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# Geometry of color perception. Part 1: Structures and metrics of a homogeneous color space

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## Abstract

This is the first half of a two-part paper dealing with the geometry of color perception. Here we will analyze in detail the seminal 1974 work by H.L. Resnikoff, who showed that there are only two possible geometric structures and Riemannian metrics on the perceived color space  $\mathcal{P}$  compatible with the set of Schrödinger's axioms completed with the hypothesis of homogeneity. We will recast Resnikoff's model into a more modern colorimetry setting and motivate the need of a psychophysical experiment to confute or confirm the homogeneity axiom for  $\mathcal{P}$ . The second part of this paper will show how this problem can be circumvented by reformulating Resnikoff's theory in terms of Jordan algebras and quantum mechanical concepts.

## 1 Introduction and state of the art

This first half of a two-part paper provides a thorough review and a critical analysis of the pioneering work of H.L. Resnikoff on color perception developed within the papers [13, 14, 15] and the book [12]. These works are amongst the major inspirations for a modern program of re-foundation of colorimetry that will be discussed in the second part of this paper, in which it will be shown how to recast Resnikoff's model in a quantum-like theory free from ambiguities.

Even if it may be surprising at first glance, Resnikoff belong to a vast ensemble of mathematicians, physicists, biologists, philosophers and even poets who were fascinated by the concept of color. The list is impressive, ranging from Plato to Wittgenstein, passing through Aristotle, Descartes, Hook, Newton, Euler, Young, Helmholtz, Maxwell, Grassmann, Riemann, Goethe, Schopenhauer, Locke, Weber, Fechner, Dalton, Hering and, last but not least, Schrödinger.

In fact, it is the research about the mathematical analogies between color (and, more general, optics) and the oscillating behavior of quantum particles that led Schrödinger to propose the famous equation which bears his name in quantum mechanics [19]. As it will be recalled in section 2, Schrödinger performed a synthesis of the most important findings about the mathematical theory of color perception in a coherent set of axioms, introducing one of his own. This can be thought as a psycho-physical counterpart of what Maxwell did for electromagnetism.

The experiments of Wright and Guild, see e.g. [24], could have paved the way for a further development in the mathematical understanding of color perception, however, the recently founded Commission Intérrationale de l'Éclairage (CIE), took a more practical path by building up geometrically flat color spaces which had the advantage to be much more easy to handle for engineering purposes. While the  $XYZ$  space still stands today as a handy color space for colorimetry, its developments until recent years, see e.g. [17], lacked of mathematical rigor and introduced ad-hoc parameters to adapt newly discovered phenomena to the existing color space structures, instead of modifying their geometry to fit the new observations.

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Resnikoff's papers and book were written in the middle of the '70s of the twentieth century, about the same period when researchers in relativistic quantum field theory developed the *standard model* of fundamental physical interactions and when some first attempts to fuse quantum mechanics and general relativity into a single theory have been proposed. This *zeitgeist* could explain the reason why Resnikoff decided to use techniques which are quite common in theoretical physics (as Riemannian geometry, homogeneous spaces, Lie groups and algebras) to study color perception. In this sense, his achievements could be considered a very elegant example of 'theoretical psycho-physics'.

In spite of its extreme originality and deepness, Resnikoff's work remained practically unnoticed until today, probably due to the fact that the mathematical knowledge needed to understand the meaning of his findings is quite vast and does not belong to the typical mathematical background of scientists working on colorimetry.

One of the aims of this paper is to rewrite Resnikoff's results in more modern and pedagogical terms, thus making them accessible to a wider range of researchers in colorimetry, vision science and image processing.

A fundamental point in Resnikoff's theory is the selection of a suitable group of transformations  $G$  with respect to which the perceived color space is homogeneous. Resnikoff's theory is supposed to be linear, however, as he himself declared, the linearity of these maps is far from being obvious or proven by experiments.

This remark will give us the opportunity to propose a novel psycho-physical experiment to either confute or confirm linearity. Moreover, it will also motivate the reformulation of Resnikoff's theory based on Jordan algebras and quantum mechanical structures that will be discussed in the second half of this two-part paper, with the evident advantage to be free from the need of specifying  $G$  explicitly.

The paper is structured as follows: in the next section Schrödinger's classical axioms for the space of perceived colors will be recalled, along with the notation and the nomenclature that will be used in the rest of the paper. Then, the homogeneity axiom will be introduced, along with the analysis of its consequences.

The final part of the paper will deal with the invariant Riemannian metrics on the space of perceived colors that can be singled out in a unique way and their relationship with existing color distances.

## 2 Review of Schrödinger's axioms for the space of perceived colors

Let us begin by introducing the notation and nomenclature that will be used in the paper.

- $\Lambda = [\lambda_{\min}, \lambda_{\max}]$  denotes the *visual spectrum*, the extrema of  $\Lambda$  are left unspecified because their numerical value is not important and because there is no agreement about their precise value. Typically, one chooses  $\lambda_{\min} = 380\text{nm}$  (extreme violet of the visual spectrum) and  $\lambda_{\max} = 780\text{nm}$  (extreme red).
- $\mathbf{x} : \Lambda \rightarrow \mathbb{R}^+$ : is the physical light function measured by an optical transducer, which will be called *physical light*.
- In standard colorimetric tests,  $\mathbf{x}$  is presented in a dim room to the ideal standard observer, with an aperture angle of either 2 degrees (which correspond to foveal vision), or 10 degrees (which allows extra-foveal vision). However, as we will see,  $\mathbf{x}$  can also be presented as a small central area seen against a uniform (or uniformly illuminated) background. In this latter case we talk about color in (uniform) context. Experiments of color in non-uniform context are still very rare, see e.g. [16], [8].

- *Color matching*: the typical way to compare the perception of two physical lights is to divide in two the field of view (creating what is called a *bipartite field*) and putting the two color stimuli side-by-side. This can be performed with or without the presence of a context, but in the latter case, the context must be the same for both color stimuli. It is important to notice that it has been verified experimentally that many spectrally different physical lights are perceived as the same color by humans [24].

- Since light stimuli have finite energy, i.e.  $\int_{\Lambda} \mathbf{x}(\lambda)^2 d\lambda < +\infty$ , a physical light can be considered as an element of  $L^2(\Lambda) \subset L^1(\Lambda)$ , where the inclusion holds because the Lebesgue measure of  $\Lambda$  is finite. Since light stimuli are real-valued,  $L^2(\Lambda)$  will be considered as a real vector space and we set

$$L_+^2(\Lambda) = \{x \in L^2(\Lambda), x(\lambda) \geq 0\}. \quad (1)$$

- A *spectral light stimulus*  $\mathbf{x}_{\lambda_0}(\lambda)$  relative to the wavelength  $\lambda_0$  is a quasi-monochromatic visible radiation for which there exist two values  $k, \varepsilon > 0$  such that  $\mathbf{x}_{\lambda_0}(\lambda)$  can be approximated by the following expression

$$\mathbf{x}(\lambda) = \begin{cases} k & \text{if } \lambda_0 - \varepsilon < \lambda < \lambda_0 + \varepsilon \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

- The so-called *metamer equivalence*  $\sim$  can be stated like this:

$$\mathbf{x}, \mathbf{y} \in L_+^2(\Lambda), \mathbf{x} \sim \mathbf{y} \iff \mathbf{x} \text{ and } \mathbf{y} \text{ are perceived as identical,}$$

the name is well posed, since  $\sim$  has been empirically proven to be an equivalence relation [4].

- The *space of perceived colors* is defined as:

$$\mathcal{P} = L_+^2(\Lambda) / \sim, \quad (3)$$

the metamer equivalent class of  $\mathbf{x}$  will be simply denoted by  $x$ :

$$x = [\mathbf{x}]_{\sim}. \quad (4)$$

The algebraic operations of sum (superposition of lights) and multiplication by a *positive* scalar can be passed from  $L^2(\Lambda)$  to  $\mathcal{P}$  simply by defining:

$$\lambda_1 x + \lambda_2 y = [\lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}]_{\sim}, \quad \forall \lambda_1, \lambda_2 \in \mathbb{R}^+, \forall \mathbf{x}, \mathbf{y} \in L_+^2(\Lambda), \quad (5)$$

where  $\lambda \mathbf{x}$  is the physical light  $\mathbf{x} \in L_+^2(\Lambda)$  modulated in its intensity by the coefficient  $\lambda \in \mathbb{R}^+$  and  $\mathbf{x} + \mathbf{y}$  simply means superposition of physical lights. The 0 of  $\mathcal{P}$  is the equivalent class of physical lights whose intensity is so small that they do not activate the retinal photoreceptors.

