



**HAL**  
open science

# Asymptotically optimal pointwise and minimax change-point detection for general stochastic models with a composite post-change hypothesis

Serguei Pergamenchtkov, Alexander Tartakovsky

► **To cite this version:**

Serguei Pergamenchtkov, Alexander Tartakovsky. Asymptotically optimal pointwise and minimax change-point detection for general stochastic models with a composite post-change hypothesis. *Journal of Multivariate Analysis*, 2019. hal-02334919

**HAL Id: hal-02334919**

**<https://hal.science/hal-02334919>**

Submitted on 27 Oct 2019

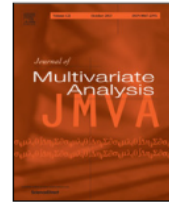
**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: [www.elsevier.com/locate/jmva](http://www.elsevier.com/locate/jmva)

# Asymptotically optimal pointwise and minimax change-point detection for general stochastic models with a composite post-change hypothesis

Serguei Pergamenchikov<sup>a,b</sup>, Alexander G. Tartakovsky<sup>c,d,\*</sup>

<sup>a</sup> Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS-Université de Rouen Normandie, France

<sup>b</sup> National Research Tomsk State University, International Laboratory of Statistics of Stochastic Processes and Quantitative Finance, Tomsk, Russia

<sup>c</sup> Space Informatics Laboratory, Moscow Institute of Physics and Technology, Moscow, Russia

<sup>d</sup> AGT StatConsult, Los Angeles, CA, USA

## ARTICLE INFO

### Article history:

Received 27 March 2019

Received in revised form 22 August 2019

Accepted 22 August 2019

Available online xxx

### AMS 2010 subject classifications:

primary 62L10

62L15

secondary 60G40

60J05

60J20

### Keywords:

Asymptotic optimality

Changepoint detection

Composite post-change hypothesis

Quickest detection

Weighted Shiryaev–Roberts procedure

## ABSTRACT

A weighted Shiryaev–Roberts change detection procedure is shown to approximately minimize the expected delay to detection as well as higher moments of the detection delay among all change-point detection procedures with the given low maximal local probability of a false alarm within a window of a fixed length in pointwise and minimax settings for general non-i.i.d. data models and for the composite post-change hypothesis when the post-change parameter is unknown. We establish very general conditions for models under which the weighted Shiryaev–Roberts procedure is asymptotically optimal. These conditions are formulated in terms of the rate of convergence in the strong law of large numbers for the log-likelihood ratios between the “change” and “no-change” hypotheses, and we also provide sufficient conditions for a large class of ergodic Markov processes. Examples related to multivariate Markov models where these conditions hold are given.

© 2019 Elsevier Inc. All rights reserved.

## 1. Introduction and basic notation

A substantial part of the development of quickest (sequential) change-point detection has been directed towards establishing optimality and asymptotic optimality of certain detection procedures such as CUSUM, Shiryaev, Shiryaev–Roberts, EWMA and their mutual comparison in various settings (Bayesian, minimax, etc.). See, e.g., [1,2,5,8,9,11–13,15–17,19–23,26–30]. This article is concerned with the problem of minimizing the moments of the detection delay,  $\mathcal{R}_{\nu,\theta}^r(\tau) = E_{\nu,\theta} [(\tau - \nu)^r | \tau > \nu]$ , in pointwise (i.e., for all change points  $\nu$ ) and minimax (i.e., for a worst change point) settings among all procedures for which the probability of a false alarm  $P_\infty(k \leq \tau < k + m | \tau \geq k)$  is fixed and small. Hereafter  $\tau$  is a detection procedure (stopping time),  $\nu$  is a point of change, and  $\theta$  is a post-change parameter.

To be more specific, observations  $X_1, X_2, \dots$  are random variables on a probability space  $(\Omega, \mathcal{F})$ , which may change statistical properties at an unknown point in time  $\nu \in \{0, 1, 2, \dots\}$ . For a fixed change point  $\nu = k$  and a parameter

\* Corresponding author.

E-mail address: [agt@phystech.edu](mailto:agt@phystech.edu) (A.G. Tartakovsky).

<https://doi.org/10.1016/j.jmva.2019.104541>

0047-259X/© 2019 Elsevier Inc. All rights reserved.

$\theta \in \Theta$ , there is a measure  $P_{k,\theta}$  on this space, which in the case of no change ( $\nu = \infty$ ) will be denoted by  $P_\infty$ . Let  $E_{k,\theta}$  denote the expectation under  $P_{k,\theta}$  when  $\nu = k < \infty$ , and let  $E_\infty$  denote the same when there is no change, i.e.,  $\nu = \infty$ . We use the convention that  $X_\nu$  is the last pre-change observation. Write  $\mathbf{X}^n = (X_1, \dots, X_n)$  for the concatenation of the first  $n$  observations. Joint probability densities of  $\mathbf{X}^n$  are taken with respect to a  $\sigma$ -finite measure and denoted by  $p_{k,\theta}(\mathbf{X}^n) = p(\mathbf{X}^n | \nu = k, \theta)$  when the change point  $\nu = k$  is fixed and finite (i.e., joint post-change density) and  $p_\infty(\mathbf{X}^n) = p(\mathbf{X}^n | \nu = \infty)$  stands for the pre-change joint density (when the change never occurs). Let  $\{\psi_n(X_n | \mathbf{X}^{n-1})\}_{n \geq 1}$  and  $\{f_{\theta,n}(X_n | \mathbf{X}^{n-1})\}_{n \geq 1}$  be two sequences of conditional densities of  $X_n$  given  $\mathbf{X}^{n-1}$  with respect to some non-degenerate  $\sigma$ -finite measure. We are interested in the general non-i.i.d. case that

$$p_{\nu,\theta}(\mathbf{X}^n) = p_\infty(\mathbf{X}^n) = \prod_{i=1}^n \psi_i(X_i | \mathbf{X}^{i-1}) \quad \text{for } \nu \geq n, \quad (1.1)$$

$$p_{\nu,\theta}(\mathbf{X}^n) = \prod_{i=1}^{\nu} \psi_i(X_i | \mathbf{X}^{i-1}) \times \prod_{i=\nu+1}^n f_{\theta,i}(X_i | \mathbf{X}^{i-1}) \quad \text{for } \nu < n.$$

In other words,  $\{\psi_n(X_n | \mathbf{X}^{n-1})\}_{n \geq 1}$  and  $\{f_{\theta,n}(X_n | \mathbf{X}^{n-1})\}_{n \geq 1}$  are the conditional pre-change and post-change densities, respectively, so that if the change occurs at time  $\nu = k$ , then the conditional density of the  $(k+1)$ th observation changes from  $\psi_{k+1}(X_{k+1} | \mathbf{X}^k)$  to  $f_{\theta,k+1}(X_{k+1} | \mathbf{X}^k)$ . Note that the post-change densities may depend on the change point  $\nu$ , i.e.,  $f_{\theta,n}(X_n | \mathbf{X}^{n-1}) = f_{\theta,n}^{(\nu)}(X_n | \mathbf{X}^{n-1})$  for  $n > \nu$ . We omit the superscript  $\nu$  for brevity.

In many applications, the pre-change distribution is known, but the parameter  $\theta$  of the post-change distribution is unknown. In this case, the post-change hypothesis " $H_k^\theta: \nu = k, \theta = \vartheta$ ",  $\vartheta \in \Theta$  is composite.

Obviously, the general non-i.i.d. model given by (1.1) implies that under the measure  $P_\infty$  the conditional density of  $X_n$  given  $\mathbf{X}^{n-1}$  is  $\psi_n(X_n | \mathbf{X}^{n-1})$  for all  $n \geq 1$  and under  $P_{k,\theta}$ , for any  $0 \leq k < \infty$ , the conditional density of  $X_n$  is  $\psi_n(X_n | \mathbf{X}^{n-1})$  if  $n \leq k$  and is  $f_{\theta,n}(X_n | \mathbf{X}^{n-1})$  if  $n > k$ .

A sequential detection procedure is a stopping (Markov) time  $\tau$  for an observed sequence  $\{X_n\}_{n \geq 1}$ , i.e.,  $\tau$  is an extended integer-valued random variable, such that the event  $\{\tau \leq n\}$  belongs to the  $\sigma$ -algebra  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ . We denote by  $\mathcal{M}$  the set of all stopping times. A false alarm is raised whenever the detection is declared before the change occurs, i.e., when  $\tau \leq \nu$ . (Recall that  $X_{\nu+1}$  is the first post-change observation.) The goal of the quickest change-point detection problem is to develop a detection procedure that guarantees a stochastically small delay to detection  $\tau - \nu$  provided that there is no false alarm (i.e.,  $\tau > \nu$ ) under a given (typically low) risk of false alarms.

The present paper extends the results of Pergamenchtchikov and Tartakovsky [17] to the case of the composite post-change hypothesis when the parameter  $\theta$  is unknown. To this end, we need to develop crucially new synthesis and analysis methods compared to the case of known post-change distribution. Specifically, we show that the mixture version of the Shiryaev–Roberts procedure (referred in this paper as the weighted SR procedure), which is a natural generalization of the Shiryaev–Roberts (SR) procedure in the case of the composite hypothesis, is asymptotically optimal in the class of procedures with the prescribed maximal conditional probability of false alarm when it is small, minimizing moments of the detection delay pointwise (for all change points) as well as in the minimax sense (for the worst change point and the worst parameter value). While basic ideas and methods are similar to those used in the recent publications by Pergamenchtchikov and Tartakovsky [17] and Tartakovsky [27] the results related to asymptotic optimality by no means trivially follow from these works, especially in the minimax setting. Also, since verification of a general condition that guarantees asymptotic optimality related to the  $r$ -complete-type convergence of the properly normalized log-likelihood ratio to a finite number locally in the vicinity of a true parameter value is a difficult task, an important contribution is obtaining a set of sufficient conditions for optimality for a wide class of Markov processes (see Section 5).

The rest of the paper is organized as follows. In Section 2, we specify the weighted (mixture) Shiryaev–Roberts (WSR) procedure and formulate pointwise and minimax optimization problems of interest. In Section 3, we consider the Bayesian version of the problem in the class of procedures with the given weighted probability of false alarm. Based on the recent results of Tartakovsky [27] we establish asymptotic pointwise and minimax properties of the WSR procedure. These results allow us to establish the main theoretical results in Section 4 regarding asymptotic optimality in the class of procedures with the local false alarm probability constraint (in a fixed window). In Section 5, we find certain sufficient conditions for asymptotic optimality for the class of ergodic Markov processes. In Section 6, we provide examples of detecting changes in multivariate Markov models. In Section 7, we present the results of Monte Carlo simulations for a particular example of detecting a change in the correlation coefficient of the Markov Gaussian process that allow us to compare the performance of the weighted SR with that of the SR procedure as well as to verify the precision of the first-order asymptotic approximations that follow from the theory.

## 2. Problem formulation and the detection procedure

Let  $W$  be a probability distribution on  $\Theta$ , which is assumed to be quite arbitrary satisfying the condition  $(C_W)$  For any  $\delta > 0$ , the distribution  $W(u)$  is positive on  $\{u \in \Theta : |u - \theta| < \delta\}$  for every  $\theta \in \Theta$ :

$$W\{u \in \Theta : |u - \theta| < \delta\} > 0 \quad \forall \delta > 0, \theta \in \Theta.$$

This condition means that we do not consider parameter values of  $\theta$  from  $\Theta$  of the measure null. See Remark 1 in Section 8 for a discussion of the choice of the distribution function  $W(\theta)$ .

Define the weighted (average) likelihood ratio (LR)

$$\Lambda_n^k(W) = \int_{\Theta} \prod_{i=k+1}^n \frac{f_{\theta,i}(X_i|\mathbf{X}^{i-1})}{\psi_i(X_i|\mathbf{X}^{i-1})} dW(\theta), \quad n > k$$

and the weighted SR statistic

$$R_n^W = \sum_{k=1}^n \Lambda_n^k(W) \equiv \int_{\Theta} R_n(\theta) dW(\theta), \quad n \geq 1, \quad R_0^W = 0, \tag{2.1}$$

where

$$R_n(\theta) = \sum_{k=1}^n \prod_{i=k}^n \frac{f_{\theta,i}(X_i|\mathbf{X}^{i-1})}{\psi_i(X_i|\mathbf{X}^{i-1})} \tag{2.2}$$

is the SR statistic tuned to  $\theta \in \Theta$ . In this paper, we consider the *Weighted Shiryaev–Roberts* (WSR) detection procedure (or mixture SR) given by the stopping time

$$T_a = \inf \{n \geq 1 : \log R_n^W \geq a\}, \tag{2.3}$$

where  $a > -\infty$  is a threshold controlling for the false alarm risk. In definitions of stopping times we always set  $\inf\{\emptyset\} = +\infty$ .

For any  $0 < \beta < 1$ ,  $m \geq 1$ , and  $\ell \geq 1$  introduce the class of change detection procedures that upper-bounds the local conditional probability of false alarm (LCPFA)  $P_{\infty}(\tau < k + m | \tau \geq k) = P_{\infty}(k \leq \tau < k + m | \tau \geq k)$  in the time interval  $[k, k + m - 1]$  of the length  $m$ :

$$\mathcal{H}(\beta, \ell, m) = \left\{ \tau \in \mathcal{M} : \sup_{1 \leq k \leq \ell} P_{\infty}(\tau < k + m | \tau \geq k) \leq \beta \right\}, \tag{2.4}$$

where  $\mathcal{M}$  is a class of all Markov times. For  $r \geq 1$  and  $\theta \in \Theta$ , we consider the risk associated with the conditional  $r$ th moment of the detection delay

$$\mathcal{R}_{v,\theta}^r(\tau) = E_{v,\theta} [(\tau - v)^r | \tau > v] \tag{2.5}$$

and the following three problems (pointwise and minimax).

*Pointwise Optimization Problem:*

$$\inf_{\tau \in \mathcal{H}(\beta, \ell, m)} \mathcal{R}_{v,\theta}^r(\tau) \quad \text{for every } v \geq 0, \theta \in \Theta. \tag{2.6}$$

*Minimax Optimization Problem:*

$$\inf_{\tau \in \mathcal{H}(\beta, \ell, m)} \sup_{0 \leq v < \infty} \mathcal{R}_{v,\theta}^r(\tau) \quad \text{for every } \theta \in \Theta. \tag{2.7}$$

*Double Minimax Optimization Problem:*

$$\inf_{\tau \in \mathcal{H}(\beta, \ell, m)} \sup_{\theta \in \Theta} \sup_{0 \leq v < \infty} I_{\theta}^r \mathcal{R}_{v,\theta}^r(\tau). \tag{2.8}$$

The function  $I(\theta) = I_{\theta}$ , which characterizes the distance between pre- and post-change distributions, and the parameters  $\ell$  and  $m$  will be specified later.

