



HAL
open science

The analytic theory of vectorial Drinfeld modular forms

Federico Pellarin

► **To cite this version:**

| Federico Pellarin. The analytic theory of vectorial Drinfeld modular forms. 2023. hal-02334842v3

HAL Id: hal-02334842

<https://hal.science/hal-02334842v3>

Preprint submitted on 5 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE ANALYTIC THEORY OF VECTORIAL DRINFELD MODULAR FORMS

F. PELLARIN

ABSTRACT. In this text we generalize the notion of Drinfeld modular form for the group $\Gamma := \mathrm{GL}_2(\mathbb{F}_q[\theta])$ to a vector-valued setting, where the target spaces are certain modules over positive characteristic Banach algebras over which are defined what we call the 'representations of the first kind'. Under quite reasonable restrictions, we show that the spaces of such modular forms are finite-dimensional, endowed with certain generalizations of Hecke operators, with differential operators à la Serre etc. The crucial point of this work is the introduction of a 'field of uniformizers', a field extension of the valued field of formal Laurent series $\mathbb{C}_\infty((u))$ where u is the usual uniformizer for Drinfeld modular forms, in which we can study the expansions at the cusp infinity of our modular forms and which is wildly ramified and not discretely valued. Examples of such modular forms are given through the construction of Poincaré and Eisenstein series.

After the discussion of these fundamental properties, the paper continues with a more detailed analysis of the special case of modular forms associated to a restricted class of representations ρ_Σ^* of Γ which has more importance in arithmetical applications. More structure results are given in this case, and a harmonic product formula is obtained which allows, with the help of conjectures on the structure of an \mathbb{F}_p -algebra of A -periodic multiple sums, multiple Eisenstein series etc., to produce conjectural formulas for Eisenstein series. Other properties such as integrality of coefficients of Eisenstein series, specialization at roots of unity etc. are included as well.

CONTENTS

1. Introduction	3
1.1. Motivation through three examples	4
1.2. The field of uniformizers	6
1.3. Description of the basic objects	7
1.4. How to interpret the example	10
1.5. Results of the text	12
2. Preliminaries	17
2.1. Rings, fields, modules	18
2.2. Analytic functions with values in non-archimedean Banach algebras	23
2.3. Drinfeld modules and exponential functions	27
2.4. Some functions associated with the Carlitz module	34

Date: July 7, 2023.

1991 Mathematics Subject Classification. MSC2020 classification 11G09.

Key words and phrases. Drinfeld modular forms, representations of the Drinfeld modular group.

3. Field of uniformizers	39
3.1. Some algebras and fields	40
3.2. Tame series	54
4. Quasi-periodic functions, representations of the first kind	64
4.1. Quasi-periodic functions	65
4.2. Representations of the first kind	69
4.3. Hecke operators	76
4.4. Remarks on representations	78
5. Structure results for modular forms	81
5.1. Changing the coefficient field	81
5.2. Finiteness results	83
5.3. Poincaré series	90
5.4. Eisenstein series	100
6. Differential operators on modular forms, Perkins' series	101
6.1. Higher derivatives on tame series	102
6.2. Divided higher derivatives of ρ -quasi-periodic functions	104
6.3. Serre's derivatives	107
6.4. Application to Perkins' series	107
7. Eisenstein series for ρ_{Σ}^*	112
7.1. Link between Eisenstein series and Poincaré series	113
7.2. The v -valuation of Eisenstein series	114
7.3. Application to modular forms of weight one for ρ_{Σ}^*	116
7.4. Rationality and integrality of coefficients	118
7.5. Negative weight modular forms and functional identities	120
7.6. Some applications to quasi-modular and \mathfrak{v} -adic modular forms	125
8. Modular forms for the representations ρ_{Σ}^*	133
8.1. Structure of weak modular forms	134
8.2. Strongly regular modular forms	137
8.3. More structure properties	140
9. Harmonic product and Eisenstein series	147
9.1. Existence of the harmonic product	149
9.2. Explicit formulas	154
9.3. A conjecture on multiple sums	161
10. Perspectives on algebraic properties of Eisenstein series	161
10.1. Multiple Eisenstein series	162
10.2. A conjecture for zeta values in Tate algebras	165
10.3. A modular analogue	169
Addendum	169
References	169

1. INTRODUCTION

The aim of this volume is to introduce new perspectives in the *analytic theory of Drinfeld modular forms* for the Drinfeld modular group $\mathrm{GL}_2(\mathbb{F}_q[\theta])$. This theory was initiated in the foundational works by Goss starting from his Ph. D. Thesis (see [35]) and continued in the works of Gekeler, notably in [24]. Probably these are the main foundational papers. The theory is in expansion since then thanks to the work of several authors. For 'analytic theory' we mean a collection of algebraic or analytic results coming from the study of expansions at the cusp infinity, such as finiteness of dimensions of spaces of modular forms, Hecke operators, differential operators, congruences.

Modular forms with values in positive characteristic fields ⁽¹⁾ such as

$$\mathbb{C}_\infty := \mathbb{F}_q(\widehat{(\theta^{-1})})^{sep},$$

are at the center of an active domain of research with deep developments in several directions, for more general groups $\mathrm{GL}_n(A)$ ($n \geq 2$), with A ring of functions over a smooth projective geometrically irreducible curve regular away from an infinity point, its congruence subgroups, leading to an algebraic and analytic theory of modular forms and to compactification problems as in the works of Pink and Basson, Breuer and Pink [69, 8], Gekeler [27, 28, 29, 30], Häberli [40], Hartl and Yu [41]. The arithmetic theory of Drinfeld modular forms, if compared with that of classical modular forms, also has a different flavor. We mention the investigations related to Galoisian representations and the cohomological theory of crystals by Böckle [9, 10] and aspects of P -adic continuous families of Drinfeld modular forms by Hattori [42] and Nicole and Rosso [50, 51]. These works illustrate how the theory ramifies deeply in a multitude of directions but the list of reference we give is far from being representative. More references can be found in the above mentioned works.

In the present volume, we voluntarily restrict our attention to the simplest case of the *Drinfeld modular group*

$$\Gamma := \mathrm{GL}_2(\mathbb{F}_q[\theta])$$

and we follow yet another direction of research which, as far as we can see, has not been deeply investigated yet. We want to begin the study of analytic properties of modular forms associated with an extension of the notion of type, initially introduced by Gekeler in [24]. We replace it with a class of representations of Γ .

It is well known that the theory of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ is deeply enriched by considering characters and multiplier systems, and Drinfeld modular forms do not make exception to this principle. The *type* of a Drinfeld modular form for the group Γ can be viewed as a one-dimensional representation of Γ . In this paper we are interested in certain higher dimensional representations of this group that we call 'representations of the first kind' and our basic observation is that they are naturally contained in certain rigid analytic families at the infinity place.

The reader that wants to immediately skip to the description of the results contained here can read §1.5 directly; just below in §1.1, we shall introduce simple explicit examples

¹With $\mathbb{F}_q(\widehat{(\theta^{-1})})^{sep}$ a separable closure of the local field $\mathbb{F}_q(\widehat{(\theta^{-1})})$ and $\widehat{}$ denoting the completion.

that, playing the role of Ariadne's thread, can be helpful entering into the details of this work.

1.1. Motivation through three examples. One of the principal initial reasons for our endeavor comes from remarks on analytic families of modular forms first raised in the paper [56], and later, in [61]. We shall give three explicit examples in various directions that can help the reader to understand our viewpoint.

1.1.1. *Serre's example.* Consider a prime number $p > 2$. We recall the fundamental example of the p -adic family of Eisenstein series (for $\mathrm{SL}_2(\mathbb{Z})$ in the settings and notations of Serre [73, §1.6]; we will not recall all the definitions of the involved objects here). This can be viewed as a family of formal series

$$k \mapsto G_k^* = \frac{1}{2}\zeta^*(1-k) + \sum_{n \geq 1} \sigma_{k-1}^*(n) \mathbf{q}^n \in \mathbb{Q}_p + \mathbf{q}\mathbb{Z}_p[[\mathbf{q}]],$$

in $\mathbb{Z}_p[[\mathbf{q}]]$ with \mathbf{q} an indeterminate (which is often identified with the uniformizer at infinity of the modular group), with the parameter $k = (s, u)$ in the topological group

$$\mathbf{X}_p = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$$

submitted to the condition that u is a multiple of the class of 2. We do not recall the definition of ζ^* and σ_k^* in detail here (read Serre, loc. cit.), but ζ^* is essentially Kubota-Leopold's p -adic zeta function and σ_k^* is the arithmetic function obtained from the arithmetic function σ_k when we drop the divisors that are multiple of p , so that a \mathbf{X}_p -exponentiation can be defined. If we choose any k as above, it can be proved that the corresponding value G_k^* is a p -adic modular form of weight k in Serre's sense. There is an injective group map $\mathbb{Z} \rightarrow \mathbf{X}_p$; if we choose $k \in 2\mathbb{N}^*$ (where $\mathbb{N}^* = \mathbb{Z}_{\geq 1}$) the value of G^* at k is the image in $\mathbb{Q}_p[[\mathbf{q}]]$ of a modular form of weight k for the congruence subgroup $\Gamma_0(p)$ of $\mathrm{SL}_2(\mathbb{Z})$. Indeed in this case $\zeta^*(1-k) = (1-p^{k-1})\zeta(1-k) \in \mathbb{Q}$ so that and we can view G_k^* in $\mathbb{Q}[[\mathbf{q}]]$ and

$$G_k^* = G_k - p^{k-1}G_k|_V,$$

where if $f = \sum_{n \geq 0} f_n \mathbf{q}^n$, $f|_V := \sum_{n \geq 0} f_n \mathbf{q}^{pn}$, and where $G_w \in \mathbb{Q} + \mathbf{q}\mathbb{Z}[[\mathbf{q}]]$ is the \mathbf{q} -expansion of one of the various normalizations of the Eisenstein series of weight $w \in (2\mathbb{N}^*)$. In synthesis, the p -adic Eisenstein family, for even integer values of the parameter, specializes to modular forms for $\Gamma_0(p)$. The level p is therefore fixed, and the weight, non-constant, varies in the topological group \mathbf{X}_p . If p is an irregular prime it can happen that $\zeta^*(1-k) = 0$ for some integral values of k . In this case, G_k^* is a cusp form.

