Strong Ramsey Games in Unbounded Time

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Abstract

For two graphs B and H the strong Ramsey game $\mathcal{R}(B, H)$ on the board B and with target H is played as follows. Two players alternately claim edges of B. The first player to build a copy of H wins. If none of the players win, the game is declared a draw. A notorious open question of Beck [4–6] asks whether the first player has a winning strategy in $\mathcal{R}(K_n, K_k)$ in bounded time as $n \to \infty$. Surprisingly, in a recent paper [16] Hefetz, Kusch, Narins, Pokrovskiy, Requilé and Sarid constructed a 5-uniform hypergraph $\mathcal{H}$ for which they proved that the first player does not have a winning strategy in $\mathcal{R}(K_n^{(5)}, \mathcal{H})$ in bounded time. They naturally ask whether the same result holds for graphs. In this paper we make further progress in decreasing the rank.

In our first result, we construct a graph G (in fact $G = K_6 \setminus K_4$) and prove that the first player does not have a winning strategy in $\mathcal{R}(K_n \sqcup K_n, G)$ in bounded time. As an application of this result we deduce our second result in which we construct a 4-uniform hypergraph G' and prove that the first player does not have a winning strategy in $\mathcal{R}(K_n^{(4)}, G')$ in bounded time. This improves the result in the paper above.

By compactness, an equivalent formulation of our first result is that the game $\mathcal{R}(K_\omega \sqcup K_\omega, G)$ is a draw. Another reason for interest on the board $K_\omega \sqcup K_\omega$ is a folklore result that the disjoint union of two finite positional games both of which are first player wins is also a first player win. An amusing corollary of our first result is that at least one of the following two natural statements is false: (1) for every graph H, $\mathcal{R}(K_\omega, H)$ is a first player win; (2) for every graph H if $\mathcal{R}(K_\omega, H)$ is a first player win, then $\mathcal{R}(K_\omega \sqcup K_\omega, H)$ is also a first player win. Surprisingly, we cannot decide between the two.

Keywords: Positional games, Ramsey games, Strong games
2010 MSC: 05C57, 91A43, 91A46

1. Introduction

1.1. Background

Positional games were first studied by Hales and Jewett [13] and Erdős and Selfridge [10]. The general setting was given by Berge [8], but the field was shaped by the numerous works of Beck since the early 80s including [1–7]. Though the most natural positional games are the strong ones, in which both players compete to achieve the same objective, the theory has largely deviated from this direction due to the prohibitive difficulty of strong games. Many weak variants have been developed and have proved more suitable for study, like Maker-Breaker games, for which much is known (see e.g. [6]). However, the strong games remain very poorly understood.

In the present paper we study instances of a particular type of strong game – the strong Ramsey game. This was first introduced by Harary [14] for cliques and later for arbitrary graphs [15]. In the strong Ramsey game, or simply the *Ramsey game*, $\mathcal{R}(B, H)$ on a finite or infinite graph B, called the *board*, with *target graph* H, two players, **P1** and

P2, alternate to take previously unclaimed edges of B, starting with **P1**. The first player to claim an isomorphic copy of H *wins* and if none does so in finite time, the game is declared a *draw*. More generally, one might take B and H to be r-uniform hypergraphs. This setting was already mentioned by Beck and Csirmaz and Beck in [1, 7].

For fixed graphs B, H, in $\mathcal{R}(B, H)$, a very general strategy stealing argument due to Nash shows that **P2** cannot have a winning strategy. Moreover, for any fixed target graph H it follows from Ramsey's theorem [18] that $\mathcal{R}(K_n, H)$ cannot end in a draw for n sufficiently large. Therefore, in $\mathcal{R}(K_n, H)$ **P1** has a winning strategy for n sufficiently large. The strategy given by this argument is not explicit, and indeed almost no examples of explicit strategies are known. In particular, no explicit strategy has been exhibited for $\mathcal{R}(K_n, K_k)$ with $k \geq 5$ and n large.

This fact makes it difficult to attack most natural questions in the field. For instance a notorious open question popularised by Beck [4–6] asks if, for fixed k, in the game $\mathcal{R}(K_n, K_k)$, **P1** has a winning strategy in bounded time as $n \to \infty$. An easy compactness argument shows that this is equivalent to the game $\mathcal{R}(K_\omega, K_k)$ being a **P1**-win. The answer is conjectured to be in the affirmative [4, 5, 17] but no progress has been made on the problem for $k \geq 5$. Beck emphasised the importance of the question when he listed the question among his "7 most humiliating open problems" [6]. In the opposite direction, another notorious open question asks if for every fixed target graph H in the game $\mathcal{R}(K_n, H)$, **P1** has a winning strategy in bounded time as $n \to \infty$ or, equivalently, $\mathcal{R}(K_\omega, H)$ is a **P1** win. In a recent paper [16], Hefetz, Kusch, Narins, Pokrovskiy, Requilé and Sarid addressed the natural generalisation to hypergraphs, and changed the intuition about this phenomenon completely. Surprisingly, in [16] they exhibited a target 5-uniform hypergraph $\mathcal{H}$ for which they constructed an explicit drawing strategy for **P2** in the game $\mathcal{R}\left(K_\omega^{(5)}, \mathcal{H}\right)$. This result provides strong evidence that just strategy stealing and Ramsey-type arguments are insufficient to attack Beck's conjecture.

However, the corresponding question for graphs, as asked in [16], still remains open. As we explain in Section 1.3, as the rank decreases this question becomes much harder. In this paper we make further progress in decreasing it. In our first result, Theorem 1.2 (see Corollary 1.3), we exhibit a graph G (in fact $G = K_6 \setminus K_4$, see Figure 1), for which we prove that in $\mathcal{R}(K_n \sqcup K_n, G)$ **P1** does not have a winning strategy in bounded time. In order to do so we build on the work of [16]. However, there is a serious obstacle in adapting the strategy developed in [16] from 5-uniform hypergraphs to graphs. As we further explain in Section 1.3, although the strategy developed in [16] is a strong game drawing strategy, the core of the argument is a weak game fast winning strategy that is inapplicable to graphs. Therefore, most of our effort is spent developing a much more intricate and elaborate strategy for graphs reflecting characteristic difficulties of strong games, which we discuss further in Section 5.

Turning to hypergraphs, in our second result, Theorem 1.7 (see Corollary 1.8), we exhibit a 4-uniform hypergraph G', obtained by adding 2 new vertices and including them in all edges of G, for which we prove that in $\mathcal{R}\left(K_n^{(4)}, G'\right)$ **P1** does not have a winning strategy in bounded time. In order to do so we cover the board $K_n^{(4)}$ with copies of the board $K_{n-2}^{(2)}$, noting that $K_n^{(4)} = \bigcup_{X \neq Y \in V(K_n^{(4)})} K_{n-2}^{X,Y}$, where $K_{n-2}^{X,Y}$ is the set of hyperedges containing X and Y, naturally identified with K_{n-2}. While in the graph setting the strategy is considerably more involved, in the hypergraph setting the strategy is much more difficult to analyse.

It is not clear whether one should expect $\mathcal{R}(K_n, G)$ to admit a bounded time winning strategy for **P1**, but it is known (see [9] for a simple proof) that for having a winning strategy on K_n is equivalent to having one on $K_n \sqcup K_n$. Thus, by compactness, our result refutes one of two natural conjectures extending simple finite board facts to infinite boards. Namely, it is not the case that on the one hand for every graph H $\mathcal{R}(K_\omega, H)$ is a **P1**-win and on the other hand for every graph H if $\mathcal{R}(K_\omega, H)$ is a **P1**-win, then $\mathcal{R}(K_\omega \sqcup K_\omega, H)$ is also a **P1**-win.

1.2. Results

In this section we state our results.

Definition 1.1. *Let $G = K_6 \setminus K_4$ be the graph with edge set $E(K_6) \setminus E(K_4)$ (see Figure 1).*

In the graph setting the central result is as follows.

Theorem 1.2. *The game $\mathcal{R}(K_\omega \sqcup K_\omega, G)$ is a draw.*

As mentioned in the previous section, one can also formulate the result for finite boards. The proof implies the following.
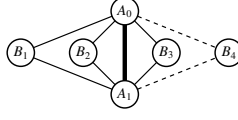
Figure 1: The graph G. The dashed lines form the pair, the solid lines form the core and the base is thickened.

Corollary 1.3. *In the game $\mathcal{R}(K_n \sqcup K_n, G)$ **P1** does not have a strategy which guarantees him a win in less than $2n - O(1)$ total moves, as $n \to \infty$.*

Remark 1.4. *Note that we obtain an explicit drawing strategy for the first player in $\mathcal{R}(K_\omega, G)$.*

An amusing corollary of Theorem 1.2 is the following.

