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► **To cite this version:**

B Berdjane, S Pergamenshchikov. SEQUENTIAL  $\delta$ -OPTIMAL CONSUMPTION AND INVESTMENT FOR STOCHASTIC VOLATILITY MARKETS WITH UNKNOWN PARAMETERS. *Theory of Probability and its Applications*, 2016, 60, pp.533-560. <10.1137/S0040585X97986588>. <hal-02334869>

**HAL Id: hal-02334869**

**<https://hal.science/hal-02334869v1>**

Submitted on 27 Oct 2019

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## SEQUENTIAL $\delta$ -OPTIMAL CONSUMPTION AND INVESTMENT FOR STOCHASTIC VOLATILITY MARKETS WITH UNKNOWN PARAMETERS\*

B. BERDJANE<sup>†</sup> AND S. PERGAMENSHCHIKOV<sup>‡</sup>

**Abstract.** We consider an optimal investment and consumption problem for a Black–Scholes financial market with stochastic volatility and unknown stock price appreciation rate. The volatility parameter is driven by an external economic factor modeled as a diffusion process of Ornstein–Uhlenbeck type with unknown drift. We use the dynamical programming approach and find an optimal financial strategy which depends on the drift parameter. To estimate the drift coefficient we observe the economic factor  $Y$  in an interval  $[0, T_0]$  for fixed  $T_0 > 0$ , and use sequential estimation. We show that the consumption and investment strategy calculated through this sequential procedure is  $\delta$ -optimal.

**Key words.** sequential analysis, truncate sequential estimate, Black–Scholes market model, stochastic volatility, optimal consumption and investment, Hamilton–Jacobi–Bellman equation

DOI. 10.1137/S0040585X97986588

**1. Introduction.** We deal with the finite-time optimal consumption and investment problem in a Black–Scholes financial market with stochastic volatility (see, e.g., [7]). We consider the same power utility function for both consumption and terminal wealth. The volatility parameter in our situation depends on some economic factor, modeled as a diffusion process of Ornstein–Uhlenbeck type. The classical approach to this problem goes back to Merton [22].

By applying results from the stochastic control, explicit solutions have been obtained for financial markets with nonrandom coefficients (see, e.g., [13], [15], [29], [26], and references therein). Since then, the consumption and investment problems has been extended in many directions [27]. One of the important generalizations considers financial models with stochastic volatility, since empirical studies of stock-price returns show that the estimated volatility exhibits random characteristics (see, e.g., [28] and [10]).

The pure investment problem for such models is considered in [31] and [25]. In these papers, authors use the dynamic programming approach and show that the nonlinear HJB (Hamilton–Jacobi–Bellman) equation can be transformed into a quasi-linear PDE. The similar approach has been used in [16] for optimal consumption–investment problems with the default risk for financial markets with nonrandom coefficients. Furthermore, in [5], by making use of the Girsanov measure transformation

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\*Received by the editors March 2, 2015; revised June 15, 2015. This work was supported by the grant by the Government of Russian Federation grant No 14.A.12.31.0007, by the Russian Science Foundation project No 14-49-00079, National Research University “MPEI” (Moscow, Russia), by the International Laboratory of Statistics of Stochastic Processes and Quantitative Finance by Russian National Research Tomsk State. Originally published in the Russian journal *Teoriya Veroyatnostei i ee Primeneniya*, 60 (2015), No. 4, pp. 628–659.

<http://www.siam.org/journals/tvp/60-4/T98757.html>

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the authors study a pure optimal consumption problem for stochastic volatility markets. In [2] and [9] the authors use dual methods.

Usually, the classical existence and uniqueness theorem for the HJB equation is shown by the linear PDE methods (see, for example, chapter VI.6 and appendix E in [6]). In this paper we use the approach proposed in [4] and used in [1]. The difference between our work and these two papers is that, in [4], authors consider a pure jump process as the driven economic factor. The HJB equation in this case is an integro-differential equation of the first order. In our case it is a highly non linear PDE of the second order. In [1] the same problem is considered where the market coefficients are known, and depend on a diffusion process with bounded parameters. The result therein does not allow the Gaussian Ornstein–Uhlenbeck process. Similarly to [4] and [1] we study the HJB equation through the Feynman–Kac representation. We introduce a special metric space in which the Feynman–Kac mapping is a contraction. Taking this into account we show the fixed-point theorem for this mapping and we show that the fixed-point solution is the classical unique solution for the HJB equation in our case.

In the second part of our paper, we consider both the stock price appreciation rate and the drift of the economic factor to be unknown. To estimate the drift of a process of Ornstein–Uhlenbeck type we require sequential analysis methods (see [23] and [19, sections 17.5-6]). The drift parameter will be estimated from the observations of the process  $Y$ , in some interval  $[0, T_0]$ . It should be noted that in this case the usual likelihood estimator for the drift parameter is a nonlinear function of observations and it is not possible to calculate directly a nonasymptotic upper bound for its accuracy. To overcome this difficulty we use the truncated sequential estimate from [14] which enables us a nonasymptotic upper bound for mean accuracy estimation. After that we deal with the optimal strategy in the interval  $[T_0, T]$ , under the estimated parameter. We show that the expected absolute deviation of the objective function for the given strategy is less than some known fixed level  $\delta$ , i.e., the strategy calculated through the sequential procedure is  $\delta$ -optimal. Moreover, in this paper we find the explicit form for this level. This allows to keep small the deviation of the objective function from the optimal one by controlling the initial endowment.

The paper is organized as follows: in sections 2, 3 we introduce the market model, state the optimization problem and give the related HJB equation. Section 4 is set for definitions. The solution of the optimal consumption and investment problem is given in sections 5–7. In section 8 we consider unknown the drift parameter  $\alpha$  for the economic factor  $Y$  and use a truncated sequential method to construct its estimate  $\hat{\alpha}$ . We obtain an explicit upper for the deviation  $\mathbf{E}|\hat{\alpha} - \alpha|$  for any fixed  $T_0 > 0$ . Moreover, considering the optimal consumption investment problem in the finite interval  $[T_0, T]$ , we show that the strategy calculated through this truncation procedure is  $\delta$ -optimal. Similar results are given in section 8.3 when, in addition of using  $\hat{\alpha}$ , we consider an estimate  $\hat{\mu}$  of the unknown stock price appreciation rate. A numerical example is given in section 9 and auxiliary results are reported into the appendix.

**2. Market model.** Let  $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$  be a standard and filtered probability space with two standard independent  $(\mathcal{F}_t)_{0 \leq t \leq T}$  adapted Wiener processes  $(W_t)_{0 \leq t \leq T}$  and  $(U_t)_{0 \leq t \leq T}$  taking their values in  $\mathbf{R}$ . Our financial market consists of one *riskless* money market account  $S_0 = (S_0(t))_{0 \leq t \leq T}$  and one *risky stock*

$S = (S(t))_{0 \leq t \leq T}$  governed by the following equations:

$$(2.1) \quad \begin{aligned} dS_0(t) &= r S_0(t) dt, \\ dS(t) &= S(t)\mu dt + S(t)\sigma(Y_t) dW_t, \end{aligned}$$

with  $S_0(0) = 1$  and  $S(0) = s > 0$ . In this model  $r \in \mathbf{R}_+$  is the *riskless bond interest rate*,  $\mu$  is the *stock price appreciation rate* and  $\sigma(y)$  is *stock-volatility*. For all  $y \in \mathbf{R}$  the coefficient  $\sigma(y) \in \mathbf{R}_+$  is a nonrandom continuous bounded function and satisfies

$$\inf_{y \in \mathbf{R}} \sigma(y) = \sigma_1 > 0.$$

We assume also that  $\sigma(y)$  is differentiable and has bounded derivative. Moreover, we assume that the stochastic factor  $Y$  valued in  $\mathbf{R}$  is of Ornstein–Uhlenbeck type. It has a dynamics governed by the following stochastic differential equation:

$$(2.2) \quad dY_t = \alpha Y_t dt + \beta dU_t,$$

where the initial value  $Y_0$  is a nonrandom constant,  $\alpha < 0$  and  $\beta > 0$  are fixed parameters. We denote by  $(Y_s^{t,y})_{s \geq t}$  the process  $Y$  starts at  $Y_t = y$ , i.e.,

$$Y_s^{t,y} = ye^{\alpha(s-t)} + \int_t^s \beta e^{\alpha(s-v)} dU_v.$$

In this paper we consider the optimization problem on the time interval  $[T_0, T]$ , where  $0 \leq T_0 < T$  are fixed time limits. Let  $X_t$  be the investor capital at a time  $t \in [T_0, T]$ . We denote by  $\varphi_t \in \mathbf{R}$  the fraction of the capital invested in stocks ( $S$ ) and  $1 - \varphi_t$  is the share of the capital invested in the risk-free asset ( $S_0$ ).

The strategy of the investor at the time  $t \in [T_0, T]$  consists, firstly, in the choice of the proportion  $\varphi_t$ , and secondly, in the choice consumption rate  $\varsigma_t$  ( $\varsigma_t \geq 0$ ). Then, according to the model (2.1) the capital evolution is given by the following equation:

$$(2.3) \quad dX_t = \mu \varphi_t X_t dt + \sigma(Y_t) \varphi_t X_t dW_t + r(1 - \varphi_t) X_t dt - \varsigma_t dt, \quad T_0 \leq t \leq T,$$

where  $X_{T_0} = x > 0$  is the initial endowment. Note, that for the model (2.1) the risk premium is the  $\mathbf{R} \rightarrow \mathbf{R}$  function defined as

$$(2.4) \quad \theta(y) = \frac{\mu - r}{\sigma(y)}.$$

If instead of the pair  $(\varphi_t, \varsigma_t)$  one considers the strategy  $(\pi_t, c_t)$ , where  $\pi_t = \sigma(Y_t)\varphi_t$  and  $c_t = \varsigma_t/X_t$ , then the wealth process satisfies the following stochastic differential equation

$$(2.5) \quad dX_t = X_t(r + \pi_t\theta(Y_t) - c_t) dt + X_t\pi_t dW_t, \quad X_{T_0} = x.$$

Now we describe the set of all admissible strategies. A portfolio control (financial strategy)  $\vartheta = (\vartheta_t)_{T_0 \leq t \leq T} = ((\pi_t, c_t))_{T_0 \leq t \leq T}$  is said to be *admissible* if it is  $(\mathcal{F}_t)_{T_0 \leq t \leq T}$  progressively measurable with values in  $\mathbf{R} \times [0, \infty)$ , such that the equation (2.5) has a unique strong a.s. positive continuous solution  $(X_t^\vartheta)_{T_0 \leq t \leq T}$  on  $[0, T]$ . We denote the set of *admissible portfolios controls* by  $\mathcal{V}$ .

In this paper we consider the power utility functions  $x^\gamma$  for  $0 < \gamma < 1$  for the consumption and for the terminal wealth. The goal is to maximize the expected

utilities from the consumption on the time interval  $[T_0, T]$ , for fixed  $T_0$ , and from the terminal wealth at maturity  $T$ . Then for any  $x, y \in \mathbf{R}$ , and  $\vartheta \in \mathcal{V}$  the value function is defined by

$$J(T_0, x, y, \vartheta) := \mathbf{E}_{T_0, x, y} \left( \int_{T_0}^T c_t^\gamma (X_t^\vartheta)^\gamma dt + (X_T^\vartheta)^\gamma \right),$$

where  $\mathbf{E}_{T_0, x, y}$  is the conditional expectation  $\mathbf{E}(\cdot | X_{T_0} = x, Y_{T_0} = y)$ . Our goal is to maximize this function, i.e., to calculate

$$(2.6) \quad J(T_0, x, y, \vartheta^*) = \sup_{\vartheta \in \mathcal{V}} J(T_0, x, y, \vartheta).$$

For the sequel we will use the notation  $J^*(T_0, x, y)$  or simply  $J_{T_0}^*$  instead of  $J(T_0, x, y, \vartheta^*)$ .  $\blacksquare$

In the case when the parameters  $\alpha$  and  $\mu$  are unknown we assume that  $\alpha_2 \leq \alpha \leq \alpha_1$  and  $|\mu| \leq \mu_*$ , where  $\alpha_2 < \alpha_1 < 0$  and  $\mu_* > 0$  are known fixed constants.

Our goal is to find a strategy for approaching in a some sense, to the optimal, i.e., we will seek  $\delta$ -optimal strategies in the sense of the following definition.

**DEFINITION 2.1.** *A strategy  $\vartheta \in \mathcal{V}$  is  $\delta$  optimal if*

$$\sup_{\alpha_2 \leq \alpha \leq \alpha_1} \sup_{|\mu| \leq \mu_*} \mathbf{E} \left( J(T_0, x, Y_{T_0}, \tilde{\vartheta}) - J^*(T_0, x, Y_{T_0}) \right) \leq \delta.$$

*Remark 2.1.* Note that if the parameters  $\alpha$  and  $\mu$  are known, then we consider the problem (2.6) with  $T_0 = 0$ . It should be note also that for known theses parameters the same problem as (2.6) is solved in [1], but the economic factor  $Y$  considered there is a general diffusion process with bounded coefficients. In the present paper  $Y$  is an Ornstein–Uhlenbeck process, so the drift is not bounded, but we take advantage of the fact that  $Y$  is Gaussian and not correlated to the market, which is not the case in [1].