1.1.2. *Goss' example.* Inspired by the above example of Serre, and based on earlier works of Petrov [66, 67], Goss [39] looked for an analogue picture in the settings of Drinfeld modular forms for $\Gamma = \mathrm{GL}_2(A)$, where $A := \mathbb{F}_q[\theta]$. One first choses $P \in A$ monic and irreducible; an analogue of a prime number. While p -adic modular forms are, in Serre's approach, formal series in \mathbf{q} , here Goss considers a new indeterminate u ⁽²⁾ that can be identified with the usual parameter at infinity for the Drinfeld modular group (more details

²Not to be mixed up with Serre's coordinate u in \mathbf{X}_p .

about it will be given in the present paper; the definition is recalled in full detail in (1.3)). Then he proceeds constructing families of formal series in u that retrace some properties of the above G_k^* . Recall that in Serre's example it was used, crucially that there is an action of \mathbb{Z} on formal series in \mathbf{q} induced by \mathbb{G}_m (indeed, there is the operator $(\cdot)|_V$); here there is a similar action of A on formal series in u (it is not given by \mathbb{G}_m but by its Drinfeldian companion which is Carlitz's module; see later). If $a \in A$, we write $u_a \in A[[u]]$ for the image of u under the action of a (in Serre's settings we would have written $\mathbf{q}_p = \mathbf{q}^p$). Also, there is a simple way to define exponentiation a^k of a monic polynomial $a \in A$ by an element k of Goss' topological group

$$\mathbf{X}_P = \mathbb{Z}_p \times \mathbb{Z}/(q^{\deg_\theta(P)} - 1)\mathbb{Z}.$$

This can be boiled down to construct the map $\mathbf{X}_P \rightarrow \widehat{A}_P[[u]]$ (where \widehat{A}_P is the P -completion of A , see [39] for details and more generality):

$$k \mapsto f_k = \sum_{\substack{a \in A \\ a \text{ monic}}} a^k u_a \in \widehat{A}_P[[u]].$$

It is not difficult to show, in a way similar to Serre's, that the special values corresponding to $k \in \mathbb{N}^*$ with $k \equiv 1 \pmod{q-1}$ are the u -expansions of Drinfeld cusp forms of weight k for the full Drinfeld modular group Γ and by [39, Theorem 2], the above is a P -adic family of Drinfeld modular forms in the sense of Serre. Again the level of the group is constant (full level) and the weights vary. There is a substantial difference in the comparison with Serre's example. Goss observed that there is no non-zero constant term in these u -expansions of forms; the elements of the family are all cusp forms. In clear, there is no occurrence of any analogue of Kubota-Leopoldt zeta function in Goss' construction.

1.1.3. *A basic ∞ -adic example.* Serre's and Goss' examples are relative to the choice of a finite place p of \mathbb{Q} or P of $K := \mathbb{F}_q(\theta)$, but in the Drinfeldian setting, it is possible to also work with the choice of the place infinity ∞ of K . In the present volume we are mainly concerned with this aspect of the theory. Less known is the existence of certain non-trivial ∞ -adic families. Here is an explicit example. We denote by \mathbb{F}_q^{ac} an algebraic closure of \mathbb{F}_q .

We consider a Dirichlet character

$$\chi : (A/PA)^\times \rightarrow (\mathbb{F}_q^{ac})^\times$$

of level $P \in A$ (monic and irreducible) that we extend to A in the usual way; now the parameter of the ∞ -adic family that we construct can be specialized to χ (so P varies). We define the following rigid analytic function over the Drinfeld half-plane $\Omega := \mathbb{C}_\infty \setminus K_\infty \rightarrow \mathbb{C}_\infty$ (where $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$):

$$(1.1) \quad g_\chi(z) := -L(1; \chi) + \sum'_{a, b \in A} (az + b)^{-1} \chi(b),$$

where the dash indicates a sum avoiding $a = 0$ and where $L(1; \chi)$ is a Dirichlet L -value $\sum_{a \in A, \text{ monic}} \chi(a) a^{-1}$ which is also a special value of Goss' abelian L -function. It is not

difficult to show that given χ , g_χ is a Drinfeld modular form of weight 1 for the principal congruence subgroup $\Gamma(P)$ of Γ , thought neither is an Eisenstein series, nor a cusp form.

Is there an analytic space having the Dirichlet characters χ as points, in such a way that we can associate to the isobaric family g_χ a u -series expansion, knowing that the level now varies as a function of χ ? The answer is affirmative, and it is essentially contained in the papers [55, 61]. The present work proposes to widely extend the theory. The characters χ can indeed be viewed as closed points of the rigid analytic affine line $\mathbb{A}_{\mathbb{C}_\infty}^{1,an}$ (and more generally we will encounter higher dimensional affine spaces as parameter spaces). Not only, but there exists an ∞ -adic analogue of the Kubota-Leopold's zeta function that allows to interpolate the constant terms of all the g_χ ; a class of zeta values in Tate algebras discussed, for example, in [2, 5, 55]. Additionally, although the functions g_χ are not Eisenstein series, the analytic function that interpolates them (a family) is a coefficient of a certain *vector Eisenstein series of weight 1 with values in Tate algebras for the full group Γ , associated to a certain representation of Γ* , and other entries of it sometimes deliver the special values of Goss' family. The reader will find a systematic study of these structures in this paper. For instance, see our §7.

1.2. The field of uniformizers. To study congruences or Serre's p -adic analytic families of modular forms useful tool is provided by the series expansions at a cusp. For instance, Gekeler's seminal paper [24] uses 'Fourier series' of modular forms (we say ' u -expansions') in an essential way. Many times, readers take for granted the existence of such expansions.

One bad news is that this viewpoint is no longer sufficient and in particular, it does not extend to the example (1.1). It is more difficult to expand at a cusp the modular forms that we are interested in. The good news is that there exists a *field of uniformizers* \mathfrak{K} (Definition 3.33) in which we can embed all the coordinates of our Eisenstein series with values in modular forms, or more generally, our modular forms.

Recall that a Drinfeld modular form for the group Γ in the sense of [35, 24] can be identified with an element of the v -valued field

$$\mathbb{C}_\infty((u)),$$

where v is the u -adic valuation, which is discrete. More precisely, a Drinfeld modular form f has a u -expansion

$$f = \sum_{i \geq 0} f_i u^i, \quad f_i \in \mathbb{C}_\infty.$$

A coordinate f of one of our vector modular forms with values in Tate algebras (including the modular form interpolating (1.1)) can be identified, in unique way, with a formal series

$$(1.2) \quad f = f_0 + \sum_{i > 0} f_i u^i,$$

where f_0 is an element of the completion \mathbb{K} of the fraction field of a Tate algebra for the Gauss' norm, and the coefficients f_i are entire functions $\mathbb{C}_\infty \rightarrow \mathbb{K}$ that we call *tame series*, and that in general, are not constant functions. The field of uniformizers will be described in §3, a rather important section of our work. We note that \mathfrak{K} is a valued field extension

of $\mathbb{C}_\infty((u))$, but the extension of the valuation on it is no longer discrete and the extension of valuations is wildly ramified.

We present a spectrum of aspects which is limited to the construction of the analytic uniformization at the cusp infinity and some consequences going from the proof that vector spaces of modular forms are of finite dimension, to the construction of explicit examples of Eisenstein and Poincaré series and the analysis of their expansions in \mathfrak{k} . This already offered challenges and rich pictures and we decided to confine our attention only to those aspects which are tangible by an appropriate generalization of the viewpoint of Gekeler's seminal paper [24].

The volume presents the foundations to study new aspects of Drinfeld modular forms and to do this, it presents new tools with an elementary approach. It is enriched with several questions, problems and conjectures. Among other crucial aspects that we do not develop here we mention the interpretation of modular forms of our settings as sections of algebraic vector bundles and their links with the theory of harmonic cocycles à la Teitelbaum.

1.3. Description of the basic objects. Let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, let \mathbb{F}_q be the finite field with q elements and characteristic p , and θ an indeterminate over \mathbb{F}_q . All along this text, we denote by A the \mathbb{F}_q -algebra $\mathbb{F}_q[\theta]$. We set $K = \mathbb{F}_q(\theta)$. On K , we consider the multiplicative valuation $|\cdot|$ defined by $|a| = q^{-\deg_\theta(a)}$, a being in K , so that $|\theta| = q$. Let $K_\infty := \mathbb{F}_q((1/\theta))$ be the local field which is the completion of K for this absolute value, let K_∞^{sep} be a separable algebraic closure of K_∞ , let \mathbb{C}_∞ be the completion of K_∞^{sep} for the unique extension of $|\cdot|$ to K_∞^{sep} . Then, the field \mathbb{C}_∞ is at once algebraically closed and complete for $|\cdot|$ with valuation group $q^{\mathbb{Q}}$ and residual field $\mathbb{F}_q^{\text{sep}}$, an algebraic closure of \mathbb{F}_q .

The 'Drinfeld half-plane' $\Omega = \mathbb{C}_\infty \setminus K_\infty$, with the usual rigid analytic structure in the sense of [22, Definition 4.3.1], carries an action of $\Gamma = \text{GL}_2(A)$ and $\tilde{\Gamma} = \text{PGL}_2(A)$ by homographies: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}$, and $z \in \Omega$,

$$\gamma(z) := \frac{az + b}{cz + d}.$$

Denote by

$$J_{\begin{pmatrix} * & * \\ c & d \end{pmatrix}}(z) = cz + d$$

the usual factor of automorphy $\Gamma \times \Omega \rightarrow \mathbb{C}_\infty^\times$. Let us consider $w, m \in \mathbb{Z}$; then, if $w \equiv 2m \pmod{q-1}$, the map $(\gamma, z) \mapsto J_\gamma(z)^w \det(\gamma)^{-m}$ defines a factor of automorphy for $\tilde{\Gamma}$. There is a bijection between these factors of automorphy and the couples $(w, m) \in \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ submitted to the above congruence.

We thus suppose that $w \in \mathbb{Z}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ are such that $w \equiv 2m \pmod{q-1}$. We recall the definition of Drinfeld modular forms (as considered by Gekeler and Goss, see [24, Definition (5.7)]).

Definition 1.1. A *Drinfeld modular form of weight $w \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$* for the group Γ is a rigid analytic function $\Omega \xrightarrow{f} \mathbb{C}_\infty$ such that

$$f(\gamma(z)) = J_\gamma(z)^w \det(\gamma)^{-m} f(z) \quad \forall z \in \Omega, \quad \forall \gamma \in \tilde{\Gamma}$$

and such that additionally, there exists $0 < c < 1$ with the property that if $z \in \Omega$ is such that $|u(z)| \leq c$, where

$$(1.3) \quad u(z) = \frac{1}{\tilde{\pi}} \sum_{a \in A} \frac{1}{z - a},$$

$\tilde{\pi} \in \mathbb{C}_\infty \setminus K_\infty$ being a fundamental period of Carlitz's module ⁽³⁾, then there is a uniformly convergent series expansion

$$(1.4) \quad f(z) = \sum_{n \geq 0} f_n u(z)^n, \quad f_n \in \mathbb{C}_\infty.$$

We say that a function f in (1.4) is *regular at the infinity cusp*.

Note that (1.4) is not the only formulation of the regularity at the infinity cusp ⁽⁴⁾. We can restate (1.4) equivalently by asking that the set of real numbers $|f(z)|$ is bounded if we choose $z \in \Omega$ such that $|u(z)|$ is small.

