



HAL
open science

Lot sizing problem with multi-mode replenishment and batch delivery

Ayse Akbalik, Christophe Rapine

► **To cite this version:**

Ayse Akbalik, Christophe Rapine. Lot sizing problem with multi-mode replenishment and batch delivery. *Omega*, 2018, 81, pp.123-133. 10.1016/j.omega.2017.10.005 . hal-02336558

HAL Id: hal-02336558

<https://hal.science/hal-02336558>

Submitted on 28 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Lot sizing problem with multi-mode replenishment and batch delivery

Ayse Akbalik, Christophe Rapine

Université de Lorraine, Ile du Saulcy, Metz, F-57012, France

Abstract

We study the single-item uncapacitated lot sizing problem with multi-mode replenishment and batch deliveries (ULS-MMB). Specifically, we consider that each replenishment mode has a Full Truck Load (FTL) cost structure and incurs a fixed ordering cost plus a fixed cost per batch. This problem arises in practice when a retailer places the order with different suppliers in each period. We show that this problem is NP-hard even for a single period and under very restricted cost parameters. We then show that ULS-MMB can be transformed into a lot sizing problem with only one replenishment mode per period, that is, ULS-MMB is a special case of lot-sizing with time-varying batch sizes. This simple observation allows us to improve some results already known in the literature of multi-mode replenishment. We propose a very efficient 2-approximation algorithm and establish that the problem admits an FPTAS. Finally, we show that the problem restricted to two modes with divisible batch sizes can be solved in polynomial time.

Keywords: Lot sizing, multi-mode, batch delivery, approximation algorithm, polynomial time algorithm.

1. Introduction

Determination of the best combination of suppliers and transportation modes, allocation of orders among them represent a very crucial issue for retailers in their replenishment process. The multi-mode replenishment from several suppliers has become a very common practice in supply chains in order to benefit from the cost effectiveness and the flexibility (see Aissaoui *et al.* [2]). A systematic approach is then needed for such complex decisions. In this paper we focus on a very classical production planning model, used as well as for the replenishment systems : lot sizing problem.

We consider the single-item uncapacitated lot-sizing problem where a retailer can place the orders with different suppliers, that is, considering the possibility of multiple sourcing for its replenishments. Each supplier incurs a specific procurement cost, and we assume that any quantity can be ordered from any supplier in each period. In the following, we indifferently call a replenishment mode *a supplier* or simply *a mode*. This problem can also be seen as a one vendor-one buyer problem with different transportation modes available to ship the units between them. One particularity of this paper is to consider that the quantities ordered from a supplier are replenished by batches, representing typically the capacity of the vehicles (*i.e.* trucks, containers, ...) used to ship the units, where a fixed cost is to be paid for each vehicle used.

More and more theoretical studies and practical cases arise on multi-mode transportation domain, showing the significant benefits that can be obtained by combining different modes of replenishment. According to Jaruphongsa *et al.* [13], having the possibility to use multiple suppliers and/or shipment modes in inventory replenishment may be more cost effective than selecting a particular supplier or shipment mode. In their paper Jaruphongsa *et al.* [13] provide an example of Third-Party Logistics (TPL) provider offering several alternative delivery modes with different costs and lead times. As mentioned in Ekşioğlu [12], “*the models and methods proposed for this problem can also be used as a submodule in MRP systems for requirement planning with multi-mode replenishment option*”. In Aissaoui *et*

Email addresses: ayse.akbalik@univ-lorraine.fr (Ayse Akbalik), christophe.rapine@univ-lorraine.fr (Christophe Rapine)

al. [2], the authors mention that multiple sourcing is a very interesting option for a retailer, particularly when one of its suppliers can not satisfy the whole demand, for different reasons such as price discount offers or possible limitations on quality, delivery, price, etc.

We consider in the following that the retailer has to satisfy his demands over a time horizon of T periods, with the possibility to use up to M different modes in each period. Shortages are not allowed, that is, demand d_t in period t has to be satisfied from the available units in stock or/and by ordering units in period t from one or more suppliers. Carrying inventory at retailer level from period t to $t + 1$ incurs a holding cost of h_t per unit. Ordering some units from supplier i in period t incurs a fixed setup cost f_{it} and a fixed cost per batch k_{it} , in addition to a unit procurement cost p_{it} . We also denote by B_i the size of the batches used by supplier i to ship the ordered quantities. Notice that we restrict ourselves to the case of a stationary batch size for each supplier, though this size may vary from one supplier to another. The ordering cost assumed for each supplier is known in the literature as a Full Truck Load (FTL) cost structure. It is also called stepwise cost or multiple setup cost. The total procurement cost $r_{it}(x)$ for an amount x ordered in period t using mode i is given by :

$$r_{it}(0) = 0 \text{ and } r_{it}(x) = f_{it} + p_{it}x + \lceil x/B_i \rceil k_{it} \text{ for } x > 0$$

We can formulate problem ULS-MMB as follows :

$$(\text{ULS-MMB}) \begin{cases} \min & \sum_{t=1}^T (\sum_{i=1}^M r_{it}(x_{it}) + h_t s_t) \\ \text{s.t.} & s_{t-1} + \sum_{i=1}^M x_{it} = d_t + s_t \quad \forall t = 1, \dots, T \\ & x_{it} \in \mathbb{R}_+ \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T \\ & s_t \in \mathbb{R}_+ \quad \forall t = 1, \dots, T \end{cases}$$

In the formulation, x_{it} represents the quantity ordered from supplier i in period t and s_t the stock level at retailer at the end of period t . Wlog, we assume no initial inventory, that is $s_0 = 0$ and null lead time. This formulation is non-linear due to the FTL procurement cost $r_{it}()$. Note that the classical fixed charge cost structure is a special case of $r_{it}(x)$ with the assumption of $k_{it} = 0$.

Contributions of the article. This paper has several theoretical contributions to the field. First, it establishes the complexity of problem ULS-MMB, not yet stated in the literature. ULS-MMB is shown to be NP-hard under very restricted conditions. It also shows that the existing results for ULS-MMB for stationary procurement costs can be improved, even under time-dependent costs, using a very simple transformation to a single-mode problem. A worst case analysis for the primal-dual heuristic proposed by Ekşioğlu [12] is given to show that this latter can behave arbitrarily bad. After having established the NP-hardness of the problem, and the lack of approximation algorithm in the literature, some positive results are proposed. The article presents a very efficient algorithm with a worst case guarantee of 2, and problem ULS-MMB is shown to admit an FPTAS using a known result from Chubanov *et al.* [11]. Finally, a polynomial time algorithm is proposed, based on a new decomposition of the time horizon, for the special case where 2 modes are available and their batch sizes are divisible. This one also represents a significant improvement of the existing pseudo-polynomial time algorithm proposed by Jaruphongsa *et al.* [13]. Notice that our results can be transposed to the single-item single-mode lot sizing problem with time-dependent batches, where the batch sizes are restricted to only two different values, one being a multiple of the other. Table 1, in Section 2, gives an overview of the results of this paper, compared with those proposed in Jaruphongsa *et al.* [13], Ekşioğlu [12] and Bai and Xu [6], the most relevant papers in the literature.

Organization of the paper. In Section 2 relevant papers studying lot sizing problems with batch delivery and multi-mode replenishment are discussed. Section 3 is devoted to establish the NP-hardness of ULS-MMB. We give a transformation of the multi-mode problem to the single mode problem in Section 4. This latter allows us to use stronger dominance properties. In Section 5, we give a worst case analysis for the primal-dual heuristic proposed by Ekşioğlu [12]. In the same section we propose a 2-approximation algorithm and we give the insights of an FPTAS proposed by Chubanov *et al.* [11] which can be applied to our ULS-MMB. In Section 6 we give a polynomial time algorithm for the special case of ULS-MMB with only two modes and divisible batch sizes. We then discuss open problems and some possible extensions of this work in Section 7.

Papers	Parameters				Other hypotheses	Complexity results
This study	k_i	-	B_i	M	null production cost, $p = 0$	NP-hard
This study	k_i, p_i	-	B_i	M	unit fixed cost per batch, $k_i = 1$	NP-hard
This study	f_{it}, k_{it}, p_{it}	h_t	B_{it}	M		FPTAS
Jaruphongsa <i>et al.</i> [13]	f_i, p_i	h_t	B_i	$M = 2$	fixed charge cost	$O(T^2)$
This study	f_{it}, p_{it}	h_t	B_{it}	$M = 2$	fixed charge cost	$O(T \log(T))$
Jaruphongsa <i>et al.</i> [13]	f_i, k_i, p_i	h_t	B_i	$M = 2$	one supplier: fixed charge cost one supplier: multiple setup cost	$O(T^3)$
Jaruphongsa <i>et al.</i> [13]	f_i, k_i, p_i	h_t	B_i	$M = 2$	divisible batches ($B_1 = \mu B_2$)	pseudo-pol., $O(\mu^2 T^4)$
This study	f_{it}, k_{it}, p_{it}	h_t	B_{it}	$M = 2$	divisible batches ($B_1 = \mu B_2$)	polynomial, $O(T^7)$
Ekşioğlu [12]	f_i, p_i	h_t	B_i	M	fixed charge cost	$O(MT^2)$
This study	f_{it}, p_{it}	h_t	B_{it}	M	fixed charge cost	$O(MT \log(MT))$
Bai and Xu [6]	f_{it}, k_{it}, p_{it}	h_t	B_i	M	incremental quantity discount with only one breakpoint	$O(MT^2)$
Bai and Xu [6]	f_{it}, k_{it}, p_{it}	h_t	B_i	M	all unit quantity discount with only one breakpoint	$O(MT^3 + T^2)$
Bai and Xu [6]	f_{it}, k_{it}, p_{it}	h_t	B_i	M	incremental quantity discount one mode has multiple setup costs	$O(T^4 + MT)$

Table 1: Comparative table for existing and new results for ULS-MMB. We index each parameter by i , t , or it to indicate that, it is mode, time, mode&time dependent, respectively.

