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# ON SOME BERNSTEIN PROCESSES SIMILAR TO COX-INGERSOLL-ROSS ONES

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ABSTRACT. We present some results on Bernstein processes, which are Brownian diffusions that appear in Euclidean Quantum Mechanics. We express the distributions of these processes with the help of those of Bessel processes. We then determine two solutions of the dual equation of the heat equation with potential.

These results first appeared in the first author's PhD thesis (Rouen, 2013).

Keywords: Bernstein processes, Bessel processes, interest rate models, heat equation.

## 1. INTRODUCTION

*Bernstein processes* or reciprocal diffusions combine two dynamic behaviors : forward and backward. These processes are useful tools of stochastic quantum mechanics, and also find applications in various other fields.

We give in this paper an explicit expression for the distribution density of a special Bernstein process. This process is similar to the Cox-Ingersoll-Ross process of financial mathematics. Indeed, in this work, we show that each *one-factor affine interest rate model* (in the sense of Leblanc-Scaillet [3]) can be described using such a Bernstein process. See also [4].

The groundbreaking idea of replacing the complex Schrödinger equation by forward and backward heat equations in duality goes back to Schrödinger ([6], pp. 144-153).

Two solutions of the dual equation have been computed by Lescot, Quintard and Zambrini [4] by using the Gaussian character of Ornstein-Uhlenbeck process and Brownian motion. In this paper, this goal is reached by using another class of diffusion which is that of *Bessel processes*. Indeed, in our case, the Bernstein process defined by a square root of a Cox-Ingersoll-Ross process (CIR) can be represented in terms of a Bessel process ([2], p.314 (2)), and the density of this Bessel process is well known [5], p.441 (1.4)). So, by using the link between the two families of diffusions, we are able to determine the density of the law of this Bernstein process.

Let us first give some definitions and recall some preliminary results.

**Definition 1.1.** Let  $\theta > 0$  denote a parameter. A Bernstein process  $z$  is a process satisfying a stochastic differential equation of the form

$$dz(t) = \theta dw(t) + \tilde{B}(t, z(t))dt$$

relatively to the canonical filtration of the Brownian motion  $w$  and the dual stochastic differential equation

$$d_*z(t) = \theta dw_*(t) + \tilde{B}_*(t, z(t))dt$$

relatively to the canonical decreasing filtration of another Brownian motion  $w_*$  which is independent of  $w$ , where

$$\tilde{B} := \theta^2 \frac{\frac{\partial \eta}{\partial q}}{\eta},$$

$$\tilde{B}_* := -\theta^2 \frac{\frac{\partial \eta_*}{\partial q}}{\eta_*},$$

and, for each given  $t > 0$ , the law of  $z(t)$  is  $\eta(t, q)\eta_*(t, q)dq$ . The function  $\eta$  is assumed to be an everywhere positive solution to the backward heat equation with potential  $V$ :

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V\eta \quad (\mathcal{C}_1^{(V)}).$$

Similarly,  $\eta_*$  is assumed to be everywhere positive and a solution to

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V\eta_* \quad (\mathcal{C}_2^{(V)}).$$

**Definition 1.2.** [1],p.454 (2.1) The Bessel function  $J_\lambda$  with index  $\lambda \in \mathbf{C}$  is defined by

$$J_\lambda(z) = \left(\frac{z}{2}\right)^\lambda \sum_{n=0}^{\infty} \frac{(-z^2)^n}{2^{2n} n! \Gamma(n + \lambda + 1)}.$$

This function satisfies the Bessel equation with parameter  $\lambda \in \mathbf{C}$

$$(1.1) \quad z^2 \ddot{\omega} + z\dot{\omega} + (z^2 - \lambda^2)\omega = 0.$$

**Definition 1.3.** The Bessel modified function  $I_\nu$  with index  $\nu \in \mathbf{R}$  is defined by

$$I_\nu(z) = i^{-\nu} J_\nu(iz).$$

This function satisfies the linear differential equation of the second order

$$(1.2) \quad z^2 \ddot{I}_\nu(z) + z\dot{I}_\nu(z) - (z^2 + \nu^2)I_\nu(z) = 0, \text{ that is}$$

