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Symmetries of partial differential equations and stochastic processes in mathematical physics and in finance

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Abstract. In 1971, B. Kent Harrison and Frank B. Estabrook introduced a method to determine the symmetries of partial differential equations (PDEs). These last years, the determination of the symmetries of PDEs in Mathematical Physics, in Mathematical Biology and in Financial Mathematics has proved useful. The computations effected in all these cases let appear a remarkable degree of similarity between them. So with the same aim in mind, we develop a general framework for the computation of the symmetries with this method, we give properties of isovectors for a rather general type of PDE's and some results on the Lie algebra itself. Finally we present three examples for which all the results we exposed hold.

1. The method of isovectors

We shall give an overview of joint work with Helene Quintard and Jean-Claude Zambrini ([1]) and of L. Valade's ongoing PhD Thesis.

The method of isovectors was introduced in [2] in order to classify up to equivalence (systems of) partial differential equations appearing in mathematical physics.

Given a system of partial differential equations, after if necessary making a change of variable(s) and/or unknown function(s), we can express it as the vanishing of a family of first-order differential forms. An isovector is then defined as a vector field in all the variables preserving the differential ideal generated by the forms.

For the one-dimensional heat equation, the symmetries were determined using a different language) by Bluman and Cole ([3]).

Olver's *prolongation method* ([4]) provides a somewhat different approach.

Let us now give some details. We shall consider an equation (\mathcal{E}) of the shape

$$\frac{\partial u}{\partial t} = G \left(t, q, u, \frac{\partial u}{\partial q}, \dots, \frac{\partial^{n-1} u}{\partial q^{n-1}} \right) + \lambda \frac{\partial^n u}{\partial q^n}$$

for $\lambda \neq 0$, $n \geq 2$, $t \in J$ (an interval of \mathbf{R}) and $q \in O$ (an open set in \mathbf{R}).

In order to study the symmetries of the equation, we shall temporarily consider $u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial q}, \dots,$

$\frac{\partial^{n-1} u}{\partial q^{n-1}}$ as *independent* variables.



We shall take as state space $M := J \times O \times \mathbf{R}^{n+1}$, the generic point of which will be denoted by $(t, q, u, A, B_1, \dots, B_{n-1})$.

All of our computations will take place in the differential algebra $\wedge T^*M$.

We set :

$$\begin{aligned}\alpha &:= du - Adt - B_1dq \\ \gamma &:= (A - G)dtdq - \lambda dtdB_{n-1}\end{aligned}$$

and, for $1 \leq i \leq n - 2$,

$$\beta_i = dtdB_i - B_{i+1}dtdq;$$

we now have

$$d\alpha = -dAdt - dB_1dq$$

and

$$d\beta_i = -dB_{i+1}dtdq.$$

(\mathcal{E}) is equivalent to the simultaneous vanishing of these forms on a 2-dimensional submanifold of M .

We define I as the ideal of $\wedge T^*M$ generated by α , $d\alpha$, the β_i , the $d\beta_i$, and γ .

Lemma 1. $d\gamma \in I$.

Proof:

$$d\gamma = dAdtdq - dGdtdq = -d\alpha dq - G_u d u d t d q - \sum_{i=1}^{n-1} G_{B_i} dB_i d t d q$$

but $d u d t d q = \alpha d t d q$ and $dB_i d t d q = d\beta_{i-1}$ for all i between 1 and $n - 1$.

So $d\gamma \in I$ ■

Therefore I is closed under d (i.e. $d(I) \subseteq I$), hence is a differential ideal of $\wedge T^*(M)$.

We shall denote by \mathcal{G} the *isovector algebra* of (\mathcal{E}); this is the set of vector fields $N \in TM$ such that

$$\mathcal{L}_N(I) \subseteq I.$$

Due to the formal properties ([2] p.654) of the Lie derivative, these isovectors constitute a Lie algebra for the usual bracket of vector fields on M .

We shall write each $N \in \mathcal{G}$ as

$$N = N^t \frac{\partial}{\partial t} + N^q \frac{\partial}{\partial q} + N^u \frac{\partial}{\partial u} + N^A \frac{\partial}{\partial A} + \sum_{i=1}^{n-1} N^{B_i} \frac{\partial}{\partial B_i};$$

then we define

$$\tilde{N} = -N^t \frac{\partial}{\partial t} - N^q \frac{\partial}{\partial q} + N^u.$$

Theorem 1. For each $N \in \mathcal{G}$, N^t depends only on t , N^q depends only on t and q and N^u depends only on t , q and u .

Theorem 2. Under the additional hypothesis $\frac{\partial^2 G}{\partial B_1 \partial B_{n-1}} = 0$, for each $N \in \mathcal{G}$, N^u is affine in u .

Now we can set $N^u = l(t, q) + um(t, q)$.

There exist two other functions $f(t)$ and $w(t, q)$ such that:

$$\begin{aligned}N^t &= -f(t) \\ N^q &= -w(t, q) \\ N^A &= B_1 w_t + l_t + um_t + A f_t + Am \\ N^{B_i} &= B_i m + B_i w_q + l_q + um_q \quad \forall i \in \{1, \dots, n-1\}.\end{aligned}$$

Detailed proofs of the previous two theorems will be given in a subsequent paper.

