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Ruin probabilities for a Lévy-driven generalized Ornstein–Uhlenbeck process

Yuri Kabanov · Serguei Pergamenschikov

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Abstract We study the asymptotic of the ruin probability for a process which is the solution of linear SDE defined by a pair of independent Lévy processes. Our main interest is the model describing the evolution of the capital reserve of an insurance company selling annuities and investing in a risky asset. Let $\beta > 0$ be the root of the cumulant-generating function H of the increment of the log price process V_1 . We show that the ruin probability admits the exact asymptotic $Cu^{-\beta}$ as the initial capital $u \rightarrow \infty$ assuming only that the law of V_T is non-arithmetic without any further assumptions on the price process.

Keywords Ruin probabilities · Dual models · Price process · Renewal theory · Distributional equation · Autoregression with random coefficients · Lévy process

Mathematics Subject Classification (2000) 60G44

JEL Classification G22 · G23

1 Introduction

The general ruin problem can be formulated as follows. We are given a family of scalar processes X^u with the initial values $u > 0$. The object of interest is the exit probability of X^u from the positive half-line as a function of u . More formally, let $\tau^u := \inf\{t: X_t^u \leq 0\}$. The question is to determine the function $\Psi(u, T) := \mathbf{P}(\tau^u \leq T)$ (the ruin probability on a finite interval $[0, T]$) or $\Psi(u) := \mathbf{P}(\tau^u < \infty)$ (the ruin probability on $[0, \infty[$). In the particular case where $X^u = u + X^0$ the function $\Psi(u)$ is the tails of distribution function of random variable $\sup_{t \geq 0} (-X^0)$, respectively.

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The exact solution of the problem is available only in a few rare cases. For instance, for $X^u = u + W$ where W is the Wiener process $\Psi(u, T) = \mathbf{P}(\sup_{t \leq T} W_t \geq u)$ and it remains to recall that the explicit formula for the distribution of the supremum of the Wiener process was obtained already in the Louis Bachelier thesis of 1900 which is, probably, the first ever mathematical study on continuous stochastic processes. Another example is the well-known explicit formula for $\Psi(u)$ in the Lundberg model of the ruin of insurance company with exponential claims, i.e. when $X^u = u + P$ and P is a compound Poisson process with drift and exponential jumps. Of course, for more complicated cases the explicit formulae are not available and only asymptotic results or bounds can be obtained as it is done, e.g., in the Lundberg–Cramér theory. In particular, if $\mathbf{E} P_1 > 0$ and the size of jumps are random variables satisfying the Cramér condition (i.e. with finite exponential moments), then $\Psi(u)$ is exponentially decreasing as $u \rightarrow \infty$.

In this paper we consider the ruin problem for a rather general model, suggested by Paulsen in [32], in which X^u (sometimes called the generalized Ornstein–Uhlenbeck process) is given as the solution of linear stochastic equation

$$X_t^u = u + P_t + \int_{]0, t]} X_{s-}^u dR_s, \quad (1.1)$$

where R and P are independent Lévy processes with their Lévy triplets (a, σ^2, Π) and (a_P, σ_P^2, Π_P) , respectively.

There is a growing interest in models of this type because they describe the evolution of reserves of insurance companies investing in a risky asset with the price process S . In the financial-actuarial context R is interpreted as the **relative price process** with $dR_t = dS_t/S_{t-}$, i.e. the price process S is the stochastic (Doléans) exponential $\mathcal{E}(R)$. The equation (1.1) means that the (infinitesimal) increment of the capital reserve dX_t^u is the sum of the increment dP_t due to the insurance business activity and the increment due to the risky placement which is the product of number of owned shares X_{t-}^u/S_{t-} and the price increment of a share dS_t , that is $X_{t-}^u dR_t$.

In this model the **log price process** $V = \ln \mathcal{E}(R)$ is also a Lévy process with the triplet (a_V, σ^2, Π_V) . Recall that the behavior of the ruin probability in such models is radically different from that in the classical actuarial models. For instance, if the price of the risky asset follows a geometric Brownian motion, that is, $R_t = at + \sigma W_t$, and the risk process P is as in the Lundberg model, then $\Psi(u) = O(u^{1-2a/\sigma^2})$, $u \rightarrow \infty$, if $2a/\sigma^2 > 1$, and $\Psi(u) \equiv 1$ otherwise, [16], [23], [36].

We exclude degenerate cases by assuming that $\Pi(]-\infty, -1]) = 0$ (otherwise $\Psi(u) = 1$ for all $u > 0$, see discussion in Section 2) and P is not a subordinator (otherwise $\Psi(u) = 0$ for all $u > 0$ because $X^u > 0$, see (3.2), (3.1)). Also **we exclude** the case $R = 0$ well studied in the literature, see [26].

We are especially interested in the case where the process P describing the “business part” of the model has only upward jumps (in other words, P is spectrally positive). In the classical actuarial literature such models are referred to as the annuity insurance models (or models with negative risk sums), [18], [38], while in modern sources they serve also to describe the capital reserve of a venture company investing in development of new technologies and selling innovations; sometimes they are referred to as the dual models, [1] – [3], [5], etc.

The mentioned specificity of models with negative risk sums leads to a continuous down-crossing of the zero level by the capital reserve process. This allows us to obtain the exact (up to a multiplicative constant) asymptotic of the ruin probability under weak assumptions on the price dynamics.

Let $H : q \mapsto \ln \mathbf{E} e^{-qV_1}$ be the cumulant-generating function of the increment of log price process V on the interval $[0, 1]$. The function H is convex and its effective domain $\text{dom } H$ is a convex subset of \mathbb{R} containing zero.

If the distribution of jumps of the business process has not too heavy tails, the asymptotic of the ruin probability $\Psi(u)$ as $u \rightarrow \infty$ is determined by the strictly positive root β of H , assumed existing and laying in the interior of $\text{dom } H$. Unfortunately, the existing results are overloaded by numerous integrability assumptions on processes R and P while the law $\mathcal{L}(V_T)$ of the random variable V_T is required to contain an absolute continuous component where T is independent random variable uniformly distributed on $[0, 1]$, see, e.g., Th. 3.2 in [34] whose part (b) provides an information how heavier tails may change the asymptotic.

The aim of our study is to obtain the exact asymptotic of the exit probability in this now classical framework under the weakest conditions. Our main result has the following easy to memorize formulation.

Theorem 1.1 *Suppose that H has a root $\beta > 0$ not laying on the boundary of $\text{dom } H$ and $\int |x|^\beta I_{\{|x|>1\}} \Pi_P(dx) < \infty$. Then*

$$0 < \liminf_{u \rightarrow \infty} u^\beta \Psi(u) \leq \limsup_{u \rightarrow \infty} u^\beta \Psi(u) < \infty. \quad (1.2)$$

If, moreover, P jumps only upward and the distribution $\mathcal{L}(V_1)$ is non-arithmetic¹, then $\Psi(u) \sim C_\infty u^{-\beta}$ where $C_\infty > 0$.

In our argument we are based, as many other authors, on the theory of distributional equations as presented in the paper by Goldie, [17]. Unfortunately, Goldie's theory does not give a clear answer when the constant defining the asymptotic of the tail of the solution of an affine distributional equation is strictly positive. The striking simplicity of our formulation is due to a recent progress in this theory, namely, due to the criterion by Guivarc'h and Le Page, [20], which simple proof can be found in the paper [10] by Buraczewski and Damek. This criterion gives a necessary and sufficient condition for the strict positivity of the constant in the Kesten–Goldie theorem determining the rate of decay of the tail of solution at infinity. Its obvious corollary allows us to simplify radically the proofs and get rid of additional assumptions presented in the earlier papers, see [24], [4], [30] – [35] and references therein. Our technique involves only affine distributional equations and avoids more demanding Letac-type equations.

The question whether the concluding statement of the theorem holds when P has downward jumps remains open.

The structure of the paper is the following. In Section 2 we formulate the model and provide some prerequisites from Lévy processes. Section 3 contains a well-know reduction of the ruin problem to the study of asymptotic behavior of a stochastic integral (called in the actuarial literature continuous perpetuity, see [13]). In Section 4 we prove moment inequalities for maximal functions of stochastic integrals needed to analyze the limiting behavior of an exponential functional in Section 5. The latter section is concluded by the proof of the main result and some comments on its formulation. In Section 6 we establish Theorem 6.4 on the ruin with probability one using the technique suggested in [36]. This theorem implies, in particular, that in the classical model with negative risk sums and investments in the risky asset with price following a geometric Brownian motion the ruin is imminent if $a \leq \sigma^2/2$, [23]. In Section 7 we discuss examples.

¹ That is, the distribution is not concentrated on a set $\mathbb{Z}d = \{0, \pm d, \pm 2d, \dots\}$.

Our presentation is oriented towards the reader with preferences in the Lévy processes rather than in the theory of distributional equations (called also implicit renewal theory). That is why in Section 8 (Appendix) we provide a rather detailed information on the latter covering the arithmetic case. In particular, we give a proof of a version of the Grincevičius theorem under slightly weaker conditions as in the original paper.

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2 Preliminaries from the theory of Lévy processes

Let (a, σ^2, Π) and (a_P, σ_P^2, Π_P) be the Lévy triplets of the processes R and P corresponding to the standard² truncation function $h(x) := xI_{\{|x| \leq 1\}}$. Putting $\bar{h}(x) := xI_{\{|x| > 1\}}$ we can write the canonical decomposition of R in the form

$$R_t = at + \sigma W_t + h * (\mu - \nu)_t + \bar{h} * \mu_t \quad (2.1)$$

where W is a standard Wiener process, the Poisson random measure $\mu(dt, dx)$ is the jump measure of R having the deterministic compensator of the form $\nu(dt, dx) = dt\Pi(dx)$. For notions and results see [22], Ch. 2, [8], and also [12], Chs. 2 and 3.

As in [22], we use $*$ for the standard notation of stochastic calculus for integrals with respect to random measures. For instance,

$$h * (\mu - \nu)_t = \int_0^t \int h(x)(\mu - \nu)(ds, dx).$$

We hope that the reader will be not confused that $f(x)$ may denote the whole function f or its value at x ; the typical example is $\ln(1+x)$ explaining why such a flexibility is convenient. The symbols $\Pi(f)$ or $\Pi(f(x))$ stands for the integral of f with respect to the measure Π .

Recall that

$$\Pi(x^2 \wedge 1) := \int (x^2 \wedge 1)\Pi(dx) < \infty$$

and the condition $\sigma = 0$ and $\Pi(|h|) < \infty$ is necessary and sufficient for R to have trajectories of (locally) finite variation, see Prop. 3.9 in [12].

The process P describing the actuarial (“business”) part of the model admits a similar representation:

$$P_t = a_P t + \sigma_P W_t^P + h * (\mu^P - \nu^P)_t + \bar{h} * \mu_t^P. \quad (2.2)$$

The Lévy processes R and P generate the filtration $\mathbf{F}^{R,P} = (\mathcal{F}_t^{R,P})_{t \geq 0}$.

Standing assumption S.0 *The Lévy measure Π is concentrated on the interval $] - 1, \infty[$; σ^2 and Π do not vanish simultaneously; the process P is not a subordinator.*

Recall that if Π charges $] - 1, \infty[$, then the ruin happens at the instant τ of the first jump of the Poisson process $I_{\{x \leq -1\}} * \mu$ of strictly positive intensity. Indeed, the independence of processes P and R implies that their trajectories have no common instants of jumps (except a null set). Note that $\tau = \inf\{t \geq 0 : xI_{\{x \leq -1\}} * \mu_t \leq -1\} < \infty$, when $\Pi(]-\infty, -1]) > 0$, and $\Delta R_\tau \leq -1$. According to (1.1) $\Delta X_\tau = X_{\tau-} \Delta R_\tau$, that is $X_\tau = X_{\tau-}(\Delta R_\tau + 1)$. It follows that $\tau^u \leq \tau < \infty$.

² Other truncation functions are also used in the literature, see, e.g., [34].

If Π does not charge $] -1, \infty[$, but P is a subordinator, that is an increasing Lévy process, the ruin never happens. According to [12], Prop. 3.10, the process P is not a subordinator if and only if $\sigma_P^2 > 0$, or one of the following three conditions hold:

- 1) $\Pi_P(] - \infty, 0]) > 0$,
- 2) $\Pi_P(] - \infty, 0]) = 0$, $\Pi_P(xI_{\{x>0\}}) = \infty$,
- 3) $\Pi_P(] - \infty, 0]) = 0$, $\Pi_P(xI_{\{x>0\}}) < \infty$, $a_P - \Pi_P(xI_{\{0<x\leq 1\}}) < 0$.