The *type* in Definition 1.1 corresponds to a representation

$$(1.5) \quad \Gamma \xrightarrow{\det^{-m}} \mathrm{GL}_1(\mathbb{F}_q), \quad m \in \mathbb{Z}/(q-1)\mathbb{Z}.$$

In dimension > 1 it happens that certain representations of Γ naturally have non-trivial analytic deformations, and this makes it natural to consider functions with values in positive-dimensional Tate algebras or in similar ultrametric Banach algebras. We consider $\Sigma \subset \mathbb{N}^*$ a finite subset. Let $\mathbb{F}_q(t_\Sigma)$ be the field of rational fractions with coefficients in \mathbb{F}_q in the set of independent variables $t_\Sigma := (t_i : i \in \Sigma)$. We choose a representation

$$(1.6) \quad \Gamma \xrightarrow{\rho} \mathrm{GL}_N(\mathbb{F}_q(t_\Sigma)).$$

Let $w \in \mathbb{Z}$ be such that the map $(\gamma, z) \mapsto J_\gamma(z)^w \rho(\gamma)$ defines a factor of automorphy

$$\tilde{\Gamma} \times \Omega \rightarrow \mathrm{GL}_N(\mathbb{F}_q(t_\Sigma)).$$

The necessary and sufficient condition for this is that

$$(1.7) \quad \rho(\mu I_2) = \mu^{-w} I_N, \quad \mu \in \mathbb{F}_q^\times,$$

as it comes out after a simple computation.

We consider the field

$$\mathbb{K}_\Sigma = \mathbb{C}_\infty(t_\Sigma)^\wedge = \widehat{\mathbb{C}_\infty(t_\Sigma)}$$

(the completion for the Gauss norm) ⁽⁵⁾ so that $\mathrm{GL}_N(\mathbb{F}_q(t_\Sigma)) \subset \mathrm{GL}_N(\mathbb{K}_\Sigma)$. We denote by $\|\cdot\|$ the multiplicative valuation of \mathbb{K}_Σ , extending $|\cdot|$ of \mathbb{C}_∞ . We further extend this to a norm on matrices with entries in \mathbb{K}_Σ in the usual way by taking the supremum of

³Our analogue of $2\pi i$, see (2.11).

⁴Note also that classically, a *modular form for* $\mathrm{SL}_2(\mathbb{Z})$ (or for a subgroup of $\mathrm{SL}_2(\mathbb{R})$ which is commensurable with it) can be also defined as a holomorphic function $f : \mathcal{H} = \{z = x + \sqrt{-1}y \in \mathbb{C} : x, y \in \mathbb{R}, y > 0\} \rightarrow \mathbb{C}$ satisfying a well known family of functional relations and such that, if $z = x + \sqrt{-1}y$ with $x, y \in \mathbb{R}$, there exists $c \in \mathbb{R}$ such that $f(x + iy) = \mathcal{O}(y^c + y^{-c})$ (compare with Miyake's [48, Theorem 2.1.4]).

⁵Observe the notation $(\cdot)^\wedge$ that will be used when the other notation will lead to a too large hat.

the multiplicative valuations of the entries. In §2.2 we discuss the notion of rigid analytic functions with values in \mathbb{K}_Σ . Taking this notion into account:

Definition 1.2. A rigid analytic function

$$\Omega \xrightarrow{f} \mathbb{K}_\Sigma^{N \times 1}$$

such that

$$(1.8) \quad f(\gamma(z)) = J_\gamma(z)^w \rho(\gamma) f(z) \quad \forall z \in \Omega, \quad \forall \gamma \in \tilde{\Gamma},$$

is called *modular-like of weight w for ρ* . Additionally, we say that such a function $f = {}^t(f_1, \dots, f_N)$ is:

- (1) A *weak modular form of weight w for ρ* if there exists $M \in \mathbb{Z}$ such that $\|u(z)^M f(z)\|$ is bounded as $0 < |u(z)| < c$ for some $c < 1$.
- (2) A *modular form of weight w (for ρ)* if $\|f(z)\|$ is bounded as $0 < |u(z)| < c$ for some $c < 1$.
- (3) A *cuspidal form of weight w* if $\|f(z)\| \rightarrow 0$ as $u(z) \rightarrow 0$.

Let B be a \mathbb{C}_∞ -sub-algebra of \mathbb{K}_Σ . We suppose that ρ as in (1.6) has image in $\mathrm{GL}_N(B)$. We denote by $M_w^!(\rho; B)$ (resp. $M_w(\rho; B)$, $S_w(\rho; B)$) the B -modules of weak modular forms (resp. modular forms, cuspidal forms) of weight w for ρ such that their images are contained in $B^{N \times 1}$. We have that

$$S_w(\rho; B) \subset M_w(\rho; B) \subset M_w^!(\rho; B).$$

If $B = \mathbb{C}_\infty$, $N = 1$ and $\rho = \det^{-m}$, these \mathbb{C}_∞ -vector spaces coincide with the corresponding spaces of 'classical' Drinfeld modular forms of weight w , type m in the framework of Definition 1.1.

To be relevant, Definition 1.2 must deliver certain primordial properties such as the finite dimensionality of the modules $M_w(\rho; B)$, or their invariance under the action of variants of Hecke operators. We are far from being able to return satisfactory answers in such a level of generality. However, there is a class of representations, called *representations of the first kind*, introduced and discussed in §4.2, which looks suitable for our investigation because they contain a variety of arithmetically interesting examples. An explicit example of such representations is, with t a variable, the one which associates to a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the matrix

$$(1.9) \quad \rho_{\chi_t}(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q[t]),$$

where χ_t is the unique \mathbb{F}_q -algebra morphism $\mathbb{F}_q[\theta] \rightarrow \mathbb{F}_q[t]$ sending θ to t . Another interesting example is the contragredient (or dual) representation

$$\rho_{\chi_t}^* := {}^t \rho_{\chi_t}^{-1},$$

investigated in [55, 63]; in the latter case, we have already explicitly described the module structure of $M_w(\rho_t^*; \mathbb{T})$ (the values are in $\mathbb{T}^{2 \times 1}$ where

$$\mathbb{T} := \widehat{\mathbb{C}_\infty[t]}_{\|\cdot\|}$$

is the Tate algebra completion of $\mathbb{C}_\infty[t]$ for the Gauss valuation $\|\cdot\|$ extending the valuation of \mathbb{C}_∞) and proved that these \mathbb{T} -modules are endowed with endomorphisms given by a natural generalization of Hecke operators.

1.4. How to interpret the example. The very first non-trivial example of our Eisenstein series, first observed in [55], is the following function $\Omega \rightarrow \mathbb{T}^{2 \times 1}$ of the variable $z \in \Omega$:

$$(1.10) \quad \mathcal{E}(1; \rho_{\chi_t}^*) = \sum'_{a, b \in A} (az + b)^{-1} \begin{pmatrix} \chi_t(a) \\ \chi_t(b) \end{pmatrix}.$$

It is an analytic function $\Omega \rightarrow \mathbb{T}^{2 \times 1}$ following our §2.2. It is not at all difficult to see that

$$\mathcal{E}(1; \rho_{\chi_t}^*) \in M_1(\rho_{\chi_t}^*; \mathbb{T}).$$

We look at the second coordinate, which can be rewritten, after an elementary manipulation, as

$$(1.11) \quad \mathbf{g} := -\zeta_A(1; \chi_t) + \sum_{\substack{a \in A \\ a \text{ monic}}} \sum_{b \in A} (az - b)^{-1} \chi_t(b),$$

with the negative of the ζ -value in Tate algebra

$$(1.12) \quad \zeta_A(1; \chi_t) = \sum_{\substack{a \in A \\ a \text{ monic}}} a^{-1} \chi_t(a) = \prod_P \left(1 - \frac{\chi_t(P)}{P}\right)^{-1} \in \mathbb{T}^\times$$

as a constant term, where the product runs over the irreducible monic polynomials of A (introduced in [55]). Let $P \in A$ be monic and irreducible and consider one of its roots $\xi \in (\mathbb{F}_q^{ac})^\times$. The evaluation ev_ξ of t at ξ allows to identify $\text{ev}_\xi \circ \chi_t$ with χ a certain Dirichlet character of level P . Hence, viewing (1.1), g_χ can be identified the evaluation ev_ξ of the function \mathbf{g} in (1.11).

We have collected the functions g_χ in an analytic family \mathbf{g} but we did not yet identify the sum (1.11) with some kind of u -expansion as in (1.2). The strong point of Definition 1.2 is its simplicity but in practice it does not allow to do computations with Drinfeld modular forms. If we compare with Definition 1.1, we still need a valuation at the infinity cusp, available at least in the case of classical Drinfeld modular forms by considering the order in u in (1.4). This problem is already mentioned in [63].

Although rather technical in general, the construction of the valuation can be made more transparent, in the example we are discussing, with the use of *Perkins' series* (see §6.4) ⁽⁶⁾. The simplest Perkins series is:

$$\psi(1; \chi_t) := \sum_{b \in A} (z - b)^{-1} \chi_t(b).$$

⁶We mention that Perkins' investigations have also important connections with the notion of *quasi-periodic functions* of Gekeler (as in [25]).

It can be seen as a meromorphic function $\mathbb{C}_\infty \rightarrow \mathbb{T}$ of the variable z and allows to rewrite (1.11) in the following way:

$$(1.13) \quad -\zeta_A(1; \chi_t) + \sum_{\substack{a \in A \\ a \text{ monic}}} \psi(1; \chi_t)_a,$$

where the function $\psi(1; \chi_t)_a$ is defined by replacing the variable z with az (in agreement with the notation u_a). Perkins proved formulas such as

$$(1.14) \quad \psi(1; \chi_t) = \omega(t)^{-1} \tilde{\pi} u(z) \sum_{j \geq 0} \exp_C \left(\frac{\tilde{\pi} z}{\theta^{j+1}} \right) t^j, \quad z \in \mathbb{C}_\infty \setminus A.$$

In the right-hand side we have Carlitz's exponential \exp_C and the Anderson-Thakur function $\omega(t)$ (all these items will be reviewed in §2). We postpone presenting the definition of the field of uniformizers \mathfrak{K} but this is a very first example of one of its elements. More explicitly, $\psi(1; \chi_t)$ is a series $\sum_i f_i u^i$ with a unique non-zero monomial, of the form $f_1 u^1$ where

$$f_1 = \frac{\tilde{\pi}}{\omega(t)} \sum_{j \geq 0} \exp_C \left(\frac{\tilde{\pi} z}{\theta^{j+1}} \right) t^j.$$

Our settings are such that the additive valuation v extending that of $\mathbb{C}_\infty((u))$, evaluated on the left-hand side of (1.14), equals

$$v(u) + v(f_1) = 1 + v(\exp_C(\tilde{\pi} z / \theta)) = 1 - \frac{1}{q},$$

and the leading term of the tame series f_1 is proportional to $\exp_C(\tilde{\pi} z / \theta)$. It can be proved that the series expansion of \mathbf{g} in \mathfrak{K} is:

$$\mathbf{g} = f_0 + \sum_{i > 0} f_i u^i,$$

with $f_0 = -\zeta_A(1; \chi_t)$ and f_1 as above. The other coefficients f_2, f_3, \dots are progressively more and more difficult to compute and there is no easily recognizable pattern that can help in that task. Yet, it is easy to verify that the coefficients are elements of $\frac{\tilde{\pi}}{\omega(t)} A[t]$ (the constant term f_0 too, but this is a non-trivial property). In general, this process allows to compute v -valuations of the entries of our Eisenstein series and more generally, of modular forms.