2. Literature review

In the literature there are several studies on the supplier selection problem coupled with lot sizing decisions. Basnet and Leung [7] study a multi-product and multi-supplier lot sizing problem (LSP) assuming a classical fixed charge cost structure. They propose an enumerative algorithm and a heuristic. Recently, Cárdenas-Barrón *et al.* [9] improve this work proposing a more efficient heuristic based on reduce-and-optimize approach. Mansini *et al.* [17] consider a multi-product, multi-supplier LSP, assuming quantity discount schemes with truckload shipments. They propose mixed integer programming formulation and develop linear programming based heuristic. Zhao and Klabjan [23] propose valid inequalities for the single-item LSP with supplier selection, both for uncapacitated and capacitated cases, where they assume a fixed charge cost structure for the replenishment and production. Choudhary and Shankar [10] study the single-item LSP with supplier and carrier selection, assume all-unit quantity discount cost structure for the replenishment, and propose a mixed integer linear program. The same authors also provide a summary of quantitative studies in the literature proposed for this model. Lee *et al.* [14] provide an up-to-date literature review on the lot sizing and supplier selection under quantity discounts. They give a mixed integer program for this single-item LSP, assuming all-unit quantity discounts together with fixed costs per vehicle for the transportation part, and they use genetic algorithm to solve it. A detailed and comprehensive state-of-the-art can be found in Aissaoui *et al.* [2] on the supplier selection and lot sizing modeling.

Note that the previous studies from the literature consider either a classical fixed charge cost structure or various quantity discount schemes for the single or multi-item lot sizing with supplier selection. Some of them also integrate fixed transportation costs within quantity discount cost structure, see Lee *et al.* [14] and Mansini *et al.* [17]. In all those studies mixed integer programming formulations are proposed, followed by either some heuristics or valid inequalities to solve the related problems. In our study, we particularly focus on complexity analysis and polynomial time (exact or approximate) algorithms for the multi-supplier (or multi-mode) single-item lot sizing problem with batch delivery.

Concerning the lot sizing problem with multi-suppliers under multiple setup cost structure, as we assumed in this paper, we have only found the three following studies: Jaruphongsa *et al.* [13], Ekşioğlu [12] and Bai and Xu [6]. The theoretical results from those three papers are detailed below and a comparative study is given in Table 1 to better show our contributions in the literature. Jaruphongsa *et al.* [13] consider the special case of ULS-MMB with only two modes and a stationary procurement cost, that is $M = 2$ and $r_{it}() = r_i()$ for all t and i . Under those hypotheses, the authors

establish some dominance properties, and propose polynomial time algorithms. If the two modes have an FTL cost structure, the authors propose a pseudo-polynomial time algorithm assuming divisible batch sizes. The complexity of this latter depends on the value of the (integer) ratio of the batch sizes. We show in Section 6 that this problem can be solved in polynomial time, even with time-varying cost parameters. Ekşioğlu [12] also considers ULS-MMB with stationary costs, but with a number $M \geq 2$ of modes. If all the modes have a fixed charge cost structure, the author proposes a polynomial time algorithm in $O(MT^2)$ using some dominance properties proposed by Jaruphongsa *et al.* [13]. The same author proposes a primal-dual heuristic for the case with M modes and batch deliveries, that is, for FTL cost structures. In Section 5 we show that this algorithm can behave arbitrarily bad. Bai and Xu [6] consider some special cases of ULS-MMB with incremental and all-unit quantity discount cost structures and multiple setup costs. Note that the authors consider only one breakpoint in their incremental and all-unit quantity discount cost functions. Polynomial time algorithms are given for different extensions of ULS-MMB with M suppliers. See Table 1 for the related complexity results. Notice that in none of these studies the complexity status of the problem ULS-MMB is established. We show in Section 3 that the problem is NP-hard even for stationary cost parameters.

In Section 4 we show that ULS-MMB is equivalent to the classical single-item (single-mode) uncapacitated lot sizing problem with batch delivery. For a recent study on the ULS with batch production, readers are referred to Akbalik and Rapine [5]. Note that in Akbalik and Rapine [5] the batch sizes are assumed to be time-dependent. The authors propose a complexity classification and polynomial time algorithms for different cases. For the capacitated LSP with batch delivery refer to Akbalik and Rapine [4] and to van Vyve [20] for polynomial time algorithms and to Akbalik and Pochet [3] for valid inequalities and polyhedral studies. In Akbalik and Rapine [4] and Akbalik and Rapine [5], the authors give a complete state-of-the-art on the LSP with stepwise cost structure.

3. Complexity result

In this section, we establish that ULS-MMB remains NP-hard even for a single period problem and under very restricted cost functions. More precisely, we state the following theorem :

Theorem 1. *Problem ULS-MMB is NP-hard even restricted to a single period, with null setup and null holding costs and either :*

- (a) *null unit procurement cost, that is, the procurement cost from the supplier i when ordering a quantity x is equal to $r_i(x) = \lceil x/B_i \rceil k_i$, or*
- (b) *unitary fixed cost per batch, that is, the procurement cost from the supplier i when ordering a quantity x is equal to $r_i(x) = \lceil x/B_i \rceil + p_i x$.*

Proof. The polynomial time reductions are made from two special cases of the uncapacitated lot-sizing problem with batch deliveries and time-varying batch sizes (ULS- B_t). In this lot-sizing problem, a single mode is available, and the procurement cost follows an FTL cost structure. However, the size B_t of a batch may vary from one period to another. We consider the particular class of instances of ULS- B_t , with null holding cost and a positive demand occurring only at the last period T of the planning. We transform such an instance I of ULS- B_t into an instance $\tau(I)$ of ULS-MMB with only one period but T suppliers, supplier i being associated with period i in instance I . Specifically, the cost of ordering x units using supplier i in instance $\tau(I)$, is equal to the cost of ordering x units in period i in instance I . Clearly, any solution for instance I can be transformed into a solution for instance I' , without modifying its cost, and vice-versa. Hence, transformation τ is a polynomial reduction.

To complete the proof, we need that the particular cases of ULS- B_t , with null holding cost and a positive demand only at the end of the time horizon, to be NP-hard under the two restricted cost structures (a) and (b) given in the theorem. We use the following NP-hardness results from the literature of the lot-sizing problem :

- (a) Theorem 2 in Akbalik and Rapine [5] establishes, using a reduction from the Unbounded Knapsack Problem (Lueker [16]), that ULS- B_t is NP-hard even if the only positive demand occurs at the last period, and the setup, holding and unit procurement costs are null for all the periods.
- (b) Theorem 3 in Akbalik and Rapine [5] establishes, using a reduction from the Money Change Problem (Böcker and Lipták [8]), that ULS- B_t is NP-hard, even if the only positive demand occurs at the last period, the setup and holding costs are null, and the fixed costs per batch are all equal to 1.

This completes the proof of Theorem 1. □

Notice that Theorem 1 implies in particular that ULS-MMB is NP-hard for stationary procurement costs, like the problem considered in Ekşioğlu [12]. This complexity result holds if the number of modes M is part of the instance. When M is fixed, the complexity of the problem is open. However, in Section 6, we show that the problem restricted to 2 modes with divisible batch sizes is polynomially solvable.

4. Transformation into a single-mode problem

In this section, we observe that ULS-MMB can be viewed as a special case of the lot-sizing problem with only one replenishment mode, namely the uncapacitated lot-sizing problem with batch deliveries and time-varying batch sizes (ULS-B_t). This simple observation allows us to improve some results from the literature on multi-mode lot-sizing, simply by applying the well-known results from the single-mode lot-sizing field.

Specifically, an instance I of ULS-MMB can be transformed in linear time into an instance I' of ULS-B_t over MT periods as follows: To each period t of I corresponds M periods of I' , indexed as $\{M(t-1)+1, \dots, Mt\}$. The demand is null for all these M periods except for period Mt where demand d_t occurs. The holding cost between those periods is null, and the cost to carry units from period Mt to $(M+1)t$ is h_t . Finally, the procurement cost for a period $t' = M(t-1)+i$, $i = 1, \dots, M$, is equal to r_{it} . Notice that the supplier-dependent procurement costs (f_{it}, k_{it}, p_{it}) now appear independently over the periods of I' . This new problem is a “classical” lot sizing problem with only one supplier (one mode) over a time horizon of MT periods. See Figure 1 for a schematic representation of the transformation.

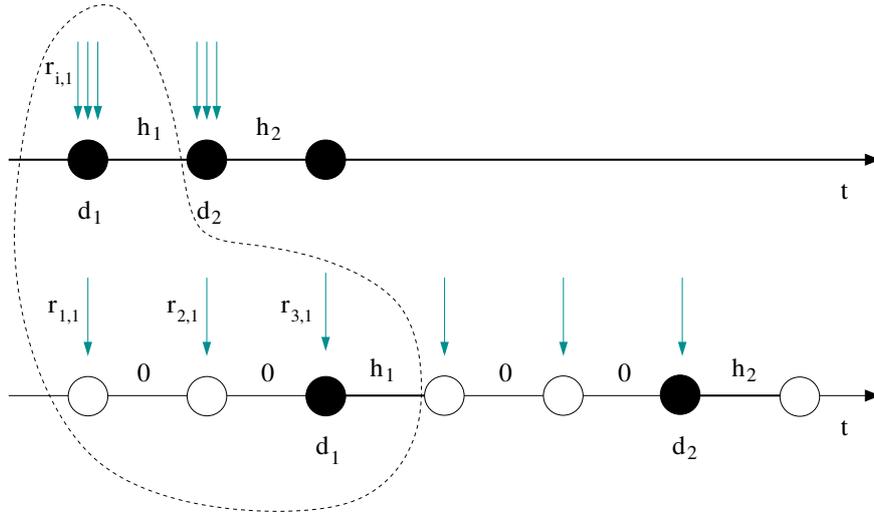


Figure 1: Transformation of ULS-MMB into ULS-B_t

There are several natural properties that one could derive from this transformation. For the special case studied in Jaruphongsas *et al.* [13] where the procurement cost does not contain a fixed cost per batch ($k_{it} = 0$), the classical results for the uncapacitated lot sizing problem with a fixed charge cost structure can be used. In particular the zero-inventory-ordering (ZIO) property is dominant and only one mode is used in each ordering period. Wagelmans *et al.* [22] is one of the three studies proposing an algorithm in $O(T \log T)$ time for linear variable costs. We have the following immediate result:

Theorem 2. *Problem ULS-MMB can be solved in time $O(MT \log(MT))$ for fixed charge procurement costs $r_{it}(x) = f_{it} + p_{it}x$, that is, when fixed costs per batch are null.*

As already reported in the literature review, Ekşioğlu [12] proposes a specific algorithm of time complexity $O(MT^2)$ for this case. Also notice that her algorithm assumes stationary procurement costs, while our simple reduction can handle time-varying cost parameters.

