



(BOUNDED) TRAVELING COMBUSTION FRONTS WITH DEGENERATE KINETICS

Nathaël Alibaud, Gawtum Namah

► To cite this version:

Nathaël Alibaud, Gawtum Namah. (BOUNDED) TRAVELING COMBUSTION FRONTS WITH DEGENERATE KINETICS. 2019. <hal-02337115>

HAL Id: hal-02337115

<https://hal.science/hal-02337115v1>

Preprint submitted on 29 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

(BOUNDED) TRAVELING COMBUSTION FRONTS WITH DEGENERATE KINETICS

NATHAËL ALIBAUD AND GAWTUM NAMAH

ABSTRACT. We consider the propagation of a flame front in a solid periodic medium. The model is governed by a free boundary system in which the front's velocity depends on the temperature via a kinetic rate which may degenerate. We show the existence of travelling wave solutions which are bounded and global. Previous results by the same authors (cf. [1]) were obtained for essentially positively lower bounded kinetics or eventually which have some very weak degeneracy. Here we consider general degenerate kinetics, including in particular those of Arrhenius type which are commonly used in physics.

1. INTRODUCTION

In this paper, we consider a flame front in a solid periodic medium $\mathbb{R}_x \times \mathbb{R}_y$ where the fresh region is a hypograph $\{x < \xi(y, t)\}$ with a temperature $T = T(x, y, t)$. The evolution of (ξ, T) will be governed by the free boundary system

$$(1) \quad \begin{cases} T_t - \Delta T = 0, & x < \xi(y, t), t > 0, \\ \xi_t + R(y)K(T)\sqrt{1 + \xi_y^2} = \frac{\xi_{yy}}{1 + \xi_y^2}, & x = \xi(y, t), t > 0, \end{cases}$$

subject to the boundary conditions

$$(2) \quad \begin{cases} \frac{\partial T}{\partial \nu} = V_n, & x = \xi(y, t), \\ T(x, y, t) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases}$$

where $\nu := (1, -\xi_y)/\sqrt{1 + \xi_y^2}$ is the outward unit normal and V_n is the normal velocity of the front. The second equation of (1) states that the front propagates with a normal velocity V_n given by

$$V_n = -R(y)K(T) - \kappa,$$

where κ is the mean curvature and RK is the forcing term. The latter depends on the temperature through a *kinetic* which is generally of Arrhenius type *i.e.* of the form

$$(3) \quad K(T) = Ae^{-\frac{B}{T}}$$

for some physical positive constants A and B . The heterogeneity is given through the function R which represents the combustion rate. It is typically a periodic step function for a striated medium obtained from a superposition of different materials, which is the most common situation. The heterogeneity may as well come from A , B or other intrinsic parameters such as the diffusivity, etc. Here we will assume that all these parameters are independent of the striations and normalized to one except for $R = R(y)$. We opted for such a simple problem to shed light on the

Date: October 29, 2019.

2010 Mathematics Subject Classification. Primary 35R35, 80A25, 35C07, 35B10; Secondary 35B27, 80M35.

Key words and phrases. Combustion, free boundary problems, front propagation, bounded traveling wave solutions, degenerate Arrhenius kinetics.

main mathematical difficulties but extensions are possible. For more details and references about the physical model, see [2], [5] and [1].

In this work we will focus on the existence of traveling wave solutions (TWS) to (1)-(2) *i.e.* fronts having a constant profile $\psi = \psi(y)$ and moving with a constant speed $c > 0$. This comes to looking for solutions of the form

$$\xi(y, t) = -ct + \psi(y) \text{ and } T(x, y, t) = u(x + ct, y).$$

It is convenient to fix the front through the change of variable $x + ct \mapsto x$ which leads to the problem of finding a triplet (c, ψ, u) satisfying

$$(4) \quad \begin{cases} cu_x - \Delta u = 0, & x < \psi(y), \\ \frac{\partial u}{\partial \nu} = \frac{c}{\sqrt{1 + \psi_y^2}}, & x = \psi(y), \\ u(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases}$$

and

$$(5) \quad -c + R(y)K(u)\sqrt{1 + \psi_y^2} = \frac{\psi_{yy}}{1 + \psi_y^2}, \quad x = \psi(y).$$

This problem has been dealt in [1] by the present authors for a positively lower bounded combustion rate R and for a positive kinetic K which can eventually very weakly degenerate at zero. This means that K may vanish but not too fast. The worst situation that we were able to consider is when

$$K(u) \sim \frac{1}{\ln \frac{1}{u}} \text{ as } u \rightarrow 0.$$

The existence of a TWS was then proved in [1] where the speed $c > 0$ and the profile $\psi = \psi(y)$ is globally defined and bounded. Our aim is to generalize this result to much more degenerate kinetics including those of Arrhenius type (3), which is what seems to be the most realistic. More precisely, we just assume that there are constants R_m , R_M , and K_M such that

$$(6) \quad \begin{cases} R : \mathbb{T} \rightarrow \mathbb{R} \text{ is measurable with } 0 < R_m \leq R(y) \leq R_M, \text{ and} \\ K : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is continuous, nondecreasing with } 0 < K(u) < K_M, \end{cases}$$

where \mathbb{T} is the torus \mathbb{R}/\mathbb{Z} . Let us now define a solution of (4) in the variational sense.

Definition 1. Let $\Omega := \{(x, y); x < \psi(y), y \in \mathbb{T}\}$. Then given $c \in \mathbb{R}$ and $\psi \in W^{1,\infty}(\mathbb{T})$, a function u will be called a variational solution of (4) if $u \in H^1(\Omega)$ and satisfies

$$(7) \quad c \int_{\Omega} u_x w + \int_{\Omega} \nabla u \nabla w = \int_{x=\psi(y)} \frac{cw}{\sqrt{1 + \psi_y^2}} \text{ for all } w \in H^1(\Omega).$$

We now state our main result.