However, solving the optimization problems (2.6)–(2.8) for any LCPFA  $\beta < 1$  is practically impossible, especially for the general non-i.i.d. model (1.1). For this reason, we will focus on the asymptotic problems when the LCPFA  $\beta$  goes to 0. Our goal is to show that the WSR detection procedure  $T_a$  with a suitable threshold  $a = a_{\beta}$  is first-order asymptotically pointwise and minimax optimal in class  $\mathcal{H}(\beta, \ell_{\beta}, m_{\beta})$ , where  $\ell_{\beta}$  and  $m_{\beta}$  tend to infinity as  $\beta \rightarrow 0$  with a suitable rate.

In addition, in the next section, we consider a Bayesian-type problem of minimizing the risk (2.5) in a class of procedures with the given weighted probability of false alarm. The solution of this problem is used for obtaining the main optimality results in Section 4 in asymptotic versions of optimization problems (2.6)–(2.8) as  $\beta \rightarrow 0$ .

### 3. Asymptotic optimality under weighted PFA constraint

#### 3.1. The non-i.i.d. case

In order to solve asymptotic (as  $\beta \rightarrow 0$ ) optimization problems (2.6)–(2.8) it is constructive to consider a Bayesian-type class of change detection procedures that upper-bounds a weighted probability of false alarm

$$\text{PFA}(\tau) = \sum_{k=0}^{\infty} P_{k,\theta}(\tau \leq k)P(v = k) = \sum_{k=0}^{\infty} P_{\infty}(\tau \leq k)P(v = k),$$

ARTICLE IN PRESS

4

S. Pergamenchtchikov and A.G. Tartakovsky / Journal of Multivariate Analysis xxx (xxxx) xxx

assuming that the change point  $\nu$  is a random variable independent of the observations with prior distribution  $P(\nu = k)$ ,  $k \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ , and the optimization problems

$$\inf_{\{\tau: \text{PFA}(\tau) \leq \alpha\}} \mathcal{R}_{k,\theta}^r(\tau), \quad \forall k \in \mathbb{Z}_+, \theta \in \Theta, \tag{3.1}$$

$$\inf_{\{\tau: \text{PFA}(\tau) \leq \alpha\}} \sup_{k \geq 0} \mathcal{R}_{k,\theta}^r(\tau), \quad \forall \theta \in \Theta, \tag{3.2}$$

$$\inf_{\{\tau: \text{PFA}(\tau) \leq \alpha\}} \sup_{\theta \in \Theta} \sup_{k \geq 0} I_\theta^r \mathcal{R}_{k,\theta}^r(\tau), \tag{3.3}$$

where  $0 < \alpha < 1$  is a prespecified (usually relatively small) number.

This idea was used in [17] for establishing asymptotic optimality of the SR procedure in the case of a simple post-change hypothesis.

In what follows, for our purposes it suffices to assume that the prior probability distribution  $P(\nu = k)$  of the change point  $\nu$  is geometric with the parameter  $0 < \varrho < 1$ , i.e., for  $k \in \mathbb{Z}_+$ ,

$$P(\nu = k) = \pi_k(\varrho) = \varrho (1 - \varrho)^k, \tag{3.4}$$

so that  $\text{PFA}(\tau) = \sum_{k=0}^\infty \varrho (1 - \varrho)^k P_\infty(\tau \leq k)$ .

Now, for some fixed  $0 < \varrho, \alpha < 1$ , define the following Bayesian class of change-point detection procedures with the weighted PFA not greater than the given number  $\alpha$ :

$$\Delta(\alpha, \varrho) = \{\tau \in \mathcal{M} : \text{PFA}(\tau) \leq \alpha\}. \tag{3.5}$$

For a fixed  $\theta \in \Theta$ , introduce the log-likelihood ratio (LLR) process  $\{Z_n^k(\theta)\}_{n \geq k+1}$  between the hypotheses  $H_k^\theta$  ( $k \in \mathbb{Z}_+$ ) and  $H_\infty$ :

$$Z_n^k(\theta) = \sum_{j=k+1}^n \log \frac{f_{\theta,j}(X_j | \mathbf{X}^{j-1})}{\psi_j(X_j | \mathbf{X}^{j-1})}. \tag{3.6}$$

Assume that there is a positive and finite number  $I_\theta$  such that the normalized LLR  $n^{-1}Z_{n+k}^k(\theta)$  converges to  $I_\theta$   $r$ -completely, i.e.,

$$\sum_{n=1}^\infty n^{r-1} P_{k,\theta} \{ |n^{-1}Z_{n+k}^k(\theta) - I_\theta| > \varepsilon \} < \infty, \quad \forall \varepsilon > 0. \tag{3.7}$$

Then it follows from [26] that in the Bayesian setting, when one wants to minimize the  $r$ th moment of the delay to detection (3.1) and when the parameter  $\theta$  is known, the asymptotically (as  $\alpha \rightarrow 0$  and  $\varrho = \varrho_\alpha \rightarrow 0$ ) optimal detection procedure in class (3.5) is the Shiryaev–Roberts detection procedure that raises an alarm at the first time such that the SR statistic  $R_n(\theta)$  exceeds threshold  $(1 - \varrho)/\varrho\alpha$ . This result was extended by Tartakovsky [27] to the case where  $\theta$  is unknown. Below we will use the results obtained in [27] to show that the WSR procedure (2.3) is asymptotically optimal in problems (3.1)–(3.3) under condition (3.7) and some other conditions. These Bayesian-type asymptotic optimality results are the key for proving asymptotic optimality properties in class  $\mathcal{H}(\beta, \ell, m)$  with the constraint on the maximal LCPFA defined in (2.4).

The following condition is sufficient for obtaining the asymptotic lower bounds for all positive moments of the detection delay in class  $\Delta(\alpha) = \Delta(\alpha, \varrho_\alpha)$ :

(A<sub>1</sub>) Assume that there exists a positive and finite number  $I_\theta > 0$  such that for any  $k \geq 0$  and  $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} P_{k,\theta} \left\{ \frac{1}{N} \max_{1 \leq n \leq N} Z_{k+n}^k(\theta) \geq (1 + \varepsilon)I_\theta \right\} = 0, \quad \theta \in \Theta. \tag{3.8}$$

Indeed, by Lemma 1 in [27], we have that if condition (A<sub>1</sub>) is satisfied and  $\varrho_\alpha \rightarrow 0$  as  $\alpha \rightarrow 0$ , then for every  $\nu \geq 0, \theta \in \Theta$ , and  $r \geq 1$

$$\liminf_{\alpha \rightarrow 0} \frac{1}{|\log \alpha|^r} \inf_{\tau \in \Delta(\alpha)} \sup_{\nu \geq 0} \mathcal{R}_{\nu,\theta}^r(\tau) \geq \liminf_{\alpha \rightarrow 0} \frac{1}{|\log \alpha|^r} \inf_{\tau \in \Delta(\alpha)} \mathcal{R}_{\nu,\theta}^r(\tau) \geq \frac{1}{I_\theta^r}. \tag{3.9}$$

Note that condition (A<sub>1</sub>) holds whenever  $Z_n^k(\theta)/(n - k)$  converges almost surely to  $I_\theta$  under  $P_{k,\theta}$ :

$$\frac{1}{n} Z_{k+n}^k(\theta) \xrightarrow[n \rightarrow \infty]{P_{k,\theta}\text{-a.s.}} I_\theta, \quad \theta \in \Theta. \tag{3.10}$$

This is always true for i.i.d. data models with

$$I_\theta = E_{0,\theta}[Z_1^0(\theta)] = \int \log \left[ \frac{f_\theta(x)}{\psi(x)} \right] f_\theta(x) d\mu(x)$$

being the Kullback–Leibler information number.

ARTICLE IN PRESS

The first question we ask is how to select the threshold in the WSR procedure to imbed it into class  $\Delta(\alpha, \varrho)$ . The following lemma answers this question.

**Lemma 1.** For all  $a > 0$  and any prior distribution of  $\nu$  with finite mean  $\bar{\nu} = \sum_{j=1}^{\infty} j P(\nu = j)$ , the PFA of the WSR procedure  $T_a$  given by (2.3) satisfies the inequality

$$PFA(T_a) \leq \bar{\nu} e^{-a},$$

so that if

$$a = a_\alpha = \log(\bar{\nu}/\alpha)$$

then  $PFA(T_{a_\alpha}) \leq \alpha$ . Therefore, if the prior distribution of the change point is geometric, then  $T_{a_\alpha} \in \Delta(\alpha, \varrho)$  for any  $0 < \alpha, \varrho < 1$  if  $a_\alpha = \log[(1 - \varrho)/\varrho\alpha]$ .

**Proof.** Note that under  $P_\infty$  the WSR statistic  $R_n^W$  is a submartingale with mean  $E_\infty[R_n^W] = n$ . Thus, by Doob's submartingale inequality, for  $j \in \{1, 2, \dots\}$

$$P_\infty(T_a \leq j) = P_\infty\left(\max_{1 \leq i \leq j} R_i^W \geq e^a\right) \leq j e^{-a} \tag{3.11}$$

and  $P_\infty(T_a \leq 0) = 0$ . Hence, for any prior distribution (not necessarily geometric)

$$PFA(T_a) = \sum_{j=1}^{\infty} P(\nu = j) P_\infty(T_a \leq j) \leq \bar{\nu} e^{-a}.$$

Therefore, assuming that  $\bar{\nu} < \infty$ , we obtain that setting  $a = a_\alpha = \log(\bar{\nu}/\alpha)$  implies  $PFA(T_{a_\alpha}) \leq \alpha$ . If particularly, the prior distribution is geometric, then  $T_{a_\alpha} \in \Delta(\alpha, \varrho)$  for any  $0 < \alpha, \varrho < 1$  when  $a_\alpha = \log[(1 - \varrho)/\varrho\alpha]$ .  $\square$

In order to study asymptotic approximations to risks of the WSR procedure and for establishing its asymptotic optimality, we impose the following left-tail condition:

**(A<sub>2</sub>(r))** There exists a positive continuous  $\Theta \rightarrow \mathbb{R}$  function  $I(\theta) = I_\theta$  such that for every  $\theta \in \Theta$ , for any  $\varepsilon > 0$ , and for some  $r \geq 1$

$$\mathcal{Y}_r(\varepsilon, \theta) := \lim_{\delta \rightarrow 0} \sum_{n=1}^{\infty} n^{r-1} \sup_{\nu \geq 0} P_{\nu, \theta} \left( \frac{1}{n} \inf_{|u-\theta| < \delta} Z_{\nu+n}^\nu(u) < I_\theta - \varepsilon \right) < \infty.$$

To check this condition it is sufficient to check the following condition:

**(A<sub>2</sub><sup>\*</sup>(r))** There exists a positive continuous  $\Theta \rightarrow \mathbb{R}$  function  $I(\theta)$  such that for every compact set  $\mathbf{K} \subseteq \Theta$ , for any  $\varepsilon > 0$ , and for some  $r \geq 1$

$$\mathcal{Y}_r^*(\varepsilon, \mathbf{K}) := \sup_{\theta \in \mathbf{K}} \mathcal{Y}_r(\varepsilon, \theta) < \infty.$$

In what follows, we assume that the parameter  $\varrho$  is a function of  $\alpha$  such that

$$\lim_{\alpha \rightarrow 0} \varrho_\alpha = 0, \quad \lim_{\alpha \rightarrow 0} \frac{|\log \varrho_\alpha|}{|\log \alpha|} = 0. \tag{3.12}$$

Moreover, let  $k^*$  be a function of  $\alpha$  such that

$$\lim_{\alpha \rightarrow 0} k_\alpha^* = \infty, \quad \lim_{\alpha \rightarrow 0} \alpha \varrho_\alpha k_\alpha^* = 0. \tag{3.13}$$

Note that if in the WSR procedure defined in (2.1)–(2.3) the threshold  $a = a(\alpha, \varrho_\alpha)$  is selected as

$$a_\alpha = (1 - \varrho_\alpha)/(\varrho_\alpha \alpha),$$

where  $\varrho_\alpha$  satisfies condition (3.12), then  $a_\alpha \sim |\log \alpha|$  and, by Lemma 1,  $PFA(T_{a_\alpha}) \leq \alpha$ , i.e., this choice of the threshold guarantees that  $T_{a_\alpha} \in \Delta(\alpha, \varrho_\alpha) = \Delta(\alpha)$  for every  $0 < \alpha < 1$ . Hereafter, notation  $b_c \sim \tilde{b}_c$  as  $c \rightarrow c_0$  means that  $\lim_{c \rightarrow c_0} (b_c/\tilde{b}_c) = 1$ , i.e.,  $b_c = \tilde{b}_c(1 + o(1))$ , where  $o(1) \rightarrow 0$  as  $c \rightarrow c_0$ .

The following theorem establishes first-order asymptotic optimality of the WSR procedure in class  $\Delta(\alpha, \varrho_\alpha) = \Delta(\alpha)$ .

**Theorem 1.** Assume that right-tail and left-tail conditions (A<sub>1</sub>) and (A<sub>2</sub>(r)) hold for some  $0 < I_\theta < \infty$  and the parameter  $0 < \varrho = \varrho_\alpha < 1$  of the geometric prior distribution satisfies conditions (3.12).

(i) Then, for all  $\theta \in \Theta$  and all fixed  $\nu \in \mathbb{Z}_+$ , as  $a \rightarrow \infty$ ,

$$\mathcal{R}_{\nu, \theta}^r(T_a) \sim \left( \frac{a}{I_\theta} \right)^r. \tag{3.14}$$

ARTICLE IN PRESS

If  $a = a_\alpha$  is so selected that  $\text{PFA}(T_{a_\alpha}) \leq \alpha$  and  $\log a_\alpha \sim |\log \alpha|$  as  $\alpha \rightarrow 0$ , in particular  $a_\alpha = \log[(1 - \varrho_\alpha)/\varrho_\alpha \alpha]$ , then, for all  $\theta \in \Theta$  and all fixed  $\nu \in \mathbb{Z}_+$ , as  $\alpha \rightarrow 0$ ,

$$\inf_{\tau \in \Delta(\alpha)} \mathcal{R}_{\nu, \theta}^r(\tau) \sim \left( \frac{|\log \alpha|}{I_\theta} \right)^r \sim \mathcal{R}_{\nu, \theta}^r(T_{a_\alpha}). \tag{3.15}$$

(ii) If  $k^* = k_\alpha^*$  satisfies conditions (3.13) and if  $a = a_\alpha$  is so selected that  $\text{PFA}(T_{a_\alpha}) \leq \alpha$  and  $\log a_\alpha \sim |\log \alpha|$  as  $\alpha \rightarrow 0$ , in particular as  $a_\alpha = \log[(1 - \varrho_\alpha)/\varrho_\alpha \alpha]$ , then, for all  $\theta \in \Theta$ , as  $\alpha \rightarrow 0$ ,

$$\inf_{\tau \in \Delta(\alpha)} \max_{0 \leq \nu \leq k_\alpha^*} \mathcal{R}_{\nu, \theta}^r(\tau) \sim \left( \frac{|\log \alpha|}{I_\theta} \right)^r \sim \max_{0 \leq \nu \leq k_\alpha^*} \mathcal{R}_{\nu, \theta}^r(T_{a_\alpha}). \tag{3.16}$$

Thus, the WSR procedure  $T_{a_\alpha}$  is first-order asymptotically pointwise optimal and minimax in class  $\Delta(\alpha, \varrho_\alpha) = \Delta(\alpha)$  with respect to the moments of the detection delay up to order  $r$ .

