in the general case:

Proof of Theorem 4.1 (under Conditions 1.2 and 1.5). We first claim that, as a consequence of Theorem 5.1

$$(5.4) \quad \mathbb{P}(f_{\eta,r} \in \text{Cross}_\ell(2r, r)) \geq 1 - r^2/R^2(\log r)^{-1}.$$

Indeed, by Proposition 3.6,

$$|\mathbb{P}(f_\eta \in \text{Cross}_\ell(2r, r)) - \mathbb{P}(f_{\eta,r} \in \text{Cross}_\ell(2r, r))|$$

is at most

$$c_1 t^2 r^{3-\alpha} (\log r)^2 ((\log r)^2 + (\log t)^2) + c_1 r^2 e^{-c_2 t}$$

for all large enough $t > 0$, where $\alpha > 0$ is a constant we can set arbitrarily large. Setting $t = (\log r)^2$ and $\alpha > 1 + 2/\gamma$, the above tends to zero, and so the claim follows from Theorem 5.1.

The next step is to show that

$$(5.5) \quad \mathbb{P}(f_r \in \text{Cross}_\ell(2R, R)) \rightarrow 1$$

as $R \rightarrow \infty$. For this we will introduce a discrete percolation model on the edges of (a translated copy of) the lattice $r\mathbb{Z}^2$, as follows.

Define the index sets

$$I^h = \{1, \dots, \lceil 2R/r \rceil\} \quad \text{and} \quad I^v = \{1, \dots, \lfloor R/r \rfloor\},$$

and let L^h and L^v be respectively a unit line-segment in the horizontal and vertical directions from the origin. Then define a set of horizontal edges $\mathcal{T}^h = \{e_{i,j}^h\}_{i \in I^h, j \in I^v}$ where

$$e_{i,j}^h = r(i - 1/2, j - 1/2) + L^h,$$

and similarly a set of vertical edges $\mathcal{T}^v = \{e_{i,j}^v\}_{i \in I^h, j \in I^v}$ where

$$e_{i,j}^v = r(i - 1/2, j - 1/2) + L^v.$$

Notice that $\mathcal{T} = \mathcal{T}^h \cup \mathcal{T}^v$ is then a translation of the grid

$$r\mathbb{Z}^2 \cap [r\lceil 2R/r \rceil, r\lfloor R/r \rfloor].$$

We then define a percolation model on \mathcal{T} by declaring a horizontal edge $e_{(i,j)}^h \in \mathcal{T}^h$ to be open for the field $g \in \{f_r, f_{\eta,r}\}$ if and only if

$$\{g(\cdot - r(i, j)) \in \text{Cross}_\ell(2r, r)\}.$$

and, similarly, a vertical edge $e_{(i,j)}^v \in \mathcal{T}^v$ to be open for the field $g \in \{f_r, f_{\eta,r}\}$ if and only if

$$\{g(\cdot - r(i, j)) \in \text{Cross}_\ell(r, 2r)\}.$$

Let \mathcal{C} be the event that the percolation model on \mathcal{T} has a left-right crossing. The key property that the construction offers is that

$$(5.6) \quad \{g \in \mathcal{C}\} \subset \{g \in \text{Cross}_\ell(2R, R)\}.$$

Now, by (5.4) and the symmetry of the shot noise field, each edge in \mathcal{T} is open for $f_{\eta,r}$ with probability exceeding $1 - R^{-(2-2\gamma)}(\log R)^{-1}$. Since there are $O((R/r)^2) = O(R^{2-2\gamma})$ edges in \mathcal{T} , by the union bound the probability that all edges are open for $f_{\eta,r}$ tends to one. Moreover, by Lemma 5.2, there is a coupling of f_r , $f_{\eta,r}$, and a Poisson point process $\mathcal{P}_{c_1\eta}$ of intensity $c_1\eta$, such that each edge in \mathcal{T} is open for $f_{\eta,r}$ if and only if it is open for f_r except for the edges intersecting the set

$$(5.7) \quad \bigcup_{i \in \mathcal{P}_{c_1\eta}} (\{i\} + B(r)).$$

Since

$$\eta r^2 = R^{-\kappa} R^{2\gamma} \rightarrow 0,$$

an edge $e \in \mathcal{T}$ intersects (5.7) with probability tending to zero. Hence by standard properties of sub-critical percolation, if edges of \mathcal{T} are closed if and only if they intersect (5.7) then the event \mathcal{C} occurs with probability tending to one. Thus \mathcal{C} occurs for f_r with probability tending to one, and hence we deduce (5.5) from (5.6).