5. Approximation algorithms

We have shown in Section 3 that ULS-MMB is NP-hard. Thus, there is no hope to find exact polynomial time algorithms, unless $P = NP$. The design of approximation algorithms is one common approach to deal with NP-hard problems. Recall that a polynomial time algorithm is said to have a performance guarantee of λ (or is a λ -approximation) if for any instance, it delivers a solution of cost not greater than λ times the optimum value of the instance. In this section, we first show that the primal-dual heuristic proposed by Ekşioğlu [12] for problem ULS-MMB with stationary procurement costs at each retailer can perform arbitrarily bad compared to the optimal solution. Then, we investigate if approximation algorithms exist for problem ULS-MMB, and more generally for problem ULS- B_t .

5.1. The primal-dual heuristic of Ekşioğlu [12]

The approach proposed by Ekşioğlu [12] is based on the extended formulation (Q) given below. In this formulation, variables $x_{it\tau}$ represent the amount of products ordered using mode i at period t to (partially) fulfill the demand of period τ . Binary variables y_{it} indicate if mode i is used in period t , while integer variables z_{it} correspond to the number of batches delivered using mode i in period t . Finally, parameters $c_{it\tau} \equiv p_{it} + h_t + \dots + h_{\tau-1}$ represent the variable cost for ordering one unit of product in period t using mode i and carrying it in stock till period τ .

$$(Q) \left\{ \begin{array}{l} \min \quad \sum_{t=1}^T \sum_{i=1}^M \left[\sum_{\tau=t}^T c_{it\tau} x_{it\tau} + f_{it} y_{it} + k_{it} z_{it} \right] \\ \text{s.t.} \quad \sum_{i=1}^M \sum_{t=1}^{\tau} x_{it\tau} = d_{\tau} \quad \forall \tau = 1, \dots, T \quad (C1) \\ x_{it\tau} \leq d_{\tau} y_{it} \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T \quad \forall \tau = t, \dots, T \quad (C2) \\ \sum_{\tau=t}^T x_{it\tau} \leq B_i z_{it} \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T \quad (C3) \\ x_{it\tau} \in \mathbb{R}_+ \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T, \forall \tau = t, \dots, T \\ y_{it} \in \{0, 1\} \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T \\ z_{it} \in \mathbb{Z}_+ \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T \end{array} \right.$$

Associating dual variables v_{τ} to constraint (C1), the dual of the linear relaxation of (Q) can be written in a compact way, see Ekşioğlu [12], as follows :

$$(D) \left\{ \begin{array}{l} \max \quad \sum_{\tau=1}^T d_{\tau} v_{\tau} \\ \text{s.t.} \quad \sum_{\tau=t}^T d_{\tau} \max\{0, v_{\tau} - (c_{it\tau} + \frac{k_{it}}{B_i})\} \leq f_{it} \quad \forall i = 1, \dots, M, \forall t = 1, \dots, T \quad (D1) \\ v_{\tau} \in \mathbb{R}_+ \quad \forall \tau = 1, \dots, T, \end{array} \right.$$

The heuristic sequentially increases the value of the v_{τ} variables, starting from $\tau = 1$, until constraint (D1) becomes saturated, to find a dual solution. Then, a primal solution is constructed by selecting mode i in period t if the corresponding constraint (D1) is saturated in the dual solution, such that the complementary slackness condition holds for this constraint¹.

Primal-dual method is a powerful technique to develop polynomial time algorithms with proven performance guarantee. However, the solution produced by the heuristic of Ekşioğlu [12] can perform arbitrarily bad. In addition, the gap of the linear relaxation of the extended formulation (Q) can not be bounded. To see this, consider the very simple following instance : we have only one period, with a demand of one unit to fulfill. There are 2 modes available : the first mode has a batch size of 1 and a fixed cost per batch of λ , where $\lambda \geq 1$ is a given parameter. The second mode has a batch size of λ^2 and also a fixed cost per batch of λ^2 . Both unit ordering costs and setup costs are null for

¹The algorithm is a bit more involved, using a backward induction. But, for the case of a single period, as developed in our bad example, this description is sufficient.

the 2 modes. Clearly, the optimal policy consists in ordering the demand using one batch of mode 1, resulting in a cost $\text{OPT} = \lambda$. On our instance, the dual (D) boils down to :

$$\begin{aligned} \max \quad & v_1 \\ \text{s.t.} \quad & v_1 \leq \lambda \quad (\text{mode 1}) \\ & v_1 \leq 1 \quad (\text{mode 2}) \\ & v_1 \in \mathbb{R}_+ \end{aligned}$$

It can be easily checked that the optimal value of the linear relaxation of (Q) is equal to $Q^* = 1$, since solution $v_1^* = 1$ is dual feasible, and solution $x_{2,1,1} = 1, y_{2,1} = 1$ and $z_{2,1} = 1/\lambda^2$ is primal feasible. In the solution produced by the primal-dual heuristic, the unit of demand is ordered using mode 2, since the corresponding constraint is saturated in the dual. Its value is $H = \lambda^2$. This simple example shows that the value Q^* of the linear relaxation can be arbitrarily small compared to the optimal integer value OPT of (Q), and that the value of the solution produced by the heuristic can in its turn be arbitrarily large compared to OPT . We investigate in the following if approximation algorithms exist for problem ULS-MMB.

5.2. A simple approximation algorithm

To propose an efficient approximation, our basic idea is to substitute the procurement cost function $r_{it}(x)$ for mode i in period t , which is neither continuous nor concave, by a much simpler cost function $a_{it}(x)$, namely a fixed charge cost structure of the form $f'_{it} + p'_{it}(x)$. The resulting multi-mode replenishment problem can be efficiently solved to optimality in time complexity $O(MT \log(MT))$, see Section 4, Theorem 2. To obtain a lower bound, that is, for the resulting lot-sizing problem being a relaxation, we need that $a_{it}(x) \leq r_{it}(x)$ for any amount x . The first candidate, one may think of, is certainly function $\bar{r}(x) = f_{it} + (p_{it} + k_{it}/B_i)x$, where the fixed cost per batch is evenly split over all the batch size B_i . However, the optimal value of the relaxation may be arbitrarily small compared to the optimum, as shown by the bad example of the previous section.

Instead, we replace each FTL procurement cost $r_{it}(x)$ by the affine procurement cost $a_{it}(x)$ with a setup cost $f'_{it} = (f_{it} + k_{it})/2$ and a unit procurement cost $p'_{it} = (k_{it}/B_i + p_{it})/2$. For any quantity x , the affine cost a_{it} is not only a lower bound of the procurement cost $r_{it}(x)$, but we have the following (sandwich) inequalities (illustrated in Figure 2):

$$a_{it}(x) \leq r_{it}(x) \leq 2a_{it}(x) \quad (1)$$

To see this, notice that on one hand, for any quantity x , we have $k_i \lceil x/B_i \rceil \leq k_i(x/B_i + 1)$, which clearly implies that $r_{it}(x) \leq 2a_{it}(x)$. On the other hand, for any $x > 0$, we clearly have $k_i \leq k_i \lceil x/B_i \rceil$ and $k_i x/B_i \leq k_i \lceil x/B_i \rceil$. It implies that $a_{it}(x) \leq r_{it}(x)$ also holds.

The algorithm solves to optimality the relaxation where procurement costs r_{it} 's are replaced by affine costs a_{it} 's. Due to inequalities (1), the solution obtained has a performance guarantee of 2. We have the following result :

Theorem 3. *Problem ULS-MMB can be approximated with a performance guarantee of 2 in time complexity $O(MT \log(MT))$.*

This result holds for time-varying cost parameters. Notice that problem ULS-B $_i$ can be approximated with a guarantee of 2 by the same algorithm, in time $O(T \log T)$.

5.3. Problem ULS-MMB belongs to FPTAS

In this section we investigate if far better approximation performances can be obtained. Recall that an FPTAS is a family of approximation algorithms such that the problem can be approximated within a performance guarantee of ε for any given $\varepsilon > 0$, in polynomial time of the instance size and of $1/\varepsilon$. According to Chubanov *et al.* [11] the first FPTAS for the capacitated lot sizing problem is developed by van Hoesel and Wagelmans [21], where the authors consider concave (convex) production and backloging cost functions. To show that problem ULS-MMB admits an FPTAS, we use the general result of Chubanov *et al.* [11]. In their paper, Chubanov *et al.* [11] propose an FPTAS for a very broad class of lot-sizing problems, simply assuming monotone cost structures for procurement and holding costs. One additional assumption is that each cost function is computable in polynomial time for any quantity. Recently, Ng *et al.* [18] propose a better FPTAS for this case, faster than the one proposed in Chubanov *et al.* [11].

