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# RIEMANN'S NON-DIFFERENTIABLE FUNCTION IS INTERMITTENT

ALEXANDRE BORITCHEV, DANIEL ECEIZABARRENA, AND VICTOR VILAÇA DA ROCHA

ABSTRACT. Riemann's non-differentiable function, introduced in the middle of the 19th century as a purely mathematical pathological object, is relevant in the study of the binormal flow, as shown recently by De La Hoz and Vega. From this physical point of view, the function is therefore related to turbulent phenomena.

We rigorously study the fine intermittent nature of this function on small scales. To do so, we define the flatness, an analytic quantity measuring it, in two different ways: one in the physical space and the other one in the Fourier space. We prove that both expressions diverge logarithmically as the relevant scale parameter tends to 0.

The regularity of Riemann's non-differentiable function is a classical subject, heavily linked to its small-scale behaviour. However, our subtle asymptotics for a classical hydrodynamical quantity are new and sharp.

## 1. INTRODUCTION AND MOTIVATION

Riemann's non-differentiable function

$$f(t) = \sum_{n=1}^{\infty} \frac{\sin(n^2 t)}{n^2}, \quad t \in [0, 2\pi].$$

is a celebrated example of a continuous but almost nowhere differentiable function. It was introduced by the eponymous mathematician in the 1860s [35], and since then it has been widely studied from an analytic perspective [12, 19, 20, 22, 25]. Interesting by itself as one of the first examples of an analytically pathological function, there is evidence that it appears naturally in a physical context. Indeed, according to [23], its complex-valued analogue

$$R(t) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n^2 t}}{n^2}, \quad t \in [0, 1], \quad (1)$$

seems to be a fair representative of some particular temporal trajectories in the dynamics of vortex filaments. These are described by the Vortex Filament equation, also known as the Binormal Flow equation, a nonlinear PDE strongly linked to the nonlinear Schrödinger equation. These dynamics are associated to turbulent phenomena, so the study of properties of  $R$  that are related to turbulence is quite natural. The objective of our work is to give sharp quantitative estimates that measure one of these, the intermittency of  $R$ , following the ideas of Frisch [16, Chapter 8].

In Subsections 1.1 to 1.3, we introduce the physical setting where Riemann's non-differentiable function appears naturally, as well as its relationship with the concept of intermittency in the

context of turbulence. After that, in Subsection 2.1 we set notation, and in Subsection 2.2 we define two quantities measuring intermittency. One of them is based on frequency high-pass filtering; the other one makes use of the so-called structure functions, a popular tool in the study of turbulence. Our main result is the following:

**Theorem.** *Riemann's non-differentiable function (1) is intermittent from the point of view of both high-pass filtering and structure functions.*

In Subsection 2.2, a quantitative and sharp version of this result is given in Theorem 1.

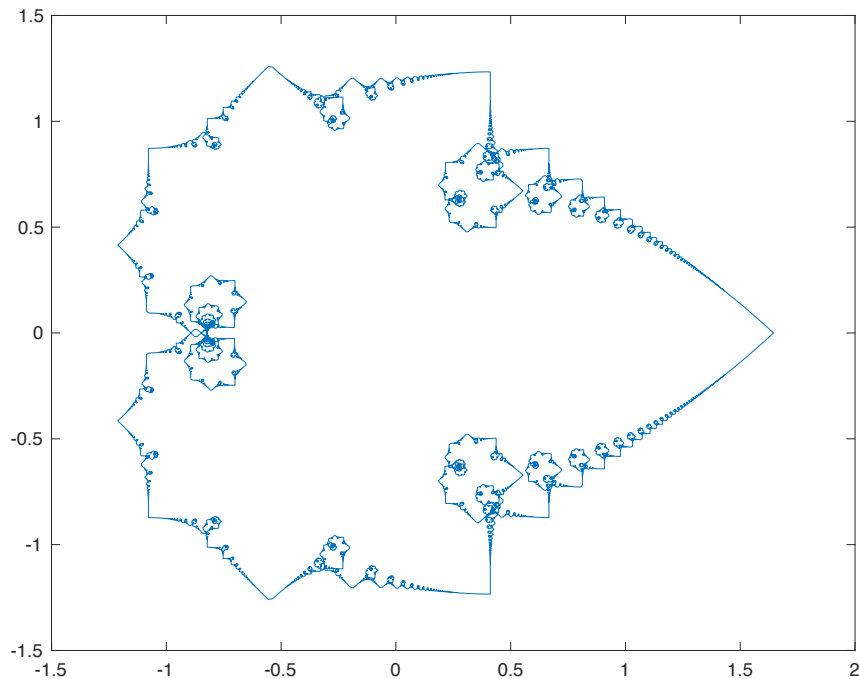


FIGURE 1. The image of Riemann's non-differentiable function  $R$ .

**1.1. The Vortex Filament Equation and Riemann's non-differentiable function.** The vortex filament equation (VFE) is

$$\mathbf{X}_t = \mathbf{X}_x \wedge \mathbf{X}_{xx}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (\text{VFE})$$

where  $\mathbf{X} = \mathbf{X}(t, x) \in \mathbb{R}^3$  is a curve parametrized by arclength  $x$  and time  $t$ . Here,  $\wedge$  is the usual cross product. Considering the Frenet-Serret frame  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ , where  $\mathbf{T} = \mathbf{X}_x$  is the tangent vector,  $\mathbf{N} = \mathbf{T}_x / \|\mathbf{T}_x\|$  is the normal vector and  $\mathbf{B} = \mathbf{T} \wedge \mathbf{N}$  is the binormal vector, the equation can be equivalently written as

$$\mathbf{X}_t = \kappa \mathbf{B}.$$

This is why (VFE) is also called Binormal Flow equation. Here,  $\kappa(t, x)$  is the curvature which, together with the torsion  $\tau(t, x)$ , satisfies the Frenet-Serret relations

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}_x = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

The equation (VFE), first obtained by Da Rios in 1906 [11], and rederived in 1965 by Arms and Hama [1] from the Euler equation, is an asymptotic model for the dynamics of a space curve: the vortex filament. A remarkable result about this equation was given by Hasimoto in 1972 [21], where he proved that the transformation

$$\Psi(t, x) = \kappa(t, x) e^{i \int_0^x \tau(t, \sigma) d\sigma} \quad (2)$$

solves the nonlinear Schrödinger equation

$$i\Psi_t + \Psi_{xx} + \frac{1}{2} (|\Psi|^2 + A(t)) \Psi = 0. \quad (\text{NLS})$$

Here,  $A$  is a real, time dependent function that depends on  $\kappa, \tau$  and their derivatives (see [2]). This way, the Hasimoto transformation defines a correspondence between (VFE), the equation of the original problem, and the well-known nonlinear Schrödinger equation. This supplies a method to find solutions of (VFE), since they can be produced from particular solutions to (NLS), as explained in [2].