Under the first assumption of **S.0** $\Delta R > -1$ and the stochastic exponential, solution of the linear equation $dZ = Z_- dR$, $Z_0 = 1$, has the form

$$\mathcal{E}_t(R) = e^{R_t - \frac{1}{2}\sigma^2 t + \sum_{s \leq t} (\ln(1 + \Delta R_s) - \Delta R_s)}.$$

In the context of financial models it stands for the price of a risky asset (e.g., stock). The log price $V := \ln \mathcal{E}(R)$ is a Lévy process and can be written in the form

$$V_t = at - \frac{1}{2}\sigma^2 t + \sigma W_t + h * (\mu - \nu)_t + (\ln(1 + x) - h) * \mu_t. \quad (2.3)$$

Its Lévy triplet is (a_V, σ^2, Π_V) where

$$a_V = a - \frac{\sigma^2}{2} + \Pi(h(\ln(1 + x)) - h)$$

and $\Pi_V = \Pi \varphi^{-1}$, $\varphi : x \mapsto \ln(1 + x)$.

The cumulant-generating function $H : q \rightarrow \ln \mathbf{E} e^{-qV_1}$ of the random variable V_1 admits an explicit expression. Namely,

$$H(q) := -a_V q + \frac{\sigma^2}{2} q^2 + \Pi(e^{-q \ln(1+x)} - 1 + qh(\ln(1+x))). \quad (2.4)$$

Its effective domain $\text{dom } H = \{q : H(q) < \infty\}$ is the set $\{J(q) < \infty\}$ where

$$J(q) := \Pi(I_{\{|\ln(1+x)| > 1\}} e^{-q \ln(1+x)}) = \Pi(I_{\{|\ln(1+x)| > 1\}} (1+x)^{-q}). \quad (2.5)$$

Its interior is the open interval $]q, \bar{q}[$ with

$$\underline{q} := \inf\{q \leq 0 : J(q) < \infty\}, \quad \bar{q} := \sup\{q \geq 0 : J(q) < \infty\}.$$

Being a convex function, H is continuous and admits finite right and left derivatives on $]q, \bar{q}[$. If $\bar{q} > 0$, then the right derivative

$$D^+ H(0) = -a_V - \Pi(\bar{h}(\ln(1+x))) < \infty,$$

though it may be equal to $-\infty$.

In formulations of our asymptotic results we shall always assume that $\bar{q} > 0$ and the equation $H(q) = 0$ has a root $\beta \in]0, \bar{q}[$. Since H is not constant, such a root is unique. Clearly, it exists if and only if $D^+ H(0) < 0$ and $\limsup_{q \uparrow \bar{q}} H(q)/q > 0$. In the case where $\underline{q} < 0$ the condition $D^- H(0) > 0$ is necessary to ensure that $H(q) < 0$ for sufficiently small in absolute value $q < 0$.

If $J(q) < \infty$, then the process $m = (m_t(q))_{t \leq 1}$ with

$$m_t(q) := e^{-qV_t - tH(q)} \quad (2.6)$$

is a martingale and

$$\mathbf{E} e^{-qV_t} = e^{tH(q)}, \quad t \in [0, 1]. \quad (2.7)$$

In particular, we have that $H(q) = \ln \mathbf{E} e^{-qV_1} = \ln \mathbf{E} M^q$ where $M := e^{-V_1}$. For the above properties see, e.g., Th. 25.17 in [37].

Note that

$$\mathbf{E} \sup_{t \leq 1} e^{-qV_t} < \infty \quad \forall q \in]q, \bar{q}[. \quad (2.8)$$

Indeed, let $q \in]0, \bar{q}[$. Take $r \in]1, \bar{q}/q[$. Then $\mathbf{E} m_1^r(q) = e^{H(qr) - rH(q)} < \infty$. By virtue of the Doob inequality the maximal function $m_1^*(q) := \sup_{t \leq 1} m_t(q)$ belongs to L^r and it remains to observe that $e^{-qV_t} \leq C_q m_t(q)$ where the constant $C_q = \sup_{t \leq 1} e^{tH(q)}$. Similar arguments work for $q \in]q, 0[$.

3 Ruin problem: a reduction

Let us introduce the process

$$Y_t := - \int_{]0, t]} \mathcal{E}_{s-}^{-1}(R) dP_s = - \int_{]0, t]} e^{-V_{s-}} dP_s. \quad (3.1)$$

Due to independence of P and R the joint quadratic characteristic $[P, R]$ is zero, and the straightforward application of the product formula for semimartingales shows that the process

$$X_t^u := \mathcal{E}_t(R)(u - Y_t) \quad (3.2)$$

solves the non-homogeneous linear equation (1.1), i.e. the solution of the latter is given by this stochastic version of the Cauchy formula.

The strict positivity of $\mathcal{E}(R) = e^V$ implies that $\tau^u = \inf\{t \geq 0 : Y_t \geq u\}$.

The following lemma is due to Paulsen, see [32].

Lemma 3.1 *If $Y_t \rightarrow Y_\infty$ almost surely as $t \rightarrow \infty$ where Y_∞ is a finite random variable unbounded from above, then for all $u > 0$*

$$\bar{G}(u) \leq \Psi(u) = \frac{\bar{G}(u)}{\mathbf{E}(\bar{G}(X_{\tau^u}) | \tau^u < \infty)} \leq \frac{\bar{G}(u)}{\bar{G}(0)}, \quad (3.3)$$

where $\bar{G}(u) := \mathbf{P}(Y_\infty > u)$.

In particular, if $\Pi_P(]-\infty, 0]) = 0$, then $\Psi(u) = \bar{G}(u)/\bar{G}(0)$.

Proof. Let τ be an arbitrary stopping time with respect to the filtration $(\mathcal{F}_t^{P,R})$. As we assume that the finite limit Y_∞ exists, the random variable

$$Y_{\tau, \infty} := \begin{cases} - \lim_{N \rightarrow \infty} \int_{] \tau, \tau + N]} e^{-(V_t - V_\tau)} dP_t, & \tau < \infty, \\ 0, & \tau = \infty, \end{cases}$$

is well defined. On the set $\{\tau < \infty\}$

$$Y_{\tau, \infty} = e^{V_\tau}(Y_\infty - Y_\tau) = X_\tau^u + e^{V_\tau}(Y_\infty - u). \quad (3.4)$$

Let ξ be a $\mathcal{F}_\tau^{P,R}$ -measurable random variable. Since the Lévy process Y starts afresh at τ , the conditional distribution of $Y_{\tau,\infty}$ given $(\tau, \xi) = (t, x) \in \mathbb{R}_+ \times \mathbb{R}$ is the same as the distribution of Y_∞ . It follows that

$$\mathbf{P}(Y_{\tau,\infty} > \xi, \tau < \infty) = \mathbf{E} \bar{G}(\xi) \mathbf{1}_{\{\tau < \infty\}}.$$

Thus, if $\mathbf{P}(\tau < \infty) > 0$, then

$$\mathbf{P}(Y_{\tau,\infty} > \xi, \tau < \infty) = \mathbf{E}(\bar{G}(\xi) | \tau < \infty) \mathbf{P}(\tau < \infty).$$

Noting that $\Psi(u) := \mathbf{P}(\tau^u < \infty) \geq \mathbf{P}(Y_\infty > u) > 0$, we deduce from here using (3.4) that

$$\begin{aligned} \bar{G}(u) &= \mathbf{P}(Y_\infty > u, \tau^u < \infty) = \mathbf{P}(Y_{\tau^u,\infty} > X_{\tau^u}^u, \tau^u < \infty) \\ &= \mathbf{E}(\bar{G}(X_{\tau^u}^u) | \tau^u < \infty) \mathbf{P}(\tau^u < \infty) \end{aligned}$$

implying the equality in (3.3). The result follows since $X_{\tau^u}^u \leq 0$ on the set $\{\tau^u < \infty\}$ and, in the case where $\Pi_P(\cdot - \infty, 0] = 0$, the process X^u crosses zero in a continuous way, i.e. $X_{\tau^u}^u = 0$ on this set. \square

In view of the above lemma the proof of Theorem 1.1 is reduced to establishing of the existence of finite limit Y_∞ and finding the asymptotic of the tail of its distribution.

4 Moments of the maximal function

In this section we prove a simple but important result implying the existence of moments of the random variable Y_1^* . Here and in the sequel we use the standard notation of stochastic calculus for the maximal function of a process: $Y_t^* := \sup_{s \leq t} |Y_s|$.

Before the formulation we recall the Novikov inequalities, [29], also referred to as the Bichteler–Jacod inequalities, see [9], [28], providing bounds for moments of the maximal function I_1^* of stochastic integral $I = g * (\mu^P - \nu^P)$ where $g^2 * \nu_1^P < \infty$. In dependence of the parameter $\alpha \in [1, 2]$ they have the following form:

$$\mathbf{E} I_1^{*p} \leq C_{p,\alpha} \begin{cases} \mathbf{E} (|g|^\alpha * \nu_1^P)^{p/\alpha}, & \forall p \in]0, \alpha], \\ \mathbf{E} (|g|^\alpha * \nu_1^P)^{p/\alpha} + \mathbf{E} |g|^p * \nu_1^P, & \forall p \in [\alpha, \infty[. \end{cases} \quad (4.1)$$

Let U be a càdlàg process adapted with respect to a filtration under which the semimartingale P has deterministic triplet (a_P, σ_P^2, Π_P) and let $\Upsilon_t := \int_{[0,t]} U_{s-} dP_s$.

Lemma 4.1 *If $p > 0$ is such that $\Pi_P(|\bar{h}|^p) < \infty$, $K_p := \mathbf{E} U_1^{*p} < \infty$, then $\mathbf{E} \Upsilon_1^{*p} < \infty$.*

Proof. The elementary inequalities $|x + y|^p \leq |x|^p + |y|^p$ for $p \in]0, 1]$ and

$$|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p) \quad \text{for } p > 1$$

allows us to treat separately the integrals corresponding to each term in the representation

$$P_t = a_P t + \sigma_P W_t^P + h * (\mu^P - \nu^P)_t + \bar{h} * \mu_t^P,$$

that is, assuming that other terms are zero.

The case of the integral with respect to dt is obvious (we dominate U by U^*). The estimation for integral with respect to dW^P is reduced, by applying the Burkholder–Davis–Gundy inequality reduces to the estimation of the integral with respect to dt .

Let $p < 1$. In the more detailed notations $f * \mu_1^P = \sum_{\{s \leq 1: \Delta P_s > 0\}} f(s, \Delta P_s)$ and $U_- = (U_{t-})$. Therefore,

$$\mathbf{E}(|U_-|\bar{h}| * \mu_1^P)^p \leq \mathbf{E}|U_-|^p |\bar{h}|^p * \mu_1^P = \mathbf{E}|U_-|^p |\bar{h}|^p * \nu_1^P \leq \Pi_P(|\bar{h}|^p) K_p.$$

Using the Novikov inequality (with $\alpha = 2$) we have:

$$\mathbf{E}(U_- h * (\mu^P - \nu^P))_1^{*p} \leq C_{p,2} (\Pi_P(h^2))^{p/2} \mathbf{E} \left(\int_0^1 U_t^2 dt \right)^{p/2} \leq C_{p,2} (\Pi_P(h^2))^{p/2} K_p.$$

Let $p \in]1, 2[$. By the Novikov inequality with $\alpha = 1$ and we have:

$$\mathbf{E}(U_- \bar{h} * (\mu^P - \nu^P))_1^{*p} \leq C_{p,1} \left((\mathbf{E}(|U_-|\bar{h}| * \nu_1^P)^p + \mathbf{E}|U_-|^p |\bar{h}|^p * \nu_1^P) \right) \leq \tilde{C}_{p,1} K_p,$$

where $\tilde{C}_{p,1} := C_{p,1} \left((\Pi_P(|\bar{h}|))^p + \Pi_P(|\bar{h}|^p) \right)$.