**Proof.** Asymptotic approximation (3.14) in assertion (i) follows from Theorem 3 in [27] and asymptotic approximations (3.15) from Theorem 4 in [27].

It remains to prove asymptotic approximations (3.16) in assertion (ii). If  $a_\alpha = \log[(1 - \varrho_\alpha)/\varrho_\alpha \alpha]$ , then using inequality (3.11) we obtain that for any  $0 \leq \nu \leq k_\alpha^*$ ,

$$P_{\nu, \theta}(T_{a_\alpha} > \nu) = P_\infty(T_{a_\alpha} > \nu) \geq P_\infty(T_{a_\alpha} > k_\alpha^*) \geq 1 - k_\alpha^* e^{-a_\alpha} = 1 - \frac{\alpha \varrho_\alpha k_\alpha^*}{1 - \varrho_\alpha}. \tag{3.17}$$

The second condition in (3.13) implies that  $P_\infty(T_{a_\alpha} > \nu) \rightarrow 1$  as  $\alpha \rightarrow 0$  for all  $0 \leq \nu \leq k_\alpha^*$ . Since  $\mathcal{R}_{\nu, \theta}^r(T_{a_\alpha}) = E_{\nu, \theta}[(T_{a_\alpha} - \nu)^+]^r / P_\infty(T_{a_\alpha} > \nu)$ , inequality (3.9) along with equalities (3.15) imply (3.16) for  $k^* = k_\alpha^*$  satisfying conditions (3.13). This completes the proof.  $\square$

**Remark 1.** While for the sake of simplicity we consider the geometric prior distribution with the small parameter  $\varrho_\alpha$ , all the asymptotic results hold true for an arbitrary prior distribution  $\pi_k^\alpha$  such that the mean value of the change point  $E[\nu] = \sum_{k=1}^\infty k \pi_k^\alpha$  approaches infinity as  $\alpha \rightarrow 0$ , assuming that conditions (3.12) and (3.13) hold with  $\varrho_\alpha$  replaced by  $(\sum_{k=1}^\infty k \pi_k^\alpha)^{-1}$ .

It is also interesting to ask whether the WSR procedure is asymptotically optimal with respect to the following double minimax criterion

$$\sup_{\theta \in \Theta} \max_{0 \leq \nu \leq k_\alpha^*} I_\theta^r \mathcal{R}_{\nu, \theta}^r(\tau) \longrightarrow \text{minimum over } \tau \in \Delta(\alpha), \alpha \rightarrow 0$$

(see (3.3)). The following theorem gives an affirmative answer for compact subsets  $\Theta_1 \subset \Theta$ .

**Theorem 2.** Assume that the right-tail condition (A<sub>1</sub>) is satisfied for some  $0 < I_\theta < \infty$ , the parameter  $0 < \varrho = \varrho_\alpha < 1$  of the geometric prior distribution satisfies conditions (3.12), and conditions (3.13) hold for  $k^* = k_\alpha^*$ . Assume that for every  $\varepsilon > 0$  and some  $r \geq 1$

$$\sup_{\theta \in \Theta_1} \Upsilon_r(\varepsilon, \theta) < \infty, \quad \inf_{\theta \in \Theta_1} I_\theta > 0, \tag{3.18}$$

where  $\Theta_1$  is compact. If  $a = a_\alpha$  is so selected that  $\text{PFA}(T_{a_\alpha}) \leq \alpha$  and  $\log a_\alpha \sim |\log \alpha|$  as  $\alpha \rightarrow 0$ , in particular as  $a_\alpha = \log[(1 - \varrho_\alpha)/\varrho_\alpha \alpha]$ , then

$$\inf_{\tau \in \Delta(\alpha)} \sup_{\theta \in \Theta_1} \max_{0 \leq \nu \leq k_\alpha^*} I_\theta^r \mathcal{R}_{\nu, \theta}^r(\tau) \sim |\log \alpha|^r \sim \sup_{\theta \in \Theta_1} \max_{0 \leq \nu \leq k_\alpha^*} I_\theta^r \mathcal{R}_{\nu, \theta}^r(T_{a_\alpha}), \quad \alpha \rightarrow 0. \tag{3.19}$$

**Proof.** Using inequalities (3.9), we obtain that under condition (A<sub>1</sub>) the following asymptotic lower bound holds:

$$\inf_{\tau \in \Delta(\alpha)} \sup_{\theta \in \Theta_1} I_\theta^r \mathcal{R}_{\nu, \theta}^r(\tau) \geq |\log \alpha|^r (1 + o(1)), \quad \alpha \rightarrow 0. \tag{3.20}$$

To prove the theorem it suffices to show that the right-hand side in (3.20) is attained for the risk

$$\sup_{\theta \in \Theta_1} \max_{0 \leq \nu \leq k_\alpha^*} I_\theta^r \mathcal{R}_{\nu, \theta}^r(T_{a_\alpha})$$

of the WSR procedure. Using inequality (A.25) in [27], we obtain that for an arbitrary  $0 < \varepsilon < I_\theta$

$$\sup_{\theta \in \Theta_1} I_\theta^r \sup_{\nu \geq 0} E_{\nu, \theta}[(T_a - \nu)^+]^r \leq \sup_{\theta \in \Theta_1} I_\theta^r \left( 1 + \frac{a}{I_\theta - \varepsilon} \right)^r + r 2^{r-1} \sup_{\theta \in \Theta_1} I_\theta^r \Upsilon_r(\varepsilon, \theta).$$

By condition (3.18), the second term on the right side is finite, which immediately implies that

$$\sup_{\theta \in \Theta_1} I_\theta^r \sup_{\nu \geq 0} E_{\nu, \theta}[(T_a - \nu)^+]^r \leq a^r (1 + o(1)), \quad a \rightarrow \infty.$$

Next, using inequality (3.17), we obtain that for  $\nu \leq k_\alpha^*$

ARTICLE IN PRESS

$$\mathcal{R}_{v,\theta}^r(T_{a_\alpha}) = \frac{E_{v,\theta}[(T_{a_\alpha} - v)^+]^r}{P_\infty(T_{a_\alpha} > v)} \leq \frac{E_{v,\theta}[(T_{a_\alpha} - v)^+]^r}{1 - k_\alpha^* e^{-a_\alpha}} = \frac{(1 - \varrho_\alpha) E_{v,\theta}[(T_{a_\alpha} - v)^+]^r}{1 - \varrho_\alpha - \alpha \varrho_\alpha k_\alpha^*},$$

which along with the previous inequality and conditions (3.12)–(3.13) yields the upper bound

$$\sup_{\theta \in \Theta_1} \max_{0 \leq v \leq k_\alpha^*} I_\theta^r \mathcal{R}_{v,\theta}^r(T_{a_\alpha}) \leq |\log \alpha|^r (1 + o(1)), \quad \alpha \rightarrow 0$$

and the proof is complete. □

**Remark 2.** Let

$$T_B^*(\theta) = \inf \{n \geq 1 : R_n(\theta) \geq B\}$$

be the stopping time of the SR detection procedure tuned to  $\theta$ , where  $R_n(\theta)$  is the SR statistic defined in (2.2). If the least favorable value of the parameter  $\theta_*$  that maximizes the risk

$$\sup_{\theta \in \Theta_1} \max_{0 \leq v \leq k_\alpha^*} I_\theta^r \mathcal{R}_{v,\theta}^r(T_{B_\alpha}^*(\theta)) = \max_{0 \leq v \leq k_\alpha^*} I_{\theta_*}^r \mathcal{R}_{v,\theta_*}^r(T_{B_\alpha}^*(\theta_*))$$

can be found (at least approximately within a small term), then the SR rule  $T_B^*(\theta_*)$  is asymptotically double minimax. This rule is easier to implement and it should have even smaller maximal risk than the WSR rule.

3.2. The case of LLR with independent increments

We now show that condition  $(A_2(r))$  can be substantially relaxed in the case where observations are independent, but not necessarily identically distributed, i.e.,  $f_{\theta,i}(X_i | \mathbf{X}^{i-1}) = f_{\theta,i}(X_i)$  and  $\psi_i(X_i | \mathbf{X}^{i-1}) = \psi_i(X_i)$  in (1.1). More generally, we may assume that the increments  $\Delta Z_i(\theta) = \log[f_{\theta,i}(X_i | \mathbf{X}^{i-1}) / \psi_i(X_i | \mathbf{X}^{i-1})]$  of the LLR  $Z_n^k(\theta) = \sum_{i=k+1}^n \Delta Z_i(\theta)$  are independent, which is always the case if the observations are independent. This slight generalization is important for certain examples with dependent observations that lead to the LLR with independent increments.

**Theorem 3.** Assume that the LLR process  $\{Z_{k+n}^k(\theta)\}_{n \geq 1}$  has independent, not necessarily identically distributed increments under  $P_{k,\theta}$ ,  $k \geq 0$ . Suppose that condition (3.8) holds and the following condition is satisfied

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} P_{v,\theta} \left( \frac{1}{n} \int_{\Gamma_\delta} Z_{\ell+n}^\ell(\vartheta) dW(\vartheta) < I_\theta - \varepsilon \right) = 0, \quad \varepsilon > 0, \ell \geq v, \theta \in \Theta. \tag{3.21}$$

If the parameter  $\varrho = \varrho_\alpha$  of the geometric prior distribution goes to zero as  $\alpha \rightarrow 0$  at rate defined in (3.12), then relations (3.15) and (3.16) hold for all  $r > 0$ , i.e., the WSR procedure is asymptotically optimal with respect to all positive moments of the detection delay.

**Proof.** Let  $N = N(a, \varepsilon, \theta) = 1 + \lfloor a / (I_\theta - \varepsilon) \rfloor$ . Hereafter  $\lfloor x \rfloor$  denotes the integer number less than or equal to  $x$ . We begin with showing that the asymptotic upper bound

$$\mathcal{R}_{v,\theta}^r(T_a) \leq \left( \frac{a}{I_\theta} \right)^r (1 + o(1)), \quad a \rightarrow \infty \tag{3.22}$$

holds for all  $r \geq 1$  under condition (3.21). To this end, note that we have the following chain of equalities and inequalities:

$$\begin{aligned} E_{v,\theta} [(T_a - v)^+]^r &= \sum_{\ell=0}^{\infty} \int_{\ell N}^{(\ell+1)N} r t^{r-1} P_{v,\theta}(T_a - k > t) dt \leq N^r + \sum_{\ell=1}^{\infty} \int_{\ell N}^{(\ell+1)N} r t^{r-1} P_{v,\theta}(T_a - k > t) dt \\ &\leq N^r + \sum_{\ell=1}^{\infty} \int_{\ell N}^{(\ell+1)N} r t^{r-1} P_{v,\theta}(T_a - k > \ell N) dt = N^r \left( 1 + \sum_{\ell=1}^{\infty} [(\ell+1)^r - \ell^r] P_{v,\theta}(T_a - k > \ell N) \right) \\ &\leq N^r \left( 1 + \sum_{\ell=1}^{\infty} r(\ell+1)^{r-1} P_{v,\theta}(T_a - k > \ell N) \right) \leq N^r \left( 1 + r 2^{r-1} \sum_{\ell=1}^{\infty} \ell^{r-1} P_{v,\theta}(T_a - k > \ell N) \right). \end{aligned} \tag{3.23}$$

Consider the intervals (cycles) of the length  $N$  and let  $K_n(v, N) = K_n = v + nN$  and

$$\lambda_{v+nN}^{v+(n-1)N}(W) := \int_{\Gamma_\delta} Z_{v+nN}^{v+(n-1)N}(\vartheta) dW(\vartheta) = \sum_{i=K_{n-1}+1}^{K_n} \int_{\Gamma_\delta} \Delta Z_i(\vartheta) dW(\vartheta).$$

Since for any  $n \geq 1$

$$\log R_{v+nN}^W \geq \log \Lambda_{v+nN}^{v+(n-1)N}(W) \geq \log \int_{\Gamma_\delta} \exp \left\{ Z_{v+nN}^{v+(n-1)N}(\vartheta) \right\} dW(\vartheta)$$



and, by Jensen's inequality,

$$\int_{\Gamma_\delta} \exp \left\{ Z_{\nu+nN}^{\nu+(n-1)N}(\vartheta) \right\} dW(\vartheta) = W(\Gamma_\delta) \int_{\Gamma_\delta} \exp \left\{ Z_{\nu+nN}^{\nu+(n-1)N}(\vartheta) \right\} \frac{dW(\vartheta)}{W(\Gamma_\delta)} \geq W(\Gamma_\delta) \exp \left\{ \frac{\lambda_{\nu+nN}^{\nu+(n-1)N}(W)}{W(\Gamma_\delta)} \right\},$$

it follows that

$$\log R_{\nu+nN}^W \geq \frac{\lambda_{K_n}^{K_n-1}(W)}{W(\Gamma_\delta)} - |\log W(\Gamma_\delta)|, \quad n \geq 1.$$

Therefore,

$$\begin{aligned} P_{\nu,\theta}(T_a - \nu > \ell N) &= P_{\nu,\theta}(\log R_n^W < a \text{ for } n \in \{1, \dots, \nu + \ell N\}) \leq P_{\nu,\theta}(\log R_{\nu+nN}^W < a \text{ for } n \in \{1, \dots, \ell\}) \\ &\leq P_{\nu,\theta} \left( \frac{\lambda_{K_n}^{K_n-1}(W)}{W(\Gamma_\delta)} < a + |\log W(\Gamma_\delta)| \text{ for } n \in \{1, \dots, \ell\} \right). \end{aligned}$$

Since the increments of the LLR are independent, the random variables  $\lambda_{K_n}^{K_n-1}(W)$ ,  $n \in \{1, \dots, \ell\}$ , are independent, and hence,

$$\begin{aligned} P_{\nu,\theta}(T_a - \nu > \ell N) &\leq P_{\nu,\theta} \left( \frac{\lambda_{K_n}^{K_n-1}(W)}{W(\Gamma_\delta)} < a + |\log W(\Gamma_\delta)| \text{ for } n \in \{1, \dots, \ell\} \right) \\ &= \prod_{n=1}^{\ell} P_{\nu,\theta} \left( \frac{\lambda_{K_n}^{K_n-1}(W)}{W(\Gamma_\delta)} < a + |\log W(\Gamma_\delta)| \right) \\ &\leq \prod_{n=1}^{\ell} P_{\nu,\theta} \left( \frac{\lambda_{K_n}^{K_n-1}(W)}{NW(\Gamma_\delta)} < I_\theta - \varepsilon + |\log W(\Gamma_\delta)|/N \right). \end{aligned}$$

It follows that for a sufficiently large  $a > 0$ , for which  $|\log W(\Gamma_\delta)|/N \leq \varepsilon/2$ ,

$$P_{\nu,\theta}(T_a - \nu > \ell N) \leq \prod_{n=1}^{\ell} P_{\nu,\theta} \left( \frac{\lambda_{K_n}^{K_n-1}(W)}{NW(\Gamma_\delta)} < I_\theta - \varepsilon/2 \right).$$

By condition (3.21), for a sufficiently large  $a$  there exists a small  $\delta_a$  such that

$$P_{\nu,\theta} \left( \frac{1}{NW(\Gamma_\delta)} \int_{\Gamma_\delta} Z_{K_n}^{K_n-1}(\vartheta) dW(\vartheta) < I_\theta - \varepsilon/2 \right) \leq \delta_a, \quad n \geq 1.$$

Therefore, for any  $\ell \geq 1$ ,

$$P_{\nu,\theta}(T_a - \nu > \ell N) \leq \delta_a^\ell.$$

Combining this inequality with (3.23) and using the fact that  $L_{r,a} = \sum_{\ell=1}^{\infty} \ell^{r-1} \delta_a^\ell \rightarrow 0$  as  $a \rightarrow \infty$  for any  $r > 0$ , we obtain

$$\mathcal{R}_{\nu,\theta}^r(T_{a_\alpha}) = \frac{E_{\nu,\theta}[(T_{a_\alpha} - \nu)^+]^r}{P_\infty(T_a > \nu)} \leq \frac{\left(1 + \frac{\log a}{I_\theta - \varepsilon}\right)^r + r2^{r-1}L_{r,a}}{1 - \nu e^{-a}} = \left(\frac{a}{I_\theta - \varepsilon}\right)^r (1 + o(1)), \quad a \rightarrow \infty. \quad (3.24)$$

Since  $\varepsilon \in (0, I_\theta)$  is an arbitrary number, this implies the upper bound (3.22).