Finally we show that

$$(5.8) \quad \mathbb{P}(f \in \text{Cross}_\ell(2R, R)) \rightarrow 1.$$

Again, by Proposition 3.6,

$$|\mathbb{P}(f \in \text{Cross}_\ell(2R, R)) - \mathbb{P}(f_r \in \text{Cross}_\ell(2R, R))|$$

is at most

$$c_1 t^2 R^2 r^{1-\alpha} (\log r)^2 ((\log r)^2 + (\log t)^2) + c_1 R^2 e^{-c_2 t}.$$

Setting $t = (\log r)^2$ and $\alpha > 1 + 2/\gamma$, the above tends to zero, and so the claim follows from (5.5).

Since (5.8) is the analogue of Theorem 4.4, the conclusion of Theorem 4.1 then follows by the same bootstrap argument as we used in the bounded Mills ratio case. \square

Remark 5.3. A careful analysis of this proof shows that, rather than requiring the super-polynomial decay of g (and its first two derivatives), it is sufficient that the kernel decays polynomially with exponent $\alpha > 1 + \frac{2}{\gamma}$. On the other hand, the constraint on γ in (5.2) is also fixed by the proof (in particular, by the need to fix $\kappa \in (2\gamma, 1 - \zeta)$), and combining these we have

$$\alpha > 7 + \frac{4}{c_{\text{Arm}}},$$

i.e. the required exponent α depends on the arm decay exponent at the zero level.

Since the value of c_{Arm} that comes out of the proof of Theorem 3.10 is difficult to quantify (although in principle it could be made explicit), it is difficult to extract an explicit polynomial decay condition that we could substitute in Condition 1.2 in place of super-polynomial decay. Nevertheless, even under the most optimistic assumption that c_{Arm} takes its predicted value from critical percolation theory ($c_{\text{Arm}} = 5/48$), the required decay exponent would need to be large ($\alpha > 227/5 \approx 45$).

APPENDIX A.

A.1. Concentration inequalities. In this section we prove concentration inequalities for shot noise fields with unbounded marks, based on the work of Reynaud-Bouret [RB03] on Poisson stochastic integrals.

Our result concerns shot noise fields in arbitrary dimension, i.e. we consider the field

$$f(x) = \sum_{i \in \mathcal{P}} Y_i^\mu g(x - i)$$

where $g(x) \in L^1(\mathbb{R}^d)$ is a kernel, \mathcal{P} is a Poisson point process on \mathbb{R}^d with unit intensity, and $\{Y_i^\mu\}$ are i.i.d. random variables drawn from a mark distribution μ with finite mean. We assume that the kernel g decays polynomially with exponent $\alpha > d$, i.e., there exists a $c_1 > 0$ such that

$$(A.1) \quad g(x) \leq c_1 (1 + |x|)^{-\alpha}.$$

We also assume that the mark distribution decays stretched-exponentially with exponent γ , i.e., there are constants $\gamma > 0$ and $c_2, u_0 \geq 0$ such that

$$(A.2) \quad \mu((u, \infty)) \leq c_2 \exp(-u^\gamma), \quad u \geq u_0.$$

To state our result we define the auxiliary function

$$[x] = \begin{cases} (1 + \log |x|)^{\frac{2}{\gamma}}, & |x| \geq 1, \\ 1, & |x| < 1. \end{cases}$$

Moreover, for a function $h : \mathbb{R}^d \rightarrow \mathbb{R}$, we introduce the auxiliary functions

$$\tilde{h}(x) = [x]h(x) \quad \text{and} \quad \hat{h}(x) = \sup_{y \in B} |\tilde{h}(x + y)|,$$

where $B = [-1/2, 1/2]^d$. Finally we define the positive constant

$$\kappa = 2 + c_2 \int_{\mathbb{R}^d} \exp(-(1 + \log |x|)^2/2) dx.$$

Proposition A.1 (Concentration with unbounded marks). *Suppose that g and μ satisfy (A.1) and (A.2). Then the following hold:*

(1) For all $t, s \geq 1$,

$$(A.3) \quad \mathbb{P}\left(\|f\|_{\infty, sB} \geq t^{\frac{1}{\gamma}}\left(\|\hat{g}\|_{L^1} + \sqrt{2t}\|\hat{g}\|_{L^2} + \frac{t}{3}\|\hat{g}\|_{\infty}\right)\right) \leq s^2\kappa e^{-t/2}.$$