We hope that at this point the reader has a good view of our theory of tame series and the field of uniformizers, and its various consequences. Although it is difficult to explicitly compute series expansions of our modular forms, the existence of the field \mathfrak{K} provides an environment in which computations are virtually possible. Thanks to this formalism we are able, without much additional effort, to reach most of the results of the first part of the present paper. The reader may find the preliminary material §2, 3 and 4 heavy but this reflect the complexity of the given settings. It is perhaps possible to get rid of the field \mathfrak{K} and work more directly, starting with Definition 1.2 but \mathfrak{K} is the natural field in which one can study series expansions at infinity of our modular forms and also allows to

introduce notions of rationality and integrality of the coefficients etc. for modular forms. The difficulty of multiplying formal series in \mathfrak{K} mirrors the complexity of the behavior at the cusp infinity of Drinfeld modular forms in our generalized setting.

We also take the opportunity to point out that, all along the present paper, a 'modular form' is understood to be a vector-valued modular form associated to one of these representations, while a 'scalar modular form' is one of such forms associated to a one-dimensional such representation.

1.5. Results of the text. The volume is organized in ten sections. These sections can be roughly divided in three principal parts.

- I. Sections 2 to 6. We present the foundations of the theory: field of uniformizers, Hecke operators, Serre's derivatives finiteness results.
- II. Sections 7 and 8. We study modular forms for the representations ρ_Σ^* . We discuss the structure of *strongly regular modular forms*.
- III. Sections 9 and 10. We discuss arguments related to the harmonic product for multiple sums that we apply to a sort of analogue of stuffle product in the theory of classical multiple zeta values, for Eisenstein series, and we present open problems.

Part III can be read quite independently of the previous ones. Reading Part II is possible without reading all proofs in Part I. The following synthesis summarizes the content of the paper and our results (more precise statements will be formulated along the text). We proceed in the order suggested by Parts I to III.

Content of Part I. The key environment is the *field of uniformizers* \mathfrak{K} (remember §1.4) with valuation v , additive non-discrete valuation group $\mathbb{Z}[\frac{1}{p}]$, residual field $\cup_\Sigma \mathbb{K}_\Sigma$, valuation ring \mathfrak{D} and maximal ideal \mathfrak{M} , to which the entire §3 is devoted. The field \mathfrak{K} is constructed explicitly in §3 by taking the completion of the fraction field of an integral ring of entire functions that we call the *ring of tame series*. The next result is proved:

Theorem A. *Let $\Sigma \subset \mathbb{N}^*$ be a finite subset and $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ be a representation of the first kind, let $w \in \mathbb{Z}$ be such that $(\gamma, z) \mapsto J_\gamma(z)^w \rho(\gamma)$ is a factor of automorphy for $\tilde{\Gamma}$. The following properties hold.*

- (1) *There is a natural embedding of \mathbb{K}_Σ -vector spaces $M_w^!(\rho; \mathbb{K}_\Sigma) \xrightarrow{\iota_\Sigma} \mathfrak{K}^{N \times 1}$.*
- (2) *The image by ι_Σ of the \mathbb{K}_Σ -vector space of modular forms $M_w(\rho; \mathbb{K}_\Sigma)$ can be identified with $\iota_\Sigma(M_w^!(\rho; \mathbb{K}_\Sigma)) \cap \mathfrak{D}^{N \times 1}$.*
- (3) *The vector space of cusp forms $S_w(\rho; \mathbb{K}_\Sigma)$ can be identified with the sub-vector space of $M_w(\rho; \mathbb{K}_\Sigma)$ which is sent to $\mathfrak{M}^{N \times 1}$ by the embedding ι_Σ .*
- (4) *We have that $\mathbb{C}_\infty((u))$ naturally embeds in \mathfrak{K} and v restricts to the u -adic valuation.*
- (5) *The vector spaces $M_w(\rho; \mathbb{K}_\Sigma), S_w(\rho; \mathbb{K}_\Sigma)$ are endowed with Hecke operators $T_{\mathfrak{a}}$ associated to ideals \mathfrak{a} of A , which provide a totally multiplicative system of endomorphisms reducing, in the case $\Sigma = \emptyset$, to the classical Hecke operators acting on classical scalar Drinfeld modular forms and cusp forms.*
- (6) *We have \mathbb{K}_Σ -linear maps $\partial_w^{(n)} : M_w(\rho; \mathbb{K}_\Sigma) \rightarrow S_{w+2n}(\rho \det^{-n}; \mathbb{K}_\Sigma)$, defined for all $n \geq 0$ and generalizing Serre's derivatives.*

The slogan is therefore: modular forms are weak modular forms with entries in the ring of integers of the field of uniformizer, and cusp forms are modular forms with entries that are in the maximal ideal. The corresponding results in the body of the text are more precise and cover a wider spectrum of applications. The main examples of modular forms (construction of Poincaré series etc.) and the basic results concerning the spaces $M_w(\rho; \mathbb{K}_\Sigma)$ and $S_w(\rho; \mathbb{K}_\Sigma)$ are contained in §5. Parts (1), (2), (3) will be proved in Theorem 4.12 and (4) is an obvious consequence of the above (so, when $\rho = \mathbf{1}$ is the trivial representation (sending every element of Γ to $1 \in \mathrm{GL}_1$), our construction specialises to the known setting, and $M = \bigoplus_w M_w(\mathbf{1}; \mathbb{C}_\infty)$ is the well known algebra of \mathbb{C}_∞ -valued Drinfeld modular forms for Γ (of type 0 in Gekeler's terminology). We will introduce Poincaré series in §5.3 as a first non-trivial class of modular forms. Part (5) is our Theorem 4.13; the proof is very simple, thanks to the flexibility of the use of the field of uniformizers, and we can say the same about part (6), which corresponds to our Theorem 6.8.

A non-complete field \mathbb{L}_Σ intermediate between \mathbb{K}_Σ and the fraction field of \mathbb{T}_Σ will be needed in the next Theorem; it is defined in §2.1.4.

Theorem B. *The following properties hold, for ρ a representation of the first kind.*

- (1) *For all $w \in \mathbb{Z}$, the \mathbb{L}_Σ -vector space $M_w(\rho; \mathbb{L}_\Sigma)$ has finite dimension. The dimension is zero if $w < 0$.*
- (2) *The dimension of the space $M_1(\rho; \mathbb{L}_\Sigma)$ does not exceed the dimension of the \mathbb{L}_Σ -vector space of common eigenvectors in $\mathbb{L}_\Sigma^{N \times 1}$ of all the matrices $\rho(\gamma)$ with γ in the Borel subgroup of Γ .*

The matrices $\rho(\gamma)$ have all the eigenvalues equal to 1. Note that (1) of Theorem B only deals with modular forms with values in \mathbb{L}_Σ . One reason for this restriction comes from the fact that we use, in the proof, a specialisation property at roots of unity which is unavailable in the general case of \mathbb{K}_Σ -valued functions. This result corresponds to Theorem 5.5.

Content of Part II. As we have mentioned, a scalar Drinfeld modular form for Γ as in Definition 1.1 has a unique u -expansion (1.4) in $\mathbb{C}_\infty[[u]]$ and combining part (2) of Theorem A and Proposition 3.31, one sees that every entry f of a given element of $M_w(\rho; \mathbb{K}_\Sigma)$ has a uniquely determined series expansion

$$f = \sum_{i \geq 0} f_i u^i$$

where for all $i \geq 0$, f_i is an entire function $\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma$ of the variable $z \in \Omega$ of tame series described in §3.2 (and additionally, f_0 is constant in \mathbb{K}_Σ). This generalizes the case of Definition 1.1, where the coefficients f_i are all constant functions, in \mathbb{C}_∞ . It is in general very difficult to describe the coefficients f_i but we make some attempts. For instance, something can be done with Eisenstein series for the representations ρ_Σ^* (see §7) by using the already mentioned *Perkins' series* as in §6.4; see Proposition 7.2.

We fix a subset $\Sigma \subset \mathbb{N}^*$ of cardinality s and we consider, for all $i \in \Sigma$ (${}^t(\cdot)$ denotes transposition),

$$\rho_{t_i}^*(\gamma) = {}^t \begin{pmatrix} a(t_i) & b(t_i) \\ c(t_i) & d(t_i) \end{pmatrix}^{-1},$$

and

$$(1.15) \quad \rho_\Sigma^* := \bigotimes_{i \in \Sigma} \rho_{t_i}^*.$$

This is indeed a representation of the first kind of degree s where $N = 2^{|\Sigma|}$, in the sense of our Definition 4.2. Additionally, ρ_Σ^* is an irreducible representation of Γ in $\mathrm{GL}_N(\mathbb{F}_q[t_\Sigma])$ (see [57] or our Lemma 7.11). An important feature of this class of representations is that it allows to construct certain Eisenstein series in §7. If $s \equiv w \pmod{q-1}$ and $w > 0$ we have the Eisenstein series of weight w :

$$(1.16) \quad \mathcal{E}(w; \rho_\Sigma^*)(z) := \sum_{(a,b) \in A^2 \setminus \{(0,0)\}} (az + b)^{-w} \bigotimes_{i \in \Sigma} \begin{pmatrix} a(t_i) \\ b(t_i) \end{pmatrix},$$

a definition that extends (1.10) to several variables t_Σ , which is a non-zero holomorphic function $\Omega \rightarrow \mathbb{T}_\Sigma^{N \times 1}$, where \mathbb{T}_Σ is the completion $\widehat{\mathbb{C}_\infty[t_\Sigma]}$ of the polynomial algebra $\mathbb{C}_\infty[t_\Sigma]$ with respect to the Gauss norm $\|\cdot\|$, that is, the standard Tate algebra in the variables t_Σ (hence \mathbb{K}_Σ is the completion of the fraction field of \mathbb{T}_Σ for $\|\cdot\|$). These series also generalize the usual scalar Eisenstein series for Γ (case of $\Sigma = \emptyset$). We have

$$\mathcal{E}(w; \rho_\Sigma^*) \in M_w(\rho_\Sigma^*; \mathbb{T}_\Sigma) \setminus S_w(\rho_\Sigma^*; \mathbb{T}_\Sigma).$$

Writing

$$\mathcal{E}(w; \rho_\Sigma^*) = {}^t(\mathcal{E}_1, \dots, \mathcal{E}_N) \in \mathfrak{D}_\Sigma^{N \times 1}$$

we can prove that

$$\mathcal{E}_1, \dots, \mathcal{E}_{N-1} \in \mathfrak{M}, \quad \mathcal{E}_N \in \mathfrak{D} \setminus \mathfrak{M}$$

(we recall that \mathfrak{D} and \mathfrak{M} are respectively the valuation ring and the maximal ideal of the field of uniformizers).

It turns out that

$$\mathcal{E}_N \equiv -\zeta_A(1; \sigma_\Sigma) \pmod{\mathfrak{M}},$$

which is a generalization of (1.11) and (1.13), where

$$(1.17) \quad \zeta_A(n; \sigma_\Sigma) := \sum_{a \in A^+} a^{-n} \sigma_\Sigma(a) = \prod_P \left(1 - \frac{\sigma_\Sigma(P)}{P^n} \right)^{-1} \in \mathbb{T}_\Sigma^\times, \quad n \in \mathbb{N}^*$$

with $\sigma_\Sigma(a) = \prod_{i \in \Sigma} \chi_{t_i}(a)$, the eulerian product running over the irreducible monic polynomials of A (generalization of (1.12)), are the zeta values in Tate algebras studied in [2, 3, 5] as well as in other papers. It can be proved [2, §2.1] that $\zeta_A(n; \sigma_\Sigma)$ extends to an entire function of the variables t_Σ .