Setting  $a_\alpha = \log[(1 - \varrho_\alpha)/\varrho_\alpha \alpha]$  or more generally  $a_\alpha \sim |\log \alpha|$  in (3.24) yields the upper bound (for all  $r \geq 1$ , all  $\nu \geq 0$ , and all  $\theta \in \Theta$ )

$$\mathcal{R}_{\nu,\theta}^r(T_{a_\alpha}) \leq \left(\frac{|\log \alpha|}{I_\theta}\right)^r (1 + o(1)), \quad \alpha \rightarrow 0.$$

Applying this upper bound together with the lower bound (3.9) (which holds due to condition (3.8)) proves (3.15).

The proof of (3.16) is essentially analogous to that in the proof of Theorem 1(ii) above.  $\square$

#### 4. Asymptotic optimality under local PFA constraint

We now proceed with the pointwise and minimax problems (2.6)–(2.8) in the class of procedures with given LCPFA  $\mathcal{H}(\beta, \ell, m)$  defined in (2.4). Note that the asymptotic optimality results of the previous section are essential, since asymptotic optimality in class  $\mathcal{H}(\beta, \ell, m)$  is obtained by embedding this class in class  $\Delta(\alpha, \varrho)$  with specially selected parameters  $\varrho$  and  $\alpha$ .

ARTICLE IN PRESS

4.1. The non-i.i.d. case

For any  $0 < \beta < 1$ ,  $m \geq 1$ , and  $\ell \geq 1$ , define

$$\alpha_1 = \alpha_1(\beta, m) = \beta + (1 - \varrho_{1,\beta})^{m+1} \tag{4.1}$$

and

$$\alpha_2 = \alpha_2(\beta, \ell, m) = \frac{\beta(1 - \varrho_{2,\beta})^{\ell+m}}{1 + \beta}, \tag{4.2}$$

where  $\varrho_{2,\beta} = \check{\delta}_\beta \varrho_{1,\beta}$  and the functions  $0 < \varrho_{1,\beta} < 1$  and  $0 < \check{\delta}_\beta < 1$  are such that

$$\lim_{\beta \rightarrow 0} (\varrho_{1,\beta} + \check{\delta}_\beta) = 0, \quad \lim_{\beta \rightarrow 0} \frac{|\log \varrho_{1,\beta}| + |\log \check{\delta}_\beta|}{|\log \beta|} = 0. \tag{4.3}$$

For example, we can take

$$\varrho_{1,\beta} = \frac{1}{1 + |\log \beta|}, \quad \check{\delta}_\beta = \frac{\check{\delta}^*}{1 + |\log \beta|} \quad \text{with } 0 < \check{\delta}^* < 1.$$

To find asymptotic lower bounds for the problems (2.6) and (2.7) in addition to condition (A<sub>1</sub>) we impose the following condition related to the growth of the window size  $m$  in the LCPFA:

(H<sub>1</sub>) The size of the window  $m$  in (4.1) is a function of  $\beta$ , i.e.  $m = m_\beta$ , such that

$$\lim_{\beta \rightarrow 0} \frac{|\log \alpha_{1,\beta}|}{|\log \beta|} = 1, \tag{4.4}$$

where  $\alpha_{1,\beta} = \alpha_1(\beta, m_\beta)$ .

The following theorem establishes asymptotic lower bounds.

**Theorem 4.** Assume that conditions (A<sub>1</sub>) and (H<sub>1</sub>) hold. Then, for any  $\ell \geq 1$ ,  $\nu \in \mathbb{Z}_+$ ,  $\theta \in \Theta$ , and  $r \geq 1$

$$\liminf_{\beta \rightarrow 0} \frac{1}{|\log \beta|^r} \inf_{\tau \in \mathcal{H}(\beta, \ell, m_\beta)} \sup_{\nu \geq 0} \mathcal{R}_{\nu, \theta}^r(\tau) \geq \liminf_{\beta \rightarrow 0} \frac{1}{|\log \beta|^r} \inf_{\tau \in \mathcal{H}(\beta, \ell, m_\beta)} \mathcal{R}_{\nu, \theta}^r(\tau) \geq \frac{1}{I_\theta^r}. \tag{4.5}$$

**Proof.** First, recall that under condition (A<sub>1</sub>) the lower bounds (3.9) hold for any  $r \geq 1$  and  $\nu \geq 0$ . Second, we show that for any  $0 < \beta < 1$ ,  $m \geq |\log(1 - \beta)| / [|\log(1 - \varrho_{1,\beta})|] - 1 := m_0$  and  $\ell \geq 1$ , the following inclusion holds:

$$\mathcal{H}(\beta, \ell, m) \subseteq \Delta(\alpha_1, \varrho_{1,\beta}), \tag{4.6}$$

where  $\alpha_1 = \alpha_1(\beta, m) < 1$  for every  $m > m_0$  and  $\beta \in (0, 1)$ . Indeed, let  $\tau$  be from  $\mathcal{H}(\beta, \ell, m)$ . Then, using definition of class  $\mathcal{H}(\beta, \ell, m)$ , we obtain that  $P_\infty(\tau \leq m) \leq \beta$ . Therefore, taking in (3.4)  $\varrho = \varrho_{1,\beta}$ , we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \pi_k(\varrho_{1,\beta}) P_\infty(\tau \leq k) &= \sum_{k=0}^m \pi_k(\varrho_{1,\beta}) P_\infty(\tau \leq k) + \sum_{k=m+1}^{\infty} \pi_k(\varrho_{1,\beta}) P_\infty(\tau \leq k) \\ &\leq \beta + \sum_{k=m+1}^{\infty} \pi_k(\varrho_{1,\beta}) = \beta + (1 - \varrho_{1,\beta})^{m+1} = \alpha_1, \end{aligned}$$

i.e.,  $\tau \in \Delta(\alpha_1, \varrho_{1,\beta})$  where  $\alpha_1 < 1$  for  $m > m_0$ .

Inclusion (4.6) implies that for all  $\nu \geq 0$  and for a sufficiently small  $\beta > 0$

$$\inf_{\tau \in \mathcal{H}(\beta, \ell, m)} \mathcal{R}_{\nu, \theta}^r(\tau) \geq \inf_{\tau \in \Delta(\alpha_1, \varrho_{1,\beta})} \mathcal{R}_{\nu, \theta}^r(\tau).$$

Now, lower bounds (4.5) follow from lower bounds (3.9) and condition (H<sub>1</sub>). □

To establish asymptotic optimality properties of the WSR procedure with respect to the risks  $\mathcal{R}_{\nu, \theta}^r(\tau)$  (for all  $\nu \geq 0$ ) and  $\sup_{\nu \geq 0} \mathcal{R}_{\nu, \theta}^r(\tau)$  in class  $\mathcal{H}(\beta, \ell, m)$  we need the uniform  $r$ -complete convergence condition (A<sub>2</sub>( $r$ )) as well as the following condition:

(H<sub>2</sub>) Parameters  $\ell$  and  $m$  are functions of  $\beta$ , i.e.  $\ell = \ell_\beta$  and  $m = m_\beta$ , such that

$$\lim_{\beta \rightarrow 0} \frac{|\log \alpha_{2,\beta}|}{|\log \beta|} = 1, \quad \lim_{\beta \rightarrow 0} \alpha_{2,\beta} \varrho_{2,\beta} k_\beta^* = 0, \tag{4.7}$$

where  $\alpha_{2,\beta} = \alpha_2(\beta, \ell_\beta, m_\beta)$  and  $k_\beta^* = \ell_\beta + m_\beta$ .

ARTICLE IN PRESS

10

S. Pergamenchtchikov and A.G. Tartakovsky / Journal of Multivariate Analysis xxx (xxxx) xxx

The conditions (4.4) and (4.7) hold, for example, if

$$m_\beta = \lfloor |\log \beta| / \varrho_{1,\beta} \rfloor, \quad \ell_\beta = \check{\alpha} m_\beta, \quad \check{\alpha} > 0.$$

Denote by  $T_\beta$  the WSR procedure  $T_{a_\beta}$  with threshold  $a_\beta$  given by

$$a_\beta = \log \left( \frac{1 - \alpha_{2,\beta}}{\varrho_{2,\beta} \alpha_{2,\beta}} \right). \tag{4.8}$$

**Theorem 5.** If conditions  $(H_1)$  and  $(H_2)$  hold, then, for any  $0 < \beta < 1$ , the WSR procedure  $T_\beta$  with threshold  $a_\beta$  given by (4.8) belongs to class  $\mathcal{H}(\beta, \ell_\beta, m_\beta)$ . Assume in addition that conditions  $(A_1)$  and  $(A_2(r))$  are satisfied. Then

$$\inf_{\tau \in \mathcal{H}(\beta, \ell_\beta, m_\beta)} \mathcal{R}_{\nu, \theta}^r(\tau) \sim \left( \frac{|\log \beta|}{I_\theta} \right)^r \sim \mathcal{R}_{\nu, \theta}^r(T_\beta), \quad \nu \in \mathbb{Z}_+, \theta \in \Theta \tag{4.9}$$

and

$$\inf_{\tau \in \mathcal{H}(\beta, \ell_\beta, m_\beta)} \max_{0 \leq \nu \leq k_\beta^*} \mathcal{R}_{\nu, \theta}^r(\tau) \sim \left( \frac{|\log \beta|}{I_\theta} \right)^r \sim \max_{0 \leq \nu \leq k_\beta^*} \mathcal{R}_{\nu, \theta}^r(T_\beta), \quad \theta \in \Theta. \tag{4.10}$$

Therefore, the WSR procedure  $T_\beta$  is first-order asymptotically pointwise optimal and minimax in class  $\mathcal{H}(\beta, \ell_\beta, m_\beta)$ , minimizing moments of the detection delay up to order  $r$  for all parameter values  $\theta \in \Theta$ .

**Proof.** First note that the conditions (4.3) and (4.7) imply the properties (3.12)–(3.13). Now, by Lemma 1, the WSR procedure  $T_{a_\alpha(\varrho)}$  with threshold  $a_\alpha(\varrho) = \log[(1 - \alpha)/\varrho\alpha]$  belongs to class  $\Delta(\alpha, \varrho)$  for any  $0 < \alpha, \varrho < 1$ . Hence,  $T_\beta \in \Delta(\alpha_{2,\beta}, \varrho_{2,\beta})$ . Note that for any  $0 < \beta < 1$ ,  $m \geq 1$  and  $\ell \geq 1$ , the following inclusion holds:

$$\Delta(\alpha_{2,\beta}, \varrho_{2,\beta}) \subseteq \mathcal{H}(\beta, \ell, m). \tag{4.11}$$

Indeed, by definition of class  $\Delta(\alpha, \varrho)$ , we have that for any  $0 < \alpha, \varrho < 1$  and any  $i \geq 1$

$$\alpha \geq \varrho \sum_{k=i}^{\infty} (1 - \varrho)^k P_\infty(\tau \leq k) \geq \varrho P_\infty(\tau \leq i) \sum_{k=i}^{\infty} (1 - \varrho)^k = P_\infty(\tau \leq i)(1 - \varrho)^i,$$

which implies that

$$\sup_{\tau \in \Delta(\alpha, \varrho)} P_\infty(\tau \leq k_\beta^*) \leq \alpha(1 - \varrho)^{-k_\beta^*}.$$

Let  $\tau \in \Delta(\alpha_{2,\beta}, \varrho_{2,\beta})$ . Then, taking into account the latter inequality and using definition of  $\alpha_{2,\beta}$  in (4.2), we obtain that

$$\sup_{1 \leq k \leq \ell_\beta} P_\infty(\tau < k + m_\beta | \tau \geq k) \leq \sup_{1 \leq k \leq \ell_\beta} \frac{P_\infty(\tau < k + m_\beta)}{P_\infty(\tau \geq k)} \leq \frac{P_\infty(\tau < k_\beta^*)}{1 - P_\infty(\tau < k_\beta^*)} \leq \frac{\alpha_{2,\beta} (1 - \varrho_{2,\beta})^{-k_\beta^*}}{1 - \alpha_{2,\beta} (1 - \varrho_{2,\beta})^{-k_\beta^*}} = \beta,$$

i.e.,  $\tau$  belongs to  $\mathcal{H}(\beta, \ell_\beta, m_\beta)$ .

Using inclusion (4.11), we obtain that the stopping time  $T_\beta$  belongs to  $\mathcal{H}(\beta, \ell_\beta, m_\beta)$  for any  $0 < \beta < 1$ .

Next, in view of definition of  $a_\beta$  in (4.8) and of the form of the function  $\varrho_{2,\beta}$  in (4.3) we obtain, using condition  $(H_2)$ , that  $\lim_{\beta \rightarrow 0} a_\beta / |\log \beta| = 1$ . Thus, by (3.14) in Theorem 1,

$$\lim_{\beta \rightarrow 0} \frac{1}{|\log \beta|^r} \mathcal{R}_{\nu, \theta}^r(T_\beta) = \frac{1}{I_\theta^r}, \quad \nu \in \mathbb{Z}_+, \theta \in \Theta.$$

Comparing to the lower bound (4.5) implies (4.9).