Theorem 2. *Under (6), there exists a TWS (c, ψ, u) of (1)-(2) with $c > 0$, $\psi \in W^{2,\infty}(\mathbb{T})$ and $0 \leq u \in H^1(\Omega)$. More precisely ψ satisfies (5) *a.e.* and u is a variational solution of (4).*

Technically speaking, it is not obvious that the front remains bounded since the forcing term can *a priori* vanish as u goes to zero. It is indeed well known that for pure geometric propagations of the form

$$-c + H(y)\sqrt{1 + \psi_y^2} = \frac{\psi_{yy}}{1 + \psi_y^2}, \quad x = \psi(y)$$

with H not positively lower bounded, one can as well end up with unbounded traveling fronts which in addition may not be globally defined. For more details see [3], where the existence of so-called *generalized TWS* are discussed by the use of variational techniques. Here, we not only provide a new existence result of TWS for (1)-(2) with general degenerate kinetics but we also rigorously show that the front is globally defined and bounded.

The rest of the paper is organized as follows. To prove Theorem 2 we will need three key lemmas stated and proved in Sections 2, 3 and 4 respectively. In these preliminary results, we will establish an *a priori* positive lower bound for the speed c (Lemma 3) and a lower bound for the front's temperature (Lemma 10) based on some adequate monotonicity property of the temperature (Lemma 6). The proof of Theorem 2 will then be done in Section 5.

2. POSITIVE LOWER BOUND FOR THE SPEED

Here is our first key lemma.

Lemma 3 (Lower bound for c). *Assume (6) and let $c > 0$, $\psi \in W^{2,\infty}(\mathbb{T})$ satisfy (5) a.e. and $u \in H^1(\Omega)$ a variationnal solution of (4). Then*

$$c \geq R_m \int_0^1 K(s) ds.$$

To prove Lemma 3 we need some technical results.

Lemma 4. *Let (c, ψ, u) be given as in Lemma 3. Then $\int_{\mathbb{T}} u(\psi(y), y) dy = 1$.*

Proof. Consider any arbitrary $L > 0$ and take

$$w(x, y) := \begin{cases} 0 & \text{if } x \leq -L, \\ x + L & \text{if } -L \leq x \leq -L + 1, \\ 1 & \text{if } -L + 1 \leq x \leq \psi(y) \end{cases}$$

in (7), which is in $H^1(\Omega)$ since ψ is bounded. This gives

$$(8) \quad \int_{\mathbb{T}} \left(\int_{-L}^{-L+1} c(x+L)u_x dx + \int_{-L+1}^{\psi(y)} cu_x dx + \int_{-L}^{-L+1} u_x dx \right) dy = c.$$

After an integration by parts, the sum of the first two terms gives

$$c \int_{\mathbb{T}} u(\psi(y), y) dy - c \int_{\mathbb{T}} \int_{-L}^{-L+1} u dx dy$$

where the second integral vanishes when $L \rightarrow +\infty$ since

$$0 \leq c \int_{\mathbb{T}} \int_{-L}^{-L+1} u dx dy \leq c \|u\|_{L^2((-L, -L+1) \times \mathbb{T})}.$$

Likewise the third term of (8) goes to zero when $L \rightarrow +\infty$ and we conclude the proof by passing to the limit in (8). \square

We proceed by a second technical lemma.

Lemma 5. *Let (c, ψ, u) as in Lemma 3. Then $\int_{\mathbb{T}} K(u(\psi(y), y) dy \geq \int_0^1 K(s) ds$.*

Proof. For any $\varepsilon > 0$, define $K_\varepsilon(u) := \frac{1}{\varepsilon} \int_{(u-\varepsilon)^+}^u K(s) ds$. Then K_ε is Lipschitz and satisfies (6). In particular we can take $w(x, y) := K_\varepsilon(u(x, y))$ in (7). We thus have

$$c \int_{\Omega} u_x K_\varepsilon(u) + \int_{\Omega} \nabla u \nabla K_\varepsilon(u) = \int_{x=\psi(y)} \frac{c K_\varepsilon(u)}{\sqrt{1+\psi_y^2}},$$

which can be rewritten as

$$c \int_{\Omega} J_\varepsilon(u)_x dx dy + \int_{\Omega} K'_\varepsilon(u) |\nabla u|^2 dx dy = \int_{\mathbb{T}} c K_\varepsilon(u(\psi(y), y)) dy$$

where $J_\varepsilon(u) := \int_0^u K_\varepsilon(s) ds$. As the second term is nonnegative by (6) and since

$$c \int_{\Omega} J_\varepsilon(u)_x dx dy = c \int_{\mathbb{T}} \left(\int_{-\infty}^{\psi(y)} J_\varepsilon(u)_x dx \right) dy = c \int_{\mathbb{T}} J_\varepsilon(u(\psi(y), y)) dy,$$

we deduce that

$$\int_{\mathbb{T}} K_\varepsilon(u(\psi(y), y)) dy \geq \int_{\mathbb{T}} J_\varepsilon(u(\psi(y), y)) dy \geq J_\varepsilon \left(\int_{\mathbb{T}} u(\psi(y), y) dy \right) = \int_0^1 K_\varepsilon(s) ds$$

thanks to Jensen's inequality and Lemma 4. We then complete the proof by using the fact that $K_\varepsilon \uparrow K$ as $\varepsilon \downarrow 0$. \square

We are now ready to prove Lemma 3.

Proof of Lemma 3. An integration of (5) over \mathbb{T} gives

$$c = \int_{\mathbb{T}} R(y) K(u(\psi(y), y)) \sqrt{1+\psi_y^2} dy.$$

We thus have $c \geq R_m \int_{\mathbb{T}} K(u(\psi(y), y)) dy$ and we conclude by Lemma 5. \square

3. A MONOTONICITY RESULT

Here is our second key lemma.

Lemma 6 (Monotonicity of u). *Let $c > 0$, $\psi \in W^{2,\infty}(\mathbb{T})$ and $u \in H^1(\Omega)$ satisfy (4). Then the function*

$$x \in (-\infty, \max \psi] \mapsto \min_{y: (x,y) \in \overline{\Omega}} u(x, y) \text{ is nondecreasing.}$$

Remark 7. Recall that u is defined on $\overline{\Omega} = \{x \leq \psi(y)\}$, which explains the above interval of definition.

Remark 8. The min above can be understood in the classical sense since a variational solution $u \in H^1(\Omega)$ of (4) is at least in $C(\overline{\Omega})$ thanks to classical elliptic regularity results (see e.g. [9] and the references therein or [1, Appendix A.4]).

