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Sequential Robust Estimation for Nonparametric Autoregressive Models

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Abstract: We construct a robust truncated sequential estimators for the pointwise estimation problem in nonparametric autoregression models with smooth coefficients. For Gaussian models we propose an adaptive procedure based on the constructed sequential estimators. The minimax nonadaptive and adaptive convergence rates are established. It turns out that in this case these rates are the same as for regression models..

Keywords: Adaptive estimation; Nonparametric autoregression; Robust efficiency; Sequential kernel estimator.

Subject Classifications: primary 62G07,62G08; secondary 62G20.

1 INTRODUCTION

One of the standard linear models in the general theory of time series is the autoregressive model (see, for example, [1] and the references therein). Natural extensions for such models are nonparametric au-

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autoregressive models which are defined by

$$y_k = S(x_k)y_{k-1} + \xi_k, \quad 1 \leq k \leq n, \quad (1.1)$$

where $S(\cdot)$ is unknown function, the design $x_k = k/n$, y_0 is a constant and the noise $(\xi_k)_{1 \leq k \leq n}$ are i.i.d. unobservable centered random variable $\mathcal{N}(0, 1)$.

It should be noted that the varying coefficient principle is well known in the regression analysis. It permits the use of a more complex forms for regression coefficients and, therefore, the models constructed via this method are more adequate for applications (see, for instance, [8], [22]). In this paper we consider the varying coefficient autoregressive models (1.1). There is a number of papers which consider these models such as [6], [7] and [4]. In all these papers, the authors propose some asymptotic (as $n \rightarrow \infty$) methods for different identification studies without considering optimal estimation issues. To our knowledge, for the first time the minimax estimation problem for the model (1.1) has been treated in [3] and [23] in the nonadaptive case, i.e. for the known regularity of the function S . Then, in [2] it is proposed to use the sequential analysis method for the adaptive pointwise estimation problem in the case when the unknown Hölder regularity is less than one, i.e when the function S is not differentiable. In the case of the model (1.1), the adaptive pointwise estimation is possible only in the sequential framework. That is why, in this paper, we study sequential estimation methods for the smooth function S . We consider the pointwise estimation at a fixed point $z_0 \in]0; 1[$ in the two cases: when the Hölder regularity is known and when the Hölder regularity is unknown, i.e. the adaptive estimation. In the first case we consider this problem in the robust setting, i.e. we assume that the distribution of the random variables $(\xi_j)_{j \geq 1}$ in (1.1) belongs to some functional class and we consider the estimation problem with respect to robust risks which have an additional supremum over all distributions from some fixed class. For non-parametric regression models such risks was introduced in [13] for the pointwise estimation problem and in [15] for the quadratic risks. Later, for the quadratic risks the same approach was used in [19] for regression model in continuous time. Motivated by this facts, we consider the adaptive estimation problem for the Gaussian models (1.1). More precisely, we assume that the function S belongs to a Hölder class with some unknown regularity $1 < \beta \leq 2$. Unfortunately, we can not use directly the sequential procedure from [2] for the adaptive estimation of such functions. Since to obtain an optimal rate for the function with $\beta > 1$ we have to take into account the Taylor expansion of the function S at z_0 of the order 1. To study the Taylor expansion for sequential procedures one needs to control the behavior of the stopping time. Indeed, one needs to keep the stopping time near of the number of observations. This can not be done by the procedure from [2] since one needs to use the unknown function S . In this paper we construct a sequential adaptive estimate for smooth functions and we find an adaptive minimax convergence rate for smooth functions. In Section 2, we present the standard notations used in the sequel of the paper. We describe in details the statement of the problem and main results in Section 3. In Section 4, we will study some properties of kernel estimators for the model (1.1) and in Section 5, we study properties of stopping time for the constructed sequential procedure. Section 6 is devoted to the asymptotic upper bound and lower bound for the risk of the sequential kernel estimators. In Section 7, we illustrate the obtained results by numerical examples. Finally, we give an appendix which contains some technical results.

2 SEQUENTIAL PROCEDURES

2.1 Main Conditions

We assume that in the model (1.1) the i.i.d. random variables $(\xi_k)_{1 \leq k \leq n}$ have a density p (with respect to the Lebesgue measure) from the functional class \mathcal{P}_ς defined as

$$\mathcal{P}_\varsigma := \left\{ p \geq 0 : \int_{-\infty}^{+\infty} p(x) dx = 1, \quad \int_{-\infty}^{+\infty} x p(x) dx = 0, \right. \\ \left. \int_{-\infty}^{+\infty} x^2 p(x) dx = 1 \quad \text{and} \quad \sup_{k \geq 1} \frac{1}{\varsigma^k (2k-1)!!} \int_{-\infty}^{+\infty} |x|^{2k} p(x) dx \leq 1 \right\}, \quad (2.1)$$

where $\varsigma \geq 1$ is some fixed parameter. Note that the $(0, 1)$ -Gaussian density belongs to \mathcal{P} . In the sequel we denote this density by p_0 . It is clear that for any $q > 0$

$$\mathbf{s}_q^* = \sup_{p \in \mathcal{P}_\varsigma} \mathbf{E}_p |\xi_1|^q < \infty, \quad (2.2)$$

where \mathbf{E}_p is the expectation with respect to the density p from \mathcal{P}_ς . To obtain the stable (uniformly with respect to the function S) model (1.1), we assume that for some fixed $0 < \varepsilon < 1$ and $L > 0$ the unknown function S belongs to the ε -stability set

$$\Theta_{\varepsilon, L} = \left\{ S \in \mathbf{C}_1([0, 1], \mathbb{R}) : \|S\| \leq 1 - \varepsilon \quad \text{and} \quad \|\dot{S}\| \leq L \right\}, \quad (2.3)$$

where $\mathbf{C}_1[0, 1]$ is the Banach space of continuously differentiable $[0, 1] \rightarrow \mathbb{R}$ functions and $\|S\| = \sup_{0 \leq x \leq 1} |S(x)|$. Similarly to [13] and [3] we make use of the family of the *weak stable local Hölder classes* at the point z_0

$$\mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*) = \left\{ S \in \Theta_{\varepsilon, L} : |\Omega_h(z_0, S)| \leq \epsilon_n^* h^\beta \right\}, \quad (2.4)$$

where

$$\Omega_h(z_0, S) = \int_{-1}^1 (S(z_0 + uh) - S(z_0)) du,$$

the positive parameter h is defined later and $\beta = 1 + \alpha$ is the regularity parameter with $0 < \alpha < 1$. Moreover, we assume that the *weak Hölder constant* ϵ_n^* goes to zero, i.e. $\epsilon_n^* \rightarrow 0$ as $n \rightarrow \infty$. Moreover, we define the corresponding *strong stable local Hölder class* at the point z_0 as

$$\mathcal{H}^{(\beta)}(\varepsilon, L, L^*) = \left\{ S \in \Theta_{\varepsilon, L} : \Omega^*(z_0, S) \leq L^* \right\}, \quad (2.5)$$

where

$$\Omega^*(z_0, S) = \sup_{x \in [0, 1]} \frac{|\dot{S}(x) - \dot{S}(z_0)|}{|x - z_0|^\alpha}.$$

We assume that the regularity $\underline{\beta} \leq \beta \leq \bar{\beta}$, where $\underline{\beta} = 1 + \underline{\alpha}$ and $\bar{\beta} = 1 + \bar{\alpha}$ for some fixed parameters $0 < \underline{\alpha} < \bar{\alpha} < 1$.

Remark 2.1. Note that for the regression models the weak Hölder class was introduced in [13] for the efficient pointwise estimation. It is clear that it is more large than usual one with the same Hölder constant, i.e.

$$\mathcal{H}^{(\beta)}(\varepsilon, L, \epsilon_n^*) \subseteq \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*).$$

It should be noted also that for diffusion processes the local weak Hölder class was used in [12] and [14] for sequential and truncated sequential efficient pointwise estimation respectively. Moreover, in [16] these sequential pointwise efficient estimators were used to construct adaptive efficient model selection procedures in L_2 for diffusion processes.

2.2 Nonadaptive Procedure

First, we study a nonadaptive estimation problem for the function S from the functional class (2.4) of the known regularity $\beta = 1 + \alpha$. As we will see later to construct an efficient sequential procedure we need to use S as a procedure parameter. So we propose to use the first ν observations for the auxiliary estimation of $S(z_0)$. In this step we use the usual kernel estimate, i.e.

$$\widehat{S}_\nu = \frac{1}{A_\nu} \sum_{j=1}^{\nu} Q(u_j) y_{j-1} y_j, \quad A_\nu = \sum_{j=1}^{\nu} Q(u_j) y_{j-1}^2, \quad (2.6)$$

where the kernel $Q(\cdot)$ is the indicator function of the interval $[-1; 1]$; $u_j = (x_j - z_0)/h$ and h is some positive bandwidth. In the sequel for any $0 \leq k < m \leq n$ we set

$$A_{k,m} = \sum_{j=k+1}^m Q(u_j) y_{j-1}^2, \quad (2.7)$$

i.e. $A_\nu = A_{0,\nu}$. It is clear that to estimate $S(z_0)$ on the basis of the kernel estimate with the kernel Q we can use the observations $(y_j)_{k_* \leq j \leq k^*}$, where

$$k_* = [nz_0 - nh] + 1 \quad \text{and} \quad k^* = [nz_0 + nh]. \quad (2.8)$$

Here $[a]$ is the integral part of a number a . So for the first estimation we chose ν as

$$\nu = \nu(h, \alpha) = k_* + \iota, \quad (2.9)$$

where

$$\iota = \iota(h, \alpha) = [\tilde{\varepsilon}nh] + 1 \quad \text{and} \quad \tilde{\varepsilon} = \tilde{\varepsilon}(h, \alpha) = h^\alpha / \ln n.$$

Next, similarly to [2], we use a some kernel sequential procedure based on the observations $(y_j)_{\nu \leq j \leq n}$. To transform the kernel estimator in the linear function of observations and we replace the number of observations n by the following stopping time

$$\tau_H = \inf\{k \geq \nu + 1 : A_{\nu,k} \geq H\}, \quad (2.10)$$

where $\inf\{\emptyset\} = n$ and the positive threshold H will be chosen as a positive random variable measurable with respect to the σ -field $\{y_1, \dots, y_\nu\}$. Therefore, we get

$$S_h^* = \frac{1}{H} \left(\sum_{j=\nu+1}^{\tau_H-1} Q(u_j) y_{j-1} y_j + \varkappa_H Q(u_{\tau_H}) y_{\tau_H-1} y_{\tau_H} \right) \mathbf{1}_{(A_{\nu,n} \geq H)}, \quad (2.11)$$

where the correcting coefficient \varkappa_H on the set $\{A_{\nu,\tau_H} \geq H\}$ is defined as

$$A_{\nu,\tau_H-1} + \varkappa_H Q(u_{\tau_H}) y_{\tau_H-1}^2 = H$$

and $\varkappa_H = 1$ on the set $\{A_{\nu,\tau_H} < H\}$.