These Eisenstein series seem to be at the crossroad of several interesting features that we gather in the next result (but see the text for more precise results). To begin, we must point out that in §7.2, we construct an indexation $(\mathcal{E}^J)_{J \subset \Sigma}$ of the entries \mathcal{E}_i of an Eisenstein

series $\mathcal{E} = \mathcal{E}(w; \rho_\Sigma^*)$ by the subsets J of Σ . With this indexation, the first entry \mathcal{E}_1 of \mathcal{E} equals \mathcal{E}^\emptyset and the last entry \mathcal{E}_N equals \mathcal{E}^Σ . We have the next result.

Theorem C. *The following properties hold for the Eisenstein series $\mathcal{E}(w; \rho_\Sigma^*)$:*

- (1) *If $w = 1$ and $J \subsetneq \Sigma$ is such that $|J| = (m-1)(q-1) + l$ with $m > 0$ and $1 \leq l \leq q-1$ or $m = 0$ and $l = q-1$ then we have the v -valuation $v(\mathcal{E}^J) = 1 - q^{-m}(q-l) > 0$ and $v(\mathcal{E}^\Sigma) = 0$.*
- (2) *If $w > 0$, $\mathcal{E}(w; \rho_\Sigma^*)$ is \mathfrak{v} -integrally definable (it has an integrality property of the coefficients that is described in our Definition 7.7) for valuations \mathfrak{v} of $K(\mathbb{t}_\Sigma)$ associated with a non-zero prime ideal \mathfrak{p} of A , and this for all but finitely many \mathfrak{p} .*
- (3) *Evaluating the first entry of $\mathcal{E}(w; \rho_\Sigma^*)$ at $t_i = \theta^{q^{k_i}}$ for all $i \in \Sigma$ with $k_i \in \mathbb{N}$ yields, up to a scalar factor, a Drinfeld quasi-modular form in the sense of [14] with an A -expansion as in [67] and all these series occur in this process.*

Part (1) can be generalized to some cases in which $\ell_q(w) < q$ (the sum of the digits of the q -ary expansion of w is $< q$) thanks to Theorem 6.16, a result that describes the v -valuation of Perkins series as in §6.4. The question of the computation of these v -valuations in full generality, related to the computation of the v -valuation of all Perkins' series is, we should say, not easy, and still open. It is related to a similar question on v -valuations of Perkins' series and therefore of generalizations of Goss' polynomials. The recent work of Gekeler [31] suggests us that this is accessible but difficult.

Part (2) generalizes the properties of integrality of the coefficients of the u -expansion of scalar Eisenstein series as in [24, (6.3)]. Note that our result is more recondite in the case $\Sigma \neq \emptyset$. Indeed a notion of integrality of the coefficients of a series $\sum_i f_i u^i$ with coefficients f_i which are tame series has to be introduced, and this is exactly what we do, and it is not a triviality. Hence, Theorem C would not be meaningful without our investigations of §3. As for part (3), it was motivated by, and answers, a question by Goss (in a 2013 letter to A. Petrov, [37]). A quick description of properties related to \mathfrak{v} -adic modular forms is given in §7.6.5.

In §7.5 we will explore the arithmetic structure of negative weight modular forms for ρ_Σ and deduce, by duality with Eisenstein series, a weak form of the functional identities [2, Theorem 1].

In general, we do not control the dimensions and we are unable to construct bases of the spaces $M_w(\rho; \mathbb{K}_\Sigma)$ except when $w = 1$ and $\rho = \rho_\Sigma^*$. We have proved:

Theorem D. *If $|\Sigma| \equiv 1 \pmod{q-1}$ the vector space $M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ is one-dimensional, generated by $\mathcal{E}(1; \rho_\Sigma^*)$.*

This is Theorem 7.5. Part (2) of Theorem B (see Theorem 5.12) also includes an upper bound for the dimensions of the \mathbb{L}_Σ -vector spaces, and implies a positive answer to the question raised by [63, Problem 1.1] thanks to Theorem D. The proofs of (2) of Theorem B and of Theorem D are easy but use a natural isomorphism between (scalar) Drinfeld modular forms for congruence subgroups of Γ and spaces of *automorphic functions* (harmonic cocycles) over the Bruhat-Tits tree of Ω , and the same specialisation properties in terms of the variables t_i used in the proof of (1). When we do this with the entries of the elements

of $M_1(\rho; \mathbb{L}_\Sigma)$ span scalar Drinfeld modular forms of weight one for congruence subgroups of Γ . The proof of this result is thus based on a crucial earlier remark of Gekeler (which can be found in Cornelissen's paper [19]).

From §8 on, the paper almost exclusively focuses on structure properties of modular forms for the representations ρ_Σ^* . We introduce here the notion of *strongly regular modular form* (see Definition 8.4). A strongly regular modular form $f = {}^t(f_1, \dots, f_N)$ (transpose) is a Drinfeld modular form (in our generalized setting) which satisfies certain conditions on the v -valuations of its entries. Theorem 8.7 allows a completely explicit structure description for these modular forms which can be stated as follows (more precise results can be found in the text).

Theorem E. *Every strongly regular modular form associated to the representation ρ_Σ^* can be constructed combining 'elementary' Eisenstein series $\mathcal{E}(1; \rho_{t_i})$ and $\mathcal{E}(q; \rho_{t_i})$ for $i \in \Sigma$ by using the Kronecker product, and scalar Eisenstein series. In particular, the $M \otimes_{\mathbb{C}_\infty} \mathbb{K}_\Sigma$ -module of \mathbb{K}_Σ -valued strongly regular modular forms is free of rank $N = 2^s$ where $s = |\Sigma|$.*

The advantage of working with strongly regular modular forms is that to study them we do not need the full strength of the tools developed in Part I of this text, namely, the field of uniformizers and the theory of quasi-periodic matrix functions. To prove Theorem E, we only need appropriate generalizations of the arguments of [63].

The continuous $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear automorphism τ of \mathbb{K}_Σ extending the automorphism $c \mapsto c^q$ of \mathbb{K}_Σ induces injective $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear maps

$$(1.18) \quad M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma) \xrightarrow{\tau} M_{qw}(\rho_\Sigma^*; \mathbb{K}_\Sigma)$$

and we have a similar property with \mathbb{K}_Σ replaced with \mathbb{L}_Σ . We show, in Theorem 8.9 that for every w there exists $k \in \mathbb{N}$ such that $\tau^k(f)$ is *strongly regular* for every $f \in M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$. This shows that Drinfeld modular forms in $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ are not too distant from strongly regular modular forms and this allows to deduce:

Theorem F. *The \mathbb{K}_Σ -vector spaces $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ have finite dimensions.*

Note that the functions of Theorem F have values in $\mathbb{K}_\Sigma^{N \times 1}$, not just in $\mathbb{L}_\Sigma^{N \times 1}$ so that the methods of proof of Theorems B and D do not apply for Theorem F. After Theorem E for every modular form $f \in M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ there is k such that $\tau^k(f)$ can be constructed combining Eisenstein series, and the coefficients in the construction are in \mathbb{K}_Σ . In full generality, it seems difficult to overcome the use of the field \mathfrak{K} and prove Theorem F for any representation of the first kind.

Content of Part III. This work ends with §9 and §10 which are more speculative and contain a description of further perspectives of research. This part can be read quite independently of the previous ones. We present here the harmonic product for multiple sums, the interaction with multiple sums à la Thakur, multiple Eisenstein series, and we propose conjectures based on identities between Eisenstein series and many explicit formulas.

In §9 we prove (see Theorem 9.4) a variant of a harmonic product formula for certain A -periodic multiple sums and we apply it to compute several explicit formulas relating

Eisenstein series for ρ_{Σ}^* . Some of these formulas have been conjectured in earlier works. In §9 we state Conjecture 10.4, where we evoke the potential existence of an \mathbb{F}_p -algebra of *multiple Eisenstein series* and an \mathbb{F}_p -isomorphism with an \mathbb{F}_p -algebra of *multiple zeta values in Tate algebras*. Additionally, we speculate that a multiple Eisenstein series is a modular form for ρ_{Σ}^* in our settings if and only if the multiple zeta values in Tate algebras corresponding to it, which also is related to its constant term, is *eulerian* following our Definition 10.3. We describe in §10.2 a conjecture on certain identities involving zeta values in Tate algebras a particular case of which has been recently proved by Hung Le and Ngo Dac in [43] and we end the work with analogue conjectural identities involving our Eisenstein series $\mathcal{E}(w; \rho_{\Sigma}^*)$. These identities are so complicated that are essentially undetectable by numerical experiments. They do not seem to have analogues in the classical setting of \mathbb{C} -valued vector-valued Eisenstein series for the group $\mathrm{SL}_2(\mathbb{Z})$.

Acknowledgements. The author is very grateful to the reviewer for a careful reading, a detailed report and very useful comments that allowed to strongly improve many parts of the paper. The author thanks the Editorial Board of the *Memoirs of the American Mathematical Society* for the careful handling of the submission. The author also thanks the Max Plank Institute of Bonn, where he worked on the very first draft for a month in Spring 2019. Many thanks to Florian Breuer, Oğuz Gezmiş, Ernst-Ulrich Gekeler, Shin Hattori, Tuan Ngo Dac, Rudy Perkins and Dinesh Thakur for readings, corrections, discussions, and suggestions on earlier versions of the manuscript. A deep thought to David Goss, that encouraged the development of this theory. The author misses his comments and enthusiastic encouragements very much. A final acknowledgement to Guido Pellarin, father of the author. The writing of this paper began few days after he passed away, but the author felt being encouraged by him all along this journey, so this work is dedicated to his memory.

2. PRELIMINARIES

Most commonly used notation.

- $\mathbb{N} = \{0, 1, \dots\}$ the set of natural integers.
- $\mathbb{N}^* = \{1, \dots\}$ the set of positive natural integers.
- $B^{M \times N}$: M -row, N -column arrays with coefficients in the set B .
- I_r : the $r \times r$ identity matrix.
- \sqcup disjoint union.
- $\mathrm{Diag}(*, \dots, *)$ diagonal matrix.
- $\ell_q(n)$ sum of the digits of the base- q expansion of the positive integer n .
- \mathbb{F}_q finite field with $q = p^e$ elements, where p is a prime number and $e > 0$.
- $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$, $K_{\infty} = \mathbb{F}_q((\frac{1}{\theta}))$, $\mathbb{C}_{\infty} = \widehat{K_{\infty}^{sep}}$.
- $\Gamma = \mathrm{GL}_2(A)$.
- $\tilde{\Gamma} = \Gamma / \mathbb{F}_q^{\times} = \mathrm{PGL}_2(A)$.
- $\mathbf{1}$ the trivial representation sending Γ to $1 \in \mathbb{F}_q^{\times} = \mathrm{GL}_1(A)$.
- $J_{\gamma}(z)$ the usual factor of automorphy.