Proof. Let us take $x_0 \in (-\infty, \max \psi)$ and consider the problem

$$(9) \quad \begin{cases} c w_x - \Delta w = 0, & \text{in } \Omega_0, \\ \frac{\partial w}{\partial \nu} = \frac{c}{\sqrt{1+\psi_y^2}}, & \text{on } \partial\Omega_0 \cap \{x > x_0\}, \\ w = w_0, & \text{on } \partial\Omega_0 \cap \{x = x_0\}, \end{cases}$$

where

$$\Omega_0 := \Omega \cap \{x > x_0\} \text{ and } w_0 := \min_y u(x_0, y).$$

Then the constant function w_0 is a subsolution of (9) whereas u is a supersolution. By comparison,

$$(10) \quad u(x, y) \geq w_0 \text{ for all } (x, y) \in \overline{\Omega}_0$$

and it follows that

$$\min_{y:(x,y) \in \overline{\Omega}_0} u(x, y) \geq w_0 = \min_{y:(x,y) \in \overline{\Omega}_0} u(x_0, y) \text{ for all } x \in [x_0, \max \psi].$$

□

Remark 9. Note that $\partial\Omega_0$ may not be Lipschitz at truncated points. Nevertheless the comparison result used in the above proof holds by standard arguments [4, 8]. For completeness, a short verification is given in Appendix B.

4. POSITIVE LOWER BOUND FOR THE TEMPERATURE

In this section we prove a general result concerning the temperature. Consider for this sake the following problem

$$(11) \quad \begin{cases} c u_x - \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{c}{\sqrt{1+\psi_y^2}}, & x = \psi(y), \\ u(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases}$$

where $c \in \mathbb{R}$ and $\psi \in W^{2,\infty}(\mathbb{T})$ satisfy, for some given c_0, c_M and a continuous function G ,

$$(12) \quad \begin{cases} 0 < c_0 \leq c \leq c_M, \\ |\psi_{yy}| \leq G(\psi_y) \text{ a.e. and } \min_{y \in \mathbb{T}} \psi(y) = 0. \end{cases}$$

We then claim our last key result.

Lemma 10 (Lower bound on u). *Let $G \in C(\mathbb{R})$ and $0 < c_0 \leq c_M$ be given. Then there exists $\alpha > 0$ such that for all $c \in \mathbb{R}$ and $\psi \in W^{2,\infty}(\mathbb{T})$ verifying (12), the unique variationnal solution $u \in H^1(\Omega)$ of (11) satisfies*

$$\min_{y \in \mathbb{T}} u(0, y) \geq \alpha.$$

Proof. Let us define

$$\alpha := \inf_{c, \psi} \left\{ \min_y u(0, y); u \in H^1(\Omega) \text{ solution of (11)} \right\},$$

the infimum being taken on all $c \in \mathbb{R}$ and $\psi \in W^{2,\infty}(\mathbb{T})$ satisfying (12). The min is to be understood in the classical sense by Remark 8. We have to show that α is positive. Consider for this sake a minimizing triplet (c_n, ψ_n, u_n) such that

$$(13) \quad \begin{cases} c_n \text{ and } \psi_n \in W^{2,\infty}(\mathbb{T}) \text{ satisfy (12),} \\ u_n \in H^1(\Omega_n := \{x < \psi_n(y)\}) \text{ solves (11)} \\ \text{and } \lim_{n \rightarrow \infty} \min_y u_n(0, y) = \alpha. \end{cases}$$

Since $u_n \in C(\overline{\Omega}_n)$ by Remark 8 and \mathbb{T} and $[c_0, c_M]$ are compact, there exist $(y_n)_n$ and \bar{y} in \mathbb{T} as well as $\bar{c} \in [c_0, c_M]$ such that

$$(14) \quad \begin{aligned} \min_y u_n(0, y) &= u_n(0, y_n), \\ y_n &\rightarrow \bar{y} \text{ and } c_n \rightarrow \bar{c} \text{ when } n \rightarrow \infty. \end{aligned}$$

Consider now the limiting domain

$$\Omega_\infty := \cup_n \text{Int}(\cap_{k \geq n} \Omega_k),$$

where Int is the interior. Note that Ω_∞ is open. Note also that $(0, \bar{y}) \in \overline{\Omega}_\infty$ (the closure of Ω_∞) since each ψ_n was chosen such that $\psi_n \geq 0$, so that for all $\varepsilon > 0$, $(-\varepsilon, \bar{y}) \in \Omega_\infty$. The analysis which follows will depend on whether $(0, \bar{y})$ is on $\partial\Omega_\infty$ (the boundary of Ω_∞) or in Ω_∞ . In the sequel, $B(y, r) := \{z \in \mathbb{T}; d(y, z) < r\}$ will denote a ball of \mathbb{T} with $d(y, z) := \text{dist}(P^{-1}(y), P^{-1}(z))$ and where $P : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ is the usual projection. We denote similarly by $B((x, y), r)$ the balls of $\mathbb{R} \times \mathbb{T}$.

First case : $(0, \bar{y}) \in \partial\Omega_\infty$.

Claim 1. *In this case we claim that there exist an increasing sequence $(n_p)_p$ in \mathbb{N} and $(w_p)_p$ in \mathbb{T} such that*

$$(15) \quad w_p \rightarrow \bar{y} \text{ and } \psi_{n_p}(w_p) \rightarrow 0 \text{ as } p \rightarrow \infty.$$

Indeed $B((0, \bar{y}), \frac{1}{p})$ intersects $\Omega_\infty^c = \cap_n (\cup_{k \geq n} \Omega_k^c)$ for any integer $p \geq 1$, so that there exists a sequence (x_p, y_p) such that $(x_p, y_p) \in B((0, \bar{y}), \frac{1}{p}) \cap (\cup_{k \geq p} \Omega_k^c)$. Since $B((0, \bar{y}), \frac{1}{p})$ is open and (x_p, y_p) is the limit of some sequence in $\cup_{k \geq p} \Omega_k^c$, there exist $n_p \geq p$ and $(\tilde{x}_p, \tilde{y}_p)$ such that $(\tilde{x}_p, \tilde{y}_p) \in B((0, \bar{y}), \frac{1}{p}) \cap \Omega_{n_p}^c$. Taking $w_p = \tilde{y}_p$ gives (15). Indeed $w_p \in B(\bar{y}, \frac{1}{p})$ and $0 \leq \psi_{n_p}(w_p) \leq \tilde{x}_p \leq \frac{1}{p}$ by construction. Moreover the sequence n_p can be chosen increasing up to taking subsequences if necessary since $n_p \geq p$ goes to infinity as $p \rightarrow \infty$. This proves Claim 1.