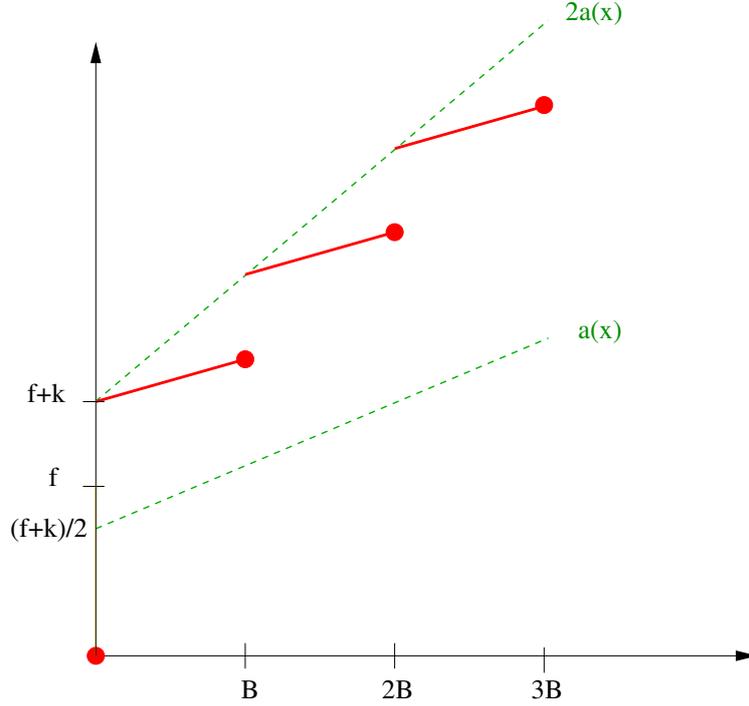


Figure 2: Illustration of the sandwich inequalities.

Observe that for ULS-MMB, computing $r_t^*(X)$, the minimal cost to order a quantity X in period t from the set of suppliers, is an NP-hard problem. However, applying the linear time transformation detailed in Section 4, we obtain an instance of problem ULS- B_t where each procurement cost can clearly be evaluated in constant time. As a consequence, problem ULS-MMB admits an FPTAS.

Theorem 4. *Problem ULS-MMB (and ULS- B_t) admits an FPTAS*

We show, in the next section that, the problem ULS-MMB restricted to 2 modes whose batch sizes are multiple one of the other can be solved in polynomial time.

6. Divisible batch sizes

We consider in this section the special case where only 2 modes are available. For short, we denote by A and B the size of the batches for the first and the second mode, respectively. We say that a batch is an A -batch, respectively a B -batch, if its capacity is of A units, respectively B units. For the sake of clarity, we also index the parameters associated to each mode by A and B , instead of indices 1 and 2. In addition, we consider that the batch sizes are divisible, that is $B = \mu A$ with μ an integer. This special case may correspond to the practical situation where two types of trucks, one small and one large, can be used for shipments; or to the case where one shipment is made by truck (one container size) and the other shipment is made for instance by train or by barge (representing a large number of containers). For a special case of this problem with stationary procurement costs, Jaruphongsa *et al.* [13] propose a pseudo-polynomial time algorithm, whose time complexity depends on input parameter μ . We show in this section that this problem is polynomially solvable. We do not make any particular assumptions on the procurement cost of each mode, which is considered time-varying and which can also have speculative motives. Since we consider linear holding costs, notice that we can assume wlog that the holding cost is null, by defining for each mode i a new variable ordering cost $\tilde{p}_{it}(x) = p_{it}(x) + (h_i + h_{i+1} + \dots + h_T)x$, see for instance Wagelmans *et al.* [22].

Classically, the algorithm computes the optimal cost $\mathcal{C}(u, v)$ of each possible subplan (u, v) . Recall that (u, v) is a subplan if both periods u and v are regeneration points (that is, with a null entering stock level), and each period $t \in \{u + 1, \dots, v - 1\}$ has a positive entering stock level. However, the algorithm is quite involved, since the computation of the cost of a subplan requires to decompose it, in its turn, into what we call σ -intervals. Before defining this notion, let us give some insights of the algorithm. It relies on two basic ideas. The first one is to notice that, we know almost everything about the ordering decisions of an optimal policy, at least modulo A . In particular, there is only a small (polynomial) number of possibilities for ordering A -batches. Using this property, the second idea is to reduce the problem to a single-mode lot-sizing problem, with only batches of size B to order. Since we have to solve this problem repeatedly for all possible subplans (u, v) , we need an efficient algorithm. In the literature, the uncapacitated lot-sizing problem with batch deliveries (ULS-B) and FTL cost structure has been studied by Li *et al.* [15]. Notice that this problem is a particular case of ULS-MMB studied in this article, with only one available mode. They propose a polynomial time algorithm in time complexity $O(T^3 \log T)$ to solve it. However, we only need to solve instances of ULS-B restricted to some subplans. The algorithm of Li *et al.* [15] has a linear time complexity in $O(T)$ on a given subplan, but it requires a precomputation step in $O(T^3 \log T)$ on the whole time horizon. Since instances slightly change in our algorithm, we can not precompute all the quantities only once. Instead, we can use another result of Li *et al.* [15]: They prove that if only full batches can be ordered, the problem can be solved in time complexity $O(T \log T)$. On a given subplan, at most one batch can be fractional (see Property 1 below), thus, one can guess in linear time $O(T)$ the period where the fractional batch is ordered, discard it from the demand, and solve the remaining instance in time $O(T \log T)$. As a consequence, the optimal policy on a given subplan can be computed in time $O(T^2 \log T)$. In Appendix, we reduce this complexity from $O(T^2 \log T)$ to $O(T^2)$ time by an algorithmic improvement, and we call it the *single-mode* algorithm.

We start by giving some structural properties of optimal policies. Recall that a non-empty batch is said to be *fractional* if it is not saturated at its capacity, otherwise we say that the batch is *full*. We have the following classical property:

Property 1. *There exists an optimal policy such that, in any of its subplans, at most one fractional batch is ordered.*

Proof. This property is also stated in Jaruphongsa *et al.* [13]. It relies on the fact that for any given policy, we can build a network flow problem where there is an arc of capacity A , respectively B , between the source and the node representing period t , for each A -batch, respectively B -batch, ordered in this period. For concave costs, it is well known that there exists an optimal flow with no circuit of free arcs, see Ahuja *et al.* [1]. Hence, inside a subplan we can have at most one free arc, corresponding to a fractional batch. \square

In the following, we consider a subplan (u, v) in a fixed optimal policy π . Our goal is to compute its optimal cost $\mathcal{C}(u, v)$. We denote by y the quantity ordered in its single fractional batch, letting $y = 0$ if the optimal policy uses only full batches inside the subplan. Notice that we know, modulo A , the fractional quantity y ordered. Indeed, since any other batch is a full batch, of either A or B units, we must have $y \equiv D_{u,v-1} \pmod{A}$. We denote by \bar{y} the quantity $D_{u,v-1} \pmod{A}$. Thus, if the only fractional batch is an A -batch, we have $y = \bar{y}$. Otherwise, if the fractional batch is a B -batch, then y belongs to the set $\{\bar{y}, A + \bar{y}, \dots, (\mu - 1)A + \bar{y}\}$. Notice that the size of this set can not be polynomially bounded in the size of the instance. However, we will show that, if we need to determine the value of y , only two cases are to be considered in an optimal policy: either $y = \bar{y}$, or $y = (\mu - 1)A + \bar{y}$. That is, either the fractional batch is almost empty, or it is almost full.

6.1. Properties of σ -intervals

In this section, we introduce the notion of σ -intervals and give some structural properties, needed to develop our polynomial time algorithm. Recall that through this section we consider a subplan (u, v) . Let \bar{t} be the period where a fractional batch is ordered in an optimal policy. By convention we set $\bar{t} = v$ if only full batches are ordered inside the subplan (u, v) . Notice that we know, inside the subplan, the stock level modulo A of each period t , depending whether the fractional period takes place before or after t . More precisely, denoting by $X_{u,t}$ the cumulative quantity ordered through periods u up to t , the flow conservation implies that $X_{u,t} = D_{u,t} + s_t$ for each period $t \geq u$. Since $X_{u,t} \equiv 0$

(mod A) if $t < \bar{t}$ and $X_{u,t} \equiv \bar{y} \pmod{A}$ if $\bar{t} \leq t < v$, we must have that $s_t \equiv -D_{u,t} \pmod{A}$ if $t < \bar{t}$ and $s_t \equiv \bar{y} - D_{u,t} \pmod{A}$ if $\bar{t} \leq t < v$. We define :

$$\sigma_t^- = -D_{u,t} \pmod{A} \quad \text{and} \quad \sigma_t^+ = (\bar{y} - D_{u,t}) \pmod{A} \quad (2)$$

Clearly all the quantities σ_t^- and σ_t^+ can be computed in linear time inside a subplan. As a consequence of the previous discussion, if we know that the stock level is less than A at a period t , then, we can assert that $s_t \in \{\sigma_t^-, \sigma_t^+\}$. This motivates the following definition :

Definition 1. A period t is called a σ -period if its entering stock level is lower than A . If k and l are two consecutive σ -periods, then the set $\{k, k+1, \dots, l-1\}$ is called a σ -interval. We denote by $[k, l]$ such an interval.

Since a regeneration point is in particular a σ -period, the subplan (u, v) can clearly be decomposed into σ -intervals. Notice that we have only 3 possibilities for the entering stock levels of the periods k and l , depending whether the fractional period of the subplan, if any, occurs before, inside, or after the σ -interval. Namely, we have $(s_{k-1}, s_{l-1}) \in \{(\sigma_{k-1}^-, \sigma_{l-1}^-), (\sigma_{k-1}^-, \sigma_{l-1}^+), (\sigma_{k-1}^+, \sigma_{l-1}^+)\}$. For each of the 3 possible couples (s_{k-1}, s_{l-1}) , assume that we can compute the optimal cost of the σ -interval $[k, l]$. We denote these costs respectively $\mathcal{C}_{[k,l]}^{(-,-)}$, $\mathcal{C}_{[k,l]}^{(-,+)}$ and $\mathcal{C}_{[k,l]}^{(+,+)}$. Given the optimal costs of all possible σ -intervals, the optimal cost of the subplan (u, v) can be computed as a shortest path in the graph, whose nodes are the σ_t values, and whose arcs are

$$\begin{aligned} & \{(\sigma_{k-1}^-, \sigma_{l-1}^-) \text{ with weight } \mathcal{C}_{[k,l]}^{(-,-)} \mid 0 \leq k < l \leq T+1\} \\ \cup & \{(\sigma_{k-1}^-, \sigma_{l-1}^+) \text{ with weight } \mathcal{C}_{[k,l]}^{(-,+)} \mid 0 \leq k < l \leq T+1\} \\ \cup & \{(\sigma_{k-1}^+, \sigma_{l-1}^+) \text{ with weight } \mathcal{C}_{[k,l]}^{(+,+)} \mid 0 \leq k < l \leq T+1\} \end{aligned}$$

Since this directed graph is acyclic and has $O(T^2)$ arcs, a shortest path can be computed in time $O(T^2)$.