Self-similar solutions, studied by Gutiérrez, Rivas and Vega in [18], and later by Banica and Vega in [2], are of the form

$$\mathbf{X}(t, x) = \sqrt{t} G\left(x/\sqrt{t}\right), \quad t > 0, x \in \mathbb{R},$$

where  $G$  is a regular function defined on  $\mathbb{R}$ , obtained from the tangent vector of  $\mathbf{X}$  in the Frenet-Serret system. These solutions are regular for  $t > 0$ , but they develop a singularity in the shape of a corner at time 0, which means that  $\mathbf{X}(0, x)$  represents the union of two non-parallel half-lines, a V-shaped initial datum. This is the behaviour of filaments of air in a delta wing during a flight [24]. Also, a corner can be created after a reconnection of two different filaments, as can be observed in the rear of a plane or in the study of superfluid helium [33]. In [3], Banica and Vega generalise the results for the initial V-shaped datum to any curve which is regular except at one point where it has a corner, and they show existence and uniqueness of solutions under suitable conditions. Later, in [4], they analyse initial data given by polygonal lines, extending the one-corner problem to a many-corner one, where they also prove existence and uniqueness.

We are interested in a similar generalisation, when the initial datum is given by a closed, regular and planar  $M$ -sided polygon, studied in [23]. This means that  $\tau(0, x) = 0$ , so the initial datum in (NLS) is  $\psi(0, x) = \kappa(0, x)$ . Through the Hasimoto transformation (2), a corner-shaped singularity for the Vortex Filament equation (VFE) becomes a Dirac delta in the curvature, so the initial datum is given by a periodic sum of Dirac deltas,

$$\Psi_M(0, x) = \sum_{k \in \mathbb{Z}} \delta\left(x - 2\pi \frac{k}{M}\right), \quad x \in \mathbb{R}. \quad (3)$$

In [23], the Galilean invariance of (NLS) is used to determine

$$\Psi_M(t, x) = \widehat{\Psi}_M(t, 0) \sum_{k \in \mathbb{Z}} e^{-i(Mk)^2 t + iMkx}.$$

Moreover, evaluation at scaled rational times  $t_{p,q} = (2\pi/M^2)(p/q)$ , where  $p, q \in \mathbb{Z}$  are coprime, gives

$$\Psi_M(t_{p,q}, x) = \frac{2\pi}{Mq} \Psi_M(t_{p,q}, 0) \sum_{k \in \mathbb{Z}} \sum_{m=0}^{q-1} G(-p, m, q) \delta\left(x - \frac{2\pi k}{M} - \frac{2\pi m}{Mq}\right), \quad (4)$$

where  $G$  stands for the generalised quadratic Gauss sums

$$G(a, b, c) = \sum_{m=0}^{c-1} e^{2\pi i(am^2 + bm)/c}, \quad a, b \in \mathbb{Z}, \quad c \in \mathbb{N}.$$

The solution (4) is a mathematical representation of the Talbot effect, a celebrated optical phenomenon discovered in 1836 [34] and rigorously described for the first time in 1881 [32]. This effect is based on the diffraction of waves, and it happens when light crosses a diffraction grating, a plate with uniformly distributed parallel vertical slits. Interference creates narrow copies of the grating in every rational multiple of a fixed distance, where the pattern is reproduced exactly. Applications have been discovered in several fields such as imaging or lithography, and accordingly, further theoretical study has been carried out (see, for instance, [5, 31]). It is visually represented by a Talbot carpet [6].

The structure of the Talbot effect matches the evolution of the polygonal filament if slits of the grating and corners of the filament are identified. Indeed, since according to the Hasimoto transformation  $\Psi_M$  shares support with the curvature of  $\mathbf{X}_M$ , from (4) one can deduce that the corresponding filament  $\mathbf{X}_M(t_{p,q}, x)$  is an  $Mq$ -sided, not necessarily planar polygon. To see this, we must recall that the filament is first defined in  $(0, 2\pi)$  and then extended periodically to  $\mathbb{R}$ . In (4), a value of  $k$  corresponds to the interval  $(2\pi k/M, 2\pi(k+1)/M)$ , and when  $k$  is fixed, the sum in  $m$  goes from 0 to  $q-1$  because the value  $q$  corresponds to the interval with  $k+1$  and  $m=0$ . Since  $(0, 2\pi)$  is covered by the values  $k=0, 1, \dots, M-1$ , and for each of them every  $m=0, 1, \dots, q-1$  corresponds to a Dirac delta, the filament has  $Mq$  corners at time  $t_{p,q}$ . The resulting polygon is in general not planar.

Also in [23], the authors consider the temporal trajectories of the corners of the initial filament, represented by  $\mathbf{X}_M(t, 2\pi k/M)$  for every fixed  $k \in \mathbb{Z}$ . However, they only study  $\mathbf{X}_M(t, 0)$ , since the translation invariance of (NLS) together with the space periodicity of the initial datum (3) imply that all of them have the same structure. Then, they show numerically that  $\mathbf{X}_M(t, 0)$  is extremely similar to the image of the function

$$\phi(t) = \sum_{k \in \mathbb{Z}} \frac{e^{-4\pi^2 ik^2 t} - 1}{-4\pi^2 k^2},$$

and after a proper rescaling depending on the number of initial sides  $M$ , it seems to converge to  $\phi$  as  $M \rightarrow \infty$ . Therefore,  $\phi$  is intrinsically related to the evolution of vortex filaments following the binormal flow, and its geometric importance as a representative of a physical trajectory is

highlighted. This function is a slight modification of Riemann's non-differentiable function (1), since

$$\phi(t) = -\frac{i}{2\pi^2} R(-2\pi t) + it + \frac{1}{12}.$$

This gives evidence of an intrinsic physical and geometric nature of Riemann's non-differentiable function.