Corollary 1.5. *It is not the case that on the one hand for every graph H $\mathcal{R}(K_\omega, H)$ is a **P1**-win and on the other hand for every graph H if $\mathcal{R}(K_\omega, H)$ is a **P1**-win, then $\mathcal{R}(K_\omega \sqcup K_\omega, H)$ is also a **P1**-win.*

In the hypergraph setting we will work with the following hypergraph.

Definition 1.6. *Let G' be the 4-uniform hypergraph obtained from our graph G as $V(G') = V(G) \sqcup \{X, Y\}$ and $E(G') = \{e \sqcup \{X, Y\}, e \in E(G)\}$.*

Below is the main result on hypergraphs.

Theorem 1.7. *The game $\mathcal{R}\left(K_\omega^{(4)}, G'\right)$ is a draw.*

As before, the proof implies the following version of the result for finite boards.

Corollary 1.8. *In the game $\mathcal{R}\left(K_n^{(4)}, G'\right)$ **P1** does not have a strategy which guarantees him a win in less than $2n - O(1)$ total moves, as $n \to \infty$.*

1.3. Overview

Graph setting. Throughout the paper, unless otherwise stated, all configurations are considered after **P1**'s move and before **P2**'s move. The graph G is formed by a *base* and four *pairs* of edges connected to it (see Figure 1). We decompose the graph G into a pair and a *core* which is formed by the base and the other three pairs.

We construct the drawing strategy for **P2** in the game $\mathcal{R}(K_\omega \sqcup K_\omega, G)$ in three stages. We denote the copy of K_ω in which **P1** takes his first edge by K^1 and the other copy by K^2.

In the first stage **P2** builds a core in K^2. While doing this **P2** ensures that the following two statements hold. On the one hand, during the entire stage **P2** remains ahead of **P1** in building G in K^2, which is intuitively possible, since **P2** is the first player in K^2. In particular, at the end of the first stage **P1** does not have a threat in K^2. On the other hand, at the end of the stage **P1** has at most $|E(G)| - 1$ edges in K^1. At the beginning of the second stage **P2** checks if **P1** has a threat in K^1. If this is the case, **P2** blocks **P1**'s (possibly infinite) threats as long as **P1** keeps making new ones in K^1. The graph G is such that **P1** cannot force a win by making such consecutive threats. If at some point **P1** does not have a threat in K^1, then, in the third stage, **P2** aims to build the pair in K^2 and complete G. He does so by making a (possibly infinite) series of threats from a well-chosen endpoint of the base to new vertices. The choice is such that **P1**'s responses to **P2**'s threats cannot be part of a threat of **P1**.

Roughly speaking, in [16] the strategy is divided into the same three stages: building a core, blocking threats and then making threats. However, there is a severe obstruction in transferring the first stage of the strategy from 5-uniform hypergraphs to graphs. Indeed, in [16] the target hypergraph has a non-trivial and identifiable core that admits a weak game fast building strategy (with number of moves equal to the size of the core). The construction of such a core relies heavily on the high rank. Unfortunately, for graphs this is not feasible anymore because the construction of any non-trivial identifiable core can be delayed by **P1**. Therefore, in the graph setting one should abandon such weak game strategy and face the essential difficulty of strong game strategies, by allowing delay and making sure to block the other player while building one's own target graph.

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Hypergraph setting. As it was discussed in the introduction, the board $K_\omega^{(4)}$ can be viewed as $\bigcup_{X,Y \in V\left(K_\omega^{(4)}\right)} K_\omega^{X,Y}$, where $K_\omega^{X,Y} = \{e \in E(K_\omega^{(4)}), X, Y \in e\}$ identifies naturally with the board K_ω. Note that a copy of G' is contained in exactly one of the boards $K_\omega^{X,Y}$ and identifies with a graph G in this board.

The drawing strategy of **P2** in the game $\mathcal{R}(K_\omega^{(4)}, G')$ still follows the same three stages. In the first stage **P2** builds a core in a board $K_\omega^{X,Y}$ (corresponding to K^2 from the graph setting) disjoint from **P1**'s first hyperedge by using a very simplified version of the first stage of the strategy for $\mathcal{R}(K_\omega \sqcup K_\omega, G)$, which exploits the larger number of symmetries in higher rank. **P2** ensures that at the end of the first stage **P1** does not have a threat in any board $K_\omega^{X,Y}$, except possibly exactly one of the six boards containing **P1**'s first hyperedge. The second and third stages remain unchanged. However, in the hypergraph setting we need a more subtle analysis than in the graph setting, as the different boards intersect. The key observation is that for $\{X_1, Y_1\} \cap \{X_2, Y_2\} = \varnothing$ the boards $K_\omega^{X_1,Y_1}$ and $K_\omega^{X_2,Y_2}$ share only one hyperedge. In this respect it is clear that higher rank simplifies the problem.

1.4. Organisation of the paper

The rest of the paper is organised as follows. In Section 2 we establish the key lemmas which are used repeatedly in the proof. In Section 3 we construct a drawing strategy for **P2** in $\mathcal{R}(K_\omega \sqcup K_\omega, G)$. In Section 4 we derive a drawing strategy for **P2** in $\mathcal{R}\left(K_\omega^{(4)}, G\right)$. In Section 5 we make some concluding remarks and state some open questions.

2. Preliminaries

2.1. Setup and notation

For this section and the next one we consider the game $\mathcal{R}(K_\omega \sqcup K_\omega, G)$, where the graph G is defined below. Denote the two disjoint copies of K_ω constituting the board by K^1 and K^2 and assume without loss of generality that **P1**'s first edge is taken in K^1.

For technical reasons, we consider the version of the game in which **P1** stops playing after a finite number of moves, and the game continues until **P2** builds a copy of G. In order to show that **P2** has a drawing strategy, it is enough to show that at the end **P1** does not have a copy of G.

Definition 2.1. *Let $G = (V, E)$ be the graph with vertex set $V = \{A_0, A_1, B_1, B_2, B_3, B_4\}$ and edge set $E = \{A_0A_1, A_0B_i, A_1B_i \mid i = 1, \ldots, 4\}$ (see Figure 1). With a slight abuse of notation we shall refer to any isomorphic copy of G as G.*

We call the edge A_0A_1 the base *of G. Note that any automorphism of G fixes the base, so it is well defined.*

For the following set of definitions we consider the game at a certain stage. Recall from Section 1.3 that all configurations are considered after **P1**'s move and before **P2**'s move.

Definition 2.2. *We say that a given graph G is **P1**-free if **P1** has no edges in G and we say it is **P2**-free if **P2** has no edges in G.*

Definition 2.3. *For a given graph G, let $e_{\mathbf{P1}}(G)$ be the number of edges in G taken by **P1** if G is **P2**-free; otherwise, define it to be 0. Define similarly $e_{\mathbf{P2}}(G)$. For a given vertex A let $\deg_{\mathbf{P1}}(A)$ and $\deg_{\mathbf{P2}}(A)$ be the number of edges that contain A taken by **P1** and **P2**, respectively.*

Note that in the course of the game for a given G the quantities $e_{\mathbf{P1}}(G)$ and $e_{\mathbf{P2}}(G)$ may decrease, becoming 0.

Definition 2.4. *We say that a vertex F is **P1**-free if $\deg_{\mathbf{P1}}(F) = 0$, we say it is **P2**-free if $\deg_{\mathbf{P2}}(F) = 0$ and we say it is* free *if $\deg_{\mathbf{P1}}(F) = \deg_{\mathbf{P2}}(F) = 0$.*

Definition 2.5. *We call an edge A_0A_1 taken by **P2** in K^2 a* potential base *if there exist two vertices B_1, B_2 in K^2 such that **P2** has the edges A_0B_1, A_0B_2, A_1B_1 and A_1B_2 and there exists a special vertex $X \in \{A_0, A_1\}$ such that:*

  *(i) **P1** does not have a triangle X, T_1, T_2;*

  *(ii) **P1** does not have a 4-cycle X, C_1, C_2, C_3 with the edge XC_2 not taken by **P2**.*

Remark 2.6. *If the edge A_0A_1 is a potential base with special vertex X, then if later in the game **P1** constructs an additional star $XF_1, XF_2, \ldots$ from X to **P1**-free vertices $F_1, F_2, \ldots$ together with exactly r extra edges, then he has at most r triangles sharing a common edge that contain X.*

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2.2. Lemmas

In this section we shall present the main technical result, Lemma 2.7, which states that if the game has reached certain configurations, then **P2** has a drawing strategy. This result will be used recurrently in conjunction with the strategy constructed in Section 3. In order to check one of the hypotheses of Lemma 2.7 we also establish Lemma 2.9.