We proceed by another claim:

Claim 2. *For all $\varepsilon > 0$ and $p_0 \in \mathbb{N}$, there exist $y \in B(\bar{y}, \varepsilon)$ and $p \geq p_0$ such that*

$$(16) \quad |\psi'_{n_p}(y)| < \varepsilon.$$

Indeed if the above does not hold, then there exist $\varepsilon > 0$ and $p_0 \in \mathbb{N}$ such that for any $p \geq p_0$,

$$|\psi'_{n_p}(y)| \geq \varepsilon \text{ for all } y \in B(\bar{y}, \varepsilon).$$

Up to taking a subsequence, we can suppose that for all p , $\psi'_{n_p}(y) \geq \varepsilon$ in $B(\bar{y}, \varepsilon)$. For ε small enough, identifying $B(\bar{y}, \varepsilon)$ with the interval $I := P^{-1}(B(\bar{y}, \varepsilon))$ in \mathbb{R} , we have in I

$$\begin{aligned} \psi_{n_p}(\bar{y} - \varepsilon) &= \psi_{n_p}(w_p) + \int_{w_p}^{\bar{y} - \varepsilon} \psi'_{n_p}(y) dy \\ &\leq \psi_{n_p}(w_p) - \varepsilon(w_p - (\bar{y} - \varepsilon)). \end{aligned}$$

This is not possible because as $\psi_{n_p} \geq 0$ by definition, the LHS is ≥ 0 whereas the RHS goes to $-\varepsilon^2 < 0$ in the limit $p \rightarrow \infty$. This proves Claim 2.

Now (16) implies that there exist a subsequence $(n_{p_k})_k$ and another sequence $(\tilde{w}_k)_k$ such that

$$\tilde{w}_k \rightarrow \bar{y} \text{ and } \psi'_{n_{p_k}}(\tilde{w}_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Recalling (15) and renaming the sequences from the beginning of the proof for simplicity, we have proved that we have $(w_n)_n$ and $(\tilde{w}_n)_n$ such that

$$(17) \quad w_n \rightarrow \bar{y}, \tilde{w}_n \rightarrow \bar{y} \text{ with } \psi_n(w_n) \rightarrow 0 \text{ and } \psi'_n(\tilde{w}_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This leads to our next claim below concerning the fronts.

Claim 3. *There exists ε_0 such that $(\psi_n)_n$ is bounded in $W^{2,\infty}(B(\bar{y}, \varepsilon_0))$. In particular $\psi_n \rightarrow \bar{\psi}$ uniformly in $B(\bar{y}, \varepsilon_0)$, up to some subsequence, for some $\bar{\psi} \in W^{2,\infty}(B(\bar{y}, \varepsilon_0))$.*

Indeed the existence of ε_0 as well as the uniform bound for ψ'_n in $B(\bar{y}, \varepsilon_0)$ are provided by Lemma 12 in Appendix B and (17). The bounds for ψ_n follow from (17) again and the ones for ψ''_n from (12). With these bounds in hand, Ascoli-Arzelà theorem completes the proof of Claim 3.

To conclude, we will pass to the limit in the boundary problem satisfied by the temperature, cf. (11). In the case where $(0, \bar{y}) \in \partial\Omega_\infty$, we only need to handle the boundary condition on the moving interface $\{x = \psi_n(y), y \in B(\bar{y}, \varepsilon_0)\}$. Before hand, it is convenient to extend u_n onto $\mathbb{R} \times \mathbb{T}$ by

$$(18) \quad \tilde{u}_n(x, y) := \begin{cases} u_n(x, y) & \text{if } (x, y) \in \Omega_n, \\ u_n(2\psi_n(y) - x, y) & \text{otherwise.} \end{cases}$$

Consider now the cylinder $\tilde{Q} := \{(x, y); y \in B(\bar{y}, \varepsilon_0)\}$. We then have

Claim 4. *There exists $0 \leq \tilde{u} \in C^1(\tilde{Q})$ such that*

$$\tilde{u}_n \rightarrow \tilde{u} \text{ locally uniformly in } \tilde{Q} \text{ as } n \rightarrow \infty,$$

up to a subsequence, with

$$\frac{\partial \tilde{u}}{\partial \nu} = \frac{\bar{c}}{\sqrt{1 + \bar{\psi}_y^2}} \text{ on } \{x = \bar{\psi}(y), y \in B(\bar{y}, \varepsilon_0)\}.$$

To show this claim, apply Lemma 11 of Appendix A with $B_0 := B(\bar{y}, \varepsilon_0)$ and $B_1 := \emptyset$. Note that (23) holds by (14) and Claim 3.

To conclude we have $u_n(0, y_n) \rightarrow \tilde{u}(0, \bar{y}) = \alpha$ as $n \rightarrow \infty$, cf. (13), (14), (17), as well as Claims 3 and 4, and it remains now to show that $\tilde{u}(0, \bar{y}) > 0$. But as $\tilde{u} \geq 0$ around $(0, \bar{y})$ and $\frac{\partial \tilde{u}}{\partial \nu}(0, \bar{y}) > 0$, we end up with $\tilde{u}(0, \bar{y}) > 0$. This completes the proof of Lemma 10 when $(0, \bar{y}) \in \partial\Omega_\infty$.

Second case : $(0, \bar{y}) \in \Omega_\infty$.