Indeed, we have  $I_\nu(z) = i^{-\nu} J_\nu(iz)$ ,  $\dot{I}_\nu(z) = i^{-\nu} i \dot{J}_\nu(iz)$  et  $\ddot{I}_\nu(z) = i^{-\nu} i^2 \ddot{J}_\nu(iz)$ . Equation (1.1) gives

$$(iz)^2 i^\nu i^{-2} \ddot{I}_\nu(z) + iz i^\nu i^{-1} \dot{I}_\nu(z) + (-z^2 - \nu^2) i^\nu I_\nu(z) = 0$$

$$z^2 \ddot{I}_\nu(z) + z\dot{I}_\nu(z) - (z^2 + \nu^2)I_\nu(z) = 0,$$

which yields (1.2).

We refer to [1] for details on Bessel functions.

**Definition 1.4.** For all  $\delta \geq 0$  and  $x_0 \geq 0$ , the unique solution of the stochastic differential equation

$$Y_t = x_0 + \delta t + 2 \int_0^t \sqrt{|Y_s|} dw(s)$$

(starting from  $x_0$ ) is called squared Bessel process of dimension  $\delta$ . This process will be denoted by  $BESQ_{x_0}^\delta$ .

We refer to [2] and [5] for a survey on Bessel processes.

**Definition 1.5.** The Cox-Ingersoll-Ross (CIR) family of diffusions, also known as square root diffusions, solve

$$dX_t = (a + bX_t)dt + c\sqrt{X_t}dw_t,$$

where  $X_0 = x_0$ ,  $a \geq 0$ ,  $b \in \mathbf{R}$ ,  $c > 0$ , and  $(w_t)_{t \geq 0}$  is a standard Brownian motion.

**Proposition 1.6.** [5], p.441 (1.4) If  $x_0 = 0$ , then the density  $q_t^\delta$  of the law of  $Y_t$  is given by

$$(1.3) \quad q_t^\delta(0, y) = (2t)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2t}\right), \quad t > 0.$$

If  $\mathbf{x}_0 > \mathbf{0}$ , then the density  $q_t^\delta$  of the law of  $Y_t$  is given by

$$(1.4) \quad q_t^\delta(x_0, y) = \frac{1}{2t} \left(\frac{y}{x_0}\right)^{\frac{\nu}{2}} \exp\left(-\frac{x_0 + y}{2t}\right) I_\nu\left(\frac{\sqrt{x_0 y}}{t}\right), \quad t > 0,$$

where  $\nu := \frac{\delta}{2} - 1$ .

## 2. AFFINE INTEREST RATE MODELS

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q)$  be a filtered probability space, and let  $(w(t))_{t \geq 0}$  be an  $(\mathcal{F}_t)_{t \geq 0}$  standard Brownian motion.

A one-factor affine interest rate model is characterized by the instantaneous rate  $r(t)$ , satisfying the stochastic differential equation

$$dr(t) = \sqrt{\alpha r(t) + \beta} dw(t) + (\phi - \lambda r(t)) dt$$

under the risk-neutral probability  $Q$  [3], p.351).

Assuming  $\alpha \neq 0$ , let us set, with the notations of Lescot–Quintard–Zambrini [4],

$$\tilde{\phi} = \phi + \frac{\lambda\beta}{\alpha},$$

$$\delta = \frac{4\tilde{\phi}}{\alpha},$$

$$A = \frac{\alpha^4}{128}(\delta - 1)(\delta - 3),$$

and

$$B = \frac{\lambda^2}{8}.$$

The following result of Lescot–Quintard–Zambrini [4] shows that the diffusion term of a one-factor affine interest rate model is a Bernstein process. So this means that we are able to describe the law of the interest rate process by using this Bernstein process.