**3. Hamilton–Jacobi–Bellman equation.** Now we introduce the HJB equation for the problem (2.6). To this end, for any two times differentiable  $[0, T] \times \mathbf{R}_+ \times \mathbf{R} \rightarrow \mathbf{R}$  function  $f$  we denote by  $Df(t, x, y)$  and  $D^2f(t, x, y)$  the following vectors of the partial derivatives:

$$Df(t, x, y) = \left( \frac{\partial}{\partial x} f(t, x, y), \frac{\partial}{\partial y} f(t, x, y) \right)',$$

$$D^2f(t, x, y) = \left( \frac{\partial^2}{\partial x^2} f(t, x, y), \frac{\partial^2}{\partial y^2} f(t, x, y) \right)'$$

Here the prime denotes the transposition. Let now  $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2$ ,  $\mathbf{M} = (M_1, M_2) \in \mathbf{R}^2$  and  $\nu = (\nu_1, \nu_2) \in \mathbf{R} \times \mathbf{R}_+$  be fixed parameters. For these parameters we set

$$H_0(x, y, \mathbf{q}, \mathbf{M}, \nu) := (r + \nu_1 \theta(y) - \nu_2) x q_1 + \alpha y q_2 + \frac{1}{2} M_1 \nu_1^2 x^2 + \frac{\beta^2}{2} M_2 + (\nu_2 x)^\gamma.$$

In this case the HJB equation has the following form:

$$(3.1) \quad \frac{\partial}{\partial t} z(t, x, y) + \sup_{\nu \in \mathbf{R} \times \mathbf{R}_+} H_0(x, y, Dz(t, x, y), D^2z(t, x, y), \nu) = 0,$$

$$z(T, x, y) = x^\gamma.$$

Note, that for  $x > 0$ ,  $\mathbf{q}_1 > 0$ , and  $\mathbf{M}_1 < 0$

$$\sup_{\nu \in \mathbf{R} \times \mathbf{R}_+} H_0(x, y, \mathbf{q}, \mathbf{M}, \nu) = xrq_1 + \alpha yq_2 + \frac{1}{q_*} \left( \frac{\gamma}{q_1} \right)^{q_*-1} + \frac{|\theta(y)q_1|^2}{2|M_1|} + \frac{\beta^2}{2} M_2,$$

where  $q_* = (1 - \gamma)^{-1}$ . To study this equation we represent  $z(t, x, y)$  as

$$(3.2) \quad z(t, x, y) = x^\gamma h(t, y).$$

It is easy to deduce that the function  $h$  satisfies the following quasi-linear PDE:

$$(3.3) \quad \begin{aligned} & \frac{\partial}{\partial t} h(t, y) + Q(y)h(t, y) + \alpha y \frac{\partial}{\partial y} h(t, y) \\ & + \frac{\beta^2}{2} \frac{\partial^2}{\partial y^2} h(t, y) + \frac{1}{q_*} \left( \frac{1}{h(t, y)} \right)^{q_*-1} = 0, \\ & h(T, y) = 1, \end{aligned}$$

where

$$(3.4) \quad Q(y) = \gamma \left( r + \frac{\theta^2(y)}{2(1-\gamma)} \right).$$

Note that, using the conditions on  $\sigma(y)$ , the function  $Q(y)$  is bounded differentiable and has bounded derivative. Therefore, we can set

$$(3.5) \quad Q_* = \sup_{y \in \mathbf{R}} Q(y) \quad \text{and} \quad Q_1^* = \sup_{y \in \mathbf{R}} \left| \frac{dQ(y)}{dy} \right|.$$

Our goal is to study equation (3.3). By making use of the probabilistic representation for the linear PDE (the Feynman–Kac formula) we show in Proposition 5.4, that the solution of this equation is the fixed-point solution for a special mapping of the integral type which will be introduced in the following section.

**4. Useful definitions.** First, to study equation (3.3) we introduce a special functional space. Let  $\mathcal{X}$  be the set of uniformly continuous functions on  $\mathcal{K} := [T_0, T] \times \mathbf{R}$  with values in  $[1, \infty)$  such that

$$(4.1) \quad \|f\|_\infty = \sup_{(t,y) \in \mathcal{K}} |f(t, y)| \leq \mathbf{r}^*,$$

where  $\mathbf{r}^* = (\tilde{T} + 1) e^{Q_* \tilde{T}}$  and  $\tilde{T} = T - T_0$ . Now, we define a metric  $\varrho_*(\cdot, \cdot)$  in  $\mathcal{X}$  as follows: for any  $f, g$  in  $\mathcal{X}$

$$(4.2) \quad \varrho_*(f, g) = \|f - g\|_* = \sup_{(t,y) \in \mathcal{K}} e^{-\varkappa(T-t)} |f(t, y) - g(t, y)|.$$

Here  $\varkappa = Q_* + \zeta + 1$  and  $\zeta$  is some positive parameter which will be specified later. We define now the process  $\eta$  by its dynamics

$$(4.3) \quad d\eta_s = \alpha \eta_s ds + \beta d\tilde{\mathbf{U}}_s, \quad \eta_0 = Y_0,$$

So,  $(\eta_t)_{t \geq 0}$  has the same distribution as  $(Y_t)_{t \geq 0}$ . Here  $(\tilde{\mathbf{U}}_t)_{t \geq 0}$  is a standard Brownian motion independent of  $(\mathbf{U}_t)_{t \geq 0}$ . Let us now define the  $\mathcal{X} \rightarrow \mathcal{X}$  Feynman–Kac mapping  $\mathcal{L}$ :

$$(4.4) \quad \mathcal{L}_f(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathcal{H}_f(t, s, y) ds,$$

where  $\mathcal{G}(t, s, y) = \exp\{\int_t^s Q(\eta_u^{t,y}) du\}$  and

$$(4.5) \quad \mathcal{H}_f(t, s, y) = \mathbf{E} \left[ (f(s, \eta_s^{t,y}))^{1-q_*} \mathcal{G}(t, s, y) \right]$$

and  $(\eta_s^{t,y})_{t \leq s \leq T}$  is the process  $\eta$  starting at  $\eta_t = y$ . To solve the HJB equation we need to find the fixed-point solution for the mapping  $\mathcal{L}$  in  $\mathcal{X}$ , i.e.,

$$(4.6) \quad h = \mathcal{L}h.$$

To this end we construct the following iterated scheme. We set  $h_0 \equiv 1$ ,

$$(4.7) \quad h_n(t, y) = \mathcal{L}h_{n-1}(t, y), \quad n \geq 1,$$

and study the convergence of this sequence in  $\mathcal{K}$ . Actually, we will use the existence argument of a fixed point, for a contracting operator in a complete metric space.

**5. Solution of the HJB equation.** We give in this section the existence and uniqueness result, of a solution for the HJB equation (3.3). For this, we show some properties of the Feynman–Kac operator  $\mathcal{L}$ .

PROPOSITION 5.1. *The operator  $\mathcal{L}$  is “stable” in  $\mathcal{X}$  that is*

$$\mathcal{L}f \in \mathcal{X} \quad \forall f \in \mathcal{X}.$$

*Proof.* Obviously, that for any  $f \in \mathcal{X}$  we have  $\mathcal{L}f \geq 1$ . Moreover, setting

$$(5.1) \quad \tilde{f}_s = f(s, \eta_s^{t,y}),$$

we represent  $\mathcal{L}f(t, y)$  as

$$(5.2) \quad \mathcal{L}f(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathbf{E} \left[ (\tilde{f}_s)^{1-q_*} \mathcal{G}(t, s, y) \right] ds.$$

Therefore, taking into account that  $\tilde{f}_s \geq 1$  and  $q_* \geq 1$  we get

$$(5.3) \quad \mathcal{L}f(t, y) \leq e^{Q_*(T-t)} + \int_t^T \frac{1}{q_*} e^{Q_*(s-t)} ds \leq \mathbf{r}^*,$$

where the upper bound  $\mathbf{r}^*$  is defined in (4.1). Now we have to show that  $\mathcal{L}f$  is a uniformly continuous function on  $\mathcal{K}$  for any  $f \in \mathcal{X}$ . For any  $f \in \mathcal{X} \cap C^{1,1}(\mathcal{K})$  we consider equation (3.3), i.e.,

$$(5.4) \quad \begin{aligned} & \frac{\partial}{\partial t} u(t, y) + Q(y)u(t, y) + \alpha y \frac{\partial}{\partial y} u(t, y) \\ & + \frac{\beta^2}{2} \frac{\partial^2}{\partial y^2} u(t, y) + \frac{1}{q_*} \left( \frac{1}{f(t, y)} \right)^{q_*-1} = 0, \\ & u(T, y) = 1. \end{aligned}$$

Setting here  $\tilde{u}(t, y) = u(T_0 + T - t, y)$  we obtain a uniformly parabolic equation for  $\tilde{u}$  with initial condition  $\tilde{u}(T_0, y) = 1$ . Moreover, we know that  $Q$  has bounded derivative. We deduce that for any  $f$  from  $\mathcal{X} \cap C^{1,1}(\mathcal{K})$ , Theorem 5.1 from [17, p. 320] with

$0 < l < 1$  provides the existence of a unique solution of (5.4) belonging to  $\mathbf{C}^{1,2}(\mathcal{K})$ . Applying the Itô formula to the process

$$\left( u(s, \eta_s^{t,y}) \exp \left\{ \int_t^s Q(\eta_v^{t,y}) dv \right\} \right)_{t \leq s \leq T}$$

and taking into account equation (5.4) we get

$$(5.5) \quad u(t, y) = \mathcal{L}_f(t, y).$$

Therefore, the function  $\mathcal{L}_f(t, y) \in \mathbf{C}^{1,2}(\mathcal{K})$ , i.e.,  $\mathcal{L}_f \in \mathcal{X}$  for any  $f \in \mathcal{X} \cap C^{1,1}(\mathcal{K})$ .

Moreover, for any  $f \in \mathcal{X}$  there exists a sequence  $(f_n)_{n \geq 1}$  from  $\mathcal{X} \cap C^{1,1}(\mathcal{K})$  such that

$$\sup_{(t,y) \in \mathcal{K}} |f_n(t, y) - f(t, y)| \rightarrow 0, \quad n \rightarrow \infty.$$

This implies

$$\sup_{(t,y) \in \mathcal{K}} |\mathcal{L}_{f_n}(t, y) - \mathcal{L}_f(t, y)| \rightarrow 0, \quad n \rightarrow \infty.$$

So  $\mathcal{L}_f(t, y)$  is uniformly continuous on  $\mathcal{K}$ , i.e.,  $\mathcal{L}_f \in \mathcal{X}$ . Proposition 5.1 is proved.

PROPOSITION 5.2. *The mapping  $\mathcal{L}$  is a contraction in the metric space  $(\mathcal{X}, \varrho_*)$ , i.e., for any  $f, g$  from  $\mathcal{X}$*

$$(5.6) \quad \varrho_*(\mathcal{L}_f, \mathcal{L}_g) \leq \lambda \varrho_*(f, g),$$

where

$$(5.7) \quad \lambda = \frac{1}{\zeta + 1}, \quad \zeta > 0.$$

Actually, as shown in Corollary 6.1, an appropriate choice of  $\zeta$  gives a supergeometric convergence rate for the sequence  $(h_n)_{n \geq 1}$  defined in (4.7), to the limit function  $h(t, y)$ , which is the fixed point of the operator  $\mathcal{L}$ .

*Proof.* First note, that for any  $a > b \geq 1$  and  $q > 0$

$$b^{-q} - a^{-q} \leq q(b - a).$$

Using this bound one can obtain that for any  $f$  and  $g$  from  $\mathcal{X}$  and for any  $y \in \mathbf{R}$

$$\begin{aligned} |\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)| &\leq \frac{1}{q_*} \mathbf{E} \int_t^T \mathcal{G}(t, s, y) \left| (\tilde{f}_s)^{1-q_*} - (\tilde{g}_s)^{1-q_*} \right| ds \\ &\leq \gamma \mathbf{E} \int_t^T \mathcal{G}(t, s, y) |\tilde{f}_s - \tilde{g}_s| ds. \end{aligned}$$

We recall that  $\tilde{f}_s = f(s, \eta_s^{t,y})$  and  $\tilde{g}_s = g(s, \eta_s^{t,y})$ . Taking into account here that  $\mathcal{G}(t, s, y) \leq e^{Q_*(s-t)}$  we obtain

$$|\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)| \leq \int_t^T e^{Q_*(s-t)} \mathbf{E} |\tilde{f}_s - \tilde{g}_s| ds.$$

Taking into account in the last inequality, that

$$(5.8) \quad |\tilde{f}_s - \tilde{g}_s| \leq e^{\varkappa(T-s)} \varrho_*(f, g) \quad \text{a.s.}$$

we get for all  $(t, y)$  in  $\mathcal{K}$

$$(5.9) \quad \left| e^{-\varkappa(T-t)} (\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)) \right| \leq \frac{1}{\varkappa - Q_*} \varrho_*(f, g).$$

Taking into account the definition of  $\varkappa$  in (4.2), we obtain inequality (5.6). Hence Proposition 5.2 is proved.