- $\Omega = \mathbb{C}_\infty \setminus K_\infty$ the Drinfeld half-plane.
- u the uniformizer at infinity of Ω .
- S_w, M_w , spaces of cusp forms and modular forms of weight w .
- Σ a finite subset of \mathbb{N}^* .
- \mathbb{T}_Σ Tate algebra in the variables $t_\Sigma = (t_i : i \in \Sigma)$.
- \mathbb{K}_Σ the completion of the fraction field of \mathbb{T}_Σ (see §2.1.3).
- \mathbb{L}_Σ a certain intermediate field $\mathbb{T}_\Sigma \subset \mathbb{L}_\Sigma \subset \mathbb{K}_\Sigma$ (see §2.1.4).
- \mathfrak{K} field of uniformizers, with valuation v , valuation ring \mathfrak{O} , maximal ideal \mathfrak{M} , residual field $\cup_\Sigma \mathbb{K}_\Sigma$.
- $B^\circ \langle\langle \underline{e} \rangle\rangle^b$ the B -module of tame series with coefficients in B .
- ω the function of Anderson and Thakur.

Overview of the section. In this section we collect the basic objects over which we are going to build our theory. In §2.1.3 and 2.1.4 we describe the fundamental fields $\mathbb{K}_\Sigma, \mathbb{L}_\Sigma$, depending on choices of finite subsets Σ of $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ (and already used in the introduction). They serve to introduce, in §2.2, a class of analytic functions with values in certain non-archimedean countably cartesian Banach algebras, such as \mathbb{K}_Σ (\mathbb{L}_Σ is not complete but it is a filtered union of such algebras). For example, Proposition 2.11 is a useful analogue in our settings of Liouville's Theorem stating that a bounded entire function is constant. In §2.3 the reader will find the basic tools related to the exponential and the logarithm of a Drinfeld module, and allied functions. In §2.4 we discuss other relevant functions, notably certain generalizations of Anderson and Thakur omega function, and generalizations of the entire map $\chi_t : \mathbb{C}_\infty \rightarrow \mathbb{T}$ that interpolates the map $A \ni a \mapsto a(t) \in \mathbb{F}_q[t]$. These functions arise naturally when one studies quasi-periodic matrix functions in §4. In turn, these quasi-periodic matrix functions are essential to construct and analyze expansions of our modular forms.

2.1. Rings, fields, modules. For the general settings on valued rings and fields and local fields, we refer to the author's [60, §2], from which we borrow the basic notation, and the books [16, 75].

We present here some basic tools that we need, on ultrametric Banach vector spaces and algebras (§2.1.1), Tate and affinoid algebras (§2.1.2). In §2.1.3 and §2.1.4 we introduce certain ultrametric fields $\mathbb{K}_\Sigma, \mathbb{L}_\Sigma$ (the former complete the latter not), crucial to us as they constitute the target spaces of the entries of the vectorial modular forms we discuss in this paper. The level of generality of this presentation is quite broad (it can be useful for other works). Later in the present volume we only consider the cases of $L = \mathbb{C}_\infty$ or L a local field containing \mathbb{F}_q .

We consider a field L containing \mathbb{F}_q , valued with multiplicative valuation

$$L \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}.$$

We also choose an additive valuation

$$L \xrightarrow{v} \mathbb{R} \cup \{\infty\}$$

with the property that $|\cdot| = c^{-v(\cdot)}$, for some $c > 1$. We denote by $\mathcal{O}_L, \mathcal{M}_L$ and k_L respectively the valuation ring, the maximal ideal, and the residual field $\mathcal{O}_L/\mathcal{M}_L$ of L .

The set \mathcal{L} , whose elements are the subfields F of L over which the restriction of $|\cdot|$ is trivial, is non-empty (it contains \mathbb{F}_q) and it is ordered by inclusion. Hence there is a unique maximal subfield \tilde{k}_L over which the restriction of the valuation is trivial. This subfield is algebraically closed in L and the inclusion map $\tilde{k}_L \rightarrow \mathcal{O}_L$ induces an inclusion $\tilde{k}_L \rightarrow k_L$.

Lemma 2.1. *If L is complete and if there is a ring map $k_L \rightarrow \mathbb{F}_q^{ac}$, then $\tilde{k}_L \cong k_L$.*

Proof. For all $x \in \mathcal{O}_L$ there is $m > 0$ such that $\varpi(x) := \lim_{n \rightarrow \infty} x^{p^{mn}}$ exists; it does not depend on m . This defines a ring map $k_L \rightarrow \tilde{k}_L$ inverse of the above map $\tilde{k}_L \rightarrow k_L$. \square

We suppose that L is complete and that k_L embeds in an algebraic closure of \mathbb{F}_q and we identify k_L with the maximal subfield of L over which the valuation is trivial. If $x \in \mathcal{O}_L$ we denote by \bar{x} its image in k_L by the morphism of reduction modulo \mathcal{M}_L .

2.1.1. Banach L -vector spaces and algebras.

Definition 2.2. A Banach L -vector space $(B, |\cdot|_B)$ is the datum of an L -vector space B together with a map

$$|\cdot|_B : B \rightarrow \mathbb{R}_{\geq 0}$$

such that

- (1) for all $x, y \in B$, $|x + y|_B \leq \max\{|x|_B, |y|_B\}$,
- (2) for all $x \in B$ and $\lambda \in L$, $|\lambda x|_B = |\lambda| |x|_B$,
- (3) for any $x \in B$, $|x|_B = 0$ if and only if $x = 0$,

and such that B is complete for the metric induced by $|\cdot|_B$.

We say that two Banach L -vector spaces $(B_1, |\cdot|_{B_1})$ and $(B_2, |\cdot|_{B_2})$ are *isometrically isomorphic* if there exists an isomorphism of vector spaces $\varphi : B_1 \rightarrow B_2$ such that $|\varphi(x)|_{B_2} = |x|_{B_1}$ for all $x \in B_1$.

The spaces $c_I(L)$. Let I be a countable set. We denote by $c_I(L)$ the set of sequences $(x_i)_{i \in I} \in L^I$ such that $x_i \rightarrow 0$ where the limit is for the Fréchet filter of I , that is, the filter of the complements of finite subsets of I (we shall more simply write $i \rightarrow \infty$). The set $c_I(L)$ is an L -vector space. We set $\|(x_i)_{i \in I}\| = \sup_{i \in I} \{|x_i|\}$ for $(x_i)_{i \in I} \in c_I(L)$. Then, the supremum is a maximum and $(c_I(L), \|\cdot\|)$ carries a structure of Banach L -vector space. Note that $\|c_I(L)\| = |L|$, where

$$|L| := \{r \in \mathbb{R}_{\geq 0} : \exists x \in L \text{ such that } |x| = r\};$$

the image of $\|\cdot\|$ equals the image of $|\cdot|$ in $\mathbb{R}_{\geq 0}$.

Definition 2.3. A Banach L -vector space B is *countably cartesian* if it is isometrically isomorphic to a space $c_I(L)$ with I countable. Let $\mathcal{B} = (b_i)_{i \in I}$ be a family of elements of B . We say that \mathcal{B} is an *orthonormal basis* if $|b_i|_B = 1$ for all i and if every element $f \in B$ can be expanded in a unique way in a series

$$(2.1) \quad f = \sum_{i \in I} f_i b_i, \quad f_i \in L, \quad f_i \rightarrow 0,$$

so that $|f|_B = \max_i |f_i|$.

Compare with [15, Chapter 2].

Definition 2.4. A Banach L -vector space $(B, |\cdot|_B)$ with a structure of commutative L -algebra with unit is said to be a *Banach L -algebra* if $|1|_B = 1$ and $|\cdot|_B$ is sub-multiplicative: for all $x, y \in B$, $|xy|_B \leq |x|_B |y|_B$. We identify L with $L \cdot 1 \subset B$. A Banach L -algebra is *countably cartesian* if the underlying Banach L -vector space is so.

2.1.2. *Tate algebras, affinoid algebras.* All along the present work we make use of the following basic notations. Let R be a commutative ring with unit. We choose once and for all independent variables t_i with $i \in \mathbb{N}^*$ and we work in the R -algebra $R[t_i : i \in \mathbb{N}^*]$. If Σ is a finite subset of \mathbb{N}^* of cardinality $|\Sigma| = s$ we denote by \underline{t}_Σ the family of variables $(t_i : i \in \Sigma)$. Then, $R[\underline{t}_\Sigma]$ denotes the R -algebra $R[t_i : i \in \Sigma]$ in the s variables \underline{t}_Σ , embedded in $R[t_i : i \in \mathbb{N}^*]$ in the canonical way. If $\Sigma = \{i\}$ is a singleton, then we often simplify our notations writing $t = t_i$.

We consider Σ a finite subset of \mathbb{N}^* and a sub-multiplicative norm $|\cdot|'$ on $L[\underline{t}_\Sigma]$ which restricts to $|\cdot|$ on $L \subset L[\underline{t}_\Sigma]$ (L is identified with a subalgebra of $L[\underline{t}_\Sigma]$). We denote by

$$\widehat{L[\underline{t}_\Sigma]}_{|\cdot|'}, \quad \left(\text{or } L[\underline{t}_\Sigma]^{\wedge}_{|\cdot|'} \right)$$

the completion of $L[\underline{t}_\Sigma]$ for $|\cdot|'$ (⁷). It is a Banach L -algebra in the sense of Definition 2.4.

For example, we can take $|\cdot|' = \|\cdot\|$ the *Gauss valuation* over $L[\underline{t}_\Sigma]$, that is, the unique norm of $L[\underline{t}_\Sigma]$ which extends $|\cdot|$, such that

$$\|\underline{t}_\Sigma^i\| = 1$$

for all $\underline{i} = (i_j : j \in \Sigma) \in \mathbb{N}^\Sigma$, where

$$\underline{t}_\Sigma^i = \prod_{j \in \Sigma} t_j^{i_j}.$$

It is easy to see that $\|\cdot\|$ is multiplicative (to see this it suffices to compute images in the residual field). In this case we write

$$\mathbb{T}_{L,\Sigma} := \widehat{L[\underline{t}_\Sigma]}_{\|\cdot\|}.$$

We usually drop the reference to L if it is algebraically closed or if its choice is clear in the context, hence writing in a more compact way \mathbb{T}_Σ . This is the *Tate algebra* (or *standard affinoid algebra*) of dimension $s = |\Sigma|$. If $\Sigma = \{i\}$ is a singleton we prefer the simpler notation \mathbb{T}_L or \mathbb{T} for this algebra, with variable t . Note that if $\Sigma' \subset \Sigma$ then the canonical embedding $L[\underline{t}_{\Sigma'}] \subset L[\underline{t}_\Sigma]$ induces an embedding $\mathbb{T}_{L,\Sigma'} \subset \mathbb{T}_{L,\Sigma}$.