Using again the Novikov inequality but with $\alpha = 2$ we obtain that

$$\mathbf{E}(U_- h * (\mu^P - \nu^P))_1^{*p} \leq C_{p,2} \mathbf{E}(U_-^2 h^2 * \nu_1^P)^{p/2} \leq C_{p,2} (\Pi_P(h^2))^p K_p.$$

Finally, let $p \geq 2$. Using the Novikov inequality with $\alpha = 2$, we have:

$$\begin{aligned} \mathbf{E}(U_- x * (\mu^P - \nu^P))_1^{*p} &\leq C_{p,2} (\Pi_P(|x|^2))^{p/2} \mathbf{E} \left(\int_0^1 U_t^2 dt \right)^{p/2} \\ &\quad + C_{p,2} \Pi_P(|x|^p) \mathbf{E} \int_0^1 |U_t|^p dt \\ &\leq C_{p,2} \left((\Pi_P(|x|^2))^{p/2} + \Pi_P(|x|^p) \right) K_p. \end{aligned}$$

Combining the above estimate we conclude that $\mathbf{E} \gamma_1^{*p} \leq C K_p$ for some constant C . \square

5 Convergence of Y_t

Using Lemma 4.1 the almost sure convergence of Y_t given by (3.1) to a finite random variable Y_∞ can be easily established under very weak assumptions ensuring also that Y_∞ solves an affine distributional equation and is unbounded from above. Namely, we have the following:

Proposition 5.1 *If there is $p > 0$ such that $H(p) < 0$ and $\Pi_P(|\bar{h}|^p) < \infty$, then Y_t converge a.s. to a finite random variable Y_∞ unbounded from above. Its law $\mathcal{L}(Y_\infty)$ is the unique solution of the distributional equation*

$$Y_\infty \stackrel{d}{=} Y_1 + M_1 Y_\infty, \quad Y_\infty \text{ independent of } (M_1, Y_1), \quad (5.1)$$

where $M_1 := e^{-V_1}$.

Proof. If the hypotheses hold for some p , they hold also for smaller values. We assume without loss of generality that $p < 1$ and $H(p+) < \infty$. For any integer $j \geq 1$ we have the identity

$$Y_j - Y_{j-1} = M_1 \dots M_{j-1} Q_j, \quad .$$

where (M_j, Q_j) are independent random vectors with components

$$M_j := e^{-(V_j - V_{j-1})}, \quad Q_j := - \int_{]j-1, j]} e^{-(V_v - V_{j-1})} dP_v \quad (5.2)$$

having distributions $\mathcal{L}(M_j) = \mathcal{L}(M_1)$ and $\mathcal{L}(Q_j) = \mathcal{L}(Y_1)$. By assumption, the value $\rho := \mathbf{E}M_1^p = e^{H(p)} < 1$ and $\mathbf{E}|Y_1|^p < \infty$ in virtue of (2.8) and Lemma 4.1. Since $\mathbf{E}M_1 \dots M_{j-1} |Q_j| = \rho^j \mathbf{E}|Y_1|^p$ we have that $\mathbf{E} \sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty$ and, therefore, $\sum_{j \geq 1} |Y_j - Y_{j-1}|^p < \infty$ a.s. But then also $\sum_{j \geq 1} |Y_j - Y_{j-1}| < \infty$ a.s. and, therefore, the sequence Y_n converges almost surely to the random variable $Y_\infty := \sum_{j \geq 1} (Y_j - Y_{j-1})$.

Put

$$\Delta_n := \sup_{n-1 \leq v \leq n} \left| \int_{]n-1, v]} e^{-V_s} dP_s \right|, \quad n \geq 1.$$

Note that

$$\mathbf{E} \Delta_n^p = \mathbf{E} \prod_{j=1}^{n-1} M_j^p \sup_{n-1 \leq v \leq n} \left| \int_{]n-1, v]} e^{-(V_s - V_{n-1})} dP_s \right|^p = \rho^{n-1} \mathbf{E} Y_1^{*p} < \infty.$$

For any $\varepsilon > 0$ we get using the Chebyshev inequality that

$$\sum_{n \geq 1} \mathbf{P}(\Delta_n > \varepsilon) \leq \varepsilon^{-p} \mathbf{E} Y_1^{*p} \sum_{n \geq 1} \rho^{n-1} < \infty.$$

By the Borel–Cantelli lemma $\Delta_n(\omega) \leq \varepsilon$ for all $n \geq n_0(\omega)$ for each $\omega \in \Omega$ except a null-set. This implies the convergence $Y_t \rightarrow Y_\infty$ a.s., $t \rightarrow \infty$.

Let us consider the sequence

$$Y_{1,n} := Q_2 + M_2 Q_3 + \dots + M_2 \dots M_n Q_{n+1}$$

converging almost surely to a random variable $Y_{1,\infty}$ distributed as Y_∞ . Passing to the limit in the obvious identity $Y_n = Q_1 + M_1 Y_{1,n-1}$ we obtain that $Y_\infty = Q_1 + M_1 Y_{1,\infty}$. For finite n the random variables $Y_{1,n}$ and (M_1, Q_1) are independent, $\mathcal{L}(Y_{1,n}) = \mathcal{L}(Y_n)$. Hence, $Y_{1,\infty}$ and (M_1, Q_1) are independent, $\mathcal{L}(Y_{1,\infty}) = \mathcal{L}(Y_\infty)$ and $\mathcal{L}(Y_\infty) = \mathcal{L}(Q_1 + M_1 Y_{1,\infty})$. This is exactly the properties abbreviated by (5.1).

Note that our hypothesis insures the uniqueness of the affine distributional equation (5.1). Indeed, any its (finite) solution \tilde{Y}_∞ can be realized on the same probability space as Y_∞ as a random variable independent on the sequence (M_j, Q_j) . Then

$$\mathcal{L}(\tilde{Y}_\infty) = \mathcal{L}(Q_1 + M_1 \tilde{Y}_\infty) = \mathcal{L}(Q_1 + M_1 Q_2 + \dots + M_1 \dots M_{n-1} Q_n + M_1 \dots M_n \tilde{Y}_\infty).$$

Since the product $M_1 \dots M_n \rightarrow 0$ in L^p as $n \rightarrow \infty$, hence, in probability, the residual term also tends to zero in probability. Thus, $\mathcal{L}(\tilde{Y}_\infty) = \mathcal{L}(Y_\infty)$.

It remains to check that Y_∞ is unbounded from above. For this it is useful the following simple observation.

Lemma 5.2 *If the random variables Q_1 and Q_1/M_1 are unbounded from above, then Y_∞ is also unbounded from above.*

Proof. Since Q_1/M_1 is unbounded from above and independent on $Y_{1,\infty}$, we have that $\mathbf{P}(Y_{1,\infty} > 0) = \mathbf{P}(Y_\infty > 0) = \mathbf{P}(Q_1/M_1 + Y_{1,\infty} > 0) > 0$. Take arbitrary $u > 0$. Then

$$\begin{aligned} \mathbf{P}(Y_\infty > u) &\geq \mathbf{P}(Q_1 + M_1 Y_{1,\infty} > u, Y_{1,\infty} > 0) \geq \mathbf{P}(Q_1 > u, Y_{1,\infty} > 0) \\ &= \mathbf{P}(Q_1 > u) \mathbf{P}(Y_{1,\infty} > 0) > 0 \end{aligned}$$

and the lemma is proven. \square

Notation: $\mathcal{J}_\theta := \int_{[0,1]} e^{-\theta V_v} dv$, $Q_\theta := - \int_{[0,1]} e^{-\theta V_v} dP_v$ where $\theta = \pm 1$.

Lemma 5.3 $\mathcal{L}(Q_{-1}) = \mathcal{L}(Q_1/M_1)$.

Proof. We have:

$$\begin{aligned} \int_{]0,1]} \sum_{k=1}^n e^{V_{k/n} - I_{[(k-1)/n, k/n]}(v)} dP_v &= \sum_{k=1}^n e^{V_{k/n}} (P_{k/n} - P_{(k-1)/n}), \\ e^{V_1} \int_{]0,1]} \sum_{k=1}^n e^{-V_{k/n} - I_{[(k-1)/n, k/n]}(v)} dP_v &= \sum_{k=1}^n e^{V_1 - V_{k/n}} (P_{k/n} - P_{(k-1)/n}). \end{aligned}$$

Note that V and P are independent, the increments $P_{k/n} - P_{(k-1)/n}$ are independent and identically distributed, and $\mathcal{L}(V_1 - V_{k/n}) = \mathcal{L}(V_{(n-k)/n})$. Thus, the right-hand sides of the above identities have the same distribution. The result follows because the left-hand sides tend in probability, respectively, to $-Q_{-1}$ and $-Q_1/M_1$. \square

Thus, Y_∞ is unbounded from above if so are the stochastic integrals Q_θ . Lemma 5.4 below shows that Q_θ are unbounded from above if the ordinary integrals \mathcal{J}_θ are unbounded from above. For the latter property we prove necessary and sufficient conditions in terms of defining characteristics (Lemma 5.7). The case where these conditions are not fulfilled we treat separately (Lemma 5.8).

Lemma 5.4 *If \mathcal{J}_θ is unbounded from above, so is Q_θ .*

Proof. We argue using the following observation: if $f(x, y)$ is a measurable function and ξ, η are independent random variables with distributions \mathbf{P}_ξ and \mathbf{P}_η , then the distribution of $f(\xi, \eta)$ is unbounded from below if the distribution of $f(\xi, y)$ is unbounded from below on a set of y of positive measure \mathbf{P}_η .

In the case $\sigma_P^2 > 0$, we use the representation

$$Q_\theta = -\sigma_P \int_{]0,1]} e^{-\theta V_v} dW_v^P + \int_{]0,1]} e^{-\theta V_v} d(\sigma_P W_v^P - P_v).$$

Applying the above observation (with $\xi = W^P$ and $\eta = (R, P - \sigma_P W^P)$) and taking into account that the Wiener integral of a strictly positive deterministic function is a nonzero Gaussian random variable, we get that Q_θ is unbounded.

Consider the case where $\sigma_P^2 = 0$.

For $\varepsilon > 0$ we denote by ζ^ε the locally square integrable martingale with

$$\zeta_t^\varepsilon := e^{-\theta V_t} I_{\{|x| \leq \varepsilon\}} x * (\mu^P - \nu^P)_t. \quad (5.3)$$

Since $\langle \zeta^\varepsilon \rangle_1 = e^{-2\theta V_1} I_{\{|x| \leq \varepsilon\}} x^2 * \nu_1^P \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have that $\sup_{t \leq 1} |\zeta_t^\varepsilon| \rightarrow 0$ in probability.

Note that

$$Q_\theta = (\Pi_P(xI_{\{\varepsilon \leq |x| \leq 1\}}) - a_P)\mathcal{J}_\theta - \zeta_1^\varepsilon - e^{-\theta V^-} I_{\{|x| > \varepsilon\}} x * \mu_1^P.$$

Take $N > 1$. Since \mathcal{J}_θ is unbounded from above, there is $N_1 > N + 1$ such that the set $\{N \leq \mathcal{J}_\theta \leq N_1, \inf_{t \leq 1} e^{-V_t} \geq 1/N_1\}$ is non-null. Then

$$I^\varepsilon := \{N \leq \mathcal{J}_\theta \leq N_1, \inf_{t \leq 1} e^{-V_t} \geq 1/N_1, |\zeta_1^\varepsilon| \leq 1\}$$

is also a non-null set for all sufficiently small $\varepsilon > 0$.

The process P is not a subordinator and, therefore, we have only three possible cases.