Now, to obtain an efficient estimate we need to use the all n observations, i.e. asymptotically for sufficiently large n the stopping time $\tau_H \approx n$. Similarly to [18], one can show that $\tau_H \approx \gamma(S) H$ as $H \rightarrow \infty$, where

$$\gamma(S) = 1 - S^2(z_0). \quad (2.12)$$

Therefore, to use the all asymptotically observations we have to chose H as the number of observations divided by $\gamma(S)$. But in our case we use $k^* - k_*$ observations to estimate $S(z_0)$, Therefore, to obtain optimal estimate we need to define H as $(k^* - k_*)/\gamma(S)$. Taking into account that $k^* - k_* \approx 2nh$ and that $\gamma(S)$ is unknown we define the threshold H as

$$H = H(h, \alpha) = \phi nh, \quad \phi = \phi(h, \alpha) = \frac{2(1 - \tilde{\epsilon})}{\gamma(\tilde{S}_\nu)}, \quad (2.13)$$

where \tilde{S}_ν is the projection of the estimator \hat{S}_ν in the interval $] -1 + \varepsilon, 1 - \varepsilon[$, i.e.

$$\tilde{S}_\nu = \min(\max(\hat{S}_\nu, -1 + \varepsilon), 1 - \varepsilon).$$

In this paper we chose the bandwidth h in the following form

$$h = h(\beta) = (\kappa_n)^{\frac{1}{2\beta+1}}, \quad (2.14)$$

where the sequence κ_n is positive, such that

$$\kappa_* = \liminf_{n \rightarrow \infty} n\kappa_n > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} n^\delta \kappa_n = 0 \quad (2.15)$$

for any $0 < \delta < 1$.

2.3 Adaptive Procedure

We will construct an adaptive minimax sequential estimation for the function S from the functional class (2.5) of the unknown regularity β . To this end we will use the modification of the adaptive Lepskii method proposed in [2] based on the sequential estimators (2.11). We set

$$d_n = \frac{n}{\ln n} \quad \text{and} \quad N(\beta) = (d_n)^{\frac{\beta}{2\beta+1}}. \quad (2.16)$$

Moreover, we chose the bandwidth h in the form (2.14) with $\kappa_n = 1/d_n$, i.e. we set

$$\check{h} = \check{h}(\beta) = \left(\frac{1}{d_n} \right)^{\frac{1}{2\beta+1}}. \quad (2.17)$$

We define the grids on the intervals $[\underline{\beta}, \bar{\beta}]$ and $[\underline{\alpha}, \bar{\alpha}]$ as

$$\beta_k = \underline{\beta} + \frac{k}{m}(\bar{\beta} - \underline{\beta}) \quad \text{and} \quad \alpha_k = \underline{\alpha} + \frac{k}{m}(\bar{\alpha} - \underline{\alpha}) \quad (2.18)$$

for $0 \leq k \leq m$ with $m = \lfloor \ln d_n \rfloor + 1$, and we set

$$N_k = N(\beta_k) \quad \text{and} \quad \check{h}_k = \check{h}(\beta_k).$$

Replacing in (2.9) and (2.13) the parameters h and α we define

$$\check{\nu}_k = \nu(\check{h}_k, \alpha_k) \quad \text{and} \quad \check{H}_k = H(\check{h}_k, \alpha_k).$$

Now using these parameters in the estimators (2.6) and (2.11) we set $\check{S}_k = S_{\check{h}_k}^*(z_0)$ and

$$\check{\omega}_k = \max_{0 \leq j \leq k} \left(|\check{S}_j - \check{S}_k| - \frac{\check{\lambda}}{N_j} \right), \quad (2.19)$$

where

$$\check{\lambda} > \check{\lambda}_* = 4\sqrt{2} \left(\frac{\bar{\beta} - \underline{\beta}}{(2\underline{\beta} + 1)(2\bar{\beta} + 1)} \right)^{1/2}.$$

In particular, if $\underline{\beta} = 1$ and $\bar{\beta} = 2$ we get $\check{\lambda}_* = 4(2/15)^{1/2}$. We also define the optimal index as

$$\check{k} = \max \left\{ 0 \leq k \leq m : \check{\omega}_k \leq \frac{\check{\lambda}}{N_k} \right\}. \quad (2.20)$$

The adaptive estimator is now defined as

$$\hat{S}_{a,n} = S_{\check{h}_{\check{k}}}^* \quad \text{and} \quad \check{h}_{\check{k}} = \check{h}_{\check{k}}. \quad (2.21)$$

Remark 2.2. It should be noted that in the difference from the usual adaptive pointwise estimation (see, for example, [20], [11], [2]) the threshold $\check{\lambda}$ in (2.19) does not depend on the parameters $L > 0$ and $L^* > 0$ of the Hölder class (2.5).

3 MAIN RESULTS

3.1 Robust Efficient Estimation

The problem is to estimate the function $S(\cdot)$ at a fixed point $z_0 \in]0, 1[$, i.e. the value $S(z_0)$. For this problem we make use of the risk proposed in [3]. Namely, for any estimate $\tilde{S} = \tilde{S}_n(z_0)$ (i.e. any measurable with respect to the observations $(y_k)_{1 \leq k \leq n}$ function) we define the following robust risk

$$\mathcal{R}_n(\tilde{S}_n, S) = \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} |\tilde{S}_n(z_0) - S(z_0)|, \quad (3.1)$$

where $\mathbf{E}_{S,p}$ is the expectation taken with respect to the distribution $\mathbf{P}_{S,p}$ of the vector (y_1, \dots, y_n) in (1.1) corresponding to the function S and the density p from \mathcal{P}_ζ .

With the help of the function $\gamma(S)$ defined in (2.12), we describe the sharp lower bound for the minimax risks with the normalizing coefficient

$$\varphi_n = n^{\frac{\beta}{2\beta+1}}. \quad (3.2)$$

Theorem 3.1. For any $0 < \varepsilon < 1$

$$\underline{\lim}_{n \rightarrow \infty} \inf_{\tilde{S}} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \gamma^{-1/2}(S) \varphi_n \mathcal{R}_n(\tilde{S}_n, S) \geq \mathbf{E}|\eta|, \quad (3.3)$$

where η is a Gaussian random variable with the parameters $(0, 1/2)$.

Now we give the upper bound for the minimax risk of the sequential kernel estimator defined in (2.11).

Theorem 3.2. The estimator (2.11) with the parameters (2.13) – (2.14) and $\kappa_n = n^{-1}$ satisfies the following inequality

$$\overline{\lim}_{n \rightarrow \infty} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \gamma^{-1/2}(S) \varphi_n \mathcal{R}_n(S_h^*, S) \leq \mathbf{E}|\eta|,$$

where η is a Gaussian random variable with the parameters $(0, 1/2)$.

Remark 3.1. Theorems 3.1 and 3.2 imply that the estimator (2.11), with the parameters (2.14) is asymptotically robust efficient with respect to class \mathcal{P}_ζ .

3.2 Adaptive Estimation

Now we consider the Gaussian model (1.1), i.e. assume that the random variables $(\xi_j)_{j \geq 1}$ are $\mathcal{N}(0, 1)$. The problem is to estimate the function S at a fixed point $z_0 \in]0, 1[$, i.e. the value $S(z_0)$. For any estimate \tilde{S}_n of $S(z_0)$ (i.e. any measurable with respect to the observations $(y_k)_{1 \leq k \leq n}$ function), we define the adaptive risk for the functions S from $\mathcal{H}^{(\beta)}(\varepsilon, L, L^*)$ as

$$\mathcal{R}_{a,n}(\tilde{S}_n) = \sup_{\beta \in [\underline{\beta}; \bar{\beta}]} \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} N(\beta) \mathbf{E}_S |\tilde{S}_n - S(z_0)|, \quad (3.4)$$

where $N(\beta)$ is defined in (2.16), $\mathbf{E}_S = \mathbf{E}_{S, p_0}$ is the expectation taken with respect to the distribution $\mathbf{P}_S = \mathbf{P}_{S, p_0}$.

First we give the lower bound for the minimax risk. We show that with the convergence rate $N(\beta)$ the lower bound for the minimax risk is strictly positive.

Theorem 3.3. There exists $L_0^* > 0$ such that for all $L^* > L_0^*$, the risk (3.4) admits the following lower bound:

$$\liminf_{n \rightarrow \infty} \inf_{\tilde{S}_n} \mathcal{R}_{a,n}(\tilde{S}_n) > 0,$$

where the infimum is taken over all estimators \tilde{S}_n .

The proof of this theorem is given in [2].

Now we give the upper bound for the minimax risk of the sequential kernel estimator defined in (2.11). To obtain an upper bound for the adaptive risk (3.4) of the procedure (2.21) we need to study the family $(S_h^*)_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}}$.

Theorem 3.4. *The sequential procedure (2.11) with the bandwidth h defined in (2.14) for $\kappa_n = \ln n/n$ satisfies the following property*

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} (\Upsilon_n(h))^{-1} \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} |S_h^* - S(z_0)| < \infty$$

where $\Upsilon_n(h) = h^\beta + (nh)^{-1/2}$.

Using this theorem we can establish the minimax property for the procedure (2.21).