(2) Assume that μ is symmetric. Then, for all $t \geq 1$,

$$(A.4) \quad \mathbb{P}\left(|f(0)| \geq t^{\frac{1}{\gamma}}\left(\sqrt{2t}\|\tilde{g}\|_{L^2} + \frac{t}{3}\|\tilde{g}\|_{\infty}\right)\right) \leq \kappa e^{-t/2}.$$

Suppose furthermore that g is C^1 -smooth, and there exists a $c_3 > 0$ such that, for all $i = 1, \dots, d$,

$$\partial_i g(x) \leq c_3(1 + |x|)^{-\alpha-1}.$$

Then for all $t, s \geq 1$,

$$(A.5) \quad \mathbb{P}\left(\|f\|_{\infty, sB} \geq dt^{\frac{1}{\gamma}}\left(\sqrt{2t}\|\tilde{g}\|_{L^2} + \frac{t}{3}\|\tilde{g}\|_{\infty} + \|\widehat{\nabla}g\|_{L^1} + \sqrt{2t}\|\widehat{\nabla}g\|_{L^2} + \frac{t}{3}\|\widehat{\nabla}g\|_{\infty}\right)\right) \leq s^2(d+1)\kappa e^{-t/2}.$$

To illustrate these bounds, we present the following family of examples:

Example A.2. Fix $\alpha > 0$ and consider the family of kernels

$$g(x) = g_R(x) = |x|^{-\alpha}\mathbf{1}_{\{|x|>R\}}, \quad R > 1.$$

Note that, as $R \rightarrow \infty$,

$$\|g\|_{\infty} \sim R^{-\alpha}, \quad \|g\|_{L^1} \sim R^{d-\alpha}, \quad \|g\|_{L^2} \sim R^{d/2-\alpha},$$

$$\|\nabla g\|_{\infty} \sim R^{-\alpha-1}, \quad \|\nabla g\|_{L^1} \sim R^{d-\alpha-1} \quad \text{and} \quad \|\nabla g\|_{L^2} \sim R^{d/2-\alpha-1},$$

and the corresponding norms of \tilde{g}, \hat{g} and $\widehat{\nabla}g$ also decay with the same respective powers (up to logarithmic factors). In particular, the dominating terms among the norms in (A.5) are of order $R^{d-\alpha-1}$ and $R^{d/2-\alpha}$ (up to logarithmic factors).

Now suppose that (A.2) holds, and set $t = (\log R)^2$ and $s = R^{\beta}$ for some $\beta > 0$. Then there exist constants $c_1, c_2, c_3 > 0$ such that, for all $R \geq 1$,

$$\mathbb{P}(\|f\|_{\infty, sB} \geq c_1(R^{d-\alpha-1} + R^{d/2-\alpha})(\log R)^{c_2}) \leq c_3 R^{2\beta} \exp(-(\log R)^2/2).$$

In particular, this probability decays faster than any polynomial as $R \rightarrow \infty$.

Remark A.3. If the mark distribution μ is bounded rather than decaying stretched-exponentially, then (A.3)–(A.5) hold with g replacing \tilde{g} (with \hat{h} defined in terms of \hat{h} instead of \tilde{h}), and 1 replacing $t^{1/\gamma}$.

Proof of Proposition A.1. Let $t \geq 1$. We begin by defining a suitable event that allows to truncate the mark distribution. Denote by $(x_i)_{i \geq 1}$ some enumeration of the points of \mathcal{P} , and remark that $(x_i, Y_i)_{i \geq 1}$ has the law of a Poisson point process \mathcal{P}' with intensity measure $dx\mu(dm)$ on \mathbb{R}^d , called the *marked point process*. Define $\tilde{Y}_i = Y_i/([x_i]t^{1/\gamma})$ and $Z = \{(x, m) : |m| \leq [x]t^{1/\gamma}\}$, and define the truncation event

$$\Omega = \{|\tilde{Y}_i| \leq 1 \text{ for all } i \geq 1\} = \{\mathcal{P}' \cap Z^c = \emptyset\}.$$