1) $\Pi_P(\cdot - \infty, 0] > 0$. Then $\Pi_P(\cdot - \infty, -\varepsilon_0] > 0$ for some $\varepsilon_0 > 0$. Due to independence, the intersection of I^ε with the set

$$\{|I_{\{x < -\varepsilon\}} x * \mu_1^P| \geq N_1(a_P^+ N_1 + N), I_{\{x > \varepsilon\}} * \mu_1^P = 0\}$$

is non-null when $\varepsilon \in]0, \varepsilon_0[$. On this intersection we have that

$$Q_\theta \geq -a_P \mathcal{J}_\theta - \zeta_1^\varepsilon - e^{-\theta V^-} I_{\{x < -\varepsilon\}} x * \mu_1^P \geq -a_P^+ N_1 - 1 + a_P^+ N_1 + N \geq N - 1.$$

2) $\Pi_P(\cdot - \infty, 0] = 0$, $\Pi_P(h) = \infty$. Diminishing in the need ε to ensure the inequality $\Pi_P(xI_{\{x > \varepsilon\}}) \geq N_1(a_P^+ N_1 + N)$, we have on the non-null set $I^\varepsilon \cap \{I_{\{x > \varepsilon\}} * \mu_1^P = 0\}$ that

$$Q_\theta = -a_P \mathcal{J}_\theta - \zeta_1^\varepsilon + e^{-\theta V^-} I_{\{x > \varepsilon\}} * \nu_1^P \geq -a_P^+ N_1 - 1 + a_P^+ N_1 + N \geq N - 1.$$

3) $\Pi_P(\cdot - \infty, 0] = 0$, $\Pi_P(h) < \infty$, and $\Pi_P(h) - a_P > 0$. Then on the non-null set $\{\mathcal{J}_\theta \geq N\} \cap \{I_{\{x > 0\}} * \mu_1^P = 0\}$ we have that

$$Q_\theta = (\Pi_P(h) - a_P)\mathcal{J}_\theta \geq (\Pi_P(h) - a_P)N.$$

Since N is arbitrary, in all three cases Q_θ is unbounded from above. \square

Remark 5.5 If $\mathcal{J}_1 I_{\{V_1 < 0\}}$ is unbounded from above, so is $Q_1 I_{\{V_1 < 0\}}$.

Remark 5.6 The proof above shows that in the case where $\sigma_P = 0$ there is a constant $\kappa > 0$ such that if the set $\{\mathcal{J}_\theta > N\}$ is non-null, then $Q_\theta > \kappa N$ on its $\mathcal{F}_1^{R,P}$ -measurable non-null subset. The statement remains valid with obvious changes if the integration over the interval $[0, 1]$ is replaced by the integral over arbitrary finite interval $[0, T]$.

Lemma 5.7 (i) The random variable \mathcal{J}_1 is unbounded from above iff $\sigma^2 + \Pi(\cdot - 1, 0] > 0$ or $\Pi(xI_{\{0 < x \leq 1\}}) = \infty$.

(ii) The random variable \mathcal{J}_{-1} is unbounded from above iff $\sigma^2 + \Pi(\cdot, \infty] > 0$ or $\Pi(xI_{\{x < 0\}}) = -\infty$.

Proof. In the case where $\sigma^2 > 0$ the “if” parts of the statements are obvious: W is independent of the jump part of V and the distribution of the random variable $\int_0^1 e^{-\sigma \theta W_v} g(v) dv$, where $g > 0$ is a deterministic function, has a support unbounded from above.

So, suppose that $\sigma^2 = 0$ and consider the “if” parts separately.

(i) Let $\Pi(\cdot - 1, 0] > 0$, i.e. $\Pi(\cdot - 1, -\varepsilon] > 0$ for some $\varepsilon \in]0, 1[$. Then the process V given by (2.3) admits the decomposition

$$V_t = at + h * (\mu - \nu)_t + (\ln(1+x) - h) * \mu_t = (a - \Pi(xI_{\{-1 < x \leq -\varepsilon\}}))t + V_t' + V_t'',$$

where

$$\begin{aligned} V_t' &:= I_{\{-\varepsilon < x \leq 1\}} x * (\mu - \nu)_t + (\ln(1+x) - x) I_{\{-\varepsilon < x \leq 1\}} * \mu_t \\ &\quad + \ln(1+x) I_{\{x > 1\}} * \mu_t, \\ V_t'' &:= \ln(1+x) I_{\{-1 < x \leq -\varepsilon\}} * \mu_t. \end{aligned}$$

The processes V' and V'' are independent. The decreasing process V'' has jumps of the size not less than $|\ln(1-\varepsilon)|$ and the number of jumps on the interval $[0, t]$ is a Poisson random variable with parameter $t\Pi(\cdot - 1, -\varepsilon] > 0$. Hence, V_t'' is unbounded from below for any $t \in]0, 1[$. In particular, for any $N > 0$, the set where $e^{-V''} \geq N$ on the interval $[1/2, 1]$ is non-null. The required property follows from these considerations.

Let $\Pi(h(x)I_{\{x > 0\}}) = \infty$. We assume without loss of generality that $\Pi(\cdot - 1, 0] = 0$. In this case, the process V has only positive jumps. Take arbitrary $N > 1$ and choose $\varepsilon > 0$ such that $\Pi(xI_{\{\varepsilon < x \leq 1\}}) > 2N$ and $\Pi(I_{\{0 < x \leq \varepsilon\}} \ln^2(1+x)) \leq 1/(32N^2)$. We have the decomposition

$$V_t = ct + V_t^{(1)} + V_t^{(2)} + V_t^{(3)},$$

where the processes

$$\begin{aligned} V_t^{(1)} &:= I_{\{0 < x \leq \varepsilon\}} \ln(1+x) * (\mu - \nu), \\ V_t^{(2)} &:= I_{\{\varepsilon < x \leq 1\}} \ln(1+x) * (\mu - \nu), \\ V_t^{(3)} &:= I_{\{x > 1\}} \ln(1+x) * \mu \end{aligned}$$

are independent and the constant

$$c := a + \Pi((\ln(1+x) - x)I_{\{0 < x \leq 1\}}) < \infty.$$

By the Doob inequality $P(\sup_{t \leq 1} V_t^{(1)} < N/2) > 1/2$. The processes $V^{(2)}$ and $V^{(3)}$ have no jumps on $[0, 1]$ on a non-null set. In the absence of jumps the trajectory of $V^{(2)}$ is the linear function

$$y_t = -\Pi(\ln(1+x)I_{\{\varepsilon < x \leq 1\}})t \leq -2Nt.$$

It follows that $\sup_{1/2 \leq t \leq 1} V_t \leq c - N/2$ on the set of positive probability. This implies that \mathcal{J}_1 is unbounded from above.

(ii) Let $\Pi(\cdot]0, \infty) > 0$, i.e. $\Pi(\cdot] \varepsilon, \infty) > 0$ for some $\varepsilon > 0$. Then

$$V_t = at + h * (\mu - \nu)_t + (\ln(1+x) - h) * \mu_t = (a - \Pi(hI_{\{x > \varepsilon\}}))t + \tilde{V}_t' + \tilde{V}_t'',$$

where

$$\begin{aligned} \tilde{V}_t' &:= I_{\{x \leq \varepsilon\}} h * (\mu - \nu)_t + (\ln(1+x) - h) I_{\{x \leq \varepsilon\}} * \mu_t, \\ \tilde{V}_t'' &:= \ln(1+x) I_{\{x > \varepsilon\}} * \mu_t. \end{aligned}$$

The processes \tilde{V}' and \tilde{V}'' are independent. The increasing process \tilde{V}'' has jumps of the size not less than $\ln(1+\varepsilon)$ and the number of jumps on the interval $[0, t]$ is a Poisson random variable with parameter $t\Pi(\cdot] \varepsilon, \infty) > 0$. Hence, \tilde{V}_t'' is unbounded from above for any $t \in]0, 1[$. In particular, for any $N > 0$, the set where $e^{\tilde{V}''} \geq N$ on the interval $[1/2, 1]$ is non-null. These facts imply the required property.

It remains to consider the case $\Pi(xI_{\{x < 0\}}) = -\infty$ and $\Pi(]0, \infty[) = 0$. The process V has only negative jumps. Take arbitrary $N > 1$ and choose $\varepsilon \in]0, 1/2[$ such that

$$-\Pi(\ln(1+x)I_{\{-1/2 < x \leq -\varepsilon\}}) > 2N, \quad \Pi(I_{\{-\varepsilon < x < 0\}} \ln^2(1+x)) \leq 1/(32N^2).$$

This time we use the representation

$$V_t = \tilde{c}t + \tilde{V}_t^{(1)} + \tilde{V}_t^{(2)} + \tilde{V}_t^{(3)},$$

where the processes

$$\begin{aligned} \tilde{V}_t^{(1)} &:= I_{\{-\varepsilon < x < 0\}} \ln(1+x) * (\mu - \nu), \\ \tilde{V}_t^{(2)} &:= I_{\{-1/2 < x \leq -\varepsilon\}} \ln(1+x) * (\mu - \nu), \\ \tilde{V}_t^{(3)} &:= I_{\{-1 < x \leq -1/2\}} \ln(1+x) * \mu \end{aligned}$$

are independent and the constant

$$\tilde{c} := a + \Pi(\ln(1+x)I_{\{-1/2 < x < 0\}} - h).$$

By the Doob inequality $\mathbf{P}(\sup_{t \leq 1} \tilde{V}_t^{(1)} < N/2) > 1/2$. The processes $\tilde{V}^{(2)}$ and $\tilde{V}^{(3)}$ have no jumps on $[0, 1]$ with strictly positive probability. In the absence of jumps the trajectory of $\tilde{V}^{(2)}$ is the linear function

$$y = -\Pi(\ln(1+x)I_{\{-1/2 < x \leq -\varepsilon\}})t \geq 2Nt.$$

It follows that $\sup_{1/2 \leq t \leq 1} V_t \leq \tilde{c} + N/2$ on a non-null set. This implies that J_{-1} is unbounded from above.

The “only if” parts of the lemma are obvious. \square

Summarizing, we conclude that Q_1 and Q_{-1} (hence, Y_∞) are unbounded from above if $\sigma^2 > 0$, or $\sigma_P^2 > 0$, or $\Pi(|h|) = \infty$, or $\Pi(]-1, 0]) > 0$ and $\Pi(]0, \infty[) > 0$. The remaining cases are treated in the following:

Lemma 5.8 *Let $\sigma = 0$, $\Pi(|h|) < \infty$, $\sigma_P = 0$. If $\Pi(]-1, 0]) = 0$ or $\Pi(]0, \infty[) = 0$, then the random variable Y_∞ is unbounded from above.*

Proof. By our assumptions $V_t = ct + L_t$ where the constant $c := a - \Pi(h)$, $\Pi \neq 0$, and $L_t := \ln(1+x) * \mu_t$. The assumption $\beta > 0$ implies that $V_1 < 0$ with strictly positive probability and V cannot be increasing or decreasing process. So, there are two cases which we consider separately.

(i) $c < 0$ and $\Pi(]0, \infty[) > 0$. Take any $T > 1$. Then $\int_{[0, T]} e^{-V_t} dt \geq T/e$ on the non-null set $\{L_T \leq 1\}$. By virtue of Remark 5.6 on a non-null $\mathcal{F}_T^{R, P}$ -measurable subset $\Gamma_T \subseteq \{L_T \leq 1\}$ we have that $-\int_{[0, T]} e^{-V_t} dP_t \geq K_T$ where $K_T \rightarrow \infty$ as $T \rightarrow \infty$. For every $T > 1$

$$\mathbf{P}(\Gamma_T \cap \{L_{T+1} - L_T \geq |c|(T+1)\}) = \mathbf{P}(\Gamma_T) \mathbf{P}(L_{T+1} - L_T \geq |c|(T+1)) > 0.$$

Let ζ^ε be the square integrable martingale given by (5.3) with $\theta = 1$. Take $N > 1$ sufficiently large and $\varepsilon > 0$ sufficiently small to ensure that the set $\Gamma_T^{\varepsilon, N}$ defined as the intersection

$\Gamma_T \cap \{L_{T+1} - L_T \geq |c|(T+1)\}$, $\{\sup_{s \in [T, T+1]} e^{-V_s} \leq N, \inf_{s \in [T, T+1]} e^{-V_s} \geq 1/N\}$, and $\{|\zeta_{T+1}^\varepsilon - \zeta_T^\varepsilon| \leq 1\}$ is non-null.

Let us consider the representation

$$Y_\infty = - \int_{[0, T]} e^{-V_{t-}} dP_t + a_P^\varepsilon \int_{]T, T+1]} e^{-V_{t-}} dt - \zeta_{T+1}^\varepsilon + \zeta_T^\varepsilon \\ - I_{]T, \infty[} [e^{-V_{-}} x I_{\{|x| > \varepsilon\}} * \mu_{T+1}^P + e^{-V_{T+1}} Y_{T+1, \infty}].$$

Take arbitrary $y < 0$ such that the set $\{Y_{T+1, \infty} > y\}$ is non-null.

Since the process P is not a subordinator with $\sigma_P = 0$, it must satisfy one of the characterizing conditions 1), 2), 3) of Section 2. Let us consider them consecutively.