PROPOSITION 5.3. *The fixed point equation  $\mathcal{L}_h = h$  has a unique solution in  $\mathcal{X}$ .*

*Proof.* Indeed, using the contraction of the operator  $\mathcal{L}$  in  $\mathcal{X}$  and the definition of the sequence  $(h_n)_{n \geq 1}$  in (4.7) we get, that for any  $n \geq 1$

$$(5.10) \quad \varrho_*(h_n, h_{n-1}) \leq \lambda^{n-1} \varrho_*(h_1, h_0),$$

i.e., the sequence  $(h_n)_{n \geq 1}$  is fundamental in  $(\mathcal{X}, \varrho_*)$ . The metric space  $(\mathcal{X}, \varrho_*)$  is complete since it is included in the Banach space  $C^{0,0}(\mathcal{K})$ , and  $\|\cdot\|_\infty$  is equivalent to  $\|\cdot\|_*$  defined in (4.2). Therefore, this sequence has a limit in  $\mathcal{X}$ , i.e., there exists a function  $h$  from  $\mathcal{X}$  for which

$$\lim_{n \rightarrow \infty} \varrho_*(h, h_n) = 0.$$

Moreover, taking into account that  $h_n = \mathcal{L}_{h_{n-1}}$  we obtain, that for any  $n \geq 1$

$$\varrho_*(h, \mathcal{L}_h) \leq \varrho_*(h, h_n) + \varrho_*(\mathcal{L}_{h_{n-1}}, \mathcal{L}_h) \leq \varrho_*(h, h_n) + \lambda \varrho_*(h, h_{n-1}).$$

The last expression tends to zero as  $n \rightarrow \infty$ . Therefore,  $\varrho_*(h, \mathcal{L}_h) = 0$ , i.e.,  $h = \mathcal{L}_h$ . Proposition 5.2 implies immediately that this solution is unique.

We are ready to state the result about the solution of the HJB equation.

PROPOSITION 5.4. *The HJB equation (3.3) has a unique solution which is the solution  $h$  of the fixed-point problem  $\mathcal{L}_h = h$ .*

*Proof.* First, note that in view of Lemma 10.5, the function  $\mathcal{L}_h(t, y)$  is 1/2-Hölderian with respect to  $t$  on  $|y| < n$  for any  $n \geq 1$ . Therefore, choosing in (5.4) the function  $f = f_n(t, y) = u(t, \tilde{y}_n)$  (where  $\tilde{y}_n$  is the projection of  $y$  into  $[-n, n]$ ) we obtain through Theorem 5.1 from [17, p. 320] and Lemma 10.5, that the equation (5.4) has a unique solution  $u_n(t, y)$ . It is clear that the function

$$u(t, y) = \sum_{n \geq 1} u_n(t, y) \mathbf{1}_{\{n-1 < |y| \leq n\}}$$

is the solution to equation (5.4) for  $f = u(t, y)$ . Taking into account the representation (5.5) and the fixed point equation  $\mathcal{L}_h = h$  we obtain, that the solution of equation (5.4) is

$$u = \mathcal{L}_h = h.$$

Therefore, the function  $h$  satisfies equation (3.3). Moreover, this solution is unique since  $h$  is a unique solution of the fixed point problem.

Choosing in (5.4) the function  $f = u$  and taking into account the representation (5.5) and the fixed point equation  $\mathcal{L}_h = h$  we obtain, that the solution of equation (5.4) is

$$u = \mathcal{L}_h = h.$$

Therefore, the function  $h$  satisfies equation (3.3). Moreover, this solution is unique since  $h$  is the unique solution of the fixed point problem. Proposition 5.4 is proved.

*Remark 5.1.* (1) We can find in [21] an existence and uniqueness proof for a more general quasilinear equation but therein, authors did not give a way to calculate this solution, whereas in our case, the solution is the fixed point function for the Feynman–Kac operator. Moreover our method allows to obtain the super geometric convergence rate for the sequence approximating the solution, which is a very important property in practice. In [3] author shows an existence and uniqueness result where the global result is deduced from a local existence and uniqueness theorem.

The application of contraction mapping or fixed-point theorem to solve nonlinear PDE is not new see, e.g., [8] and [24] where the term “generalised solution” is used for quasilinear/semilinear PDE, and the fixed point of the Feynman–Kac representation is discussed.

**6. Supergeometric convergence rate.** For the sequence  $(h_n)_{n \geq 1}$  defined in (4.7), and  $h$  the fixed point solution for  $h = \mathcal{L}_h$ , we study the behavior of the deviation

$$\Delta_n(t, y) = h(t, y) - h_n(t, y).$$

In the following theorem we make an appropriate choice of  $\zeta$  for the contraction parameter  $\lambda$  to get the super-geometric convergence rate for the sequence  $(h_n)_{n \geq 1}$ .

**THEOREM 6.1.** *The fixed point problem  $\mathcal{L}_h = h$  admits a unique solution  $h$  in  $\mathcal{X}$  such that for any  $n \geq 1$  and  $\zeta > 0$*

$$(6.1) \quad \sup_{(t,y) \in \mathcal{K}} |\Delta_n(t, y)| \leq \mathbf{B}^* \lambda^n,$$

where  $\mathbf{B}^* = e^{\varkappa \tilde{T}} (1 + \mathbf{r}^*) / (1 - \lambda)$  and  $\varkappa$  is given in (4.2).

*Proof.* Proposition 5.3 implies the first part of this theorem. Moreover, from (5.10) it is easy to see, that for each  $n \geq 1$

$$\varrho_*(h, h_n) \leq \frac{\lambda^n}{1 - \lambda} \varrho_*(h_1, h_0).$$

Thanks to Proposition 5.1 all the functions  $h_n$  belong to  $\mathcal{X}$ , i.e., by the definition of the space  $\mathcal{X}$

$$\varrho_*(h_1, h_0) \leq \sup_{(t,y) \in \mathcal{K}} |h_1(t, y) - 1| \leq 1 + \mathbf{r}^*.$$

Taking into account that

$$\sup_{(t,y) \in \mathcal{K}} |\Delta_n(t, y)| \leq e^{\varkappa \tilde{T}} \varrho_*(h, h_n),$$

we obtain the inequality (6.1). Hence Theorem 6.1 is proved.

Now we can minimize the upper bound (6.1) over  $\zeta > 0$ . Indeed,

$$\mathbf{B}^* \lambda^n = \mathbf{C}^* \exp\{\mathbf{g}_n(\zeta)\},$$

where  $\mathbf{C}^* = (1 + \mathbf{r}^*) e^{(Q_* + 1)\tilde{T}}$  and

$$\mathbf{g}_n(x) = x\tilde{T} - \log x - (n - 1) \log(1 + x).$$

Now we minimize this function over  $x > 0$ , i.e.,

$$\min_{x>0} \mathbf{g}_n(x) = x_n^* \tilde{T} - \log x_n^* - (n-1) \log(1+x_n^*),$$

where

$$x_n^* = \frac{\sqrt{(\tilde{T}-n)^2 + 4\tilde{T}} + n - \tilde{T}}{2\tilde{T}}.$$

Therefore, for  $\zeta = \zeta_n^* = x_n^*$  we obtain the optimal upper bound (6.1).

**COROLLARY 6.1.** *The fixed point problem has a unique solution  $h$  in  $\mathcal{X}$  such that for any  $n \geq 1$*

$$(6.2) \quad \sup_{(t,y) \in \mathcal{K}} |\Delta_n(t,y)| \leq \mathbf{U}_n^*,$$

where  $\mathbf{U}_n^* = \mathbf{C}^* \exp\{\mathbf{g}_n^*\}$ . Moreover, one can check directly that for any  $0 < \delta < 1$

$$\mathbf{U}_n^* = O(n^{-\delta n}), \quad n \rightarrow \infty.$$

This means that the convergence rate is more rapid than any geometric one, i.e., it is supergeometric.

**7. Known parameters.** We consider our optimal consumption and investment problem in the case of markets with known parameters. The following theorem is the analogous of Theorem 3.4 in [1]. The main difference between the two results is that the drift coefficient of the process  $Y$  in [1] must be bounded and so does not allow the Ornstein–Uhlenbeck process. Moreover, the economic factor  $Y$  is correlated to the market by the Brownian motion  $\mathbf{U}$ , which is not the case in the present paper, since we consider the process  $\mathbf{U}$  independent of  $W$ .

**THEOREM 7.1.** *The optimal value of  $J(T_0, x, y, \vartheta)$  for the optimization problem (2.6) is given by*

$$J_{T_0}^* = J(T_0, x, y, \vartheta^*) = \sup_{\vartheta \in \mathcal{V}} J(T_0, x, y, \vartheta) = x^\gamma h(T_0, y),$$

where  $h(t, y)$  is a unique solution of equation (3.3). Moreover, for all  $T_0 \leq t \leq T$  an optimal financial strategy  $\vartheta^* = (\pi^*, c^*)$  is of the form

$$(7.1) \quad \pi_t^* = \pi^*(Y_t) = \frac{\theta(Y_t)}{1-\gamma}, \quad c_t^* = c^*(t, Y_t) = (h(t, Y_t))^{-q^*}.$$

The optimal wealth process  $(X_t^*)_{T_0 \leq t \leq T}$  satisfies the following stochastic equation:

$$(7.2) \quad dX_t^* = \mathbf{a}^*(t, Y_t) X_t^* dt + X_t^* \mathbf{b}^*(Y_t) dW_t, \quad X_{T_0}^* = x,$$

where

$$(7.3) \quad \mathbf{a}^*(t, y) = \frac{|\theta(y)|^2}{1-\gamma} + r - (h(t, y))^{-q^*}, \quad \mathbf{b}^*(y) = \frac{\theta(y)}{1-\gamma}.$$

The solution  $X_t^*$  can be written as

$$(7.4) \quad X_s^* = X_t^* \exp \left\{ \int_t^s \mathbf{a}^*(v, Y_v) dv \right\} \mathcal{E}_{t,s},$$

where  $\mathcal{E}_{t,s} = \exp\{\int_t^s \mathbf{b}^*(Y_v) dW_v - (1/2) \int_t^s |\mathbf{b}^*(Y_v)|^2 dv\}$ .

The proof of the theorem follows the same arguments, as Theorem 3.4 in [1], so it is omitted.

**8. Unknown parameters.** In this section we consider the Black–Scholes market with unknown stock price appreciation rate  $\mu$  and the unknown drift parameter  $\alpha$  of the economic factor  $Y$ . We observe the process  $Y$  in the interval  $[0, T_0]$  (in this case  $T_0 > 0$ ), and use sequential methods to estimate the drift. After that, we will deal with the consumption–investment optimization problem on the finite interval  $[T_0, T]$  and look for the behavior of the optimal value function  $J^*(T_0, x, y)$  under the estimated parameters. We define the value function  $\widehat{J}_{T_0}^*$  the estimate of  $J_{T_0}^*$

$$(8.1) \quad \widehat{J}_{T_0}^* := \mathbf{E}_{T_0} \left( \int_{T_0}^T (\widehat{c}_t^*)^\gamma (\widehat{X}_t^*)^\gamma dt + (\widehat{X}_T^*)^\gamma \right),$$

$\mathbf{E}_{T_0}$  is the conditional expectation  $\mathbf{E}(\cdot | \mathcal{F}_{T_0})$  is a simplified notation for  $X_t^{\widehat{\theta}^*}$  and from (7.4) we write

$$(8.2) \quad \widehat{X}_s^* = \widehat{X}_t^* \exp \left\{ \int_t^s \widehat{\mathbf{a}}^*(v, Y_v) dv \right\} \widehat{\mathcal{E}}_{t,s},$$

where  $\widehat{\mathcal{E}}_{t,s} = \exp\{\int_t^s \widehat{\mathbf{b}}^*(Y_v) dW_v - (1/2) \int_t^s |\widehat{\mathbf{b}}^*(Y_v)|^2 dv\}$ . The functions  $\widehat{\mathbf{a}}^*(t, y)$  and  $\widehat{\mathbf{b}}^*(t, y)$  are defined as

$$(8.3) \quad \widehat{\mathbf{a}}^*(t, y) = \frac{|\widehat{\theta}(y)|^2}{1-\gamma} + r - (\widehat{h}(t, y))^{-q_*}, \quad \widehat{\mathbf{b}}^*(y) = \frac{\widehat{\theta}(y)}{1-\gamma}, \quad \widehat{\theta}(y) = \frac{\widehat{\mu} - r}{\sigma(y)}.$$

The estimated consumption process is  $\widehat{c}_t^* = \widehat{c}^*(t, Y_t) = (\widehat{h}(t, Y_t))^{-q_*}$  and  $\widehat{h}(t, y)$  is a unique solution for  $h = \widehat{\mathcal{L}}_h$ . The operator  $\widehat{\mathcal{L}}$  is defined by

$$(8.4) \quad \widehat{\mathcal{L}}_f(t, y) = \mathbf{E} \widehat{\mathcal{G}}(t, T, y) + \frac{1}{q_*} \int_t^T \mathbf{E} \left( [f(s, \widehat{\eta}_s^{t,y})]^{1-q_*} \widehat{\mathcal{G}}(t, s, y) \right) ds,$$

where  $\widehat{\mathcal{G}}(t, s, y) = \exp\{\int_t^s \widehat{Q}(\widehat{\eta}_u^{t,y}) du\}$ . The process  $(\widehat{\eta}_s^{t,y})_{t \leq s \leq T}$  has the following dynamics:

$$(8.5) \quad d\widehat{\eta}_s^{t,y} = \widehat{\alpha} \widehat{\eta}_s^{t,y} ds + \beta d\widetilde{\mathbf{U}}_s, \quad \widehat{\eta}_t^{t,y} = y.$$

Here  $\widehat{\alpha}$  and  $\widehat{\mu}$  are some estimates for the parameters  $\alpha$  and  $\mu$  which will be specified later.