Recalling that $\Omega_\infty = \cup_n \text{Int}(\cap_{k \geq n} \Omega_k)$, there exist now n_0 and an open $O \subset \mathbb{R}^2$ such that

$$(19) \quad (0, \bar{y}) \in O \subseteq \Omega_n \text{ for all } n \geq n_0.$$

Roughly speaking, we will show that u_n converges to some nontrivial $\tilde{u} \geq 0$ which satisfies the first equation of (11) in O . This will imply that $\alpha = \tilde{u}(0, \bar{y}) > 0$ by an argument of propagation of maximum. The overall idea to get a nontrivial limit is to work eventually in a larger open \mathcal{O} with $O \subseteq \mathcal{O} \subseteq \Omega_\infty$, so that the nontrivial boundary condition of (11) holds on some part of $\partial\mathcal{O}$.

To construct \mathcal{O} , take $(z_n)_n$ in \mathbb{T} such that z_n is a minimizer of ψ_n . Considering a subsequence if necessary, we can assume that

$$(20) \quad \psi_n(z_n) = \psi'_n(z_n) = 0 \text{ and } z_n \rightarrow \bar{z} \text{ as } n \rightarrow \infty,$$

for some $\bar{z} \in \mathbb{T}$. The above property looks like (17) and we have the following analogous of Claim 3 whose proof is similar.

Claim 5. *There exist ε_0 and $\bar{\psi} \in W^{2,\infty}(B(\bar{z}, \varepsilon_0))$ such that $(\psi_n)_n$ is bounded in $W^{2,\infty}(B(\bar{z}, \varepsilon_0))$ and $\psi_n \rightarrow \bar{\psi}$ uniformly in $B(\bar{z}, \varepsilon_0)$, up to some subsequence.*

Using now (19), there exist $\eta > 0$ and $\varepsilon_1 > 0$ such that $(-\eta, \eta) \times B(\bar{y}, \varepsilon_1) \subset \Omega_n$ for all $n \geq n_0$. Recall that $\Omega_n = \{x < \psi_n(y)\}$ so that we have

$$\psi_n(y) \geq \eta, \text{ for all } n \geq n_0 \text{ and } y \in B(\bar{y}, \varepsilon_1).$$

Using in addition (20) and Claim 5, we note in particular that $\bar{y} \neq \bar{z}$. This enables us to choose ε_0 and ε_1 (smaller if needed) so that $B(\bar{z}, \varepsilon_0) \cap B(\bar{y}, \varepsilon_1) = \emptyset$. We can now define \mathcal{O} (resp. $\tilde{\mathcal{O}}$) as follows:

$$\mathcal{O} \text{ (resp. } \tilde{\mathcal{O}}) := \{(x, y); x < \chi(y) \text{ (resp. } x < \tilde{\chi}(y))\},$$

where

$$\chi(y) \text{ (resp. } \tilde{\chi}(y)) := \begin{cases} \bar{\psi}(y) \text{ (resp. } +\infty) & \text{if } y \in B(\bar{z}, \varepsilon_0), \\ \eta & \text{if } y \in B(\bar{y}, \varepsilon_1), \\ 0 & \text{otherwise.} \end{cases}$$

As previously announced, we need to identify both the PDE and the boundary condition of the limiting temperature. We propose

Claim 6. *There exists $0 \leq \tilde{u} \in C^1(\tilde{\mathcal{O}}) \cap C^2(\mathcal{O})$ such that*

$$\tilde{u}_n \rightarrow \tilde{u} \text{ locally uniformly in } \tilde{\mathcal{O}} \text{ as } n \rightarrow \infty,$$

up to a subsequence, where

$$(21) \quad \begin{cases} \bar{c} \tilde{u}_x - \Delta \tilde{u} = 0, \text{ in } \mathcal{O}, \\ \frac{\partial \tilde{u}}{\partial \nu} = \frac{\bar{c}}{\sqrt{1+\bar{\psi}_y^2}}, \quad x = \bar{\psi}(y), \quad y \in B(\bar{z}, \varepsilon_0). \end{cases}$$

To show this claim, apply again Lemma 11 with now $B_0 := B(\bar{z}, \varepsilon_0)$, η as above and $B_1 := B(\bar{y}, \varepsilon_1)$, so (23) is ensured by Claim 5 and (24) holds by what precedes.

Once again $u_n(0, y_n) \rightarrow \tilde{u}(0, \bar{y}) = \alpha$ and it remains to show that $\tilde{u}(0, \bar{y}) > 0$. For this sake, note that \tilde{u} is not identically zero because $\bar{c} > 0$ in the boundary condition of (21). By the strong maximum principle, $\tilde{u}(0, \bar{y}) > 0$ since it cannot achieve its minimum in the connected open \mathcal{O} otherwise it will vanish everywhere in \mathcal{O} , cf. [6, Sec. 6.4.2]. This completes the proof of Lemma 10. \square

5. PROOF OF THEOREM 2

Now we are ready to prove our main result. For this sake, let us consider Eqns. (4)-(5) with $K(u)$ replaced by the truncated function

$$K_n(s) := \max \left\{ K(s), \frac{1}{n} \right\}.$$

By [1, Thm. 3.2], the latter system admits a solution (c_n, ψ_n, u_n) for any integer $n \geq 1$, with $c_n > 0$, $\psi_n \in W^{2,\infty}(\mathbb{T})$ and $0 \leq u_n \in H^1(\{x < \psi_n(y)\})$. Since ψ_n is defined up to an additive constant, we can choose it such that $\min_y \psi_n(y) = 0$. Now from Lemma 3, we know that

$$c_n \geq c_0 := R_m \int_0^1 K(s) ds > 0 \text{ for all } n.$$

Likewise we have the following upper bound by [5, Thm. 2.1] and Assumption (6):

$$c_n \leq c_M := R_M K_M \text{ for all } n.$$

From the front's equation (5), we then have

$$|(\psi_n)_{yy}| \leq 2c_M(1 + \psi_{ny}^2)^{\frac{3}{2}}.$$

Hence, applying Lemma 10 with $G(h) := 2c_M(1 + h^2)^{\frac{3}{2}}$ leads to the existence of an $\alpha > 0$, α depending only on c_0 , c_M and G , such that

$$\min_y u_n(0, y) \geq \alpha \text{ for all } n.$$

Now as $x \mapsto \min_y u(x, y)$ is nondecreasing by Lemma 6, we have $u_n(\psi_n(y), y) \geq \alpha$, and consequently by (6) and the definition of K_n , we end up with

$$K_n(u_n(\psi_n(y), y)) \geq K_n(\alpha) \geq K(\alpha) \text{ for all } n \text{ and } y \in \mathbb{T}.$$