**Proposition 2.1.** (Lescot–Quintard–Zambrini [4]) *Let  $X_t$  be defined by*

$$X_t = \alpha r(t) + \beta.$$

*The process  $X$  is then Cox-Ingersoll-Ross diffusion in the sense of Definition 1.5. Define  $Z(t)$  by*

$$Z(t) = \sqrt{\alpha r(t) + \beta}.$$

*Then  $Z$  is a Bernstein process with*

$$\theta = \frac{\alpha}{2}$$

*and the potential*

$$V(t, q) = \frac{A}{q^2} + Bq^2.$$

**Lemma 2.2.** [2] p.314 (2) *Let  $\mathbf{X}_0 = \mathbf{x}_0$ , then*

$$X(t) = e^{-\lambda t} Y(s),$$

*where  $s = \frac{\alpha^2(e^{\lambda t} - 1)}{4\lambda}$  and  $Y$  is BESQ $_{x_0}^\delta$ .*

Our main result is the following : we determine two solutions of the dual equation ( $\mathcal{C}_2^{(V)}$ ), when the potential  $V$  is written as

$$V(t, q) = \frac{A}{q^2} + Bq^2.$$

Firstly we compute the density  $\rho_t(q)$  of the law of  $Z_t$ . Here, two cases arise :  $\mathbf{x}_0 = \mathbf{0}$  or  $\mathbf{x}_0 > \mathbf{0}$ .

**Proposition 2.3.** *If  $\mathbf{x}_0 = \mathbf{0}$ , then the density  $\rho_t(q)$  of the law of  $Z_t$  is written as*

$$\rho_t(q) = \alpha^{-\delta} 2^{\frac{\delta}{2}+1} \lambda^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} q^{\delta-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t} q^2}{\alpha^2(e^{\lambda t} - 1)}\right) \mathbb{1}_{\{q>0\}}.$$

*Proof.* For fixed  $t$  and for all  $g$  bounded continuous,

$$\begin{aligned} E(g(X_t)) &= E[g(\exp(-\lambda t)Y(s))] \\ &\stackrel{(1.3)}{=} \int_0^\infty g(\exp(-\lambda t)y) (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2s}\right) dy \\ &= (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} \int_0^\infty g(\exp(-\lambda t)y) y^{\frac{\delta}{2}-1} \exp\left(-\frac{y}{2s}\right) dy. \end{aligned}$$

By the change of variable  $x = \exp(-\lambda t)y$ , we get

$$\begin{aligned} E(g(X_t)) &= (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} \int_0^\infty g(x) [x \exp(\lambda t)]^{\frac{\delta}{2}-1} \exp\left(-\frac{e^{\lambda t}x}{2s}\right) \exp(\lambda t) dx \\ &= \int_0^\infty g(x) (2s)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\frac{\delta}{2}-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(-\frac{e^{\lambda t}x}{2s}\right) dx. \end{aligned}$$

Then the density  $\ell_{X_t}$  of the law of  $X_t$  is given by

$$\ell_{X_t}(x) = \alpha^{-\delta} (2\lambda)^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} x^{\frac{\delta}{2}-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t}x}{\alpha^2(e^{\lambda t} - 1)}\right) \mathbb{1}_{\{x>0\}}.$$

For fixed  $t$  and for all bounded continuous  $\varphi$ , we have

$$\begin{aligned} E[\varphi(Z_t)] &= E[\varphi(\sqrt{X_t})] \\ &= \int_0^\infty \varphi(\sqrt{x}) \ell_{X_t}(x) dx. \end{aligned}$$