Some geometric results were obtained in [12] and also in [10], where the box-counting dimension of some trajectories related to  $R$  and other similar Fourier series were analysed. Further results regarding  $\phi$  were obtained by the second author in [13, 14, 15]. More precisely, in [14] the Hausdorff dimension of the trajectory represented by its image is analysed, and in [15] it is proved that this trajectory does not have a tangent anywhere. On the physical side, the fact that the trajectories of vortex filaments following the binormal flow are related to turbulent phenomena justifies our interest for the relationship between Riemann's non-differentiable function and turbulence-related properties, such as intermittency and multifractality.

**1.2. Turbulence and intermittency.** In a series of papers in 1941, Kolmogorov gave a precise description of the properties of turbulent flows, for which the Reynolds number is large enough. These articles, usually referred to as the K41 Theory, are seminal for the theory of turbulence. The idea is that at small scales and away from the boundaries, the behaviour of a turbulent flow is universal: it does not depend on the fluid under consideration, nor on the geometry of the particular mechanisms producing the turbulence. The K41 Theory assumes that the flows are homogeneous (statistically invariant under translations) and isotropic (statistically invariant under rotations). Among the results of these papers, K41 hypotheses imply the self-similarity of both the high-pass filters of the velocity and the velocity increments.

Although this theory gives remarkable results, such as the Kolmogorov four-fifths law [28], two main issues have been pointed out: the lack of universality, emphasized by Landau in 1944 and reported later in [29] (it was just a footnote in the first version of the book), and the lack of self-similarity for the flows at small scales (in the dissipation range), highlighted by several experiments. The intermittency of a flow is formally a measure of this lack of self-similarity, and several ways to define it can be found in the literature. In [16, Sections 8.2 and 8.3], using the probabilistic description of turbulence, Frisch gives two definitions of intermittency for random stationary functions.

The first idea is to deal with high-pass filters in the Fourier space. Let the velocity of the flow  $v$  be a stationary random function, which can be expressed in terms of its Fourier transform

$$v(t) = \int_{\mathbb{R}} e^{i\omega t} \hat{v}(\omega) d\omega.$$

Deleting the low frequencies, the high-pass filtered signal  $v_{>\Omega}$  is defined by

$$v_{>\Omega}(t) = \int_{|\omega|>\Omega} e^{i\omega t} \hat{v}(\omega) d\omega, \quad \text{for } \Omega > 0.$$

This filter is used to define the flatness

$$F(\Omega) = \frac{\langle (v_{>\Omega}(t))^4 \rangle}{\langle (v_{>\Omega}(t))^2 \rangle^2}, \quad (5)$$

a tool to measure the intermittency of a signal. Here  $\langle \cdot \rangle$  denotes an ensemble average, defined as the mean over many realizations of the flow with different initial conditions and forcing. This could as well be a temporal average if we assume the ergodicity of the flow, or a spatial one if we use its homogeneity. In the study of turbulence, these three definitions are usually considered to be equivalent. We remark that, due to the stationarity of the signal  $v$ , the flatness  $F$  does not depend on time.

The key point in this definition is that  $F$  is constant for self-similar flows. Indeed, if  $v(t)$  has self-similar high-pass filters with a scaling exponent  $h$ , then  $v_{>\lambda\Omega} \stackrel{\text{law}}{=} \lambda^h v_{>\Omega}$  for any  $\lambda > 0$ . As a consequence,  $F(\lambda\Omega) = F(\Omega)$  for every  $\lambda > 0$ . Therefore, the more the flatness  $F$  grows with  $\Omega$ , the less  $v(t)$  is self-similar at small scales, and thus by definition the more the flow  $v(t)$  is intermittent.

**Remark 1.1.** *Frisch [16, Section 8.2] suggests that, in the setting above, the inverse of the flatness measures the time the studied signal is “on”. This remark leads to a more visual definition of intermittency: a signal is said to be **intermittent** if it displays activity during only a fraction of time, which decreases with the scale under consideration.*

The second idea for a definition of intermittency is to use the structure functions, which play a key role in the literature of turbulence (see [16, 25] and also [7, 8, 9] for rigorous results for Burgers turbulence). Let  $v$  be the velocity of a turbulent flow. For  $r, l \in \mathbb{R}^3$ , we define the longitudinal velocity increment  $\delta v(r, l)$  between two points separated by  $l$  by

$$\delta v(r, l) = (v(r + l) - v(r)) \cdot \frac{l}{\|l\|},$$

For  $p \geq 1$  and  $\ell > 0$ , we can now define the  $p^{\text{th}}$  structure function  $S_p$  by

$$S_p(\ell) = \langle (\delta v(\ell))^p \rangle, \quad (6)$$

where there is no dependence on  $r$  by homogeneity of the turbulence. Moreover, the above only depends on  $\ell = \|l\|$  by isotropy. Then, the flatness with respect to structure functions, which we call  $G$ , is

$$G(\ell) = \frac{S_4(\ell)}{S_2(\ell)^2} = \frac{\langle (\delta v(\ell))^4 \rangle}{\langle (\delta v(\ell))^2 \rangle^2}. \quad (7)$$

Once again, the key point is that  $G$  is constant for self-similar flows. Indeed, if  $\delta v(\ell)$  is self-similar with scaling exponent  $h$ , then  $\delta v(r, \lambda\ell) \stackrel{\text{law}}{=} \lambda^h \delta v(r, \ell)$  for any  $\lambda > 0$ . We obtain thus  $G(\lambda\ell) = G(\ell)$  for all  $\lambda > 0$ . Therefore, we have the same characterization as for the flatness  $F$ : the more  $G$  grows while  $\ell$  goes to 0, the further  $v$  is from being self-similar at small scales, and thus by definition the more  $v$  is intermittent.

**Remark 1.2.** *Through the hypothesis of the self-similarity of the velocity increments, the K41 Theory gives the exponent  $h = 1/3$  and thus*

$$S_p(\ell) \underset{\ell \rightarrow 0}{\sim} \ell^{\frac{p}{3}}.$$

*Several experiments, some of which are listed in [16], show that the above does not hold when  $p \neq 3$  (in particular when  $p$  is large). The Kolmogorov four-fifths law is concerned with the special case  $p = 3$ , treated without the hypothesis of self-similarity: assuming homogeneity, isotropy and finite dissipation, then*

$$S_3(\ell) = -\frac{4}{5}\varepsilon\ell,$$

*where  $\varepsilon$  is the mean energy dissipation per unit mass.*

We mentioned that Riemann's non-differentiable function represents a trajectory of the filament in several experiments related to turbulent flows. The goal of this paper is thus to analyse the intermittency of Riemann's function adapting the two definitions of flatness given above.