To prove (4.10) it suffices to show that

$$\limsup_{\beta \rightarrow 0} \frac{\max_{0 \leq \nu \leq k_\beta^*} \mathcal{R}_{\nu, \theta}^r(T_\beta)}{|\log \beta|^r} \leq \frac{1}{I_\theta^r}, \quad \theta \in \Theta. \tag{4.12}$$

Note that

$$\max_{0 \leq \nu \leq k_\beta^*} \mathcal{R}_{\nu, \theta}^r(T_\beta) \leq \frac{\max_{0 \leq \nu \leq k_\beta^*} E_{\nu, \theta} [(T_\beta - \nu)^+]^r}{\min_{0 \leq \nu \leq k_\beta^*} P_\infty(T_\beta > \nu)},$$

where as  $\beta \rightarrow 0$

$$\min_{0 \leq \nu \leq k_\beta^*} P_\infty(T_\beta > \nu) = P_\infty(T_\beta > k_\beta^*) \rightarrow 1.$$

ARTICLE IN PRESS

Also, by inequality (A.25) in [27], for an arbitrary  $0 < \varepsilon < I_\theta$ ,

$$\sup_{v \geq 0} E_{v,\theta} [(T_\beta - v)^+]^r \leq \left(1 + \frac{a_\beta}{I_\theta - \varepsilon}\right)^r + r2^{r-1} \Upsilon_r(\varepsilon, \theta).$$

Note that the second condition in (4.3) and the first condition in (4.7) imply that  $a_\beta \sim |\log \beta|$  as  $\beta \rightarrow 0$ . As a result, we obtain as  $\beta \rightarrow 0$

$$\max_{0 \leq v \leq k_\beta^*} \mathcal{R}_{v,\theta}^r(T_\beta) \leq \frac{\sup_{v \geq 0} E_{v,\theta} [(T_\beta - v)^+]^r}{P_\infty(T_\beta > k_\beta^*)} = \left(\frac{|\log \beta|}{I_\theta}\right)^r (1 + o(1)).$$

This obviously yields the upper bound (4.12) and the proof is complete.  $\square$

The following theorem establishes minimax properties of the WSR procedure with respect to the risk

$$\sup_{\theta \in \Theta_1} \max_{0 \leq v \leq k_\beta^*} I_\theta^r \mathcal{R}_{v,\theta}^r(\tau)$$

for compact subsets  $\Theta_1$  of  $\Theta$ . It is absolutely similar to Theorem 2 and its proof follows almost immediately from Theorems 2 and 5, and for this reason it is omitted.

**Theorem 6.** Assume that the right-tail and the left-tail conditions  $(\mathbf{A}_1)$  and (3.18) are satisfied for some  $0 < I_\theta < \infty$  and that conditions  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$  are satisfied as well. Then

$$\inf_{\tau \in \mathcal{H}(\beta, \ell_\beta, m_\beta)} \sup_{\theta \in \Theta_1} \max_{0 \leq v \leq k_\beta^*} I_\theta^r \mathcal{R}_{v,\theta}^r(\tau) \sim |\log \beta|^r \sim \sup_{\theta \in \Theta_1} \max_{0 \leq v \leq k_\beta^*} I_\theta^r \mathcal{R}_{v,\theta}^r(T_\beta) \text{ as } \beta \rightarrow 0.$$

4.2. The case of LLR with independent increments

As in Section 3.2, we now consider a particular (still quite general) case where the LLR process has independent increments. The following theorem is similar to Theorem 3.

**Theorem 7.** Assume that the LLR process  $\{Z_{k+n}^k(\theta)\}_{n \geq 1}$  has independent, not necessarily identically distributed increments under  $P_{k,\theta}$ ,  $k \geq 0$ . Suppose that conditions  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$ ,  $(\mathbf{A}_1)$ , and (3.21) are satisfied. Then asymptotic relations (4.9) and (4.10) hold for all  $r > 0$ , i.e., the WSR procedure  $T_\beta$  is first-order asymptotically pointwise optimal and minimax in class  $\mathcal{H}(\beta, \ell_\beta, m_\beta)$  with respect to all positive moments of the detection delay for all  $\theta \in \Theta$ .

**Proof.** The proof follows almost immediately from Theorems 3 and 5 and is omitted.  $\square$

5. Uniform concentration inequalities and sufficient conditions of asymptotic optimality for Markov processes

Condition  $(\mathbf{A}_1)$  is usually not difficult to check since it follows from the almost sure convergence of the log-likelihood ratio (3.10). However, verification of the left-tail  $r$ -complete-type convergence condition  $(\mathbf{A}_2(r))$  may be a challenge. In this section, we obtain sufficient conditions for a class of homogeneous Markov processes in order to verify condition  $(\mathbf{A}_2(r))$  in particular examples.

Let  $(X_n^\theta)_{n \geq 1}$  be a time homogeneous Markov process with values in a measurable space  $(\mathcal{X}, \mathcal{B})$  defined by a family of the transition probabilities  $(P^\theta(x, A))_{\theta \in \Theta}$  for some fixed parameter set  $\Theta \subseteq \mathbb{R}^p$ . In the sequel, we denote by  $\mathbf{E}_x^\theta(\cdot)$  the expectation with respect to this probability. In addition, we assume that this process is geometrically ergodic, i.e.,

$(\mathbf{B}_1)$  For any  $\theta \in \Theta$  there exist a probability measure  $\lambda$  on  $(\mathcal{X}, \mathcal{B})$  and the Lyapunov  $\mathcal{X} \rightarrow [1, \infty)$  function  $\mathbf{V}^\theta$  with  $\lambda^\theta(\mathbf{V}^\theta) < \infty$ , such that for some positive constants  $0 < R < \infty$  and  $\kappa > 0$ ,

$$\sup_{n \geq 0} e^{\kappa n} \sup_{x \in \mathbb{R}} \sup_{\theta \in \Theta} \sup_{0 < F \leq \mathbf{V}^\theta} \frac{1}{\mathbf{V}^\theta(x)} |\mathbf{E}_x^\theta [F(X_n)] - \lambda^\theta(F)| \leq R.$$

Now, for some  $q > 0$ , we set

$$v_q^*(x) = \sup_{n \geq 0} \sup_{\theta \in \Theta} \mathbf{E}_x^\theta [\mathbf{V}^\theta(X_n)]^q. \tag{5.1}$$

Let  $g$  be a measurable  $\Theta \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  function such that the following integrals exist

$$\tilde{g}(\theta, x) = \int_{\mathcal{X}} g(\theta, y, x) P^\theta(x, dy), \quad \lambda^\theta(\tilde{g}) = \int_{\mathcal{X}} \tilde{g}(u) \lambda^\theta(du). \tag{5.2}$$

$(\mathbf{B}_2)$  Assume that  $g$  satisfies the Hölder condition of the power  $0 < \gamma \leq 1$  with respect to the first variable, i.e., there exists a measurable positive  $\mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  function  $h$  such that for any  $x, y$  from  $\mathcal{X}$

$$\sup_{u, \theta \in \Theta} \frac{|g(u, y, x) - g(\theta, y, x)|}{|u - \theta|^\gamma} \leq h(y, x) \tag{5.3}$$

ARTICLE IN PRESS

1 and the corresponding integrals  $\tilde{h}(\theta, x)$  and  $\lambda^\theta(\tilde{h})$  exist for any  $\theta \in \Theta$ , where  $\tilde{h}(\theta, x)$  is defined as  $\tilde{g}$  in (5.2) and  $|\cdot|$  is the  
 2 Euclidean norm in  $\mathbb{R}^p$ .

3 **(B<sub>3</sub>)** Assume that the functions  $g$  and  $h$  are such that  $|\tilde{g}(\theta, x)| \leq \mathbf{V}^\theta(x)$  and  $|\tilde{h}(\theta, x)| \leq \mathbf{V}^\theta(x)$  for all  $\theta \in \Theta$  and  $x \in \mathcal{X}$ .

4 Now, for any measurable  $\Theta \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  function  $g$  for which there exist the integrals (5.2), we introduce the  
 5 deviation processes

$$6 \quad W_n^g(u, \theta) = n^{-1} \sum_{j=1}^n g(u, X_j, X_{j-1}) - \lambda^\theta(\tilde{g}), \quad \tilde{W}_n^g(\theta) = W_n^g(\theta, \theta). \quad (5.4)$$

7 Similarly to (5.1), we define for some  $q > 0$

$$8 \quad g_q^*(x) = \sup_{n \geq 1} \sup_{\theta \in \Theta} \mathbf{E}_x^\theta |g(\theta, X_n, X_{n-1})|^q, \quad h_q^*(x) = \sup_{n \geq 1} \sup_{\theta \in \Theta} \mathbf{E}_x^\theta |h(X_n, X_{n-1})|^q. \quad (5.5)$$

9 Proposition 1 of [17] implies the following result.

10 **Proposition 1.** Assume that conditions **(B<sub>1</sub>)–(B<sub>3</sub>)** hold. Then for any  $q \geq 2$ , for which  $v_q^*(x) < \infty$ ,  $g_q^*(x) < \infty$ , and  $h_q^*(x) < \infty$ ,  
 11 one has

$$12 \quad \sup_{n \geq 2} n^{q/2} \sup_{\theta \in \Theta} \sup_{x \in \mathcal{X}} \frac{\mathbf{E}_x^\theta |\tilde{W}_n^g(\theta)|^q + \mathbf{E}_x^\theta |\tilde{W}_n^h(\theta)|^q}{(1 + v_q^*(x) + g_q^*(x) + h_q^*(x))} < \infty. \quad (5.6)$$

13 **Proposition 2.** Assume that conditions **(B<sub>1</sub>)–(B<sub>3</sub>)** hold and  $v_q^*(x) < \infty$ ,  $g_q^*(x) < \infty$ , and  $h_q^*(x) < \infty$  for some  $q \geq 2$ . Then for  
 14 any  $\varepsilon > 0$  there exists  $\delta_0 > 0$  such that

$$15 \quad \sup_{0 < \delta \leq \delta_0} \sup_{n \geq 2} n^{q/2} \sup_{\theta \in \Theta} \sup_{x \in \mathcal{X}} \frac{\mathbf{P}_x^\theta (\sup_{|u-\theta| < \delta} |W_n^g(u, \theta)| > \varepsilon)}{(1 + v_q^*(x) + g_q^*(x) + h_q^*(x))} < \infty. \quad (5.7)$$

16 **Proof.** First note that

$$17 \quad \sup_{|u-\theta| < \delta} |W_n^g(u, \theta)| \leq \delta^\gamma (\lambda^\theta(\tilde{h}) + \tilde{W}_n^h(\theta)) + |\tilde{W}_n^g(\theta)|.$$

18 Therefore, for any

$$19 \quad 0 < \delta \leq \left( \frac{\varepsilon}{2(1 + \lambda^\theta(\tilde{h}))} \right)^{1/\gamma} := \delta_0$$

20 we obtain that

$$21 \quad \mathbf{P}_x^\theta \left( \sup_{|u-\theta| < \delta} |W_n^g(u, \theta)| > \varepsilon \right) \leq \mathbf{P}_x^\theta (|\tilde{W}_n^g(\theta)| > \varepsilon/2) + \mathbf{P}_x^\theta (|\tilde{W}_n^h(\theta)| > 1).$$

22 Applying the bound (5.6) we obtain (5.7). Hence Proposition 2 follows.  $\square$

23 We return to the change detection problem for Markov processes, assuming that the sequence of observations  $(X_n)_{n \geq 1}$   
 24 is a Markov process, such that  $(X_n)_{1 \leq n \leq \nu}$  is a homogeneous process with the transition (from  $x$  to  $y$ ) density  $\psi(y|x)$ . In the  
 25 sequel, we denote by  $\check{\mathbf{P}}$  the distribution of this process when  $\nu = \infty$ , i.e., when the Markov process  $(X_j)_{j \geq 1}$  has transition  
 26 density  $\psi(y|x)$ . The expectation with respect to this distribution we denote by  $\check{\mathbf{E}}$ . Moreover,  $(X_n)_{n > \nu}$  is homogeneous  
 27 positive ergodic with the transition density  $f_\theta(y|x)$  and the ergodic (stationary) distribution  $\lambda^\theta$ . The densities  $\psi(y|x)$  and  
 28  $f_\theta(y|x)$  are calculated with respect to a sigma-finite positive measure  $\mu$  on  $\mathcal{B}$ .

29 In this case, we can represent the process  $Z_n^k(\theta)$  defined in (3.6) as

$$30 \quad Z_n^k(\theta) = \sum_{j=k+1}^n g(\theta, X_j, X_{j-1}), \quad g(\theta, y, x) = \log \frac{f_\theta(y|x)}{\psi(y|x)}.$$

31 Therefore, in this case,

$$32 \quad \tilde{g}(\theta, x) = \int_{\mathcal{X}} g(\theta, y, x) f_\theta(y|x) \mu(dy).$$

33 Moreover, if we assume that density  $f_u(y|x)$  is continuously differentiable with respect to  $u$  in a compact set  $\mathbf{K} \subseteq \Theta$ , then  
 34 the inequality (5.3) holds with  $\gamma = 1$  and for any function  $h(y, x)$  for which

$$35 \quad \sup_{u \in \Theta} \max_{1 \leq j \leq p} |\partial g(u, y, x) / \partial u_j| \leq h(y, x).$$

36 **(C<sub>1</sub>)** Assume that there exists a set  $C \in \mathcal{B}$  with  $\mu(C) < \infty$  such that

ARTICLE IN PRESS

(C1.1)  $f_* = \inf_{\theta \in \mathbf{K}} \inf_{x, y \in C} f_\theta(y|x) > 0$ .

(C1.2) For any  $\theta \in \mathbf{K}$  there exists  $\mathcal{X} \rightarrow [1, \infty)$  Lyapunov's function  $\mathbf{V}^\theta$  such that

- $\mathbf{V}^\theta(x) \geq \tilde{g}(\theta, x)$  and  $\mathbf{V}^\theta(x) \geq \tilde{h}(\theta, x)$  for any  $\theta \in \mathbf{K}$  and  $x \in \mathcal{X}$ ,
- $\sup_{\theta \in \Theta} \sup_{x \in C} \mathbf{V}^\theta(x) < \infty$ .
- For some  $0 < \rho < 1$  and  $D > 0$  and for all  $x \in \mathcal{X}$  and  $\theta \in \Theta$ ,

$$\mathbf{E}_x^\theta[\mathbf{V}^\theta(X_1)] \leq (1 - \rho)\mathbf{V}^\theta(x) + D\mathbb{1}_{\{C\}}(x). \tag{5.8}$$

(C2) Assume that there exists  $q > 2$  such that

$$\sup_{k \geq 1} \check{\mathbf{E}}[g_q^*(X_k)] < \infty, \quad \sup_{k \geq 1} \check{\mathbf{E}}[h_q^*(X_k)] < \infty, \quad \sup_{k \geq 1} \check{\mathbf{E}}[v_q^*(X_k)] < \infty,$$

where the function  $v_q^*(x)$  is defined in (5.1),  $g_q^*(x)$  and  $h_q^*(x)$  are given in (5.5).