By the change of variable  $q = \sqrt{x}$ , we get

$$\begin{aligned} E[\varphi(Z_t)] &= \int_0^\infty \varphi(q) \alpha^{-\delta} (2\lambda)^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} q^{\delta-2} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t} q^2}{\alpha^2(e^{\lambda t} - 1)}\right) 2q dq \\ &= \int_0^\infty \varphi(q) \alpha^{-\delta} 2^{\frac{\delta}{2}+1} \lambda^{\frac{\delta}{2}} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1} q^{\delta-1} \exp\left(\frac{\lambda\delta}{2}t\right) \exp\left(\frac{-2\lambda e^{\lambda t} q^2}{\alpha^2(e^{\lambda t} - 1)}\right) dq. \end{aligned}$$

This proves the result.  $\square$

Let  $\eta(t, q)$  be defined by

$$\eta(t, q) = \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2}\right) q^{\frac{\delta-1}{2}}.$$

The function  $\eta$  is a solution of

$$\theta^2 \frac{\partial \eta}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta}{\partial q^2} + V\eta \quad (\mathcal{C}_1^{(V)}),$$

with

$$V(t, q) = \frac{A}{q^2} + Bq^2.$$

**Proposition 2.4.** *The function  $\eta_*(t, q)$  defined by*

$$\begin{aligned} \eta_*(t, q) &= \frac{\rho_t(q)}{\eta(t, q)} \\ &= \frac{2^{\frac{\delta}{2}+1}}{\alpha^\delta} \frac{\lambda^{\frac{\delta}{2}}}{\Gamma\left(\frac{\delta}{2}\right)} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} q^{\frac{\delta-1}{2}} \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh\left(\frac{\lambda t}{2}\right)}\right), \end{aligned}$$

satisfies the dual equation

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V\eta_* \quad (\mathcal{C}_2^{(V)}).$$

*Proof.* Let

$$C = \alpha^{-\delta} 2^{\frac{\delta}{2}+1} \lambda^{\frac{\delta}{2}} \Gamma\left(\frac{\delta}{2}\right)^{-1}$$

and

$$C_1 = C(e^{\lambda t} - 1)^{-\frac{\delta}{2}}.$$

We have

$$\frac{\partial \eta_*}{\partial t} = \left( -\frac{\delta}{2} (\lambda e^{\lambda t}) + \frac{\lambda\delta}{4} - \frac{\lambda q^2}{\alpha^2} \left( -\frac{2\lambda e^{\lambda t}}{(e^{\lambda t} - 1)^2} \right) \right) C(e^{\lambda t} - 1)^{-\frac{\delta}{2}} q^{\frac{\delta-1}{2}} \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh\left(\frac{\lambda t}{2}\right)}\right),$$

$$\frac{\partial \eta_*}{\partial q} = \left( \left( \frac{\delta-1}{2} \right) q^{\frac{\delta-3}{2}} + q^{\frac{\delta-1}{2}} \left( -\frac{2\lambda q}{\alpha^2 \tanh\left(\frac{\lambda t}{2}\right)} \right) \right) C_1 \exp\left(\frac{\lambda\delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh\left(\frac{\lambda t}{2}\right)}\right),$$

$$\begin{aligned} \frac{\partial^2 \eta_*}{\partial q^2} &= \left( \left( \frac{\delta-1}{2} \right) \left( \frac{\delta-3}{2} \right) q^{\frac{\delta-5}{2}} + \left( \frac{\delta-1}{2} \right) q^{\frac{\delta-3}{2}} \left( -\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) + q^{\frac{\delta-1}{2}} \left( -\frac{2\lambda}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) \right. \\ &\quad \left. + \left( \frac{\delta-1}{2} \right) q^{\frac{\delta-3}{2}} \left( -\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right) + q^{\frac{\delta-1}{2}} \left( -\frac{2\lambda q}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right)^2 \right) C_1 \exp\left( \frac{\lambda \delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right), \end{aligned}$$