**1.3. Turbulence, multifractality and Riemann's non-differentiable function.** Multifractality is a concept deeply related to fully developed turbulence in fluids. Experiments have shown that the velocity of a turbulent fluid follows a very erratic pattern, so a classification of its analytic regularity might offer relevant information about its evolution. Assuming the latter is described by some function  $f$ , such a classification is offered by the so-called spectrum of singularities, which we define in the following lines.

For  $\alpha > 0$ , a function  $f$  is said to be locally  $\alpha$ -Hölder regular in a point  $t_0$ , which is denoted  $f \in C^\alpha(t_0)$ , if there exists a polynomial  $P_{t_0}$  of degree at most  $\lfloor \alpha \rfloor$  such that  $|f(t_0 + h) - P_{t_0}(h)| \leq C|h|^\alpha$  for small enough  $h \in \mathbb{R}$ . Let the Hölder exponent of  $f$  at  $t_0$  be

$$H_f(t_0) = \sup \{ \alpha : f \in C^\alpha(t_0) \}.$$

Then, the spectrum of singularities is defined for each  $\alpha > 0$  as the Hausdorff dimension of the set of points  $t_0$  having Hölder exponent  $\alpha$ ,

$$d(\alpha) = \dim_{\mathcal{H}} \{ t_0 : H_f(t_0) = \alpha \}.$$

The spectrum of singularities is difficult to measure in experiments, so in an attempt to overcome this, Frisch and Parisi [17] conjectured a formula relating it with the structure functions (6). In this deterministic context, Jaffard [25, 26] adapts the definition of structure functions as

$$S_p(f, \ell) = \int |f(t + \ell) - f(t)|^p dt. \quad (8)$$

Unlike the structure functions from the K41 Theory, defined as moments of order  $p$  of a stationary random process, this definition uses  $L^p$  norms, and thus absolute values in the integral. If one assumes that  $S_p(f, \ell)$  scales like  $|\ell|^{\zeta(p)}$  when  $\ell \rightarrow 0$ , the Frisch-Parisi conjecture states that the spectrum is the Legendre transform of  $\zeta$ ,

$$d(\alpha) = \inf_p (\alpha p - \zeta(p) + 1). \quad (9)$$



However, restrictions may arise in the range of validity of (9). A more general and complete version of the conjecture was given by Jaffard [26], who showed that these restrictions can be avoided if  $\zeta(p)$  is substituted by  $\eta(p) = \sup\{s \mid f \in B_p^{s/p, \infty}\}$ , where  $B_p^{s/p, \infty}$  stands for the usual Besov spaces. Indeed,  $\eta(p)$  generalises  $\zeta(p)$ . There are examples when the Frisch-Parisi conjecture (9) does not hold, but it has been proved for self-similar functions [27]. In these cases, it is said that  $f$  satisfies the multifractal formalism. In [26], partial results were obtained for generic functions.

Multifractal analysis of Riemann's non-differentiable function was carried out by Jaffard [25] as a natural continuation of its regularity analysis. Indeed, after the differentiability problem was finally solved by Gerver in [19, 20] following a partial result by Hardy in [22], the next natural step was to study the local Hölder regularity which, as explained above, is central in the determination of the spectrum of singularities. After an important initial contribution by Duistermaat [12], Jaffard proved that

$$d(\alpha) = \begin{cases} 4\alpha - 2, & \alpha \in [1/2, 3/4], \\ 0, & \alpha = 3/2, \\ -\infty, & \text{otherwise,} \end{cases}$$

showing that  $R$  is a multifractal function, since there is a continuum of Hölder exponents with non-trivial Hausdorff dimension. His proof is based on wavelet theory. He also checked the validity of the Frisch-Parisi conjecture (9), therefore proving that  $R$  satisfies the multifractal formalism. This suggested a turbulent nature of  $R$ , as was later observed in the context of the evolution of vortex filaments.

## 2. STATEMENT OF THE RESULT

**2.1. Setting and notation.** In this paper, we work with the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ , and unless otherwise stated we consider complex-valued functions defined on  $\mathbb{T}$ . For  $p \geq 1$ , we denote by  $L^p$  the Lebesgue space on the circle,  $L^p(\mathbb{T})$ , and by  $\ell^p$  the Lebesgue sequence space  $\ell^p(\mathbb{Z})$ . We work with functions such that the corresponding Fourier series

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e_n(t)$$

are absolutely convergent, where  $(e_n)_{n \in \mathbb{Z}}$  is the orthonormal basis of  $L^2(\mathbb{T})$  defined by  $e_n(t) = e^{2\pi i n t}$  and  $a_n \in \mathbb{C}$  are the Fourier coefficients of  $f$ . In particular, this implies that  $f$  is continuous, and therefore  $f \in L^p(\mathbb{T})$  for every  $p \in [1, +\infty]$ .

In the case of Riemann's non-differentiable function, we use the notation

$$R(t) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n^2 t}}{n^2} = \sum_{k=1}^{\infty} \frac{\sigma_k}{k} e^{2\pi i k t},$$

where  $\sigma_k$  is defined by

$$\sigma_k = \begin{cases} 1, & \text{if } k \text{ is the square of an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

For two positive functions  $f$  and  $g$ , we write  $f \lesssim g$  to denote that there exists a constant  $C > 0$  such that  $f \leq Cg$ . We also write  $f \simeq g$  to denote that  $f \lesssim g$  and  $g \lesssim f$ . If the constants involved depend on some parameter  $\alpha$ , we write  $f \lesssim_\alpha g$  and  $f \simeq_\alpha g$ .

For any  $a, b \in \mathbb{R}$  such that  $a < b$ , we write  $\sum_{n=a}^b = \sum_{n \in [a, b] \cap \mathbb{Z}}$  and  $\sum_{n>a}^b = \sum_{n \in (a, b] \cap \mathbb{Z}}$ .

**2.2. Flatness and main result.** In the deterministic setting of Riemann's non-differentiable function, definitions (5) and (7) need to be modified. The standard way to do so is to substitute  $p$ -moments by the  $p$ -th powers of  $L^p$  norms, as was done when going from (6) to (8).