Definition 1.

$$\mathcal{J}_0 := \{N \in \mathcal{G} | N^t = N^q = 0\}.$$

Theorem 3. \mathcal{J}_0 is an ideal of \mathcal{G} .

Proof: In fact if we consider $N \in \mathcal{G}$ and $M \in \mathcal{J}_0$ then we have

$$[N, M]^t = [N, M](t) = N(M(t)) - M(N(t)) = N(M^t) - M(N^t) = -M^t \frac{\partial N^t}{\partial t} = 0$$

because $M^t = 0$ and N^t only depends on t .

Similarly,

$$[N, M]^q = [N, M](q) = N(M(q)) - M(N(q)) = N(M^q) - M(N^q) = -M^t \frac{\partial N^q}{\partial t} - M^q \frac{\partial N^q}{\partial q} = 0$$

because $M^t = M^q = 0$ and N^q only depends on t and q .

Now the result is proved. ■

From now we assume the hypothesis of theorem 2 to be satisfied.

Lemma 2. We pose $N^u = l + um$ and $N'^u = l' + um'$. Then

$$[N, N']^u = [N, N'](u) = N(N'^u) - N'(N^u) = N(l') + (l + um)m' + uN(m') - N'(l) - (l' + um')m - uN'(m)$$

Hence

$$l_{[N, N']} = N(l') - N'(l) + lm' - l'm$$

and

$$m_{[N, N']} = N(m') - N'(m).$$

Proposition 1. $N \mapsto -\tilde{N}$ is a morphism of Lie algebras.

Proof: The argument in [1] is still true here. ■

Definition 2. We define:

$$\mathcal{J} = \{N \in \mathcal{G} | m = N^t = N^q = 0\}$$

and

$$\mathcal{H} = \{N \in \mathcal{G} | l = 0\}.$$

Theorem 4. \mathcal{H} is a subalgebra of \mathcal{G} , \mathcal{J} is an abelian ideal of \mathcal{G} and the sum $\mathcal{J} \oplus \mathcal{H}$ is direct; in particular \mathcal{H} is isomorphic to a subalgebra of $\frac{\mathcal{G}}{\mathcal{J}}$.

Proof: For $N \in \mathcal{H}$ and $N' \in \mathcal{H}$, $[N, N']^u = [N, N'](u) = N(N'^u) - N'(N^u) = umm' + uN(m') - um'm - uN'(m)$. Hence $l_{[N, N']} = 0$ and then $[N, N'] \in \mathcal{H}$; hence \mathcal{H} is a subalgebra of \mathcal{G} .

\mathcal{J} is an ideal of \mathcal{G} according to the same reasoning as Theorem 3. Moreover it is abelian because $\forall N \in \mathcal{J}$ and $\forall N' \in \mathcal{J}$, $[N, N'] = 0$.

Considering $N \in \mathcal{J} \cap \mathcal{H}$, we have $N^t = N^q = N^u = 0$ and according to the expression of N^A and N^{B_i} for $1 \leq i \leq n - 1$ in Theorem 2, we obtain $N = 0$; hence $\mathcal{J} \cap \mathcal{H} = \{0\}$. ■

We call $\frac{\mathcal{G}}{\mathcal{J}}$ the isovector algebra of the equation (\mathcal{E}).

Theorem 5. Let us assume that either $G = 0$ or $n = 2$ and G is of the form $G = c(t)B_1 + V(t, q)u$ (this is the case in the examples below). Then $\mathcal{G} = \mathcal{J} \oplus \mathcal{H}$; in particular, $\frac{\mathcal{G}}{\mathcal{J}} \cong \mathcal{H}$.

2. Examples

- (i) We can apply the method and these result to find symmetries of the Black-Scholes equation This is the most famous equation in Mathematical Finance:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0$$

We set $q = \ln(S)$ and $u(t, q) = C(t, \exp(q)) = u(t, S)$. Now the equation becomes

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial q^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial u}{\partial q} - ru = 0$$

According to our notations $G = -(r - \frac{\sigma^2}{2})B_1 + ru$ and $\lambda = -\frac{1}{2}\sigma^2$.

Here it turns out ([5]) that \mathcal{H} has dimension 6, and is isomorphic to the algebra $\mathcal{H}_{0,0}$ below. This is not surprising inasmuch that both the Black-Scholes equation and the HJB equation with $V = 0$ can be reduced to the heat equation. Nevertheless our computation does not depend upon that reduction, and would actually suggest it; notably, the quantities $r + \frac{\sigma^2}{2}$ and $r - \frac{\sigma^2}{2}$ appear in a natural way.

- (ii) Now we present the *backward heat equation with potential V*:

$$\theta^2 \frac{\partial u}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 u}{\partial q^2} + Vu.$$

Here we have $\lambda = -\theta^2/2$ and $G = \frac{1}{\theta^2}Vu$.