**8.1. Sequential procedure.** We assume the unknown parameter  $\alpha$  taking values in some bounded interval  $[\alpha_2, \alpha_1]$ , with  $\alpha_2 < \alpha < \alpha_1 < 0$ . We define  $\widehat{\alpha}$  as the projection onto the interval  $[\alpha_2, \alpha_1]$  of the sequential estimate  $\alpha^*$ :

$$(8.6) \quad \widehat{\alpha} = \text{Proj}_{[\alpha_2, \alpha_1]} \alpha^*, \quad \alpha^* = \frac{\int_0^{\tau_H} Y_t dY_t}{H} \mathbf{1}_{\{\tau_H \leq T_0\}},$$

where  $\tau_H = \inf\{t \geq 0: \int_0^t Y_s^2 ds \geq H\}$  and for all  $x \in \mathbf{R}$

$$\text{Proj}_{[\alpha_2, \alpha_1]}(x) = \alpha_2 \mathbf{1}_{\{x \leq \alpha_2\}} + x \mathbf{1}_{\{\alpha_2 \leq x \leq \alpha_1\}} + \alpha_1 \mathbf{1}_{\{x \geq \alpha_1\}}.$$

Furthermore, we introduce the function  $\epsilon(\cdot)$ , which will serve later for the  $\delta$ -optimality: ■

$$(8.7) \quad \epsilon(T_0) = \sqrt{\frac{\beta^2}{\beta_2(T_0 - T_0^{5/6})} + \frac{\alpha_2^2}{\beta^{12}} \left( \frac{\mathbf{k}(3)}{T_0^2} \right)},$$

Here  $H = \beta_2(T_0 - T_0^\varepsilon)$ ,  $\beta_2 = \beta^2/2|\alpha_2|$ ,  $\varepsilon = 5/6$  and

$$\mathbf{k}_1(m) = 2^{2m-1} (Y_0^{2m} + (2m-1)!! \beta_1^m) \quad \text{and} \quad \beta_1 = \frac{\beta^2}{2|\alpha_1|},$$

with  $\mathbf{k}_1(m) = 2^{2m-1} (Y_0^{2m} + (2m-1)!! \beta_1^m)$  and  $\beta_1 = \beta^2/2|\alpha_1|$ . The proposition below gives  $\hat{\alpha}$  the truncated sequential estimate of  $\alpha$  and gives a bound for the expected deviation  $\mathbf{E}|\bar{\alpha}|$ , where  $\bar{\alpha} = \hat{\alpha} - \alpha$ .

PROPOSITION 8.1. *For any  $0 < T_0 < T$*

$$\sup_{\alpha_2 \leq \alpha \leq \alpha_1} \mathbf{E}|\bar{\alpha}| \leq \epsilon(T_0).$$

*Proof.* Note first that  $\mathbf{E}|\bar{\alpha}| \leq \mathbf{E}|\alpha^* - \alpha|$ , so it is enough to show that  $\mathbf{E}|\alpha^* - \alpha| \leq \epsilon(T_0)$ . Moreover, we know from [19, Chap. 17], that the maximum likelihood estimate of  $\alpha$  is given by

$$\frac{\int_0^{T_0} Y_t dY_t}{\int_0^{T_0} Y_t^2 dt}.$$

To estimate  $\alpha$  we use the sequential maximum likelihood estimate proposed in [19] and [23]

$$\tilde{\alpha} = \frac{\int_0^{\tau_H} Y_t dY_t}{\int_0^{\tau_H} Y_t^2 dt} = \alpha + \beta \frac{\int_0^{\tau_H} Y_t d\mathbf{U}_t}{H}.$$

Taking into account that  $\int_0^\infty Y_t^2 dt = +\infty$  a.s., we obtain that  $\tilde{\alpha} \sim \mathcal{N}(\alpha, \beta^2/H)$  and hence

$$\mathbf{E}|\tilde{\alpha} - \alpha|^2 = \frac{\beta^2}{H}.$$

The problem with the previous estimate is that  $\tau_H$  may be greater than  $T_0$ . To overcome this difficulty we use the truncated modification of the sequential estimate  $\hat{\alpha}$  from [14], i.e.,  $\alpha^* = \tilde{\alpha} \mathbf{1}_{\{\tau_H \leq T_0\}}$ . We observe that

$$\begin{aligned} \alpha^* - \alpha &= (\alpha^* - \alpha) \mathbf{1}_{\{\tau_H \leq T_0\}} + (\alpha^* - \alpha) \mathbf{1}_{\{\tau_H > T_0\}} \\ &= \beta \frac{\int_0^{\tau_H} Y_t d\mathbf{U}_t}{H} \mathbf{1}_{\{\tau_H \leq T_0\}} - \alpha \mathbf{1}_{\{\tau_H > T_0\}}. \end{aligned}$$

So

$$\begin{aligned} \mathbf{E}(\alpha^* - \alpha)^2 &= \frac{\beta^2}{H^2} \mathbf{E} \left( \int_0^{\tau_H} Y_t d\mathbf{U}_t \mathbf{1}_{\{\tau_H \leq T_0\}} \right)^2 + \alpha^2 \mathbf{P}(\tau_H > T_0) \\ &\leq \frac{\beta^2}{H^2} \mathbf{E} \left( \int_0^{\tau_H} Y_t d\mathbf{U}_t \right)^2 + \alpha^2 \mathbf{P}(\tau_H > T_0) \\ (8.8) \quad &\leq \frac{\beta^2}{H} + \alpha^2 \mathbf{P} \left( \int_0^{T_0} Y_t^2 dt < H \right). \end{aligned}$$

Moreover, by the Itô formula

$$dY_t^2 = 2Y_t dY_t + \beta^2 dt = (2\alpha Y_t^2 + \beta^2) dt + 2\beta Y_t d\mathbf{U}_t.$$

From there we deduce that

$$\int_0^{T_0} (2\alpha Y_t^2 + \beta^2) dt = Y_{T_0}^2 - Y_0^2 - 2\beta \int_0^{T_0} Y_t d\mathbf{U}_t.$$

Taking into account that  $\alpha_2 \leq \alpha \leq \alpha_1 < 0$  and using the Markov's inequality, we get for any integer  $m > 0$

$$\begin{aligned} \mathbf{P}\left(\int_0^{T_0} Y_t^2 dt < H\right) &= \mathbf{P}\left(\int_0^{T_0} (2\alpha Y_t^2 + \beta^2) dt > 2\alpha H + \beta^2 T_0\right) \\ &= \mathbf{P}\left(Y_{T_0}^2 - Y_0^2 - 2\beta \int_0^{T_0} Y_t d\mathbf{U}_t > 2\alpha H + \beta^2 T_0\right) \\ &\leq \frac{\mathbf{E}(Y_{T_0}^2 - Y_0^2 - 2\beta \int_0^{T_0} Y_t d\mathbf{U}_t)^{2m}}{(2\alpha_2 H + \beta^2 T_0)^{2m}}. \end{aligned}$$

Here  $2\alpha H + \beta^2 T_0 > 0$ , i.e.,  $0 < H < \beta^2 T_0$ . For the stochastic integral  $\xi_t = \int_0^t \beta e^{\alpha(t-v)} d\mathbf{U}_v$  we get for any  $m \in \mathbf{N}_*$

$$\mathbf{E} \xi_t^{2m} = (2m-1)!! [\mathbf{E} \xi_t^2]^m \leq (2m-1)!! \beta_1^m.$$

Furthermore, in view of  $Y_{T_0} = Y_0 e^{\alpha T_0} + \xi_{T_0}$  we obtain

$$\mathbf{E} Y_{T_0}^{2m} \leq 2^{2m-1} (\mathbf{E}(Y_0 e^{\alpha T_0})^{2m} + \mathbf{E} \xi_{T_0}^{2m}) \leq \mathbf{k}_1(m).$$

Moreover, we have (see, e.g., [18, Lemma 4.12])

$$\mathbf{E}\left(\int_0^{T_0} Y_t d\mathbf{U}_t\right)^{2m} \leq (m(2m-1))^m T_0^{m-1} \int_0^{T_0} \mathbf{E} Y_s^{2m} ds \leq \mathbf{k}_2(m) T_0^m,$$

where  $\mathbf{k}_2(m) = (m(2m-1))^m \mathbf{k}_1(m)$ . We conclude that

$$\mathbf{P}\left(\int_0^{T_0} Y_t^2 dt < H\right) \leq \frac{3^{2m-1} (Y_0^{2m} + \mathbf{k}_1(m) + (2\beta)^{2m} \mathbf{k}_2(m) T_0^m)}{(2\alpha_2 H + \beta^2 T_0)^{2m}}.$$

We set  $H = \beta_2 (T_0 - T_0^\varepsilon)$  for some  $\varepsilon$ , we obtain

$$\mathbf{P}\left(\int_0^{T_0} Y_t^2 dt < H\right) \leq \frac{1}{(\beta^2)^{2m}} \frac{\mathbf{k}(m)}{T_0^{m(2\varepsilon-1)}}.$$

Replacement in (8.8) gives

$$\mathbf{E}(\alpha^* - \alpha)^2 \leq \frac{\beta^2}{\beta_2(T_0 - T_0^\varepsilon)} + \frac{\alpha^2}{\beta^{4m}} \frac{\mathbf{k}(m)}{T_0^{m(2\varepsilon-1)}}.$$

We fix  $\varepsilon = 5/6$  and  $m = 3$  so that  $m(2\varepsilon - 1) = 2$ , which gives  $\varepsilon^2(T_0)$  and then the desired result. Proposition 8.1 is proved.

**8.2. Known stock price appreciation rate  $\mu$ .** We consider in this section the consumption-investment problem for markets with known  $\mu$  and unknown  $\alpha$ , i.e., in this case  $\hat{\mu} = \mu$  and, therefore,  $\hat{\theta}(y) = \theta(y)$  in (8.3). To state the approximation result we set

$$\begin{aligned} \mathbf{h}_1 &= \frac{1 + 2\gamma + \zeta_0}{1 + \zeta_0} \frac{\tilde{T}}{|\alpha_1|} (2Q_1^* \tilde{T} + \gamma h_1^*), \\ \Gamma &= \left(q_* \tilde{T} (\sqrt{\tilde{c}})^\gamma + (\tilde{T} + 1) (\sqrt{\tilde{c} q_*})^\gamma\right) \frac{1}{\varkappa^\gamma} e^{\gamma \varkappa \tilde{T}}, \end{aligned} \tag{8.9}$$

Here  $\zeta_0 > 0$ ,  $\tilde{c} = 4\tilde{T}e^{2c_0\tilde{T}}$ ,

$$c_0 = \theta_*^4 + \theta_*^2(2|r| + 3) + (|r| + 1)^2 \quad \text{and} \quad \theta_* = \frac{\mu_* + |r|}{\sigma_1(1 - \gamma)}.$$

Moreover,  $\tilde{d}$  is the upper bound (8.14) and

$$(8.10) \quad h_1^* = \left( \tilde{T}Q_1^* + \frac{Q_1^*\tilde{T}^2}{q_*} \right) e^{Q_*\tilde{T}} + \frac{3}{q_*} \sqrt{\frac{2|\alpha_2|}{\beta^2(1 - e^{2\alpha_2})}} e^{Q_*\tilde{T}\tilde{T}}.$$

**THEOREM 8.1.** *For any  $0 < T_0 < T$  and any  $m \geq 1$*

$$(8.11) \quad \sup_{\alpha_2 \leq \alpha \leq \alpha_1} \sup_{|\mu| \leq \mu_*} \mathbf{E} \left| \widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0}) \right| \leq \delta_m,$$

where

$$\delta_m = \delta_m(x, T_0) = \Gamma \mathbf{h}_1^\gamma x^\gamma \left( (2\iota_0)^\gamma + \left( \frac{(2m-1)!! \beta^{2m}}{(2|\alpha_1|)^m} \right)^{\gamma/(2m)} \right) \epsilon^\gamma(T_0),$$

$\iota_0 = \beta/\sqrt{2|\alpha_1|}$  and  $\epsilon(T_0)$  is defined in (8.7).