The Campbell-Mecke formula bounds the probability of the complement as

$$(A.6) \quad \mathbb{P}(\Omega^c) \leq \sum_i \mathbb{P}(|\tilde{Y}_i| > 1) = \int_{\mathbb{R}^d} \mathbb{P}(Y_1 > [x]t^{1/\gamma}) dx \leq c_2 \int_{\mathbb{R}^d} \exp(-[x]^{\gamma}t) dx$$

$$(A.7) \quad \leq c_2 \int_{\mathbb{R}^d} \exp\left(-\frac{t}{2}(1 + [x]^{\gamma})\right) dx \leq c_2 e^{-t/2} \int_{\mathbb{R}^d} \exp(-[x]^{\gamma}/2) dx \leq c_2 \bar{c} e^{-t/2}.$$

We next recall the results of Reynaud-Bouret [RB03] on Poisson stochastic integrals. Let \mathcal{P}'' be a Poisson point process with intensity $dx\mu(dm)\mathbf{1}_{\{(x,m) \in Z\}}$ on $\mathbb{R}^d \times \mathbb{R}$. For $h : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable bounded, introduce the auxiliary function

$$\bar{h}(x, m) = mh(x)([x]t^{1/\gamma})^{-1}, \quad (x, m) \in \mathbb{R}^d \times \mathbb{R},$$

and define

$$I_h = \sum_{(x,m) \in \mathcal{P}''} \bar{h}(x, m).$$

Using the abbreviations $\|h\|_p = \|h\|_{L^p(\mathbb{R}^d)}$ and $\|\bar{h}\|_p = \|\bar{h}\|_{L^p(\mathbb{R}^d \times \mathbb{R})}$ for $p \in [1, \infty]$, [RB03, Proposition 7] states that

$$(A.8) \quad \mathbb{P}\left(I_h \geq \mathbb{E}(I_h) + \sqrt{2t}\|\bar{h}\|_2 + \frac{t}{3}\|\bar{h}\|_\infty\right) \leq e^{-t}.$$

Since $\|\bar{h}\|_2 \leq \|h\|_2$ and $\|\bar{h}\|_\infty \leq \|h\|_\infty$, it yields

$$(A.9) \quad \mathbb{P}\left(I_h \geq \mathbb{E}(I_h) + \sqrt{2t}\|h\|_2 + \frac{t}{3}\|h\|_\infty\right) \leq e^{-t}.$$

We are now ready to prove the claims (A.3)–(A.5), beginning with (A.4). First note that, for all $u \geq 0$,

$$\begin{aligned} \mathbb{P}(f(0) \geq t^{1/\gamma}u) &= \mathbb{P}\left(\sum_{(x_i, Y_i) \in \mathcal{P}'} Y_i t^{-1/\gamma} g(x_i) \geq u\right) = \mathbb{P}\left(\sum_{(x_i, Y_i) \in \mathcal{P}'} \tilde{Y}_i \tilde{g}(x_i) \geq u\right) \\ &\leq \mathbb{P}\left(\left\{\sum_{(x_i, Y_i) \in \mathcal{P}'} \tilde{Y}_i \tilde{g}(x_i) \geq u\right\} \cap \Omega\right) + \mathbb{P}(\Omega^c) \\ &\leq \mathbb{P}\left(\sum_{(x_i, Y_i) \in \mathcal{P}''} \tilde{Y}_i \tilde{g}(x_i) \geq u\right) + \mathbb{P}(\Omega^c) \\ &= \mathbb{P}(I_{\tilde{g}} \geq u) + \mathbb{P}(\Omega^c), \end{aligned}$$

the last inequality can be proved by discretising \mathcal{P} . Note also that $\mathbb{E}(I_{\tilde{g}}) = 0$ if the mark distribution μ is symmetric. Abbreviating $u_1 = \sqrt{2t}\|\tilde{g}\|_2 + \frac{t}{3}\|\tilde{g}\|_\infty$ and applying (A.9) gives that

$$\mathbb{P}(I_{\tilde{g}} \geq u_1) \leq e^{-t}.$$

Hence, doing the same with $h = -\tilde{g}$,

$$\mathbb{P}(|f(0)| \geq t^{\frac{1}{\gamma}}u_1) \leq \mathbb{P}(f(0) \geq t^{\frac{1}{\gamma}}u_1) + \mathbb{P}(-f(0) \geq t^{\frac{1}{\gamma}}u_1) \leq 2e^{-t} + \mathbb{P}(\Omega^c).$$

Combining with (A.6) yields (A.4).

