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# **Klein-Gordon and Schrödinger equations for a free particle in the rest frame**

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## **Abstract**

A system of two equations is found that has solutions which coincide with the solutions of the Klein-Gordon equation in the rest frame. This system includes the Schrödinger equation for a free neutral spinless particle. Using the Schrödinger equation as an additional condition for solving the Klein-Gordon equation in the rest frame leads to two Helmholtz equations. Helmholtz equations can be solved by specifying a particle model and boundary conditions. One of the Helmholtz equations leads to discreteness of the rest masses of relativistic particles.

**Keywords** Relativistic particle; Helmholtz equation; Principle of correspondence; Discreteness of the rest mass spectrum

## **Introduction**

The main successes of quantum mechanics in the quantitative description of nonrelativistic systems are connected with the Schrödinger equation. Often this equation is regarded as the nonrelativistic limit of the Klein-Gordon equation [1-6]. When considering this limit, as a rule, in the solutions of the Klein-Gordon equation, the speed of light tends to infinity. In [7], the relationship of the Klein-Gordon and Schrödinger equations is analyzed in detail and it is concluded that the Klein-Gordon equation cannot be reduced to the Schrödinger equation in any limit. We can agree with the author of [7], if we keep in mind an arbitrary reference frame. But it must be noted that the Klein-Gordon equation must strictly go over into the Schrödinger equation only at a particle velocity of zero, that is, in the particle's rest frame.

The objective of this paper is to find a solution joint to the Klein-Gordon and Schrödinger equations in the rest frame for a neutral spinless particle.

## **The rest frame in relativistic quantum mechanics**

In classical relativistic mechanics, the particle's rest frame is a frame of reference where the particle velocity  $\mathbf{v}$  is equal to zero. In relativistic quantum mechanics, its own reference frame was used in [8-12]. In quantum mechanics, as in quantum field theory, there are problems with determining the eigenvalues of the velocity

operator [13-15]. In various formulations of quantum theory, it is believed that the particle velocity has the same direction as the wave vector  $\mathbf{k}$ . In any case, strictly speaking, the situation with the speed of a single particle does not satisfy the principle of correspondence between classical and quantum mechanics. There are formulations of quantum mechanics (primarily the De Broglie-Bohm approaches [16-19]), where the particle velocity is interpreted as the group velocity of the wave packet

$$\mathbf{v} = \frac{1}{\hbar} \cdot \frac{\partial E}{\partial \mathbf{k}} . \quad (1)$$

When performing the relativistic dispersion relation

$$E = c \hbar \left( \frac{m^2 c^2}{\hbar^2} + \mathbf{k}^2 \right)^{1/2} \quad (2)$$

particle velocity in the relativistic version of the De Broglie-Bohm approach is written as

$$\mathbf{v} = c \mathbf{k} \left( \frac{m^2 c^2}{\hbar^2} + \mathbf{k}^2 \right)^{-1/2} . \quad (3)$$

Here,  $\hbar$  is the Planck constant,  $c$  is the speed of light in vacuum,  $m$  is the rest mass of the particle,  $E = E(\mathbf{k})$  is the dispersion relation.

That is,  $\mathbf{v} \rightarrow 0$  in the reference frame, where  $\mathbf{k} \rightarrow 0$ . Therefore, when passing from an arbitrary reference frame to its own reference frame, all the characteristics of the particle, including the wave function, do not contain the wave vector  $\mathbf{k}$  in their record. If we find the wave function in the rest frame (not containing the vector  $\mathbf{k}$ ), then in other reference systems this function contains the vector  $\mathbf{k}$  in its notation, which allows this function to be transformed according to irreducible representations of the Lorentz group. Such a situation should be fulfilled in other approaches when fulfilling the principle of correspondence between classical and quantum mechanics.

### **Joint solution of the Klein-Gordon and Schrödinger equations in the rest frame**

Let's look at a system of two equations

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi \quad , \quad (4)$$

$$\frac{\partial^2 \Psi}{\partial t^2} + 2i\omega \frac{\partial \Psi}{\partial t} + \omega^2 = 0 \quad . \quad (5)$$

Here  $\omega = mc^2/\hbar$ ,  $\nabla^2$  is the Laplace operator.

Equation (4) is the Schrödinger equation for a free particle. By a free particle we mean a particle that does not interact with other particles, but interacts with a vacuum. Equation (5) is chosen so that when solving the system of equations (4, 5) by the substitution method, the Klein-Gordon equation

$$\frac{\hbar^2}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = \hbar^2 \nabla^2 \Psi - m^2 c^2 \Psi \quad (6)$$

is fulfilled. Indeed, from (4)

$$\frac{\partial \Psi}{\partial t} = i \frac{\hbar}{2m} \nabla^2 \Psi \quad . \quad (7)$$

Substituting (7) into (5), we obtain equation (6). Thus, the solutions of the system of equations (4.5) are also solutions of the equation (6) (but not vice versa).