In the following, we consider a fixed σ -interval $[k, l]$ inside the subplan (u, v) . We denote in a generic way σ_{k-1} and σ_{l-1} the entering stock level at periods k and l , respectively, and $\mathcal{C}_{[k,l]}$ the optimal cost of the interval. Notice that the entering stock level of each period inside a σ -interval is greater than A by definition. As a consequence, for a given policy, we can move the ordering of A units wherever we wish inside a σ -interval, backward or forward, keeping a feasible policy. This simple observation will allow us to prove the properties of this section, using interchange arguments. We first show that we know quite precisely the structure of the interval for the A -batches. We have the following property :

Property 2. Inside a σ -interval, there is at most one period where full A -batches are ordered.

Proof. Assume that in an optimal policy, there exists 2 periods t and t' , $k \leq t < t' < l$, where some full A -batches are ordered. We can consider an alternative solution where the ordering of one full A -batch of period t' is anticipated in period t . Since the stock level is at least of A units between periods t and t' , we can also consider the alternative policy, postponing one full batch of period t to period t' . The setup cost f_A being already paid in periods t and t' , the cost of these two solutions differs only by the cost of ordering one full A -batch, respectively in period t or in period t' (recall that we assume wlog that the holding cost is null). For the initial policy to be optimal, these costs must be equal, that is, ordering a full A -batch in period t has the same cost as ordering it in period t' . Hence, the solution ordering all the full A -batches in the first period t must also be an optimal solution. The property follows. \square

In the following, we denote by t_A such an ordering period in the σ -interval $[k, l]$, if it exists (otherwise by convention we let $t_A = l$). We also denote by α the number of full A -batches ordered in that period (and thus inside the σ -interval, due to Property 2). We say that the period t_A is *minimal* if $\alpha < \mu$, that is, if less than μ full A -batches are ordered in the interval. We have the following property:

Property 3. If period t_A is not minimal, then no full B -batch is ordered at that period.

Proof. Assume that $\alpha \geq \mu$, and that a full B -batch is ordered in period t_A . Hence B units are ordered using one full B -batch, and B other units are ordered using μ full A -batches. Clearly, for the policy to be optimal, the two

corresponding costs must be equal. Hence we can consider an alternate optimal policy replacing each full B -batches in period t_A by μ full A -batches. The property follows. \square

Hence, if period t_A is not minimal, only the fractional batch (either an A -batch or a B -batch) may be ordered in this period in addition to the full A -batches. Also notice, due to Properties 1 and 2, that only one another period in the σ -interval may order some units using mode A : This corresponds to the case where the fractional batch is an A -batch and $\bar{t} \neq t_A$. In addition, we know quite precisely the quantity ordered in the fractional batch :

Property 4. *If some full A -batches are ordered inside the σ -interval, then the fractional quantity y ordered belongs to $\{\bar{y}, (\mu - 1)A + \bar{y}\}$.*

Proof. Recall that we define \bar{y} as a shorthand for $(y \bmod A)$. We have only 2 cases to consider: either the fractional batch is an A -batch or a B -batch. In the first case, we have $y = \bar{y}$. In the case of a fractional B -batch, assume that we have $y = qA + \bar{y}$, with $1 \leq q \leq \mu - 1$. Once again, we can use a simple interchange argument. Since the stock level is at least of A units inside the σ -interval, it is always feasible to order A additional units in the fractional B -batch, cancelling one full A -batch ordered, or in the opposite way, to order one additional full A -batch in period t_A , decreasing of A units the quantity ordered in the fractional B -batch. Since the chosen policy is optimal, these two alternate solutions must also be optimal. Hence, we can decide to anticipate the ordering of A units in period t_A or in the fractional period, whichever the one occurring first in the policy, preserving the σ -interval structure. We can pursue this way, till the fractional B -batch either contains less than A units (in this case we have by definition $y = \bar{y}$), or becomes almost full, that is, it is not possible to add A units in the batch (in this case we have by definition $y = (\mu - 1)A + \bar{y}$). The property follows. \square

6.2. A simple illustration

Figure 3 illustrates the different notations introduced in the previous section. We consider the σ -interval $[1, 7]$, that is, $k = 1$ and $l = 7$. The batch sizes are $A = 10$ and $B = 30$, hence $\mu = B/A = 3$. We consider on this example that the demands on the interval are given by the vector $d = (43, 12, 25, 40, 14, 41)$, and that the entering stock levels at period k and l are equal, respectively, to $s_0 = 6$ and $s_6 = 4$. Notice that both of them are less than the batch size $A = 10$ (see Definition 1). The number of units to produce inside the σ -interval is necessarily equal to $(D_{k,l-1} + \sigma_{l-1} - \sigma_{k-1}) = 173$. We can compute $\bar{y} = 173 \bmod A = 3$ and $\alpha \bmod \mu = 2$.

In Figure 3, a dominant production plan $x = (60, 0, 30, 50, 0, 33)$ is represented. Observe that periods 1 and 3 are full B -batch periods, period 4 is a full A -batch period and period 6 is a fractional period, ordering an A -batch of 3 units. The stock levels inside the σ -interval are equal to $s = (23, 11, 16, 26, 12)$, and verify that $s_t > A$ for all $t \in \{k, \dots, l - 2\}$. For this dominant production plan, all the previous properties hold:

- The only fractional batch occurs in $\bar{t} = 6$ (Property 1)
- There is only one period $t_A = 4$ where full A -batches are ordered (Property 2)
- As t_A is not minimal ($\alpha = 5 > \mu = 3$), no B -batch is ordered in t_A (Property 3)
- The fractional quantity ordered is $y = 3 = \bar{y}$, due to the fact that the fractional batch is an A -batch (Property 4)

6.3. A polynomial time algorithm

For a given subplan (u, v) , the algorithm computes the optimal cost $\mathcal{C}_{[k,l]}$ of each possible σ -interval $[k, l]$. Then the optimal cost of the subplan can be classically obtained in time complexity $O(T^2)$ by a shortest path algorithm as explained before. Recall that our basic idea is to exploit the structural properties of a dominant policy to enumerate the possibilities for ordering the full A -batches and reduce the problem to a single-mode lot-sizing problem, with only batches of size B . We detail in this section the computation of the cost $\mathcal{C}[k, l]$ in the case where a fractional batch is ordered inside the σ -interval, that is $\sigma_{k-1} = \sigma_{k-1}^-$ and $\sigma_{l-1} = \sigma_{l-1}^+$. The case when only full batches are ordered inside the interval can be easily deduced.

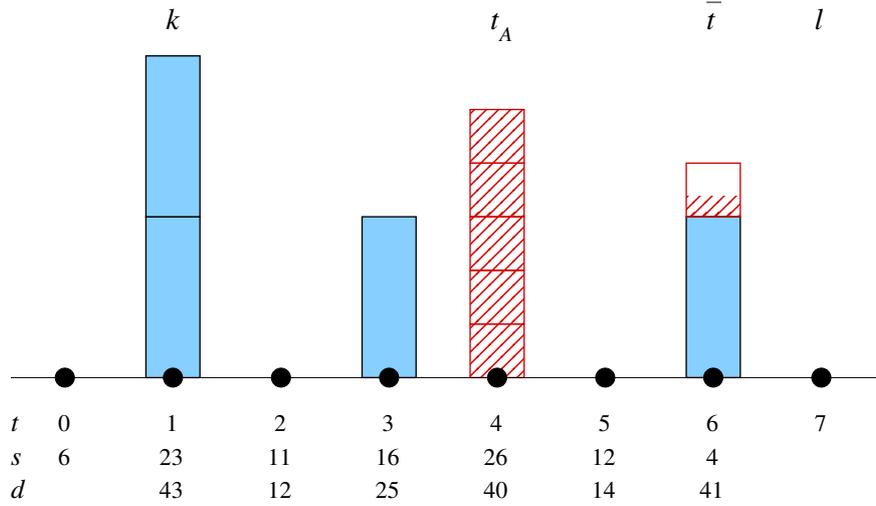


Figure 3: An example of dominant production plan for the σ -interval $[1, 7]$, with $A = 10$ and $B = 30$. A -batches are represented by dashed red boxes, B -batches by plain blue boxes.

Assume that we know inside the σ -interval all the periods when some A -batches are ordered, and, for each of these periods, how many units are ordered using mode A . Then we can discard these units, by decreasing accordingly the demands at these periods (and at the following periods if needed), to solve a lot-sizing problem with a single mode, namely B . Recall that this problem can be solved in time complexity $O(T^2)$ by our *single-mode* algorithm. The optimal cost of the interval is thus the cost of the single mode policy returned by the *single-mode* algorithm, plus the cost incurred by the units discarded using mode A .

To guess in which periods mode A is used, we basically enumerate all the possibilities. Clearly, such an approach can be efficient only if there is a small (polynomially bounded) number of possibilities to consider. Due to Property 2, at most one period, namely t_A , may order full A -batches, and eventually another period may order a fractional A -batch. However, guessing how many full A -batches are ordered may seem challenging if t_A is not minimal, see previous section. We show below that this number can also be determined by the *single-mode* algorithm. We consider the following cases, depending if some full A -batches are ordered, and if the fractional batch is an A -batch or a B -batch :

case 1. No full A -batch is ordered inside the σ -interval. If the optimal policy does not use mode A inside the interval, we can compute the optimal cost $\mathcal{C}_{[k,l]}$ directly with the *single-mode* algorithm, considering that only mode B is available. Otherwise, the units ordered using mode A necessarily correspond to a fractional batch. By definition, we know that exactly \bar{y} units are ordered in the fractional batch. Hence, we can enumerate all the $O(T)$ possibilities for the period \bar{t} , and for each one we can solve a single mode lot-sizing problem, where quantity \bar{y} has been discarded from the demands. Hence, $\mathcal{C}_{[k,l]}$ can be computed in time complexity $O(T^3)$ in this case.