**Definition 2.1.** For  $f : \mathbb{T} \rightarrow \mathbb{C}$  and  $N \in \mathbb{N}$ , we define the **high-pass filter** as the projection of  $f$  on Fourier modes above  $N$ , defined by

$$f_{\geq N}(t) = \sum_{|n| \geq N} a_n e_n(t), \quad (10)$$

where  $a_n$  are the Fourier coefficients of  $f$ . Similarly, the **low-pass filter** is the projection of  $f$  on Fourier modes below  $N$ , defined by

$$f_{\leq N}(t) = \sum_{|n| \leq N} a_n e_n(t), \quad (11)$$

The **flatness** of  $f$  in the sense of high-pass filtering is given by

$$F_f(N) = \frac{\|f_{\geq N}\|_{L^4(\mathbb{T})}^4}{\|f_{\geq N}\|_{L^2(\mathbb{T})}^4}.$$

We say that  $f$  is **intermittent in the sense of high-pass filtering** if

$$\lim_{N \rightarrow +\infty} F_f(N) = +\infty.$$

**Remark 2.2.** The filters (10) and (11) can also be defined with strict inequalities analogously:

$$f_{>N}(t) = \sum_{|n| > N} a_n e_n(t), \quad f_{<N}(t) = \sum_{|n| < N} a_n e_n(t).$$

**Definition 2.3.** Let  $p \geq 1$ ,  $f : \mathbb{T} \rightarrow \mathbb{C}$  a bounded and measurable function and  $\ell \in [0, 1]$ . The **structure functions** of  $f$  are defined by

$$S_{f,p}(\ell) = \int_{\mathbb{T}} |f(t+\ell) - f(t)|^p dt. \quad (12)$$

The **flatness** of  $f$  in the sense of structure functions is

$$G_f(\ell) = \frac{S_{f,4}(\ell)}{S_{f,2}(\ell)^2}.$$

We say that  $f$  is **intermittent in the sense of structure functions** if

$$\lim_{\ell \rightarrow 0} G_f(\ell) = +\infty.$$

**Remark 2.4.** If there is no risk of confusion regarding  $f$ , we write  $S_p(\ell)$  instead of  $S_{f,p}(\ell)$ .

**Remark 2.5.** In (10),  $N$  measures small perturbations in high-frequency Fourier oscillations, while in (12),  $\ell$  measures small variations in space. Both capture the small-scale behaviour of  $R$ . The result in Theorem 1 is consistent with this fact. Indeed, if we identify  $N$  with  $\ell^{-1}$ , both  $F_R(N)$  and  $G_R(\ell^{-1})$  measure the same phenomenon.

The asymptotics of both definitions of flatness are given by the following theorem:

**Theorem 1.** Let  $R$  be Riemann's non-differentiable function (1). There exist  $N_0 \in \mathbb{N}$  and  $0 < \ell_0 < 1$  such that for  $N > N_0$  and  $\ell < \ell_0$ , we have

$$F_R(N) \simeq \log N, \quad G_R(\ell) \simeq \log(\ell^{-1}).$$

Consequently,  $R$  is intermittent in the sense of both high-pass filtering and structure functions.

**Remark 2.6.** The set  $R([0,1])$  is not self-similar, but the asymptotic behaviour of  $R$  [12] and also Figure 1 reveal at least the presence of some approximate self-similar structure. Therefore, if intermittency is a measure of the lack of self-similarity,  $R$  should have weak intermittent properties, and its flatness should show this. The logarithmic growth of both  $F_R$  and  $G_R$  in Theorem 1 agrees with this interpretation.

In Section 3, we prove the high-pass filter part of Theorem 1 and in Section 4, we prove the structure function part.

### 3. INTERMITTENCY IN THE SENSE OF HIGH-PASS FILTERS

To prove the part of Theorem 1 concerning high-pass filters, we will use the Littlewood-Paley decomposition of  $R$ , as well as a result of Zalcwasser [36] on the  $L^4$  norm of the sum of square-phased exponentials. Both results are stated in Appendices A.1 and A.2.

We estimate the  $L^2$  norm of the high-pass filter first.

**Lemma 3.1.** For every  $N \geq 2$ ,

$$\|R_{\geq N}\|_{L^2(\mathbb{T})} \simeq N^{-3/4}.$$

*Proof.* By Plancherel's theorem, we get

$$\|R_{\geq N}\|_{L^2(\mathbb{T})}^2 = \int_0^1 \left| \sum_{n=N}^{\infty} \frac{\sigma_n}{n} e^{2\pi i n t} \right|^2 dt = \sum_{n=N}^{\infty} \frac{\sigma_n}{n^2} = \sum_{n=\sqrt{N}}^{\infty} \frac{1}{n^4} \simeq \int_{\sqrt{N}}^{\infty} \frac{dt}{t^4} = \frac{1}{N^{3/2}}.$$

□

To compute the  $L^4$ -norm of the high-pass filter, one may try to use Plancherel's theorem again, since  $\|f\|_4^4 = \|f^2\|_2^2$  holds. However,

$$R_{\geq N}^2(t) = \sum_{k=2N}^{\infty} \left( \sum_{n=N}^{k-N} \frac{\sigma_n \sigma_{k-n}}{n(k-n)} \right) e^{2\pi i k t}, \quad (13)$$

whose Fourier coefficients are related to  $\sum_{n=N}^{k-N} \sigma_n \sigma_{k-n}$ , the number of ways in which  $k$  can be written as a sum of two squares both of which are greater than  $N$ . The study of such sums is a classical problem in number theory and can be very technical. Instead, the Fourier series of  $R$  can be decomposed in frequency pieces that act almost independently by the Littlewood-Paley decomposition. We use this technique to prove the following lemma.