Suppose that $\Pi_P(\cdot - \infty, 0] > 0$. Then there is $\varepsilon_0 > 0$ such that $\Pi_P(\cdot - \infty, -\varepsilon_0] > 0$. Due to the independence, the intersection of $\Gamma_T^{\varepsilon, N}$ with the set

$$\tilde{\Gamma}_T^{\varepsilon, N} := \{I_{]T, \infty[} [I_{\{x < -\varepsilon\}} * \mu_{T+1}^P \geq -(1/\varepsilon)N^2 a_P^\varepsilon, I_{]T, \infty[} [I_{\{x > \varepsilon\}} * \mu_{T+1}^P = 0]\}$$

is non-null when $\varepsilon \in]0, \varepsilon_0[$.

Due to independence, the intersection of $\Gamma_T^{\varepsilon, N} \cap \tilde{\Gamma}_T^{\varepsilon, N}$ and $\{Y_{T+1, \infty} > y\}$ also is a non-null set. But on this intersection we have inequality $Y_\infty \geq K_T - 1 + y$ implying that Y_∞ is unbounded from above.

Suppose that $\Pi_P(\cdot - \infty, 0] = 0$, $\Pi_P(h) = \infty$. Thus, for sufficiently small $\varepsilon > 0$ we have $a_P^\varepsilon > 0$. On the non-null set

$$\Gamma_T^{\varepsilon, N} \cap \{I_{]T, \infty[} [I_{\{x > \varepsilon\}} * \mu_{T+1}^P = 0]\} \cap \{Y_{T+1, \infty} > y\}$$

the inequality $Y_\infty \geq K_T - 1 + y$ holds and we conclude as above.

Finally, suppose that $\Pi_P(\cdot - \infty, 0] = 0$, $\Pi_P(h) < \infty$, and $\Pi_P(h) - a_P > 0$. In this case we can use the representation

$$Y_\infty = - \int_{[0, T]} e^{-V_{t-}} dP_t + (\Pi_P(h) - a_P) \int_{]T, T+1]} e^{-V_{t-}} dt \\ - I_{]T, \infty[} [e^{-V_{-}} x I_{\{x > 0\}} * \mu_{T+1}^P + e^{-V_{T+1}} Y_{T+1, \infty}].$$

On the non-null set $\Gamma_T^{\varepsilon, N} \cap \{I_{]T, \infty[} [I_{\{x > 0\}} * \mu_{T+1}^P = 0]\} \cap \{Y_{T+1, \infty} > y\}$ we have that $Y_\infty \geq K_T + y$ implying that Y_∞ is unbounded from above.

(ii) $c > 0$ and $\Pi(\cdot - 1, 0] > 0$. In this case there are $\gamma, \gamma_1 \in]0, 1[$, $\gamma < \gamma_1$, such that the three sets $\{I_{]-1, -\gamma[} * \mu_1 = 0\}$, $\{I_{[-\gamma, -\gamma_1[} * \mu_{1/2} = I_{[-\gamma, -\gamma_1[} * \mu_1 = N\}$, and $\{\ln(1+x)I_{]-\gamma_1, 0[} * \mu_1 \geq -1\}$ are non-null. Due to independence, their intersection A_N is also non-null.

On A_N we have the bounds

$$c + N \ln(1 - \gamma) - 1 \leq V_1 \leq c + N \ln(1 - \gamma_1)$$

and

$$\mathcal{J}_1 := \int_{[0, 1]} e^{-V_{t-}} dt \geq e^{-c} \int_{[0, 1/2]} e^{-\ln(1+x)*\mu_t} dt \geq \frac{1}{2} e^{-c} (1 - \gamma_1)^{-N}.$$

In virtue of Remark 5.6 there is a constant κ_N an $\mathcal{F}_1^{R, P}$ -measurable non-null subset B_N of A_N such that $Q_1 \geq \kappa_N$ on B_N and $\kappa_N \rightarrow \infty$ as $N \rightarrow \infty$.

Take $T = T_N > 0$ such that $cT + N \ln(1 - \gamma) - 2 \geq 0$. Then the set

$$\{I_{]1, 1+T[}[\ln(1+x) * \mu_{1+T} \geq -1\}$$

is non-null and its intersection with B_N is also non-null. On this intersection $e^{-V_{1+T}} \leq 1$ and

$$c_1(N) \leq V_{t-} \leq c_2(N)$$

where $c_1(N) := c + N \ln(1 - \gamma) - 2$, $c_2(N) := c(T + 1) + N \ln(1 - \gamma_1)$.

With this we accomplish the arguments by considering the cases corresponding to the properties 1), 2), and 3) with obvious modifications. \square

With the above lemma the proof of Proposition 5.1 is complete. \square

Proof of the main theorem. First, we relate the notations and hypotheses of Theorem 1.1 with those used in results from the implicit renewal theory summarized in Theorem 8.6 of Appendix. The hypothesis that $H(\beta) = 0$ means that $\mathbf{E}M^\beta = 1$ with $M = M_1 = e^{-V_1}$. Also, $\mathbf{E}M^{\beta+\varepsilon} < \infty$ for some $\varepsilon > 0$, since β does not belong to the boundary of the effective domain of the function H . In view of (2.8) and Lemma 4.1 we have that $\mathbf{E}|Q|^\beta < \infty$ where $Q = Q_1 = \int_{]0, 1]} e^{-V_v} dv$. Proposition 5.1 provides us the information that the almost sure limit Y_∞ of the process Y given by (3.1) does exist, it is finite, unbounded from above, and has the law which solving the distributional equation $\mathcal{L}(Y_\infty) = \mathcal{L}(Q + MY_\infty)$ which can be written in the form (8.1). Thus, all the condition of Theorem 8.6 are fulfilled. The latter gives the statements on the asymptotic behavior of the tail function $\bar{G}(u) = P(Y_\infty > u)$ as $u \rightarrow \infty$. The reference to Lemma 3.1 allows us to transform them into statements on the asymptotic behavior of the ruin probability $\Psi(u)$ and complete the proof. \square

Remark 5.9 The constant C_∞ in Theorem 1.1 is of the form $C_\infty = C_+/\bar{G}(0)$ where C_+ is given in (8.3).

Remark 5.10 Note that the hypothesis $\beta \in \text{int dom } H$ can be replaced by the slightly weaker assumption $\mathbf{E}e^{-\beta V_1} V_1^- < \infty$.

Remark 5.11 The hypothesis $\mathcal{L}(V_1)$ is non-arithmetic also can be replaced by a weaker one: one can assume that $\mathcal{L}(V_T)$ is non-arithmetic for some $T > 0$. Indeed, due to the identity $\ln \mathbf{E}e^{-\beta V_T} = TH(\beta)$ the root β does not depend on the choice of the time unit.

The following lemma shows that the condition on $\mathcal{L}(V_1)$ can be formulated in terms of the Lévy triplets.

Lemma 5.12 *The (non-degenerate) distribution of V_1 is arithmetic if and only if $\sigma = 0$, $\Pi(\mathbb{R}) < \infty$, and there is $d > 0$ such that Π_V is concentrated on the lattice $\Pi(h) - a + \mathbb{Z}d$.*

Proof. Recall that $\sigma_V = \sigma$ and $\Pi_V = \Pi\varphi^{-1}$ where $\varphi : x \mapsto \ln(1+x)$. So, we have $\Pi_V(\mathbb{R}) = \Pi(\mathbb{R})$. If $\sigma_V > 0$ or $\Pi_V(\mathbb{R}) = \infty$, the distribution of V_1 has a density, see Prop. 3.12 in [12]. If $\sigma = 0$ and $0 < \Pi_V(\mathbb{R}) < \infty$, then V is a compound Poisson process with drift $c = a - \Pi(h)$ and distribution of jumps $F_V := \Pi_V/\Pi_V(\mathbb{R})$. In such a case $\mathcal{L}(V_1)$ is concentrated on the lattice $\mathbb{Z}d$ if and only if Π_V is concentrated on the lattice $-c + \mathbb{Z}d$. \square

Remark 5.13 The property that Y_∞ is unbounded from above can be deduced from much more general Th.1 on support of the exponential functionals from the paper [6]. However, the results for the supports of \mathcal{J}_θ and Q_θ and arguments presented here have its own interest and can be used also without assuming, as in [6], that the limit Y_∞ exists.

6 Ruin with probability one

In this section we give conditions under which the ruin is imminent whatever is the initial reserve.

Recall the following ergodic property of the autoregressive process $(X_n^u)_{n \geq 1}$ with random coefficients which is defined recursively by the relations

$$X_n^u = A_n X_{n-1}^u + B_n, \quad n \geq 1, \quad X_0^u = u, \quad (6.1)$$

where $(A_n, B_n)_{n \geq 1}$ is a sequence of i.i.d. random variables in \mathbb{R}^2 (see [36], Prop. 7.1 and, for a deeper result, [14]).

Lemma 6.1 *Suppose that $\mathbf{E}|A_n|^\delta < 1$ and $\mathbf{E}|B_n|^\delta < \infty$ for some $\delta \in]0, 1[$. Then for any $u \in \mathbb{R}$ the sequence X_n^u converges in L^δ (hence, in probability) to the random variable*

$$X_\infty^0 = \sum_{n=1}^{\infty} B_n \prod_{j=1}^{n-1} A_j$$

and for any bounded uniformly continuous function f

$$\frac{1}{N} \sum_{n=1}^N f(X_n^u) \rightarrow \mathbf{E}f(X_\infty^0) \quad \text{in probability as } N \rightarrow \infty. \quad (6.2)$$

Corollary 6.2 *Suppose that $\mathbf{E}|A_n|^\delta < 1$ and $\mathbf{E}|B_n|^\delta < \infty$ for some $\delta \in]0, 1[$.*

(i) *If $\mathbf{P}(X_\infty^0 < 0) > 0$, then $\inf_{n \geq 1} X_n^u < 0$.*

(ii) *If $A_1 > 0$ and B_1/A_1 is unbounded from below, then $\inf_{n \geq 1} X_n^u < 0$.*

Proof. We get (i) just by the straightforward application of (6.2) to the function

$$f(x) := I_{\{x < -1\}} + x I_{\{-1 \leq x < 0\}}.$$

The statement (ii) follows from (i). Indeed, put $X_\infty^{0,1} := \sum_{n=2}^{\infty} B_n \prod_{j=2}^{n-1} A_j$. Then

$$X_\infty^0 = B_1 + A_1 X_\infty^{0,1} = A_1 (X_\infty^{0,1} + B_1/A_1).$$

Since B_1/A_1 and $X_\infty^{0,1}$ are independent and the random variable B_1/A_1 is unbounded from below, $\mathbf{P}(X_\infty^0 < 0) > 0$. \square

Let M_j and Q_j be the same as in (5.2).

Proposition 6.3 *Suppose that $\mathbf{E}M_1^{-\delta} < 1$ and $\mathbf{E}M_1^{-\delta}|Q_1|^\delta < \infty$ for some $\delta \in]0, 1[$. If Q_1 is unbounded from above, then $\Psi(u) \equiv 1$.*

Proof. The process X^u solving the equation (1.1) and restricted to the integer values of the time scale admits the representation

$$X_n^u = e^{V_n - V_{n-1}} X_{n-1}^u + e^{V_n} \int_{]n-1, n]} e^{-V_t} dP_t, \quad n \geq 1, \quad X_0^u = u.$$

That is, X_n^u is given by (6.1) with $A_n = M_n^{-1}$ and $B_n = -M_n^{-1}Q_n$. The result follows from the statement (ii) of Corollary 6.2. \square

Now we give more specific conditions of the ruin with probability one in terms of the triplets.