In the two following cases, we consider that at least one full A -batch is ordered inside the interval $[k, l]$. Recall that we also assume that the fractional batch of the subplan (u, v) is ordered inside the interval. We denote by z the quotient of the euclidian division of $(D_{k,l-1} + \sigma_{l-1} - \sigma_{k-1}) \bmod B$ by A . That is, we can write the total amount of units $(D_{k,l-1} + \sigma_{l-1} - \sigma_{k-1})$ to order in the σ -interval as $\beta B + zA + \bar{y}$, with $z < \mu$, for β some integer. Clearly, quantity z can be computed in constant time for each σ -interval, given that all the cumulative demands $D_{t,t'}$ have been precomputed.

case 2. The fractional batch is an A batch. In this case, A -batches may be ordered in only two periods inside the interval : either in period t_A , where all the full A -batches are ordered, or in period \bar{t} , where the fractional A -batch is ordered, with possibly $t_A = \bar{t}$. By definition, we know that exactly \bar{y} units are ordered in the fractional batch. In addition, if the ordering period t_A is minimal, necessarily, exactly z full A -batches are ordered in this period. Hence,

for each possible couple of periods t_A and \bar{t} , we can discard the quantities zA and \bar{y} ordered using mode A from the demands, and compute an optimal policy using only mode B , like in case 1. Notice that we have $O(T^2)$ possibilities, each one requiring the computation of a policy with the *single-mode* algorithm, in time $O(T^2)$.

Now, let us turn our attention to the remaining situation where the ordering period t_A is not minimal. We know from Property 3 that only full A -batches are ordered at this period, and eventually the fractional A -batch. However, we can not polynomially bound the number of values to consider for α , the number of full A -batches ordered in this period. We can only assert that α modulo μ must be equal to z , that is $\alpha = q\mu + z$ for some integer q . Instead of enumerating all the possible values for q , our idea is to let the *single-mode* algorithm determine it for us. For this, again, we consider a lot-sizing problem where only mode B is available, where quantity \bar{y} is discarded from the demands at period \bar{t} , but only quantity zA (and not αA) is discarded from period t_A . The difference with the situation when t_A is minimal is that we modify the cost of ordering a B -batch in period t_A . Specifically, we set $\tilde{p}'_{B,t_A} = \tilde{p}_{A,t_A}$ and $k'_{B,t_A} = \mu(k_{A,t_A})$, with no fixed ordering cost at this period, $f'_{B,t_A} = 0$. That is, the cost of a full B -batch at period t_A corresponds to the price of ordering μ full A -batches. Since the remaining demand to satisfy inside the σ -interval is a multiple of B , only full B -batches are ordered in an optimal solution. Clearly, the solution given by the *single-mode* algorithm can be converted into a solution to the multiple mode problem with the same cost by replacing each full B -batch ordered in period t_A by μ full A -batches. Again, we have $O(T^2)$ possibilities to consider for the periods (t_A, \bar{t}) , each one requiring a computational effort of $O(T^2)$.

Hence, the overall time complexity to evaluate $\mathcal{C}_{[k,l]}$ in this case is in $O(T^4)$. However, we can reduce this complexity to $O(T^3)$ by noticing that we can decide *a posteriori* where the fractional period \bar{t} occurs in the interval. More precisely, consider an optimal policy, where the fractional batch is ordered in period \bar{t} . Since the stock level is greater than or equal to A inside the σ -interval, we can anticipate or delay the ordering of the fractional batch to any other period, keeping a feasible solution². Hence, \bar{t} must correspond to the period minimizing the cost of ordering \bar{y} units using mode A . Let us define

$$t^*(\bar{y}) = \arg \min_{t=k, \dots, l-1} \{f_{A,t} + k_{A,t} + \tilde{p}_{A,t}\bar{y}\}$$

Notice that this period can be determined in linear time for a given σ -interval. In an optimal policy, we claim that \bar{t} is equal either to $t^*(\bar{y})$ or to t_A . Indeed, the fractional batch must be ordered in the period minimizing its cost, which is either $t^*(\bar{y})$ if the setup cost for mode A has to be paid, or t_A to take advantage that the setup cost has already been paid for the full A -batches. Thus, we can modify the algorithm by enumerating the possible values only for period t_A , each time considering only two possibilities, namely $t^*(\bar{y})$ and t_A , for discarding the quantity \bar{y} from the demands. As a consequence, the overall time complexity to evaluate $\mathcal{C}_{[k,l]}$ is reduced to $O(T^3)$.

We use the same example given in Subsection 6.2 to illustrate this case. Recall that the fractional batch is an A -batch with $\bar{t} = 6$ and $\bar{y} = 3$; and also period t_A is not minimal. We use the equivalence $\alpha = q\mu + z$ to find the value $z = 2$. In order to use the single-mode algorithm, we have to discard the value $zA = 20$ from the demand of period t_A , and the fractional quantity $\bar{y} = 3$ is discarded from the demand of period $\bar{t} = 6$. The remaining quantities to be considered are shown in Figure 4 with the new demand vector $d = [43, 12, 25, 20, 14, 38]$. The dominant production plan found by the single-mode algorithm after the demand reduction is illustrated with the new $x = [60, 0, 30, 30, 0, 30]$.

case 3. The fractional batch is a B -batch. Due to Property 2, an optimal policy orders using mode A only in period t_A . In addition, Property 4 states that there are only two possibilities for the quantity y ordered in the fractional batch. As a consequence, if we denote by $\bar{\alpha}$ the number α of full A -batches ordered modulo μ , we have either $\bar{\alpha} = z$ (if $y = \bar{y}$) or $\bar{\alpha} = (z + 1) \bmod \mu$ (if $y = B - A + \bar{y}$). If period t_A is minimal, that is $\alpha < \mu$, we have only two possible values to consider for α . For each possible period t_A and for each of the two possible values for α , we can discard this quantity from the demands, and compute an optimal policy using only mode B . The corresponding optimal cost of the interval can hence be obtained in time $O(T^3)$.

Otherwise, if the A -ordering period is not minimal, we know from Property 3 that only A -batches are ordered at this period, and eventually the fractional B -batch. Again, we can not enumerate all the possible values for α , but as in the previous case we let the *single-mode* algorithm determine it in its optimal solution. For each period t_A and for each

²Though we may not keep the structure of the interval, that is, we may create a period with a stock level lower than A .

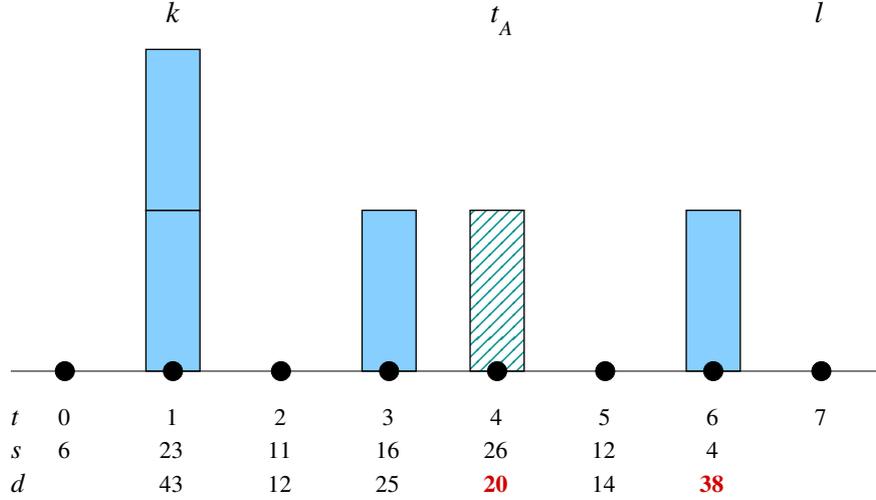


Figure 4: Dominant production plan found by the single-mode algorithm after the reduction to a single-mode problem.

value of $\bar{\alpha} \in \{z, (z+1) \bmod \mu\}$, we discard quantity $\bar{\alpha}$ from the demands in period t_A , and modify the procurement cost of mode B such that ordering a full B -batch in period t_A has the same cost as ordering μ full A -batches, see case 2. The only difference is the way we deal with the fractional batch.

If we assume that period \bar{t} coincides with t_A , we simply discard the fractional quantity ordered from the demands (recall that we have only 2 possibilities to consider for the fractional quantity y). The reason is that the ordering cost of the fractional B -batch would not be accounted correctly in the discarded instance in period t_A otherwise. For a fixed period t_A , this sub-case requires only $O(T^2)$ operations. If we assume that period \bar{t} is different from t_A , we let the *single-mode* algorithm determines it. Again, this subcase requires only $O(T^2)$ operations for a given period t_A . An awkward situation may arise if the policy returned by the *single-mode* algorithm on the discarded instance orders a fractional batch precisely in period t_A . However, this fractional quantity can be ordered at the same cost (or lower) in our original instance using A -batches. Thus, it turns out that the optimal policy in fact does not contain a fractional B -batch, and hence we can ignore the policy constructed since the optimal cost can be determined according to case 2.

All 3 cases require the same computational effort in $O(T^3)$. The optimal cost $\mathcal{C}_{[k,l]}$ of a σ -interval $[k, l]$ can hence be computed in time $O(T^3)$, by selecting the minimal cost returned in each case. Since we have $O(T^4)$ σ -intervals to consider for the overall planning horizon, the time complexity of the algorithm is in $O(T^7)$. We can state the following theorem :

Theorem 5. *Problem ULS-MMB restricted to 2 modes and divisible batch sizes can be solved in time complexity $O(T^7)$.*

One can easily check that this algorithm can also be applied for the lot sizing problem with a time-dependent batch sizes, where we are restricted to only two values of batch sizes A or B in a given period, where A divides B .