The Tate algebra $\mathbb{T}_{L,\Sigma}$ is isomorphic to the sub- L -algebra of the formal series

$$(2.2) \quad f = \sum_{\substack{i_j \geq 0 \forall j \in \Sigma \\ \underline{i} = (i_j : j \in \Sigma)}} f_{\underline{i}} \underline{t}_\Sigma^i \in L[[\underline{t}_\Sigma]]$$

⁷The last notation is introduced for graphical convenience, in those circumstances where the hat in the first displayed formula is too large.

which satisfy

$$\lim_{\min\{i_j:j \in \Sigma\} \rightarrow \infty} f_{\underline{i}} = 0.$$

Thus, we have, for f a formal series of $\mathbb{T}_{L,\Sigma}$ expanded as above and non-zero, that

$$\|f\| = \sup_{\underline{i}} |f_{\underline{i}}| = \max_{\underline{i}} |f_{\underline{i}}|$$

and $\mathbb{T}_{L,\Sigma}$ is countably cartesian (Definition 2.4). It is well known that $\mathbb{T}_{L,\Sigma}$ is a ring which is Noetherian and it is also a unique factorisation domain, normal, of Krull dimension s (see [22, Theorem 3.2.1] for a wider treatise, see [15]). We will also use the L -sub-algebra $\mathbb{E}_{L,\Sigma}$ of $\mathbb{T}_{L,\Sigma}$ of the series f as above with the property that for all $r \in |L^\times|$,

$$\lim_{\min\{i_j:j \in \Sigma\} \rightarrow \infty} |f_{\underline{i}}| r^{i_1 + \dots + i_n} = 0.$$

If L is complete and algebraically closed, this can be identified with the L -algebra of *entire functions* in the variables t_Σ . If Σ is a singleton $\{i\}$, we will write \mathbb{E}_L or \mathbb{E} for this algebra, and we will use the variable $t = t_i$.

An *affinoid L -algebra* \mathcal{A} is the datum of a topological L -algebra \mathcal{A} together with a surjective L -algebra morphism

$$(2.3) \quad \mathbb{T}_{L,\Sigma} \xrightarrow{\psi} \mathcal{A}, \quad \Sigma \subset \mathbb{N}^*, \quad \Sigma \text{ finite.}$$

Every affinoid L -algebra comes equipped with a Banach L -algebra structure, with the norm

$$\|g\| = \inf_{\psi(f)=g} \|f\|, \quad g \in \mathcal{A}.$$

The kernel of ψ is closed and we have the next result where we assume that L is algebraically closed.

Lemma 2.5. *Every affinoid L -algebra is countably cartesian.*

Proof. We consider \mathcal{A} an affinoid algebra, with ψ and $\mathbb{T}_{L,\Sigma}$ as in (2.3). If L is algebraically closed and \mathfrak{J} is an ideal of $\mathbb{T}_{L,\Sigma}$, by [12, §1.3 Theorem 6], there exists an orthonormal basis $(b_i)_{i \in I}$ of $\mathbb{T}_{L,\Sigma}$ and a subset $J \subset I$ such that $(b_j)_{j \in J}$ is an orthonormal basis of \mathfrak{J} . Then, $(\psi(b_i))_{i \in I \setminus J}$ defines an orthonormal basis of \mathcal{A} . \square

The general case is also true, where L is not necessarily algebraically closed.

Remark 2.6. Note that if \mathcal{A} is the affinoid algebra associated to an affinoid subset of $\mathbb{P}_L^{1,an}$ (with $\mathbb{P}_L^{1,an}$ the rigid analytic affine line over L), with its spectral norm, then it is countably cartesian also as an easy consequence of the Mittag-Leffler decomposition [22, Proposition 2.2.6].

2.1.3. The completion $\mathbb{K}_{L,\Sigma}$ of the fraction field of $\mathbb{T}_{L,\Sigma}$. Let L be a valued field, complete, containing \mathbb{F}_q . The fraction field of $\mathbb{T}_{L,\Sigma}$ is not complete, unless $\Sigma = \emptyset$. We write $\mathbb{K}_{L,\Sigma}$ for its completion. It is easy to see that this is also equal to the completion of $L(t_\Sigma)$, for the extension of the Gauss norm. If L is a local field, so that $L = \mathbb{F}((\pi))$ with \mathbb{F} a finite field containing \mathbb{F}_q and π a uniformizer, then $\mathbb{K}_\Sigma = k_L(t_\Sigma)((\pi))$. The residual field $k_{\mathbb{K}_{L,\Sigma}}$ of $\mathbb{K}_{L,\Sigma}$ is $k_L(t_\Sigma)$. If $\Sigma' \subset \Sigma$, we have an isometric embedding $\mathbb{K}_{L,\Sigma'} \subset \mathbb{K}_{L,\Sigma}$.

Lemma 2.7. *Let Σ' be a subset of Σ . Let $\mathcal{B} = (b_i)_{i \in I}$ be a family of elements of $\mathcal{O}_{\mathbb{K}_{L,\Sigma}}$ such that $(\bar{b}_i)_{i \in I}$ is a basis of the $k_L(\underline{t}_{\Sigma'})$ -vector space $k_L(\underline{t}_{\Sigma})$. Then, every element f of $\mathbb{K}_{L,\Sigma}$ can be expanded, in a unique way, as a converging series*

$$f = \sum_{i \in I} f_i b_i, \quad f_i \in \mathbb{K}_{\Sigma'}, \quad f_i \rightarrow 0,$$

and $\|f\| = \max_{i \in I} \|f_i\|$.

In the above lemma I is countable (this follows from the fact that k_L is countable). If we choose $\Sigma' = \emptyset$ we see that Lemma 2.7 implies that $\mathbb{K}_{L,\Sigma}$ is countably cartesian as in Definition 2.4. In other words, the Banach L -vector space $\mathbb{K}_{L,\Sigma}$ is endowed with an orthonormal basis providing us with an isometric isomorphism with a Banach L -space $c_I(L)$. The proof that we present is essentially the same as Serre's in [72, Lemma 1, Proposition 1].

Proof of Lemma 2.7. One sees easily that $\|\mathbb{K}_{L,\Sigma}\| = |L|$, therefore it suffices to show the lemma for $f \in \mathbb{K}_{L,\Sigma}$ with $\|f\| = 1$. Let us consider $\alpha \in L[\underline{t}_{\Sigma}]$ with $\|\alpha\| = 1$. We can decompose (in a unique way) $\alpha = \alpha_0 + \alpha_1$ with $\alpha_i \in L[\underline{t}_{\Sigma}]$, $\bar{\alpha}_1 \in k_L[\underline{t}_{\Sigma}] \setminus \{0\}$, and $\|\alpha_0\| < 1$. For any multi-index $\underline{k} = (k_i : i \in \Sigma) \in \mathbb{N}^{\Sigma}$ we have, in $\mathbb{K}_{L,\Sigma}$ (with $\underline{t}_{\Sigma}^{\underline{k}} = \prod_{i \in \Sigma} t_i^{k_i}$):

$$\underline{t}_{\Sigma}^{\underline{k}} \alpha^{-1} = \frac{\underline{t}_{\Sigma}^{\underline{k}}}{\alpha_1} \left(1 - \frac{\alpha_0}{\alpha_1} + \frac{\alpha_0^2}{\alpha_1^2} - \dots \right)$$

(the series converges because $\|\alpha_0\| < 1$). For every \underline{k} and $j \geq 0$, the image of $\underline{t}_{\Sigma}^{\underline{k}} \alpha_1^{-j}$ in $k_L(\underline{t}_{\Sigma})$ for the reduction map can be expanded in the basis $(\bar{b}_i)_{i \in I}$. We deduce that any element $f = \frac{\beta}{\alpha} \in L(\underline{t}_{\Sigma})$, $\alpha \neq 0$, can be expanded as a convergent series:

$$f = \sum_{i \in I} f_i b_i, \quad f_i \rightarrow 0, \quad f_i \in \mathbb{K}_{\Sigma'}.$$

This expansion is unique because otherwise, there would exist a non-trivial relation

$$0 = \sum_{i \in I} f_i b_i$$

such that for some $i \in I$, $\|f_i\| = 1$, in contradiction with the fact that $(\bar{b}_i)_{i \in I}$ is a basis of $k_L(\underline{t}_{\Sigma})$ over $k_L(\underline{t}_{\Sigma'})$. This means that there is an isometric embedding $L(\underline{t}_{\Sigma}) \rightarrow c_I(\mathbb{K}_{L,\Sigma'})$. Completing, we are left with an isometric isomorphism of Banach L -vector spaces $\mathbb{K}_{L,\Sigma} \cong c_I(\mathbb{K}_{L,\Sigma'})$ which terminates the proof. \square

2.1.4. The non-complete fields $\mathbb{L}_{L,\Sigma}$. Let Σ, L, \dots as in §2.1.3. In this paper we also need certain fields intermediate between the fraction field of $\mathbb{T}_{L,\Sigma}$ and $\mathbb{K}_{L,\Sigma}$. For any $d \in k_L[\underline{t}_{\Sigma}] \setminus \{0\}$ we have the affinoid L -algebra (completion for the Gauss norm) $\widehat{\mathbb{T}_{L,\Sigma}[d^{-1}]}$ which is a Banach L -sub-algebra of $\mathbb{K}_{L,\Sigma}$ which also is countably cartesian. We consider

$$\mathbb{L}_{L,\Sigma} = \bigcup_{d \in k_L[\underline{t}_{\Sigma}] \setminus \{0\}} \widehat{\mathbb{T}_{L,\Sigma}[d^{-1}]}.$$

Lemma 2.8. $\mathbb{L}_{L,\Sigma}$ is a subfield of $\mathbb{K}_{L,\Sigma}$.

Proof. The relation of divisibility in $k_L[t_\Sigma]$ induces a filtration of $\mathbb{L}_{L,\Sigma}$ by Banach L -subalgebras of the form $\widehat{\mathbb{T}_{L,\Sigma}[d^{-1}]}$ so that $\mathbb{L}_{L,\Sigma}$ is an L -sub-algebra of $\mathbb{K}_{L,\Sigma}$. We still need to show that every non-zero element f of $\mathbb{L}_{L,\Sigma}$ is invertible; we follow the same ideas of Lemma 2.7; there is no loss of generality if we suppose that $\|f\| = 1$. There exists $d \in k_L[t_\Sigma] \setminus \{0\}$ such that $f \in \mathbb{T}_{L,\Sigma}[d^{-1}]^\wedge$. We can write $f = \alpha_1 - \alpha_0$ where $\alpha_1 \in k_L[t_\Sigma][d^{-1}] \setminus \{0\}$ and where $\alpha_0 \in \mathbb{T}_{L,\Sigma}[d^{-1}]^\wedge$ is such that $\|\alpha_0\| < 1$. Therefore, in $\mathbb{K}_{L,\Sigma}$:

$$\frac{1}{f} = \frac{1}{\alpha_1} \left(1 - \frac{\alpha_0}{\alpha_1}\right)^{-1} = \frac{1}{\alpha_1} \sum_{i \geq 0} \left(\frac{\alpha_0}{\alpha_1}\right)^i$$

and the series converges in $\mathbb{T}_{L,\Sigma}[\tilde{d}^{-1}]^\wedge \subset \mathbb{L}_{L,\Sigma}$, for some element $\tilde{d} \in k_L[t_\Sigma]$. \square

Note that $\mathbb{L}_{L,\Sigma}$ contains the fraction field of \mathbb{T}_Σ and is not complete, unless $\Sigma = \emptyset$. The fields $\mathbb{L}_{L,\Sigma}$ and $\mathbb{K}_{L,\Sigma}$ both have residual field $k_L(t_\Sigma)$ and $\mathbb{K}_{L,\Sigma}$ is the completion of $\mathbb{L}_{L,\Sigma}$ for the Gauss norm.