and

$$\begin{aligned} V(t, q) \eta_*(t, q) &= \left( \frac{A}{q^2} + Bq^2 \right) \eta_*(t, q) \\ &= \left( \frac{\alpha^4}{2^7} (\delta-1)(\delta-3) q^{-2} + \frac{\lambda^2}{2^3} q^2 \right) \eta_*(t, q) \\ &= \left( \alpha^{-\delta+4} 2^{\frac{\delta}{2}-6} \lambda^{\frac{\delta}{2}} (\delta-1)(\delta-3) q^{\frac{\delta-5}{2}} + \alpha^{-\delta} 2^{\frac{\delta}{2}-2} \lambda^{\frac{\delta}{2}+2} q^{\frac{\delta+3}{2}} \right) \Gamma\left(\frac{\delta}{2}\right)^{-1} (e^{\lambda t} - 1)^{-\frac{\delta}{2}} \\ &\quad \times \exp\left( \frac{\lambda \delta t}{4} - \frac{\lambda q^2}{\alpha^2 \tanh(\frac{\lambda t}{2})} \right). \end{aligned}$$

It is then straightforward to see that, for all  $(t, q) \in \mathbf{R}_+^* \times \mathbf{R}_+^*$ ,

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V \eta_* \quad (\mathcal{C}_2^{(V)}).$$

□

**Proposition 2.5.** *If  $\mathbf{x}_0 > \mathbf{0}$ , then the density  $\rho_t(q)$  of the law  $Z_t$  is given by*

$$\rho_t(q) = \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\lambda t(\frac{\delta}{4} + \frac{1}{2})}}{(e^{\lambda t} - 1)} I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) q^{\frac{\delta}{2}} \exp\left(-\frac{2\lambda}{\alpha^2} \left(\frac{z_0^2 + e^{\lambda t} q^2}{(e^{\lambda t} - 1)}\right)\right) \mathbb{1}_{\{q>0\}}.$$

*Proof.* We have

$$X_t = e^{-\lambda t} Y(s), \quad s = \frac{\alpha^2}{4\lambda} (e^{\lambda t} - 1)$$

and

$$\begin{cases} Z_t &= \sqrt{X_t} \\ Z_0 &= \sqrt{x_0}. \end{cases}$$

For fixed  $t$ , for all  $g$  bounded continuous, using (1.4), we get

$$\begin{aligned} E(g(Z_t)) &= E\left(g\left(\sqrt{e^{-\lambda t} Y(s)}\right)\right) \\ &= \int_0^\infty g(\sqrt{e^{-\lambda t} y}) \frac{1}{2s} \left(\frac{y}{x_0}\right)^{\frac{\nu}{2}} \exp\left(-\frac{x_0 + y}{2s}\right) I_\nu\left(\frac{\sqrt{x_0 y}}{s}\right) dy. \end{aligned}$$

By the change of variable  $z = \sqrt{e^{-\lambda t} y}$ , we obtain

$$E(g(Z_t)) = \int_0^\infty g(z) \frac{1}{2s} \left(\frac{e^{\lambda t} z^2}{z_0^2}\right)^{\frac{\nu}{2}} \exp\left(-\frac{z_0^2 + e^{\lambda t} z^2}{2s}\right) I_\nu\left(\frac{z_0 z e^{\frac{\lambda t}{2}}}{s}\right) 2z e^{\lambda t} dz.$$

We then find the density  $\rho_t(q)$  of the law de  $Z_t$  by replacing  $s$  and  $\nu$  by their values. □

**Proposition 2.6.** *The function  $\eta_*(t, q)$  defined by*

$$\begin{aligned}\eta_*(t, q) &= \frac{\rho_t(q)}{\eta(t, q)} \\ &= \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) q^{\frac{1}{2}} \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \left(\frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right)\right)\end{aligned}$$

satisfies the dual equation  $(\mathcal{C}_2^{(V)})$ .