We proceed by setting

$$H_n(y) := R(y)K_n(u_n(\psi_n(y), y)).$$

Since $K(\alpha) > 0$ by (6), the above uniform positive lower boundedness of $(H_n)_n$ enables us to use the results of [1] to pass to the limit as $n \rightarrow \infty$. Indeed an application of [1, Lemmas 2.5 & 2.7] gives the existence of (H, c, ψ, u) limit of (H_n, c_n, ψ_n, u_n) , up to a subsequence, where

$$\begin{cases} c u_x - \Delta u = 0, & \text{in } \{x < \psi(y)\}, \\ \frac{\partial u}{\partial \nu} = \frac{c}{\sqrt{1+\psi_y^2}}, & x = \psi(y), \\ u(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases}$$

and

$$-c + H(y)\sqrt{1+\psi_y^2} = \frac{\psi_{yy}}{1+\psi_y^2}, \quad x = \psi(y),$$

in the sense and with the regularity claimed in Theorem 2. It remains to identify $H(y)$. As $H_n \rightharpoonup H$ in L^∞ weak- $*$ (see [1]), we have

$$\int_{\mathbb{T}} H_n(y)\varphi(y) dy \rightarrow \int_{\mathbb{T}} H(y)\varphi(y) dy \text{ for all } \varphi \in C_c^\infty(\mathbb{T}).$$

But by [1, Lemma 2.8], we have

$$u_n(\psi_n(y), y) \rightarrow u(\psi(y), y) \text{ for a.e. } y \in \mathbb{T},$$

so that, by the definition of H_n , the assumptions in (6), and Lebesgue's theorem, we also have

$$\int_{\mathbb{T}} H_n(y)\varphi(y) dy \rightarrow \int_{\mathbb{T}} R(y)K(u(\psi(y), y))\varphi(y) dy \text{ for all } \varphi \in C_c^\infty(\mathbb{T}).$$

Finally we end up with $H(y) = R(y)K(u(\psi(y), y))$ and the proof of Theorem 2 is complete. \square

APPENDIX A. A CONVERGENCE RESULT FOR THE TEMPERATURE

Let us now state and prove Lemma 11, see below, that we have admitted during the proof of Lemma 10. As before, consider $(c_n)_n$ in \mathbb{R} , $(\psi_n)_n$ in $W^{2,\infty}(\mathbb{T})$ and $(u_n)_n$ in $H^1(\Omega_n = \{x < \psi_n(y)\})$ such that

$$(22) \quad \begin{cases} c_n (u_n)_x - \Delta u_n = 0, & \text{in } \Omega_n \\ \frac{\partial u_n}{\partial \nu} = \frac{c_n}{\sqrt{1+\psi_{ny}^2}}, & x = \psi_n(y), \\ u_n(x, y) \rightarrow 0, & \text{as } x \rightarrow -\infty, \end{cases}$$

as well as

$$(23) \quad \begin{cases} 0 < c_n \leq c_M \text{ and } c_n \rightarrow \bar{c} > 0 \text{ as } n \rightarrow \infty, \\ (\psi_n)_n \text{ is bounded in } W^{2,\infty}(B_0), \text{ and} \\ \psi_n \rightarrow \bar{\psi} \text{ uniformly in } B_0 \text{ as } n \rightarrow \infty, \end{cases}$$

for some reals c_M and \bar{c} , nonempty open $B_0 \subset \mathbb{T}$ and $\bar{\psi} \in W^{2,\infty}(B_0)$. Let moreover $\eta > 0$ and $B_1 \subset \mathbb{T}$ be another open, eventually empty, such that

$$(24) \quad B_0 \cap B_1 = \emptyset \text{ and } (-\eta, \eta) \times B_1 \subset \cap_n \Omega_n.$$

Define then

$$U_n := \{(x, y); x < \chi_n(y)\},$$

where

$$\chi_n(y) := \begin{cases} \psi_n(y) & \text{if } y \in B_0, \\ \eta & \text{if } y \in B_1, \\ 0 & \text{otherwise.} \end{cases}$$

Likewise we define the limiting domain

$$U := \{(x, y); x < \chi(y)\},$$

with χ the same as χ_n with ψ_n replaced by $\bar{\psi}$. We further consider the extended domain

$$\tilde{U} := \{(x, y); x < \tilde{\chi}(y)\},$$

where

$$\tilde{\chi}(y) := \begin{cases} +\infty & \text{if } y \in B_0, \\ \eta & \text{if } y \in B_1, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that \tilde{u}_n is defined in (18) and belongs at least to $C \cap H^1(\mathbb{R} \times \mathbb{T})$ because $u_n \in C \cap H^1(\Omega_n)$ and $\psi_n \in W^{1,\infty}(\mathbb{T})$.

Lemma 11. *Under the above assumptions, there is $0 \leq \tilde{u} \in C^1(\tilde{U}) \cap C^2(U)$ such that*

$$\tilde{u}_n \rightarrow \tilde{u} \text{ locally uniformly in } \tilde{U} \text{ as } n \rightarrow \infty,$$

up to a subsequence, where

$$(25) \quad \begin{cases} \bar{c} \tilde{u}_x - \Delta \tilde{u} = 0, & \text{in } U, \\ \frac{\partial \tilde{u}}{\partial \nu} = \frac{\bar{c}}{\sqrt{1 + \bar{\psi}_y^2}}, & x = \bar{\psi}(y), y \in B_0. \end{cases}$$

Proof. We proceed in several steps.

Step 1: A priori estimates.

We claim that

$$(26) \quad \begin{cases} 0 \leq u_n(x, y) \leq e^{c_n x} & \forall (x, y) \in \Omega_n \\ \text{and } \int_{\Omega_n} |\nabla u_n|^2 \leq c_n. \end{cases}$$

The first estimate follows since the functions 0 and $e^{c_n x}$ are respectively sub and supersolution of (22), see *e.g.* [1, Appendix A.2]. For the second estimate, take $w := u_n$ in (7), which is a weak formulation of (22) as well, to see that

$$c_n \int_{\Omega} (u_n^2/2)_x + \int_{\Omega} |\nabla u_n|^2 = \int_{x=\psi(y)} u_n \frac{c_n}{\sqrt{1 + (\psi'_n)^2}} dy = c_n$$

by Lemma 4. The proof of (26) is complete since the first term is equal to $c_n \int_{\mathbb{T}} u_n^2(\psi_n(y), y)/2 dy \geq 0$.