We turn to (A.3). Similarly to above, for all $u \geq 0$,

$$\begin{aligned} \mathbb{P}\left(\sup_{x \in B} |f(x)| \geq t^{1/\gamma}u\right) &= \mathbb{P}\left(\sup_{x \in B} \left| \sum_{(x_i, Y_i) \in \mathcal{P}'} \tilde{Y}_i \tilde{g}(x + x_i) \right| \geq u\right) \\ &\leq \mathbb{P}\left(\sum_{(x_i, Y_i) \in \mathcal{P}'} |\tilde{Y}_i| \sup_{x \in B} |\tilde{g}(x + B)| \geq u\right) \\ &= \mathbb{P}\left(\sum_{(x_i, Y_i) \in \mathcal{P}'} |\tilde{Y}_i| \hat{g}(x) \geq u\right) \\ &\leq \mathbb{P}(I_{\hat{g}} \geq u) + \mathbb{P}(\Omega^c) \end{aligned}$$

Abbreviating $u_2 = \|\hat{g}\|_1 + \sqrt{2t}\|\hat{g}\|_2 + \frac{t}{3}\|\hat{g}\|_\infty$ and applying (A.8) gives that

$$\mathbb{P}(I_{\hat{g}} \geq u_2) \leq e^{-t}.$$

Combining with (A.6) yields (A.3) for $s = 1$. For larger s , cut sB in $[s^2]$ cubes B_k homothetic to a subset of B . We have, for $u \geq 0$,

$$\mathbb{P}(\|f\|_{\infty, sB} \geq u) \leq \sum_{k=1}^{[s^2]} \mathbb{P}(\|f\|_{\infty, B_k} \geq u) \leq s^2 \mathbb{P}(\|f\|_{\infty, B} \geq u)$$

by translational invariance. This yields (A.3) for general $s \geq 1$.

Finally, let us prove (A.5). For $x \in sB$, $|f(x)| \leq |f(0)| + \frac{\sqrt{d}}{2} \sup_{x \in sB} |\nabla f(x)|$. Hence we have, for $u \geq 0$,

$$\begin{aligned} \mathbb{P}(\|f\|_{\infty, sB} \geq u) &\leq \mathbb{P}\left(|f(0)| \geq \frac{u}{2} \text{ or } \exists 1 \leq i \leq d : \sup_{sB} |\partial_i f| \geq \frac{u}{d}\right) \\ &\leq \mathbb{P}\left(|f(0)| > \frac{u}{2}\right) + \sum_{i=1}^d \mathbb{P}\left(\|\partial_i f\|_{\infty, sB} \geq \frac{u}{d}\right). \end{aligned}$$

Setting

$$u = dt^{\frac{1}{\gamma}} \left(\sqrt{2t} \|\tilde{g}\|_{L^2} + \frac{t}{3} \|\tilde{g}\|_{\infty} + \|\widehat{\nabla}g\|_{L^1} + \sqrt{2t} \|\widehat{\nabla}g\|_{L^2} + \frac{t}{3} \|\widehat{\nabla}g\|_{\infty} \right)$$

and applying (A.4) to the field f and (A.3) to the field $\partial_i f$ (which is a shot noise field with kernel $\partial_i g$) we get the result. \square

A.2. Bounded density of shot noise fields. In this section we give examples of shot noise fields (1.1) which satisfy the bounded density assumption in Condition 1.5 (which implies the weaker Condition 1.4).

Proposition A.4. *Consider the shot noise field (1.1) with arbitrary mark distribution μ and kernel either*

$$(A.10) \quad g(x) = (1 + |x|)^{-\alpha}$$

for some $\alpha > 2$, or

$$(A.11) \quad g(x) = \exp(-|x|^\alpha) \quad \text{or} \quad g(x) = \exp(-(1 + |x|^2)^{\alpha/2})$$

for some $\alpha \in (0, 1)$. Then Condition 1.5 is satisfied.

Remark A.5. In the case of the kernel (A.10), the hypothesis $\alpha > 2$ is only used to ensure that g is integrable (and hence $f(0)$ is well defined), but is not used elsewhere in the proof.

Proof. Recall from Remark 1.6 that the characteristic function of $(f(0), \nabla f(0))$ is

$$\varphi(u, v) = \exp\left(\int (\exp(im[ug(x) + \langle v, \nabla g(x) \rangle]) - 1) dx \mu(dm)\right).$$

By the discussion in Remark 1.6, it is sufficient to prove that $|\varphi(u, v)|^\lambda$ is integrable on $\mathbb{R} \times \mathbb{R}^2$ for some $\lambda \in (0, 1)$.