**Lemma 3.2.** *There exists  $N_0 \in \mathbb{N}$  such that*

$$\|R_{\geq N}\|_{L^4(\mathbb{T})} \gtrsim N^{-3/4} (\log N)^{1/4}, \quad \forall N \geq N_0.$$

*Proof.* According to Appendix A.1, the Littlewood-Paley decomposition of  $R_{\geq N}$  is

$$R_{\geq N}(t) = \sum_{j=1}^{\infty} \Delta_j R_{\geq N}(t), \quad (14)$$

where the Littlewood-Paley pieces are

$$\Delta_0 R_{\geq N}(t) = \sum_{1 \leq n < A} \frac{\sigma_n}{n} e^{2\pi i n t} \quad \text{and} \quad \Delta_j R_{\geq N}(t) = \sum_{A^j \leq n < A^{j+1}} \frac{\sigma_n}{n} e^{2\pi i n t}, \quad \forall j \in \mathbb{N}.$$

The value of  $A$  will be chosen later (see (21)). We define  $j(N)$  as the index corresponding to the piece containing the  $N$ -th Fourier coefficient, the only one satisfying  $A^{j(N)} \leq N < A^{j(N)+1}$ . Then,  $\Delta_j R_{\geq N} = 0$  for every  $j < j(N)$ . By the Littlewood-Paley theorem, we may write the inequality

$$\|R_{\geq N}\|_{L^4(\mathbb{T})} \simeq \left\| \left( \sum_{j \geq j(N)} |\Delta_j R_{\geq N}|^2 \right)^{1/2} \right\|_{L^4(\mathbb{T})} \geq \|\Delta_{i(N)} R\|_{L^4(\mathbb{T})}, \quad (15)$$

where  $i(N) = j(N) + 1$ , using that  $\Delta_{i(N)} R = \Delta_{i(N)} R_{\geq N}$ . The choice of  $i(N)$  comes from the fact that it is the first complete Littlewood-Paley piece after  $j(N)$ , which is truncated as a consequence of the high-pass filter.

Let us estimate  $\|\Delta_{i(N)} R\|_{L^4(\mathbb{T})}$ . As in (13), we use Plancherel's theorem to write

$$\|\Delta_{i(N)} R\|_{L^4}^4 = \|(\Delta_{i(N)} R)^2\|_{L^2}^2 = \left\| \left( \sum_{n=A^{i(N)}}^{A^{i(N)+1}} \frac{\sigma_n}{n} e^{2\pi i n t} \right)^2 \right\|_{L^2}^2 \simeq \sum_{k=2A^{i(N)}}^{2A^{i(N)+1}} \left| \sum_n \frac{\sigma_n \sigma_{k-n}}{n(k-n)} \right|^2, \quad (16)$$

where the index  $n$  must satisfy  $A^{i(N)} \leq n \leq A^{i(N)+1}$  and  $A^{i(N)} \leq k-n \leq A^{i(N)+1}$ . In both cases,  $n \simeq A^{i(N)}$  and  $k-n \simeq A^{i(N)}$ . Hence, we can take the denominators outside the sum:

$$\|\Delta_{i(N)} R\|_{L^4}^4 \simeq \frac{1}{A^{4i(N)}} \sum_{k=2A^{i(N)}}^{2A^{i(N)+1}} \left| \sum_n \sigma_n \sigma_{k-n} \right|^2 = \frac{1}{A^{4i(N)}} \left\| \sum_{n=A^{i(N)}}^{A^{i(N)+1}} \sigma_n e^{2\pi i n t} \right\|_{L^4}^4. \quad (17)$$

This is a sum of exponentials with squared phases, whose  $L^p$  norms were computed by Zalcwasser [36]. First, the triangle inequality gives

$$\left\| \sum_{n=A^{i(N)/2}}^{A^{(i(N)+1)/2}} e^{2\pi i n^2 t} \right\|_{L^4} \lesssim \left\| \sum_{n=1}^{A^{(i(N)+1)/2}} e^{2\pi i n^2 t} \right\|_{L^4} + \left\| \sum_{n=1}^{A^{i(N)/2}} e^{2\pi i n^2 t} \right\|_{L^4}, \quad (18)$$

so by Zalcwasser's theorem in Appendix A.2 we get, for large enough  $N$ ,

$$\begin{aligned} \|\Delta_{i(N)}R\|_{L^4} &\lesssim \frac{1}{A^{i(N)}} \left( \left( A^{i(N)+1} \log A^{(i(N)+1)/2} \right)^{1/4} + \left( A^{i(N)} \log A^{i(N)/2} \right)^{1/4} \right) \\ &\lesssim \frac{1}{A^{i(N)}} \left( A^{i(N)} \log A^{i(N)/2} \right)^{1/4} \lesssim A^{-3i(N)/4} \left( \log A^{i(N)} \right)^{1/4}. \end{aligned} \quad (19)$$

On the other hand, using the reverse triangle inequality in (18), we get

$$\left\| \sum_{n=A^{i(N)/2}}^{A^{(i(N)+1)/2}} e^{2\pi i n^2 t} \right\|_{L^4} \gtrsim \left\| \sum_{n=1}^{A^{(i(N)+1)/2}} e^{2\pi i n^2 t} \right\|_{L^4} - \left\| \sum_{n=1}^{A^{i(N)/2}} e^{2\pi i n^2 t} \right\|_{L^4}.$$

Let us denote the constants in Zalcwasser's theorem for  $p = 4$  by  $0 < c < C$ . We get

$$\begin{aligned} \|\Delta_{i(N)}R\|_{L^4} &\gtrsim \frac{1}{A^{i(N)}} \left( \left( c A^{i(N)+1} \log A^{(i(N)+1)/2} \right)^{1/4} - \left( C A^{i(N)} \log A^{i(N)/2} \right)^{1/4} \right) \\ &= \frac{1}{A^{i(N)}} \left( A^{i(N)} \log A^{i(N)/2} \right)^{1/4} \left( (c A(1 + i(N)^{-1}))^{1/4} - C^{1/4} \right) \\ &\gtrsim A^{-3i(N)/4} \left( \log A^{i(N)} \right)^{1/4} \left( c^{1/4} A^{1/4} - C^{1/4} \right). \end{aligned} \quad (20)$$

Finally, choose  $A$  so that

$$(cA)^{1/4} - C^{1/4} = 1. \quad (21)$$

Observe that in the proof of (19) and (20) we may replace  $i(N)$  by any  $j \geq i(N)$ , so we have proved that

$$\|\Delta_j R\|_{L^4} \simeq A^{-3j/4} \left( \log A^j \right)^{1/4}, \quad \forall j \geq i(N). \quad (22)$$

Coming back to  $j = i(N)$ , from (15), since  $A^{i(N)-1} \leq N < A^{i(N)}$ , we get

$$\|R_{\geq N}\|_{L^4(\mathbb{T})} \gtrsim N^{-3/4} (\log N)^{1/4}.$$

□

Lemma 3.2 suffices to prove that the flatness of Riemann's non-differentiable function tends to infinity. However, we can be more precise and show that the lower bound in the lemma is sharp.