Lemma 2.7. *Assume that before P2's turn, the game has the following properties:*

(a) **P1** *has at most six edges in* K^1*;*

(b) *for any G in* K^2*,* $e_{P1}(G) \le 5$*;*

(c) **P2** *has a potential base* $A_0 A_1$*.*

Then, P2 has a drawing strategy.

Proof. Let $\mathcal{E}_0^1$ and $\mathcal{E}_0^2$ be the set of initial edges of **P1** in K^1 and K^2 respectively. Without loss of generality let A_0 be the special vertex of the potential base $A_0 A_1$. The line of play of **P2** is divided in three stages as follows. In the first stage, **P2** takes edges from A_1 to free vertices F_i for $i = 1, \ldots, k$ until at some point **P1** does not take the edge $A_0 F_k$ (this necessarily happens, since **P1** makes only a finite number of moves). Then **P2** takes $A_0 F_k$. Let $\mathcal{E}_1^2 = \{A_0 F_i, 1 \le i < k\}$ be the star taken by **P1** and $\mathcal{E}_2$ be the set consisting of the last two edges taken by **P1** (if he did not stop playing).

Claim. One can guarantee that after a finite number of moves, called second stage, the configuration (before **P2**'s move) satisfies the following two properties.

- For each G in K^1 we have that $e_{P1}(G) \le 7$.

- The set of edges **P1** has taken in K^2 is $\mathcal{E}_0^2 \sqcup \mathcal{E}_1^2 \sqcup \mathcal{E}^2$ for some $\mathcal{E}^2$ with $|\mathcal{E}^2| \le 2$.

Proof of the claim. We consider three cases for the configuration at the end of the first stage (before **P2**'s move).

Case I. P1 has exactly 8 edges in K^1 forming a copy G^1 of G with its base $C_0 C_1$ present but the edge $C_0 D_1$ absent.

As $|\mathcal{E}_0^1| \le 6$, we have that $\mathcal{E}_2$ is contained in G^1. **P2** takes the edge $C_0 D_1$ and, while **P1** keeps taking edges $C_{\epsilon_n} D_n$ to **P1**-free vertices D_n, **P2** keeps taking the edges $C_{1-\epsilon_n} D_n$ for $\epsilon_n \in \{0, 1\}$. Eventually **P1** either stops or takes a different kind of edge ε. Let $\mathcal{E} = \{\varepsilon\}$. Note that $e_{P1}(G) \le 7$ for all G in K^1. Indeed, this immediately follows from the fact that for all vertices $C \in V(K^1) \setminus \{C_0, C_1\}$ we have that $\deg_{P1}(C) \le 3$.

Case II. P1 has exactly 8 edges in K^1 forming a copy G^1 of G without its base $C_0 C_1$.

As $|\mathcal{E}_0^1| \le 6$, we have that $\mathcal{E}_2$ is contained in G^1. **P2** takes $C_0 C_1$ and then **P1** takes an edge ε. Let $\mathcal{E} = \{\varepsilon\}$. Note that $e_{P1}(G) \le 7$ for all G in K^1.

Case III. For any G in K^1, $e_{P1}(G) \le 7$.

In this case we directly skip to the third stage and set $\mathcal{E} = \mathcal{E}_2$.

In all cases, let $\mathcal{E}^2 = \mathcal{E} \cap K^2$ and note that the set of edges **P1** has taken in K^2 is $\mathcal{E}_0^2 \sqcup \mathcal{E}_1^2 \sqcup \mathcal{E}^2$, with $|\mathcal{E}^2| \le 2$, as claimed. $\quad\square$

In the third stage **P2** takes edges from A_1 to free vertices F_i for $k < i \le l$ until at some point **P1** does not take the edge $A_0 F_l$. Then **P2** takes $A_0 F_l$ and constructs a G. The game stops after **P2**'s move. Let $\mathcal{E}_3^2 = \{A_0 F_i, k < i < l\}$ and $\mathcal{E}_4$ be the set consisting of the last edge taken by **P1** (if he did not stop playing).

Claim. At the end of the third stage **P1** does not have a G.
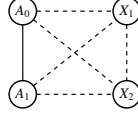
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Figure 2: The 2Δ-configuration.

Proof of the claim. Notice that by the first Claim for any G in K^1 we have $e_{\mathbf{P1}}(G) \leq 7 + 1$, since $|\mathcal{E}_4| \leq 1$. We claim that the same holds in K^2. Indeed, **P1** cannot have a G in K^2 that does not intersect $\mathcal{E}_1^2 \cup \mathcal{E}_3^2$, as $e_{\mathbf{P1}}(G) \leq 5 + 3$ by (b) and $|\mathcal{E}_4| + |\mathcal{E}^2| \leq 3$. On the other hand, **P1** cannot have a G in K^2 that contains some $A_0 F_i$, as neither of the vertices A_0 and F_i can be in the base. This is because **P1** has at most 3 triangles sharing an edge that contain vertex A_0 by Remark 2.6, and $\deg_{\mathbf{P1}}(F_i) \leq 1 + 3$. □

Clearly, the last claim implies that **P2** has a drawing strategy. □

Note that **P2** can also adopt a slightly different strategy in the first stage. Say that **P2** has the triangle $A_0 A_1 B_1$ and assume that **P2** takes the edges from A_1 to free vertices F_i for $i = 1, \ldots, 5$, while **P1** takes the edges $A_0 F_i$. Then **P2** can win in three moves by taking three of the five edges $B_1 F_i$. Thus, in the original version of the game $\mathcal{R}(K_\omega \sqcup K_\omega, G)$ (in which **P1** does not stop playing after a finite time) **P2** can always build the core of G in his drawing strategy, as claimed in Section 1.3.

We next provide a quick way to check that condition (c) of Lemma 2.7 holds.

Definition 2.8. *At a certain stage of the game, and for a given edge $A_0 A_1$ taken by **P2** in K^2, we call an edge taken by **P1** good for $A_0 A_1$ if every 4-cycle A_i, C_1, C_2, C_3 with $A_i C_2$ not taken by **P2** and every triangle A_i, C_1, C_2 that contain this edge also contains an edge taken by **P2**. In particular, all edges of **P1** in K^1 are good for any edge of **P2** in K^2. We call bad for $A_0 A_1$ an edge of **P1** in K^2 that is not good for $A_0 A_1$.*

Equivalently, an edge $X_1 X_2$ is good for $A_0 A_1$ if and only if it is in K^1 or

$$\{X_1, X_2\} \cap \{A_0, A_1\} = \varnothing, \qquad \textbf{P2 has } A_0 X_1 \text{ or } A_0 X_2, \qquad \textbf{P2 has } A_1 X_1 \text{ or } A_1 X_2. \qquad (*)$$

We call a 2Δ-configuration for an edge $A_0 A_1$ the edges $A_0 X_1, A_0 X_2, A_1 X_1, A_1 X_2, X_1 X_2$ (see Figure 2) for any two vertices X_1, X_2 disjoint from A_0, A_1.

Lemma 2.9. *Assume that before his turn **P2** has an edge $A_0 A_1$ in K^2 such that:*

(a) ***P1** has at most 5 bad edges for $A_0 A_1$;*

(b) ***P1** does not have a 2Δ-configuration for $A_0 A_1$;*

(c) ***P2** has edges $A_0 B_1, A_0 B_2, A_1 B_1, A_1 B_2$ for some vertices B_1, B_2.*

Then $A_0 A_1$ is a potential base i.e. condition (c) of Lemma 2.7 holds.

Proof. Assume for a contradiction that $A_0 A_1$ is not a potential-base. Let us call a triangle or 4-cycle as in Definition 2.5 an *obstruction* for X ($X \in \{A_0, A_1\}$). By condition (c), **P1** has an obstruction for each of A_0 and A_1. Yet, since **P2** has $A_0 A_1$, a single obstruction cannot contain both A_0 and A_1. So **P1** has two separate obstructions, one containing A_0 but not A_1, and one containing A_1 but not A_0. The obstruction containing A_0 has degree 2 at A_0, so the obstruction containing A_1 has at most $5 - 2$ edges by condition (a), since all edges in an obstruction are bad. Thus, both obstructions are triangles, they contain A_0 and A_1 and they share a common edge by condition (a), which contradicts condition (b), as $A_0 A_1$ is taken by **P2**. □
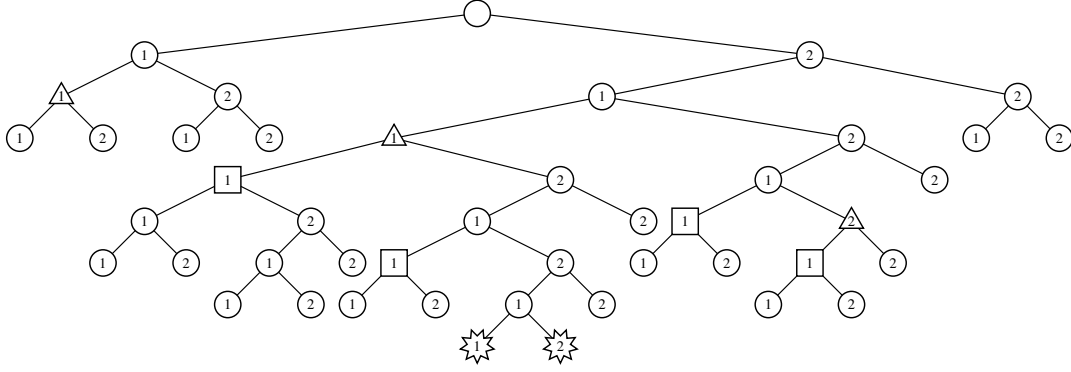
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Figure 3: The case tree. Each vertex represents the case whose label is given by appending the labels of vertices along the path from the root to it. The two special end-cases are in stars. The marked split-cases are in triangles if **P1** lost 1 edge and in squares if he lost 2.