We first find suitable bounds on $|\varphi(u, v)|$ for $|u| \vee |v| > 1$. Introduce a positive constant $\rho = \rho_{u, v}$ (to be defined later) and $u_\theta = (\cos(\theta), \sin(\theta))$. Assume that there is $m_0 > 0$ such that $\mu([m_0, m_0 + 1]) > 0$ (the reasoning is the same if μ is concentrated on $(-\infty, 0)$). Then we have

$$(A.12) \quad \begin{aligned} |\varphi(u, v)| &= \exp\left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} (\cos(umg(x) + \langle v, m\nabla g(x) \rangle) - 1) dx \mu(dm)\right) \\ &\leq \exp\left(\int_{m_0}^{m_0+1} \int_{\rho}^{\infty} \int_0^{2\pi} (\cos(umh(r) + \langle v, mu_\theta h'(r) \rangle) - 1) r dr d\theta \mu(dm)\right), \end{aligned}$$

where $h(|x|) = g(x)$ (well-defined since g is isotropic in all the cases we consider). Recall the zeroth Bessel function, defined for $s \geq 0$ by

$$J_0(s) = \int_0^{2\pi} \cos(s \cos(\theta)) d\theta = \int_0^{2\pi} \exp(i\langle v, u_\theta \rangle) d\theta \quad \text{for any } v \in \mathbb{R}^2 \text{ with } |v| = s.$$

Then (A.12) equals

$$\begin{aligned} & \exp \left(\int_{m_0}^{m_0+1} \int_{\rho}^{\infty} (\cos(umh(r))J_0(m|v||h'(r)|) - 2\pi)rdr\mu(dm) \right) \\ &= \exp \left(\underbrace{2\pi \int_{m_0}^{m_0+1} \int_{\rho}^{\infty} (\cos(umh(r)) - 1)rdr}_{I_1} \right. \\ & \quad \left. + \underbrace{\int_{m_0}^{m_0+1} \int_{\rho}^{\infty} \cos(umh(r))(J_0(m|v||h'(r)|) - J_0(0))rdr\mu(dm)}_{I_2} \right). \end{aligned}$$

Let us bound I_1 and I_2 separately. In the rest of the proof, c denotes a positive constant (depending only on α, m_0, λ) whose value might change from line to line.

For I_1 , recall that $\cos(s) - 1 \leq -s^2/3$ and $\cos(s) > 1/2$ for $0 < s \leq s_0$, where s_0 is some positive constant. Defining

$$\rho_u := \inf\{\rho : |u|(m_0 + 1)h(r) \leq s_0, r \geq \rho\},$$

we deduce that, for $\rho \geq \rho_u$,

$$I_1 \leq -c|u|^2 \int_{\rho}^{\infty} h(r)^2 r dr.$$

For I_2 , we first recall that $J_0(t) - J_0(0) \leq -ct^2$ for $0 \leq t \leq t_0$, where t_0 is some positive constant. Consider now

$$\rho_v := \inf\{\rho : |v|(m_0 + 1)h'(r) \leq t_0; r \geq \rho\}.$$

Then we have for $\rho = \rho_{u,v} := \max\{\rho_u, \rho_v\} \geq c \max\{|u|^{1/\alpha}, |v|^{1/(\alpha+1)}\}$

$$I_2 \leq -c|v|^2 \int_{\rho}^{\infty} h'(r)^2 r dr.$$

Recall that it is sufficient to prove that $|\varphi(u, v)|^\lambda$ is integrable on $\mathbb{R} \times \mathbb{R}^2$ for some $\lambda \in (0, 1)$. We will actually prove that $B_i := \int_{D_i} |\varphi(u, v)|^\lambda < \infty$ for $i = 1, 2$, where

$$D_1 := \{u \in \mathbb{R}, v \in \mathbb{R}^2 : 1 < |v|, 0 < |u| : \rho_u \leq \rho_v\}$$

$$D_2 := \{u \in \mathbb{R}, v \in \mathbb{R}^2 : 1 < |u|, 0 < |v| : \rho_v \leq \rho_u\},$$

which is also sufficient since $|\varphi(u, v)| \leq 1$.