Theorem 6.4 *Suppose that $0 \in \text{int dom } H$ and $\Pi_P(|\bar{h}|^\varepsilon) < \infty$ for some $\varepsilon > 0$. If $a_V + \Pi(\bar{h}(\ln(1+x))) \leq 0$, then $\Psi(u) \equiv 1$.*

Proof. Note that $D^-H(0) = -a_V - \Pi(\bar{h}(\ln(1+x)))$. If $D^-H(0) > 0$, then for all $q < 0$ sufficiently close to zero $H(q) < 0$, that is $\mathbf{E}M_1^q < 1$. By virtue of Lemma 5.3 the law $\mathcal{L}(M_1^{-1}Q_1) = \mathcal{L}(Q_{-1})$. If $\Pi_P(|\bar{h}|^\varepsilon) < \infty$ for some $\varepsilon > 0$, then Lemma 4.1 implies that $\mathbf{E}|Q_{-1}|^q < \infty$ for sufficiently small $q > 0$. To get the result we can use Proposition 6.3. Indeed, by virtue of Lemmata 5.4 and 5.7(i) the random variable Q_1 is unbounded from above except, eventually, the case where $\sigma^2 = 0$, $\sigma_P^2 = 0$, $\Pi(|h|) < \infty$, and $\Pi(\cdot - 1, 0] = 0$, $\Pi(\cdot, \infty) < 0$. Recall that in this special case $V_t = ct + L_t$ where $c := a - \Pi(h)$ and $L_t := \ln(1+x) * \mu_t$. Note that

$$X_n^0 = \int_{[0,n]} e^{V_n - V_t} dP_t \stackrel{d}{=} \int_{[0,n]} e^{V_t} dP_t =: -\widehat{Y}_n,$$

where the equality in law holds in virtue of Lemma 5.3 (the latter is formulated for the interval $[0, 1]$ but its extension to arbitrary one is obvious). The random variable \widehat{Y}_n is defined by the same formula as Y_n with V replaced by $-V$. As in Proposition 5.3 we show that converges to a finite value \widehat{Y}_∞ in probability. It follows that $\mathcal{L}(X_n^0) = \mathcal{L}(-\widehat{Y}_n)$. As in Lemma 5.8(i) we can show that \widehat{Y}_n is unbounded from above.

In the case where $D^-H(0) = 0$ we consider, following [36], the discrete-time process $(\tilde{X}_n^u)_{n \in \mathbb{N}}$ where $\tilde{X}_n^u = X_{T_n}$ and the descending ladder times T_n of the random walk $(V_n)_{n \in \mathbb{N}}$ which are defined as follows: $T_0 := 0$,

$$T_n := \inf\{k > T_{n-1} : V_k - V_{T_{n-1}} < 0\}.$$

Since $J(q) = \Pi(I_{\{|\ln(1+x)| > 1\}}(1+x)^{-q}) < \infty$ for any $q \in]q, \bar{q}[$, we have that $\Pi(\ln^2(1+x)) < \infty$. The formula (2.3) can be written as

$$V_t = \sigma W_t + \ln(1+x) * (\mu - \nu)_t,$$

i.e. V is a square integrable martingale, $\mathbf{E}V_1 = 0$ and $\mathbf{E}V_1^2 < \infty$.

According to Theorem 1a in Ch. XII.7 of Feller's book [15] and the remark preceding the citing theorem, the above properties imply that there is a finite constant c such that

$$\mathbf{P}(T_1 > n) \leq cn^{-1/2}. \quad (6.3)$$

It follows, in particular, that the differences $T_n - T_{n-1}$ are well-defined and form a sequence of finite independent random variables distributed as T_1 . The discrete-time process $\tilde{X}_n^u = X_{T_n}^u$ has the representation

$$\tilde{X}_n^u = e^{V_{T_n} - V_{T_{n-1}}} \tilde{X}_{n-1}^u + e^{V_{T_n}} \int_{]T_{n-1}, T_n]} e^{-V_t} dP_t, \quad n \geq 1, \quad \tilde{X}_0^u = u,$$

and solves the linear equation

$$\tilde{X}_n^u = \tilde{A}_n \tilde{X}_{n-1}^u + \tilde{B}_n, \quad n \geq 1, \quad X_0^u = u,$$

where

$$\tilde{A}_n := e^{V_{T_n} - V_{T_{n-1}}}, \quad \tilde{B}_n := e^{V_{T_n}} \int_{]T_{n-1}, T_n]} e^{-V_t} dP_t,$$

and $\tilde{B}_1/\tilde{A}_1 = -Y_{T_1}$ where Y is given by (3.1).

By construction, $\tilde{A}_1^\delta < 1$ for any $\delta > 0$.

Using the definition of Q_j given by (5.2) we have that

$$|\tilde{B}_1| \leq \sum_{j=1}^{T_1} e^{V_{T_1} - V_{j-1}} |Q_j| \leq \sum_{j=1}^{T_1} |Q_j|.$$

According to Lemma 4.1 $\mathbf{E}|Q_1|^p < \infty$ for some $p \in]0, 1[$. Taking $r \in]0, p/5[$ and defining the sequence $l_n := \lfloor n^{4r} \rfloor$, we have, using the Chebyshev inequality and (6.3), that

$$\begin{aligned} \mathbf{E}|\tilde{B}_1|^r &\leq 1 + r \sum_{n \geq 1} n^{r-1} \mathbf{P}\left(\sum_{j=1}^{T_1} |Q_j| > n\right) \\ &\leq 1 + r \sum_{n \geq 1} n^{r-1} \mathbf{P}\left(\sum_{j=1}^{l_n} |Q_j| > n\right) + r \sum_{n \geq 1} n^{r-1} \mathbf{P}\left(T_1 > l_n\right) \\ &\leq 1 + r \mathbf{E}|Q_1|^p \sum_{n \geq 1} l_n n^{r-1-p} + rc \sum_{n \geq 1} n^{r-1} l_n^{-1/2} < \infty. \end{aligned}$$

To apply Corollary 6.2(ii) it remains to check that Y_{T_1} is unbounded from above. Since $\{Q_1 > N, V_1 < 0\} \subseteq \{Y_{T_1} > N\}$, it is sufficient to check that the probability of the set in the left-hand side is strictly positive for all $N > 0$, or, by virtue of Remark 5.5, that

$$\mathbf{P}(\mathcal{J}_1 > N, V_1 < 0) > 0 \quad \forall N > 0. \quad (6.4)$$

Let $\sigma^2 > 0$. The conditional distribution of the process $(W_s)_{s \leq 1}$ given $W_1 = x$ coincides with the (unconditional) distribution of the Brownian bridge $B^x = (B_s^x)_{s \leq 1}$ with $B_s^x = W_s + s(x - W_1)$. Using this we easily get that for any bounded positive function g and any $y, M \in \mathbb{R}$ the probability

$$\mathbf{P}\left(\int_0^1 e^{-\sigma W_v} g(v) dv > y, W_1 < M\right) > 0,$$

cf. with Lemma 4.2 in [23]. This implies (6.4).

Suppose that $\sigma^2 = 0$, but $\Pi(]-1, 0]) > 0$, i.e. $\Pi(]-1, -\varepsilon]) > 0$ for some $\varepsilon \in]0, 1[$. In the decomposition $V = V^{(1)} + V^{(2)}$, where

$$\begin{aligned} V_t^{(1)} &= I_{\{-1 < x \leq -\varepsilon\}} \ln(1+x) * \mu_t, \\ V_t^{(2)} &= (a - \Pi(hI_{\{-1 < x \leq -\varepsilon\}}))t + I_{\{x > -\varepsilon\}} h * (\mu - \nu)_t \\ &\quad + I_{\{x > -\varepsilon\}} (\ln(1+x) - h) * \mu_t, \end{aligned}$$

the processes $V^{(1)}$ and $V^{(2)}$ are independent. The process $V^{(1)}$ is decreasing by negative jumps whose absolute value are larger or equal than $|\ln(1 - \varepsilon)|$ and the number of jumps on the interval $[0, 1/2]$ has the Poisson distribution with parameter $(1/2)\Pi(]-1, -\varepsilon]) > 0$.

Thus, $\mathbf{P}(V_{1/2}^{(1)} < -n) > 0$ for any real n . It follows that

$$\begin{aligned} \mathbf{P}(\mathcal{J}_1 > N, V_1 < 0) &\geq \mathbf{P}\left(\int_0^1 e^{-V_t} dt > N, V_1 < 0, V_{1/2}^{(1)} < -n\right) \\ &\geq \mathbf{P}\left(e^n \int_{1/2}^1 e^{-V_t^{(2)}} dt > N, V_1^{(2)} < n, V_{1/2}^{(1)} < -n\right) \\ &= \mathbf{P}\left(\int_{1/2}^1 e^{-V_t^{(2)}} dt > Ne^{-n}, V_1^{(2)} < n\right) \mathbf{P}(V_{1/2}^{(1)} < -n). \end{aligned}$$

The right-hand side is strictly positive for sufficiently large n and (6.4) holds.

The case where $\Pi(xI_{\{0 < x \leq 1\}}) = \infty$ is treated similarly as in the last part of the proof of Lemma 5.7(i).

The exceptional case is treated by a reduction to Corollary 6.2(i). \square

The above theorem implies that in the classical model with negative risk sums (where $\sigma_P = 0$, the jumps of P are positive and form a compound Poisson process, $\Pi_P(|x|) < \infty$, trend is negative, i.e. $a_P - \Pi_P(x) < 0$) and investments into a risky asset with the price following a geometric Brownian motion (that is, $\Pi = 0$ and $\sigma \neq 0$), the ruin is imminent if $a_V = a - \sigma^2/2 \leq 0$.

7 Examples

Example 1. Let us consider the model with negative risk sums in which the Lévy measure $\Pi_P(dx) = \lambda F_P(dx)$ where the constant $\lambda > 0$ and the probability distribution $F_P(dx)$ is concentrated on $]0, \infty[$, and

$$a_P^0 := \lambda \int_{[0,1]} x F_P(dx) - a_P.$$

The process P admits the representation as sum of an independent Wiener process with drift and a compound Poisson process:

$$P_t = -a_P^0 t + \sigma_P W_t^P + \sum_{j=1}^{N_t^P} \xi_j. \quad (7.1)$$

where the Poisson process N^P with intensity λ_P is independent of the sequence $(\xi_j)_{j \geq 1}$ of positive i.i.d. random variables with common distribution F_P .

Suppose that the price process is a geometric Brownian motion

$$\mathcal{E}_t(R) = e^{V_t} = e^{(a - \sigma^2/2)t + \sigma W_t},$$

that is, $\sigma \neq 0$, $\Pi = 0$.

For this model $\underline{q} = -\infty$, $\bar{q} = \infty$. The condition $D^+ H(0) < 0$ is reduced to the inequality $\sigma^2/2 < a$ and the function $H(q) = (\sigma^2/2 - a + q\sigma^2/2)q$ has the root $\beta = 2a/\sigma^2 - 1 > 0$. Suppose that $\sigma_P^2 + (a_P^0)^+ > 0$. By Theorem 1.1 the exact asymptotic $\Psi(u) \sim C_\infty u^{-\beta}$, as $u \rightarrow \infty$, holds if $\mathbf{E}\xi_1^{\beta+1} < \infty$. Since the exponential distribution has the above property, we recover, as a very particular case the asymptotic result of [23] where it was assumed that $\sigma_P^2 = 0$ and $a_P^0 > 0$.

If $\sigma_P^2 + (a_P^0)^+ > 0$, $\sigma^2/2 \geq a$, and $\mathbf{E}\xi_1^\epsilon < \infty$ for some $\epsilon > 0$, then Theorem 6.4 implies that $\Psi(u) \equiv 1$.

The models with the price process given by a geometric Brownian motion were intensively studied using the representation of Ψ as a solution of integro-differential equations. To the reader interested not only in asymptotical results but also in a behavior of the ruin probabilities for finite values of the initial capital we recommend a very detailed study [7] with a number of simulation results.

Example 2. Let the process P be again given by (7.1) and suppose that the price process has a jump component, namely,

$$\mathcal{E}_t(R) = \exp \left\{ (a - \sigma^2/2)t + \sigma W_t + \sum_{j=1}^{N_t} \ln(1 + \eta_j) \right\},$$

where the Poisson process N with intensity $\lambda > 0$ is independent on the sequence $(\eta_j)_{j \geq 1}$ of i.i.d. random variables with common distribution F not concentrated at zero and such that $F(-\infty, -1] = 0$, see [27], Ch. 7. That is, the log price process is represented as

$$V_t = (a - \sigma^2/2)t + \sigma W_t + \ln(1 + x) * \mu_t,$$

where $\Pi(dx) = \lambda F(dx)$. The function H is given by the formula

$$H(q) = (\sigma^2/2 - a + q\sigma^2/2)q + \lambda(\mathbf{E}(1 + \eta_1)^{-q} - 1).$$

Suppose that $\mathbf{E}(1 + \eta_1)^{-q} < \infty$ for all $q > 0$. Then $\bar{q} = \infty$.