**Lemma 3.3.** *There exists  $N_0 \in \mathbb{N}$  such that*

$$\|R_{\geq N}\|_{L^4(\mathbb{T})} \lesssim N^{-3/4} (\log N)^{1/4}, \quad \forall N \geq N_0.$$

*Proof.* Applying the triangle inequality in the Littlewood-Paley decomposition (14), we write

$$\|R_{\geq N}\|_{L^4} \leq \|\Delta_{j(N)}R_{\geq N}\|_{L^4} + \sum_{j \geq i(N)} \|\Delta_j R\|_{L^4}. \quad (23)$$

Using (22) we can estimate  $\|\Delta_j R\|_{L^4}$  for  $j \geq i(N)$ . To deal with the index  $j(N)$ , following the arguments in (16), (17), (18) and (19) and using  $A^{j(N)} \leq N < A^{j(N)+1}$ , we write

$$\begin{aligned} \|\Delta_{j(N)} R_{\geq N}\|_{L^4} &\simeq \frac{1}{N} \left\| \sum_{n=N}^{A^{j(N)+1}} \sigma_n e^{2\pi i n t} \right\|_{L^4} \lesssim \frac{1}{N} \left( \left\| \sum_{n=1}^{A^{j(N)+1}} \sigma_n e^{2\pi i n t} \right\|_{L^4} + \left\| \sum_{n=1}^N \sigma_n e^{2\pi i n t} \right\|_{L^4} \right) \\ &\lesssim \frac{1}{N} (N \log N)^{1/4} = N^{-3/4} (\log N)^{1/4}. \end{aligned} \quad (24)$$

On the other hand, using (22), we bound

$$\sum_{j \geq i(N)} \|\Delta_j R\|_{L^4} \lesssim \sum_{j \geq i(N)} A^{-3j/4} (\log A^j)^{1/4} \simeq \sum_{j \geq i(N)} j^{1/4} A^{-3j/4}. \quad (25)$$

By Hölder's inequality one can write

$$\sum_{j \geq i(N)} j^{1/4} A^{-3j/4} \leq \left( \sum_{j \geq i(N)} j A^{-3j/4} \right)^{1/4} \left( \sum_{j \geq i(N)} A^{-3j/4} \right)^{3/4}. \quad (26)$$

The second sum is geometric and equals

$$\frac{A^{-3i(N)/4}}{1 - A^{-3/4}} \simeq A^{-3i(N)/4}.$$

The first one can be computed differentiating power series. Indeed, for  $|r| < 1$  we write

$$\begin{aligned} \sum_{j \geq i(N)} j r^j &\leq \sum_{j \geq i(N)} (j+1) r^j = \frac{d}{dr} \sum_{j \geq i(N)} r^{j+1} = \frac{d}{dr} \frac{r^{i(N)+1}}{1-r} \\ &= \frac{i(N) r^{i(N)}}{1-r} \left( 1 + \frac{1}{i(N)} \left( 1 + \frac{r}{1-r} \right) \right) \\ &\leq \frac{2}{1-r} i(N) r^{i(N)}. \end{aligned}$$

The last inequality is satisfied when  $i(N) > 1 + r/(1-r)$ , which holds when  $N > A^{1+r/(1-r)}$ . Choosing  $r = A^{-3/4}$ , from (25) and (26), we get

$$\sum_{j \geq i(N)} \|\Delta_j R\|_{L^4} \lesssim i(N)^{1/4} A^{-3i(N)/4} \simeq N^{-3/4} (\log N)^{1/4}.$$

Combining this with (23) and (24), we finally obtain

$$\|R_{\geq N}\|_{L^4} \lesssim N^{-3/4} (\log N)^{1/4},$$

for large enough  $N$ . □

From Lemmas 3.1, 3.2 and 3.3, the proof of the high-pass filter part of Theorem 1 is immediate.

*Proof of Theorem 1 (Part 1).* For large enough  $N$  determined by Lemmas 3.2 and 3.3, we may write

$$F_R(N) = \frac{\|R_{\geq N}\|_{L^4(\mathbb{T})}^4}{\|R_{\geq N}\|_{L^2(\mathbb{T})}^4} \simeq \frac{N^{-3} \log N}{N^{-3}} = \log N,$$

and therefore  $\lim_{N \rightarrow \infty} F_R(N) = +\infty$ . □

#### 4. INTERMITTENCY IN THE SENSE OF STRUCTURE FUNCTIONS

We begin by observing that for  $t, \ell \in [0, 1]$ , making the elementary change of variables  $t \mapsto t - \ell/2$ , the structure functions  $S_p(\ell)$  can be described in terms of the increment function  $I(\ell, t)$ ,

$$I(\ell, t) = R(t + \ell/2) - R(t - \ell/2) = 2i \sum_{k \geq 1} \frac{\sin(\pi k \ell)}{k} \sigma_k e_k(t),$$

so that if we write  $I(\ell) = I(\ell, \cdot)$ , we have

$$S_p(\ell) = \|I(\ell)\|_p^p.$$

**Lemma 4.1.** *For  $0 < \ell < 1/2$ ,*

$$S_2(\ell) \simeq \ell^{3/2}.$$

*Proof.* By Parseval's theorem,

$$S_2(\ell) \simeq \sum_{k \geq 1} \frac{\sin^2(\pi k \ell)}{k^2} \sigma_k^2 = \underbrace{\sum_{n \leq (2\ell)^{-1/2}} \frac{\sin^2(\pi n^2 \ell)}{n^4}}_{A_2(\ell)} + \underbrace{\sum_{n > (2\ell)^{-1/2}} \frac{\sin^2(\pi n^2 \ell)}{n^4}}_{B_2(\ell)}.$$

On the one hand, since  $\sin(t)/t \simeq 1$  for  $|t| \leq \pi/2$ ,

$$A_2(\ell) \simeq \sum_{n \leq (2\ell)^{-1/2}} \frac{(\pi n^2 \ell)^2}{n^4} \simeq \ell^{3/2}.$$

On the other hand, we have the upper bound

$$B_2(\ell) \leq \sum_{n > (2\ell)^{-1/2}} \frac{1}{n^4} \simeq \int_{(2\ell)^{-1/2}}^{\infty} \frac{dt}{t^4} \simeq \ell^{3/2}.$$

□

**Lemma 4.2.** *There exists  $0 < \ell_0 < 1/2$  such that*

$$S_4(\ell) \simeq \ell^3 \log(\ell^{-1}), \quad \forall \ell \in (0, \ell_0).$$

*Proof.* We decompose  $I(\ell)$  in low and high frequencies so that by the triangle inequality we get

$$S_4^{1/4}(\ell) = \|I(\ell)\|_4 = \|I_{\leq \ell^{-1/2}}(\ell) + I_{> \ell^{-1/2}}(\ell)\|_4 \leq \|I_{\leq \ell^{-1/2}}(\ell)\|_4 + \|I_{> \ell^{-1/2}}(\ell)\|_4.$$