- If we want to endow the quotient  $\mathcal{P}$  of an algebraic structure, we must specify what it means to perform a linear combination of light stimuli with real coefficients. This is related to the concept of color matching in a bipartite field, as explained in detail by E. Dubois in its excellent treatise [4]. Given a color stimulus  $\mathbf{x} \in L_+^2(\Lambda)$  and a real *negative* coefficient  $\lambda$ , we interpret  $\lambda \mathbf{x}$  as the same light stimulus as  $(-\lambda)\mathbf{x}$  but *shown to the observer on the other side* of the bipartite field. This allows the definition of linear combinations with real, positive and negative, coefficients and it is the correct way to interpret the famous Wright and Guild experiments, see e.g. [24] or [17]. In these psycho-physical tests it is shown that either the superposition of three light stimuli suffice *to match* a given color, or the superposition of two light stimuli on one side of a bipartite field matches the superposition of the given color and a light stimulus on the other side. This is the way in which the sentence '*every color stimulus can be matched by a linear combination of three spectral lights stimuli*' has to be interpreted.

- The *vector space of perceived colors*: it is defined as the real span of  $\mathcal{P}$ , i.e.

$$V = \text{span}(\mathcal{P}), \quad (6)$$

where the span is performed with respect to the linear structure described above, i.e. *equality in  $V$  is color matching* and multiplication by a negative real number must be interpreted as light superposition on the other side of a bipartite visual field in a color matching experiment.

With this notation, Schrödinger's axioms, see [18], can be stated like this.

- **Axiom 1** (Newton 1704, [11]) If  $x \in \mathcal{P}$  and  $\alpha \in \mathbb{R}^+$ , then  $\alpha x \in \mathcal{P}$ ;
- **Axiom 2** (Schrödinger 1920, [18]) if  $x \in \mathcal{P}$  then it does not exist any  $y \in \mathcal{P}$ ,  $y \neq x$ , such that  $x + y = 0$ ;
- **Axiom 3** (Grassmann 1853, [7] & Helmholtz 1866 [21]) For every  $x, y \in \mathcal{P}$  and for every  $\alpha \in [0, 1]$ ,  $\alpha x + (1 - \alpha)y \in \mathcal{P}$ ;
- **Axiom 4** (Grassmann 1853, [7]) If  $x_k \in \mathcal{P}$ , then there exist  $\alpha_k \in \mathbb{R} \setminus \{0\}$ ,  $k = 1, \dots, 4$ , such that  $\sum_{k=1}^4 \alpha_k x_k = 0$ .

Let us now discuss the colorimetric and mathematical meaning of the axioms, in particular, mixing Axiom 1,2 and 4, we will come out with a finer version of Axiom 4.

Mathematically speaking, the meaning of Axiom 1 is simple:  $\mathcal{P}$  is an *infinite cone in  $V$* . However, notice that Axiom 1 is an *idealization*: when  $\alpha$  is very large, photoreceptors saturate and when  $\alpha$  is very small quantum effects and background noise will disrupt the relation input-output which persists in photopic conditions.

Axiom 2 means that no superposition of perceived lights is perceived as the absence of light (recall that the symbol  $+$  means additive synthesis of light stimuli). This assumption is *true for non-coherent light*, as normal daylight, because it is well known that, for coherent light, destructive interference can extinguish light intensity in certain spatial positions when two light beams are superposed. Mathematically speaking, since  $x + y = 0$  is equivalent to  $x = -y$ , the axiom implies that no  $x \in \mathcal{P}$  can be written as the opposite of an element  $y \in \mathcal{P}$ ,  $y \neq x$ , i.e. that  $\mathcal{P}$  does not contain 1-dimensional vector subspaces.

Axiom 3 means that the line segment which joins the perceived colors  $x$  and  $y$  consists entirely of perceived colors, thus  $\mathcal{P}$  is a *convex cone*. This condition is well known to be equivalent to be closed under conical convex combinations, i.e. linear combinations with positive coefficients between 0 and 1 whose sum is 1.

This fact, along with Axiom 1, implies that  $\mathcal{P}$  is *closed under arbitrary linear combinations with positive coefficients*, in fact, for all  $\alpha_1, \alpha_2 \in \mathbb{R}^+$  and  $x_1, x_2 \in \mathcal{P}$ ,  $\frac{1}{\alpha_1 + \alpha_2} [\alpha_1 x_1 + \alpha_2 x_2] \equiv z$  is a convex combination of elements of  $\mathcal{P}$ , so  $z \in \mathcal{P}$  thanks to Axiom 3, but then also  $(\alpha_1 + \alpha_2)z = \alpha_1 x_1 + \alpha_2 x_2 \in \mathcal{P}$  thanks to Axiom 1. By iterating this argument we have that  $\sum_{k=1}^n \alpha_k x_k \in \mathcal{P}$   $\forall \alpha_k \in \mathbb{R}^+$ ,  $x_k \in \mathcal{P}$ ,  $k = 1, \dots, n$ .

Axiom 4 means that every collection of more than three perceived colors is a linearly dependent family in the vector space  $V$  spanned by the elements of  $\mathcal{P}$ , i.e. any 4 perceived colors are linearly dependent with respect to the algebraic structure of  $V$  previously defined.

A finer version of Axiom 4 can be obtained with the following argument. First of all, notice that Axioms 1-3 prevent the  $\alpha_k$ 's to have all the same sign. In fact, let us imagine that all the coefficients  $\alpha_1, \dots, \alpha_4$  are positive, then  $\bar{x} = \sum_{k=1}^3 \alpha_k x_k \in \mathcal{P}$  (thanks to what just proven) and  $\bar{y} = \alpha_4 x_4 \in \mathcal{P}$  (thanks to Axiom 1), then  $\sum_{k=1}^4 \alpha_k x_k = 0$  implies  $\bar{x} + \bar{y} = 0$ , which is impossible

thanks to Axiom 2. A similar argument can be used when all the coefficients  $\alpha_1, \dots, \alpha_4$  are negative.

To resume, Axiom 1-4 imply this stronger version of Axiom 4.

**Axiom 4'** For all quadruple of perceived lights  $x_k \in \mathcal{P}$ ,  $k = 1, \dots, 4$ , there are coefficients  $\alpha_k \in \mathbb{R}$ , *not all with the same sign*, such that  $\sum_{k=1}^4 \alpha_k x_k = 0$ .

There are only two options coherent with Axiom 4'. The first option is that three coefficients have the same sign and the remaining one has opposite sign. Since equality in  $V$  is color matching and a negative coefficient means that the corresponding light stimulus must be shown on the other side of the bipartite field, this means that one light stimulus color matches the superposition of other three light stimuli.

In the second option, two coefficients are positives and two are negatives: this means that the superposition of two lights stimuli color match the superposition of other two lights stimuli. This case has been already established by the already quoted experiments of Wright and Guild [17].

Another direct consequence of Axiom 4 is that  $\dim(V) \leq 3$ , in particular, we call observer for which:

- $\dim(V) = 3$ : *trichromate*;
- $\dim(V) = 3$ : *dichromate*;
- $\dim(V) = 1$ : *monochromate*;
- $\dim(V) = 0$ : *blind*.

Following [2], we can define the following *projection map*

$$\begin{aligned} \pi : L_+^2(\Lambda) &\longrightarrow \mathcal{P} \\ \mathbf{x} &\longmapsto x, \end{aligned} \tag{7}$$

which implies that *infinitely many spectrally different lights coincide perceptually*.

In what follows, we will fix our attention only on the trichromatic case, i.e. from now on,  $\dim(V) = 3$ .

### 3 Resnikoff's homogeneity axiom for $\mathcal{P}$

As stated in the introduction, in [13] Resnikoff used the theory of homogeneous spaces to study the geometry and the metrics of the perceived color space  $\mathcal{P}$ . Notice that this is far from being a trivial task, since the metamer equivalence classes that make up  $\mathcal{P}$  are very difficult objects to characterize from a mathematical point of view. Thus, a theory of  $\mathcal{P}$  which bypasses the use of metamer equivalent classes is highly desirable.

Before going through the details of his analysis, it is worth recapping the definition of homogeneous space.

If  $X$  is a topological space and  $G$  is a group of transformations acting on it,  $G \times X \rightarrow X$ ,  $(g, x) \mapsto g(x)$ , then  $X$  is a *G-homogeneous space* if, for any two points  $x, y \in X$ , there exists a transformation  $g \in G$  such that  $g(x) = y$ , i.e. any two points of  $X$  can be joined by a suitable transformation  $g$  of  $G$ , otherwise stated, the action of  $G$  on  $X$  is transitive (there is only one  $G$  orbit).  $X$  is *locally homogeneous with respect to G* if this property holds only locally, i.e. if for every  $x \in X$  there is an open neighborhood  $U_x$  of  $x$  such that every  $x' \in U_x$  can be written as  $x' = g(x)$  for a certain  $g \in G$ .