*Proof.* We have

$$\begin{aligned}\frac{\partial \eta_*}{\partial t} &= \left( \left( -\frac{2\lambda^2}{\alpha^2 z_0^\nu} e^{\frac{\lambda t}{2}} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)^2} q^{\frac{1}{2}} + \frac{8\lambda^3}{\alpha^4 z_0^\nu} \frac{e^{\frac{3\lambda t}{2}}}{(e^{\lambda t} - 1)^3} q^{\frac{5}{2}} + \frac{8\lambda^3}{\alpha^4 z_0^\nu} \frac{e^{\frac{3\lambda t}{2}}}{(e^{\lambda t} - 1)^3} z_0^2 q^{\frac{1}{2}} \right) I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right. \\ &\quad \left. - \frac{8z_0 \lambda^3}{\alpha^4 z_0^\nu} e^{\lambda t} \frac{e^{\lambda t} + 1}{(e^{\lambda t} - 1)^3} q^{\frac{3}{2}} I_\nu'\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right) \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial \eta_*}{\partial q} &= \left( \left( \frac{1}{2} q^{-\frac{1}{2}} - \frac{2\lambda}{\alpha^2} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)} q^{\frac{3}{2}} \right) I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) + \frac{4\lambda z_0}{\alpha^2} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} q^{\frac{1}{2}} I_\nu'\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right) \\ &\quad \times \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \eta_*}{\partial q^2} &= \left( \left( -\frac{1}{4} q^{-\frac{3}{2}} - \frac{4\lambda}{\alpha^2} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)} q^{\frac{1}{2}} + \frac{4\lambda^2}{\alpha^4} \frac{(e^{\lambda t} + 1)^2}{(e^{\lambda t} - 1)^2} q^{\frac{5}{2}} \right) I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right. \\ &\quad + \left( \frac{4\lambda z_0}{\alpha^2} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} q^{-\frac{1}{2}} - \frac{16\lambda^2 z_0}{\alpha^4} e^{\frac{\lambda t}{2}} \frac{(e^{\lambda t} + 1)}{(e^{\lambda t} - 1)^2} q^{\frac{3}{2}} \right) I_\nu'\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \\ &\quad \left. + \frac{16\lambda^2 z_0^2}{\alpha^4} \frac{e^{\lambda t}}{(e^{\lambda t} - 1)^2} q^{\frac{1}{2}} I_\nu''\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \right) \frac{4\lambda}{\alpha^2 z_0^\nu} \frac{e^{\frac{\lambda t}{2}}}{(e^{\lambda t} - 1)} \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right),\end{aligned}$$

and

$$\begin{aligned}V(t, q) \eta_*(t, q) &= \left( \frac{A}{q^2} + Bq^2 \right) \eta_*(t, q) \\ &= \left( \frac{\alpha^4}{27} (\delta - 1)(\delta - 3) q^{-2} + \frac{\lambda^2}{23} q^2 \right) \eta_*(t, q) \\ &= \left( \lambda \frac{\alpha^2}{25} (\delta - 1)(\delta - 3) q^{-\frac{3}{2}} + \frac{\lambda^3}{2\alpha^2} q^{\frac{5}{2}} \right) \frac{e^{\frac{\lambda t}{2}}}{z_0^\nu (e^{\lambda t} - 1)} I_\nu\left(\frac{4\lambda z_0 e^{\frac{\lambda t}{2}} q}{\alpha^2 (e^{\lambda t} - 1)}\right) \\ &\quad \times \exp\left(\frac{\lambda q^2}{\alpha^2} - \frac{2\lambda}{\alpha^2} \left(\frac{z_0^2 + e^{\lambda t} q^2}{e^{\lambda t} - 1}\right)\right).\end{aligned}$$

Taking into account Equation (1.2) for the Bessel function  $I_\nu$ , it is then straightforward to see that for all  $(t, q) \in \mathbf{R}_+^* \times \mathbf{R}_+^*$ ,

$$-\theta^2 \frac{\partial \eta_*}{\partial t} = -\frac{\theta^4}{2} \frac{\partial^2 \eta_*}{\partial q^2} + V \eta_* \quad (\mathcal{C}_2^{(V)}).$$

□



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