3. A drawing strategy

3.1. Setting

In this section we present a drawing strategy for **P2** in $\mathcal{R}(K_\omega \sqcup K_\omega, G)$. We construct it by considering various cases, according to the moves of **P1**. The following trick allows us to reduce significantly the number of cases considered: instead of considering all possible moves of **P1** up to isomorphism we often give him *additional edges*, i.e. edges taken by **P1** which are not specified immediately after **P1**'s turn, but are obviously taken into account. All figures depict (part of) the configuration in K^2 only (the first edge of **P1** being in K^1), not featuring the additional edges. At a certain stage of the game, the current number of additional edges is marked as "$+n$" on the corresponding figure and the rest of **P1**'s edges are specified. The first edge of **P1** is considered specified, although it does not appear drawn. In all the figures, **P1**'s edges are drawn as dashed lines while **P2**'s edges are drawn as continuous lines. Whenever two figures are drawn for the same case, the left one corresponds to the initial state of the game before **P2**'s move under the assumption defining the present case, while the right one corresponds to the final state of the game before **P2**'s subsequent move. If only one figure is drawn, it corresponds to the final state of the game, as the initial state of the game is the same as the parent case with an additional assumption defining the present case.

The cases considered in the strategy naturally form a strict binary tree (see Figure 3), whose leaves we call *end-cases* and whose internal nodes we call *split-cases*. When possible, we will respect the convention that at each split-case we consider a particular edge **P2** is interested in, the left child (labeled by 1) corresponds to the edge being already taken by **P1**, while the right child (labeled by 2) corresponds to the edge being available for **P2** to take.

Finally, if, at a certain stage of the game, **P1** has $k + 1$ edges in total (including his first edge in K^1 and at most k other edges in K^2), we say that **P1** *lost l edges*, if $e_{\mathbf{P1}}(G) \leq k - l$ for any G in K^2. It is clear that the number of edges lost by **P2** is non-decreasing in the course of the game. For that reason we shall indicate the loss of edges as soon as it arises. More precisely, a split-case is *marked* if at the final stage of the case **P1** is certain to have lost more edges than in the parent case. These cases are denoted by triangles (resp. squares) in Figure 3 if **P1** lost 1 (resp. 2) edges.

For the reader's convenience, in Section 3.2 we make a detailed analysis of **Case 1.** This analysis contains on one hand the strategy of **P2** in **Case 1.** and on the other hand the detailed verification of the hypotheses of Lemma 2.7 in each of the end-cases. This allows us to conclude that **P2** has a drawing strategy in **Case 1.** In Section 3.3 we present the automatic procedure to check that in all end-cases of **Case 2.**, with the exception of the two special end-cases, the hypotheses of Lemma 2.7 are satisfied. In Section 3.4 we give the strategy of **P2** in **Case 2.**, leaving the mechanical checks explained in Section 3.3 to the reader. Finally, the two *special end-cases*, namely **Case 2.1.1.2.1.2.1.1.** and **Case 2.1.1.2.1.2.1.2.**, are treated separately in Section 3.5, as they do not allow a direct application of Lemma 2.7. Altogether, this allows us to conclude that **P2** has a drawing strategy.
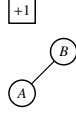
3.2. Split-case 1.

Figure 4: **General split-case.** Without loss of generality assume that the first edge **P1** takes is XY in K^1. In response, **P2** takes the edge AB in K^2. We mention that, excluding XY, **P1** has 1 unspecified additional edge (not appearing in the figure) by "+1".
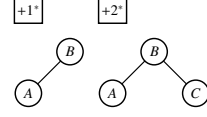

Figure 5: **Split-case 1.** Assume that the second edge taken by **P1** is either in K^1 or in K^2 and is incident with either A or B. In the latter case, we may assume that it is incident with B. We indicate that either one of these is the case by a "*" in the figures. Then **P2** takes the edge BC, where C is any free vertex in K^2.

Figure 6: **Marked split-case 1.1.** Assume that **P1** has AC. Then **P2** takes BD, where D is any free vertex in K^2. Note that at least one of the edges DA, DC is not taken by any player, so assume without loss of generality that DA is not taken. Then **P2** takes DA. **P1** lost an edge.

Figure 7: **End-case 1.1.1.** Assume that **P1** has DC. Then **P2** takes BE, where E is any free vertex in K^2. Note that at least one of the edges ED, EA is not taken by any player, so assume without loss of generality that ED is not taken. Then **P2** takes ED. The potential base is BD.

Figure 8: **End-case 1.1.2.** Assume that **P1** does not have DC. Then **P2** takes this edge. The potential base is BD.

Marked split-case 1.1. Let us begin by checking that in **Marked split-case 1.1.**, Figure 6, **P1** lost an edge. At this stage **P1** has a total of 5 edges including the first edge XY, which is in K^1. We need to check that $e_{\mathbf{P1}}(G) \leq 3$ for any G in K^2. Assume for the sake of contradiction that there exist a G in K^2 such that $e_{\mathbf{P1}}(G) \geq 4$. Then G contains all edges of **P1** except XY and contains no edges of **P2**. This forces the second edge of **P1** to be in K^2, and moreover, condition "*" forces this edge to be incident to B, say BF. In order for G to contain both AC and BF, the base X_0X_1 of G needs to satisfy without loss of generality $X_0 \in \{A, C\}$, as any edge in G is incident with the base. Then G contains the edge X_0B, as every vertex of G is connected to X_0, so X_0B needs not to be taken by **P2**. This yields the desired contradiction (see Figure 6).

In order to apply Lemma 2.7 to **End-case 1.1.1.** and **End-case 1.1.2.** and conclude that in these cases **P2** has a drawing strategy, we are now ready to check that the hypotheses of Lemma 2.7 are satisfied in these cases.

Firstly, let us check that condition (b) of Lemma 2.7 holds. Since **P1** lost an edge in **Marked split-case 1.1.**, the same holds for **End-case 1.1.1.** and **End-case 1.1.2.**. Note that the total number of edges of **P1** including the first edge XY which is in K^1 is 7 and 6, respectively, for the two end-cases. Therefore, in **End-case 1.1.1.** and **End-case 1.1.2.** each G in K^2 satisfies $e_{\mathbf{P1}}(G) \leq 5$ and $e_{\mathbf{P1}}(G) \leq 4$, respectively, as desired.

Secondly, let us check that condition (a) of Lemma 2.7 holds. This follows immediately from the fact that in **End-case 1.1.1.** and **End-case 1.1.2. P1** has 4 additional edges, so at most 5 edges in K^1.

Thirdly, let us check that condition (c) of Lemma 2.7 holds. We do so by applying Lemma 2.9 to the thickened edge BD. Thus, we are left with checking that the hypotheses of Lemma 2.9 are satisfied.

Condition (c) of Lemma 2.9 is checked directly on the figures. Note that by (*) we have that AC is good for BD, as both AB and AD are taken by **P2**. Recall that in **End-case 1.1.1.** and in **End-case 1.1.2. P1** has at most 6 and 5 edges in K^2, respectively. Therefore, in **End-case 1.1.1.** and in **End-case 1.1.2. P1** has at most 5 and 4 bad edges, respectively. It follows that condition (a) of Lemma 2.9 holds. Furthermore, in **End-case 1.1.2.** condition (b) of Lemma 2.9 also holds. Finally, we are left with checking condition (b) in **End-case 1.1.1.** i.e. that CD, together with the other 4 possible bad edges, cannot form a 2Δ-configuration. Indeed, CD cannot appear in a 2Δ-configuration for BD, since BC is taken by **P2**. This concludes the proof.
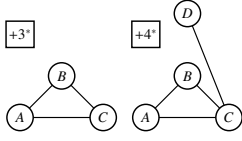
Figure 9: **Split-case 1.2.** Assume that **P1** does not have AC. Then **P2** takes this edge and then plays the edge CD, where D is a free vertex.
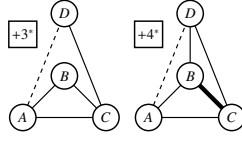
Figure 10: **End-case 1.2.1.** Assume that **P1** has AD. Then he does not have BD, as D was a **P1**-free vertex before his move. Then **P2** takes BD.
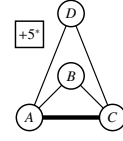The potential base is BC.