7. Conclusion and perspectives

In this study we considered the single-item uncapacitated lot sizing problem with multi-mode replenishment and with batch deliveries. Assuming more general parameters, we improved the results from the literature. The keystone is based on the transformation of the ULS-MMB to the single-mode replenishment problem, multiplying the initial number of periods T by the number of modes M and considering the initial time and mode-dependent parameters over an horizon of MT periods with only time-dependent parameters. This transformation allowed us to use strong properties proposed for the classical uncapacitated lot sizing problem with batch production. Our main contributions

can be cited as : NP-hardness result for ULS-MMB , an efficient 2-approximation algorithm, improvement of the existing results in Ekşioğlu [12] and in Jaruphongsa *et al.* [13], proposition of an FPTAS using a result from Chubanov *et al.* [11] for ULS-MMB, and a polynomial time algorithm for a special case with 2 modes and divisible batch sizes.

As a by-product, we also proposed a new algorithm to solve the single-mode problem in time complexity $O(T^4)$. This complexity is higher than the complexity of the algorithm of Li *et al.* [15], but the two approaches can be extended to slightly different cost structures. $O(T^3 \log T)$ time algorithm of Li *et al.* [15] remains valid for non-decreasing concave holding and backordering costs and non-decreasing concave Less-than-Truck Load (LTL) freight cost functions. Our $O(T^4)$ time algorithm does not take into account backordering costs, consider linear holding costs, but the procurement cost can be more general including concave variable costs (this includes Full-Truck Load cost structure with concave variable cost and also LTL freight cost functions). Also notice that our algorithm is much simpler to implement for practitioners.

There are many interesting extensions and open cases to explore as perspectives. For instance, we can wonder if a polynomial time algorithm can be proposed for the case with 3 modes and divisible batch sizes. More generally the complexity status of the problem is open if the number of batches M is fixed, that is, is not part of the inputs. In particular if only 2 modes are available, but their batch sizes are not divisible. Possible extensions of this work can also consider that some suppliers offer incremental or all unit price discounts to incite to entirely order from them, see [6]. Under an FTL cost structure for shipments, the complexity of the problem is also open.

References

- [1] R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, Prentice Hall, (1993)
- [2] N. Aissaoui, M. Haouari, E. Hassini, Supplier selection and order lot sizing modeling: a review, *Computers and Operations Research*, 34, 3516-3540 (2007)
- [3] A. Akbalik, Y. Pochet, Valid inequalities for the single-item capacitated lot sizing problem with stepwise costs, *European Journal of Operational Research* 198, 412–434 (2009)
- [4] A. Akbalik, C. Rapine, Polynomial time algorithms for the constant capacitated single-item lot sizing problem with stepwise production cost, *Operations Research Letters* 40, 390–397 (2012)
- [5] A. Akbalik and C. Rapine, The single item uncapacitated lot-sizing problem with time-dependent batch sizes: NP-hard and polynomial cases, *European Journal of Operational Research* 229, 353–363 (2013)
- [6] Q.-G. Bai and J.-T. Xu, Optimal solutions for the economic lot-sizing problem with multiple suppliers and cost structures, *Journal of Applied Mathematics and Computing* 37, 331–345 (2011)
- [7] C. Basnet, J.M.Y. Leung, Inventory lot-sizing with supplier selection, *Computers and Operations Research* 32, 1-14 (2005)
- [8] S. Böcker, Z. Lipták, A Fast and Simple Algorithm for the Money Changing Problem, *Algorithmica* 48, 413-432 (2007)
- [9] L. E. Cárdenas-Barrón, J. L. González-Velarde, G. Treviño-Garza, A new approach to solve the multi-product multi-period inventory lot sizing with supplier selection problem, *Computers & Operations Research* 64, 225–232 (2015)
- [10] D. Choudhary, R. Shankar, Joint decision of procurement lot-size, supplier selection, and carrier selection, *Journal of Purchasing and Supply Management* 19(1), 16–26 (2013)
- [11] S. Chubanov, M. Kovalyov and E. Pesch, An FPTAS for a single-item capacitated economic lot-sizing problem with monotone cost structure, *Mathematical Programming* 106(3), 453–466 (2006)
- [12] S.D. Ekşioğlu, A primal-dual algorithm for the economic lot-sizing problem with multi-mode replenishment, *European Journal of Operational Research* 197, 93–101 (2009)
- [13] W. Jaruphongsa, S. Çetinkaya and C.-Y. Lee, A dynamic lot-sizing model with multi-mode replenishments: polynomial algorithms for special cases with dual and multiple modes, *IIE Transactions* 37, 453–467 (2005)
- [14] A.H.I. Lee, H.-Y. Kang, C.-M. Lai and W.-Y. Hong, An integrated model for lot sizing with supplier selection and quantity discounts, *Applied Mathematical Modelling* 37, 4733–4746 (2013)
- [15] C.-L. Li, V.N. Hsu, W.-Q. Xiao, Dynamic lot sizing with batch ordering and truckload discounts, *Operations Research* 52 (4), 639–654 (2004)
- [16] G.S. Lueker, Two NP-Complete Problems in Nonnegative Integer Programming, Technical Report TR-178, Department of Electrical Engineering, Princeton University (1975)
- [17] R. Mansini, M. W. Savelsbergh, B. Tocchella, The supplier selection problem with quantity discounts and truckload shipping, *Omega* 40(4), 445–455 (2012)
- [18] C.T. Ng, M.Y. Kovalyov, T.C.E. Cheng, A simple FPTAS for a single-item capacitated economic lot-sizing problem with a monotone cost structure, *European Journal of Operational Research* 200, 621–624 (2010)
- [19] Y. Pochet, L.A. Wolsey, *Production Planning by Mixed Integer Programming*, Springer, 2006
- [20] M. van Vyve, Algorithms for single-item lot-sizing problems with constant batch size, *Mathematics of Operations Research* 32(3) 594–613 (2007)
- [21] C.P.M. van Hoesel, A.P.M. Wagelmans, Fully polynomial approximation schemes for single-item capacitated economic lot-sizing problems, *Mathematics of Operations Research* 26 339–357 (2001)
- [22] A. Wagelmans, S. van Hoesel and A. Kolen, Economic lot sizing: An $O(n \log n)$ algorithm that runs in linear time in the Wagner-Whitin case, *Operations Research* 40(1), 145–156 (1992)
- [23] Y. Zhao, D. Klabjan, A polyhedral study of lot-sizing with supplier selection, *Discrete Optimization* 9(2), 65–76 (2012)

Appendix A. Solving the Single-mode Problem on a subplan

We consider the problem of finding an optimal policy over a subplan (u, v) for the single-mode problem. In our algorithm proposed for divisible batch sizes, we have to repeatedly solve this problem on different σ -intervals. In the following, in order to generalize the notion of subplan, we simply assume that the stock levels s_{u-1} and s_{v-1} are fixed (typically equal to 0 in the “classical” definition of a subplan), and that at most one fractional batch occurs over the time horizon $\{u, u + 1, \dots, v - 1\}$.

In addition, we consider a slightly more general cost structure. Recall that in this paper we focused on *Full Truck Load* (FTL) procurement costs with linear variable costs, that is, dropping index i of the mode, $r_i(x) = f_i + p_i x + \lceil x/B \rceil k_i$ for any quantity $x > 0$ ordered. In the following, we only require that the procurement costs $r_i(x)$ satisfy a condition on the marginal costs of the full batches. Precisely, for any quantity $x \geq 0$, let $\Delta_i(x) \equiv r_i(x + B) - r_i(x)$ be the marginal cost of ordering one extra full batch, in addition to the quantity x yet ordered. We require these marginal costs to be non-increasing, relatively to the full batches:

$$\Delta_i(x + B) \leq \Delta_i(x) \quad \forall x \geq 0 \quad (\text{A.1})$$

The marginal cost of a full batch is cheaper and cheaper as more full batches are ordered. Notice that an FTL cost structure with concave variable costs, $r_i(x) = f_i + p_i(x) + \lceil x/B \rceil k_i$, with $p_i(\cdot)$ a concave function, satisfies Condition (A.1). Indeed, we have $\Delta_i(x + B) = p_i(x + 2B) - p_i(x + B)$ and $\Delta_i(x) = p_i(x + B) - p_i(x)$. Since $p_i(\cdot)$ is concave, for any values y and z , we have $2p_i(\frac{y+z}{2}) \geq p_i(y) + p_i(z)$. Considering $y = x$ and $z = x + 2B$, we obtain that $\Delta_i(x + B) \leq \Delta_i(x)$. This cost structure enables in particular to model economies of scale, when the supplier offers wholesale prices for large ordered quantity, which corresponds to a concave variable cost function $p(\cdot)$. Notice that this condition is also satisfied by an LTL cost structure, with a concave variable cost $p(\cdot)$, if $r_i(x) = f_i + p_i(x) + \lfloor x/B \rfloor k_i + g_i(x - B \lfloor x/B \rfloor)$. In this cost structure, the (last) partially loaded batch is charged according to a function g , called the *LTL freight cost function*. We refer for instance to [15] for a discussion of these costs.

In the following, we consider procurement costs r_i satisfying Condition (A.1). We require in addition that for any quantity x , procurement cost $r_i(x)$ can be evaluated in constant time.

Since the stock levels s_{u-1} and s_{v-1} are fixed and at most one fractional batch occurs over the (u, v) , the quantity \bar{x} ordered in the fractional batch is necessarily equal to $(D_{u,v-1} + s_{v-1} - s_{u-1}) \bmod B$. We denote by \bar{t} the fractional ordering period, if any, letting $\bar{t} = v$ if no fractional batch is ordered. We show that finding the minimal cost policy on the subplan can be performed in time complexity $O(T^2)$. We first introduce the following simple definition :

Definition 2. An ordering period t is said to be large if $x_t \geq B$, small otherwise.

In plain words, a large period orders at least one full batch. Notice that a small period corresponds necessarily to the ordering of a fractional batch. Thus, at most one ordering period can be a small period over the subplan (u, v) . On the opposite, the fractional period \bar{t} is not necessarily a small period, since full batches can be ordered in addition to the fractional batch. We have the following dominance property :

Property 5. Consider two consecutive ordering periods t and r . If both periods are large, then $s_{r-1} < B$.