*Proof.* Note that

$$\sup_{(s,y) \in \mathcal{K}} \left( |\mathbf{a}^*(s, y)|^2 + |\mathbf{b}^*(y)|^2 \right) \leq c_0.$$

Moreover, for any  $T_0 < T$

$$(8.12) \quad \begin{aligned} |\widehat{J}_{T_0}^* - J_{T_0}^*| &\leq \mathbf{E}_{T_0} \int_{T_0}^T \left| (\widehat{c}_t^*)^\gamma (\widehat{X}_t^*)^\gamma - (c_t^*)^\gamma (X_t^*)^\gamma \right| dt + \mathbf{E}_{T_0} |(\widehat{X}_T^*)^\gamma - (X_T^*)^\gamma| \\ &\leq \mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt + \mathbf{E}_{T_0} |\widehat{X}_T^* - X_T^*|^\gamma. \end{aligned}$$

Using Lemma 8.1 we get

$$|\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq \Gamma \mathbf{h}_1^\gamma x^\gamma (2\iota_0 + |Y_{T_0}|)^\gamma |\bar{\alpha}|^\gamma$$

and, therefore,

$$\mathbf{E} |\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq \Gamma x^\gamma \mathbf{h}_1^\gamma (2\iota_0)^\gamma \mathbf{E} |\bar{\alpha}|^\gamma \Gamma x^\gamma \mathbf{h}_1^\gamma \mathbf{E} (|Y_{T_0}|^\gamma |\bar{\alpha}|^\gamma).$$

By Holder's and Jensen's inequalities for  $m' = m(2 - \gamma)/\gamma > 1$  with  $m \geq 1$

$$\begin{aligned} \mathbf{E} (|Y_{T_0}|^\gamma |\bar{\alpha}|^\gamma) &\leq \left( \mathbf{E} |Y_{T_0}|^{2\gamma/(2-\gamma)} \right)^{(2-\gamma)/2} (\mathbf{E} |\bar{\alpha}|^2)^{\gamma/2} \\ &\leq \left( \mathbf{E} |Y_{T_0}|^{2\gamma m'/(2-\gamma)} \right)^{(2-\gamma)/(2m')} \epsilon^\gamma(T_0) \\ &\leq \left( \mathbf{E} |Y_{T_0}|^{2m} \right)^{\gamma/(2m)} \epsilon^\gamma(T_0). \end{aligned}$$

From [11], Lemma 1.1.1 it follows that  $\mathbf{E} |Y_{T_0}|^{2m} \leq c_m(T_0) \leq c_m(0)$ , where

$$c_m(T_0) = (2m-1)!! \beta^{2m} \left( \frac{1 - e^{2\alpha T_0}}{2|\alpha|} \right)^m.$$

We conclude that for any  $m \geq 1$

$$(8.13) \quad \mathbf{E}(|Y_{T_0}|^\gamma |\bar{\alpha}|^\gamma) \leq \left( \frac{(2m-1)!! \beta^{2m}}{(2|\alpha_1|)^m} \right)^{\gamma/(2m)} \epsilon^\gamma(T_0),$$

which gives the desired result.

*Remark 8.1.* We observe in Theorem 8.1, that the expected deviation  $\mathbf{E}|\widehat{J}_{T_0}^* - J^*(T_0, x, y)|$  can be arbitrary small, if either we observe the process  $Y$  in a wide interval  $[0, T_0]$  so that  $\mathbf{E}|\bar{\alpha}|$  be small enough, or we invest a small capital  $x$  at the initial time. That means, when the estimation interval is not wide enough, which is the case in practice, we can always find a consumption-investment strategy that belongs closer to the optimal one. For this aim, we need to be cautious in choosing the initial endowment and we need to take into account the upper bound (8.11).

LEMMA 8.1. *For any  $0 < T_0 \leq T$*

$$(8.14) \quad \mathbf{E}_{T_0} \left( \sup_{T_0 \leq s \leq T} (\widehat{X}_s^*)^2 \right) < x^2 \tilde{d}^2 \quad \text{and} \quad \tilde{d}^2 = 4e^{2\tilde{T}(A^* + (B^*)^2)},$$

where  $A^* = \sup_{(s,y) \in \mathcal{K}} \widehat{\mathbf{a}}^*(s, y)$  and  $B^* = \sup_{(s,y) \in \mathcal{K}} \widehat{\mathbf{b}}^*(s, y)$ . Moreover,

$$(8.15) \quad \sup_{T_0 \leq t \leq T} \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma \leq k_1 x^\gamma \left( \mathbf{h}_1(2\iota_0 + |Y_{T_0}|) \right)^\gamma |\bar{\alpha}|^\gamma$$

and

$$(8.16) \quad \mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \leq k_2 x^\gamma \left( \mathbf{h}_1(2\iota_0 + |Y_{T_0}|) \right)^\gamma |\bar{\alpha}|^\gamma,$$

where  $k_1 = (\sqrt{\widetilde{c}q_*})^\gamma e^{\gamma \mathfrak{K} \tilde{T}} / \mathfrak{K}^\gamma$  and  $k_2 = (\tilde{T}(\sqrt{\widetilde{c}q_*})^\gamma + \tilde{d}^\gamma q_* \tilde{T}) e^{\gamma \mathfrak{K} \tilde{T}} / \mathfrak{K}^\gamma$ .

*Proof.* It is clear from (8.2), that for the bounded function  $\widehat{\mathbf{b}}^*(y)$  the process  $(\widehat{\mathcal{E}}_{t,s})_{t \leq s \leq T}$  is a quadratic integrable martingale and by the Doob inequality

$$\begin{aligned} \mathbf{E}_{T_0} \sup_{T_0 \leq s \leq T} (\widehat{X}_s^*)^2 &\leq x^2 e^{2\tilde{T}A^*} \mathbf{E} \sup_{t \leq s \leq T} \widehat{\mathcal{E}}_{t,s}^2 \leq x^2 \cdot 4e^{2\tilde{T}A^*} \mathbf{E} \widehat{\mathcal{E}}_{t,T}^2 \\ &\leq 4x^2 e^{2\tilde{T}A^*} e^{\tilde{T}(B^*)^2}. \end{aligned}$$

This gives (8.14). We set  $\Delta_t = \widehat{X}_t^* - X_t^*$ ,  $A_s = \mathbf{a}^*(s, Y_s)$ ,  $B_s = \mathbf{b}^*(Y_s)$ ,  $\widehat{A}_s = \widehat{\mathbf{a}}^*(s, Y_s)$ , and  $\widehat{B}_s = \widehat{\mathbf{b}}^*(Y_s)$ . The functions  $\widehat{\mathbf{a}}^*(s, y)$  and  $\widehat{\mathbf{b}}^*(y)$  are defined in (8.3). It is clear that if  $\mu$  is known, (i.e.,  $\widehat{\mu} = \mu$ ) the function  $\widehat{B}_s = B_s$ . But we keep this function to use this proof in the case when the parameter  $\mu$  is unknown. Moreover we define  $\varphi_1(s) = \widehat{A}_s \widehat{X}_s^* - A_s X_s^*$  and  $\varphi_2(s) = \widehat{B}_s \widehat{X}_s^* - B_s X_s^*$ . So, from (7.2) we get

$$\begin{aligned} \Delta_t^2 &= \left( \int_{T_0}^t \varphi_1(s) ds + \int_{T_0}^t \varphi_2(s) dW_s \right)^2 \\ &\leq 2(t - T_0) \int_{T_0}^t \varphi_1^2(s) ds + 2 \left( \int_{T_0}^t \varphi_2(s) dW_s \right)^2. \end{aligned}$$

We observe that

$$\varphi_1^2(s) \leq \left( |\widehat{A}_s - A_s| |\widehat{X}_s^*| + |A_s| |\Delta_s| \right)^2 \leq 2|\widehat{A}_s - A_s|^2 |\widehat{X}_s^*|^2 + 2|A_s|^2 |\Delta_s|^2.$$

Furthermore, since  $\widehat{B}_s = B_s$  we obtain

$$\varphi_2^2(s) \leq \left( |\widehat{B}_s - B_s| |\widehat{X}_s^*| + |B_s| |\Delta_s| \right)^2 \leq |B_s|^2 |\Delta_s|^2.$$

Setting now  $g(t) = \mathbf{E}_{T_0}(\Delta_t^2)$  we obtain

$$g(t) \leq c_0 \int_{T_0}^t g(s) ds + \psi(t) \quad \text{and} \quad \psi(t) = 4\widetilde{T} \int_{T_0}^t \mathbf{E}_{T_0} \left( |\widehat{A}_s - A_s|^2 |\widehat{X}_s^*|^2 \right) ds.$$

The Gronwall–Bellman inequality yields

$$\begin{aligned} g(t) &\leq \psi(t) e^{2c_0 t} \leq x^2 \cdot 4\widetilde{T} e^{c_0 T} \int_{T_0}^t \mathbf{E}_{T_0} \left( |\widehat{A}_s - A_s|^2 |\widehat{X}_s^*|^2 \right) ds \\ &\leq \widetilde{c}^2 x^2 \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{A}_s - A_s|^2 ds \leq \widetilde{c}^2 x^2 \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s)^{-q_*} - h(s, Y_s)^{-q_*}|^2 ds \\ &\leq \widetilde{c}^2 x^2 q_* \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)|^2 ds. \end{aligned}$$

Using (10.13) and Lemma 10.6 we obtain, that for any  $T_0 \leq s \leq T$

$$\begin{aligned} \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)| &\leq \mathbf{h}_1 \mathbf{E}_{T_0} \left( e^{\varkappa(T-s)} (\ell_0 + |Y_s|) \right) |\bar{\alpha}| \\ &\leq \mathbf{h}_1 (\ell_0 + \mathbf{E}_{T_0} |Y_s|) e^{\varkappa(T-s)} |\bar{\alpha}| \\ (8.17) \quad &\leq \mathbf{h}_1 (2\ell_0 + |Y_{T_0}|) e^{\varkappa(T-s)} |\bar{\alpha}|. \end{aligned}$$

Therefore,

$$g(t) \leq x^2 \widetilde{c}^2 q_* \left( \mathbf{h}_1 (2\ell_0 + |Y_{T_0}|) \right)^2 \frac{e^{2\varkappa\widetilde{T}}}{\varkappa^2} |\bar{\alpha}|^2,$$

Hence, (8.15) holds. We show now inequality (8.16). Note that in view of (7.1) the optimal consumption  $0 \leq c_t^* \leq 1$ . Thus,

$$\begin{aligned} &\mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \\ &\leq \mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma |\widehat{X}_t^*|^\gamma dt + \int_{T_0}^T \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma dt \\ &\leq x^\gamma \widetilde{d}^\gamma \mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma dt + \widetilde{T} \sup_{T_0 \leq t \leq T} \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma. \end{aligned}$$

Using now the upper bound (8.17) and taking into account that  $\inf_{(t,y) \in \mathcal{K}} h(t, y) \geq 1$ , we obtain

$$\begin{aligned} \mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma dt &\leq q_* \int_{T_0}^T \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)|^\gamma dt \\ &\leq q_* \widetilde{T} \left( \mathbf{h}_1 (2\ell_0 + |Y_{T_0}|) \right)^\gamma \frac{e^{\gamma\widetilde{T}}}{\varkappa^\gamma} \mathbf{E}_{T_0} |\bar{\alpha}|^\gamma. \end{aligned}$$

Therefore,

$$\mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \leq k_2 x^\gamma \left( \mathbf{h}_1 (2\ell_0 + |Y_{T_0}|) \right)^\gamma \mathbf{E}_{T_0} |\bar{\alpha}|^\gamma.$$

This implies (8.16) and then Lemma 8.1.

**8.3. Unknown stock price appreciation rate  $\mu$ .** In practice, it is not realistic to consider known the stock price appreciation rate  $\mu$ . In this section, in addition to the unknown drift parameter  $\alpha$  of the economic factor process, we consider an unknown stock price appreciation rate  $\mu$ . We recall that the dynamics of the risky stock is given in (2.1). Let  $\hat{\mu}$  its estimate defined by

$$(8.18) \quad \hat{\mu} = \text{Proj}_{[-\mu_*, \mu_*]}(\tilde{\mu}), \quad \tilde{\mu} = \frac{Z_{T_0}}{T_0}, \quad Z_t = \int_0^t \frac{1}{S_t} dS_t.$$

LEMMA 8.2. For any  $0 < T_0 < T$

$$(8.19) \quad \sup_{|\mu| \leq \mu_*} \mathbf{E} |\hat{\mu} - \mu| \leq \epsilon_1(T_0) \quad \text{and} \quad \sup_{|\mu| \leq \mu_*} \mathbf{E} |\hat{\mu} - \mu|^2 \leq \epsilon_1^2(T_0),$$

where  $\epsilon_1(T_0) = \sigma^*/\sqrt{T_0}$  and  $\sigma^* = \sup_{y \in \mathbf{R}} \sigma(y)$ .