Figure 11: **End-case 1.2.2.** Assume that **P2** does not have AD. Then **P2** takes this edge. The potential base is AC.

Split-case 1.2. Note that **Split-case 1.2.** is not marked. In order to apply Lemma 2.7 to **End-case 1.2.1.** and **End-case 1.2.2.** and conclude that in these cases **P2** has a drawing strategy, we shall check that the hypotheses of Lemma 2.7 are satisfied.

In both **End-case 1.2.1.** and **End-case 1.2.2.** condition (b) of Lemma 2.7 trivially holds as **P1** has at most 5 edges in K^2. Condition (a) of Lemma 2.7 is also automatic, as **P1** has 4 and 5 additional edges, respectively.

In order to check that condition (c) of Lemma 2.7 holds, we shall check that the hypotheses of Lemma 2.9 are satisfied. Condition (c) of Lemma 2.9 is checked directly on Figures 10 and 11. Note that in **End-case 1.2.1.** the edge AD is good for BC, as AB and AC are both taken by **P2**. It follows that there are at most 4 bad edges, implying conditions (a) and (b) of Lemma 2.9. Also note that in **End-case 1.2.2.** the second edge of **P1** is either in K^1 or is in K^2 incident to B by condition "*". In either case this edge is good for AC, as both AB and BC are taken by **P2**. This allows us to conclude as in the previous case.

3.3. Applying Lemma 2.7

Let us now explain how the conditions of Lemma 2.7 are verified in the remaining end-cases with the exception of the special ones which we shall discuss further in Section 3.5. The procedure generalises the one in the example cases discussed in detail in Section 3.2.

Condition (a) of Lemma 2.7. In each of the end-cases (including the special ones), we mechanically check that **P1** has at most 5 additional edges, so he has at most 6 edges in K^1.

Condition (b) of Lemma 2.7. It is just a little harder to check that in all non-special end-cases for all G in K^2, $e_{\mathbf{P1}}(G) \leq 5$. For each marked split-case (see Figure 3) we see that **P1** loses (at least) l edges for a certain number $l \in \{1, 2\}$ which depends on the case. If **P1** loses l edges in a given split-case, it follows that he also loses l edges in all of its descendant cases. For each non-special end-case we refer to the closest ancestor marked split-case (if any) to see how many edges **P1** lost (see Figure 3) and check that condition (b) holds.

In order to establish that **P1** loses l edges in a marked split-case, we proceed as follows. Let $k + 1$ be the total number of edges taken by **P1**. We consider all non-isomorphic edges X_0X_1 in K^2, such that X_0X_1 is not taken by **P2** (so X_0X_1 could potentially become the base of a G that **P1** constructs). We count the number of edges in K^2 taken by **P1** which are of the form X_iY with YX_{1-i} not taken by **P2**, and add the edge X_0X_1 if it is taken by **P1**. This number bounds $e_{\mathbf{P1}}(G)$ for copies of G with base X_0X_1. We check that for each choice of X_0X_1 there are at most $k - l$ such edges. To exclude a large number of edges X_0X_1 from the very beginning, we first investigate $\deg_{\mathbf{P1}}(v)$ for all vertices v.

Condition (c) of Lemma 2.7. In each of the non-special end-cases, we mechanically check that **P2** has a potential base A_0A_1 – which is declared and marked in all pictures by a thickened edge. To do so it is enough to check the conditions of Lemma 2.9. Firstly, we inspect that **P1** has at most 5 bad edges for A_0A_1 (condition (a) of Lemma 2.9), using (*) to establish that some edges are good for A_0A_1 and further that the bad edges do not form a 2Δ-configuration if there are 5 of them (condition (b) of Lemma 2.9). Finally, we inspect that there exist two vertices B_1, B_2 such that **P2** has the edges $A_0B_1, A_0B_2, A_1B_1, A_1B_2$ (condition (c) of Lemma 2.9), so we can conclude.

Remark 3.1. *While the verification of the conditions of Lemma 2.7 is very easy given the strategy for **P2**, it is a central part of the paper to determine that right strategy. The idea of using additional edges and reducing drastically the number of cases helped us achieve this goal.*

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3.4. Split-case 2.

We now describe the rest of the strategy. Recall the **General split-case**, Figure 4.
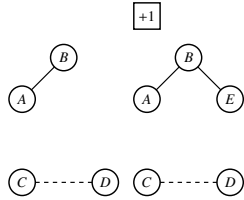

Figure 12: **Split-case 2.** Assume that the second edge taken by **P1** is *CD* with *C* and *D* free vertices in K^2. Then **P2** takes *BE*, where *E* is a free vertex in K^2.
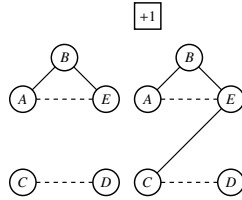

Figure 13: **Split-case 2.1.** Assume that **P1** has *AE*. Then **P2** takes *CE*.
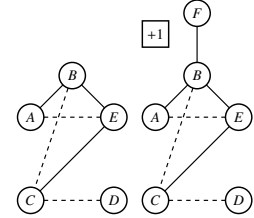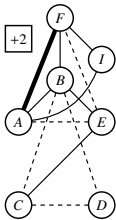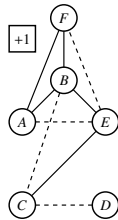

Figure 14: **Marked split-case 2.1.1.** Assume that **P1** has *BC*. Then **P2** takes *BF*, where *F* is a free vertex in K^2.
P1 lost an edge.

Figure 15: **Marked split-case 2.1.1.1.** Assume that **P1** has *EF*. Then **P2** takes *AF*.
P1 lost two edges.

Figure 16: **Split-case 2.1.1.1.1.** Assume that **P1** has *BD*. Then **P2** takes *FI*, where *I* is a free vertex in K^2.

Figure 17: **End-case 2.1.1.1.1.1.** Assume that **P1** has *AI*. Then **P2** takes *BI*.
The potential base is *BF*.

Figure 18: **End-case 2.1.1.1.1.2.** Assume that **P1** does not have *AI*. Then **P2** takes this edge.
The potential base is *AF*.

Figure 19: **Split-case 2.1.1.1.2.** Assume that **P1** does not have *BD*.

Figure 20: **Split-case 2.1.1.1.2.1.** Assume that **P1** has *AD* or *DF* and without loss of generality let this edge be *AD*. Then **P2** takes *FI*, where *I* is a free vertex in K^2.
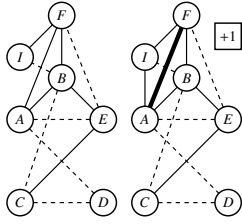
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Figure 21: **End-case 2.1.1.1.2.1.1.** Assume that **P1** has *BI*. Then **P2** takes *AI*. The potential base is *AF*.
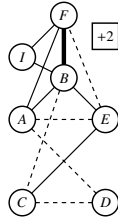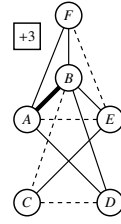

Figure 22: **End-case 2.1.1.1.2.1.2.** Assume that **P1** does not have *BI*. Then **P2** takes this edge.
The potential base is *BF*.

Figure 23: **End-case 2.1.1.1.2.2.** Assume that **P1** has none of *DA*, *DB* or *DF*. Then **P1** takes *BD* and then one of *AD* and *DF*, which is not taken. Without loss of generality, let this edge be *AD*.
The potential base is *AB*.

Figure 24: **Split-case 2.1.1.2.** Assume that **P1** does not have *EF*. Then **P2** takes this edge.

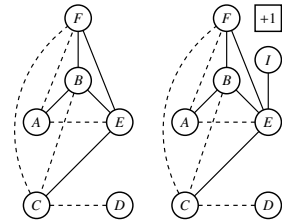Figure 25: **Split-case 2.1.1.2.1.** Assume that **P1** has *CF*.