The motivation to introduce homogeneous spaces comes from Weber-Fechner's law [24], which states that the perceived intensity, called *brightness*,  $b(x)$  of  $x \in \mathcal{P}$  is proportional to  $\log x$  (for a wide range of intensities), thus, the relative brightness  $b(x_1) - b(x_2)$  between  $x_1 \in \mathcal{P}$  and  $x_2 \in \mathcal{P}$  will be proportional to  $\log(x_1) - \log(x_2) = \log(\frac{x_1}{x_2}) = \log(\frac{\lambda x_1}{\lambda x_2})$  for all positive coefficient  $\lambda$  belonging

to the (wide) range of values for which Weber-Fechner's law is valid. This explains why the relative brightness is invariant under the simultaneous modification of light expressed by

$$x_1 \mapsto \lambda x_1, \quad x_2 \mapsto \lambda x_2, \quad \lambda > 0. \quad (8)$$

The set of all possible light intensities (we do not consider here the option of absence of light) can be identified with  $\mathbb{R}^+ = (0, +\infty)$ , which is both a topological space and a group with respect to the ordinary multiplication of positive real numbers. The very simple observation that

$$\forall x, y \in \mathbb{R}^+, \quad x \neq 0, \quad y = \frac{y}{x}x \equiv \lambda x, \quad (9)$$

shows that  $\mathbb{R}^+$  is a  $\mathbb{R}^+$ -homogeneous space. The previous remark about brightness perception implies that the relative brightness between two perceived lights is a  $\mathbb{R}^+$ -invariant function defined on  $\mathbb{R}^+$ .

What is crucial here is that, up to a selection of unit of measurement, Weber-Fechner's law defines the *unique*  $\mathbb{R}^+$ -invariant metric on  $\mathbb{R}^+$ , i.e.

$$d(x_1, x_2) = \log(x_1) - \log(x_2) = \log \frac{x_1}{x_2}, \quad x_1, x_2 \in \mathbb{R}^+. \quad (10)$$

The purpose of Resnikoff's paper [13] is to generalize this argument to the entire color space in order to single out metrics by using invariance properties of human vision. The major phenomenological evidences to believe that the entire color space is homogeneous with respect to the action of some group is color constancy [5], i.e. the robustness of the human visual system to perceive color (and not only light intensity) with respect to different illuminants. This property is essential for the stability of our vision, in fact, without it, we should relearn color relationships among objects each time we change the illumination of a scene. Color constancy is actually one of the first features that is needed in any model of robot vision, video-surveillance, to quote but a few applications.

The remainder of Resnikoff's paper [13] is devoted to the mathematical development of a theory of color perception based upon the presumed homogeneity of color space. Here we must underline that the observational arrangement considered by Resnikoff is that of color in a uniform context. As we will see, the presence of this context is crucial for the development of his theory.

The first information that we need is relative to the group of transformation with respect to  $\mathcal{P}$  is (supposed to be) homogeneous. Resnikoff proposed the group  $\text{GL}^+(\mathcal{P})$  of *orientation-preserving linear transformations of  $V$  which preserve the cone  $\mathcal{P}$* , i.e.

$$\text{GL}^+(\mathcal{P}) := \{B \in \text{GL}(V) : \det(B) > 0, \text{ and } B(x) \in \mathcal{P} \quad \forall x \in \mathcal{P}\}, \quad (11)$$

where  $\text{GL}(V)$  is the group of invertible linear operators on  $V$ , which can be identified with the group of real  $n \times n$  matrices with determinant different from zero, where  $n = \dim(V) = 3$ .  $\text{GL}(V)$  is the complementary set in  $\text{M}(3, \mathbb{R}) \simeq \mathbb{R}^9$  of  $\det^{-1}(0)$ , the counter-image of 0 by the determinant function, which is continuous in the Euclidean topology, thus  $\det^{-1}(0)$  is closed and so  $\text{GL}(V)$  is an open subset of  $\mathbb{R}^9$  with respect to the Euclidean topology.

In [13], the transformations  $B \in \text{GL}^+(\mathcal{P})$  are called 'changes of background illumination', however, here they will be simply called *changes of background*, without specifying 'illumination', because this can be reductive since a background change can be also performed by a modification of its reflectance.

Let us now introduce the idea behind the homogeneity axiom. It is known that any perceived color  $x \in \mathcal{P}$  can be transformed into any 'sufficiently near' one  $y \in \mathcal{P}$  by an appropriate change of background, see Figure 1 for a graphical representation.

For this reason Resnikoff postulates that  $\mathcal{P}$  is a local homogeneous space with respect to the group  $\text{GL}^+(\mathcal{P})$ . Notice that this is not a physical property of color, but a perceptual feature of human vision, usually referred to as chromatic induction, see e.g. [22, 16, 8] for more details about how induction can be measured.

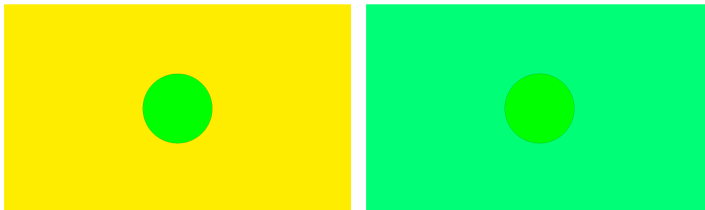


Figure 1: The two disks are exactly the same, however, the first one is perceived as yellowish because of its background.

Thus, for every  $x \in \mathcal{P}$ , there exists an open neighborhood  $U_x \subset \mathcal{P}$  such that each  $y \in U_x$  can be expressed as  $y = B(x) \in \mathcal{P}$  for some  $B \in \text{GL}^+(\mathcal{P})$ , so every point of  $\mathcal{P}$  is an interior point, i.e.  $\mathcal{P}$  is open in  $V$  and therefore it inherits the structure of a differentiable manifold from  $V^1$ . With respect to this differential structure, each transformation  $B \in \text{GL}^+(\mathcal{P})$ ,  $B : \mathcal{P} \rightarrow \mathcal{P}$ , is a diffeomorphism.

Let us now consider local homogeneity in conjunction with Axiom 3, i.e. with the convexity of  $\mathcal{P}$ : for every couple of perceived colors  $x, y \in \mathcal{P}$  there exists the line segment  $L$  that joins  $x$  to  $y$ .  $L$  is a *compact* subset of  $\mathcal{P}$  and local homogeneity assures that, for any  $z \in L$  there exists an open neighborhood  $U_z \subset \mathcal{P}$  that is a homogeneous space with respect to the group  $\text{GL}^+(\mathcal{P})$ . As we move  $z$  in  $L$ , we obtain the open covering  $\bigcup_{z \in L} U_z$  of  $L$ , and, by definition of compactness, we can extract a *finite* open covering of  $L$  from it, i.e. there exist  $x_1, \dots, x_n \in L$ ,  $n < +\infty$ , such that  $\bigcup_{k=1}^n U_{x_k}$  is an open covering of  $L$ .

Let  $B_k \in \text{GL}^+(\mathcal{P})$  be the change of background transformation which carries  $x_k$  to  $x_{k+1}$ , where  $k = 1, \dots, n-1$ ,  $x_0 \equiv x$  and  $x_n \equiv y$ , then, since  $\text{GL}^+(\mathcal{P})$  is a group, the transformation  $B \equiv B_n \circ B_{n-1} \cdots \circ B_1$  carries  $x$  to  $y$ , i.e.  $y = B(x)$ , for every couple of perceived lights  $x, y \in \mathcal{P}$ .

Thus, Axiom 3 implies that local homogeneity is equivalent to global homogeneity and, for this reason, Resnikoff is led to postulate his own fifth axiom on the structure of the color space.

**Axiom 5** (Resnikoff 1974, [13]):  $\mathcal{P}$  is a  $\text{GL}^+(\mathcal{P})$ -homogeneous space.

Homogeneous spaces have many useful properties that Resnikoff used to determine the geometrical structure of  $\mathcal{P}$  and to single out invariant measures on it. However, we postpone this analysis after an interlude in which we discuss the important issue of linearity for the transformations  $B \in \text{GL}^+(\mathcal{P})$ .

### 3.1 A psycho-physical analysis of background transformations

In this subsection we analyze in deeper detail the properties of the transformations  $B \in \text{GL}^+(\mathcal{P})$ . The fact that  $B$  is required to be orientation-preserving is needed to maintain intensity relations when perceived colors differ only by positive scalar, i.e. only in brightness. Geometrically speaking, an orientation-preserving transformation respects the direction of each generatrix of the cone  $\mathcal{P}$ .

The requirement that the cone  $\mathcal{P}$  must be preserved is natural too, because after a change of background a perceived color remains a perceivable, thus, if  $B \in \text{GL}^+(\mathcal{P})$  and  $x \in \mathcal{P}$ , then  $B(x) \in \mathcal{P}$ .

The transformation must be invertible, because once the reverse change of background is performed, the perceived color returns the original one.

However, there are two main issues related to the background transformations that must be discussed.