In the case of the potential

$$V(t, q) = \frac{C}{q^2} + Dq^2,$$

let $\mathcal{H}_{C,D} := \mathcal{H}_V$. Then for $C \neq 0$, $\mathcal{H}_{C,D} \simeq \mathcal{H}_{1,0}$ has dimension 4; for $C = 0$, $\mathcal{H}_{C,D} \simeq \mathcal{H}_{0,0}$ has dimension 6 (see [1] and [6]).

Furthermore, $\mathcal{H}_{1,0} \subseteq \mathcal{H}_{0,0}$ ([6]). In addition these Lie algebras possess canonical bases, continuous in D for fixed C , and compatible with the inclusions

$$\mathcal{H}_{C,D} \subseteq \mathcal{H}_{0,D}.$$

This computation was effected in [1] (see also [6], [7] and [8]) using the transformation $S = -\theta^2 \ln(u)$, that converts (\mathcal{E}) to the Hamilton-Jacobi-Bellman equation (\mathcal{HJB}^V) :

$$\frac{\partial S}{\partial t} = -\frac{\theta^2}{2} \frac{\partial^2 S}{\partial q^2} + \frac{1}{2} \left(\frac{\partial S}{\partial q}\right)^2 - V.$$

- (iii) Finally we have

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial q^4}.$$

Here it is evident that $\lambda = -1$ and $G = 0$.

The algebra of isovectors has been determined by Vigot ([9]) and Valade where q is x . \mathcal{H} has a basis $(X_i) (1 \leq i \leq 4)$ with brackets:

	X_1	X_2	X_3	X_4
X_1	0	0	X_1	0
X_2	0	0	$4X_2$	0
X_3	$-X_1$	$-4X_2$	0	0
X_4	0	0	0	0

One has

$$\begin{aligned} -\tilde{X}_1 &= \frac{\partial}{\partial q} \\ -\tilde{X}_2 &= \frac{\partial}{\partial t} \\ -\tilde{X}_3 &= q \frac{\partial}{\partial q} + 4t \frac{\partial}{\partial t} \\ -\tilde{X}_4 &= u \frac{\partial}{\partial u} \end{aligned}$$

Therefore $\mathcal{H} = \mathbf{R} \oplus \mathcal{L}$, with \mathcal{L} a three-dimensional solvable Lie algebra with 2-dimensional derived algebra. \mathcal{L} is isomorphic to the algebra over \mathbf{R} considered in [10] (p. 13, line 1 for $\alpha = 4$).

Once exponentiated, here is how the basis elements of $\tilde{\mathcal{H}} = \{\tilde{N} | N \in \mathcal{H}\}$ act on a solution u of the equation:

$$\begin{aligned} e^{\alpha \tilde{X}_1} u(t, q) &= u(t + \alpha, q) \\ e^{\alpha \tilde{X}_2} u(t, q) &= u(t, q + \alpha) \\ e^{\alpha \tilde{X}_3} u(t, q) &= u(e^{4\alpha} t, e^\alpha q) \\ e^{\alpha \tilde{X}_4} u(t, q) &= e^\alpha u(t, q) \end{aligned}$$

The equation is deeply related to Hochberg's pseudo-process ([11]).

3. Comments

We present here only results for one variable in space but in her thesis (see [1]), H. Quintard has determined the structure of the isovector algebra for the equation

$$\frac{\partial u}{\partial t} = \sigma \Delta u + V u.$$

for quadratic V .

Also, some works are in progress about the equation $u_t = \Delta^2 u$.

We want also to extend this method to more general equations.

For example in an equation of financial mathematics due to Frey

$$u_t + \frac{1}{2} \sigma^2 q^2 \frac{u_{qq}}{(1 - \rho q \lambda(q) u_{qq})^2} = 0$$

(In the literature q is S and it represents the price of the stock that we consider). Here ρ is a real parameter and λ a given function.

It is a nonlinear version of the Black-Scholes equation, first considered by Frey ([12]). Bobrov ([13]), in an unpublished paper, determined the isovectors; they were computed again in a different way by Valade.

Defining

$$\begin{aligned}\tilde{V}_1 &= \frac{\partial}{\partial t}, \\ \tilde{V}_2 &= q \frac{\partial}{\partial u}, \\ \tilde{V}_3 &= \frac{\partial}{\partial u},\end{aligned}$$

then, when λ is not of the form ωq^k , it appears that $\tilde{\mathcal{G}}$ is generated by \tilde{V}_1 , \tilde{V}_2 and \tilde{V}_3 ; in particular, it is abelian of dimension 3.

When $\lambda(q) \equiv \omega q^k$, $\tilde{\mathcal{G}}$ is generated by \tilde{V}_1 , \tilde{V}_2 , \tilde{V}_3 and

$$\tilde{V}_4 := -q \frac{\partial}{\partial q} + (1-k)u \frac{\partial}{\partial u}$$

and it has a far more interesting structure ([13]).

We have

$$\tilde{\mathcal{J}} = \langle \tilde{V}_2, \tilde{V}_3 \rangle .$$

We are trying to generalize this.

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