The solution of the system of equations (4.5) we begin with the solution of equation (5). Characteristic equation

$$p^2 + 2i\omega p + \omega^2 = 0 \quad (8)$$

has two roots

$$\begin{aligned} p_1 &= -(1-\sqrt{2})i\omega \\ p_2 &= -(1+\sqrt{2})i\omega \quad . \end{aligned} \quad (9)$$

Therefore, the general solution of equation (5) can be written as

$$\Psi = C_1(\vec{r})e^{p_1 t} + C_2(\vec{r})e^{p_2 t} = C_1(\vec{r})e^{-(1-\sqrt{2})i\omega t} + C_2(\vec{r})e^{-(1+\sqrt{2})i\omega t} \quad . \quad (10)$$

The coefficients  $C_1(\mathbf{r})$  and  $C_2(\mathbf{r})$  are generally complex and depend on spatial coordinates.

Solution (10) can also be written as

$$\Psi = e^{-i\omega t} \left[ \left( C_1(\vec{r})e^{\sqrt{2}i\omega t} + C_2(\vec{r})e^{-\sqrt{2}i\omega t} \right) \right] \quad . \quad (11)$$

To determine  $C_1(\mathbf{r})$  and  $C_2(\mathbf{r})$  we substitute (10, 11) into the Schrödinger equation (4). Then we get

$$\begin{aligned} & C_1(\vec{r})(1 - \sqrt{2})\omega e^{-(1-\sqrt{2})i\omega t} + C_2(\vec{r})(1 + \sqrt{2})\omega e^{-(1+\sqrt{2})i\omega t} = \\ & = -\frac{\hbar}{2m}\nabla^2 C_1(\vec{r})e^{-(1-\sqrt{2})i\omega t} - \frac{\hbar}{2m}\nabla^2 C_2(\vec{r})e^{-(1+\sqrt{2})i\omega t}. \end{aligned} \quad (12)$$

This shows that (10, 11) are solutions of equation (4) under the conditions

$$\nabla^2 C_1(\vec{r}) = Q_1^2 \cdot C_1(\vec{r}) \quad (13)$$

$$\nabla^2 C_2(\vec{r}) = Q_2^2 \cdot C_2(\vec{r}) \quad . \quad (14)$$

Here

$$Q_1^2 = 2(\sqrt{2} - 1) \cdot \frac{m^2 c^2}{\hbar^2} \quad (15)$$

$$Q_2^2 = -2(\sqrt{2} + 1) \cdot \frac{m^2 c^2}{\hbar^2} \quad . \quad (16)$$

### **Discreteness of the rest masses of relativistic spinless particles as a consequence of the joint solution of the Klein-Gordon and Schrödinger equations in the rest frame**

Thus, the solutions (10, 11) are the same for the system of two equations, which includes the Schrödinger equation, and for the Klein-Gordon equation in the rest frame when (13,14) is satisfied. Conditions (13,14) are the Helmholtz equations for  $C_1(\mathbf{r})$  and  $C_2(\mathbf{r})$ . Moreover, equation (14) has discrete solutions. These solutions are determined by the symmetry of the function  $C_2(\mathbf{r})$  with an appropriate choice of the coordinate system (for example, spherical or cylindrical) and the shape of the boundary conditions [20-24]. In [23, 24], the simplest polar model of a particle in the rest frame was considered. The Helmholtz equation similar to (14) was obtained under the assumption that a particle that does not interact with other particles can be described by the simplest wave equation. In this case, a part depending on spatial coordinates was allocated in the solution of the equation. In the polar model, it was divided into radial and azimuthal components

$$\Psi_q = R(\rho) \cdot \Phi(\varphi). \quad (17)$$

After the transformation  $q\rho=x$  the equation for  $R(\rho)$  was reduced to the Bessel equation for the function

$$F(x)=R(x/q). \quad (18)$$

When boundary conditions

$$R(a) = 0, \quad (19)$$

$$|R(\rho)| \rightarrow \infty \text{ under } \rho \rightarrow 0 \quad (20)$$

are met the solutions of the Bessel equation for a  $F_{nl}(x)$  function are linear combinations of the Bessel and Neumann functions

$$F_{nl}(x) = B_{nl}^{(N)} N_n(x) + B_{nl}^{(B)} J_n(x). \quad (21)$$

Here,  $J_n(x)$ ,  $N_n(x)$  are n-order Bessel and Neumann functions,  $B_{nl}^{(N)}$ ,  $B_{nl}^{(B)}$  are expansion coefficients. In this case, the allowed discrete values of the rest masses were given by the expression

$$m_{nl} = \pm \frac{1}{a} \frac{\hbar}{c} X_{nl}. \quad (22)$$

Here,  $X_{nl}$  are the values of the roots of the  $F_{nl}(x)$  functions in (21).

