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► **To cite this version:**

Paul Lescot. On an arithmetical question related to perfect numbers. *Mathematical Gazette*, In press.  
hal-02336970

**HAL Id: hal-02336970**

**<https://hal.science/hal-02336970>**

Submitted on 29 Oct 2019

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# ON AN ARITHMETICAL QUESTION RELATED TO PERFECT NUMBERS

## 1. INTRODUCTION AND STATEMENT OF THE PROBLEM

For  $n \geq 1$  an integer, we shall denote by  $\sigma(n)$  the sum of the (positive) divisors of  $n$ .

The line of thought leading to this note started with [3, Exemple III, p. 380], where two integers  $k$  such that  $\sigma(k^3)$  is a perfect square are given.

This suggests that we might look for numbers  $n$  such that  $\sigma(n)$  be a square. One rapidly notices that  $\sigma(66) = 144 = 12^2$  ; furthermore

$$\sigma(66) - 2.66 = 144 - 132 = 12,$$

whence

$$\sigma(66) = (\sigma(66) - 2.66)^2.$$

We were thus led to the following definition :

**Definition.** An integer  $n$  is termed **quadratically perfect** if

$$\sigma(n) = (\sigma(n) - 2n)^2.$$

As we have just established, 66 is such a number. A quick search yields three others : 1, 3 and 491536.

As a matter of fact, one has

**Lemma 1.** *The only quadratically perfect primary integers are 1 and 3.*

*Proof.* Let  $n = p^k$  ( $p$  prime) be quadratically perfect ; if  $k = 0$ ,  $n = 1$ . Let us then assume  $k \geq 1$  ; we have

$$\sigma(n) = 1 + p + \dots + p^k,$$

whence, setting  $S := \sum_{l=0}^{k-1} p^l$ , we find

$$\sigma(n) - 2n = S - p^k$$

and

$$\begin{aligned} (S - p^k)^2 &= (\sigma(n) - 2n)^2 \\ &= \sigma(n) \\ &= S + p^k. \quad (*) \end{aligned}$$

If  $k \geq 2$ , (\*) gives us, by working modulo  $p^2$ , that

$$S^2 \equiv S \pmod{p^2};$$

as  $S \equiv 1 + p \pmod{p^2}$ , it follows that

$$(1 + p)^2 \equiv 1 + p \pmod{p^2},$$

i.e.  $p^2$  divides  $p$ , a contradiction. Therefore  $k = 1$ ,  $S = 1$  and (\*) becomes

$$(1 - p)^2 = 1 + p,$$

that is  $p^2 = 3p$ ,  $p = 3$  and  $n = p^k = 3^1 = 3$ . □

## 2. THE MAIN THEOREM

On the positive side, one has

**Proposition 1.** *If  $m$  is a perfect number such that  $2m - 1$  is prime, then*

$$n := m(2m - 1)$$

*is quadratically perfect.*

*Remark.*  $m = 6$  ( $2m - 1 = 11$ ) yields  $n = 66$ , and  $m = 496$  ( $2m - 1 = 991$ ) yields  $n = 491536$ .

*Proof.* Seeing that  $m$  and  $2m - 1$  are coprime, one has

$$\begin{aligned} \sigma(n) &= \sigma(m(2m - 1)) \\ &= \sigma(m)\sigma(2m - 1) \\ &= 2m(1 + (2m - 1)) \\ &\quad \text{[as } m \text{ is perfect and } 2m - 1 \text{ is prime]} \\ &= 4m^2. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma(n) - 2n &= 4m^2 - 2m(2m - 1) \\ &= 2m, \end{aligned}$$

and

$$(\sigma(n) - 2n)^2 = (2m)^2 = 4m^2 = \sigma(n);$$

$n$  is quadratically perfect. □

A partial converse holds :

**Theorem 1.** *Let  $n$  be an even quadratically perfect number ; then there exists an even perfect number  $m$  such that  $n = m(2m - 1)$ .*

*Remark.* It is well-known (see e.g. [1,Theorem 277], or [2,pp.33-34]) that  $m$  can then be written as

$$m = 2^{p-1}(2^p - 1)$$

with  $2^p - 1$  (hence also  $p$ ) prime ;  $m = 6$  corresponds to  $p = 2$ , and  $m = 496$  to  $p = 5$ .