The first is that, from an operative point of view, it is not evident how it is possible to pass from any fixed perceived color  $x \in \mathcal{P}$  to an arbitrary other  $y \in \mathcal{P}$  with a background change. In fact,

<sup>1</sup> $V$ , as a real vector space, is naturally endowed with the Euclidean differential structure.



to the best of our knowledge, the literature about color induction from a change of background illumination is quite limited and almost entirely confined to achromatic stimuli [22], [16].

The second issue is that *it is not clear why the transformations  $B$  should be linear*. Resnikoff himself, in the paper [14] declared it to be ‘*the least verified aspect*’ of the group of transformations that he selected to act on  $\mathcal{P}$ .

Linearity means that

$$B(\alpha x + \beta y) = \alpha B(x) + \beta B(y), \quad \forall \alpha, \beta \in \mathbb{R}, \forall x, y \in \mathcal{P}, \quad (12)$$

where  $B$  is any change of background. As usual, equality means color match and terms with negative coefficients must be transferred on the other side of the equality and the resulting equation must be interpreted via a color matching procedure.

We propose hereafter a psycho-physical experiment to check the linearity of background transformations. Let us start considering *additivity*: an experiment of color matching is needed to test if  $B(x + y)$  color matches  $B(x) + B(y)$ . Fig. 2 gives a handy depiction of the experiment.

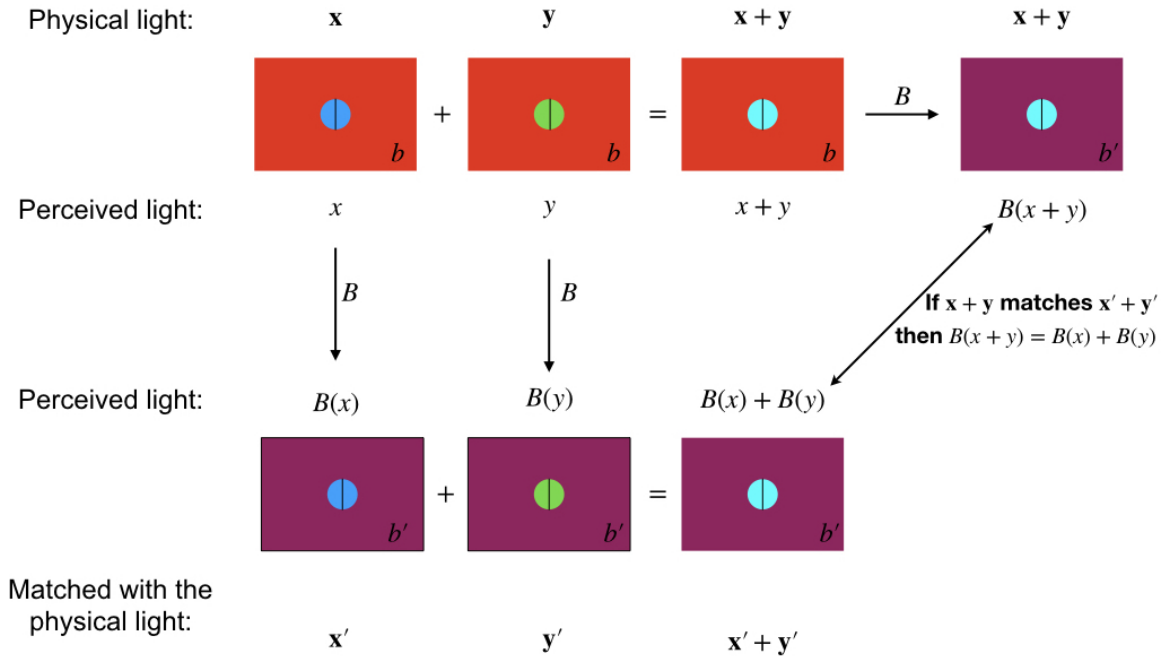


Figure 2: The experimental setup depicted in the figure can be used to check the additivity of background transformations.

- *Measure of  $B(x + y)$* . Let us choose two reference colors  $x$  and  $y$ , e.g. represented by the perception of two Munsell chips with respect to the same background  $b$ . Let  $\mathbf{x}$  and  $\mathbf{y}$  the physical light which color-matches  $x$  and  $y$ , respectively. Then, we perform a change of background, mathematically modeled by a transformation  $B \in \text{GL}^+(\mathcal{P})$ , from  $b$  to  $b'$ . The color sensation generated by the superposition  $\mathbf{x} + \mathbf{y}$  in the context  $b'$  is  $B(x + y) \in \mathcal{P}$ .
- *Measure of  $B(x) + B(y)$* . We show the Munsell chip originally associated with  $x$  in the modified context  $b'$  and we color-match the new color sensation  $B(x) \in \mathcal{P}$  with the physical light  $\mathbf{x}'$ . We repeat this procedure for  $y$ , color-matching  $B(y) \in \mathcal{P}$  with the physical light  $\mathbf{y}'$ .

If the color sensations produced by  $\mathbf{x} + \mathbf{y}$  in the background  $b'$  matches that produced by  $\mathbf{x}' + \mathbf{y}'$  in the same background, then the change of context is additive.

To test *homogeneity* we can use a similar procedure to verify if  $B(\alpha x) = \alpha B(x)$ , for at least a sufficiently large class of coefficients  $\alpha \in \mathbb{R}^+$ .

Unless psycho-physical experiments are carefully carried out to confirm or to confute the linearity hypothesis, this one remains a conjecture for Resnikoff's model.

In what follows the linearity of the transformations  $B$  will be assumed. However, it must be clear that, if the linearity hypothesis turned out to be false, then the mathematical foundation of the results that we are going to discuss will be lost.

## 4 Consequences of the homogeneity axiom on the geometrical structure of $\mathcal{P}$

First of all, we remark that  $\text{GL}^+(\mathcal{P})$  is an open subgroup of  $\text{GL}^+(V)$ , thus it inherits the structure of Lie group from it. Secondly, let us consider the subgroup of  $\text{GL}^+(\mathcal{P})$  defined by  $K = \{B \in \text{GL}^+(\mathcal{P}) : B(x) = x\}$ , where  $x \in \mathcal{P}$  is fixed.  $K$  is called the stabilizer, or isotropy subgroup of  $\text{GL}^+(\mathcal{P})$  and it is known to be *compact*.

This general result of the theory of homogeneous spaces, see e.g. [9], [10], will be fundamental during this section: if a differential manifold  $X$  is  $G$ -homogeneous space w.r.t the action  $\eta : G \times X \rightarrow X$  of the Lie group  $G$ , then the map  $\beta : G/K \rightarrow X$  defined by  $\beta(gK) = \eta(g, x)$  is a diffeomorphism for every fixed  $x \in X$ .

In our case, this means that we can write the diffeomorphic identification:

$$\mathcal{P} \simeq \text{GL}^+(\mathcal{P})/K. \quad (13)$$

By Axiom 1, the transformation  $\alpha \mapsto \alpha x$ ,  $\alpha \in \mathbb{R}^+$ , preserves  $\mathcal{P}$ , it follows that every  $B \in \text{GL}^+(\mathcal{P})$  can be uniquely expressed in the form  $\alpha B'$ , where  $\alpha \in \mathbb{R}^+$  and  $B' \in \text{SL}(\mathcal{P})$ , where  $\text{SL}(\mathcal{P})$  is the subgroup of  $\text{GL}^+(\mathcal{P})$  given by the matrices of this group with determinant 1.

It follows that  $\text{GL}^+(\mathcal{P}) = \mathbb{R}^+ \times \text{SL}(\mathcal{P})$ , i.e.

$$\mathcal{P} \simeq \mathbb{R}^+ \times \text{SL}(\mathcal{P})/K, \quad (14)$$

and that  $K$  can be identified with<sup>2</sup> a compact subgroup of  $\text{SL}(\mathcal{P})$ , which is a subgroup of  $\text{SL}(n, \mathbb{R})$ ,  $n = 1, 2, 3$ , accordingly to Axiom 4.