Let us now specialise to the kernels (A.10) and (A.11). In the case (A.10) we have $h(r) = (1 + r)^{-\alpha}$, which yields

$$I_1 \leq -cu^2\rho^{2-2\alpha} \quad \text{and} \quad I_2 \leq -c|v|^2\rho^{-2\alpha}.$$

Hence

$$|\varphi(u, v)| \leq \exp(-cu^2\rho^{2-2\alpha} - c|v|^2\rho^{-2\alpha})$$

for any $|u| \vee |v| > 1$ and ρ satisfying

$$\rho \geq c \max\{|u|^{1/\alpha}, |v|^{1/(\alpha+1)}\}.$$

Therefore we have

$$\begin{aligned} B_1 &\leq \int_{D_1} \exp(-c\lambda|u|^2\rho^{2-2\alpha} - c\lambda|v|^2\rho^{-2\alpha}) \mathbf{1}_{\{\rho \geq c|v|^{1/(\alpha+1)}\}} dudv \\ &\leq \int_{D_1} \exp(-c\lambda|u|^2|v|^{2-2\alpha} - c\lambda|v|^{2-2\frac{\alpha}{\alpha+1}}) dudv \\ B_2 &\leq \int_{D_2} \exp(-c\lambda|u|^2\rho^{2-2\alpha} - c\lambda|v|^2\rho^{-2\alpha}) \mathbf{1}_{\{\rho \geq c|u|^{1/\alpha}\}} dudv \\ &\leq \int_{D_2} \exp(-c\lambda|u|^{2/\alpha} - c\lambda|v|^2|u|^{-2}) dudv. \end{aligned}$$

Since it is easy to check that these integrals are finite for every $\lambda > 0$, the proof is complete in the case (A.10).

In the case (A.11) we will only give the full argument for $h(r) = e^{-r^\alpha}$, since the decay of $h(r)$ and $h'(r)$ as $r \rightarrow \infty$ are similar for both kernels. In this case we have

$$I_1 \leq -cu^2 \int_\rho^\infty re^{-2r^\alpha} dr \leq -cu^2 \rho^{2-\alpha} e^{-2\rho^\alpha}$$

$$I_2 \leq -c|v|^2 \int_\rho^\infty r^{2\alpha-1} e^{-2r^\alpha} dr \leq -c|v|^2 \rho^\alpha e^{-2\rho^\alpha}$$

for any $|u| \vee |v| > 1$ and ρ satisfying

$$e^{-\rho^\alpha} \leq c/|u| \quad \text{and} \quad \rho^{\alpha-1} e^{-\rho^\alpha} \leq c/|v|.$$

On D_2 , $e^{-\rho^\alpha} \geq c/|u|$, and so

$$B_2 \leq \int_{D_1} \exp(-cu^2 \rho^{2-\alpha} u^{-2} - c|v|^2 \rho^\alpha u^{-2}) dudv$$

$$\leq \int_1^\infty \exp(-c\rho^{2-\alpha}) \left(\int_0^\infty \exp\left(-c\left(\frac{|v|}{u\rho^{-\alpha/2}}\right)^2\right) dv \right) du$$

$$\leq c \int_1^\infty \exp(-c\rho^{2-\alpha}) u^2 \rho^{-\alpha} du.$$

For $2 - \alpha > \alpha$ (i.e. $\alpha < 1$),

$$\exp(-c\rho^{2-\alpha}) u^2 = \exp(-c\rho^{2-\alpha} + c\rho^\alpha) \leq c \exp(-2\rho^\alpha) = cu^{-2},$$

hence $B_2 < \infty$. On D_1 , $e^{-\rho^\alpha} = c\rho^{1-\alpha}/|v|$, and so

$$B_1 \leq \int_{D_2} \exp(-cu^2 \rho^{4-3\alpha} |v|^{-2} - c\rho^{2-\alpha}) dudv \leq \int_1^\infty \exp(-c\rho^{2-\alpha}) |v| \rho^{-2+\frac{3\alpha}{2}} dv.$$

As before, $\exp(-c\rho^{2-\alpha})$ is dominated by $\exp(c\rho^\alpha)$, and hence by any power of $|v|$, and the integral is finite, hence $(f(0); \nabla f(0))$ has a bounded joint density. □