## 3. THE PROOF OF THEOREM 1

For convenience, this will be broken up into four lemmas.

Let us write  $n = 2^a x$  ( $a \geq 1$ ,  $x$  odd) ; then

$$\begin{aligned}\sigma(n) &= \sigma(2^a)\sigma(x) \\ &= (2^{a+1} - 1)\sigma(x).\end{aligned}$$

**Lemma 2.** *There are integers  $b$ ,  $e$  and  $f$  such that  $\sigma(x) = be^2$  and*

$$bce^2 - (bc^2 + 1)f - e = 0.$$

*Proof.* Let us define  $b$  as the *square-free part* of  $2^{a+1} - 1$  ; then

$$2^{a+1} - 1 = bc^2,$$

for some  $c \in \mathbf{N}$ ,  $c \geq 1$ . As  $2^{a+1} - 1 \equiv -1 \pmod{4}$ ,  $2^{a+1} - 1$  is not a square, therefore  $b \geq 2$  ; in fact,  $b \geq 3$  as  $b$  is odd.

As  $(\sigma(n) - 2n)^2 = \sigma(n)$ , we have

$$\begin{aligned}bc^2\sigma(x) &= (2^{a+1} - 1)\sigma(x) \\ &= \sigma(n) \\ &= (\sigma(n) - 2n)^2\end{aligned}$$

whence  $c$  divides  $\sigma(n) - 2n$  :

$$\sigma(n) - 2n = \epsilon cd,$$

with  $\epsilon \in \{-1, 1\}$  and  $d \geq 1$ . It follows that

$$bc^2\sigma(x) = c^2d^2,$$

*i.e.*  $b\sigma(x) = d^2$ , and  $b$  divides  $d^2$  ;  $b$  being square-free,  $b$  divides  $d$  :  $d = be$  ( $e \geq 1$ ). Therefore  $b\sigma(x) = b^2e^2$ , *i.e.*  $\sigma(x) = be^2$ . It now appears that

$$\begin{aligned}b^2c^2e^2 - (bc^2 + 1)x &= bc^2.be^2 - 2^{a+1}x \\ &= (2^{a+1} - 1)\sigma(x) - 2n \\ &= \sigma(n) - 2n \\ &= \epsilon cd \\ &= \epsilon cbe;\end{aligned}$$

therefore,  $bc$  divides  $x$ .

Let us then set  $x = bcf$  ( $f \geq 1$ ); as  $b > 1$ ,  $bc > 1$ ,  $be^2 = \sigma(x) \geq x + f$  and

$$\begin{aligned}b^2c^2e^2 &= bc^2\sigma(x) \\ &\geq bc^2x + bc^2f \\ &\geq bc^2x + bcf \\ &= bc^2x + x,\end{aligned}$$

thus  $\epsilon = 1$ .

We now get

$$b^2c^2e^2 - (bc^2 + 1)bcf = cbe,$$

that is

$$bce^2 - (bc^2 + 1)f - e = 0.$$

□

**Lemma 3.**  $c = 1$ , i.e.  $2^{a+1} - 1$  is square-free.

*Proof.* We know that  $b > 1$ ; then

$$be^2 = \sigma(x) = \sigma(bcf) \geq bcf + cf,$$

and, using Lemma 3.1 :

$$\begin{aligned} 0 &= bce^2 - (bc^2 + 1)f - e \\ &\geq c(bcf + cf) - (bc^2 + 1)f - e \\ &= (c^2 - 1)f - e. \end{aligned}$$

Assuming that  $c > 1$ , it would follow that  $f \leq \frac{e}{c^2 - 1}$ , whence that

$$\begin{aligned} bce^2 &= (bc^2 + 1)f + e \\ &\leq (bc^2 + 1)\frac{e}{c^2 - 1} + e \\ &= \frac{e}{c^2 - 1}(bc^2 + c^2) \\ &= e(b + 1)\frac{c^2}{c^2 - 1}. \end{aligned}$$

It would now appear that

$$\begin{aligned} e &\leq \frac{b + 1}{b} \frac{c}{c^2 - 1} \\ &\leq \frac{4}{3} \frac{3}{8} = \frac{1}{2} < 1, \end{aligned}$$

a contradiction.