Let $\sigma \neq 0$. Then $\limsup_{q \rightarrow \infty} H(q)/q = \infty$. If

$$D^+ H(0) = \sigma^2/2 - a - \lambda \mathbf{E} \ln(1 + \eta_1) < 0, \quad (7.2)$$

then the root $\beta > 0$ of the equation $H(q) = 0$ does exist. Thus, if $\mathbf{E}\xi_1^\beta < \infty$, then Theorem 1.1 can be applied to get that $\Psi(u) \sim C_\infty u^{-\beta}$ where $C_\infty > 0$.

If $\mathbf{E}(1 + \eta_1)^{1-2a/\sigma^2} < 1$ (resp., $\mathbf{E}(1 + \eta_1)^{1-2a/\sigma^2} > 1$), the root β is larger (resp., smaller) than $2a/\sigma^2 - 1$, the value of the root of H in the model of Example 1 where the price process is continuous.

Let $\sigma = 0$. If

$$D^+ H(0) = -a - \lambda \mathbf{E} \ln(1 + \eta_1) < 0,$$

and

$$\limsup_{q \rightarrow \infty} q^{-1} \mathbf{E}((1 + \eta_1)^{-q} - 1) > a/\lambda,$$

then the root $\beta > 0$ also exists. Theorem 1.1 can be applied when $\mathbf{P}(\eta_1 > 0) \in]0, 1[$ and the we have the exact asymptotic if the distribution of $\ln(1 + \eta_1)$ is non-arithmetic.

Suppose that $\mathbf{E}(1 + \eta_1)^{-q} < \infty$ for all $q \in \mathbb{R}$. Then $\underline{q} = -\infty$ and $\bar{q} = \infty$. If the conditions $\sigma^2/2 - a - \lambda \mathbf{E} \ln(1 + \eta_1) \geq 0$, $\sigma^2 + \mathbf{P}(\eta_1 < 0) > 0$, and $\mathbf{E}|\xi_1|^\varepsilon < \infty$ for some $\varepsilon > 0$ hold, then $\Psi(u) \equiv 1$ in virtue of Theorems 6.4.

8 Appendix: Tails of solutions of distributional equations

8.1 Kesten–Goldie theorem

Here we present a short account of needed results on distributional equations (random equations in the terminology of [17])

$$Y_\infty \stackrel{d}{=} Q + M Y_\infty, \quad Y_\infty \text{ independent of } (M, Q), \quad (8.1)$$

where (M, Q) is a two-dimensional random variable such that $M > 0$ and $\mathbf{P}(M \neq 1) > 0$ and $\stackrel{d}{=}$ is the equality in law. This is a symbolical notation which means that we are given,

in fact, a two-dimensional distribution \mathcal{L} on $]0, \infty[\times \mathbb{R}$ not concentrated on $\{1\} \times \mathbb{R}$ and the problem is to find a probability space with random variables Y_∞ and (M, Q) on it such that Y_∞ and (M, Q) are independent, $\mathcal{L}(M, Q) = \mathcal{L}$, and $\mathcal{L}(Y_\infty) = \mathcal{L}(Q + M Y_\infty)$. The uniqueness in this problem means the uniqueness of the distribution of Y_∞ .

In the sequel (M_j, Q_j) will be an i.i.d. sequence whose generic term (M, Q) has the distribution \mathcal{L} and $Z_j := M_1 \dots M_j$, $Z_n^* := \sup_{j \leq n} Z_j$.

If there is $p > 0$ such that $\mathbf{E}M^p < 1$ and $\mathbf{E}|Q|^p < \infty$, then the solution Y_∞ of (8.1) can be easily realized on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where the sequence (M_j, Q_j) is defined — just as the limit in L^p of the series $\sum_{j \geq 0} Z_{j-1} Q_j$, see the beginning of the proof of Proposition 5.1.

The following classical result of the renewal theory is the Kesten–Goldie theorem, see Th. 4.1 in [17]:

Theorem 8.1 *Suppose that (M, Q) is such that the distribution of $\ln M$ is non-arithmetic and, for some $\beta > 0$,*

$$\mathbf{E}M^\beta = 1, \quad \mathbf{E}M^\beta (\ln M)^+ < \infty, \quad \mathbf{E}|Q|^\beta < \infty. \quad (8.2)$$

Then

$$\begin{aligned} \lim_{u \rightarrow \infty} u^\beta \mathbf{P}(Y_\infty > u) &= C_+ < \infty, \\ \lim_{u \rightarrow \infty} u^\beta \mathbf{P}(Y_\infty < -u) &= C_- < \infty, \end{aligned}$$

where $C_+ + C_- > 0$.

Theorem 8.1 leaves open the question when the constant C_+ is strictly positive. The expression

$$C_+ = \frac{\mathbf{E} \left(((Q + M Y_\infty)^+)^{\beta} - ((M Y_\infty)^+)^{\beta} \right)}{\beta \mathbf{E}M^\beta \ln M} \quad (8.3)$$

given in [17] and involving the unknown distribution Y_∞ is not helpful. How to check that the right-hand side of this formula is strictly positive? Recently, Guivarc'h and Le Page showed for the above case where the distribution of $\ln M$ is non-arithmetic that $C_+ > 0$ if and only if Y_∞ is unbounded from above, see [20] and also the paper [10] for simpler arguments. Of course, this criterion is not a result formulated in terms of the given data: it involves a property of the unknown distribution of Y_∞ , namely, that the support is unbounded. But this property can be checked in the considered model.

The remaining part of the appendix deals is a compendium of facts needed to cover also the arithmetic case.

8.2 Grincevičius theorem

The theorem below is a simplified version of Th.2(b), [19], but with a slightly weaker assumption on Q , namely, $\mathbf{E}|Q|^\beta < \infty$, used in our study. For the reader convenience we give its complete proof after recalling some concepts and facts from the renewal theory.

Theorem 8.2 *Suppose that (8.2) holds and the distribution of $\ln M$ is concentrated on the lattice $\mathbb{Z}d = \{0, \pm d, \pm 2d, \dots\}$ where $d > 0$. Then*

$$\limsup_{u \rightarrow \infty} u^\beta \mathbf{P}(Y_\infty > u) < \infty. \quad (8.4)$$

We consider the convolution-type linear operator which is well-defined for all positive as well as for (the Lebesgue) integrable functions by the formula

$$\check{\psi}(x) = \int_{-\infty}^x e^{-(x-y)} \psi(y) dy. \quad (8.5)$$

Clearly, the functions ψ and $\check{\psi}$ are integrable or not simultaneously and

$$\int_{\mathbb{R}} \check{\psi}(x) dx = \int_{\mathbb{R}} \psi(x) dx.$$

Suppose that $\psi \geq 0$ is integrable. Then $\check{\psi}(x + \delta) \geq e^{-\delta} \check{\psi}(x)$ for any $\delta > 0$ and

$$\delta \inf_{x \in [j\delta, (j+1)\delta]} \check{\psi}(x) \geq \delta e^{-\delta} \check{\psi}(j\delta) \geq e^{-2\delta} \int_{(j-1)\delta}^{j\delta} \check{\psi}(x) dx$$

implying that

$$\underline{U}(\check{\psi}, \delta) := \delta \sum_{j \in \mathbb{Z}} \inf_{x \in [j\delta, (j+1)\delta]} \check{\psi}(x) \geq e^{-2\delta} \int_{\mathbb{R}} \check{\psi}(x) dx.$$

Similarly,

$$\bar{U}(\check{\psi}, \delta) := \delta \sum_{j \in \mathbb{Z}} \sup_{x \in [j\delta, (j+1)\delta]} \check{\psi}(x) \leq e^{2\delta} \int_{\mathbb{R}} \check{\psi}(x) dx.$$

Thus, $\bar{U}(\check{\psi}, \delta) < \infty$ and $\bar{U}(\check{\psi}, \delta) - \underline{U}(\check{\psi}, \delta) \rightarrow 0$ as $\delta \rightarrow \infty$. These two properties mean, by definition, that the function $\check{\psi}$ is directly Riemann integrable. Arguing with the positive and negative parts, we obtain that if ψ is integrable, then $\check{\psi}$ is directly Riemann integrable.

We shall use in the sequel the following renewal theorem for the random walk $S_n := \sum_{i=1}^n \xi_i$ on a lattice, see Prop. 2.1, [21].

Proposition 8.3 *Let ξ_i be i.i.d. random variables taking values in the lattice $\mathbb{Z}d$, $d > 0$, and having finite expectation $m := \mathbf{E}\xi_i > 0$. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. If $x \in \mathbb{R}$ is such that $\sum_{j \in \mathbb{Z}} |F(x + jd)| < \infty$, then*

$$\lim_{n \rightarrow \infty} \mathbf{E} \sum_{k \geq 0} F(x + nd - S_k) = \frac{d}{m} \sum_{j \in \mathbb{Z}} F(x + jd).$$

Proof of Theorem 8.2. Let the solution of (8.1) be realized on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We shall use the notation (M, Q) instead of (M_1, Q_1) . As usual, the tail function $\bar{G}(u) := \mathbf{P}(Y_\infty > u)$. Define the function $g(x) := e^{\beta x} \bar{G}(e^x)$. Since Y_∞ and M are independent, $\mathbf{P}(MY_\infty > e^x) = \mathbf{E}\bar{G}(e^{x-\ln M})$. Introducing the new probability measure $\tilde{\mathbf{P}} := M^\beta \mathbf{P}$ and noting that

$$e^{\beta x} \mathbf{P}(MY_\infty > e^x) = \mathbf{E} M^\beta e^{\beta(x-\ln M)} \bar{G}(e^{x-\ln M}) = \tilde{\mathbf{E}} g(x - \ln M),$$

we obtain the following identity (called *renewal equation*):

$$g(x) = D(x) + \tilde{\mathbf{E}} g(x - \ln M), \quad (8.6)$$

where $D(x) := e^{\beta x} (\mathbf{P}(Y_\infty > e^x) - \mathbf{P}(MY_\infty > e^x))$. The Jensen inequality for the convex function $x \mapsto x \ln x$ implies that $\tilde{\mathbf{E}} \ln M = \mathbf{E} M^\beta \ln M > 0$ and, hence, $\tilde{\mathbf{E}} |\ln M| < \infty$.

Let us check that the function $x \mapsto D(x)$ is integrable. To this aim, we note that for any random variables ξ, η

$$|\mathbf{P}(\xi > s) - \mathbf{P}(\eta > s)| \leq \mathbf{P}(\eta^+ \leq s < \xi^+) + \mathbf{P}(\xi^+ \leq s < \eta^+).$$

Using the Fubini theorem we obtain that

$$\int_0^\infty \mathbf{P}(\eta^+ \leq s < \xi^+) s^{\beta-1} ds = \mathbf{E} I_{\{\eta^+ < \xi^+\}} \int_{\eta^+}^{\xi^+} s^{\beta-1} ds = \frac{1}{\beta} \mathbf{E}((\xi^+)^\beta - (\eta^+)^\beta)^+.$$

Applying this bound with $\xi := Q + MY_\infty \stackrel{d}{=} Y_\infty$ and $\eta := MY_\infty$ we get that

$$\int_{\mathbb{R}} |D(x)| dx = \int_0^\infty |\mathbf{P}(\xi > s) - \mathbf{P}(\eta > s)| s^{\beta-1} ds \leq \frac{1}{\beta} \mathbf{E}|(\xi^+)^\beta - (\eta^+)^\beta|$$

and it remains to verify that

$$\mathbf{E}|((Q + \eta)^+)^{\beta} - (\eta^+)^{\beta}| < \infty \quad (8.7)$$

when $\mathbf{E}|Q|^\beta < \infty$. But $|((Q + \eta)^+)^{\beta} - (\eta^+)^{\beta}| = \zeta_1 + \zeta_2$ with positive summands

$$\begin{aligned} \zeta_1 &:= I_{\{-Q < \eta \leq 0\}} (Q + \eta)^\beta + I_{\{0 < \eta \leq -Q\}} \eta^\beta \leq |Q|^\beta, \\ \zeta_2 &:= I_{\{Q + \eta > 0, \eta > 0\}} |(Q + \eta)^\beta - \eta^\beta|. \end{aligned}$$

If $\beta \leq 1$, then the random variable ζ_2 is also dominated by the random variable $|Q|^\beta$. If $\beta > 1$, then the inequality $|x^\beta - y^\beta| \leq \beta|x - y|(x \vee y)^{\beta-1}$ for $x, y \geq 0$ combined with the inequality $(|a| + |b|)^{\beta-1} \leq 2^{(\beta-2)^+}(|a|^{\beta-1} + |b|^{\beta-1})$ leads to the estimate

$$\zeta_2 \leq 2^{(\beta-2)^+} \beta |Q| (|\eta|^{\beta-1} + |Q|^{\beta-1}).$$

Using the independence of (M, Q) and Y_∞ , the Hölder inequality, and taking into account that $\mathbf{E}M^\beta = 1$ and $\mathbf{E}|Y_\infty|^p < \infty$ for $p \in [0, \beta]$ we get that

$$\mathbf{E}|Q||\eta|^{\beta-1} = \mathbf{E}|Q|M^{\beta-1} \mathbf{E}|Y_\infty|^{\beta-1} \leq (\mathbf{E}|Q|^\beta)^{1/\beta} \mathbf{E}|Y_\infty|^{\beta-1} < \infty.$$

Thus, (8.7) holds.