REFERENCES

- [AB18] D. Ahlberg and R. Baldasso. Noise sensitivity and Voronoi percolation. *Electron. J. Probab.*, 23, 2018.
- [Ale96] K.S. Alexander. Boundedness of level lines for two-dimensional random fields. *Ann. Probab.*, 24(4):1653–1674, 1996.
- [AT07] R. Adler and J. Taylor. *Random fields and geometry*. Springer, 2007.
- [BB05] M. Bagnoli and T. Bergstrom. Log-concave probability and its applications. *Econ. Theory*, 26:445–469, 2005.
- [BB09] F. Baccelli and B. Błaszczyszyn. *Stochastic geometry and wireless networks: Volume I Theory*. Now Publishers, Inc., 2009.
- [BB10] F. Baccelli and B. Błaszczyszyn. *Stochastic geometry and wireless networks: Volume II Applications*. Now Publishers, Inc., 2010.
- [BB15] F. Baccelli and A. Biswas. On scaling limits of power law shot-noise fields. *Stochastic Models*, 31(2):187–207, 2015.
- [BD16a] H. Biermé and A. Desolneux. Mean geometry for 2d random fields: level perimeter and level total curvature integrals. preprint HAL, No. 01370902, 2016.
- [BD16b] H. Biermé and A. Desolneux. On the perimeter of excursion sets of shot noise random fields. *Ann. Prob.*, 44(1):521–543, 2016.
- [BG17] V. Beffara and D. Gayet. Percolation of random nodal lines. *Publ. Math. IHES*, 126:131–176, 2017.
- [BKS99] I. Benjamini, G. Kalai, and O. Schramm. Noise sensitivity of Boolean functions and applications to percolation. *Publ. Math. IHES*, 90(1):5–43, 1999.
- [BM17] E. Broman and R. Meester. Phase transition and uniqueness of levelset percolation. *J. Stat. Phys.*, 167(6):1376–1400, 2017.
- [BR06] B. Bollobás and O. Riordan. *Percolation*. Cambridge University Press, 2006.
- [BST12] A. Bulinski, E. Spodarev, and F. Timmermann. Central limit theorems for the excursion set volumes of weakly dependent random fields. *Bernoulli*, 18(1):100–118, 2012.
- [Cam09] N.R. Campbell. The study of discontinuous phenomena. *Proc. Cambridge. Philos. Soc.*, 15:117–136, 1909.
- [Gri99] G.R. Grimmett. *Percolation*. Springer: Berlin, Germany, 1999.

- [Han02] D.G. Handron. Generalized billiard paths and morse theory for manifolds with corners. *Topology Appl.*, 126(1-2):83–118, 2002.
- [Har60] T.E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Camb. Phil. Soc.*, 56:13–20, 1960.
- [Kes80] H. Kesten. The critical probability of bond percolation on the square lattice equals $1/2$. *Commun. Math. Phys.*, 74:41–59, 1980.
- [LR19] R. Lachièze-Rey. Second order fluctuations of non-stabilizing functionals, application to shot noise excursions. *Ann. Appl. Probab.*, 2019.
- [MR96] R. Meester and R. Roy. *Continuum Percolation*. Cambridge University Press, Cambridge, 1996.
- [MS83] S.A. Molchanov and A.K. Stepanov. Percolation in random fields. II. *Theor. Math. Phys.*, 55(3):592–599, 1983.
- [MV19] S. Muirhead and H. Vanneuille. The sharp phase transition for level set percolation of smooth planar Gaussian fields. *Ann. I. Henri Poincaré Probab. Stat. (to appear)*, 2019.
- [OSSS05] R. O’Donnell, M. Saks, O. Schramm, and R.A. Servedio. Every decision tree has an influential variable. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’05)*, pages 31–39, 2005.
- [RB03] P. Reynaud-Bouret. Adaptative estimation of the intensity of inhomogeneous poisson processes via concentration inequalities. *Probab. Theory Related Fields*, 126:103–153, 2003.
- [Res07] S.I. Resnick. *Extreme Values, Regular Variation and Point Processes*. Springer, 2007.
- [RV19a] A. Rivera and H. Vanneuille. The critical threshold for Bargmann-Fock percolation. *Ann. Henri Lebesgue (to appear)*, 2019.
- [RV19b] A. Rivera and H. Vanneuille. Quasi-independence for nodal lines. *Ann. Henri Poincaré (to appear)*, 2019.
- [SS10] O. Schramm and J.E. Steif. Quantitative noise sensitivity and exceptional times for percolation. *Ann. Math.*, 171(2):619–672, 2010.
- [Tas16] V. Tassion. Crossing probabilities for Voronoi percolation. *Ann. Probab.*, 44(5):3385–3398, 2016.