Theorem 3.5. *The estimation procedure (2.21) satisfies the following asymptotic property*

$$\limsup_{n \rightarrow \infty} \mathcal{R}_{a,n}(\widehat{S}_{a,n}) < \infty. \quad (3.5)$$

Remark 3.2. Theorem 3.3 gives the lower bound for the adaptive risk, i.e. the convergence rate $N(\beta)$ is best for the adapted risk. Moreover, by Theorem 3.5 the adaptive estimates (2.21) possesses this convergence rate. In this case, this estimates is called optimal in the sense of the adaptive risk (3.4)

4 PROPERTIES OF \widehat{S}_ν

We start with studying the properties of the estimate (2.6). To this end for any $q > 1$ we set

$$\varrho_q^* = \frac{(12(1 + \kappa_{*}))^q}{(\varepsilon \kappa_{*})^{2q}} \left(\mathbf{b}_q^* \left(r_q^* \mathbf{s}_q^* + \mathbf{s}_{2q}^* + 1 \right) + 2(1 + L)^q r_{2q}^* \right), \quad (4.1)$$

where

$$\mathbf{r}_q^* = 2^{q-1} \left(|y_0|^q + \mathbf{s}_q^* \left(\frac{1}{\varepsilon} \right)^q \right) \quad \text{and} \quad \mathbf{b}_q^* = \frac{18^q q^{3q/2}}{(q-1)^{q/2}}.$$

Now we obtain a non asymptotic upper bound for the tail probability for the deviation

$$\widehat{\Delta}_\nu = \widehat{S}_\nu - S(z_0). \quad (4.2)$$

Lemma 4.1. *For any $q > 1$, $h > 0$ and $a > Lh$*

$$\sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{P}_{S,p} \left(|\widehat{\Delta}_\nu| > a \right) \leq \mathbf{M}_{1,q} (\ln n)^q h^{(1-\alpha)q} + \frac{\mathbf{M}_{2,q}}{[\iota(a - Lh)^2]^{q/2}},$$

where $\mathbf{M}_{1,q} = 2^q \varrho_q^*$ and $\mathbf{M}_{2,q} = 2^q \mathbf{b}_q^* \mathbf{s}_q^* r_q^*$.

Proof. First, we write the estimation error as follows

$$\widehat{\Delta}_\nu = B_\nu + \frac{1}{A_\nu} \zeta_\nu,$$

where $\zeta_\nu = \sum_{j=1}^\nu Q(u_j) y_{j-1} \xi_j$ and

$$B_\nu = \frac{1}{A_\nu} \sum_{j=1}^\nu Q(u_j) (S(x_j) - S(z_0)) y_{j-1}^2.$$

Note that $|B_\nu| \leq Lh$ for any $S \in \Theta_{\varepsilon, L}$. Putting $v = \iota/2$ we can write

$$\begin{aligned} \mathbf{P}_{S,p}(|\widehat{\Delta}_\nu| > a) &= \mathbf{P}_{S,p}(|\widehat{\Delta}_\nu| > a, A_\nu < v) + \mathbf{P}_{S,p}(|\widehat{\Delta}_\nu| > a, A_\nu \geq v) \\ &\leq \mathbf{P}_{S,p}(A_\nu < v) + \mathbf{P}_{S,p}\left(Lh + \frac{|\zeta_\nu|}{A_\nu} > a, A_\nu \geq v\right) \\ &\leq \mathbf{P}_{S,p}(A_\nu < v) + \mathbf{P}_{S,p}\left(\frac{|\zeta_\nu|}{A_\nu} > a - Lh, A_\nu \geq v\right). \end{aligned} \quad (4.3)$$

Now, for any $\mathbb{R} \rightarrow \mathbb{R}$ function f and numbers $0 \leq k \leq m-1$ we set

$$\varrho_{k,m}(f) = \frac{1}{nh} \sum_{j=k+1}^m f(u_j) y_{j-1}^2 - \frac{1}{\gamma(S)} \frac{1}{nh} \sum_{j=k+1}^m f(u_j). \quad (4.4)$$

Using this function we can estimate the first term on the left-hand side of (4.3) as

$$\begin{aligned} \mathbf{P}_{S,p}(A_\nu < v) &= \mathbf{P}_{S,p}\left(\varrho_{k_*,\nu}(Q) + \frac{1}{nh\gamma(S)} \sum_{j=k_*+1}^\nu Q(u_j) < \frac{v}{nh}\right) \\ &= \mathbf{P}_{S,p}\left(\varrho_{k_*,\nu}(Q) < -\frac{\iota}{2nh}\right) \\ &\leq \mathbf{P}_{S,p}\left(|\varrho_{k_*,\nu}(Q)| > \tilde{\varepsilon}/2\right) \leq \left(\frac{2}{\tilde{\varepsilon}}\right)^q \mathbf{E}_{S,p}|\varrho_{k_*,\nu}(Q)|^q. \end{aligned}$$

Therefore, using here Lemma 8.3 we get

$$\mathbf{P}_{S,p}(A_\nu < v) \leq 2^q R^q \varrho_q^* (\ln n)^q h^{(1-\alpha)q},$$

where the coefficient ϱ_q^* is defined in (4.1). The last term on the right-hand side of (4.3) can be estimated as

$$\begin{aligned} \mathbf{P}_{S,p}\left(\frac{1}{A_\nu}|\zeta_\nu| > a - Lh, A_\nu \geq v\right) &\leq \mathbf{P}_{S,p}(|\zeta_\nu| > v(a - Lh)) \\ &\leq \frac{1}{v^q(a - Lh)^q} \mathbf{E}_{S,p}|\zeta_\nu|^q. \end{aligned}$$

Now in view of the Burkholder inequality, it comes

$$\begin{aligned} \mathbf{E}_{S,p}|\zeta_\nu|^q &= \mathbf{E}_{S,p}\left(\sum_{j=1}^\nu Q(u_j) y_{j-1} \xi_j\right)^q \leq \mathbf{b}_q^* \mathbf{E}_{S,p}\left(\sum_{j=1}^\nu Q^2(u_j) y_{j-1}^2 \xi_j^2\right)^{q/2} \\ &= \mathbf{b}_q^* \mathbf{E}_{S,p}\left(\sum_{j=k_*+1}^{k_*+\iota} y_{j-1}^2 \xi_j^2\right)^{q/2}, \end{aligned}$$

and after applying the Hölder inequality, we obtain

$$\mathbf{E}_{S,p}|\zeta_\nu|^q \leq \mathbf{b}_q^* \iota^{q/2-1} \sum_{j=k_*+1}^{k_*+\iota} \mathbf{E}_{S,p} y_{j-1}^q \xi_j^q \leq \mathbf{b}_q^* \mathbf{s}_q^* r_q^* \iota^{q/2}.$$

Therefore,

$$\mathbf{P}_{S,p} \left(\frac{1}{A_\nu} |\zeta_\nu| > a - Lh, A_\nu \geq v \right) \leq \frac{2^q \mathbf{b}_q^* \mathbf{s}_q^* r_q^*}{\iota^{q/2} (a - Lh)^q}.$$

Hence Lemma 4.1 \square

Lemma 4.2. *Let the bandwidth h be defined by the conditions (2.14)–(2.15). Then, for all $m \geq 1$ and $0 < \delta < 1$*

$$\lim_{n \rightarrow +\infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-m} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{P}_{S,p} \left(|\widehat{\Delta}_\nu| > \delta \tilde{\varepsilon} \right) = 0.$$

Proof. By applying Lemma 4.1 for $a = \delta \tilde{\varepsilon}$, we obtain that for sufficiently large $n \geq 1$, for which $\delta \tilde{\varepsilon} > 2Lh$, and for any $m \geq 1$ and $q > m/(1 - \bar{\alpha})$

$$\begin{aligned} h^{-m} \mathbf{P}_{S,p} (|\widehat{\Delta}_\nu| > \delta \tilde{\varepsilon}) &\leq h^{-m} \mathbf{M}_{1,q} (\ln n)^q h^{(1-\alpha)q} + \frac{\mathbf{M}_{2,q} h^{-m}}{(\iota(\delta \tilde{\varepsilon} - Lh)^2)^{q/2}} \\ &\leq \mathbf{M}_{1,q} (\ln n)^q \kappa_n^{\frac{(1-\bar{\alpha})q-m}{2\bar{\beta}+1}} + \frac{2^q \mathbf{M}_{2,q} h^{-(m+q/2)}}{\delta^q n^{q/2} \tilde{\varepsilon}_n^{3q/2}} \\ &\leq \mathbf{M}_{1,q} (\ln n)^q \kappa_n^{\frac{(1-\bar{\alpha})q-m}{2\bar{\beta}+1}} + \frac{2^q \mathbf{M}_{2,q} (\ln n)^{3q/2}}{\delta^q (\kappa_n n)^{q/2}} \kappa_n^{\frac{q(1-\bar{\alpha}/2)-m}{2\bar{\beta}+1}}. \end{aligned}$$

Taking into account here the conditions (2.15) we come to Lemma 4.2. \square

5 PROPERTIES OF STOPPING TIME τ_H

First we need to study some asymptotic properties of the term (2.10).