The integrability of D allows us to transform (8.6) into the equality

$$\check{g}(x) = \check{D}(x) + \check{\mathbf{E}}\check{g}(x - \ln M).$$

Iterating it, we obtain that

$$\check{g}(x) = \sum_{n=0}^{N-1} \check{\mathbf{E}}\check{D}(x - S_n) + \check{\mathbf{E}}\check{g}(x - S_N), \quad (8.8)$$

where $S_n := \sum_{i=1}^n \xi_i$ for $n \geq 1$, (ξ_i) is a sequence of independent random variables on $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$ independent on Y_∞ such that the distribution $\mathcal{L}(\xi_i, \tilde{\mathbf{P}}) = \mathcal{L}(\ln M, \tilde{\mathbf{P}})$. In particular, $\check{\mathbf{E}}e^{-\beta\xi_i} = 1$.

By the strong law of large numbers $S_N/N \rightarrow \check{\mathbf{E}}\ln M > 0$ $\tilde{\mathbf{P}}$ -a.s., $N \rightarrow \infty$, and, therefore, $y - \ln S_N \rightarrow -\infty$ $\tilde{\mathbf{P}}$ -a.s. for every y . Since $\check{\mathbf{E}}e^{-\beta S_N} = 1$, we have by dominated convergence that

$$\check{\mathbf{E}}\check{g}(y - S_N) = \check{\mathbf{E}}e^{\beta(y - S_N)} \check{G}(e^{y - S_N}) \rightarrow 0.$$

It follows that the remainder term $\mathbf{E}\check{g}(x - S_N)$ in (8.8) tends to zero, thus,

$$\check{g}(x) = \sum_{k \geq 0} \check{\mathbf{E}}\check{D}(x - S_k). \quad (8.9)$$

Using Proposition 8.3 (with $F = \check{D}$) we obtain that for any $x > 0$

$$\lim_{n \rightarrow \infty} \check{g}(x + dn) = \frac{d}{\check{\mathbf{E}} \ln M} \sum_{j \in \mathbb{Z}} \check{D}(x + jd) \leq \bar{U}(\check{D}, d) < \infty. \quad (8.10)$$

Replacing in the integrant the function $\bar{G}(e^y)$ by its smallest value $G(e^x)$ we obtain that

$$\check{g}(x) := \int_{-\infty}^x e^{-(x-y)} e^{\beta y} \bar{G}(e^y) dy \geq \frac{1}{\beta + 1} g(x)$$

and, therefore,

$$\limsup_{u \rightarrow \infty} u^\beta \mathbf{P}(Y_\infty > u) = \limsup_{x \rightarrow \infty} g(x) \leq (\beta + 1) \limsup_{x \rightarrow \infty} \check{g}(x) < \infty.$$

Theorem 8.2 is proven. \square

8.3 Buraczewski–Damek approach

The following result, usually formulated in terms of the supremum of the random walk $S_n := \sum_{i=1}^n \ln M_i$, is well-known (see, e.g., Th. A in [25] for much more general setting).

Proposition 8.4 *If M satisfies (8.2), then*

$$\liminf_{u \rightarrow \infty} u^\beta \mathbf{P}(Z_\infty^* > u) > 0. \quad (8.11)$$

Proof. Let $F(x) := \mathbf{P}(\ln M \leq x)$, $\bar{F}(x) := 1 - F(x)$, $S_n := \sum_{i=1}^n \xi_i$ where $\xi_i := \ln M_i$. The function $\bar{H}(x) := \mathbf{P}(\sup_n S_n > x)$ admits the representation

$$\bar{H}(x) = \mathbf{P}(\xi_1 > x) + \mathbf{E} I_{\{\xi_1 \leq x\}} \bar{H}(x - \xi_1) = \bar{F}(x) + \int_{-\infty}^x \bar{H}(x - t) dF(t).$$

Putting $Z(x) := e^{\beta x} \bar{H}(x)$, $z(x) := e^{\beta x} \bar{F}(x)$, and $\tilde{\mathbf{P}} := e^{\beta \xi_1} \mathbf{P}$, we obtain from here that

$$Z(x) = z(x) + \check{\mathbf{E}} Z(x - \xi_1) I_{\{\xi_1 \leq x\}}. \quad (8.12)$$

The same arguments as were used in deriving (8.8) lead to the representation

$$Z(x) = \check{\mathbf{E}} \sum_{k \geq 0} z(x - S_k) I_{\{S_k \leq x\}}. \quad (8.13)$$

The function $\hat{z}(x) := z(x) I_{\{x \geq 0\}}$ is directly Riemann integrable. Indeed, for $j \geq 0$ we have that

$$\sup_{x \in [j\delta, (j+1)\delta]} \hat{z}(x) \leq e^{\beta(j+1)\delta} \bar{F}(j\delta) \leq e^{2\beta\delta} \int_{(j-1)\delta}^{j\delta} e^{\beta v} \bar{F}(v) dv$$

and, therefore,

$$\bar{U}(\hat{z}, \delta) = \delta z(0) + \delta \sum_{j \geq 0} \sup_{x \in [j\delta, (j+1)\delta]} z(x) \leq \delta z(0) + e^{2\beta\delta} \int_{-\delta}^{\infty} e^{\beta v} \bar{F}(v) dv.$$

In the same spirit

$$\inf_{x \in [j\delta, (j+1)\delta]} z(x) \geq e^{\beta j\delta} \bar{F}((j+1)\delta) \geq e^{-2\beta\delta} \int_{(j+1)\delta}^{(j+2)\delta} e^{\beta v} \bar{F}(v) dv$$

and

$$\underline{U}(\hat{z}, \delta) = \delta \sum_{j \geq 0} \sup_{x \in [j\delta, (j+1)\delta]} z(x) \geq e^{-2\beta\delta} \int_{\delta}^{\infty} e^{\beta v} \bar{F}(v) dv.$$

Taking into account that

$$\int_{\mathbb{R}} e^{\beta v} \bar{F}(v) dv = \frac{1}{\beta} \mathbf{E} e^{\beta \xi_1} = \frac{1}{\beta} < \infty.$$

We get from here that $\bar{U}(\hat{z}, \delta) < \infty$ and $\bar{U}(\hat{z}, \delta) - \underline{U}(\hat{z}, \delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Using the renewal theory, we obtain, if the law of ξ is non-arithmetic, that

$$\lim_{x \rightarrow \infty} e^{\beta x} \bar{H}(x) = \frac{1}{\mathbf{E} \xi} \int_0^{\infty} z(v) dv, \quad (8.14)$$

see, e.g., Ch. XI, 9, [15]. If the law of ξ is arithmetic with the step $d > 0$, then, according to Proposition 8.3 for any $x > 0$

$$\lim_{n \rightarrow \infty} e^{\beta(x+nd)} \bar{H}(x+nd) = \frac{d}{\mathbf{E} \xi} \sum_{j \in \mathbb{Z}} z(x+jd) I_{\{x+jd \geq 0\}}. \quad (8.15)$$

The equalities (8.14) and (8.15) implies the statement. \square

The proof of the result below, formulated in a form to cover our needs, follows the same line as in Lemma 2.6 of the Buraczewski–Damek paper [10] with minor changes to include also the arithmetic case.

Theorem 8.5 *Suppose that (8.2) hold. If the support of distribution of Y_{∞} is unbounded from above then*

$$\liminf_{u \rightarrow \infty} u^{\beta} \mathbf{P}(Y_{\infty} > u) > 0.$$

Proof. Let

$$\bar{Y}_n := - \sum_{j=1}^n Q_j^- Z_{j-1}, \quad Y_{n,\infty} := \sum_{j=n+1}^{\infty} Q_j \prod_{l=n+1}^{j-1} M_l$$

and let $Z_n^* := \sup_{j \leq n} Z_j$. Theorems 8.1, 8.2 imply that $\mathbf{P}(\bar{Y}_{\infty} < -u) \leq C_1 u^{-\beta}$ with $C_1 > 0$. On the other hand, by Proposition 8.4 we have that $\mathbf{P}(Z_{\infty}^* > u) \geq C_2 u^{-\beta}$ with $C_2 > 0$. Of course, in both cases the inequalities hold when u is sufficiently large.

Put $U_n := \{Z_n > u, \bar{Y}_n > -Cu\}$ where $C^{\beta} := 4C_1/C_2$. The process \bar{Y} decreases. Therefore, we have the inclusion $\{Z_n > u\} \subseteq \{\bar{Y}_{\infty} \leq -Cu\} \cup U_n$. It follows that for sufficiently large $u > 0$

$$\begin{aligned} (3/4)C_2 u^{-\beta} &\leq \mathbf{P}(Z_{\infty}^* > u) = \mathbf{P}(\cup_n \{Z_n > u\}) \leq \mathbf{P}(\bar{Y}_{\infty} \leq -Cu) + \mathbf{P}(\cup_n U_n) \\ &\leq 2C_1 C^{-\beta} u^{-\beta} + \mathbf{P}(\cup_n U_n) \end{aligned}$$

implying that $\mathbf{P}(\cup_n U_n) \geq (1/4)C_2 u^{-\beta}$.

Since $\bar{Y}_n + Z_n Y_{n,\infty} \leq Y_n + Z_n Y_{n,\infty} = Y_\infty$, we have that

$$\{Y_{n,\infty} > C + 1\} \cap U_n \subseteq \{\bar{Y}_n + Z_n Y_{n,\infty} > u\} \cap U_n \subseteq \{Y_\infty > u\} \cap U_n,$$

Note that $\mathbf{P}(Y_\infty > C + 1) = \mathbf{P}(Y_{n,\infty} > C + 1)$, because $\mathcal{L}(Y_{n,\infty}) = \mathcal{L}(Y_\infty)$. Using the independence of $Y_{n,\infty}$ and the sets $W_n := U_n \cap (\cup_{k=1}^{n-1} U_k)^c$ forming a disjoint partition of $\cup_n U_n$, we get that

$$\begin{aligned} \mathbf{P}(Y_\infty > C + 1)\mathbf{P}(\cup_n W_n) &= \sum_n \mathbf{P}(\{Y_{n,\infty} > C + 1\} \cap W_n) \\ &\leq \sum_n \mathbf{P}(\{Y_\infty > u\} \cap W_n) \leq \mathbf{P}(Y_\infty > u). \end{aligned}$$

Thus, $\mathbf{P}(Y_\infty > u) \geq (1/4)bC_2 u^{-\beta}$ where $b := \mathbf{P}(Y_\infty > C + 1) > 0$ by the assumption that the support of $\mathcal{L}(Y_\infty)$ is unbounded from above. The obtained asymptotic bound implies that $C_+ > 0$. \square

Summarizing the above results we get for function $\bar{G}(u) = \mathbf{P}(Y_\infty > u)$ the following asymptotic properties when $u \rightarrow \infty$:

Theorem 8.6 *Suppose that (8.2) holds. Then $\limsup u^\beta \bar{G}(u) < \infty$. If Y_∞ is unbounded from above, then $\liminf u^\beta \bar{G}(u) > 0$ and in the case where $\mathcal{L}(\ln M)$ is non-arithmetic $\bar{G}(u) \sim C_+ u^{-\beta}$ where $C_+ > 0$.*

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