Figure 26: **Marked split-case 2.1.1.2.1.1.** Assume that **P1** has *AF*. Then **P2** takes *EI*, where *I* is a free vertex in K^2.
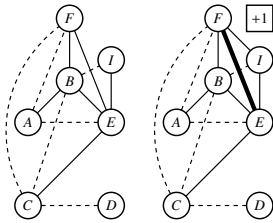**P1** lost 2 edges.

Figure 27: **End-case 2.1.1.2.1.1.1.** Assume that **P1** has *BI*. Then **P2** takes *FI*.
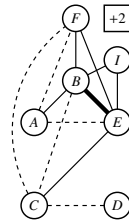The potential base is *EF*.

Figure 28: **End-case 2.1.1.2.1.1.2.** Assume that **P1** does not have *BI*. Then **P2** takes this edge.
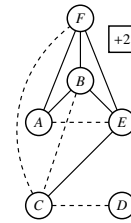The potential base is *BE*.

Figure 29: **Split-case 2.1.1.2.1.2.** Assume that **P1** does not have *AF*. Then **P2** takes this edge.
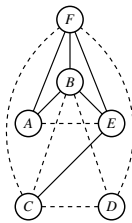

Figure 30: **Split-case 2.1.1.2.1.2.1.** Assume that **P1** has both *BD* and *DF*. Then **P2** takes *FI*, where *I* is a free vertex in K^2. If **P1** does not take *BI*, **P2** takes it and stops. Otherwise, **P1** takes *BI*, then **P2** takes *FJ*, where *J* is a free vertex in K^2. If **P1** does not take *BJ*, **P2** takes it and stops. Otherwise, **P1** takes *BJ*, then **P2** takes *FK*, where *K* is a free vertex in K^2. If **P1** does not take *BK*, **P2** takes it and stops.
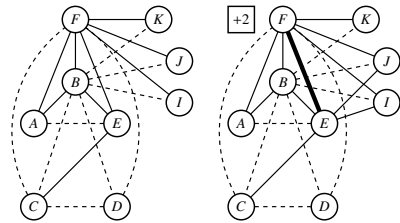

Figure 31: **Special end-case 2.1.1.2.1.2.1.1.** Assume that in the strategy described in the previous case **P1** did take *BI*, *BJ* and *BK* successively. Then **P2** takes *EI* and one of *EJ* and *EK* which remains free after **P1**'s turn. Without loss of generality, let this edge be *EJ*.
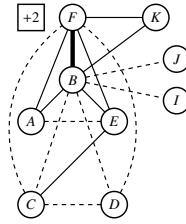This case is treated separately in Section 3.5.

Figure 32: **Special end-case 2.1.1.2.1.2.1.2.** Assume that in the strategy described in split-case **2.1.1.2.1.2.1.** **P1** did not take *BI*, then *BJ* and then *BK*. Without loss of generality we can assume that *BK* and *FK* were taken by **P2**, provided that **P2** concede both *FI* and *FJ* if they were taken by **P2** and that we give **P1** the edges *BI* and *BJ* regardless if **P1** took them (this corresponds to the case where **P1** responded the first two times, but not the third, and **P2** promises not to use the first two edges he acquired).
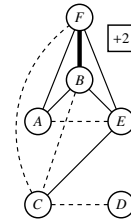
This case is treated separately in Section 3.5.

Figure 33: **End-case 2.1.1.2.1.2.2.** Assume that **P1** does not have both *BD* and *DF*, so that **P1** does not have a 2Δ-configuration for *BF*. The potential base is *BF*.
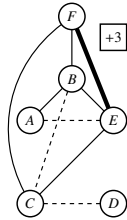
Figure 34: **End-case 2.1.1.2.2.** Assume that **P1** does not have *CF*. Then **P2** takes this edge.
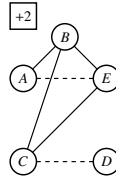
The potential base is *EF*.

Figure 35: **Split-case 2.1.2.** Assume that **P1** does not have *BC*. Then **P2** takes this edge.
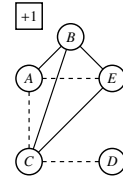
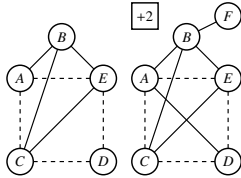Figure 36: **Split-case 2.1.2.1.** Assume that **P1** has *AC*.

Figure 37: **Marked split-case 2.1.2.1.1.** Assume that **P1** has *DE*. Then **P2** takes *AD* and then *BF*, where *F* is a free vertex in K^2. **P1** lost 2 edges.

Figure 38: **End-case 2.1.2.1.1.1.** Assume that **P1** has *EF*. Then he does not have *CF*, as *F* was a **P1**-free vertex before his move. Then **P2** takes *CF*.

The potential base is *BC*.

Figure 39: **End-case 2.1.2.1.1.2.** Assume that **P1** does not have *EF*. Then **P2** takes this edge.

The potential base is *BE*.

Figure 40: **Marked split-case 2.1.2.1.2.** Assume that **P1** does not have *DE*. Then **P2** takes this edge.

P1 lost an edge.

Figure 41: **Marked split-case 2.1.2.1.2.1.** Assume that **P1** has *BD*. Then **P2** takes *BF*, where *F* is a free vertex in K^2. **P1** lost 2 edges.

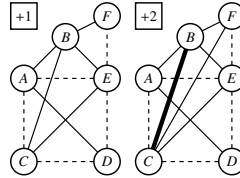Figure 42: **End-case 2.1.2.1.2.1.1.** Assume that **P1** has *EF*. Then he does not have *CF*, as *F* was a **P1**-free vertex before his move. Then **P2** takes *CF*.
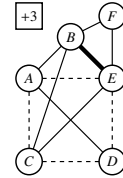
The potential base is *BC*.

Figure 43: **End-case 2.1.2.1.2.1.2.** Assume that **P1** does not have EF. Then **P2** takes this edge.
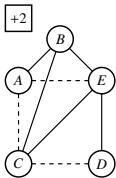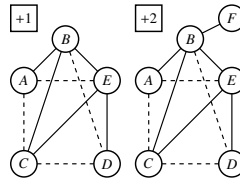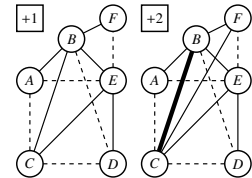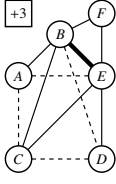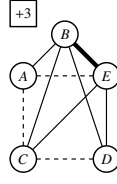The potential base is BE.

Figure 44: **End-case 2.1.2.1.2.2.** Assume that **P1** does not have BD. Then **P2** takes this edge.
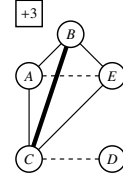The potential base is BE.

Figure 45: **End-case 2.1.2.2.** Assume **P1** does not have AC. Then **P2** takes this edge. The potential base is BC.
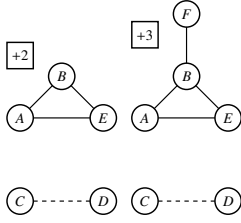

Figure 46: **Split-case 2.2.** Assume that **P1** does not have AE. Then **P2** takes this edge and plays from a **P1**-free vertex among A, B and E to a free vertex in K^2, say F. Without loss of generality assume **P2** played from B. Hence, after **P1**'s turn there is a vertex among A and E of **P1**-degree at most 1.
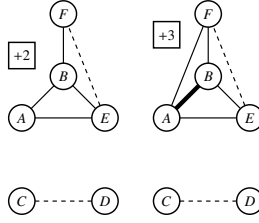

Figure 47: **End-case 2.2.1.** Assume **P1** has AF or EF. Without loss of generality assume **P1** has EF. Then he does not have AF, as F was a **P1**-free vertex before his last turn. Then **P2** takes AF.
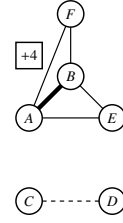The potential base is AB.

Figure 48: **End-case 2.2.2.** Assume **P1** does not have any of AF and EF. Without loss of generality assume $\deg_{\mathbf{P1}}(A) \le 1$. Then **P2** takes AF.
The potential base is AB, as $\deg_{\mathbf{P1}}(A) + \deg_{\mathbf{P1}}(B) \le 1 + 2$.

3.5. Special end-cases

In this section we present a drawing strategy for **P2** in the two special end-cases, which cannot be dealt with by directly applying Lemma 2.7.