Lemma 5.1. *Assume that the threshold H is chosen in the form (2.13) and the bandwidth h satisfies the conditions (2.14) - (2.15). Then for any $m \geq 1$*

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-m} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{P}_{S,p} (A_{\nu,n} < H) < \infty.$$

Proof. Using the definition of H in (2.13) we obtain

$$\begin{aligned} \mathbf{P}_{S,p} (A_{\nu,n} < H) &= \mathbf{P}_{S,p} \left(\frac{1}{nh} \sum_{j=\nu+1}^n Q(u_j) y_{j-1}^2 < \frac{H}{nh} \right) \\ &= \mathbf{P}_{S,p} \left(\varrho_{\nu,k^*}(Q) + \frac{1}{\gamma(S)} \sum_{j=\nu+1}^{k^*} Q(u_j) \Delta u_j < \phi \right). \end{aligned}$$

Note that

$$\sum_{j=\nu+1}^{k^*} Q(u_j) \Delta u_j = \frac{k^* - k_* - \iota}{nh} \geq 2 - \frac{\iota + 2}{nh}.$$

Taking into account that $\varepsilon^2 \leq \gamma(S) \leq 1$, we obtain

$$\left| \frac{1}{\gamma(\widehat{S}_\nu)} - \frac{1}{\gamma(S)} \right| \leq \frac{2}{\varepsilon^4} |\widehat{\Delta}_\nu|. \quad (5.1)$$

This yields

$$\begin{aligned} \mathbf{P}_{S,p}(A_{\nu,n} < H) &\leq \mathbf{P}_{S,p} \left(\sum_{j=\nu+1}^{k^*} Q(u_j) y_{j-1}^2 < H, |\widehat{\Delta}_\nu| \leq \delta \widetilde{\varepsilon} \right) + \mathbf{P}_{S,p} \left(|\widehat{\Delta}_\nu| > \delta \widetilde{\varepsilon} \right) \\ &\leq \mathbf{P}_{S,p} \left(\varrho_{\nu,k^*}(Q) < -2\widetilde{\varepsilon}_n + \frac{\widetilde{\varepsilon}_n}{\varepsilon^2} + \frac{4}{\varepsilon^4} \delta \widetilde{\varepsilon} + \frac{3}{\varepsilon^2 n h} \right) \\ &\quad + \mathbf{P}_{S,p} \left(|\widehat{\Delta}_\nu| > \delta \widetilde{\varepsilon} \right). \end{aligned}$$

Therefore for $\delta < \varepsilon^4/8$ and sufficiently large $n \geq 1$ we obtain that

$$\mathbf{P}_{S,p}(A_{\nu,n} < H) \leq \mathbf{P}_{S,p}(|\varrho_{\nu,k^*}(Q)| > \widetilde{\varepsilon}_n/2) + \mathbf{P}_{S,p}(|\widehat{\Delta}_\nu| > \delta \widetilde{\varepsilon}).$$

Lemma 8.3 and Lemma 4.2 imply Lemma 5.1. \square

Now for any weighted sequence $(w_j)_{j \geq 1}$ we set

$$Z_n = \sum_{j=\tau_H-1}^n Q(u_j) w_j y_{j-1}^2 + (1 - \varkappa_H) w_{\tau_H} Q(u_{\tau_H}) y_{\tau_H-1}^2. \quad (5.2)$$

Lemma 5.2. *Assume that the threshold H is chosen in the form (2.13) and the bandwidth h satisfies the conditions (2.14) - (2.15). Moreover, let $(w_j)_{j \geq 1}$ be a sequence bounded by a constant w^* , i.e. $\sup_{j \geq 1} |w_j| \leq w^*$. Then*

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\varsigma} \frac{\mathbf{E}_{S,p} |Z_n|}{nh} = 0. \quad (5.3)$$

Proof. It is clear that $Z_n = 0$ if $A_{\nu,n} < H$, and on the set $\{A_{\nu,n} \geq H\}$ this term can be estimated as

$$|Z_n| \leq w^* \left(\sum_{j=\nu+1}^n Q(u_j) y_{j-1}^2 - H \right) = w^*(A_{\nu,n} - H),$$

i.e. $|Z_n| \leq w^*(A_{\nu,n} - H)_+$, where $(x)_+ = \max(0, x)$. Therefore,

$$\begin{aligned} \frac{|Z_n|}{nh} &\leq w^* \left| \frac{\sum_{j=\nu+1}^n Q(u_j) y_{j-1}^2}{nh} - \phi \right| \\ &\leq |\varrho_{\nu,n}(Q)| + \left| \frac{\sum_{j=\nu+1}^n Q(u_j)}{\gamma(S)nh} - \phi \right|. \end{aligned}$$

Taking into account that $\sum_{j=\nu+1}^n Q(u_j) = k^* - k_* - \iota \leq 2nh$ we obtain

$$\begin{aligned} \frac{\mathbf{E}_{S,p}|Z_n|}{nh} &\leq \mathbf{E}_{S,p}|\varrho_{\nu,k^*}(Q)| + 2 \left| \frac{1}{\gamma(S)} - \frac{1}{\gamma(\hat{S}_\nu)} \right| + \frac{2}{\varepsilon^2} \tilde{\varepsilon} \\ &\leq \mathbf{E}_{S,p}|\varrho_{\nu,k^*}(Q)| + \frac{4}{\varepsilon^4} \mathbf{E}_{S,p} \left| \hat{\Delta}_\nu(z_0) \right| + \frac{2}{\varepsilon^2} \tilde{\varepsilon}. \end{aligned}$$

Moreover, note that

$$\begin{aligned} \mathbf{E}_{S,p} \left| \hat{\Delta}_\nu(z_0) \right| &= \mathbf{E}_{S,p} \left| \hat{\Delta}_\nu(z_0) \right| \mathbf{1}_{\{|\hat{\Delta}_\nu(z_0)| \leq \tilde{\varepsilon}\}} \\ &\quad + \mathbf{E}_{S,p} \left| \hat{\Delta}_\nu(z_0) \right| \mathbf{1}_{\{|\hat{\Delta}_\nu(z_0)| > \tilde{\varepsilon}\}} \\ &\leq \tilde{\varepsilon} + 2\mathbf{P}_{S,p} \left(\left| \hat{\Delta}_{n_0}(z_0) \right| > \tilde{\varepsilon} \right). \end{aligned}$$

Therefore, Lemma 8.3 and Lemma 4.1 imply immediately (5.3). \square

6 PROOFS

6.1 Proof of Theorem 3.1

First, similarly to the proof of Theorem 3.1 from [13] we choose the corresponding parametric functional family $S_{u,\delta}(\cdot)$ in the following form

$$S_{u,\delta}(x) = \frac{u}{\varphi_n} V_\delta \left(\frac{x - z_0}{h} \right), \quad (6.1)$$

with the function V_δ defined as

$$V_\delta(x) = \delta^{-1} \int_{-\infty}^{\infty} \tilde{Q}_\delta(u) g \left(\frac{u - x}{\delta} \right) du,$$

where $\tilde{Q}_\delta(u) = \mathbf{1}_{\{|u| \leq 1-2\delta\}} + 2\mathbf{1}_{\{1-2\delta \leq |u| \leq 1-\delta\}}$ with $0 < \delta < 1/4$ and g is some even nonnegative infinitely differentiable function such that $g(z) = 0$ for $|z| \geq 1$ and $\int_{-1}^1 g(z) dz = 1$. One can show (see [13]) that for any $b > 0$ and $0 < \delta < 1/4$ there exists $n_* = n_*(b, L, \delta) > 0$ such that for all $|u| \leq b$ and $n \geq n_*$

$$S_{u,\delta} \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \varepsilon_n^*).$$

Therefore, in this case for any $n \geq n_*$

$$\begin{aligned} \varphi_n \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \varepsilon_n^*)} \gamma^{-1/2}(S) \mathcal{R}_n(\tilde{S}_n, S) &\geq \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \varepsilon_n^*)} \gamma^{-1/2}(S) \mathbf{E}_{S,p_0} \psi_n(\tilde{S}_n, S) \\ &\geq \gamma_*(n, b) \frac{1}{2b} \int_{-b}^b \mathbf{E}_{S_{u,\delta}, p_0} \psi_n(\tilde{S}_n, S_{u,\delta}) du, \end{aligned}$$

where $\gamma_*(n, b) = \inf_{|u| \leq b} \gamma^{-1/2}(S_{u,\delta})$. The definitions (2.12) and (6.1) imply that for any $b > 0$

$$\lim_{n \rightarrow \infty} \sup_{|u| \leq b} |\gamma(S_{u,\delta}) - 1| = 0.$$

Therefore, by the same way as in the proof of Theorem 3.1 from [13] we obtain that for any $b > 0$ and $0 < \delta < 1/4$

$$\lim_{n \rightarrow \infty} \inf_{\tilde{S}} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \gamma^{-1/2}(S) \varphi_n \mathcal{R}_n(\tilde{S}_n, S) \geq I(b, \sigma_\delta), \quad (6.2)$$

where

$$I(b, \sigma_\delta) = \frac{\max(1, b - \sqrt{b})}{b} \frac{\sigma_\delta}{\sqrt{2\pi}} \int_{-\sqrt{b}}^{\sqrt{b}} e^{-\sigma_\delta^2 \frac{u^2}{2}} du,$$

with $\sigma_\delta^2 = \int_{-1}^1 V_\delta^2(u) du$. It is easy to check that $\sigma_\delta^2 \rightarrow 2$ as $\delta \rightarrow 0$. Limiting $b \rightarrow \infty$ and $\delta \rightarrow 0$ in (6.2) yield the inequality (3.3). Hence Theorem 3.1. \square

6.2 Proof of Theorem 3.2

First we set

$$\check{x}_j = \mathbf{1}_{\{\tau_H \neq j\}} + \varkappa_H \mathbf{1}_{\{\tau_H = j\}}. \quad (6.3)$$

Then taking this into account we can represent the estimate error as

$$S_h^* - S(z_0) = -S(z_0) \mathbf{1}_{(A_{\nu, n} < H)} + h^\beta B_n(h) \mathbf{1}_{(A_{\nu, n} \geq H)} + \frac{1}{\sqrt{H}} \zeta_n(h) \mathbf{1}_{(A_{\nu, n} \geq H)}, \quad (6.4)$$

where

$$B_n(h) = \frac{\sum_{j=\nu+1}^{\tau_H} \check{x}_j Q(u_j) (S(x_j) - S(z_0)) y_{j-1}^2}{h^\beta H}$$

and

$$\zeta_n(h) = \frac{\sum_{j=\nu+1}^{\tau_H} \check{x}_j Q(u_j) y_{j-1} \xi_j}{\sqrt{H}}.$$

First we study the term $B_n(h)$. To this end we introduce

$$B_n^* = \sum_{j=\nu+1}^n Q(u_j) b_j^* y_{j-1}^2, \quad b_j^* = \frac{S(x_j) - S(z_0)}{h} \mathbf{1}_{\{k_* \leq j \leq k^*\}}.$$

It is clear that for any S from $\mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)$

$$\sup_{j \geq 1} |b_j^*| \leq L.$$

Therefore, using Lemma 5.2 for the sequence (5.2) with $w_j = b_j^*$ we obtain

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \sup_{p \in \mathcal{P}_\zeta} \frac{\mathbf{E}_{S, p} |\sum_{j=\nu+1}^{\tau_H} \check{x}_j Q(u_j) b_j^* y_{j-1}^2 - B_n^*|}{nh} = 0. \quad (6.5)$$

Moreover, putting

$$f_1(u) = Q(u) \frac{S(z_0 + hu) - S(z_0)}{h},$$

we obtain

$$\frac{B_n^*}{nh} = \frac{1}{\gamma(S)} \sum_{j=1}^n f_1(u_j) \Delta u_j - \frac{1}{\gamma(S)} \sum_{j=1}^{\nu} f_1(u_j) \Delta u_j + \varrho_{\nu, n}(f_1). \quad (6.6)$$