**Theorem 8.** Assume that conditions (C1)–(C2) hold for some compact set  $\mathbf{K} \subseteq \Theta$ . Then for any  $0 < r < q/2$  condition  $(\mathbf{A}_2^*(r))$  holds with  $I(\theta) = \lambda^\theta(\tilde{g})$ .

**Proof.** First note that it follows from Theorem A.1 in the appendix that the conditions (C1) imply the property (B1). So Proposition 2 yields that there exists a positive constant  $\mathbf{C}^*$  such that for any  $x \in \mathcal{X}$

$$\mathbf{P}^\theta \left( \sup_{|u-\theta| < \delta} |W_n^g(u, \theta)| > \varepsilon | X_0 = x \right) \leq \mathbf{C}^* \mathbf{U}^*(x) n^{-q/2},$$

where  $\mathbf{U}^*(x) = 1 + v_q^*(x) + g_q^*(x) + h_q^*(x)$ . Note now that

$$\mathbf{P}_{v, \theta} \left( \frac{1}{n} \inf_{|u-\theta| < \delta} Z_{v+n}^v(u) < I_\theta - \varepsilon \right) \leq \mathbf{P}_{v, \theta} \left( \sup_{|u-\theta| < \delta} \left| \frac{1}{n} Z_{v+n}^v(u) - I_\theta \right| > \varepsilon \right).$$

In view of the homogeneous Markov property we obtain that for  $I_\theta = \lambda^\theta(\tilde{g})$  the last probability can be represented as

$$\mathbf{P}_{v, \theta} \left( \sup_{|u-\theta| < \delta} \left| \frac{1}{n} Z_{v+n}^v(u) - I_\theta \right| > \varepsilon \right) = \check{\mathbf{E}}[\Psi^\theta(X_v)],$$

where  $\Psi^\theta(x) = \mathbf{P}^\theta \left( \sup_{|u-\theta| < \delta} |W_n^g(u, \theta)| > \varepsilon | X_0 = x \right)$ . Therefore,

$$\mathbf{P}_{v, \theta} \left( \sup_{|u-\theta| < \delta} \left| \frac{1}{n} Z_{v+n}^v(u) - I_\theta \right| > \varepsilon \right) \leq \mathbf{C}^* n^{-q/2} \check{\mathbf{E}}[\mathbf{U}^*(X_v)].$$

Now condition (C2) implies  $(\mathbf{A}_2^*(r))$  for any  $0 < r < q/2$ .  $\square$

Note that condition (C1.1) does not always hold for the process  $(X_n)_{n \geq 1}$  directly. Unfortunately, this condition does not hold for the practically important autoregression process of the order more than one. For this reason, we need to weaken this requirement. Assume that there exists  $p \geq 2$  for which the process  $(\tilde{X}_{i,n})_{n \geq \tilde{\nu}}$  for  $\tilde{\nu} = \nu/p - \iota$  defined as  $\tilde{X}_{i,n} = X_{np+\iota}$  satisfies the following properties:

(C'1) Assume that there exists a set  $C \in \mathcal{B}$  with  $\mu(C) < \infty$  such that

(C'1.1)  $\tilde{f}_* = \inf_{1 \leq i \leq p} \inf_{\theta \in \mathbf{K}} \inf_{x, y \in C} \tilde{f}_{i, \theta}(y|x) > 0$ , where  $\tilde{f}_{i, \theta}(y|x)$  is the transition density for the process  $(\tilde{X}_{i,n})_{n \geq 1}$ .

(C'1.2) For any  $\theta \in \mathbf{K}$  there exists  $\mathcal{X} \rightarrow [1, \infty)$  Lyapunov's function  $\mathbf{V}^\theta$  such that

$$v^* = \max_{1 \leq j \leq p} \sup_{\theta \in \mathbf{K}} \sup_{x \in \mathcal{X}} \frac{\mathbf{E}_x^\theta[\mathbf{V}^\theta(X_j)]}{\mathbf{V}^\theta(x)} < \infty; \tag{5.9}$$

$$v_1^* = \sup_{\theta \in \mathbf{K}} \lambda^\theta(\mathbf{V}^\theta) < \infty. \tag{5.10}$$

- $\mathbf{V}^\theta(x) \geq \tilde{g}(\theta, x)$  and  $\mathbf{V}^\theta(x) \geq \tilde{h}(\theta, x)$  for  $\theta \in \mathbf{K}$  and  $x \in \mathcal{X}$  and  $\sup_{\theta \in \mathbf{K}} \sup_{x \in C} \mathbf{V}^\theta(x) < \infty$ .
- For some  $0 < \rho < 1$  and  $D > 0$  and for all  $x \in \mathcal{X}$ ,  $\theta \in \mathbf{K}$ , and  $0 \leq \iota \leq p - 1$

$$\mathbf{E}^\theta[\mathbf{V}^\theta(\tilde{X}_{i,1}) | \tilde{X}_{i,0} = x] \leq (1 - \rho)\mathbf{V}^\theta(x) + D\mathbb{1}_{\{C\}}(x). \tag{5.11}$$

**Theorem 9.** Assume that conditions (C2) and (C'1) hold. Then for any  $0 < r < q/2$  condition  $(\mathbf{A}_2^*(r))$  holds with  $I(\theta) = \lambda^\theta(\tilde{g})$ .

**Proof.** Note again that by Theorem A.1 (see the Appendix) conditions (C1) yield condition (B1) for  $(\tilde{X}_{i,n})_{n \geq \tilde{\nu}}$ , i.e., for some positive constants  $0 < R_i < \infty$  and  $\kappa_i > 0$ ,

$$\sup_{n \geq 0} e^{\kappa_i n} \sup_{x \in \mathbb{R}} \sup_{\theta \in \mathbf{K}} \sup_{0 < F \leq \mathbf{V}^\theta} \frac{1}{\mathbf{V}^\theta(x)} |D_m^\theta(x)| \leq R_i,$$

ARTICLE IN PRESS

where  $D_m^\theta(x) = \mathbf{E}^\theta [(F(\tilde{X}_{t,m}) - \lambda^\theta(F)) | \tilde{X}_{t,0} = x]$ . So, for any  $n \geq p$  we can write that  $n = mp + \iota$  for some  $0 \leq \iota \leq p - 1$  and we obtain

$$|\mathbf{E}_x^\theta(F(X_n) - \lambda^\theta(F))| = |\mathbf{E}_x^\theta D_m^\theta(X_\iota)| \leq R_\iota \mathbf{V}^\theta(X_\iota) e^{\kappa_\iota m}.$$

Now, the upper bound (5.9) implies

$$\sup_{n \geq p} e^{\kappa n} \sup_{x \in \mathbb{R}^p} \sup_{\theta \in \mathbf{K}} \sup_{0 < F \leq \mathbf{V}^\theta} \frac{1}{\mathbf{V}^\theta(x)} |\mathbf{E}_x^\theta[F(X_n) - \lambda^\theta(F)]| \leq R_*,$$

where  $R_* = v^* \max_{0 \leq \iota} R_\iota$  and  $\kappa = \min_{0 \leq \iota \leq p-1} \kappa_\iota / p$ . Thus, using bound (5.10), we obtain condition (B<sub>1</sub>) with  $R = e^{\kappa^* p} (v^* + v_1^*) + R_*$ . Using now the same argument as in the proof of Theorem 8 we obtain Theorem 9. □

6. Examples

We now present examples of detecting changes in multivariate Markov models that illustrate the general theory developed in Sections 3 and 4.

**Example 1** (Change in the Parameters of the Multivariate Linear Difference Equation). Consider the multivariate model in  $\mathbb{R}^p$  given by

$$X_n = (\check{A}_n \mathbb{1}_{\{n \leq \nu\}} + A_n \mathbb{1}_{\{n > \nu\}}) X_{n-1} + w_n,$$

where  $\check{A}_n$  and  $A_n$  are  $p \times p$  random matrices and  $(w_n)_{n \geq 1}$  is an i.i.d. sequence of Gaussian random vectors  $\mathcal{N}(0, Q_0)$  in  $\mathbb{R}^p$  with the positive definite  $p \times p$  matrix  $Q_0$ . Assume also that  $\check{A}_n = A_0 + B_n$ ,  $A_n = \theta + B_n$  and  $(B_n)_{n \geq 1}$  are i.i.d. Gaussian random matrices  $\mathcal{N}(0, Q_1)$ , where the  $p^2 \times p^2$  matrix  $Q_1 = \mathbf{E}[B_1 \otimes B_1]$  is positive definite. Hereafter, for  $p \times p$  matrices  $\mathbf{U} = (\mathbf{u}_{ij})_{1 \leq i, j \leq p}$  and  $\mathbf{V} = (\mathbf{v}_{ij})_{1 \leq i, j \leq p}$  the  $p^2 \times p^2$  matrix  $\mathbf{U} \otimes \mathbf{V}$  is

$$\mathbf{U} \otimes \mathbf{V} = (\mathbf{u}_{ij} \mathbf{v}_{kl})_{1 \leq i, j, k, l \leq p}.$$

Assume, in addition, that all eigenvalues of the matrix  $\mathbf{E}[\check{A}_1 \otimes \check{A}_1] = A_0 \otimes A_0 + Q_1$  are less than one in module. Define

$$\Theta = \{\theta \in \mathbb{R}^{p^2} : \max_{1 \leq j \leq p^4} \mathbf{e}_j(\theta \otimes \theta + Q_1) < 1\} \setminus \{A_0\},$$

where  $\mathbf{e}_j(A)$  is the  $j$ th eigenvalue of matrix  $A$ , and assume further that the matrix  $\theta \in \Theta$ . In this case, the processes  $(X_n)_{n \geq 1}$  (in the case  $\nu = \infty$ ) and  $(X_n)_{n > \nu}$  (in the case  $\nu < \infty$ ) are ergodic with the ergodic distributions given by the vectors [10]

$$\check{\zeta} = \sum_{i \geq 1} \prod_{j=1}^{i-1} \check{A}_j w_i, \quad \zeta_\theta = \sum_{i \geq 1} \prod_{j=1}^{i-1} A_j w_i,$$

i.e., the corresponding invariant measures  $\check{\lambda}$  and  $\lambda^\theta$  on  $\mathbb{R}^p$  are defined as  $\check{\lambda}(A) = P(\check{\zeta} \in \Gamma)$  and  $\lambda^\theta(A) = P(\zeta_\theta \in \Gamma)$  for any  $\Gamma \in \mathcal{B}(\mathbb{R}^p)$ . According to [3] we define the Lyapunov function as

$$\mathbf{V}^\theta(x) = v_*(1 + x^\top T(\theta)x), \quad T(\theta) = (I_{p^4} - \theta^\top \otimes \theta^\top - Q_1^\top)^{-1} \text{vec}(\check{I}_p), \tag{6.1}$$

where the symbol  $\top$  in  $(\cdot)^\top$  denotes transpose,  $v_* \geq 1$ ,  $\check{I}_m$  is the identity matrix of order  $m$ , and for the  $p \times p$  matrix  $\mathbf{V} = (\mathbf{v}_{ij})_{1 \leq i, j \leq p}$  the vector  $\text{vec}(\mathbf{V})$  is

$$\text{vec}(\mathbf{V}) = (\mathbf{v}_{11}, \dots, \mathbf{v}_{p1}, \dots, \mathbf{v}_{1p}, \dots, \mathbf{v}_{pp})^\top \in \mathbb{R}^{p^2}.$$

As shown in [3], in this case, for any  $x \in \mathbb{R}^p$  the quadratic form  $x^\top T(\theta)x \geq |x|^2$ . Hence, all eigenvalues of the matrix  $T(\theta)$  are greater than one. Let now  $\mathbf{K} \subset \Theta$  be some compact set. For some fixed  $N_* > 1$ , define the set

$$C = \{x \in \mathbb{R}^p : \max_{\theta \in \mathbf{K}} x^\top T(\theta)x \leq N_*\}.$$

By direct calculation we obtain

$$\mathbf{E}^\theta [\mathbf{V}^\theta(X_1) | X_0 = x] = \mathbf{V}^\theta(x) \left( 1 - \frac{|x|^2 - \text{tr} T(\theta)Q_0}{\mathbf{V}^\theta(x)} \right).$$

Taking into account that the function  $T(\theta)$  is continuous, we obtain that for any non-zero vector  $x \in \mathbb{R}^p$  and  $\theta \in \mathbf{K}$

$$1 \leq \mathbf{e}_{\min} \leq \frac{x^\top T(\theta)x}{|x|^2} \leq \mathbf{e}_{\max} < \infty,$$

where

$$\mathbf{e}_{\min} = \min_{\theta \in \mathbf{K}} \inf_{x \in \mathbb{R}^p, x \neq 0} \frac{x^\top T(\theta)x}{|x|^2}, \quad \mathbf{e}_{\max} = \max_{\theta \in \mathbf{K}} \sup_{x \in \mathbb{R}^p, x \neq 0} \frac{x^\top T(\theta)x}{|x|^2}.$$

ARTICLE IN PRESS

It follows that, for  $x \in C^c$ ,  $|x|^2 > N_*/\mathbf{e}_{max}$  and, therefore,

$$\frac{|x|^2 - \text{tr} T(\theta)Q_0}{\mathbf{V}^\theta(x)} \geq \frac{|x|^2}{1 + \mathbf{e}_{max}|x|^2} - \frac{\mathbf{e}_{max} \text{tr} Q_0}{\mathbf{e}_{min}|x|^2} \geq \frac{1}{2\mathbf{e}_{max}} - \frac{\mathbf{e}_{max}^2 \text{tr} Q_0}{\mathbf{e}_{min} N_*}.$$

Now we choose  $N_* > 1$  sufficiently large to obtain positive term in the right side of the last inequality. So we obtain the drift inequality (5.8) for the Lyapunov function defined in (6.1) with any coefficient  $v^* \geq 1$ . The function  $g(u, y, x)$  can be calculated for any  $x, y \in \mathbb{R}^p$  and  $u \in \Theta$  as

$$g(u, y, x) = \frac{|G^{-1/2}(x)(y - A_0x)|^2 - |G^{-1/2}(x)(y - ux)|^2}{2} = y^\top G^{-1}(x)(u - A_0)x + \frac{x^\top A_0^\top G^{-1}(x)A_0x - x^\top u^\top G^{-1}(x)ux}{2},$$

where  $G(x) = \mathbf{E}[B_1xx^\top B_1^\top] + Q_0 = Q_1 \text{Vect}(xx^\top) + Q_0$ . Taking into account that the matrices  $Q_0$  and  $Q_1$  are positive definite, we obtain that there exists some constant  $\mathbf{c}_* > 0$  for which

$$\sup_{x \in \mathbb{R}^p} |G^{-1}(x)| \leq \frac{\mathbf{c}_*}{1 + |x|^2}, \tag{6.2}$$

and we obtain that condition (B<sub>2</sub>) holds with  $\gamma = 1$  and

$$h(y, x) = \mathbf{c}_*(2\theta_{max} + |y|), \quad \theta_{max} = \max_{u \in \Theta} |u|. \tag{6.3}$$

Moreover, note that in this case

$$\tilde{g}(u, x) = \frac{1}{2} |G^{-1/2}(x)(u - A_0)x| = \frac{1}{2} x^\top (u - A_0)^\top G^{-1}(x)(u - A_0)x.$$

The bound (6.2) implies that  $g^* = \sup_{x \in \mathbb{R}^p} \sup_{\theta \in \mathbf{K}} \tilde{g}(\theta, x) < \infty$ .