In the present paper, the Helmholtz equation (14) is a consequence of the joint solution of the system of two equations and the Klein-Gordon equation and differs from the Helmholtz equation in [23, 24] by the factor  $2(1+\sqrt{2})$ . In this regard, the discrete values of the rest masses allowed by equation (14) in the polar model have the form

$$m_{ne} = \pm \frac{1}{\sqrt{2(1+\sqrt{2})}} \cdot \frac{1}{a} \cdot \frac{\hbar}{c} X_{ne}. \quad (23)$$

The value of  $a$  in (19, 22, 23), generally speaking, is arbitrary. In [23, 24], it was chosen so that formula (22) gives the value of the electron mass at  $X_{nl}=X_{01}^{(N)}$ . If

now in formula (23) the quantity  $a\sqrt{2(1+\sqrt{2})}$  is chosen so that (23) gives the value of the rest mass of an electron at  $X_{nl}=X_{01}^{(N)}$ , then all the conclusions and results of [23, 24] are valid and for formula (23) in the polar model. In particular,

the peculiarities of the values of the rest masses and lifetimes of the electron, proton, and neutron [23], as well as the results for the current-quark masses of the first generation quarks [24] are valid. At the same time, it is of interest to study the features of the mass spectrum of elementary particles with the direct use of formula (23), especially for other particle models, including the spherical model.

### **Discussions and conclusion**

The system of equations (4, 5) determines the structure of the solution (10, 11) of the Klein-Gordon equation (6), as well as additional conditions (13-16) for the spatial components  $C_1(\mathbf{r})$  and  $C_2(\mathbf{r})$ . Concretization of the solution of equation (6) when (13-16) are fulfilled should consist in the choice of a particle model (only the polar model was used in [23, 24]), as well as in the specification of boundary conditions of type (19, 20) when solving the Helmholtz equations (13, 14). While equation (14) can be considered partially investigated (it leads to discreteness of rest masses), equation (13) provides additional opportunities for studying the nature of microparticles. Thus, equations (4,5) can be considered as additional conditions for solving the Klein-Gordon equation in the rest frame and allow extracting additional information about microparticles, in particular, explaining the discreteness of the rest mass spectrum in the framework of relativistic quantum mechanics.

Note that equation (5) along with solutions of type (10, 11) has “oscillating” solutions that differ from solutions of type (10.11). These solutions are not compatible with the Schrödinger equation (4). They can be of independent interest for the study of microparticles in the rest frame. It is also of interest to use the solutions (10, 11) when creating operators in the framework of quantum field theory.

## References

1. Schoene A.Y. Journal of Mathematical Analysis and Applications **71**, 36-47 (1979).
2. Cirincione R.J., Chernoff P.R. Commun. Math. Phys. **79**, 33-46 (1981).
3. Machihara S., Nakanishi K., Ozawa T. Mathematische Annalen **322**(3),603-621 (2002).
4. Bao W., Dong X. Numer Math. **120**, 189-229 (2012).
5. Faouand E. Schratz K. Numer. Math. **126**(3), 441-469 (2014).
6. Pasquali S. Annali di Matematica Pura ed Applicata (2019).  
<https://doi.org/110.1007/s10231-018-0805-1>.
7. Wharton K.B. Found. Phys. **40**, 313–332 (2010).
8. Fanchi J.R., Collins R.E. Found. Phys. **8**, 851-876 (1979).
9. Edmonds J.D. Quantum Uncertainties, NATO ASI Series **B162**, 421-425 (1988).
10. Alba D., Lusanna L. Int. J. Mod. Phys. **A21**, 2781-2819 (2006).
11. Crater H.W., Lusanna L. Ann.Phys. **87**, 289-294 (2001).
12. Lusanna L. J. Phys.: Conf. Ser. **437**, 012015 (2013).
13. Bonge M. Int. J. Theor. Phys. **42**, 135-142 (2003).
14. Evans A.V. Annales de la Fondation Louis de Broglie **32**, 479-486 (2007).
15. Salesi G., Recami E. Found. Phys. **28**, 763-773 (2010).
16. De Broglie L., Bohm D., Hillion P., Halbwachs F., Nakabayasi T. Phys. Rev. **B129**,438-450 (1963).
17. Brown H.R., Harvey R., Martins R. Amer. J. Phys. **52**, 1130-1140 (1984)
18. Marquet P. Progress in Physics **42**, 318-322 (2016).
19. Bartley D.L. Found . Phys. **48**, 1731-1752 (2018).
20. Abramowitz M., Stegun I.A. Handbook of Mathematical Physics. Dover, New York (1965).
21. Korenev B.G. Bessel Functions and Their Applications. Taylor & Francis (2002).
22. Bayn S.S. Essentials of Mathematical Methods in Science and Engineering. Wiley (2008).
23. Salomatov V.N. Physics Essays **30**, 177-180 (2017).
24. Salomatov V.N. Physics Essays **31**, 164-166 (2018).