Using the definition of $\Omega_h(z_0, S)$ in (2.4) we can represent the first term as

$$\sum_{j=1}^n f_1(u_j) \Delta u_j = \frac{1}{h} \Omega_h(z_0, S) - \int_{u_{k^*}}^1 f_1(u) du + \sum_{j=k^*}^{k^*} \int_{u_{j-1}}^{u_j} (f_1(u_j) - f_1(u)) du.$$

Note now that for any S from $\mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)$

$$\sup_{-1 \leq u \leq 1} |f_1(u)| \leq L \quad \text{and} \quad \sup_{-1 \leq u \leq 1} |\dot{f}_1(u)| \leq L,$$

and, therefore,

$$\left| \int_{u_{k^*}}^1 f_1(u) du \right| \leq \frac{L}{nh} \quad \text{and} \quad \left| \sum_{j=k^*}^{k^*} \int_{u_{j-1}}^{u_j} (f_1(u_j) - f_1(u)) du \right| \leq \frac{2L}{nh}.$$

The last bounds imply immediately

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \left| \sum_{j=1}^n f_1(u_j) \Delta u_j \right| = 0.$$

Taking into account Lemma 8.3 in (6.6) we get

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \sup_{p \in \mathcal{P}_\zeta} \frac{\mathbf{E}_{S,p} |B_n^*|}{nh} = 0.$$

Therefore, in view of (6.5)

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} \sup_{S \in \mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} |B_n(h)| = 0.$$

To study the last term in (6.3) note that the definition of the stopping time in (2.10) implies

$$\sup_{n \geq \nu+1} \sup_{h_* \leq h \leq h^*} \sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} |\zeta_n(h)|^2 \leq 1. \quad (6.7)$$

Therefore, in view of Lemma 8.5 we obtain

$$\lim_{n \rightarrow \infty} \sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\zeta} \left| \mathbf{E}_{S,p} |\zeta_n(h)| \mathbf{1}_{(A_{\nu, n} \geq H)} - \mathbf{E} |\zeta_\infty| \right| = 0,$$

where $\zeta_\infty \sim \mathcal{N}(0, 1)$, i.e. $\mathbf{E} |\zeta_\infty| = \sqrt{2/\pi}$. Moreover, in view of Lemma 4.2 and the bound (5.1) we get

$$\lim_{n \rightarrow \infty} \sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} \left| \frac{\varphi_n^2}{\gamma(S)H} - \frac{1}{2} \right| = 0.$$

From this and Lemma 5.1 it follows Theorem 3.2. \square

6.3 Proof of Theorem 3.4

First, note that the representation (6.4) implies

$$\begin{aligned} \mathbf{E}_{S,p} |S_h^* - S(z_0)| &\leq \mathbf{P}_{S,p}(A_{\nu,n} < H) + h^\beta \mathbf{E}_{S,p} |B_n(h)| \mathbf{1}_{(A_{\nu,n} \geq H)} \\ &\quad + \mathbf{E}_{S,p} \left(\frac{|\zeta_n(h)|}{\sqrt{H}} \mathbf{1}_{(A_{\nu,n} \geq H)} \right). \end{aligned} \quad (6.8)$$

Let us show, that

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \sup_{p \in \mathcal{P}} \mathbf{E}_{S,p} |B_n(h)| < \infty. \quad (6.9)$$

Indeed, setting

$$\vartheta_j = \frac{S(x_j) - S(z_0)}{h} - \dot{S}(z_0) u_j, \quad (6.10)$$

we can represent $B_n(h)$ as

$$B_n(h) = \frac{h^{-\alpha}}{H} \dot{S}(z_0) \tilde{B}_n + \frac{h^{-\alpha}}{H} \hat{B}_n, \quad (6.11)$$

where

$$\tilde{B}_n = \sum_{j=\nu+1}^{\tau_H} \check{\varkappa}_j Q(u_j) u_j y_{j-1}^2 \quad \text{and} \quad \hat{B}_n = \sum_{j=\nu+1}^{\tau_H} \check{\varkappa}_j Q(u_j) \vartheta_j y_{j-1}^2.$$

Using now Lemma 5.2 for the sequence (5.2) with $w_j = u_j$ we obtain that

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\zeta} \frac{\mathbf{E}_{S,p} |\tilde{V}_n - \tilde{B}_n|}{nh} = 0,$$

where

$$\tilde{V}_n = \sum_{j=\nu+1}^n Q(u_j) u_j y_{j-1}^2.$$

We can represent this term as

$$\begin{aligned} \tilde{V}_n &= \varrho_{\nu,n}(Q_1) + \frac{1}{\gamma(S)} \sum_{j=\nu+1}^n Q_1(u_j) \Delta u_j \\ &= \varrho_{\nu,n}(Q_1) + \frac{1}{\gamma(S)} \left(\sum_{j=k_*+1}^{k^*} u_j \Delta u_j - \sum_{j=k_*+1}^{k_*+\iota} u_j \Delta u_j \right), \end{aligned}$$

where $Q_1(u) = Q(u) u$. Moreover, taking into account here, that

$$\left| \sum_{j=k_*+1}^{k^*} u_j \Delta u_j \right| = \left| \int_{u_{k_*}}^{u_{k^*}} u \, du + \sum_{j=k_*+1}^{k^*} \int_{u_{j-1}}^{u_j} (u_j - u) \, du \right| \leq \frac{4}{nh}$$

and

$$\left| \sum_{j=k_*+1}^{k_*+\iota} u_j \Delta u_j \right| \leq \tilde{\varepsilon} + \frac{1}{nh},$$

we obtain

$$\frac{1}{nh} \mathbf{E}_{S,p} |\tilde{V}_n| \leq \mathbf{E}_{S,p} |\varrho_{\nu,n}(Q_1)| + \frac{1}{\varepsilon^2} \left(\frac{5}{nh} + \tilde{\varepsilon} \right).$$

Now Lemma 8.3 yields

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \frac{\mathbf{E}_{S,p} |\tilde{V}_n|}{nh} = 0$$

and, therefore,

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \frac{\mathbf{E}_{S,p} |\tilde{B}_n|}{nh} = 0. \quad (6.12)$$

To estimate the last term in (6.11) note, that for any function S from $\mathcal{H}^{(\beta)}(\varepsilon, L, L^*)$ and for $k_* \leq j \leq k^*$ the coefficients (6.10) can be estimated as

$$|\vartheta_j| = \left| \int_0^{u_j} \left(\dot{S}(z_0 + hu) - \dot{S}(z_0) \right) du \right| \leq L|u_j| h^\alpha \leq L^* h^\alpha.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\alpha} \leq \alpha \leq \bar{\alpha}} h^{-\alpha} \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \sup_{p \in \mathcal{P}} \frac{1}{nh} \mathbf{E}_{S,p} |\hat{B}_n| < \infty.$$

Now the property (6.12) implies the inequality (6.9). Therefore, using Lemma 5.1 and the inequality (6.7) in (6.8), we come to Theorem 3.2. \square

6.4 Proof of Theorem 3.5

First of all, note that the coefficient ϕ defined in (2.13) will be more than one for sufficient large n for which $\tilde{\varepsilon} \leq 1/2$. So, using the representation (6.4), we get for any $1 \leq j \leq m$

$$|\check{S}_j - S(z_0)| \leq \mathbf{1}_{\Gamma_j} + (\check{h}_j)^\beta |B_n(\check{h}_j)| + \frac{1}{\sqrt{n\check{h}_j}} |\tilde{\zeta}_n(\check{h}_j)|, \quad (6.13)$$

where $\Gamma_j = \{A_{\nu,n}(\check{h}_j) < \check{H}_j\}$, $\tilde{\zeta}_n(\check{h}_j) = \zeta_n(\check{h}_j) \mathbf{1}_{(A_{\nu,n}(\check{h}_j) \geq \check{H}_j)}$, the random functions $B_n(h)$ and $\zeta_n(h)$ are defined in (6.4). We set

$$i_0 = \left\lceil \frac{m(\beta - \underline{\beta})}{\bar{\beta} - \underline{\beta}} \right\rceil.$$

This means that

$$0 \leq \beta - \beta_{i_0} < \frac{\bar{\beta} - \underline{\beta}}{m}.$$

Therefore, taking into account the definition of m in (2.18), we obtain that for any fixed integer $-\infty < l < \infty$

$$\begin{cases} 0 < \liminf_{n \rightarrow \infty} \frac{N(\beta)}{N_{i_0+l}} \leq \limsup_{n \rightarrow \infty} \frac{N(\beta)}{N_{i_0+l}} < \infty; \\ 0 < \liminf_{n \rightarrow \infty} \frac{h(\beta)}{\check{h}_{i_0+l}} \leq \limsup_{n \rightarrow \infty} \frac{h(\beta)}{\check{h}_{i_0+l}} < \infty. \end{cases} \quad (6.14)$$

These inequalities and Theorem 3.4 imply

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\beta} \leq \beta \leq \bar{\beta}} N(\beta) \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \mathbf{E}_S \varpi(i_0, z_0) < \infty, \quad (6.15)$$

where $\varpi(i_0, z_0) = |\check{S}_{i_0-1} - S(z_0)| + |\check{S}_{i_0} - S(z_0)| + |\check{S}_{i_0+1} - S(z_0)|$. Now considering the estimator $\widehat{S}_{a,n}$, one has

$$|\widehat{S}_{a,n} - S(z_0)| \leq I_1 + I_2 + \varpi(i_0, z_0), \quad (6.16)$$

where $I_1 = |\widehat{S}_{a,n} - S(z_0)| \mathbf{1}_{\{\check{k} \geq i_0+2\}}$ and $I_2 = |\widehat{S}_{a,n} - S(z_0)| \mathbf{1}_{\{\check{k} \leq i_0-2\}}$. We start with the term I_1 . We have

$$|\widehat{S}_{a,n} - S(z_0)| \mathbf{1}_{\{\check{k} \geq i_0+2\}} \leq |\widehat{S}_{a,n} - \check{S}_{i_0}| \mathbf{1}_{\{\check{k} \geq i_0+2\}} + |\check{S}_{i_0} - S(z_0)| \mathbf{1}_{\{\check{k} \geq i_0+2\}}.$$