Now, as in Example 4 in [17] choosing  $\mathbf{V}(x) = v^*[1 + (x^\top Tx)^\delta]$  with  $v^* = 1 + g^*$  and any fixed  $0 < \delta \leq 1$  yields condition (C<sub>1</sub>). Moreover, for any  $r > 0$  and  $\delta r \leq 2$

$$\sup_{x \in \mathbb{R}^p} \sup_{\theta \in \mathbf{K}} \frac{\sup_{j \geq 1} \mathbf{E}_x^\theta |X_j|^{\delta r}}{1 + |x|^{\delta r}} < \infty, \quad \sup_{j \geq 1} \check{\mathbf{E}} |X_j|^{\delta r} < \infty, \tag{6.4}$$

where  $\check{\mathbf{E}}$  denotes the expectation with respect to the distribution  $\check{\mathbf{P}}$  when  $v = \infty$ . Inequalities (6.4) imply (C<sub>2</sub>) with  $q = \delta r$ . Therefore, taking into account that  $\delta$  can be very close to zero and using Theorem 8 we get that for any  $r > 0$  and any compact set  $\mathbf{K} \subset \Theta \setminus \{A_0\}$  condition (A<sub>2</sub><sup>\*</sup>(r)) holds with  $I_\theta = \mathbf{E}^\theta[\tilde{g}(\theta, \zeta_\theta)]$ .

**Example 2** (Change in the Correlation Coefficients of the AR(p) Model). Consider the problem of detecting the change of the correlation coefficient in the pth order AR process, assuming that for  $n \geq 1$

$$X_n = a_{1,n}X_{n-1} + \dots + a_{p,n}X_{n-p} + w_n, \tag{6.5}$$

where  $a_{i,n} = a_i \mathbb{1}_{\{n \leq \nu\}} + \theta_i \mathbb{1}_{\{n > \nu\}}$  and  $(w_n)_{n \geq 1}$  are i.i.d. Gaussian random variables with  $\mathbf{E}[w_1] = 0$ ,  $\mathbf{E}[w_1^2] = 1$ . In the sequel, we use the notation  $\mathbf{a} = (a_1, \dots, a_p)^\top$  and  $\theta = (\theta_1, \dots, \theta_p)^\top$ . The process (6.5) is not Markov, but the p-dimensional process

$$\Phi_n = (X_n, \dots, X_{n-p+1})^\top \in \mathbb{R}^p \tag{6.6}$$

is Markov. Note that for  $n > \nu$

$$\Phi_n = A\Phi_{n-1} + \tilde{w}_n,$$

where

$$A = A(\theta) = \begin{pmatrix} \theta_1 & \theta_2 & \dots & \theta_p \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad \tilde{w}_n = (w_n, 0, \dots, 0) \in \mathbb{R}^p.$$

It is clear that

$$\mathbf{E}[\tilde{w}_n \tilde{w}_n^\top] = B = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Assume that the vectors  $\mathbf{a}$  and  $\theta$  belong to the set

$$\Theta = \{u \in \mathbb{R}^p : \max_{1 \leq j \leq p} |e_j(A(u))| < 1\},$$



ARTICLE IN PRESS

where  $e_j(A)$  denotes the  $j$ th eigenvalue for the matrix  $A$ . Note that, in this case, for any  $u$  from some compact set  $\mathbf{K} \subset \Theta$  and any  $y = (y_1, \dots, y_p)^T \in \mathbb{R}^p$  and  $x = (x_1, \dots, x_p)^T \in \mathbb{R}^p$  the function

$$g(u, y, x) = y_1(u - \mathbf{a})^T x + \frac{(\mathbf{a}^T x)^2 - (u^T x)^2}{2}.$$

Obviously, it follows that condition  $(\mathbf{B}_2)$  holds with  $\gamma = 1$  and

$$h(y, x) = y_1^2 + (1 + 2\theta_{\max})|x|^2,$$

where  $\theta_{\max}$  is defined in (6.3).

For any  $\theta \in \Theta$ , the process  $(\Phi_n)_{n \geq \nu+p}$  is ergodic with the normal stationary distribution

$$\varsigma_\theta = \sum_{n \geq 1} A^{n-1} \tilde{w}_n \sim \mathcal{N}(0, F), \quad F = F(\theta) = \sum_{n \geq 0} A^n B (A^T)^n. \tag{6.7}$$

Obviously, condition (C1.1) does not hold for the process (6.6). To fulfill this condition we replace this process by the embedded homogeneous Markov process  $\tilde{\Phi}_{\iota, n} = \Phi_{np+\iota}$  for some  $0 \leq \iota \leq p - 1$ . This process can be represented as

$$\tilde{\Phi}_{\iota, n} = A^p \tilde{\Phi}_{\iota, n-1} + \zeta_{\iota, n}, \quad \zeta_{\iota, n} = \sum_{j=0}^{p-1} A^j \tilde{w}_{np+\iota-j}. \tag{6.8}$$

Clearly,  $\zeta_{\iota, n}$  is Gaussian with the parameters  $(0, Q)$ , where

$$Q = Q(\theta) = \sum_{j=0}^{p-1} A^j B (A^T)^j.$$

One can check directly that this matrix is positive definite. Moreover, one can check directly that for any  $\theta \in \Theta$  and for any  $0 \leq \iota \leq p - 1$  the process (6.8) is ergodic with the same ergodic distribution given in (6.7).

Now, for any fixed  $0 < \delta \leq 1$  we define the  $\mathbb{R}^p \rightarrow \mathbb{R}$  function

$$\mathbf{V}^\theta(x) = \check{c}[1 + (x^T T x)^\delta], \quad T = T(\theta) = \sum_{l=0}^{\infty} (A^T)^{pl} A^{pl}, \tag{6.9}$$

where  $\check{c} \geq 1$  will be specified later. Let for any fixed compact set  $\mathbf{K} \subset \Theta \setminus \{\mathbf{a}\}$

$$t_{\max} = \max_{u \in \mathbf{K}} |T(u)|, \quad Q_* = \max_{u \in \mathbf{K}} |Q(u)|.$$

Obviously,  $t_{\max} > 1$ . Note that, by the Jensen inequality, for any  $0 \leq \iota < p$

$$\mathbf{E}^\theta [\mathbf{V}^\theta(\tilde{\Phi}_{\iota, 1}) \mid \tilde{\Phi}_{\iota, 0} = x] \leq \check{c} + \check{c} (x^T (A^p)^T T A^p x + \text{tr} TQ)^\delta \leq \check{c} + \check{c} (x^T (A^p)^T T A^p x t_{\max} Q_*)^\delta.$$

Also,

$$x^T T x \geq |x|^2, \quad \frac{x^T (A^p)^T T A^p x}{x^T T x} = 1 - \frac{|x|^2}{x^T T x} \leq 1 - \frac{1}{t_{\max}} = t_* < 1.$$

So, taking into account that  $(|a| + |b|)^\delta \leq |a|^\delta + |b|^\delta$  for  $0 < \delta \leq 1$ , we obtain

$$\mathbf{E}^\theta [\mathbf{V}^\theta(\tilde{\Phi}_{\iota, 1}) \mid \tilde{\Phi}_{\iota, 0} = x] \leq \check{c} + \check{c} [t_*^\delta (x^T T x)^\delta + (t_{\max} Q_*)^\delta].$$

Putting

$$N_* = \left( \frac{2(1 + t_{\max}^\delta Q_*^\delta)}{1 - t_*^\delta} \right)^{1/2\delta}, \quad \rho = (1 - t_*^\delta)/2,$$

yields that, for  $|x| \geq N_*$ ,

$$\mathbf{E}^\theta [\mathbf{V}^\theta(\tilde{\Phi}_{\iota, 1}) \mid \tilde{\Phi}_{\iota, 0} = x] \leq (1 - \rho) \mathbf{V}^\theta(x).$$

Hence, the Markov process (6.8) satisfies the drift inequality (5.11) with

$$C = \{x \in \mathbb{R}^p : |x| \leq N_*\}, \quad D = \check{c}(1 + t_{\max}^\delta N_*^{2\delta} + t_{\max}^\delta Q_*^\delta).$$

Next we need the minorizing measure in condition  $(C'_1)$  on the Borel  $\sigma$ -field in  $\mathbb{R}^p$ . To this end, we define  $\check{\nu}(\Gamma) = \text{mes}(\Gamma \cap C)/\text{mes}(C)$  for any Borel set  $\Gamma$  in  $\mathbb{R}^p$ , where  $\text{mes}(\cdot)$  is the Lebesgue measure in  $\mathbb{R}^p$ . Moreover, note that

$$\tilde{h}(\theta, x) = 1 + (\theta^T x)^2 + (1 + 2\theta_{\max})|x|^2 \leq 1 + (1 + 2\theta_{\max})|x|^2$$

and

$$\tilde{g}(\theta, x) = \frac{1}{2} [(\theta - \mathbf{a}_0)^T x]^2 \leq \theta_{\max}^2 |x|^2.$$

ARTICLE IN PRESS

**Table 1**

Operating characteristics of the WSR and SR detection procedures. Results of MS simulations with  $10^6$  runs for the probability of false alarm  $\beta = 0.01$  and the change points  $\nu = 0$  and  $\nu = 10$ . The worst change point is  $\nu = 0$ .

$\beta = 0.01, \nu = 0$							
$\theta$	$e^a$	$ADD_{\nu,\theta}(T_a)$	$LCPFA(T_a)$	$B$	$ADD_{\nu,\theta}(T_B^*)$	$LCPFA(T_B^*)$	$ADD_{\nu,\theta}^{app}(T_a)$
0.9	395	11.74	0.0080	791	11.08	0.0079	2.81
0.8	420	14.72	0.0073	791	13.72	0.0073	6.80
0.7	440	18.97	0.0070	791	17.52	0.0071	12.67
0.6	470	25.32	0.0065	791	23.15	0.0065	21.88
0.5	595	36.35	0.0049	791	31.84	0.0049	38.33
0.4	1040	59.57	0.0024	791	45.88	0.0025	72.94
$\beta = 0.01, \nu = 10$							
$\theta$	$e^a$	$ADD_{\nu,\theta}(T_a)$	$LCPFA(T_a)$	$B$	$ADD_{\nu,\theta}(T_B^*)$	$LCPFA(T_B^*)$	$ADD_{\nu,\theta}^{app}(T_a)$
0.9	395	10.05	0.0080	791	9.62	0.0079	2.81
0.8	420	12.72	0.0073	791	11.98	0.0073	6.80
0.7	440	16.59	0.0070	791	15.30	0.0071	12.67
0.6	470	22.55	0.0065	791	20.34	0.0065	21.88
0.5	595	32.96	0.0049	791	28.01	0.0049	38.33
0.4	1040	55.34	0.0024	791	40.83	0.0025	72.94

Therefore, choosing in (6.9)  $\check{c} = 1 + 2\theta_{max} + \theta_{max}^2$ , we obtain condition (C'1.2). Condition (C<sub>2</sub>) can be checked in the same way as in Example 1 for any  $r > 0$  for which  $0 < q = \delta r \leq 2$ . Therefore, taking into account that  $\delta$  can be very close to zero, Theorem 9 implies that for any  $r > 0$  and any compact set  $K \subset \Theta \setminus \{a\}$  condition (A<sub>2</sub><sup>\*</sup>(r)) holds with  $I_\theta = E^\theta[\tilde{g}(\theta, \zeta_\theta)]$ .

**7. Monte Carlo simulations**

In this section, we provide Monte Carlo (MC) simulations for the AR(1) model, which is a particular case of Example 2 for  $p = 1$ , i.e.,  $a_{1,n} = \theta_0 \mathbb{1}_{\{n \leq \nu\}} + \theta \mathbb{1}_{\{n > \nu\}}$  and  $a_{2,n} = \dots = a_{p,n} \equiv 0$  in (6.5). Let the pre-change value  $\theta_0 = 0$  and the post-change value  $\theta \in \Theta = \{\theta_1, \dots, \theta_N\}$ ,  $-1 < \theta_1 < \theta_2 < \dots < \theta_N < 1$ ,  $\theta_i \neq 0$ , and write

$$L_n^\theta(X_n, X_{n-1}) = \exp \left\{ \theta X_n X_{n-1} - \frac{\theta^2 X_{n-1}^2}{2} \right\}, \quad n \geq 1.$$

The WSR stopping time is written as

$$T_a = \inf \left\{ n \geq 1 : \log \left[ \sum_{j=1}^N W(\theta_j) R_n(\theta_j) \right] \geq a \right\},$$

where the SR statistic  $R_n(\theta)$  tuned to  $\theta$  satisfies the recursion

$$R_{n+1}(\theta) = [1 + R_n(\theta)] L_{n+1}^\theta(X_{n+1}, X_n), \quad n \geq 0, \quad R_0(\theta) = 0.$$

Thus, the WSR procedure can be easily implemented.

The information number  $I_\theta = \theta^2 / [2(1 - \theta^2)]$ , so the first-order approximation (3.14) yields the following approximate formula for the average delay to detection  $ADD_{\nu,\theta_j}(T_a) = E_{\nu,\theta}(T_a - \nu | T_a > \nu)$ :

$$ADD_{\nu,\theta}(T_a) \approx ADD_{\nu,\theta}^{app}(T_a) = \frac{2(1 - \theta^2)a}{\theta^2}. \tag{7.1}$$

In the MC simulations, we set

$$\Theta = \{-0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$$

and uniform prior  $W(\theta_j) = 1/18, j = 1, \dots, 18$ .