Consequently, there is a constant C independent of n such that

$$(27) \quad \max_{\tilde{U}} |\tilde{u}_n| \leq C \text{ and } \int_{\tilde{U}} |\nabla \tilde{u}_n|^2 \leq C,$$

thanks to (18) and (23). Note that φ_n is not assumed uniformly bounded outside B_0 , but in that region we use that $\tilde{U} \cap \{y \notin B_0\} \subset \Omega_n$ to deduce (27) from (26).

Step 2: Limiting problem.

By (27), u_n converges weakly to some \tilde{u} in $H^1_{\text{loc}}(\tilde{U})$ up to a subsequence. Taking any test $\varphi \in C_c^\infty(\tilde{U})$ in (22), we have

$$(28) \quad c_n \int_{\Omega_n} (u_n)_x \varphi + \int_{\Omega_n} \nabla u_n \nabla \varphi = c_n \int_{B_0} \varphi(\psi_n(y), y) dy.$$

The LHS can be rewritten as

$$\int_{U \cap \{y \notin B_0\}} (c_n (u_n)_x \varphi + \nabla u_n \nabla \varphi) + \int_{\tilde{U} \cap \{y \in B_0\}} (c_n (u_n)_x \varphi + \nabla u_n \nabla \varphi) \mathbf{1}_{x < \psi_n(y)},$$

since $\text{supp } \varphi \cap \{y \notin B_0\} \subset U \cap \{y \notin B_0\}$. This converges to

$$\int_{U \cap \{y \notin B_0\}} (\bar{c} \tilde{u}_x \varphi + \nabla \tilde{u} \nabla \varphi) + \int_{\tilde{U} \cap \{y \in B_0\}} (\bar{c} \tilde{u}_x \varphi + \nabla \tilde{u} \nabla \varphi) \mathbf{1}_{x < \bar{\psi}(y)},$$

because $\mathbf{1}_{x < \psi_n(y)} \rightarrow \mathbf{1}_{x < \bar{\psi}(y)}$ a.e. in $\tilde{U} \cap \{y \in B_0\}$ by the uniform convergence of ψ_n to $\bar{\psi}$ in B_0 , cf. (23). Noting that

$$\tilde{U} \cap \{x < \bar{\psi}(y), y \in B_0\} = U \cap \{y \in B_0\},$$

the LHS of (28) thus converges to $\int_U (\bar{c} \tilde{u}_x \varphi + \nabla \tilde{u} \nabla \varphi)$. Using again the uniform convergence of ψ_n to $\bar{\psi}$ in B_0 to handle the RHS, we obtain that

$$\int_U (\bar{c} \tilde{u}_x \varphi + \nabla \tilde{u} \nabla \varphi) = \bar{c} \int_{B_0} \varphi(\bar{\psi}(y), y) dy \quad \forall \varphi \in C_c^\infty(\tilde{U}).$$

This proves that $\tilde{u} \in H_{\text{loc}}^1(\tilde{U})$ is a variational solution of (25). Note that it is nonnegative by (26).

Step 3: Local uniform convergence.

When calling for Lemma 11, it was important that $\tilde{u}_n \rightarrow \tilde{u}$ locally uniformly in \tilde{U} , especially around the moving interfaces $\{x = \psi_n(y), y \in B_0\}$. Let us adapt an idea of [9] to establish this convergence. It consists in considering the problem satisfied by \tilde{u}_n in order to apply standard interior elliptic estimates, cf. also [1, Appendix A.4].

By [1, Lemma A.5],

$$b_n(\tilde{u}_n)_x - \text{div}(A_n \nabla \tilde{u}_n) = -(b_n)_x \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{T}),$$

with

$$b_n(x, y) = c_n \mathbf{1}_{\{x < \psi_n(y)\}} - c_n \mathbf{1}_{\{x > \psi_n(y)\}},$$

$$A_n(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{1}_{\{x < \psi_n(y)\}} - \begin{pmatrix} 1 + 4\psi_{ny}^2 & 2\psi_{ny} \\ 2\psi_{ny} & 1 \end{pmatrix} \mathbf{1}_{\{x > \psi_n(y)\}}.$$

We claim that there are $\Lambda \geq \lambda > 0$, $\nu \geq 0$, such that for each n and a.e. $x, y \in \tilde{U}$,

$$\Lambda \geq A_n(x, y) \geq \lambda \text{ and } \lambda^{-1} |b_n(x, y)| \leq \nu.$$

For $y \notin B_0$, this follows once again from the fact $\tilde{U} \cap \{y \notin B_0\} \subset \Omega_n$, so $b_n = c_n$ and $A_n = Id$ everywhere in that region. For $y \in B_0$, recall that $(\psi_n)_n$ is bounded in $W^{1,\infty}(B_0)$ and the proof is the same as in [1, Lemma A.7].

Now by [7, Thm. 8.24] combined with (27), $(\tilde{u}_n)_n$ is locally equi-Hölder continuous in \tilde{U} and the convergence of \tilde{u}_n to \tilde{u} is local uniform, up to taking another subsequence if necessary.

Step 4: C^1 regularity.