Moreover,

$$\begin{aligned} |\widehat{S}_{a,n} - \check{S}_{i_0}| \mathbf{1}_{\{\check{k} \geq i_0+2\}} &\leq \check{\omega}_{\check{k}} \mathbf{1}_{\{\check{k} \geq i_0+2\}} + \frac{\check{\lambda}}{N_{i_0}} \\ &\leq \frac{\check{\lambda}}{N_{\check{k}}} \mathbf{1}_{\{\check{k} \geq i_0+2\}} + \frac{\check{\lambda}}{N_{i_0}} \leq \frac{\check{\lambda}}{N(\beta)} + \frac{\check{\lambda}}{N_{i_0}}. \end{aligned}$$

The inequalities (6.14)–(6.15) imply immediately

$$\limsup_{n \rightarrow \infty} \sup_{\underline{\beta} \leq \beta \leq \bar{\beta}} N(\beta) \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \mathbf{E}_S I_1 < \infty.$$

Now we study the term I_2 . From (6.13) it follows that

$$I_2 \leq \left(\mathbf{1}_{\Gamma_{\check{k}}} + (\check{h}_{\check{k}})^\beta |B_n(\check{h}_{\check{k}})| + \frac{1}{\sqrt{n\check{h}_{\check{k}}}} |\tilde{\zeta}_n(\check{h}_{\check{k}})| \right) \mathbf{1}_{\{\check{k} \leq i_0-2\}}.$$

Therefore,

$$\mathbf{E}_S I_2 \leq m I_1^*(S) + I_2^*(S) \sum_{j=0}^{i_0-2} \check{h}_j^\beta + \Psi_n(S), \quad (6.17)$$

where $I_1^*(S) = \max_{0 \leq l \leq m} \mathbf{P}_S(\Gamma_l)$, $I_2^*(S) = \max_{0 \leq l \leq m} \mathbf{E}_S |B_n(\check{h}_l)|$ and

$$\Psi_n(S) = \frac{1}{\sqrt{n}} \sum_{j=0}^{i_0-2} \frac{1}{\sqrt{\check{h}_j}} \mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{\check{k}=j\}}.$$

Note now, that for any $0 \leq j \leq i_0$ and for sufficiently large n (for which $\ln d_n \geq m/2$) we get

$$N(\beta) \check{h}_j^\beta \leq e^{-\beta^*(i_0-j)}, \quad \beta^* = \frac{\beta(\bar{\beta} - \underline{\beta})}{(2\beta + 1)(2\bar{\beta} + 1)},$$

i.e.

$$N(\beta) \sum_{j=0}^{i_0-2} \check{h}_j^\beta \leq \frac{e^{\beta^*}}{e^{\beta^*} - 1}.$$

So, Lemma 5.1 and the inequality (6.9) yield

$$\limsup_{n \rightarrow \infty} N(\beta) \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \left(mI_1^*(S) + I_2^*(S) \sum_{j=0}^{i_0-2} \check{h}_j^\beta \right) < \infty.$$

Let us consider now the last term (6.17). To this end note that for $0 \leq j \leq i_0 - 2$

$$\{\check{k} = j\} \subseteq \{\check{\omega}_{j+1} \geq \check{\lambda}/N_{j+1}\} \subseteq \cup_{l=0}^{j+1} \{|\check{S}_l - S(z_0)| \geq \check{\lambda}/N_l\}.$$

Therefore,

$$\Psi_n(S) \leq \frac{1}{\sqrt{n}} \sum_{j=0}^{i_0-2} \frac{1}{\sqrt{\check{h}_j}} \sum_{l=0}^{j+1} \mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{|\check{S}_l - S(z_0)| \geq \check{\lambda}/N_l\}}.$$

Taking into account that $\check{h}_{j+1}/\check{h}_j \leq e$, we can rewrite the last inequality as

$$\Psi_n(S) \leq \frac{e}{\sqrt{n}} \sum_{j=1}^{i_0-1} \frac{1}{\sqrt{\check{h}_j}} \sum_{l=0}^j \mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{|\check{S}_l - S(z_0)| \geq \check{\lambda}/N_l\}}. \quad (6.18)$$

Now, taking into account the inequality (6.7), we get

$$\begin{aligned} \mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{|\check{S}_l - S(z_0)| \geq \check{\lambda}/N_l\}} &\leq \sqrt{\mathbf{P}_S(\Gamma_l)} + \mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{\check{h}_l^\beta |B_n(\check{h}_l)| \geq \check{\lambda}_1/N_l\}} \\ &\quad + \mathbf{E}_S |\zeta^*| \mathbf{1}_{\{|\zeta^*| \geq \sqrt{\ln n} \check{\lambda}_1\}} \end{aligned}$$

where $\check{\lambda}_1 = \check{\lambda}/2$ and $\zeta^* = \max_{1 \leq j \leq m} |\tilde{\zeta}_n(\check{h}_j)|$. In view of the Hölder and Chebyshev inequalities and making use of the upper bound (8.11) we obtain

$$\mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{\check{h}_l^\beta |B_n(\check{h}_l)| \geq \check{\lambda}_1/N_l\}} \leq \frac{(\mu_4^*)^{1/4} (I_2^*(S))^{3/4}}{\check{\lambda}_1^{3/4}} \left(N_l \check{h}_l^\beta \right)^{3/4}.$$

where the term $I_2^*(S)$ is defined in (6.17). Using these bounds in (6.18) we get

$$\begin{aligned} \Psi_n(S) &\leq \frac{e m^2}{\sqrt{n} \check{h}_0} \sqrt{I_1^*(S)} + \frac{e(\mu_4^*)^{1/4} (I_2^*(S))^{3/4}}{\check{\lambda}_1^{3/4}} \Upsilon_n^* \\ &\quad + \frac{e m^2}{\sqrt{n} \check{h}_0} \mathbf{E}_S |\zeta^*| \mathbf{1}_{\{|\zeta^*| \geq \check{\lambda}_1 \sqrt{\ln n}\}}, \end{aligned} \quad (6.19)$$

where

$$\Upsilon_n^* = \frac{1}{\sqrt{n}} \sum_{j=1}^{i_0-1} \left(N_j \check{h}_j^\beta \right)^{1/4} \left(N_j \check{h}_j^\alpha \right)^{1/2} v_j^* \quad \text{and} \quad v_j^* = \sum_{l=0}^j \left(\frac{N_l \check{h}_l^\beta}{N_j \check{h}_j^\beta} \right)^{3/4}.$$

Let us estimate the term Υ_n^* . To this end note, that for any $0 \leq j < i_0$ and for sufficiently large n (for which $\ln d_n \geq m/2$)

$$\left(N_j \check{h}_j^\beta \right)^{1/4} = \exp \left\{ \frac{\beta_j - \beta}{4(2\beta_j + 1)} \ln d_n \right\} \leq e^{-\beta_1^* (i_0 - j)} \quad \text{with} \quad \beta_1^* = \frac{\bar{\beta} - \beta}{8(2\bar{\beta} + 1)},$$

and

$$\left(N_j \check{h}_j^\alpha\right)^{1/2} = \exp \left\{ \frac{\beta_j - \alpha}{2(2\beta_j + 1)} \ln d_n \right\} \leq \frac{1}{\sqrt{\check{h}(\beta)}} = \frac{\sqrt{d_n}}{N(\beta)},$$

where $\check{h}(\beta)$ is defined in (2.17). Similarly for any $0 \leq l \leq j \leq i_0$ we get

$$\left(\frac{N_l \check{h}_l^\beta}{N_j \check{h}_j^\beta}\right)^{3/4} = \exp \left\{ \frac{3(\beta_l - \beta_j)(2\beta + 1)}{4(2\beta_l + 1)(2\beta_j + 1)} \ln d_n \right\} \leq e^{-\beta_2^*(j-l)}, \quad \beta_2^* = \frac{3(\bar{\beta} - \beta)}{8(2\beta + 1)}.$$

This means that the sequence $(v_j^*)_{j \geq 1}$ is bounded, i.e.

$$\sup_{j \geq 1} v_j^* \leq \frac{e^{\beta_2^*}}{e^{\beta_2^*} - 1}.$$

Therefore,

$$\lim_{n \rightarrow \infty} N(\beta) \Upsilon_n^* = 0. \quad (6.20)$$

The last term in (6.19) can be estimated through Lemma 8.10, i.e.

$$\begin{aligned} \mathbf{E}_S \zeta^* \mathbf{1}_{\{\zeta^* \geq \check{\lambda}_1 \sqrt{\ln n}\}} &\leq m \max_{1 \leq j \leq m} \mathbf{E}_S |\tilde{\zeta}_n(\check{h}_j)| \mathbf{1}_{\{|\tilde{\zeta}_n(\check{h}_j)| \geq \check{\lambda}_1 \sqrt{\ln n}\}} \\ &\leq 2m \check{\lambda}_1 \sqrt{\ln n} e^{-\frac{1}{8} \check{\lambda}_1^2 \ln n} + 2m \int_{\check{\lambda}_1 \sqrt{\ln n}}^{+\infty} e^{-z^2/8} dz. \end{aligned}$$

Therefore, for sufficiently large n (when $\check{\lambda}_1 \sqrt{\ln n} \geq 1$) we get that

$$\mathbf{E}_S \zeta^* \mathbf{1}_{\{\zeta^* \geq \check{\lambda}_1 \sqrt{\ln n}\}} \leq 2m \left(\check{\lambda}_1 \sqrt{\ln n} + 4 \right) n^{-\check{\lambda}_1^2/8}. \quad (6.21)$$

Now the definition of the parameter $\check{\lambda}$ in (2.19) yields

$$\limsup_{n \rightarrow \infty} \frac{N(\beta) m^2}{\sqrt{n \check{h}_0}} \sup_{S \in \mathcal{H}^{(\beta)}(\varepsilon, L, L^*)} \mathbf{E}_S |\zeta^*| \mathbf{1}_{\{|\zeta^*| \geq \check{\lambda}_1 \sqrt{\ln n}\}} = 0.$$

Hence Theorem 3.5. \square

7 NUMERICAL EXAMPLES

Using Scilab, we establish the following simulations to illustrate the obtained results.