End-case 2.1.1.2.1.2.1.1. Figure 31-right. Note that **P1** lost 3 edges (i.e. he has not won yet). If **P1** does not have EK, **P2** takes it and creates a G before him. Otherwise, one of **P1**'s additional edges is EK and hence **P1** lost 4 edges. Moreover, by inspection we observe that **P1** has no 4-cycles E, C_1, C_2, C_3 with the edge EC_2 not taken by **P2** and at most one triangle ET_1T_2. Let $\mathcal{E}_0$ be the initial set of edges taken by **P1** in K^2, so that $|\mathcal{E}_0| \le 11$. Then **P2** takes edges from F to free vertices L_i for $i = 1, \ldots, k$ until at some point **P1** does not take the edge EL_k. Then **P2** takes EL_k and completes a G. The game stops after **P2**'s move. Let $\mathcal{E}_1 = \{EL_i, 1 \le i < k\}$ be the star taken by **P1** and $\mathcal{E}_2$ be his last edge. We claim that **P1** does not have a G. Indeed, **P1** cannot have a G that does not intersect $\mathcal{E}_1$, as $e_{\mathbf{P1}}(G) \le 12 - 4$ since **P1** lost 4 edges and $|\mathcal{E}_0| + |\mathcal{E}_2| \le 12$. On the other hand, **P1** cannot have a G that contains some EL_i, as neither of the vertices E and L_i can be the base. This is because **P1** has at most 2 triangles containing E by the above observation and the fact that $\deg_{\mathbf{P1}}(L_i) \le 2$.

End-case 2.1.1.2.1.2.1.2. Figure 32. Note that **P1** lost 3 edges. (i.e. he has not won yet). Moreover, by inspection we observe that **P1** has either at most one 4-cycle F, C_1, C_2, C_3 with the edge FC_2 not taken by anyone and at most one triangle FT_1T_2, or no such 4-cycle and at most 2 such triangles. Let $\mathcal{E}_0$ be the initial set of edges taken by **P1** in K^2, so that $|\mathcal{E}_0| = 10$. Then **P2** takes edges from B to free vertices L_i for $i = 1, \ldots, k$ until at some point **P1** does not take the edge FL_k. Then **P2** takes FL_k and completes a G. The game stops after **P2**'s move. Let $\mathcal{E}_1 = \{FL_i, 1 \le i < k\}$ be the star taken by **P1** and $\mathcal{E}_2$ be his last edge. We claim that **P1** does not have a G. Indeed, **P1** cannot have a G that does not intersect $\mathcal{E}_1$, as $e_{\mathbf{P1}}(G) \le 11 - 3$ since **P1** lost 3 edges and $|\mathcal{E}_0| + |\mathcal{E}_2| \le 11$. On the other hand, **P1** cannot have a G that contains some FL_i, as neither of the vertices F and L_i can be the base. This is because **P1** has at most 3 triangles containing F by the above observation and the fact that $\deg_{\mathbf{P1}}(L_i) \le 2$.

This analysis concludes the proof of Theorem 1.2. $\qquad\qquad\square$

4. From graphs on two copies to hypergaphs

In this section we show that $\mathcal{R}(K_\omega^{(4)}, G')$ is a draw. Recall from Definition 1.6 that G' is the 4-uniform hypergraph obtained from the graph G by adding 2 new vertices X, Y and including them in all the edges. We call X, Y the two *centres*. Note that every isomorphism of G' fixes $\{X, Y\}$ and by abuse of notation refer to any copy of G' in $K_\omega^{(4)}$ by G'. Given the two points $\{X, Y\}$ the set of all hyperedges containing these points naturally identifies with the set of all edges of $K_\omega^{(2)}$. We call this set of hyperedges the *XY board*, which corresponds to the $K_\omega^{X,Y}$ notation from Section 1.3. We use the vocabulary introduced for G on the XY board and we denote hyperedges on the XY board by their other two vertices when X, Y are clear from the context. Note that the two centres have individual and joint degrees $9 = |E(G)| = |E(G')|$. The two points in the base have (individual) degree 5. The other vertices have degree 2. For the sake of conciseness, we use the immediate extensions of definitions for graphs to hypergraphs.

Proof of Theorem 1.7. We give the following explicit drawing strategy for **P2** divided in four stages.

In the first stage **P2** plays as follows. Without loss of generality **P1** takes the hyperedge $TUVW$. **P2** takes the hyperedge $XYAB$, where all vertices are free. **P1** takes a hyperedge which without loss of generality does not contain X. **P2** takes the hyperedge $XYBC$ where C is a free vertex and then **P1** takes a hyperedge. Then **P1** does not have both $XAYC$ and $XABC$, as C was a **P1**-free vertex before his move, say he does not have $XAYC$. Then **P2** takes $XAYC$ and **P1** takes another hyperedge. At this point on the XY board **P2** has the hyperedges AB, BC, AC and **P1** has at most 2 hyperedges. **P2** takes the hyperedge CD, where D is a free vertex and **P1** takes another hyperedge. Note that **P1** cannot have both DA and DB, as D was a **P1**-free vertex before his move, say he does not have DA. Then **P2** takes DA and **P1** takes another hyperedge. At the end of the first stage **P1** has 6 hyperedges $\mathcal{E}_0$ in total including $TUVW$ and at most 4 of them are on the XY board.

Claim. At the end of the first stage AC is a potential base $AC = A_0A_1$ (see Figure 11) with special vertex A_0 such that on the entire 4-uniform board **P1** has at most 2 hyperedges that contain A_0 but do not contain A_1.

Proof of the claim. If both vertices A and C can be the special vertex of AC, it suffices to choose A_0 to be one of them with at most 2 hyperedges taken by **P1** containing it but not the other by the pigeon-hole principle (as $TUVW$ is disjoint from AC). Otherwise, if, say, C is not special, then none of the hyperedges of the triangle or 4-cycle with a free hyperedge from C can contain A, so **P1** has at most 2 hyperedges containing A but not C. Therefore, A is the special vertex of the potential base AC. □

In the second stage we consider the XY board and proceed as in the proof of Lemma 2.7, though the proof requires more attention. **P2** takes edges from A_1 to free vertices F_i for $i = 1, \ldots, k$ until at some point **P1** does not take the edge A_0F_k. Then **P2** takes A_0F_k and **P1** takes two extra hyperedges. Let $\mathcal{E}_1 = \{A_0F_i, 1 \le i < k\}$ be the star taken by **P1** and $\mathcal{E}_2$ be the set consisting of the last two hyperedges taken by **P1**.

At the end of the second stage **P1** has $TUVW$, 7 other hyperedges and the star $\mathcal{E}_1$ (which does not have vertices in common with $TUVW$). For the third stage, consider three cases.

Case I. P1 has at least 8 hyperedges in a copy G^1 of G' containing $TUVW$ with its base present.

Assume without loss of generality that G^1 has centres TU and base C_0C_1. Note that $G^1 \cap \mathcal{E}_1 = \varnothing$ and so **P1** has exactly 8 hyperedges in G^1 and those are precisely $\mathcal{E}_0 \cup \mathcal{E}_2$. Let $C_{\epsilon_i}D_1$ be the edge absent in G^1. Then **P2** takes the edge $C_{1-\epsilon_1}D_1$ and, while **P1** keeps taking edges $C_{\epsilon_i}D_i$ to some vertices D_i, **P2** keeps taking edges $C_{1-\epsilon_i}D_i$ for $\epsilon_i \in \{0, 1\}$ and $2 \le i \le m$ until at some point **P1** either stops or takes a different kind of edge. Let $\mathcal{E}_3 = \{C_{\epsilon_i}D_i, 1 \le i \le m\}$ and let $\mathcal{E}_4$ be the set consisting of the last edge.

In the fourth stage **P2** moves back to the XY board. He takes edges from A_1 to free vertices F_i for $k < i \le l$ until at some point **P1** does not take the edge A_0F_l. Then **P2** takes A_0F_l and constructs a G. The game stops after **P2**'s move. Let $\mathcal{E}_5 = \{A_0F_i, k < i < l\}$ and $\mathcal{E}_6$ be the set consisting of the last edge taken by **P1**. We claim that **P1** does not have a G'.

Assume for a contradiction that **P1** has a G' with centres HI. Note that the XY and TU boards intersect only in the hyperedge $XYTU$, which is not of the form XYA_0F_j. Then on the HI board, **P1** has either at most 1 hyperedge of the TU board or no hyperedges of the form XYA_0F_j. Indeed if, on the contrary, the HI board contains TUJ_1K_1, TUJ_2K_2 and XYA_0F_j, then $HI = J_1K_1 = J_2K_2$, as $TU \cap XYA_0F_j = \varnothing$, a contradiction.