This step is now standard. For example, let us call for [8, Thm. 2.4.2.7] to get the regularity around the boundary $\{x = \bar{\psi}(y), y \in B_0\}$. For any arbitrary $\rho \in C_c^\infty(\tilde{U})$,

consider a nonempty $C^{1,1}$ open V such that $\overline{U} \cap \text{supp } \rho \subset \overline{V} \subset \overline{U}$ and rewrite (25) for the function $v := \tilde{u}\rho$ as follows:

$$\begin{cases} v - \Delta v = f \text{ in } V, \\ \frac{\partial v}{\partial \nu_V} = g \text{ on } \partial V, \end{cases}$$

with $f = \tilde{u}\rho - c\rho\tilde{u}_x - 2\nabla\tilde{u}\nabla\rho - \tilde{u}\Delta\rho$ and

$$g = \begin{cases} \frac{\tilde{c}\rho}{\sqrt{1+\tilde{\psi}_y^2}} + \tilde{u}\frac{\partial\rho}{\partial\nu} & \text{on } \partial V \cap \{x = \overline{\psi}(y), y \in B_0\}, \\ 0 & \text{elsewhere on } \partial V, \end{cases}$$

where ν_V is the outer unit normal of ∂V . Such a regular open V exists because $\text{supp } \rho \subset \tilde{U}$ and $\overline{\psi} \in W^{2,\infty}(B_0)$. Using then that $\tilde{u} \in L^\infty \cap H^1(\tilde{U})$, we have $f \in L^2(V)$ and $g \in H^{1/2}(\partial V)$. It follows that $v \in H^2(V)$ by [8, Thm. 2.4.2.7]. Since ρ is arbitrary, $\tilde{u} \in H_{\text{loc}}^2(U)$ thus a fortiori in $H_{\text{loc}}^2(\tilde{U})$ because $\overline{\psi} \in W^{2,\infty}(B_0)$. By Sobolev's embeddings [4, Cor. IX.14], $\nabla\tilde{u} \in L_{\text{loc}}^p(\tilde{U})$ for any $p \in [2, \infty)$, so $f \in L^p(V)$ and $g \in W^{1-1/p,p}(\partial V)$ thanks to trace results [8, Thm. 1.5.1.3]. Here we used that $\tilde{u} \in L^\infty \cap W_{\text{loc}}^{1,p}(\tilde{U})$. Applying again [8, Thm. 2.4.2.7], $v \in W^{2,p}(V)$, $\tilde{u} \in W_{\text{loc}}^{2,p}(\tilde{U})$, and by [4, Cor. IX.14], $\tilde{u} \in C^1(\tilde{U})$. To get the C^2 regularity in U , use *e.g.* [4, Rem. 26]. \square

APPENDIX B. TECHNICAL FEATURES

Here is another result used in the proofs.

Lemma 12. *Let $G \in C(\mathbb{R})$ and $R > 0$. There exists $r > 0$ such that for any $h \in W^{1,\infty}(\mathbb{T})$ and $y_0 \in \mathbb{T}$ such that $|h_y| \leq G(h)$ a.e. in \mathbb{T} and $|h(y_0)| \leq R$, we have*

$$|h(y)| \leq 2R \quad \forall y \in B(y_0, r).$$

Proof. Set $C_{2R} := \max\{\max_{[-2R, 2R]} |G|, 1\}$, let $h = h(y)$ and y_0 be as in the lemma, and define

$$r_0 := \sup\{r > 0; \sup_{B(y_0, r)} |h| \leq 2R\}.$$

The above set is not empty because $|h(y_0)| \leq R$ and h is continuous, so r_0 is well-defined in $(0, +\infty]$. We claim that $r_0 \geq R/C_{2R}$ and this will show that any $r \leq R/C_{2R}$ fits the lemma. If indeed $r_0 < R/C_{2R}$ then for all $y \in B(y_0, r_0)$,

$$|h(y)| \leq |h(y_0)| + r_0|h_y|_{L^\infty(B(y_0, r_0))} \leq R + r_0 C_{2R} < 2R,$$

but one can then choose $r_1 > r_0$ such that $\sup_{B(y_0, r_1)} |h(y)| \leq 2R$ and this contradicts the definition of r_0 . \square

It only remains to check that (10) holds, cf. Remark 9.

Proof of (10). The domain Ω_0 is not Lipschitz at $(x_0, y_0) \in \partial\Omega_0$ whenever $\psi(y_0) = x_0$ and $\psi_y(y_0) = 0$. But the variational solution $u \in H^1(\Omega)$ of (4) is in $C^1(\overline{\Omega}) \cap C^2(\Omega)$ by standard elliptic regularity; cf. the fourth step of the proof of Lemma 11. The strong maximum principle [6, Sec. 6.4.2] then implies that $\min_{\overline{\Omega}_0}(u - w_0)$ is not achieved in Ω_0 or on $\partial\Omega_0 \cap \{x > x_0\}$ where $\frac{\partial(u-w_0)}{\partial\nu} > 0$ pointwise. It is then achieved on $\partial\Omega_0 \cap \{x = x_0\}$ where $u \geq w_0$ pointwise, including at (x_0, y_0) such as above. \square

REFERENCES

- [1] N. Alibaud, G. Namah, On the propagation of a periodic flame front by an Arrhenius kinetic, *Interfaces Free Bound.*, **19** (2017) no. 3, 449–494.
- [2] C. M. Brauner, P. C. Fife, G. Namah and C. Schmidt-Lainé, Propagation of a combustion front in a striated solid medium: a homogenization analysis, *Quart. Appl. Math.* **51** (1993), no. 3, 467–493.
- [3] A. Cesaroni, M. Novaga, Long-time behavior of the mean curvature flow with periodic forcing, *Comm. Partial Differential Equations*, **38** (2013), no. 5, 780–801.
- [4] H. Brézis, *Analyse fonctionnelle : Théorie et Applications*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983.
- [5] X. Chen, G. Namah, Wave propagation under curvature effects in a heterogeneous medium, *Appl. Anal.* **64** (1997), no. 3–4, 219–233.
- [6] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, *AMS*, 1997.
- [7] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [8] Grisvard, *Boundary Value Problems in Non Smooth Domains*, Pitman, London, 1985.
- [9] R. Nittka, Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains, *J. Differential Equations* **251** (2011), no. 4–5, 860–880.

(Nathael Alibaud and Gawtum Namah) ECOLE NATIONALE SUPÉRIEURE DE MÉCANIQUE ET DES MICROTECHNIQUES, 26 CHEMIN DE L'ÉPITAPHE, AND, LABORATOIRE DE MATHÉMATIQUES DE BESANÇON, UMR CNRS 6623, UNIVERSITÉ DE BOURGOGNE FRANCHE-COMTÉ, 16 ROUTE DE GRAY, 25030 BESANÇON CEDEX, FRANCE

E-mail address: `nathael.alibaud@ens2m.fr`, `gawtum.namah@ens2m.fr`