7.1 Nonadaptive Estimation

In this section we illustrate the results obtained in the case of nonadaptive estimation. The purpose is to estimate, at a given point z_0 , the function S defined over $[0; 1]$ by

$$S(x) = (x - z_0) |x - z_0|^\alpha \quad (7.1)$$

for $z_0 = 1/\sqrt{2}$ and $\alpha = 0, 3$. Taking into account that for this function

$$\Omega_h(z_0, S) = 0$$

we obtain that for any

$$0 < \varepsilon \leq 1 - \left(\frac{1}{\sqrt{2}}\right)^\beta, \quad L \geq \beta \quad \text{and} \quad \epsilon_n^* > 0 \quad (7.2)$$

the function (7.1) belongs to the class $\mathcal{U}_n^{(\beta)}(\varepsilon, L, \epsilon_n^*)$ with $\beta = 1, 3$.

The numerical results approximate the asymptotic risk of a estimators defined in (2.11) used due to the calculation of an expectation (it performs an average for $M = 30000$ simulations) and the finite number of observations n . Here we calculate for the estimator the quantity

$$\mathbf{R}_n = \frac{1}{M} \sum_{k=1}^M \left| S_h^{*,k} - S(z_0) \right|.$$

For the standard Gaussian random variables $(\xi_j)_{j \geq 1}$ in (1.1), and by varying the number of observations n , we obtain different risks listed in the following table:

n	1000	5000	10000	20000
\mathbf{R}_n	0.034	0.021	0.017	0.012

For random variables $(\xi_j)_{j \geq 1}$ reduced from uniform random variables on $[-1, 1]$, we obtain :

n	1000	5000	10000	20000
\mathbf{R}_n	0.038	0.022	0.018	0.014

For random variables $(\xi_j)_{j \geq 1}$ centered and reduced from exponential random variables with parameter 1, we obtain :

n	1000	5000	10000	20000
\mathbf{R}_n	0.028	0.016	0.012	0.010

7.2 Numerical Result for Non Sequential Kernel Estimator

Now we give the numerical results for the kernel estimator defined as

$$\hat{S}_n(z_0) = \frac{1}{\sum_{k=1}^n Q(u_k) y_{k-1}^2} \sum_{k=1}^n Q(u_k) y_{k-1} y_k.$$

For the standard Gaussian random variables $(\xi_j)_{j \geq 1}$ in (1.1), and by varying the number of observations n , we obtain different risks listed in the following table:

n	1000	5000	10000	20000
\mathbf{R}_n	0.046	0.026	0.020	0.010

7.3 Adaptive Estimation

In this case we estimate the function (7.1) for $z_0 = 1/\sqrt{2}$ and $\alpha = 0, 7$. Obviously, this function belongs to class $\mathcal{H}^{(\beta)}(\varepsilon, L, L^*)$ with $\beta = 1, 7$, $L^* = 1$ and for any ε and L satisfying the conditions (7.2).

In the adaptive estimation we take the lower regularity $\underline{\beta} = 1.6$ and the higher regularity $\bar{\beta} = 1.8$.

We model the sequential adaptive procedure $\widehat{S}_{a,n} = S_h^*$ defined in (2.21). Numerical results approximate the asymptotic risk for this procedure by the calculation of an expectation via $M = 30000$ simulations.

$$\mathbf{R}_{a,n} = \frac{1}{M} \sum_{k=1}^M |\widehat{S}_{a,n}^k - S(z_0)|$$

By varying the number of observations n , we obtain different risks listed in the following table:

n	1000	5000	10000	20000
$\mathbf{R}_{a,n}$	0.021	0.013	0.009	0.007

8 Appendix

8.1 Concentration Properties of the Process (1.1)

In this section, we study the deviation (4.4) for the model (1.1).

Lemma 8.1. *For any $q > 1$ and $0 < \varepsilon < 1$ the random variables y_k in (1.1) satisfy the following inequality:*

$$\sup_{n \geq 1} \sup_{0 \leq k \leq n} \sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\varepsilon} \mathbf{E}_{S,p} |y_k|^q \leq r_q^*, \quad (8.1)$$

where r_q^* is defined in (4.1).

Proof. From (1.1) we obtain that for any $k \geq 1$

$$y_k = y_0 \prod_{l=1}^k S(x_l) + \sum_{i=1}^k \prod_{l=i+1}^k S(x_l) \xi_i.$$

Therefore, for $S \in \Theta_{\varepsilon, L}$ and $1 \leq k \leq n$,

$$|y_k|^q \leq 2^{q-1} \left(|y_0|^q + \left(\sum_{j=1}^k (1 - \varepsilon)^{k-j} |\xi_j| \right)^q \right).$$

Moreover, the Hölder inequality gives

$$\begin{aligned} \left(\sum_{j=1}^k (1 - \varepsilon)^{k-j} |\xi_j| \right)^q &\leq \left(\sum_{j=1}^k (1 - \varepsilon)^{k-j} \right)^{q-1} \left(\sum_{j=1}^k (1 - \varepsilon)^{k-j} |\xi_j|^q \right) \\ &\leq \left(\frac{1}{\varepsilon} \right)^{q-1} \left(\sum_{j=1}^k (1 - \varepsilon)^{k-j} |\xi_j|^q \right). \end{aligned}$$

Thus, taking into account the definition (2.2) we get for any $p \in \mathcal{P}_\zeta$

$$\mathbf{E}_{S,p} \left(\sum_{j=1}^k (1-\varepsilon)^{k-j} |\xi_j| \right)^q \leq \left(\frac{1}{\varepsilon} \right)^q \mathbf{s}_q^*.$$

Hence Lemma 8.1. \square

Now we need the following Burkholder inequality from [24].

Lemma 8.2. *Let $(M_k)_{1 \leq k \leq n}$ be a martingale. Then for any $q > 1$*

$$\mathbf{E} |M_n|^q \leq \mathbf{b}_q^* \mathbf{E} \left(\sum_{j=1}^n (M_j - M_{j-1})^2 \right)^{q/2}, \quad (8.2)$$

where the coefficient \mathbf{b}_q^* is defined in (4.1).

Now we study the deviation (4.4).

Lemma 8.3. *Let f be a $\mathbb{R} \rightarrow \mathbb{R}$ function twice continuously differentiable in $[-1, 1]$. Assume also that the bandwidth h satisfies the condition (2.14) – (2.15). Then for any $R > 0$ and $q > 1$*

$$\limsup_{n \rightarrow \infty} \sup_{k_* \leq k < m \leq k^*} \sup_{\underline{\beta} \leq \beta \leq \bar{\beta}} \sup_{R > 0} \sup_{\|f\|_1 \leq R} \frac{1}{(Rh)^q} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} |\varrho_{k,m}(f)|^q \leq \varrho_q^*, \quad (8.3)$$

where $\|f\|_1 = \|f\| + \|\dot{f}\|$ and ϱ_q^* is defined in (4.1).

Proof. First of all, note that

$$\sum_{j=k+1}^m f(u_j) y_{j-1}^2 = T_{k,m} + a_{k,m}, \quad (8.4)$$

where $T_{k,m} = \sum_{j=k+1}^m f(u_j) y_j^2$ and

$$a_{k,m} = \sum_{j=k+1}^m (f(u_j) - f(u_{j-1})) y_{j-1}^2 + f(u_k) y_k^2 - f(u_m) y_m^2.$$

Moreover, from the model (1.1) we find

$$(1 - S^2(z_0)) T_{k,m} = M_{k,m} + \check{a}_{k,m} + \sum_{j=k+1}^m f(u_j)$$

where $M_{k,m} = \sum_{j=k+1}^m (2S(x_j) y_{j-1} \xi_j + \xi_j^2 - 1) f(u_j)$ and

$$\check{a}_{k,m} = \sum_{j=k+1}^m f(u_j) S^2(x_j) y_{j-1}^2 - S^2(z_0) T_{k,m}. \quad (8.5)$$

Then we can write $\varrho_{k,m}(f)$ as follow

$$\varrho_{k,m}(f) = \frac{1}{nh \gamma(S)} (M_{k,m} + \check{a}_{k,m}) + \frac{a_{k,m}}{nh}.$$

and

$$\mathbf{E}_{S,p} |\varrho_{k,m}(f)|^q \leq \frac{3^{q-1}}{\varepsilon^{2q}} \mathbf{E}_{S,p} \left(\left(\frac{|M_{k,m}|}{nh} \right)^q + \left(\frac{|\check{a}_{k,m}|}{nh} \right)^q + \left(\frac{|a_{k,m}|}{nh} \right)^q \right). \quad (8.6)$$

where $\tilde{\kappa} = (1 + \kappa_*)/\kappa_*$. Now we note, that in view of the first condition in (2.15) for sufficient large n

$$n\kappa_n \geq \tilde{\kappa},$$

where $\tilde{\kappa} = (1 + \kappa_*)/\kappa_*$. Therefore, for sufficiently large n we get

$$\frac{1}{nh} \leq (\tilde{\kappa})^{2/(2\beta+1)} h \leq \tilde{\kappa} h, \quad (8.7)$$

Furthermore, note that $(M_{k,j})_{k < j \leq m}$ is a martingale. So, by applying the Burkholder inequality (8.2) and, taking into account that $k^* - k_* \leq 2nh$, we get

$$\begin{aligned} \mathbf{E}_{S,p} \left(\frac{1}{nh} M_{k,m} \right)^q &\leq \frac{\mathbf{b}_q^* R^q}{(nh)^q} \mathbf{E}_{S,p} \left(\sum_{j=k_*+1}^{k^*} (2S(x_j) y_{j-1} \xi_j + \xi_j^2 - 1)^2 \right)^{q/2} \\ &\leq \frac{\mathbf{b}_q^* R^q}{(nh)^{q/2+1}} \sum_{j=k_*+1}^{k^*} \mathbf{E}_{S,p} \left(2S(x_j) y_{j-1} \xi_j + \xi_j^2 - 1 \right)^q \\ &\leq \frac{4^q \mathbf{b}_q^* R^q}{(nh)^{q/2}} \left(r_q^* \mathbf{s}_q^* + \mathbf{s}_{2q}^* + 1 \right) \leq 4^q \mathbf{b}_q^* \tilde{\kappa}^q R^q \left(r_q^* \mathbf{s}_q^* + \mathbf{s}_{2q}^* + 1 \right) h^q, \end{aligned}$$

where the coefficients r_q^* and \mathbf{s}_q^* are given in (8.1) and (2.2). Note that the term (8.5) can be rewritten as

$$\begin{aligned} \check{a}_{k,m} &= S^2(z_0) \left(\sum_{j=k+1}^m (f(u_j) - f(u_{j-1})) y_{j-1}^2 + f(u_k) y_k^2 - f(u_m) y_m^2 \right) \\ &\quad + \sum_{j=k+1}^m f(u_j) (S^2(x_j) - S^2(z_0)) y_{j-1}^2. \end{aligned}$$