*Proof.* From the definition of the process  $Z$  we get

$$\tilde{\mu} - \mu = \frac{1}{T_0} \int_0^{T_0} \sigma(Y_t) dW_t,$$

This implies directly the bounds (8.19). Lemma 8.2 is proved.

Let the optimal value functions  $J^*(T_0, x, y)$  and  $\hat{J}_{T_0}^*$  its estimate given in (8.1), and let define the constants

$$k'_1 = 2\sqrt{\tilde{c}\tilde{T}} \frac{2\mu_* + r + \sigma_1 + 1}{\sigma_1^2(1-\gamma)} \quad \text{and} \quad k'_2 = \frac{e^{\varkappa\tilde{T}}}{\varkappa}.$$

Moreover, we define  $\Gamma_1 = k_3 + k_5$  and  $\Gamma_2 = k_4 + k_6$ , where

$$k_3 = (k'_1)^\gamma + \left(\sqrt{2\tilde{c}q_*k'_2\mathbf{h}_2}\right)^\gamma, \quad k_4 = \left(\sqrt{2\tilde{c}q_*k'_2\mathbf{h}_1}\right)^\gamma, \\ k_5 = \tilde{T}(k'_1)^\gamma + k_7(k'_2\mathbf{h}_2)^\gamma, \quad k_6 = k_7(k'_2\mathbf{h}_1)^\gamma, \quad k_7 = \left(\sqrt{2\tilde{c}q_*} + q_*\tilde{d}^\gamma\right).$$

Recall that  $\tilde{c} = 4e^{c_0t}\tilde{d}^2$  and  $\tilde{d}$  is given in (8.14). The constants  $\mathbf{h}_1$  is given in (8.9) and

$$(8.20) \quad \mathbf{h}_2 = \frac{\gamma(\mu_* + r) 2\tilde{T}^2}{(1-\gamma)\sigma_1^2 \iota_0}.$$

We are dealing with the following result

THEOREM 8.2. The estimate of optimal cost function  $\hat{J}_{T_0}^*$  satisfies the following inequalities:

$$(8.21) \quad |\hat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq x^\gamma \Gamma_1 (2\iota_0 + |Y_{T_0}|)^\gamma \varpi^\gamma + x^\gamma \Gamma_2 (2\iota_0 + |Y_{T_0}|)^\gamma |\bar{\alpha}|^\gamma,$$

where  $\varpi = |\hat{\mu} - \mu| + |\hat{\mu} - \mu|^2$ . Moreover, for any  $m \geq 1$

$$(8.22) \quad \sup_{\alpha_2 \leq \alpha \leq \alpha_1} \sup_{|\mu| \leq \mu_*} \mathbf{E} |\hat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq \check{\delta}_m,$$

with  $\check{\delta}_m = \check{\delta}_m(x, T_0) = x^\gamma (\Gamma_1 (3\iota_0^\gamma + |Y_0|^\gamma) \epsilon_2^\gamma(T_0) + \check{\Gamma}_m \epsilon^\gamma(T_0))$  and

$$\check{\Gamma}_m = \Gamma_2 \left( (2\iota_0)^\gamma + \left( \frac{(2m-1)!! \beta^{2m}}{(2|\alpha_1|)^m} \right)^{\gamma/2m} \right).$$

Here  $\iota_0 = \beta/\sqrt{2|\alpha_1|}$ ,  $\epsilon_2(T_0) = \epsilon_1(T_0) + \epsilon_1^2(T_0)$  is given in (8.19) and  $\epsilon(T_0)$  is defined in (8.7).

*Proof.* We follow the same arguments as in the proof of Theorem 8.1, and use Lemma 8.3 below to conclude for (8.21). Now, to show (8.22), we observe from (8.21) that

$$\begin{aligned} \mathbf{E} |\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| &\leq x^\gamma \Gamma_1 \left( (2\iota_0)^\gamma + (\mathbf{E} |Y_{T_0}|)^\gamma \right) \mathbf{E} \varpi^\gamma \\ &\quad + x^\gamma \Gamma_2 (2\iota_0)^\gamma \epsilon^\gamma(T_0) + \mathbf{E} (|Y_{T_0}|^\gamma |\bar{\alpha}|^\gamma). \end{aligned}$$

Taking into account that  $\mathbf{E} \varpi \leq \epsilon_2(T_0)$  and using the bound (8.13) we obtain (8.22). Theorem 8.2 is proved.

LEMMA 8.3. *The estimate of the wealth process  $(\widehat{X}_t^*)_{T_0 \leq t \leq T}$  satisfies the following inequalities:*

$$(8.23) \quad \begin{aligned} \sup_{T_0 \leq t \leq T} \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma &\leq x^\gamma k_3 (2\iota_0 + |Y_{T_0}|)^\gamma \varpi^\gamma \\ &\quad + x^\gamma k_4 (2\iota_0 + |Y_{T_0}|)^\gamma |\bar{\alpha}|^\gamma, \end{aligned}$$

$$(8.24) \quad \begin{aligned} \mathbf{E}_{T_0} \int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt &\leq x^\gamma k_5 (2\iota_0 + |Y_{T_0}|)^\gamma \varpi^\gamma \\ &\quad + x^\gamma k_6 (2\iota_0 + |Y_{T_0}|)^\gamma |\bar{\alpha}|^\gamma. \end{aligned}$$

*Proof.* We follow the arguments in Lemma 8.1 we set  $\Delta_t = \widehat{X}_t^* - X_t^*$ ,  $g(t) = \mathbf{E}_{T_0}(\Delta_t^2)$  we get

$$g(t) \leq c_0 \int_{T_0}^t g(s) ds + \psi(t),$$

where  $\psi(t) = 4 \mathbf{E}_{T_0} \int_{T_0}^t (|\widehat{A}_s - A_s|^2 + |\widehat{B}_s - B_s|^2) |\widehat{X}_s^*|^2 ds$ . Through the Gronwall–Bellman inequality we get

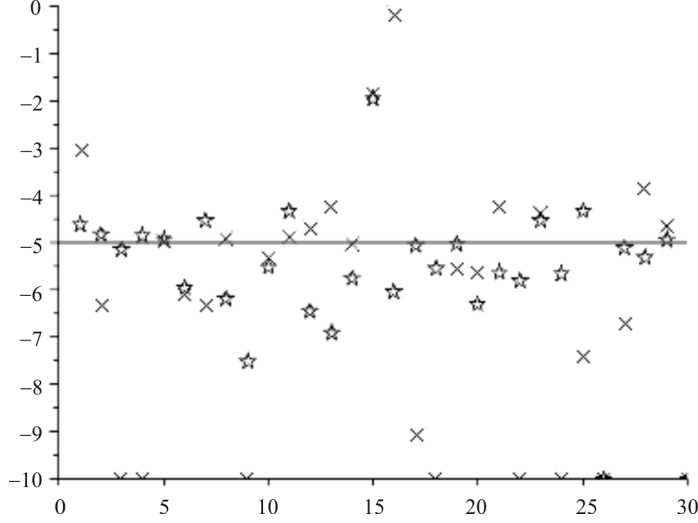
$$\begin{aligned} g(t) &\leq \psi(t) e^{c_0 t} \leq x^2 \widetilde{c} \int_{T_0}^t \mathbf{E}_{T_0} \left( |\widehat{A}_s - A_s|^2 + |\widehat{B}_s - B_s|^2 \right) ds \\ &\leq x^2 \widetilde{c} \widetilde{T} \frac{2(2\mu_* + r)^2 + \sigma_1^2}{\sigma_1^4 (1 - \gamma)^2} \varpi^2 + 2x^2 \widetilde{c} \int_{T_0}^t \mathbf{E}_{T_0} \left| \widehat{h}(s, Y_s)^{-q_*} - h(s, Y_s)^{-q_*} \right|^2 ds \\ &\leq 2x^2 \widetilde{c} \widetilde{T} \left( \frac{2\mu_2 + r + \sigma_1}{\sigma_1^2 (1 - \gamma)} \right)^2 \varpi^2 + 2x^2 \widetilde{c} q_* \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)|^2 ds. \end{aligned}$$

We use then Proposition 10.2 to get the analogous of (8.17):

$$(8.25) \quad \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)| \leq e^{\varkappa(T-s)} (2\iota_0 + |Y_{T_0}|) \widetilde{\delta},$$

where  $\widetilde{\delta} = \mathbf{h}_2 \varpi + \mathbf{h}_1 |\bar{\alpha}|$ . Then

$$\begin{aligned} g(t) &\leq 2x^2 \widetilde{c} \widetilde{T} \left( \frac{2\mu_* + r + \sigma_1}{\sigma_1^2 (1 - \gamma)} \right)^2 \varpi^2 + 2x^2 \widetilde{c} q_* \frac{e^{2\varkappa \widetilde{T}}}{\varkappa^2} (2\iota_0 + |Y_{T_0}|)^2 \widetilde{\delta}^2 \\ &\leq x^2 \left( k'_1 \varpi + k'_2 (2\iota_0 + |Y_{T_0}|) \widetilde{\delta} \right)^2, \end{aligned}$$


 FIG. 9.1. The truncated sequential estimate for  $T_0 = 5$ ,  $T_0 = 10$ .

i.e.,

$$\mathbf{E}_{T_0} |\Delta_t| \leq x \left( k'_1 \varpi + k'_2 (2\iota_0 + |Y_{T_0}|) \tilde{\delta} \right).$$

Using here the Jensen inequality for power function  $z^\gamma$  (with  $0 < \gamma < 1$ ) we obtain (8.23). Now, we show (8.24). We follow the same arguments used in Lemma 8.1 to arrive at

$$\begin{aligned} \mathbf{E} \int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt &\leq x^\gamma q_* \tilde{d}^\gamma \int_{T_0}^T \mathbf{E} |\widehat{h}(s, Y_s) - h(s, Y_s)|^\gamma ds \\ &\quad + \tilde{T} \sup_{T_0 \leq t \leq T} \mathbf{E} |\widehat{X}_t^* - X_t^*|^\gamma. \end{aligned}$$

Therefore, the upper bound (8.24) follows immediately from (8.23) and (8.25). Lemma 8.2 is proved.

**9. Simulation.** In this section we use Scilab for simulations. In Fig 1. we simulate the truncated sequential estimate  $\widehat{\alpha}$  for different values of  $T_0$ , through 30 paths of the driving process  $Y$ . The sequential estimates are represented by  $\times$  for  $T_0 = 5$  days and  $*$  for  $T_0 = 10$  days. The true drift value of the process  $Y$  is  $\alpha = -5$ . We take the bounds  $\alpha \in [-0.15, -10]$  and set  $\beta = 1$ .

In Fig 2. we simulate the limit functions  $h(t, y)$  and  $\widehat{h}(t, y)$ , under the following market settings: we set  $T_0 = 5$  and  $\tilde{T} = T - T_0 = 1$ ,  $r = 0.01$ ,  $\mu = 0.02$ . The volatility is defined by  $\sigma(y) = 0.5 + \sin^2(y)$ . The utility parameter is  $\gamma = 0.75$ . To simulate  $\widehat{h}(t, y)$ , we use a very pessimistic realization of the truncated estimate, i.e.,  $\widehat{\alpha} = -0.5$ . The true value is  $\alpha = -5$ . We see that, even in this extreme situation, the estimated function  $\widehat{h}(t, y)$  does not deviate significantly from the real value  $h(t, y)$ .

**10. Auxiliary results.** Let  $h$  the fixed point solution for  $h = \mathcal{L}h$ , where the mapping  $\mathcal{L}$  is defined in (4.4) and (4.5). Now we study the partial derivative of the  $\mathcal{H}_f(t, s, y)$  with respect to  $y$  and  $h$ .

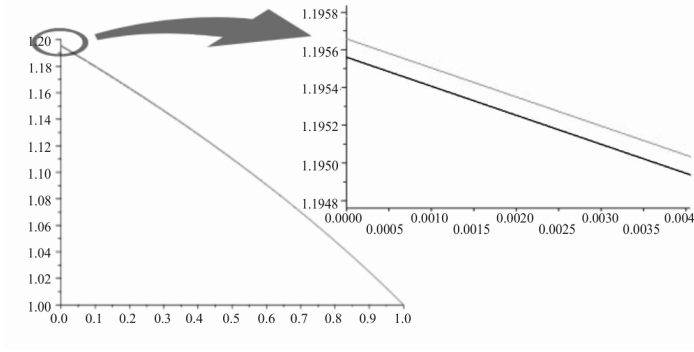


FIG. 9.2. The limit functions  $h(t, 0)$  and  $\widehat{h}(t, 0)$ .