Proof. Consider an optimal policy π . Assume for the sake of contradiction that $s_{r-1} \geq B$. It implies that for any period l between t and $r - 1$, we have $s_l \geq B$, that is, at least B units are carried in stock between the end of period t and the beginning of period r . We can consider the 2 alternate policies :

- policy π' orders B additional units in period t , and one batch less in period r , that is, $x'_t = x_t + B$ and $x'_r = x_r - B$. Policy π' is clearly feasible.
- policy π'' orders B additional units in period r , and one batch less in period t , that is, $x''_t = x_t - B$ and $x''_r = x_r + B$. Policy π'' is feasible since the stock level is greater or equal to B units in policy π for any period between t and $r - 1$

Obviously we have $x = x' + x''$, that is policy π is a convex combination of 2 feasible policies. Let us now compute the cost of each policy. We have:

$$\mathcal{C}(\pi') = \mathcal{C}(\pi) + \Delta_t(x_t) - \Delta_r(x_r - B) \quad (\text{A.2})$$

$$\mathcal{C}(\pi'') = \mathcal{C}(\pi) - \Delta_t(x_t - B) + \Delta_r(x_r) \quad (\text{A.3})$$

Now, since π is an optimal policy, we must have $\Delta_t(x_t) \geq \Delta_r(x_r - B)$ and $\Delta_r(x_r) \geq \Delta_t(x_t - B)$. However, since the procurement costs satisfy Condition (A.1), we have in particular $\Delta_t(x_t - B) \geq \Delta_t(x_t)$ and $\Delta_r(x_r - B) \geq \Delta_r(x_r)$. It implies that all these marginal costs are in fact equal, and $\mathcal{C}(\pi') = \mathcal{C}(\pi'') = \mathcal{C}(\pi)$. That is, both policies are also optimal. Considering policy π' , we can repeat this interchange argument as long as x_r and s_{r-1} are both greater than or equal to B . Indeed, each additional full batch ordered in period t is less expensive, and each additional full batch cancelled in period r is more expensive : $\Delta_t(x_t + lB) \leq \Delta_r(x_r - lB)$ for any integer l . The result follows. \square

As a consequence, if r is a large ordering period, there is only a small set of possible values for its entering stock level s_{r-1} . More precisely, let t be the preceding large ordering period (the case with r being the first large ordering period is easy to discuss in the same way). Notice that t may not be the preceding ordering period before period r , for the small ordering period \bar{t} may take place between t and r . Writing the flow conservation modulo B , we have :

- If $\bar{t} \geq r$, then $s_{r-1} = (D_{u,r-1} - s_{u-1}) \bmod B$ due to Property 5.
- If $\bar{t} \leq t$, in the same way $s_{r-1} = (D_{u,r-1} - s_{u-1} - \bar{x}) \bmod B$ due to Property 5.
- If $t < \bar{t} < r$, we claim that $s_{r-1} < 2B - 1$. Indeed, we can use the same argument as in Property 5, since an entering stock level of $(2B - 1)$ units in period r implies that at least B units are carried in stock between period t and period r . As a consequence, we have either $s_{r-1} = (D_{u,r-1} - s_{u-1} - \bar{x}) \bmod B$ or $s_{r-1} = B + (D_{u,r-1} - s_{u-1} - \bar{x}) \bmod B$

A classical approach would be to compute the cost of each possible replenishment cycle (t, r) for all possible values of the entering stock in period t , and to find the optimal cost of the subplan (u, v) using a shortest path algorithm. Since there are $O(T^2)$ possible replenishment cycles (and each one can clearly be computed in constant time), the time complexity of this approach is in $O(T^2)$. However, one difficulty is that the entering stock level of the small period can not be bounded, contrary to the entering stock level of a large period. Hence, we can only assume that $s_{\bar{t}-1} = (D_{u,\bar{t}-1} - s_{u-1}) \bmod B + \alpha B$ for α some integer. Enumerating all the possible values of α would lead to a quite inefficient algorithm. Instead, we use the following approach : we compute only the optimal cost of the replenishment cycles (t, r) such that t and r are large ordering periods. That is, if \bar{t} is a small period, it is not an extremity of the replenishment cycles we consider. We describe now how the optimal cost $\mathcal{L}(r, t)$ of such a large replenishment cycle (t, r) can be computed in constant time.

If \bar{t} does not belong to (t, r) , according to the previous discussion, we have only a small number of possible values for the entering stock levels s_{t-1} and s_{r-1} . For each possible couples of values, the quantity to order in period t is obviously fixed, equal to $D_{t,r-1} + s_{r-1} - s_{t-1}$, and thus the cost of the large cycle (t, r) can be evaluated in constant time. Let us focus on the case where $t < \bar{t} < r$, that is, the fractional period is a small period occurring between the large ordering periods t and r . As a consequence, the stock level s_{t-1} is known, and the stock level s_{r-1} can only take 2 possible values, depending whether $s_{r-1} < B$ or $B \leq s_{r-1} < 2B$. We distinguish 2 cases :

Case 1 : $x_t + s_{t-1} \geq D_{t,r-1}$

This implies that the quantity \bar{x} ordered in the fractional period is not used (assuming a FIFO discipline) to satisfy a demand before the next ordering period r . Hence, we can anticipate or delay the ordering of \bar{x} inside the interval $\{t + 1, \dots, r - 1\}$, keeping a feasible policy. Let us define for any indices a and b , $a \leq b$,

$$w_{\min}(a, b) = \min\{r_l(\bar{x}) \mid a \leq l \leq b\}$$

That is, $w_{\min}(a, b)$ corresponds to the minimal ordering cost to order a fractional batch \bar{x} over the time interval $\{a, \dots, b\}$. On one hand, in an optimal policy, \bar{t} must correspond to a period of $\{t + 1, \dots, r - 1\}$ minimizing its ordering cost. On the other hand, following the proof of Property 5, the number of full batches ordered at period t can not exceed $\lceil (D_{t,r-1} - s_{t-1})/B \rceil$ in an optimal policy. Hence, we have :

$$\mathcal{L}(t, r) = r_t(B \lceil (D_{t,r-1} - s_{t-1})/B \rceil) + w_{\min}(t + 1, r - 1)$$

Notice that the values of $w_{\min}(a, b)$ can be precomputed in time $O(T^2)$ for all possible indices $u \leq a \leq b \leq v$, since $w_{\min}(a, b + 1) = \min\{w_{\min}(a, b), r_{b+1}(\bar{x})\}$. Given these values, the cost $\mathcal{L}(t, r)$ can thus be computed in constant time.

Case 2 : $x_t + s_{t-1} < D_{t,r-1}$

Contrary to the previous case, some units ordered in the fractional batch are used to satisfy a demand before period r . As a consequence, delaying period \bar{t} may result in an infeasible policy, that is, a shortage may occur. Observe that, in this case, the number of full batches ordered in period t is equal to $\lfloor (D_{t,r-1} - s_{t-1})/B \rfloor$. Indeed, ordering less batches would result in a shortage in period $r - 1$, since, at most $B - 1$ units can be ordered in period \bar{t} . Conversely, ordering more batches would contradict our assumption that $x_t + s_{t-1} < D_{t,r-1}$. Let us define

$$\lambda(t, r) = \min\{l \mid x_t + s_{t-1} < D_{t,l-1}\} = \min\{l \mid B\lfloor (D_{t,r-1} - s_{t-1})/B \rfloor + s_{t-1} < D_{t,l-1}\}$$

That is, $\lambda(t, r)$ is the first period where a shortage occurs if the fractional batch is not ordered. Hence, we must have $\bar{t} \leq \lambda(t, r)$. Similarly to the previous case, \bar{t} must correspond to the less expensive ordering period over the interval $\{t + 1, \dots, \lambda(r, t)\}$. It results that in this case :

$$\mathcal{L}(t, r) = r_t(B\lfloor (D_{t,r-1} - s_{t-1})/B \rfloor) + w_{\min}(t + 1, \lambda(t, r))$$

Again, if the values of $\lambda()$ and $w_{\min}()$ are known, the cost $\mathcal{L}(t, r)$ can be computed in constant time. We claim that for a given period t , all the values $\lambda(t, r)$ for $t < r \leq v$ can be computed in linear time $O(T)$. As a consequence, we can again pre-compute all the values $\lambda(t, r)$ for $u \leq t < r \leq v$ in time $O(T^2)$. To see this, consider that we have determined the value of $\lambda(t, r)$ for some period r . Notice that $\lambda(t, r)$ is non-decreasing with period r for a given period t , that is, $\lambda(t, r + 1) \geq \lambda(t, r)$. This is due to the fact that the quantity ordered in period t , $B\lfloor (D_{t,r-1} - s_{t-1})/B \rfloor$, is non-decreasing with r . Hence, to find period $\lambda(t, r + 1)$, we can simply scan the first period where a shortage occurs, starting from period $\lambda(t, r)$. Notice that determining period $\lambda(t, r + 1)$ may require up to $O(T)$ operations. More precisely, we have to scan $\lambda(t, r + 1) - \lambda(t, r) + 1$ periods. Thus, determining $\lambda(t, t + 1), \dots, \lambda(t, v)$ requires to scan at most $2T$ periods altogether, which can be performed in linear time. As a conclusion, we can state the following property :

Property 6. *The optimal cost of a subplan (u, v) can be computed in time $O(T^2)$.*

Using a shortest path algorithm, one can compute the optimal cost of a policy over the whole time horizon, given the optimal values of all the possible subplans (u, v) defined by the regeneration points. This approach is valid if at most one fractional batch is used in each subplan. This is the case with FTL or LTL procurement cost structures, with concave variable costs $p_t(x)$ (see the beginning of this section), assuming that the freight function is also concave for LTL cost structure. We have the following corollary :

Theorem 6. *An optimal policy for ULS-B with FTL/LTL concave procurement costs can be computed in time $O(T^4)$.*