On the other hand, the Fourier coefficients of  $I(\ell)$  are positive for every  $1 \leq k \leq \ell^{-1}$ . This, together with Parseval's theorem, implies that

$$\|I_{\leq \ell^{-1/2}}(\ell)\|_4^4 = \|I_{\leq \ell^{-1/2}}(\ell)\|_2^2 \leq \|(I^2)_{\leq \ell^{-1}}(\ell)\|_2^2 \leq \|I(\ell)\|_2^2 = \|I(\ell)\|_4^4 = S_4(\ell).$$

In short, we have

$$\|I_{\leq \ell^{-1/2}}(\ell)\|_4 \leq S_4^{1/4}(\ell) \leq \|I_{\leq \ell^{-1/2}}(\ell)\|_4 + \|I_{> \ell^{-1/2}}(\ell)\|_4,$$

so we look for both upper and lower estimates for  $\|I_{\leq \ell^{-1}/2}(\ell)\|_4$ , but an upper bound suffices for  $\|I_{> \ell^{-1}/2}(\ell)\|_4$ .

By Parseval's theorem, we have

$$\|I_{\leq \ell^{-1}/2}(\ell)^2\|_2^2 = \sum_{k=1}^{\ell^{-1}} b_k^2, \quad b_k = \sum_m \frac{\sin(\pi m \ell) \sin(\pi(k-m)\ell)}{m(k-m)} \sigma_m \sigma_{k-m}.$$

where the index  $m$  satisfies  $1 \leq m \leq \ell^{-1}/2$  and  $1 \leq k-m \leq \ell^{-1}/2$ . As above, since  $1/2 \leq \sin(t)/t \leq 1$  for  $|t| \leq \pi/2$ , we get

$$b_k \simeq \ell^2 \sum_m \sigma_m \sigma_{k-m}.$$

Consequently, with the same restrictions on  $m$  as above,

$$\|I_{\leq \ell^{-1}/2}(\ell)^2\|_2^2 \simeq \ell^4 \sum_{k=1}^{\ell^{-1}} \left( \sum_m \sigma_m \sigma_{k-m} \right)^2 = \ell^4 \left\| \left( \sum_{k=1}^{\ell^{-1}/2} \sigma_k e_k \right)^2 \right\|_2^2 = \ell^4 \left\| \sum_{k=1}^{\sqrt{\ell^{-1}/2}} e_{k^2} \right\|_4^4$$

By Zalcwasser's theorem in Appendix A.2, for  $\ell$  small enough we get

$$\|I_{\leq \ell^{-1}/2}(\ell)\|_4^4 = \|I_{\leq \ell^{-1}/2}(\ell)^2\|_2^2 \simeq \ell^4 \ell^{-1} \log(\sqrt{\ell^{-1}}) \simeq \ell^3 \log(\ell^{-1}).$$

For the high frequency piece, using again Parseval's theorem, we can write

$$\|I_{> \ell^{-1}/2}(\ell)^2\|_2^2 = \sum_{k > \ell^{-1}} \beta_k^2, \quad \text{where} \quad \beta_k = \sum_{m, k-m > \ell^{-1}/2} \frac{\sin(\pi m \ell) \sin(\pi(k-m)\ell)}{m(k-m)} \sigma_m \sigma_{k-m}.$$

Bounding the sine trivially and using (13) with  $N = \ell^{-1}/2$ , we get

$$\|I_{> \ell^{-1}/2}(\ell)^2\|_2^2 \leq \sum_{k > \ell^{-1}} \left( \sum_{m, k-m > \ell^{-1}/2} \frac{\sigma_m \sigma_{k-m}}{m(k-m)} \right)^2 = \|(R_{\geq \ell^{-1}/2})^2\|_2^2 = \|R_{\geq \ell^{-1}/2}\|_4^4,$$

so by Lemma 3.3 we obtain that

$$\|I_{> \ell^{-1}/2}(\ell)\|_4^4 \lesssim (\ell^{-1})^{-3} \log \ell^{-1} \simeq \ell^3 \log(\ell^{-1}).$$

□

**Remark 4.3.** *The method above can be generalised to compute the asymptotic behaviour of every  $S_{2k}$ ,  $k \in \mathbb{N}$ .*

*Proof of Theorem 1 (Part 2).* By Lemmas 4.1 and 4.2, for small enough  $\ell$  we get

$$G_R(\ell) \simeq \frac{\ell^3 \log(\ell^{-1})}{(\ell^{3/2})^2} = \log(\ell^{-1}).$$

□



## APPENDIX

**A.1. The Littlewood-Paley decomposition.** We recall here the following classical result [30, Theorem 3].

**Theorem 2.** *Let  $p > 1$ ,  $A > 1$  and  $f(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t}$  a function in  $L^p(0,1)$ . Consider the decomposition*

$$f(t) = \sum_{k=1}^{\infty} \Delta_k f(t)$$

such that

$$\Delta_1 f(t) = \sum_{|n| \leq A} a_n e^{2\pi i n t}, \quad \Delta_k f(t) = \sum_{A^k < |n| \leq A^{k+1}} a_n e^{2\pi i n t}, \quad k \geq 2.$$

Then, there exist constants  $B_1, B_2 > 0$  depending on  $p$  such that

$$B_1 \leq \frac{\|(\sum_{k=1}^{\infty} |\Delta_k f|^2)^{1/2}\|_{L^p}}{\|f\|_{L^p}} \leq B_2.$$

**A.2. A theorem of Zalcwasser.** The following result, contained in [36], is crucial to our proof.

**Theorem 3.** *Let  $p > 0$ . Then, there exist  $M_p > 1$  and constants  $C_p > c_p > 0$  such that for every  $N > M_p$ ,*

$$c_p \psi_p(N) \leq \int_0^1 \left| \sum_{m=1}^N e^{2\pi i m^2 t} \right|^p dt \leq C_p \psi_p(N)$$

where

$$\psi_p(N) = \begin{cases} N^{p/2}, & p < 4, \\ N^2 \log N, & p = 4 \\ N^{p-2}, & p > 4. \end{cases}$$

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