For the trichromatic case  $n = 3$ , we have  $\dim(\text{SL}(3, \mathbb{R})) = 8$ , so

$$\begin{aligned} 3 &= \dim(\mathcal{P}) = \dim(\mathbb{R}^+ \times \text{SL}(\mathcal{P})/K) \\ &= \dim(\mathbb{R}^+) + \dim(\text{SL}(\mathcal{P})/K) = 1 + \dim(\text{SL}(\mathcal{P})) - \dim(K), \end{aligned} \quad (15)$$

but, since  $\text{SL}(\mathcal{P}) \trianglelefteq \text{SL}(3, \mathbb{R})$ ,  $\dim(\text{SL}(\mathcal{P})) \leq \dim(\text{SL}(3, \mathbb{R})) = 8$ , we have that

$$\dim(\text{SL}(\mathcal{P})) = 2 + \dim(K) \leq 8. \quad (16)$$

As proven in [13], the contractibility of  $\mathcal{P}$  implies the existence of a semi-simple Lie group  $S$  and nilpotent Lie groups  $T_{n_i}$ ,  $i = 1, \dots, k$ ,  $n_i \in \mathbb{N}$ , such that

$$\text{SL}(\mathcal{P}) \simeq S \times (T_{n_1} \times \dots \times T_{n_k}), \quad (17)$$

where the elements of  $T_{n_i}$  are upper triangular matrices of the form

$$T_{n_i} = \left\{ \begin{pmatrix} 1 & & \alpha_{\mu\nu} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : \alpha_{\mu\nu} \in \mathbb{R}^+, 1 \leq \mu < \nu \leq n_i \right\}, \quad (18)$$

<sup>2</sup>In fact, the isotropy subgroup of  $\mathbb{R}^+$  is evidently  $\{1\}$  and  $\mathbb{R}^+/\{1\} \simeq \mathbb{R}^+$ , thus the only non trivial part of the quotient operation is on a compact subgroup  $K$  of  $\text{SL}(\mathcal{P})$ .

whose dimension is  $\dim(T_{n_i}) = \frac{n_i(n_i-1)}{2}$ , thus

$$\dim(\mathrm{SL}(\mathcal{P})) = \dim(S) + \dim(T_{n_1} \times \cdots \times T_{n_k}) \quad (19)$$

and

$$\dim(S) + \sum_{i=1}^k \frac{n_i(n_i-1)}{2} = 2 + \dim(K) \leq 8. \quad (20)$$

The dimensionality constraints discussed above imply that we must consider only the following case:  $2 \leq n_i \leq 4$  for all  $i$ , in fact,  $\dim(T_2) = 1$ ,  $\dim(T_3) = 4$ ,  $\dim(T_4) = 6$ .

The groups  $T_{n_i}$  do not have compact subgroups, so  $K$  must necessarily be a subgroup of the semi-simple group  $S$  and it must verify the constraint (20). The only semi-simple groups  $S$  which are, a priori, coherent with this dimensionality constraint are the following:

$$S = \begin{cases} \emptyset, & \text{with dimension } 0 \\ \mathrm{SL}(2, \mathbb{R}), & \text{with dimension } 3 \\ \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}), & \text{with dimension } 6 \\ \mathrm{SL}(3, \mathbb{R}), & \text{with dimension } 8 \end{cases} \quad (21)$$

however, the last two options cannot be accomplished in practice. To understand the reason why, let us consider the case  $S = \mathrm{SL}(3, \mathbb{R})$ , then (20) implies  $8 + \sum_{i=1}^k \frac{n_i(n_i-1)}{2} = 2 + \dim(K) \leq 8$ , so that  $\dim(K) = 6$ , but this is not possible because the maximal compact subgroup of  $\mathrm{SL}(3, \mathbb{R})$  is  $\mathrm{SO}(3)$ , which has dimension 3, thus the dimension of  $K$  cannot be 6 and we must discard the option  $S = \mathrm{SL}(2, \mathbb{R})$ . This argument can be used to reject also the option  $S = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ , thus we are left only with the first two options: either  $S = \emptyset$  or  $S = \mathrm{SL}(2, \mathbb{R})$ , let us analyze the two cases separately.

If  $S = \emptyset$ , then  $K = \emptyset$  and (20) reduces to  $\sum_{i=1}^k \frac{n_i(n_i-1)}{2} = 2$ , with unique solution  $n_1 = n_2 = 2$ .

By recalling that  $\dim(T_{n_i}) = \frac{n_i(n_i-1)}{2}$ , it follows that  $\dim(T_2) = 1$ , so the triangular matrices  $T_2$  are labeled by one real positive parameter:  $\left\{ T_2 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, p \in \mathbb{R}^+ \right\} \simeq \mathbb{R}^+$ , hence  $\mathrm{SL}(\mathcal{P})/K \simeq T_2 \times T_2 \simeq \mathbb{R}^+ \times \mathbb{R}^+$ , thus:

$$\mathcal{P} = \mathrm{GL}^+(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathrm{SL}(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+. \quad (22)$$

Instead, if  $S = \mathrm{SL}(2, \mathbb{R})$ , then  $\dim(S) = 3$  and (20) becomes

$$\sum_{i=1}^k \frac{n_i(n_i-1)}{2} = \dim(K) - 1 \leq 5. \quad (23)$$

The compact subgroups  $K$  of  $\mathrm{SL}(2, \mathbb{R})$  are either isomorphic to the trivial subgroup  $\left\{ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ , which has dimension 0 (since it has no free parameters), or to the special orthogonal group  $\mathrm{SO}(2)$ :

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}, 0 \leq \vartheta \leq 2\pi \right\}, \quad (24)$$

which has dimension 1, since it can be parameterized by the single variable  $\vartheta$ . In the first case (23) implies

$$\sum_{i=1}^k \frac{n_i(n_i-1)}{2} = 0 - 1 = -1 \leq 5, \quad (25)$$

which cannot be satisfied by any choice of  $n_i$ . Instead, in the second case, eq. (23) implies

$$\sum_{i=1}^k \frac{n_i(n_i-1)}{2} = 1 - 1 = 0 \leq 5, \quad (26)$$

which is satisfied by choosing all the  $n_i = 0$ . This corresponds to

$$\mathcal{P} = \mathrm{GL}^+(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathrm{SL}(\mathcal{P})/K \simeq \mathbb{R}^+ \times \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2). \quad (27)$$

Summarizing: Axioms 1–5 imply that  $\mathcal{P}$  is a homogeneous space equivalent to either

$$\mathcal{P}_1 \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \quad (28)$$

or

$$\mathcal{P}_2 \simeq \mathbb{R}^+ \times \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2). \quad (29)$$

The first characterization embodies the well-known colors spaces with three separated chromatic coordinates, e.g. *LMS*, *RGB*, *XYZ*, and so on, see e.g. [6]. The second characterization is novel and it introduces the Poincaré-Lobachevsky 2-D space of constant negative curvature  $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$  in color theory.

## 5 Selection of invariant Riemannian metrics for the color spaces $\mathcal{P}_1$ and $\mathcal{P}_2$

Once Resnikoff established the only possible geometrical structure of  $\mathcal{P}$  compatible with the Axioms 1-5, he searched for Riemannian metrics on  $\mathcal{P}$  to measure color dissimilarity. As for the geometry of  $\mathcal{P}$ , he uniquely singled out the metrics thanks to an invariance principle.

We recall that a Riemannian metric  $g$  on a differentiable manifold  $M$  of dimension  $n$  is a symmetric positive-definite tensor field of type  $(0, 2)$  on  $M$ , i.e. a correspondence which assigns, smoothly with respect to each point  $x \in M$ , a scalar product  $g_x : T_x M \times T_x M \rightarrow \mathbb{R}$ ,  $(X, Y) \mapsto g_x(X, Y)$  for all  $X, Y \in T_x M$ , the tangent space to  $M$  in  $x$ . A differentiable manifold  $M$  embedded with a Riemannian metric  $g$  is called a Riemannian manifold  $(M, g)$ .

A Riemannian manifold  $(M, g)$  is also a metric space with respect to a distance canonically induced by  $g$  and defined with the help of the length of piecewise regular curves  $\gamma : [0, 1] \rightarrow M$ . If  $(M, g)$  is a connected Riemannian manifold, then, if we define the length of the curve  $\gamma$  as

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(u)\|_{\gamma(u)} du, \quad (30)$$

then the function  $d : M \times M \rightarrow \mathbb{R}^+$  defined by

$$d(x, y) = \inf\{L(\gamma), \gamma : [0, 1] \rightarrow M \text{ piecewise regular, } \gamma(0) = x, \gamma(1) = y\} \quad (31)$$

is a distance on  $M$ , called the Riemannian distance on  $M$  induced by the Riemannian metric  $g$ .

Any piecewise regular curve  $\gamma$  in  $M$  which minimizes the Riemannian distance between a pair of points  $x, y \in M$  is said to be a *geodesic* connecting the two points. Thus, the Riemannian distance  $d(x, y)$  can be defined as the length of any geodesic connecting  $x$  to  $y$ .

In the study of the color space  $M \equiv \mathcal{P}$ . Since Axioms 1-5 determine the geometric structure of  $\mathcal{P}$  as a homogeneous space, Resnikoff claimed that it is totally natural to search for a Riemannian metric on  $\mathcal{P}$  coherent with these axioms.

Let  $x, y \in \mathcal{P}$  be two perceived colors *with respect to the same background*  $b$  and let us perform a change of background, from  $b$  to  $b' \neq b$  modeled by the transformation  $B : \mathcal{P} \rightarrow \mathcal{P}$ ,  $B \in \mathrm{GL}^+(\mathcal{P})$ . Then  $x' = B(x)$  and  $y' = B(y)$  will be the lights  $\mathbf{x}$  and  $\mathbf{y}$  perceived in the new background  $b'$ .