Assume that the HI board contains a hyperedge of the star $\mathcal{E}_1 \cup \mathcal{E}_5$ taken by **P1**. Then the HI board contains at most one hyperedge from the TU board and hence it contains at most one hyperedge from $\mathcal{E}_0 \cup \mathcal{E}_2 \cup \mathcal{E}_3$, since $\mathcal{E}_0 \cup \mathcal{E}_2 \subset G^1$ is in the TU board and so is $\mathcal{E}_3$ by definition. Notice that a star in one 2-uniform board is a star in any 2-uniform board, and hence the HI board contains a star from $\mathcal{E}_1 \cup \mathcal{E}_5$. Finally, the HI board contains at most two hyperedges from $\mathcal{E}_4 \cup \mathcal{E}_6$. Altogether, on the HI board, **P1** has at most a star, and three extra hyperedges. However, by Remark 2.6, this is not enough to complete a G' – contradiction.

Otherwise, assume that G' has no hyperedge of the star $\mathcal{E}_1 \cup \mathcal{E}_5$. Further, by the proof of Lemma 2.7 we can assume that $\{H, I\} \neq \{T, U\}$. However, disregarding the star $\mathcal{E}_1 \cup \mathcal{E}_5$, the only vertices with **P1**-degree at least 9 are T, U, C_0 and C_1 and the last two have joint **P1**-degree at most $1 + 2$, since $\mathcal{E}_2 \subset G^1$ is in the TU board, which has only one hyperedge C_0C_1. So, without loss of generality, we can assume that $\{H, I\} = \{T, C_0\}$. Notice that the intersection of the TU and $HI = TC_0$ boards is a star from U when viewed in the TC_0 board and $\mathcal{E}_0 \cup \mathcal{E}_2 \subset G^1$ is in the TU board and so is $\mathcal{E}_3$ by definition. Therefore, on the TC_0 board, **P1** has only a star (from U) from $\mathcal{E}_0 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ and at most 2 extra hyperedges from $\mathcal{E}_4 \cup \mathcal{E}_6$, which is not enough to complete a G' by Remark 2.6 – contradiction.

Case II. P1 has at least 8 hyperedges in a copy of G' containing $TUVW$ with its base C_0C_1 absent.

P2 takes C_0C_1 and then, in the fourth stage, proceeds as in Case I with the same proof.

Case III. For any G' containing $TUVW$, $e_{\mathbf{P1}}(G') \leq 7$.

In this case we directly skip to the fourth stage. **P2** takes hyperedges in the XY board from A_1 to free vertices F_i for $k < i \leq l$ until at some point **P1** does not take the hyperedge A_0F_l. Then **P2** takes A_0F_l and constructs a G'. The game stops after **P2**'s move. Let $\mathcal{E}_3 = \{A_0F_i, k < i < l\}$ and $\mathcal{E}_4$ be the set consisting of the last hyperedge taken by **P1**. Assume for a contradiction that **P1** has a G' with centres HI and base C_0C_1.

Note that G' cannot contain $TUVW$, since then $e_{\mathbf{P1}}(G') \leq 8$. Also, G' cannot be disjoint from both $TUVW$ and the star $\mathcal{E}_1 \cup \mathcal{E}_3$, as $e_{\mathbf{P1}}(G') \leq |(\mathcal{E}_0 \setminus \{TUVW\}) \cup \mathcal{E}_2 \cup \mathcal{E}_4| \leq 8$. Thus, G' intersects the star $\mathcal{E}_1 \cup \mathcal{E}_3$, but does not contain $TUVW$. Recall that $\deg_{\mathbf{P1}}(F_j) \leq 1 + |\mathcal{E}_2 \cup \mathcal{E}_4| \leq 4$ implies that F_j is not a centre nor is it in a base. Therefore, without loss of generality $\{H, I, C_0\} = \{X, Y, A_0\}$. In Lemma 2.7 we already proved that G' cannot be on the XY board, so without loss of generality $(H, I, C_0) = (X, A_0, Y)$. Moreover $C_1 \neq A_1$ as XYA_0A_1 is taken by **P2**. Since without $\mathcal{E}_2 \cup \mathcal{E}_4$ we have $\deg_{\mathbf{P1}}(F_i) = 1$ for all i, G' has at most $|\mathcal{E}_2 \cup \mathcal{E}_4| \leq 3$ hyperedges from the star $\mathcal{E}_1 \cup \mathcal{E}_3$. In addition, G' has at most 3 hyperedges from $\mathcal{E}_2 \cup \mathcal{E}_4$. This means that G' contains at least 3 hyperedges from $\mathcal{E}_0$. Recall that by construction, there are at most 2 hyperedges in $\mathcal{E}_0$ that contain A_0 but do not contain A_1. To arrive at the desired contradiction it is enough to show that G' does not contain any hyperedge of the form XA_0JA_1. Indeed, if such a hyperedge is in G', then necessarily $J = C_1$ or $J = C_0$, since it needs to intersect the base C_0C_1. However, the hyperedge $XA_0C_0A_1 = XA_0YA_1$ is taken by **P2** and the hyperedge $XA_0C_1A_1$ cannot be in G' because **P2** has $XA_0C_0A_1 = XA_0YA_1$. $\quad\square$

Remark 4.1. *One can easily extend this proof to obtain r-uniform graphs with the same property for all $r \geq 4$.*

5. Concluding remarks

5.1. Key aspects

As the strategy is quite cumbersome and does not have a clear structure, it is not hard to miss the forest for the trees. Let us point out a few aspects which are important for the understanding of the dynamic of the Ramsey game in general.

The strategy reflects the significant differences between weak and strong games. The most important feature which arises in this context is the notion of delay: unlike the strategy in [16], in ours **P2** does not focus solely on building a core fast, but rather needs to delay **P1** at least as much as **P1** delays him. We now point out some parts of the strategy which best highlight this feature.

Firstly, as it is clear from Figure 3, **Case 1.** is much easier than **Case 2.**, so the second move of **P1** is crucially important. Unexpectedly, the harder **Case 2.** corresponds to two seemingly contradictory behaviours of **P1**. On the one hand, **P1** goes after **P2** in K^2, which can only have the purpose of blocking **P2**, since **P2** is the first player there. On the other hand, **P1** plays disjointly from **P2**'s edge, which is not the most natural of "blocking" moves. From another perspective, this is less surprising. Given that we prove exactly that **P2** can force a draw by moving away

from **P1** in the other copy of K_ω, it is reasonable to expect that the most efficient move of **P1** would be to also start playing "away" from **P2**.

Another striking fact is that the "right" move of **P2** in **Case 2.1.** is to go after **P1**, who is the "second player" in K^2. The purpose of this move is to constrain **P1**'s possibilities and later force **P1** to lose edges. This philosophy is even more visible in **Case 2.1.2.1.1.**. There **P2** takes an edge which is completely useless to him from a "Maker" perspective, but makes **P1** lose 2 edges, so it is as though by playing this "Breaker" move **P2** gets a net advantage of 1 edge.

These examples show that a good strong game strategy should by all means seek to interact with the other player. However, the main approach to strong games so far relies on finding fast "Maker" strategies for the corresponding weak game and transforming them into strategies for the strong games [11, 12, 16]. Thus, our strategy suggests that the Ramsey game requires more sophisticated arguments.

On a more technical note, the idea of leaving additional edges unspecified and disregarding the stage of the game when an edge was played decreases the number of cases tremendously. This approach allows us to only have the *tiny* tree in Figure 3, which can readily be analysed without the help of a computer in short time. The driving force of the drawing strategy is Lemma 2.7, which provides us with intermediate desired configurations together with a drawing strategy from these configurations.

5.2. Open problems

The present work raises various natural closely related questions which, in view of our results, do not seem as widely open and intractable as previously.

The prime question to answer is the following.

Question 5.1. *Is the game $\mathcal{R}(K_\omega, G)$ a draw?*

As we already mentioned if the answer is positive, then this would be the first such known graph. Otherwise, G would be the answer to the next question.

Question 5.2. *Does there exist a target graph H such that $\mathcal{R}(K_\omega, H)$ is a **P1**-win, but $\mathcal{R}(K_\omega \sqcup K_\omega, H)$ is not?*

Such a counterexample would prove that yet another easy and folklore fact for finite boards breaks down for infinite ones.

Another line of thought, which might not lead directly towards $\mathcal{R}(K_n, K_k)$, was suggested by Bowler [9]. He proved the natural characterisation of positional games which are a draw on infinite disjoint copies of a finite one. One can wonder if the game on G gives a draw on an infinite number of copies of K_n. We believe that this is not the case.

Conjecture 5.3. *For n sufficiently large $\mathcal{R}(\bigsqcup_\omega K_n, G)$ is a **P1**-win.*

Finally, this last conjecture seems closely related to determining if the initiative in the sense of [9] of $\mathcal{R}(K_n, K_4)$ is 6, as asked in the next question.

Question 5.4. *Let n be a fixed large integer. What is the minimal k such that **P1** has a winning strategy for $\mathcal{R}(K_n, K_4)$ such that if the second player is allowed to pass, he would not be able to do so more than k times before losing?*

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