The results are presented in Table 1 for the upper bound on the maximal local conditional probability of false alarm (LCPFA)  $\beta = 0.01$  and the number of MC runs  $10^6$ . In the table, we compare operating characteristics of the WSR rule  $T_a$  with that of the SR rule

$$T_B^*(\theta) = \inf \{ n \geq 1 : R_n(\theta) \geq B \}$$

tuned to the true value of the post-change parameter  $\theta$ , i.e., assuming that the post-change parameter  $\theta$  is known and equals one of the values shown in the table. Thresholds  $a$  and  $B$  (shown in the table) were selected in such a way that the maximal probabilities of false alarm of both rules ( $LCPFA(T_a)$  and  $LCPFA(T_B^*)$ ) were practically the same. It is seen that for relatively large values of the post-change parameter,  $\theta \geq 0.6$ , the SR rule only slightly outperforms the WSR rule, but for small parameter values (i.e., for close hypotheses) the difference becomes quite substantial. The worst change point is  $\nu = 0$ , as expected. Also, the first-order approximation (7.1) is not too accurate, especially for small and large parameter values.

## 8. Remarks

1. Despite the fact that the WSR procedure is first-order asymptotically optimal for practically arbitrary distribution  $W(\theta)$  that satisfies condition  $(C_W)$ , for practical purposes its choice may be important. In fact, selection of the weight  $W$  affects the higher-order asymptotic performance, and therefore, the real performance of the detection procedure. For example, if the set  $\Theta$  is continuous, one has to avoid  $W(\theta)$  that concentrates in the vicinity of a specific parameter value  $\theta_1$  since in this case the WSR procedure will be nearly optimal at and in the vicinity of  $\theta_1$  but will not have a good performance for other parameter values. The choice of  $W(\theta)$  is also related to the computational issue. It is reasonable to select the weight as to be in the class of conjugate priors, if possible, or to select a uniform prior if  $\Theta$  is compact. A substantial simplification occurs when  $\Theta = \{\theta_1, \dots, \theta_N\}$  is a finite discrete set. If the observations are i.i.d., then in the discrete case, it is possible to find an optimal (in a certain sense) weight using the approach proposed by Fellouris and Tartakovsky [4] for the hypothesis testing problem.

2. The traditional constraint on the false alarm risk in minimax change-point detection problems is the lower bound on the average run length to false alarm (ARL2FA)  $E_\infty[\tau] \geq \gamma \geq 1$ . This measure of false alarms makes sense when the distribution of the stopping time  $\tau$  (in our case of  $T_a$ ) is approximately geometric. This is typically the case (at least asymptotically as  $a \rightarrow \infty$ ) for i.i.d. data models [18,31]. However, apart from the i.i.d. case, there is no result on the asymptotic distribution of the stopping time  $T_a$  (as  $a \rightarrow \infty$ ), so for general non-i.i.d. models of interest in the present paper this is not necessarily true. Therefore, the usefulness of the ARL2FA is under the question, as discussed in detail in [14,24,25,28]. In fact, in general, large values of the ARL2FA do not guarantee small values of the maximal local PFA  $\sup_{k \geq 1} P_\infty(\tau < k + m | \tau \geq k)$ . But the opposite is always true since the maximal local PFA is a more stringent false alarm measure in the sense that if it is small, then the ARL2FA is necessarily large. This argument motivated us to consider the maximal local PFA instead of conventional ARL2FA.

## Acknowledgments

The work of the first author was partially supported by the RSF grant 17-11-01049 (National Research Tomsk State University), the Russian Federal Professorship program (project no. 1.472.2016/1.4) and by the research project no. 2.3208.2017/4.6 (the Ministry of Education and Science of the Russian Federation). The work of the second author was supported in part by Russian Science Foundation Grant No. 18-19-00452 and by the Russian Federation Ministry of Science and Education Arctic program at the Moscow Institute of Physics and Technology.

We would like to thank referees for useful comments that have improved the article as well as Editor-in-Chief Dr. Dietrich von Rosen for excellent handling of the manuscript. We also thank the student Valentin Spivak of the Moscow Institute of Physics and Technology for helping with MC simulations.

## Appendix. Auxiliary non-asymptotic bounds for the concentration inequalities

### Correlation inequality

The following proposition provides the important correlation inequality [6].

**Proposition A.1.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_j)_{1 \leq j \leq n}, P)$  be a filtered probability space and  $(u_j, \mathcal{F}_j)_{1 \leq j \leq n}$  be a sequence of random variables such that  $\max_{1 \leq j \leq n} E |u_j|^q < \infty$  for some  $q \geq 2$ . Define

$$\check{b}_{j,n}(q) = \left\{ E \left[ |u_j| \sum_{k=j}^n |E(u_k | \mathcal{F}_j)| \right]^{q/2} \right\}^{2/q}.$$

Then

$$E \left| \sum_{j=1}^n u_j \right|^q \leq (2q)^{q/2} \left( \sum_{j=1}^n \check{b}_{j,n}(q) \right)^{q/2}.$$

### Uniform geometric ergodicity for homogeneous Markov processes

We recall some definitions from [7] for a homogeneous Markov process  $(X_n)_{n \geq 0}$  defined on a measurable state space  $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ . Denote by  $(P^\theta(\cdot, \cdot))_{\theta \in \Theta}$  the transition probability family of this process, i.e., for any  $A \in \mathcal{B}(\mathcal{X})$ ,  $x \in \mathcal{X}$ ,

$$P^\theta(x, A) = P_x^\theta(X_1 \in A) = P^\theta(X_1 \in A | X_0 = x).$$

The  $n$ -step transition probability is  $P^{n,\theta}(x, A) = P_x^\theta(X_n \in A)$ .

We recall that a measure  $\lambda$  on  $\mathcal{B}(\mathcal{X})$  is called *invariant* (or *stationary* or *ergodic*) for this process if, for any  $A \in \mathcal{B}(\mathcal{X})$ ,

$$\lambda^\theta(A) = \int_{\mathcal{X}} P^\theta(x, A) \lambda(dx).$$

If there exists an invariant positive measure  $\lambda^\theta$  with  $\lambda^\theta(\mathcal{X}) = 1$  then the process is called *positive*. Assume that the process  $(X_n)_{n \geq 0}$  satisfies the following *minorization* condition:

(D<sub>1</sub>) There exist  $\delta > 0$ , a set  $C \in \mathcal{B}(\mathcal{X})$  and a probability measure  $\zeta$  on  $\mathcal{B}(\mathcal{X})$  with  $\zeta(C) = 1$ , such that

$$\inf_{\theta \in \Theta} \left( \inf_{x \in C} P^\theta(x, A) - \delta \zeta(A) \right) > 0$$

for any  $A \in \mathcal{B}(\mathcal{X})$ , for which  $\zeta(A) > 0$ .

Obviously, this condition implies that  $\eta = \inf_{\theta \in \Theta} \inf_{x \in C} P^\theta(x, C) - \delta > 0$ . Now we impose the *uniform drift* condition.

(D<sub>2</sub>) There exist some constants  $0 < \rho < 1$  and  $D \geq 1$  such that for any  $\theta \in \Theta$  there exist a  $\mathcal{X} \rightarrow [1, \infty)$  function  $\mathbf{V}^\theta$  and a set  $C$  from  $\mathcal{B}(\mathcal{X})$  such that

$$\mathbf{V}^* = \sup_{\theta \in \Theta} \sup_{x \in C} |\mathbf{V}^\theta(x)| < \infty$$

and

$$\sup_{\theta \in \Theta} \sup_{x \in \mathcal{X}} \left\{ \mathbf{E}_x^\theta [\mathbf{V}^\theta(X_1)] - (1 - \rho)\mathbf{V}^\theta(x) + D\mathbb{1}_{\{C\}}(x) \right\} \leq 0.$$

In this case, we call  $\mathbf{V}^\theta$  the Lyapunov function. We use the following theorem from [7].

**Theorem A.1.** Let  $(X_n)_{n \geq 0}$  be a homogeneous Markov process satisfying conditions (D<sub>1</sub>) and (D<sub>2</sub>) with the same set  $C \in \mathcal{B}(\mathcal{X})$ . Then  $(X_n)_{n \geq 0}$  is a positive uniform geometric ergodic process, i.e.,

$$\sup_{n \geq 0} e^{\kappa^* n} \sup_{x \in \mathcal{X}} \sup_{\theta \in \Theta} \sup_{0 \leq g \leq \mathbf{V}^\theta} \frac{1}{\mathbf{V}^\theta(x)} \left| \mathbf{E}_x^\theta [g(X_n)] - \lambda^\theta(\tilde{g}) \right| \leq R^*$$

for some positive constants  $\kappa^*$  and  $R^*$  which are given in [7].

## References

- [1] M. Baron, A.G. Tartakovsky, Asymptotic optimality of change-point detection schemes in general continuous-time models, *Sequential Anal.* 25 (2006) 257–296, Invited Paper in Memory of Milton Sobel.
- [2] M. Basseville, I.V. Nikiforov, *Detection of Abrupt Changes – Theory and Application*, Information and System Sciences Series, Prentice-Hall, Inc, Englewood Cliffs, NJ, USA, 1993, <http://www.irisa.fr/sisthem/kniga/> [Online].
- [3] P. Feigin, R. Tweedie, Random coefficient autoregressive processes: A Markov chain analysis of stationarity and finiteness of moments, *J. Time Series Anal.* 6 (1985) 1–14.
- [4] G. Fellouris, A.G. Tartakovsky, Almost optimal sequential tests of discrete composite hypotheses, *Statist. Sinica* 23 (2013) 1717–1741.
- [5] C.-D. Fuh, SPRT and CUSUM in hidden Markov models, *Ann. Statist.* 31 (2003) 942–977.
- [6] L. Galthouk, S. Pergamenschchikov, Uniform concentration inequality for ergodic diffusion processes observed at discrete times, *Stochastic Process. Appl.* 123 (2013) 91–109.
- [7] L. Galthouk, S. Pergamenschchikov, Geometric ergodicity for classes of homogeneous Markov chains, *Stochastic Process. Appl.* 124 (2014) 3362–3391.
- [8] M.A. Girshick, H. Rubin, A Bayes approach to a quality control model, *Ann. Math. Stat.* 23 (1952) 114–125.
- [9] D.M. Hawkins, D.H. Olwell, *Cumulative Sum Charts and Charting for Quality Improvement*, in: *Series in Statistics for Engineering and Physical Sciences*, Springer-Verlag, USA, 1998.
- [10] C. Klüppelberg, S. Pergamenschchikov, The tail of the stationary distribution of a random coefficient AR(q) process with applications to an ARCH(q) process, *Ann. Appl. Probab.* 14 (2004) 971–1005.
- [11] T.L. Lai, Information bounds and quick detection of parameter changes in stochastic systems, *IEEE Trans. Inform. Theory* 44 (1998) 2917–2929.
- [12] G. Lorden, Procedures for reacting to a change in distribution, *Ann. Math. Stat.* 42 (1971) 1897–1908.
- [13] R.L. Mason, J.C. Young, *Multivariate Statistical Process Control with Industrial Application*, SIAM, Philadelphia, PA, USA, 2001.
- [14] Y. Mei, Is average run length to false alarm always an informative criterion? *Sequential Anal.* 27 (2008) 354–376.
- [15] G.V. Moustakides, Optimal stopping times for detecting changes in distributions, *Ann. Statist.* 14 (1986) 1379–1387.
- [16] G.V. Moustakides, A.S. Polunchenko, A.G. Tartakovsky, Numerical comparison of CUSUM and Shiryaev–Roberts procedures for detecting changes in distributions, *Comm. Statist. Theory Methods* 38 (2009) 3225–3239.
- [17] S. Pergamenchtchikov, A.G. Tartakovsky, Asymptotically optimal pointwise and minimax quickest change-point detection for dependent data, *Stat. Inference Stoch. Process.* 21 (2018) 217–259.
- [18] M. Pollak, A.G. Tartakovsky, Asymptotic exponentiality of the distribution of first exit times for a class of Markov processes with applications to quickest change detection, *Theory Probab. Appl.* 53 (2009) 430–442.
- [19] M. Pollak, A.G. Tartakovsky, Optimality properties of the Shiryaev–Roberts procedure, *Statist. Sinica* 19 (2009) 1729–1739.
- [20] A.S. Polunchenko, G. Sokolov, A.G. Tartakovsky, Optimal design and analysis of the exponentially weighted moving average chart for exponential data, *Sri Lankan J. Appl. Stat.* 15 (2014) 57–80, Special Issue: Modern Statistical Methodologies in the Cutting Edge of Science.
- [21] A.S. Polunchenko, A.G. Tartakovsky, On optimality of the Shiryaev–Roberts procedure for detecting a change in distribution, *Ann. Statist.* 38 (2010) 3445–3457.
- [22] A.N. Shiryaev, On optimum methods in quickest detection problems, *Theory Probab. Appl.* 8 (1963) 22–46.
- [23] M.S. Srivastava, Y. Wu, Comparison of EWMA CUSUM and Shiryaev–Roberts procedures for detecting a shift in the mean, *Ann. Statist.* 21 (1993) 645–670.
- [24] A.G. Tartakovsky, Asymptotic performance of a multichart CUSUM test under false alarm probability constraint, in: *Proceedings of the 44th IEEE Conference Decision and Control and European Control Conference, CDC-ECC'05, Seville, SP, IEEE, Omnipress CD-ROM, 2005*, pp. 320–325.
- [25] A.G. Tartakovsky, Discussion on “Is average run length to false alarm always an informative criterion?” by Yajun Mei, *Sequential Anal.* 27 (2008) 396–405.

YJMVA: 104541

## ARTICLE IN PRESS

20

*S. Pergamenchchikov and A.G. Tartakovsky / Journal of Multivariate Analysis xxx (xxxx) xxx*

- 1 [26] A.G. Tartakovsky, On asymptotic optimality in sequential changepoint detection: Non-iid case, *IEEE Trans. Inform. Theory* 63 (2017) 3433–3450.
- 2 [27] A.G. Tartakovsky, Asymptotic optimality of mixture rules for detecting changes in general stochastic models, *IEEE Trans. Inform. Theory* 65
- 3 (2019) 1413–1429.
- 4 [28] A.G. Tartakovsky, I.V. Nikiforov, M. Basseville, *Sequential Analysis: Hypothesis Testing and Changepoint Detection*, Monographs on Statistics
- 5 and Applied Probability, Chapman & Hall/CRC Press, Boca Raton, London, New York, 2014.
- 6 [29] A.G. Tartakovsky, M. Pollak, A.S. Polunchenko, Third-order asymptotic optimality of the generalized Shiryaev–Roberts changepoint detection
- 7 procedures, *Theory Probab. Appl.* 56 (2012) 457–484.
- 8 [30] A.G. Tartakovsky, V.V. Veeravalli, General asymptotic Bayesian theory of quickest change detection, *Theory Probab. Appl.* 49 (2005) 458–497.
- 9 [31] B. Yakir, A note on the run length to false alarm of a change-point detection policy, *Ann. Statist.* 23 (1995) 272–281.

Please cite this article as: S. Pergamenchikov and A.G. Tartakovsky, Asymptotically optimal pointwise and minimax change-point detection for general stochastic models with a composite post-change hypothesis, *Journal of Multivariate Analysis* (2019) 104541, <https://doi.org/10.1016/j.jmva.2019.104541>.