The assumption that the perceptual dissimilarity between  $x$  and  $y$  is the same as that between  $x'$  and  $y'$  can be formalized in the sixth axiom.

**Axiom 6:** If  $d : \mathcal{P} \times \mathcal{P} \rightarrow [0, +\infty)$  is the Riemannian distance on  $\mathcal{P}$  that measures perceptual differences between pairs of perceived lights  $x, y \in \mathcal{P}$ , then  $d$  must satisfy

$$d(B(x), B(y)) = d(x, y), \quad \forall x, y \in \mathcal{P}, \forall B \in \mathrm{GL}^+(\mathcal{P}), \quad (32)$$

i.e.  $d$  must be  $GL^+(\mathcal{P})$ -invariant. From now on such a distance  $d$  will be called a *perceptual color metric*.

The request of  $GL^+(\mathcal{P})$ -invariance singles out in a unique way the perceptual color metric for each of the two geometrical structures of  $\mathcal{P}$ , let us see how. First of all, the diffeomorphism  $B \in GL^+(\mathcal{P})$ ,  $B : \mathcal{P} \rightarrow \mathcal{P}$ , induces a linear isomorphism on the tangent space  $T_x\mathcal{P}$ , for every  $x \in \mathcal{P}$ , via the differential map:  $d_x B : T_x\mathcal{P} \rightarrow T_{B(x)}\mathcal{P}$ , this allows us to introduce the push-forward Riemannian metric  $B_*g$  on  $\mathcal{P}$  via the equation:

$$B_*g_{B(x)}(dB(X), dB(Y)) = g_x(X, Y), \quad \forall B \in GL^+(\mathcal{P}), \forall X, Y \in T_x(\mathcal{P}). \quad (33)$$

Now it is essential to recall two results that we have discussed above: the first is that  $\mathcal{P}$  can conveniently be identified with the homogeneous space  $GL^+(\mathcal{P})/K$  and the second is that  $GL^+(\mathcal{P})$  is transitive on  $\mathcal{P}$ , so if we fix any arbitrary point  $x \in \mathcal{P}$ , we can reach any other point  $y \in \mathcal{P}$  via the action of a transformation of the group, i.e.  $y = Bx$  for some  $B \in GL^+(\mathcal{P})$ .

Let us chose, in particular, as element  $x \in \mathcal{P} \equiv GL^+(\mathcal{P})/K$  the equivalent class to which the identity transformation of  $GL^+(\mathcal{P})$  belongs, i.e. the coset  $K$  itself. In this case, by definition, we have  $B(x) = x \forall B \in K$  and, thanks to the just recalled transitivity, this  $K$ -invariance is independent on the choice of  $x$ , thus eq. (33) can be re-written as

$$B_*g_x(dB(X), dB(Y)) = g_x(X, Y), \quad \forall B \in K, \forall X, Y \in T_x(\mathcal{P}). \quad (34)$$

The quest for perceptual color metrics on  $\mathcal{P}$  is thus reduced to the much simpler task of searching for  $K$ -invariant metrics for the spaces  $\mathcal{P}_1$  and  $\mathcal{P}_2$  defined in eqs. (28) and (29), respectively.

For  $\mathcal{P}_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ , we saw in the previous section that  $K = \emptyset$ , and so  $K$ -invariance does not introduce any constraint. However, the metric must be the sum of  $\mathbb{R}^+$ -invariant metrics on each factor and all  $\mathbb{R}^+$ -invariant metrics on  $\mathbb{R}^+$  are proportional: once we have identified one such metric, all the others are positive multiples of it.

It is clear that a  $\mathbb{R}^+$ -invariant metric on  $\mathbb{R}^+$  is given by  $ds^2 = \left(\frac{dx}{x}\right)^2$ , thus **the general perceptual color metric on  $\mathcal{P}_1$  is**

$$\boxed{ds^2 = \alpha_1 \left(\frac{dx_1}{x_1}\right)^2 + \alpha_2 \left(\frac{dx_2}{x_2}\right)^2 + \alpha_3 \left(\frac{dx_3}{x_3}\right)^2}, \quad \alpha_k \in \mathbb{R}^+, k = 1, 2, 3, \quad (35)$$

which is precisely Stiles' generalization of Helmholtz's metric (this last one corresponds to the particular case  $\alpha_1 = \alpha_2 = \alpha_3 = 1$ ), see e.g. [24].

Let us now turn our attention to  $\mathcal{P}_2 = \mathbb{R}^+ \times SL(2, \mathbb{R})/SO(2)$ , in this case, as discussed in the previous section,  $K = SO(2)$ , so that the tangent space of  $\mathcal{P}_2$  at any  $x \in \mathcal{P}_2$  is

$$T_x\mathcal{P}_2 = \mathbb{R} \oplus T_K SL(2, \mathbb{R})/SO(2), \quad \forall x \in \mathcal{P}_2. \quad (36)$$

Thus, in this case,  $K$ -invariance means invariance under rotations, it follows that the perceptual color metric for this realization of  $\mathcal{P}$  must be the sum of a 1-dimensional and 2-dimensional Euclidean metrics, thus, also for  $\mathcal{P}_2$ , the perceptual color metric is unique up to the selection of units of measure on each Cartesian factor  $\mathbb{R}^+$  and  $SL(2, \mathbb{R})/SO(2)$ .

Let us give an explicit characterization of the metric on  $\mathcal{P}_2$ . First of all, we recall that  $SL(2, \mathbb{R})/SO(2)$  is isomorphic to the set  $\mathcal{N}$  of  $2 \times 2$  real symmetric positive-definite matrices with determinant 1. If we denote with  $x$  a generic  $2 \times 2$  real symmetric positive-definite matrix, then its determinant will be positive but not necessarily unitary, however, writing  $x = \det(x) \frac{x}{\det(x)}$  we can see that

$$\mathcal{P}_2 \simeq \mathbb{R}^+ \times \mathcal{N}, \quad (37)$$

i.e.  $\mathcal{P}_2$  is isomorphic to the set of  $2 \times 2$  real symmetric positive-definite matrices and  $GL^+(\mathcal{P}_2)$  can be identified with  $GL(2, \mathbb{R}) = \mathbb{R}^+ \times SL(2, \mathbb{R})$ .

The action of  $\text{GL}(2, \mathbb{R})$  on  $\mathcal{P}$  is given by  $\text{GL}(2, \mathbb{R}) \times \mathcal{P} \rightarrow \mathcal{P}$ ,  $(A, x) \mapsto Ax A^t \equiv B(x)$ , and the  $\text{GL}(2, \mathbb{R})$ -invariant distance on  $\mathcal{P}$  is<sup>3</sup>

$$ds^2 = \frac{1}{2} \text{Tr}(x^{-1} dx x^{-1} dx), \quad (38)$$

$\text{Tr}$  being the trace function.  $ds^2$  is equivalent to the Rao-Siegel metric [1, 3, 20].

Let us verify the  $\text{GL}(2, \mathbb{R})$ -invariance: first of all notice that  $B(x)^{-1} = (A^t)^{-1} x^{-1} A^{-1}$  and that, by linearity,  $dB(x) = A dx A^t$ . So

$$\begin{aligned} \text{Tr}(B(x)^{-1} dB(x) B(x)^{-1} dB(x)) &= \text{Tr}((A^t)^{-1} x^{-1} A^{-1} A dx A^t (A^t)^{-1} x^{-1} A^{-1} A dx A^t) \\ &= \text{Tr}((A^t)^{-1} x^{-1} dx x^{-1} dx A^t), \end{aligned} \quad (39)$$

by using the cyclic property of the trace we have

$$\begin{aligned} \text{Tr}(B(x)^{-1} dB(x) B(x)^{-1} dB(x)) &= \text{Tr}(A^t (A^t)^{-1} x^{-1} dx x^{-1} dx) \\ &= \text{Tr}(x^{-1} dx x^{-1} dx), \end{aligned} \quad (40)$$

$\forall B \in \text{GL}(2, \mathbb{R})$ , thus confirming the  $\text{GL}(2, \mathbb{R})$ -invariance.

## 6 The analysis of brightness perception in Resnikoff's model

A noticeable application of the perceptual color metric is the analysis of brightness perception. For the sake of clarity, we will divide the analysis in three separated subsections dedicated, respectively, to brightness difference, iso-surface brightness and the dependence of brightness on color.

### 6.1 Brightness difference

If  $\mathbf{x}$  is a physical light and  $\lambda > 0$ , then, in the configuration considered by Resnikoff (stimulus over a uniform background),  $\mathbf{x}$  and  $\lambda \mathbf{x}$  will produce two sensations of color  $x$  and  $\lambda x$ , respectively, which will differ only by their intensity, called *brightness difference*.