LEMMA 10.1. For any  $t$  and  $s$  such that  $T_0 < t \leq s \leq T$

$$(10.1) \quad \sup_{y \in \mathbf{R}} \sup_{f \in \mathcal{X}} \left| \frac{\partial}{\partial y} \mathcal{H}_f(t, s, y) \right| \leq Q_1^* \widetilde{T} e^{Q_* \widetilde{T}} + \frac{e^{Q_* \widetilde{T}}}{\nu_s},$$

where  $\nu_s^2 = \beta^2(1 - e^{2\alpha(s-t)})/2|\alpha|$ .

*Proof.* To calculate this conditional expectation note, first that

$$\eta_s = ye^{\alpha(s-t)} + \int_t^s \beta e^{\alpha(s-v)} d\widetilde{\mathbf{U}}_v = ye^{\alpha(s-t)} + \xi_s.$$

Since  $\eta$  it is a *Gaussian process*, for any  $t < v_1 < \dots < v_k < s$  and for any bounded  $\mathbf{R}^k \rightarrow \mathbf{R}$  function  $G$

$$(10.2) \quad \mathbf{E}(G(\eta_{v_1}, \dots, \eta_{v_k}) | \eta_s = z) = \mathbf{E}G(\mathbf{B}_{v_1}, \dots, \mathbf{B}_{v_k}),$$

where  $\mathbf{B}_v = \eta_v - k(v)\eta_s + k(v)z$  and the coefficient  $k(v)$  is chosen so that

$$\mathbf{E}[(\xi_v - k(v)\xi_s)\xi_s] = 0, \quad \dots \quad k(v) = \frac{\mathbf{E}(\xi_v \xi_s)}{\mathbf{E}\xi_s^2} = e^{\alpha(s-v)} \frac{1 - e^{2\alpha(v-t)}}{1 - e^{2\alpha(s-t)}} \leq 1.$$

The conditional expectation with respect to  $\eta_s$  permits represent  $\mathcal{H}_f$  as

$$(10.3) \quad \begin{aligned} \mathcal{H}_f(t, s, y) &= \int_{\mathbf{R}} \widehat{\mathcal{H}}_f(s, y, z) \mathbf{p}(z, y) dz, \\ \mathbf{p}(z, y) &= \frac{1}{\nu_s \sqrt{2\pi}} \exp \left\{ -\frac{(z - \mu(y))^2}{2\nu_s^2} \right\}, \end{aligned}$$

where  $\mu(y) = \mathbf{E}\eta_s = ye^{\alpha(s-t)}$  and  $\nu_s^2 = \mathbf{D}\eta_s$ . Since  $\mathbf{B}_s = z$  we get

$$(10.4) \quad \begin{aligned} \widehat{\mathcal{H}}_f(s, y, z) &= \mathbf{E} \left( (f(s, \eta_s^{t,y}))^{1-q_*} \exp \left\{ \int_t^s Q(\eta_u^{t,y}) du \right\} \middle| \eta_s = z \right) \\ &= \mathbf{E} \left( (f(s, z))^{1-q_*} \exp \left\{ \int_t^s Q(\mathbf{B}_u) du \right\} \right) \leq e^{Q_*(s-t)}. \end{aligned}$$

From here it follows that

$$\left| \frac{\partial}{\partial y} \widehat{\mathcal{H}}_f(s, y, z) \right| \leq \left| \int_t^s \frac{\partial Q(\mathbf{B}_u)}{\partial y} du \right| \widehat{\mathcal{H}}_f(s, y, z) \leq Q_1^*(s-t)e^{Q_*(s-t)} \leq Q_1^* \widetilde{T} e^{Q_* \widetilde{T}}.$$

Now from (10.3) we obtain

$$\begin{aligned} \frac{\partial \mathcal{H}_f(t, s, y)}{\partial y} &= \int_{\mathbf{R}} \frac{\partial \widehat{\mathcal{H}}_f(s, y, z)}{\partial y} \mathbf{p}(z, y) \, dz \\ &\quad + \int_{\mathbf{R}} \widehat{\mathcal{H}}_f(s, y, z) \frac{(z - \mu(y))\mu'(y)}{\nu_s^2} \mathbf{p}(z, y) \, dz. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \frac{\partial \mathcal{H}_f(t, s, y)}{\partial y} \right| &\leq Q_1^*(s-t)e^{Q_*(s-t)} + e^{Q_*(s-t)} \frac{\mu'(y)}{\nu_s^2} \int_{\mathbf{R}} |z - \mu(y)| \mathbf{p}(z, y) \, dz \\ &\leq Q_1^*(s-t)e^{Q_*(s-t)} + \frac{e^{(Q_*+\alpha)(s-t)}}{\nu_s^2} \frac{2\nu_s}{\sqrt{2\pi}} \leq Q_1^* \tilde{T} e^{Q_* \tilde{T}} + \frac{e^{Q_* \tilde{T}}}{\nu_s}. \end{aligned}$$

Hence Lemma 10.1 is proved.

LEMMA 10.2. *For any  $y \in \mathbf{R}$ , a unique solution of the fixed point equation  $h = \mathcal{L}_h$  is differentiable with respect to  $y$ , and its partial derivative is bounded:*

$$(10.5) \quad \sup_{T_0 \leq t \leq T, y \in \mathbf{R}} \left| \frac{\partial}{\partial y} h(t, y) \right| \leq h_1^*,$$

where  $h_1^*$  is given in (8.10).

*Proof.* It is obviously sufficient to show that  $\mathcal{L}_h(t, y)$  is differentiable with respect to  $y$ , and its partial derivative is bounded

$$\sup_{T_0 \leq t \leq T, y \in \mathbf{R}} \left| \frac{\partial}{\partial y} \mathcal{L}_f(t, y) \right| \leq h_1^*.$$

From the definition of  $\mathcal{L}_f$  in (4.4), for all  $f \in \mathcal{X}$  and for all  $t \in [T_0, T]$  and  $y \in \mathbf{R}$  we get

$$\frac{\partial}{\partial y} \mathcal{L}_f(t, y) = \mathbf{E} \frac{\partial}{\partial y} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \frac{\partial}{\partial y} \mathcal{H}_f(t, s, y) \, ds.$$

Using Lemmas 10.1 and 10.4, we get

$$\begin{aligned} \sup_{T_0 \leq t \leq T, y \in \mathbf{R}} \left| \frac{\partial}{\partial y} \mathcal{L}_f(t, y) \right| &\leq \tilde{T} Q_1^* e^{Q_* \tilde{T}} + \frac{1}{q_*} \int_t^T Q_1^* \tilde{T} e^{Q_* \tilde{T}} \, ds + \frac{1}{q_*} \int_t^T \frac{e^{Q_* \tilde{T}}}{\nu_s} \, ds \\ &\leq \tilde{T} Q_1^* e^{Q_* \tilde{T}} + \frac{Q_1^* \tilde{T}^2}{q_*} e^{Q_* \tilde{T}} + \frac{e^{Q_* \tilde{T}}}{q_*} \int_t^T \frac{1}{\nu_s} \, ds. \end{aligned}$$

To estimate  $\int_t^T (1/\nu_s) \, ds$  we observe that  $2|\alpha|(s-t) \leq 2|\alpha|\tilde{T}$  so

$$\begin{aligned} \nu_s^2 &= \beta^2 \frac{1 - e^{2\alpha(s-t)}}{2|\alpha|(s-t)} (s-t) \geq \beta^2 \frac{1 - e^{2\alpha}}{2|\alpha|} (s-t) \quad \text{if } s-t \leq 1, \\ \nu_s^2 &= \beta^2 \frac{1 - e^{2\alpha(s-t)}}{2|\alpha|} \geq \beta^2 \frac{1 - e^{2\alpha}}{2|\alpha|} \quad \text{if } s-t \geq 1. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \int_t^T \frac{1}{\nu_s} ds &\leq \sqrt{\frac{2|\alpha|}{\beta^2(1-e^{2\alpha})}} \int_t^{t+1} \frac{1}{\sqrt{s-t}} ds + \sqrt{\frac{2|\alpha|}{\beta^2(1-e^{2\alpha})}} \int_{t+1}^T ds \\ &\leq 2\sqrt{\frac{2|\alpha|}{\beta^2(1-e^{2\alpha})}} + \sqrt{\frac{2|\alpha|}{\beta^2(1-e^{2\alpha})}} \tilde{T} \leq 3\sqrt{\frac{2|\alpha|}{\beta^2(1-e^{2\alpha})}} \tilde{T}. \end{aligned}$$

Taking into account that  $\alpha_2 \leq \alpha \leq \alpha_1 < 0$  we obtain the desired result. Lemma 10.2 is proved.

Now we study the partial derivatives of the function  $\mathcal{G}(t, s, y)$  defined in (4.4). To this end we need the following general result.

LEMMA 10.3. *Let  $F = F(y, \omega)$  be a  $\mathbf{R} \times \Omega \rightarrow \mathbf{R}$  random bounded function such that for some nonrandom constant  $c^*$*

$$\left| \frac{d}{dy} F(y, \omega) \right| \leq c^* \quad a.s.$$

Then

$$\frac{d}{dy} \mathbf{E} F(y, \omega) = \mathbf{E} \frac{d}{dy} F(y, \omega).$$

This lemma follows immediately from the Lebesgue dominated convergence theorem.

LEMMA 10.4. *For any  $t < s$  the function  $\mathcal{G}$  satisfies the following properties:*

$$(10.6) \quad \sup_{y \in \mathbf{R}} \left| \frac{\partial \mathcal{G}(t, s, y)}{\partial y} \right| \leq (s-t) Q_1^* e^{Q^*(s-t)}, \quad \frac{\partial}{\partial y} \mathbf{E} \mathcal{G}(t, s, y) = \mathbf{E} \frac{\partial}{\partial y} \mathcal{G}(t, s, y).$$

*Proof.* We have immediately

$$\frac{\partial \mathcal{G}(t, s, y)}{\partial y} = \mathcal{G}(t, s, y) \mathbf{G}(t, s, y),$$

where  $\mathbf{G}(t, s, y) = \int_t^s Q_1(\eta_u^{t,y})(\partial \eta_u^{t,y} / \partial y) du$  and  $Q_1(z) = dQ(z)/dz$ .

Now Lemma 10.3 implies directly Lemma 10.4.

LEMMA 10.5. *For any  $f \in \mathcal{X}$  having bounded partial derivatives with respect to  $y \in \mathbf{R}$  and for any  $N > 0$  we have*

$$\sup_{|y| < N} \sup_{0 \leq t_1 < t_2 \leq T} \frac{|\mathcal{L}_f(t_2, y) - \mathcal{L}_f(t_1, y)|}{\sqrt{t_2 - t_1}} < \infty.$$

*Proof.* One can check directly that

$$\sup_{|y| < N} \sup_{0 \leq t_1 < t_2 \leq s \leq T} \frac{\mathbf{E} |\eta_s^{t_2, y} - \eta_s^{t_1, y}|}{\sqrt{t_2 - t_1}} < \infty.$$

This upper bound implies directly Lemma 10.5.

We recall that to the process  $(\eta_s)_{0 \leq s \leq T}$  is defined in (4.3) and  $(\hat{\eta}_s)_{0 \leq s \leq T}$  defined in (8.5).

LEMMA 10.6. For any  $t$  and  $s$  such that  $T_0 \leq t \leq s \leq T$

$$(10.7) \quad \mathbf{E}_{T_0} |\hat{\eta}_s^{t,y}| \leq \iota_0 + |y| = \frac{\beta}{\sqrt{2|\alpha_1|}} + |y| := \tilde{m}(y)$$

and

$$(10.8) \quad \mathbf{E}_{T_0} \int_t^T |\hat{\eta}_s^{t,y} - \eta_s^{t,y}| dt \leq \frac{\tilde{T}\tilde{m}(y)}{|\alpha_1|} |\bar{\alpha}|.$$

Moreover, for the known parameter  $\mu$  and unknown parameter  $\alpha$

$$(10.9) \quad \mathbf{E}_{T_0} |\hat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| \leq \tilde{T}Q_1^* e^{Q_*(T-t)} \frac{\tilde{m}(y)}{|\alpha_1|} |\bar{\alpha}|,$$

where  $Q^*$  and  $Q_1^*$  are defined in (3.5), and  $\hat{\mathcal{G}}(t, s, y)$  is given in (8.4).