We recall, that the function f and its derivative \dot{f} are bounded by R . Therefore, taking into account that for all $S \in \Theta_{\varepsilon,L}$ and $k_* \leq j \leq k^*$ the deviation $|S(x_j) - S(z_0)| \leq L|x_j - z_0| \leq Lh$, we obtain

$$\begin{aligned} |\check{a}_{k,m}| &\leq R \left(\left(\frac{1}{nh} + Lh \right) \sum_{j=k+1}^m y_j^2 + y_k^2 + y_m^2 \right) \\ &\leq \tilde{\kappa} R \left((L+1)h \sum_{j=k+1}^m y_j^2 + y_k^2 + y_m^2 \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S,p} \left(\frac{1}{nh} |\check{a}_{k,m}| \right)^q &\leq 2^{q-1} \tilde{\kappa}^q R^q r_{2q}^* \left(2^q (1+L)^q h^q + \frac{2^q}{(nh)^q} \right) \\ &\leq (4R(1+L))^q r_{2q}^* h^q. \end{aligned}$$

Similarly,

$$\sup_{S \in \Theta_{\varepsilon, L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_{S, p} \left(\frac{1}{nh} |a_{k, m}| \right)^q \leq 3 \cdot 2^{q-1} \tilde{\kappa}^q R^q r_{2q}^* h^q.$$

Then, taking this into account in (8.6) we obtain the upper bound (8.3). Hence Lemma 8.3. \square

8.2 Uniform Limit Theorem

In this section we study the following sequence

$$\tilde{\zeta}_n(h) = \zeta_n(h) \mathbf{1}_{(A_{\nu, n} \geq H)}, \quad (8.8)$$

where $\zeta_n(h)$ defined by (6.4), the bandwidth h is defined (2.14) and the threshold H is given in (2.13). We will make use of the following result.

Lemma 8.4. (cf. [10], p. 90-91) *Let $0 < \delta < 1$ and $r > 0$. Assume that $(\mathbf{m}_k)_{k \geq 1}$ is a martingale difference with respect to the filtration $(\mathcal{F}_k)_{k \geq 1}$ such that*

$$|\mathbf{m}_k| \leq \delta r^{1/2} \quad \text{and} \quad \sum_{k=1}^{\infty} \mathbf{E}(\mathbf{m}_k^2 | \mathcal{F}_{k-1}) \geq r.$$

Let

$$\tau = \inf \left\{ k \geq 1 : \sum_{j=1}^k \mathbf{E}(\mathbf{m}_j^2 | \mathcal{F}_{j-1}) \geq r \right\}.$$

There exists a function $\rho : (0, +\infty) \rightarrow [0, 2]$ not depending on distribution of the martingale difference, such that $\lim_{x \rightarrow 0} \rho(x) = 0$ and

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P} \left(\frac{1}{r^{1/2}} \sum_{k=1}^{\tau} \mathbf{m}_k \leq x \right) - \Phi(x) \right| \leq \rho(\delta),$$

where Φ is the standard normal distribution function.

Lemma 8.5. *The sequence (6.4) satisfies the following limiting property:*

$$\tilde{\zeta}_n \implies \zeta \sim \mathcal{N}(0, 1) \quad \text{uniformly in } p \in \mathcal{P}_\zeta \quad \text{and} \quad S \in \Theta_{\varepsilon, L}.$$

Proof. First for some $0 < \delta < 1$ we set

$$\mathbf{m}_j = Q(u_j) \check{y}_{j-1} \check{\xi}_j \mathbf{1}_{\{\nu < j \leq n\}} + \delta \check{\xi}_j \mathbf{1}_{\{j > n\}},$$

where $\check{y}_j = y_j \mathbf{1}_{\{|y_j| \leq \delta^2 \check{H}^{1/2}\}}$, $\check{H} = \mathbf{E}_p \check{\xi}_1^2 H$ and $\check{\xi}_j = \xi_j \mathbf{1}_{\{|\xi_j| \leq \delta^{-1}\}} - \mathbf{E}_p \xi_1 \mathbf{1}_{\{|\xi_1| \leq \delta^{-1}\}}$. It is clear that the sequence $(\mathbf{m}_{\nu+j})_{j \geq 0}$ is a martingale difference with respect to $(\mathcal{G}_j)_{j \geq 0}$, where \mathcal{G}_j is σ -field generated by the observations $\{y_1, \dots, y_{\nu+j}\}$. Now we set

$$\check{\zeta}_{\check{H}} = \frac{1}{\sqrt{\check{H}}} \sum_{j=1}^{\check{\tau}_H} \mathbf{m}_{\nu+j},$$

where $\check{\tau}_H = \inf \left\{ k \geq 1 : \sum_{j=1}^k \mathbf{E}(\mathbf{m}_{\nu+j}^2 | \mathcal{G}_{j-1}) \geq \check{H} \right\}$. Note, that $\check{\tau}_H = \tau_H$ on the set $\{A_{\nu,n} \geq H\}$ for any $H > 0$. Lemma 8.4 implies that $\check{\zeta}_{\check{H}}$ goes in distribution to $\mathcal{N}(0, 1)$ uniformly in $p \in \mathcal{P}_\zeta$ and $S \in \Theta_{\varepsilon,L}$ as $\delta \rightarrow 0$. Now we set

$$\check{\Omega}_n = \cap_{j=\nu}^{k^*} \{y_j = \check{y}_j\}.$$

Using Lemma 8.1 through the Markov inequality we obtain that

$$\sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{P}_{S,p}(\check{\Omega}_n^c) \leq \frac{(k^* - \nu) r_4^*}{\delta^8 \mathbf{E}_p \check{\xi}_1^2 (nh)^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, note that on the set $\check{\Omega}_n \cap \{A_{\nu,n} \geq H\}$

$$\left(\frac{H}{\check{H}}\right)^{1/2} \tilde{\zeta}_n - \check{\zeta}_{\check{H}} = \check{\Delta}_1 + \check{\Delta}_2, \quad (8.9)$$

where

$$\check{\Delta}_1 = \frac{1}{\sqrt{\check{H}}} \sum_{j=\nu+1}^{\tau_H} (\varkappa_j - 1) Q(u_j) \check{y}_{j-1} \xi_j, \quad \check{\Delta}_2 = \frac{1}{\sqrt{\check{H}}} \sum_{j=\nu+1}^{\tau_H} Q(u_j) \check{y}_{j-1} \tilde{\xi}_j$$

and $\tilde{\xi}_j = \xi_j - \check{\xi}_j = \xi_j \mathbf{1}_{\{|\xi_j| > \delta^{-1}\}} - \mathbf{E}_p \xi_1 \mathbf{1}_{\{|\xi_1| > \delta^{-1}\}}$. Note, that

$$\mathbf{E}_{S,p}(\check{\Delta}_1^2 | \mathcal{G}_0) \leq \delta^4 \mathbf{E}_{S,p} \left(\sum_{j=\nu+1}^{k^*} (1 - \varkappa_j)^2 | \mathcal{G}_0 \right) \leq \delta^4.$$

Moreover, taking into account that $\check{y}_j^2 \leq y_j^2$, we get

$$\begin{aligned} \mathbf{E}_{S,p}(\check{\Delta}_2^2 | \mathcal{G}_0) &\leq \frac{\mathbf{E}_p \tilde{\xi}_1^2}{\check{H}} \mathbf{E}_{S,p} \left(\sum_{j=\nu+1}^{\tau_H} Q(u_j) \check{y}_{j-1}^2 | \mathcal{G}_0 \right) \\ &\leq \frac{\mathbf{E}_p \tilde{\xi}_1^2}{\check{H}} (H + \delta^4 \check{H}) = \mathbf{E}_p \tilde{\xi}_1^2 \left(\frac{1}{\mathbf{E}_p \check{\xi}_1^2} + \delta^4 \right) \end{aligned}$$

Taking into account here, that

$$\limsup_{\delta \rightarrow 0} \sup_{p \in \mathcal{P}_\zeta} \mathbf{E}_p \tilde{\xi}_1^2 = 0 \quad \text{and} \quad \limsup_{\delta \rightarrow 0} \sup_{p \in \mathcal{P}_\zeta} |\mathbf{E}_p \check{\xi}_1^2 - 1| = 0,$$

we obtain

$$\limsup_{\delta \rightarrow 0} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \max(\mathbf{E}_{S,p} \check{\Delta}_1^2, \mathbf{E}_{S,p} \check{\Delta}_2^2) = 0.$$

Therefore, Lemma 5.1 and the representation (8.9) yield for any $\mu > 0$

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{S \in \Theta_{\varepsilon,L}} \sup_{p \in \mathcal{P}_\zeta} \mathbf{P}_{S,p} \left(\left| \tilde{\zeta}_n - \check{\zeta}_{\check{H}} \right| > \mu \right) = 0.$$

Hence Lemma 8.5. \square

8.3 Properties of $\tilde{\zeta}_n(h)$

Lemma 8.6. For all $z \geq 2$

$$\sup_{n \geq 1} \sup_{h > 0} \sup_{S \in \mathbf{C}[0,1]} \mathbf{P}_S \left(\tilde{\zeta}_n(h) \geq z \right) \leq 2e^{-z^2/8}. \quad (8.10)$$

The proof of this Lemma is the same as Lemma A.5 from [2].

Using this lemma we can obtain that for any $q > 2$

$$\sup_{n \geq 1} \sup_{h > 0} \sup_{S \in \mathbf{C}[0,1]} \mathbf{E}_S |\tilde{\zeta}_n(h)|^q \leq \mu_q^*, \quad (8.11)$$

where $\mu_q^q = 2^q + 2q \int_0^\infty t^{q-1} e^{-t^2/2} dt$.

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