Accordingly to the perceptual color distance  $d$ , the brightness difference should be measured by

$$d(x, \lambda x) = \int_x^{\lambda x} ds, \quad (41)$$

where the integral is taken over a geodesic arc connecting  $x$  to  $\lambda x$  and  $d(\lambda x) = \lambda dx$ . For  $\mathcal{P}_1$ , i.e.

if we consider the distance (35) given by  $ds^2 = \sum_{k=1}^3 \alpha_k \left( \frac{dx_k}{x_k} \right)^2$ , we have:

$$d(x, \lambda x) = \int_1^\lambda \sqrt{\sum_{k=1}^3 \alpha_k \frac{dt}{t}} = \sqrt{\sum_{k=1}^3 \alpha_k} |\log(\lambda)|, \quad (42)$$

i.e. the difference in brightness is proportional to the logarithm of intensity difference, which is exactly Weber-Fechner's law [24].

For  $\mathcal{P}_2$ , i.e. if we consider the metric (38), then we have:

$$d(x, \lambda x) = \frac{1}{\sqrt{2}} \int_1^\lambda \sqrt{\text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{dt}{t}} = |\log(\lambda)|, \quad (43)$$

again, the coherence with Weber-Fechner's law is verified.

<sup>3</sup>Resnikoff considered a normalized metric without the coefficient 1/2. Here, we re-introduce this factor to show more explicitly the correspondence with a result of the second half of this two-part paper.

The problem gets much more complicated if we consider the brightness difference induced by light stimuli which differ also with respect to their chromatic attributes (hue and saturation), and not only in intensity.

If  $b(x, y)$  is the brightness difference between two arbitrary elements  $x, y \in \mathcal{P}$ , then  $b(x, y)^2 \leq d(x, y)^2$  since  $b$  measures only the brightness difference, while  $d$  measures also the hue and saturation diversity between perceived colors.

To simplify the notation, it is convenient to write the two geometrical structures of  $\mathcal{P}$  in a compact way as  $\mathcal{P} \simeq \mathbb{R}^+ \times \mathcal{M}$ , where

$$\mathcal{M} = \begin{cases} \mathbb{R}^+ \times \mathbb{R}^+ & \text{for } \mathcal{P}_1 \\ \text{SL}(2, \mathbb{R})/\text{SO}(2) & \text{for } \mathcal{P}_2. \end{cases} \quad (44)$$

Then  $x \in \mathcal{P}$  can be written as

$$x \equiv (x_1, x_2, x_3) = (\xi, u), \quad \xi \in \mathbb{R}^+ \text{ and } u \equiv (u_1, u_2) \in \mathcal{M}. \quad (45)$$

Let us first discuss the case  $\mathcal{M} = \mathbb{R}^+ \times \mathbb{R}^+$ . A convenient explicit parameterization choice that allows us to exhibit the product structure of  $\mathcal{P}_1$  by splitting the Riemannian distance in two terms, each of which corresponds to one Cartesian factor, is the following:

$$\begin{cases} \xi = (x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3})^{1/3} \text{ (a sort of geometric average of the } x_i \text{'s)} \\ u_1 = \frac{x_1^\sigma}{\xi} \\ u_2 = \frac{x_2^\sigma}{\xi}, \end{cases} \quad (46)$$

where  $\sigma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{3}$  is the arithmetic average of the  $\alpha_i$ 's.

By direct computation, it can be verified that, in these coordinates, the distance (35) verifies the equation

$$\begin{aligned} \sigma^2 ds^2 = 3\sigma \left( \frac{d\xi}{\xi} \right)^2 + \left[ \left( \alpha_1 + \frac{\alpha_1^2}{\alpha_3} \right) \left( \frac{du_1}{u_1} \right)^2 + \frac{2\alpha_1\alpha_2}{\alpha_3} \left( \frac{du_1}{u_1} \right) \left( \frac{du_2}{u_2} \right) \right. \\ \left. + \left( \alpha_2 + \frac{\alpha_2^2}{\alpha_3} \right) \left( \frac{du_2}{u_2} \right)^2 \right]. \end{aligned} \quad (47)$$

The effect of a brightness change  $x \mapsto \lambda x$ ,  $\lambda > 0$ , on the parameterization chosen above is the following:

$$(x_1, x_2, x_3) \mapsto (\lambda x_1, \lambda x_2, \lambda x_3) \implies \xi \mapsto \lambda^{\frac{\alpha_1 + \alpha_2 + \alpha_3}{3}} \xi = \lambda^\sigma \xi, \quad (48)$$

which implies

$$u_j \mapsto \frac{\lambda^\sigma x_j^\sigma}{\lambda^\sigma \xi} = u_j, \quad j = 1, 2, \quad (49)$$

i.e. the coordinates  $u_j$  are independent on the transformation  $x \mapsto \lambda x$ . Since the dependence of  $ds^2$  is split into  $\xi$  and  $u_1, u_2$ , and since, as just proven, a brightness change only affects the  $\xi$  coordinate, the parameterization (46) allows us to express a brightness change entirely in terms of the  $\xi$  coordinate.

Let us now consider  $\mathcal{P}_2$ , identified with the set of positive definite  $2 \times 2$  real symmetric matrices  $\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$ . In this case  $\mathcal{M} = \text{SL}(2, \mathbb{R})/\text{SO}(2)$  and we will show that a suitable parameterization is the following:

$$\begin{cases} \xi = \det(x)^{1/2} \\ u_1 = \frac{x_3}{x_2} \\ u_2 = \frac{\xi}{x_2}. \end{cases} \quad (50)$$

By direct computation, we obtain

$$x = \xi \begin{pmatrix} \frac{u_1^2 + u_2^2}{u_2} & \frac{u_1}{u_2} \\ \frac{u_1}{u_2} & \frac{1}{u_2} \end{pmatrix} \quad (51)$$

and the general perceptual Riemannian distance becomes

$$ds^2 = \alpha_1 \left( \frac{d\xi}{\xi} \right)^2 + \alpha_2 \left[ \frac{(du_1)^2 + (du_2)^2}{u_2^2} \right], \quad (52)$$

$\alpha_1, \alpha_2 > 0$ .

Let us consider a brightness change and its effect on this parameterization: since  $\det(\lambda x) = \lambda^2 \det(x)$ , then a change of brightness  $x \mapsto \lambda x$  induces the transformation  $\xi \mapsto \lambda \xi$ , which explains the presence of the square root in the definition of  $\xi$ . Thanks to this observation, directly from the formulae of  $u_1, u_2$  in (50), it follows that these coordinates are unaffected by a change of brightness. Thus, again, a brightness difference is expressed only in terms of the  $\xi$  coordinate when the suitable parameterization (50) is chosen.

Due to these considerations, in the Resnikoff model, it is natural to define the *relative brightness* between two perceived colors  $x, y \in \mathcal{P}$ ,  $x = (\xi, u)$ ,  $y = (\eta, v)$ , in the coordinate system (46) or (50), as follows

$$b(x, y) = \chi \int_{\eta}^{\xi} \frac{dt}{t} = \chi \log \frac{\xi}{\eta}, \quad (53)$$

where  $\chi$  is a constant introduced to make this definition coherent with the ones given above in eqs. (42) and (43), when  $x$  and  $y$  are proportional via a positive constant.

## 6.2 Surfaces of constant brightness

The equations that we introduced so far have been used by Resnikoff to study the geometry of surfaces of constant brightness, or iso-brightness surfaces.

Let  $c \equiv (\gamma, v) \in \mathcal{P}$  be a reference color and write with  $b(x, c)$  the brightness of  $x \equiv (\xi, u)$  w.r.t. that of  $c^4$ .

As usual, we start with  $\mathcal{P}_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ : according to (53), the brightness of  $x$  relative to  $c$  is

$$b(x, c) = \chi \log \frac{\xi}{\gamma}, \quad \chi \in \mathbb{R} \text{ constant}, \quad (54)$$

so the surface of constant brightness  $\beta$  is the set of  $x \in \mathcal{P}$  satisfying  $b(x, c) = \beta$ , i.e. taking into account (46),

$$\chi \log \left( \frac{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}{\gamma} \right)^{1/3} = \beta \iff x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = \gamma e^{3\beta/\chi} = \text{constant}. \quad (55)$$

These are the same surfaces of constant brightness found by Stiles, which are isomorphic to  $\mathcal{M} \simeq \mathbb{R}^+ \times \mathbb{R}^+$ , coherently with the fact that  $\mathcal{P}_1$  is the standard colorimetric space.

The most interesting results are provided by the analysis of  $\mathcal{P}_2 = \mathbb{R}^+ \times \text{SL}(2, \mathbb{R})/\text{SO}(2)$ : this time, taking into account (50), we find<sup>5</sup>

$$\chi \log \frac{\det(x)^{1/2}}{\gamma} = \beta \iff \det(x) = \gamma^2 e^{2\beta/\chi} = \text{constant}. \quad (56)$$

<sup>4</sup>A remark is necessary about this sentence: even if Resnikoff does not explicitly mention the presence of a background in [13], if we want to be coherent with the formalism introduced so far, we must interpret  $c$  and  $x$  as the color sensations induced by two *light stimuli embedded in the same fixed uniform background*.