Therefore  $c = 1$ . □

**Lemma 4.**  $b$  is prime.

*Proof.* From Lemmas 3.1 and 3.2 follows that

$$be^2 - e - (b + 1)f = 0.$$

As  $x = bcf = bf$ , one has  $\sigma(bf) = \sigma(x) = be^2$ .

We know that  $b \geq 3$ ; let us assume that  $b$  is not prime, and let  $p$  denote the smallest prime factor of  $b$ . Then

$$\sigma(b) \geq b + p + 1 \geq b + 4$$

and

$$f(b + 4) \leq f\sigma(b) \leq \sigma(bf) = be^2,$$

whence

$$f(b + 4) \leq be^2 = e + (b + 1)f,$$

and  $3f \leq e$ . It follows that

$$\begin{aligned}
3be^2 &= 3e + (3b + 3)f \\
&\leq 3e + (b + 1)e \\
&= e(b + 4),
\end{aligned}$$

hence  $3b \leq 3be \leq b + 4$ ,  $2b \leq 4$  and  $b \leq 2$ , a contradiction.

Therefore  $b$  is prime. □

**Lemma 5.**  *$b$  does not divide  $f$ .*

*Proof.* Let us assume for a moment that  $b$  divides  $f$ ; then, as

$$be^2 = e + bf + f,$$

$b$  divides  $e$ . It follows that

$$be^2 = \sigma(bf) \geq bf + f + \frac{f}{b},$$

whence

$$\frac{f}{b} \leq be^2 - bf - f = e,$$

thus  $f \leq be$ .

Now

$$\begin{aligned}
be^2 &= e + bf + f \\
&\leq e + b^2e + be
\end{aligned}$$

and

$$\begin{aligned}
be &\leq b^2 + b + 1 \\
&= b(b + 1) + 1 \\
&< b(b + 2).
\end{aligned}$$

It appears that  $e < b + 2$ , hence  $e \leq b + 1 < 2b$ ; as  $b$  divides  $e$ ,  $e = b$  and

$$(b + 1)f = be^2 - e = b^3 - b,$$

thus  $f = b(b - 1)$ .

Now we have

$$\sigma(b^2(b - 1)) = \sigma(bf) = be^2 = b^3 = b^2(b - 1) + b^2,$$

an obvious contradiction as

$$\sigma(b^2(b - 1)) \geq b^2(b - 1) + b^2 + b + 1.$$

We conclude that  $b$  does not divide  $f$ . □

**Conclusion**

We may now write

$$\begin{aligned}
 (b+1)\sigma(f) &= \sigma(b)\sigma(f) \\
 &\quad \text{[by Lemma 3.3]} \\
 &= \sigma(bf) \\
 &\quad \text{[as } b \text{ and } f \text{ are coprime, by Lemmas 3.3 and 3.4]} \\
 &= be^2 \\
 &= bf + e + f ;
 \end{aligned}$$

thus  $b+1$  divides  $e$  :  $e = (b+1)g$ , hence

$$\begin{aligned}
 (b+1)f &= e(be-1) \\
 &= (b+1)g(b(b+1)g-1)
 \end{aligned}$$

and  $f = g(b(b+1)g-1)$ . In particular,  $g$  divides  $f$ , and  $g \neq f$ . As

$$(b+1)\sigma(f) = bf + (b+1)g + f,$$

we obtain  $\sigma(f) = f + g$  ; therefore  $g = 1$  and  $f$  is prime.

Furthermore,  $f = b(b+1) - 1$ .

Now let  $m := 2^a(2^{a+1} - 1) = 2^a b$  ; then  $m$  is perfect ([1, Theorem 276], or [2, p.33]) and even,  $2m - 1 = 2^{a+1}b - 1 = (b+1)b - 1 = f$  is prime, and

$$m(2m - 1) = 2^a b f = 2^a x = n.$$

## 4. FINAL COMMENTS

It is reasonable to expect the converse of Proposition 2.1 to still hold for all odd  $m \geq 5$  ; combined with the proof of the long-standing conjecture that there is no odd perfect number, this would imply the nonexistence of odd quadratically perfect numbers other than 1 and 3.

I wish to express my gratitude to Dr. Numa Lescot for checking with the help of a computer that there is at least no such number between 5 and  $10^6$ .

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## ACKNOWLEDGEMENT

I wish to thank the (anonymous) referee, whose comments helped me improve the style and readability of the article.