*Proof.* Since  $\eta_s = \eta_t e^{\alpha(s-t)} + \int_t^s \beta e^{\alpha(s-v)} d\tilde{\mathbf{U}}_v$  we obtain for any  $\alpha_2 \leq \alpha \leq \alpha_1 < 0$

$$\begin{aligned} \mathbf{E} (\eta_s^{t,y})^2 &= y^2 e^{2\alpha(s-t)} + \beta^2 \int_t^s e^{2\alpha(t-v)} dv \leq y^2 + \frac{\beta^2}{2|\alpha_1|} \\ &\leq \left( |y| + \frac{\beta}{\sqrt{2|\alpha_1|}} \right)^2. \end{aligned}$$

This implies the bound (10.7). Moreover, setting  $\bar{\eta}_s = \hat{\eta}_s^{t,y} - \eta_s^{t,y}$ , we obtain

$$d\bar{\eta}_s = (\hat{\alpha}\hat{\eta}_s^{t,y} - \alpha\eta_s^{t,y}) ds = \alpha\bar{\eta}_s ds + \bar{\alpha}\hat{\eta}_s^{t,y} ds,$$

i.e.,  $\bar{\eta}_s = \int_t^s \bar{\alpha} e^{\alpha(s-u)} \hat{\eta}_u^{t,y} du$ . Therefore,

$$|\bar{\eta}_s| \leq |\bar{\alpha}| \int_t^s |\hat{\eta}_u^{t,y}| e^{\alpha(s-u)} du.$$

Since  $\hat{\alpha}$  is independent of the Brownian motion  $(\tilde{\mathbf{U}}_t)$ , we get

$$(10.10) \quad \begin{aligned} \mathbf{E}_{T_0} |\bar{\eta}_s| &\leq |\bar{\alpha}| \mathbf{E}_{T_0} \int_t^s |\hat{\eta}_u^{t,y}| e^{\alpha(s-u)} du \\ &\leq |\bar{\alpha}| \int_t^s e^{\alpha(s-u)} \mathbf{E}_{T_0} |\hat{\eta}_u^{t,y}| du \leq \frac{\tilde{m}(y)}{|\alpha_1|} |\bar{\alpha}|. \end{aligned}$$

Therefore, for all  $T_0 \leq t \leq T$

$$\begin{aligned} \mathbf{E}_{T_0} \int_t^T |\bar{\eta}_s| ds &\leq \mathbf{E}_{T_0} \left( \int_t^T |\bar{\alpha}| \int_t^s e^{\alpha(s-u)} |\hat{\eta}_u^{t,y}| du ds \right) \\ &\leq \tilde{T} |\bar{\alpha}| \int_t^T e^{\alpha(s-u)} \mathbf{E}_{T_0} |\hat{\eta}_u^{t,y}| du \leq \frac{\tilde{m}(y)\tilde{T}}{|\alpha_1|} |\bar{\alpha}|, \end{aligned}$$

and we come to (10.8). To get inequality (10.9) note that

$$\begin{aligned} |\hat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| &\leq e^{Q_*(T-t)} \left| \int_t^s Q(\hat{\eta}_u^{t,y}) du - \int_t^s Q(\eta_u^{t,y}) du \right| \\ &\leq e^{Q_*(T-t)} \int_t^s \sup_{y \in \mathbf{R}} \left| \frac{\partial Q(y)}{\partial y} \right| |\bar{\eta}_u| du \\ &\leq Q_1^* e^{Q_*(T-t)} \int_t^T |\bar{\eta}_u| du. \end{aligned}$$

Thus,

$$\mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| \leq Q_1^* e^{Q_*(T-t)} \int_t^T \mathbf{E}_{T_0} |\bar{\eta}_u| du.$$

Now, the bound (10.10) implies (10.9). Hence Lemma 10.6 is proved..

We study in the following proposition the behavior of  $h(t, y)$ , the solution of the fixed point problem  $h = \mathcal{L}_h$ , when using the estimate  $\widehat{\alpha}$  of the parameter  $\alpha$ . We look for a bound for the deviation

$$(10.11) \quad \bar{h}(t, y) = \widehat{h}(t, y) - h(t, y),$$

where  $\widehat{h} = \widehat{\mathcal{L}}_{\widehat{h}}$ . The operator  $\widehat{\mathcal{L}}$  is defined in (8.4). Similarly to (4.2) we define on  $\mathcal{X}$  the metric  $\widetilde{\varrho}_*$  as follows:

$$(10.12) \quad \widetilde{\varrho}_*(f, g) = \sup_{(t, y) \in \mathcal{K}} e^{-\varkappa(T-t)} \frac{|f(t, y) - g(t, y)|}{\iota_0 + |y|},$$

where we set  $\iota_0 = \beta/\sqrt{2\alpha_1}$  and  $\varkappa = Q_* + \zeta + 1$  and set  $\zeta = \zeta_0 + 2\gamma$  for some  $\zeta_0 > 0$ .

PROPOSITION 10.1. *For the known  $\mu$  and unknown  $\alpha$ , and for any  $0 < T_0 < T$*

$$(10.13) \quad \widetilde{\varrho}_*(\widehat{h}, h) \leq \mathbf{h}_1 |\widehat{\alpha}|,$$

where  $\mathbf{h}_1$  is given in (8.9).

*Proof.* We use the definition of the operator  $\mathcal{L}$  in (4.4):

$$h(t, y) = \mathcal{L}_h(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathcal{H}_h(t, s, y) ds.$$

Through (4.5) we can estimate the deviation (10.11) as

$$|\bar{h}(t, y)| \leq \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, T, y) - \mathcal{G}(t, T, y)| + I(\widehat{\alpha}),$$

where

$$I(\widehat{\alpha}) = \frac{1}{q_*} \int_t^T \mathbf{E}_{T_0} \left| \left( \widehat{h}(s, \widehat{\eta}_s^{t, y}) \right)^{1-q_*} \widehat{\mathcal{G}}(t, s, y) - \left( h(s, \eta_s^{t, y}) \right)^{1-q_*} \mathcal{G}(t, s, y) \right| ds.$$

Moreover, this term can be bounded as

$$\begin{aligned} I(\widehat{\alpha}) &\leq \frac{1}{q_*} \int_t^T \mathbf{E}_{T_0} \left( h(s, \eta_s^{t, y})^{1-q_*} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| \right) ds \\ &\quad + \frac{1}{q_*} \int_t^T \mathbf{E}_{T_0} \left| \left( \widehat{h}(s, \widehat{\eta}_s^{t, y}) \right)^{1-q_*} - \left( h(s, \eta_s^{t, y}) \right)^{1-q_*} \right| e^{Q_*(s-t)} ds \\ &\leq \int_t^T \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| ds \\ &\quad + \frac{|1-q_*|}{q_*} \int_t^T \mathbf{E}_{T_0} |\widehat{h}(s, \widehat{\eta}_s^{t, y}) - h(s, \eta_s^{t, y})| e^{Q_*(s-t)} ds. \end{aligned}$$

We use the fact that  $q_* = 1/(1-\gamma) > 1$  and the bounds (10.9) and (10.10) to deduce

$$\begin{aligned} |\bar{h}(t, y)| &\leq (1 + \widetilde{T}) \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, T, y) - \mathcal{G}(t, T, y)| \\ &\quad + \gamma \int_t^T \mathbf{E}_{T_0} |h(s, \widehat{\eta}_s^{t, y}) - h(s, \eta_s^{t, y})| e^{Q_*(s-t)} ds \\ &\quad + \gamma \int_t^T \mathbf{E}_{T_0} |\widehat{h}(s, \widehat{\eta}_s^{t, y}) - h(s, \eta_s^{t, y})| e^{Q_*(s-t)} ds. \end{aligned}$$

The upper bound (10.5) yields

$$|h(s, \widehat{\eta}_s^{t,y}) - h(s, \eta_s^{t,y})| \leq h_1^* |\widehat{\eta}_s^{t,y} - \eta_s^{t,y}|.$$

In view of the definitions of the metric  $\tilde{\varrho}_*$  in (10.12) and of the parameter  $\varkappa$  in (4.2) we get

$$\begin{aligned} \tilde{\varrho}_*(\widehat{h}, h) &\leq \frac{(1 + \widetilde{T})\widetilde{T} Q_1^*}{|\alpha_1|} \sup_{(t,y) \in \mathcal{K}} \left( \frac{\widetilde{m}(y)}{\iota_0 + |y|} e^{(Q_* - \varkappa)(T-t)} \right) |\widehat{\alpha} - \alpha| \\ &\quad + \gamma \sup_{(t,y) \in \mathcal{K}} \int_t^T h_1^* \mathbf{E}_{T_0} |\widehat{\eta}_s^{t,y} - \eta_s^{t,y}| e^{(Q_* - \varkappa)(T-t)} ds \\ &\quad + \gamma \sup_{(t,y) \in \mathcal{K}} \int_t^T \mathbf{E}_{T_0} \frac{|\widehat{h}(s, \widehat{\eta}_s^{t,y}) - h(s, \widehat{\eta}_s^{t,y})| e^{-\varkappa(T-s)}}{\iota_0 + |\widehat{\eta}_s^{t,y}|} \\ &\quad \quad \quad \times \frac{\iota_0 + |\widehat{\eta}_s^{t,y}|}{\iota_0 + |y|} e^{(Q_* - \varkappa)(s-t)} ds. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\varrho}_*(\widehat{h}, h) &\leq \Upsilon^* |\widehat{\alpha} - \alpha| + \gamma \tilde{\varrho}_*(\widehat{h}, h) \sup_{(t,y) \in \mathcal{K}} \int_t^T \frac{\iota_0 + \mathbf{E}_{T_0} |\widehat{\eta}_s^{t,y}|}{\iota_0 + |y|} e^{(Q_* - \varkappa)(s-t)} ds \\ &\leq \Upsilon^* |\widehat{\alpha} - \alpha| + \gamma \tilde{\varrho}_*(\widehat{h}, h) \sup_{(t,y) \in \mathcal{K}} \left( \frac{\iota_0 + \widetilde{m}(y)}{\iota_0 + |y|} \int_t^T e^{(Q_* - \varkappa)(s-t)} ds \right) \\ &\leq \Upsilon^* |\widehat{\alpha} - \alpha| + \frac{2\gamma}{\varkappa - Q_*} \tilde{\varrho}_*(\widehat{h}, h), \end{aligned}$$

Here  $\Upsilon^* = (2Q_1^* \widetilde{T} + \gamma h_1^*) \widetilde{T} / |\alpha_1|$ . Hence we get

$$\tilde{\varrho}_*(\widehat{h}, h) \leq \frac{\varkappa - Q_*}{\varkappa - Q_* - 2\gamma} \Upsilon^* |\widehat{\alpha}|.$$

Taking into account that  $\varkappa = Q_* + \zeta_0 + 2\gamma + 1$ , we obtain (10.13). Hence Proposition 10.1 is proved.

We consider both the stock price appreciation rate  $\mu \in [\mu_1, \mu_2]$ , and the drift  $\alpha$  of the economic factor  $Y$  to be unknown. The following lemma gives the analogous of equation (10.9).

LEMMA 10.7. *For the unknown parameters  $\mu$  and  $\alpha$  and for any  $0 < T_0 < T$*

$$(10.14) \quad \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| \leq \gamma \frac{\mu_2 + r + 1}{(1 - \gamma)\sigma_1^2} \widetilde{T} e^{Q_*(T-t)} \varpi + \widetilde{T} Q_1^* e^{Q_*(T-t)} \frac{\widetilde{m}(y)}{|\alpha_1|} |\widehat{\alpha}|,$$

where  $\varpi = |\widehat{\mu} - \mu| + |\widehat{\mu} - \mu|^2$ .

*Proof.* First note that for the function  $Q$  defined in (3.4) we can obtain the following bound

$$\widehat{Q}(z) - Q(z) = \frac{\gamma(\widehat{\theta}^2(z) - \theta^2(z))}{2(1 - \gamma)} \leq \frac{\gamma(\mu_2 + r + 1)}{(1 - \gamma)\sigma_1^2} \varpi.$$

So,

$$\begin{aligned}
|\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| &\leq \left| \exp \left\{ \int_t^s \widehat{Q}(\widehat{\eta}_u^{t,y}) du \right\} - \exp \left\{ \int_t^s Q(\eta_u^{t,y}) du \right\} \right| \\
&\quad + e^{Q_*(T-t)} \left| \int_t^s Q(\widehat{\eta}_u^{t,y}) du - \int_t^s Q(\eta_u^{t,y}) du \right| \\
&\leq \widetilde{T} e^{Q_*(T-t)} \frac{\gamma(\mu_2 + r + 1)}{(1 - \gamma)\sigma_1^2} \varpi \\
&\quad + Q_1^* e^{Q_*(T-t)} \int_t^T |\widehat{\eta}_u^{t,y} - \eta_u^{t,y}| du.
\end{aligned}$$

Through (10.8) we obtain (10.14). Hence, Lemma 10.14 is proved.

The following proposition is the analogous of Proposition 10.1. The difference is that, in the proposition below, both  $\mu$  and  $\alpha$  are unknown.

PROPOSITION 10.2. *For the unknown parameters  $\mu$  and  $\alpha$  and for any  $0 < T_0 < T$*

$$\widetilde{\varrho}_*(\widehat{h}, h) \leq \mathbf{h}_1 |\overline{\alpha}| + \mathbf{h}_2 \varpi,$$

where the metric  $\widetilde{\varrho}_*$  is given in (10.12),  $\mathbf{h}_1$  and  $\mathbf{h}_2$  are given in (8.9) and in (8.20), respectively.

*Proof.* We follow the same arguments as in the proof of Proposition 10.1 and use Lemma 10.7 for the bound of  $\mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, T, y) - \mathcal{G}(t, T, y)|$ .

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