<sup>5</sup>In Resnikoff's paper [13] there is a typo:  $\gamma$  appears without the square.



Since  $x = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$ ,  $\det(x) = x_1x_2 - x_3^2$  and so the iso-brightness equation for the model  $\mathcal{P}_2$  is  $x_1x_2 - x_3^2 = \text{constant}$ , which is the equation of a *hyperboloid*.

Interestingly, both Yilmaz [25] and Weinberg [23] show iso-brightness surfaces which are in favor of the model  $\mathcal{P}_2$ , rather than a flat surface.

### 6.3 Dependence of brightness on color

Here  $\beta$  will not be considered as a constant anymore, but a function of the color variable  $x$ . If we replace  $\beta$  with  $\beta(x)$  in the formula  $\chi \log \frac{\det(x)^{1/2}}{\gamma} = \beta$ , i.e. we write  $\frac{\chi}{2} \log \det(x) - \chi \log \gamma = \beta(x)$ , and we choose for simplicity  $\gamma = \chi/2 = 1$ , then we have

$$\beta(x) = \log \det(x) \iff e^{\beta(x)} = \det(x). \quad (57)$$

$x \in \mathcal{P}$  is a positive-definite symmetric real  $2 \times 2$  matrix, thus it can be diagonalized with two positive eigenvalues on the diagonal and, moreover, we can consider the principal square root  $x^{1/2} \in \mathcal{P}$  of  $x$ , i.e. the only positive-definite symmetric real  $2 \times 2$  matrix such that  $x^{1/2}x^{1/2} = x$ .  $x^{1/2}$  is invertible, its inverse matrix will be denoted with  $x^{-1/2}$ .

If we consider another  $y \in \mathcal{P}$ , then, if  $\mathcal{K}$  is the identity  $2 \times 2$  matrix, we have

$$\begin{aligned} x + y &= x^{1/2}x^{1/2} + \mathcal{K}y\mathcal{K} = x^{1/2}x^{1/2} + x^{1/2}x^{-1/2}yx^{-1/2}x^{1/2} \\ &= x^{1/2}[x^{1/2} + x^{-1/2}yx^{-1/2}x^{1/2}] = x^{1/2}[\mathcal{K} + x^{-1/2}yx^{-1/2}]x^{1/2}, \end{aligned} \quad (58)$$

thus, using eq. (57), we can write

$$e^{\beta(x+y)} = \det(x+y) = \det(x^{1/2}[\mathcal{K} + x^{-1/2}yx^{-1/2}]x^{1/2}). \quad (59)$$

If we set  $u = x^{-1/2}yx^{-1/2} \in \mathcal{P}$ , then

$$\begin{aligned} e^{\beta(x+y)} &= \det(x^{1/2}[\mathcal{K} + u]x^{1/2}) = \det(x^{1/2})\det(\mathcal{K} + u)\det(x^{1/2}) \\ &= \det(x^{1/2})\det(x^{1/2})\det(\mathcal{K} + u) = \det(x^{1/2}x^{1/2})\det(\mathcal{K} + u) \\ &= \det(x)\det(\mathcal{K} + u). \end{aligned} \quad (60)$$

$u$  is positive definite and symmetric by construction, thus its eigenvalues  $\lambda_1, \lambda_2$  are positive. The determinant is invariant under changes of basis, thus we can consider the basis where  $u$  is diagonal and compute:

$$\det(\mathcal{K} + u) = \det \begin{pmatrix} 1 + \lambda_1 & 0 \\ 0 & 1 + \lambda_2 \end{pmatrix} = 1 + (\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 1 + \text{Tr}(u) + \det(u),$$

so, thanks to the cyclic property of the trace and Binet's theorem, we get

$$\begin{aligned} e^{\beta(x+y)} &= \det(x)(1 + \text{Tr}(u) + \det(u)) \\ &= \det(x) \left( 1 + \text{Tr}(x^{-1/2}yx^{-1/2}) + \det(x^{-1/2}yx^{-1/2}) \right) \\ &= \det(x) \left( 1 + \text{Tr}(x^{-1/2}x^{-1/2}y) + \det(x^{-1/2})\det(x^{-1/2})\det(y) \right) \\ &= \det(x) \left( 1 + \text{Tr}(x^{-1}y) + \det(x^{-1})\det(y) \right) \\ &= \det(x) \left( \frac{1}{\det(y)} + \frac{1}{\det(y)}\text{Tr}(x^{-1}y) + \frac{1}{\det(x)} \right) \det(y) \\ &= e^{\beta(x)} \left( \frac{1}{\det(y)} + \frac{1}{\det(y)}\text{Tr}(x^{-1}y) + \frac{1}{\det(x)} \right) e^{\beta(y)}, \end{aligned} \quad (61)$$

where we used again eq. (57) in the last step. By taking the logarithm at both sides and rearranging the terms we obtain

$$\beta(x+y) = \beta(x) + \beta(y) + \log\left(\frac{1}{\det(x)} + \frac{1}{\det(y)} + \frac{\text{Tr}(x^{-1}y)}{\det(y)}\right). \quad (62)$$

Since every mapping in  $\text{GL}(\mathcal{P}) = \mathbb{R}^+ \times \text{SL}(2, \mathbb{R})/\text{SO}(2)$  can be written as a composition of transformations like this one  $x \mapsto B_a(x) = a^{1/2}xa^{1/2}$  for suitable  $a \in \mathcal{P}$ , by using again Binet's theorem, we have that  $\det(a^{1/2}xa^{1/2}) = \det(a) \det(x)$ , so

$$\beta(B_a(x)) = \beta(a^{1/2}xa^{1/2}) = \log(\det(a^{1/2}xa^{1/2})) = \log \det(a) + \log \det(x), \quad (63)$$

i.e.

$$\beta(B_a(x)) = \beta(a) + \beta(x), \quad (64)$$

which shows that under a change of background, the brightness of  $x$  with respect to a standard color  $c$  (viewed w.r.t the same background) is modified by an additive constant which depends only on the change of background, and not on the color  $x$ .

The law (64), to our knowledge, has not yet been validated by psycho-physical experiments. This shows another example of how psycho-physics and applied mathematics must be intertwined in order to study color properties.

## 7 Conclusions

This first half of a two-part paper deals with the space of perceived colors  $\mathcal{P}$ . After Schrödinger's 1920 work on theoretical colorimetry, there have been only very few attempts to investigate the structure of  $\mathcal{P}$ . Here we have decided to put our attention on the particularly noticeable work of H.L. Resnikoff, who added to Schrödinger's axioms a new one: the hypothesis of homogeneity of  $\mathcal{P}$  with respect to a group  $G$  of suitable linear transformations.

By using standard, yet very refined, methods from differential geometry and harmonic analysis, Resnikoff was able to show that the new axiomatic set implies that  $\mathcal{P}$  can only have two structures: one is  $\mathcal{P}_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ , which is isomorphic to the well-known trichromatic color spaces commonly used in color science, while the other,  $\mathcal{P}_2 = \mathbb{R}^+ \times \text{SL}(2, \mathbb{R})/\text{SO}(2)$ , is completely new.

A further axiom, the invariance of color metric on  $\mathcal{P}$  with respect to the action of  $G$ , leads to the selection of a unique Riemannian metric on  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .

Even if very few scientists paid attention to the realization  $\mathcal{P}_2$  of the perceived color space, recent theoretical and psychophysical results tend to indicate that this geometrical structure is much closer to our color perception than the flat one represented by  $\mathcal{P}_1$ .

The contribution of this first half of the two-part paper is three-fold. First, we wanted to recast Resnikoff's work into a more modern colorimetric setting, clarifying many mathematical findings of Resnikoff's work to make them more accessible.

Second, we have pointed out that the model loses its mathematical foundation if the transformations of  $G$  are not linear. To verify linearity, we have proposed a psycho-physical test that we are currently performing. In spite of its simplicity, lengthy and careful calibrations are needed to obtain precise results. Moreover, we have pointed out a quite subtle issue about the real possibility to pass from one arbitrary fixed perceived color  $x \in \mathcal{P}$  to any other one  $y \in \mathcal{P}$  with one single *physical* change of background. So, the psycho-physical interpretation of Resnikoff's fifth axiom is still open to debate.

Finally, what underlined above serves as a firm motivation for the second half of this two-part of the paper, in which the theory of Jordan algebras will allow us circumventing the explicit specification of the group  $G$  and showing a hidden relationship between color perception and quantum mechanics in the real setting. In fact, within this framework, color description does not require the explicit specification of an external group action, instead change of background illumination appear spontaneously as a consequence of the quantum setting that will be detailed.

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