2.2. Analytic functions with values in non-archimedean Banach algebras. In this subsection we suppose that L is an algebraically closed valued field with multiplicative valuation $|\cdot|$, complete with respect to this valuation, with residual field \mathbb{F}_q^{ac} . We choose $(B, |\cdot|_B)$ a Banach L -algebra which is countably cartesian in the sense of Definition 2.4.

Let X/L be a rigid analytic variety, that is, the datum of (X, T, \mathcal{O}_X) with X a set, a G -topology T and a structure sheaf \mathcal{O}_X of L -algebras. In all the following, we denote by $\mathcal{O}_{X/B}$ the presheaf of B -algebras defined, for $\mathcal{U} = (U_i)_i$ an affinoid covering of X , by

$$\mathcal{O}_{X/B}(U_i) = \mathcal{O}_X(\widehat{U_i}) \otimes_L B \cong \mathcal{O}_X(\widehat{U_i}) \otimes_L c_I(L),$$

the completion being taken for the spectral (sub-multiplicative) norm on U_i (see [15, §3.2]), and where \cong indicates an isometric isomorphism of Banach L -vector spaces.

An *analytic function* (also called *holomorphic function*) from X to B is by definition an element of $\mathcal{O}_{X/B}(X)$. Equivalently, an analytic function $f : X \rightarrow B$ is a function such that for every rational subset $Y \subset X$, the restriction $f|_Y$ is the uniform limit over Y of a sequence of elements of $\mathcal{O}_X(Y) \otimes_L B$. As an alternative notation, we choose

$$f \in \text{Hol}(X \rightarrow B).$$

Let $\mathcal{B} = (b_i : i \in I)$ be a orthonormal basis of B (countable). Every element $f \in \text{Hol}(X \rightarrow B)$ can be expanded, in a unique way, as

$$f = \sum_{i \in I} f_i b_i$$

where $f_i|_Y \rightarrow 0$ for the spectral norm associated to any rational subset Y of X (remember (2.1)). For example, we can take $B = \mathbb{K}_\Sigma$ or $B = \mathbb{T}_\Sigma[d^{-1}]^\wedge$ with $d \in k_L[t_\Sigma] \setminus \{0\}$.

Let C be a sub- L -algebra of B (not necessarily complete). We write

$$\text{Hol}_B(X \rightarrow C)$$

for the C -algebra of holomorphic (or analytic) functions from X to B such that the image is contained in C , and we omit the subscript if $B = C$ to meet with previously introduced notation. For instance, we can take $C = \mathbb{L}_\Sigma \subset \mathbb{K}_\Sigma = B$. We denote by

$$\mathcal{O}_{X/B/C}$$

the presheaf of C -algebras determined by $\mathcal{O}_{X/B/C}(Y) = \text{Hol}_B(Y \rightarrow C)$ for Y rational subset of X . Since C is an L -algebra, for every $U = \text{Spm}(\mathcal{A})$ affinoid subdomain, $\mathcal{O}_{X/B/C}(U)$ is an \mathcal{A} -module and we can define, for M a finitely generated \mathcal{A} -module, the pre-sheaf $\mathcal{M}_{B/C}$ on X by

$$\mathcal{M}_{B/C}(U) = M \otimes_{\mathcal{A}} \mathcal{O}_{X/B/C}(U).$$

Tate's acyclicity theorem (see for example [22, Theorem 4.2.2]) is easily seen to extend to this framework and we have the next result:

Lemma 2.9. *The presheaf $\mathcal{M}_{B/C}$ is a sheaf of C -algebras.*

We omit the details of the proof because the proof of the version of Tate's acyclicity theorem given in the above reference can be easily adapted to our framework, thanks to the hypothesis that B is countably cartesian. We will limit ourselves on few aspects, in the case of \mathcal{M} trivial. If $U = \text{Spm}(\mathcal{A})$ is an affinoid subdomain of X and $(U_j)_{j \in J}$ an admissible covering of U (with J a finite set), saying that $\mathcal{O}_X|_U$ is a sheaf of L -algebras amounts to saying that there is an exact sequence of L -algebras

$$0 \rightarrow \mathcal{O}_X(U) \xrightarrow{\alpha} \prod_{j \in J} \mathcal{O}_X(U_j) \xrightarrow{\beta} \prod_{j, k \in J} \mathcal{O}_X(U_j \cap U_k)$$

where α is defined by the restrictions on the U_j 's and $\beta((f_j)_{j \in J}) = (f_j|_{U_j \cap U_k} - f_k|_{U_j \cap U_k})_{j, k \in J}$. Taking $(\cdot) \widehat{\otimes}_L B$ determines an exact sequence of B -algebras because, denoting by α and β the resulting maps, with $|\cdot|_U$ the spectral norm over U , $\sup_j |\alpha(f)|_{U_j} = \max_j |\alpha(f)|_{U_j} = |f|_U$ (α is isometric) so that if $(f_j)_{j \in J}$ is an element of $\prod_j \mathcal{O}_X(U_j) \widehat{\otimes}_L B$ such that $\beta((f_j)_j) = 0$ then, writing $f_j = \sum_{i \in I} f_j^{(i)} b_i$ with $f_j^{(i)} \rightarrow 0$ as $i \rightarrow \infty$ (expansion in the orthonormal basis $(b_i)_{i \in I}$ of B), for all $i \in I$ there exists $f^{(i)} \in \mathcal{O}_X(U)$ with $\alpha(f^{(i)}) = (f_j^{(i)})_{j \in J}$ for all j , and $f^{(i)} \rightarrow 0$ for $|\cdot|_U$ and therefore, $f = \sum_i f^{(i)} b_i$ defines an element of $\mathcal{O}_X(U) \widehat{\otimes}_L B$ such that $\alpha(f) = (f_j)_j$. Now, the maps α and β define C -algebra maps between $\mathcal{O}_{X/B/C}(U)$, $\prod_j \mathcal{O}_{X/B/C}(U_j)$ etc. and the map resulting from α is injective, while the element $f \in \mathcal{O}_{X/B}(U)$ constructed above clearly belongs to $\mathcal{O}_{X/B/C}(U)$ if $f_j \in \mathcal{O}_{X/B/C}(U_j)$ for all j .

2.2.1. Structure of $\mathcal{O}_{X/B/C}$ with X a curve. We consider B a Banach L -algebra which is countably cartesian and we suppose that Λ is a partially ordered countable set, with partial order \prec , such that there is a family $(B_\lambda)_{\lambda \in \Lambda}$ of Banach sub- L -algebras of B with the following two properties:

- (1) If $\lambda \prec \lambda'$ then $B_\lambda \subset B_{\lambda'}$,

- (2) For all $\lambda, \lambda' \in \Lambda$ such that $\lambda \prec \lambda'$ there exists an orthonormal basis $(b_i)_{i \in I}$ of B (depending on λ) and subsets $J \subset J' \subset I$ with $(b_i)_{i \in J}$ an orthonormal basis of B_λ and $(b_i)_{i \in J'}$ an orthonormal basis of $B_{\lambda'}$.

We set $C = \cup_\lambda B_\lambda$. This is a sub- L -algebra of B . We have the next Lemma.

Lemma 2.10. *Let X be a rigid analytic curve over L . The following identity holds:*

$$\mathrm{Hol}_B(X \rightarrow C) = \bigcup_{\lambda \in \Lambda} \mathrm{Hol}_{B_\lambda}(X \rightarrow B_\lambda).$$

Proof. We first show the lemma when $X = \mathrm{Spm}(\mathcal{A})$ where \mathcal{A} is an integral affinoid L -algebra. If $f : X \rightarrow L$ is analytic with infinitely many zeroes, then it is identically zero. Now, let f be a global section of $\mathcal{O}_{X/B/C}$. For all $x \in X$ there exists $\lambda \in \Lambda$ such that $f(x) \in B_\lambda$. Therefore, there exists a map

$$X \xrightarrow{\Phi} \Lambda,$$

defined by associating to every $x \in X$ a choice of $\lambda \in \Lambda$ such that $f(x) \in B_\lambda$.

Since the set underlying X is uncountable (because L is uncountable, due to the fact that it is complete) while the target set is countable, there exists an infinite subset $X_0 \subset X$ and $\lambda \in \Lambda$, such that $\Phi(x) = \lambda$ for all $x \in X_0$. Then $f(X_0) \subset B_\lambda$. We expand f in an orthonormal basis $(b_i)_{i \in I}$ of B such that for some $J \subset I$, $(b_j)_{j \in J}$ is an orthonormal basis of B_λ :

$$f = \sum_{j \in J} f_j b_j + \sum_{i \in I \setminus J} f_i b_i$$

(with $f_j \rightarrow 0$ as $j \rightarrow \infty$, uniformly on X). Since for all $i \in I \setminus J$, $f_i(x) = 0$ for all $x \in X_0$, $f_i \in \mathcal{O}_X(X)$ has infinitely zeroes and therefore vanishes identically and we deduce that $f \in \mathcal{O}_{X/B_\lambda}(X)$.

Suppose now that X is an affinoid subdomain of an affinoid curve X' . Let f be in $\mathrm{Hol}_B(X' \rightarrow C)$. Then by what seen above, we can find $\lambda, \lambda' \in \Lambda$ such that $\lambda \prec \lambda'$ and $f \in \mathcal{O}_{X'/B_{\lambda'}}$, $f|_X \in \mathcal{O}_{X/B_\lambda}(X)$. Writing

$$f = \sum_{j' \in J' \setminus J} f_{j'} b_{j'} + \sum_{j \in J} f_j b_j$$

we note that for all $j \in J' \setminus J$, $f_{j'}(x) = 0$ for all $x \in X$ which is infinite, and $f_{j'}$ vanishes identically on X . This means that $f \in \mathcal{O}_{X'/B_\lambda}$. The lemma follows easily working on an admissible covering of a given rigid analytic curve. \square

2.2.2. Entire functions. We look at B -valued analytic functions on polydisks, where $(B, |\cdot|_B)$ is a Banach L -algebra which is countably cartesian. If X is the polydisk

$$D_L(0, r)^n = \{\underline{x} = (x_1, \dots, x_n) \in L^n; |x| \leq r\}$$

with $r \in |L|$ and with the usual structure sheaf of converging series, then $\mathrm{Hol}_B(X \rightarrow B)$ equals the ring of series $\sum_{\underline{i} \geq 0} f_{\underline{i}} \underline{x}^{\underline{i}}$ where $\underline{i} = (i_1, \dots, i_n)$ with $i_j \geq 0$ for all j , where $\underline{x}^{\underline{i}} = x_1^{i_1} \cdots x_n^{i_n}$, and where $f_{\underline{i}} \in B$ are such that $|f_{\underline{i}}|_B r^{i_1 + \dots + i_n} \rightarrow 0$ as $\underline{i} \rightarrow \infty$. We deduce

that the B -algebra $\text{Hol}_B(\mathbb{A}_L^{n,an} \rightarrow B)$, with $\mathbb{A}_L^{n,an}$ the analytic n -dimensional affine space over L , is equal to the B -algebra of the functions $L^n \rightarrow B$ defined by the formal series $\sum_{i \geq 0} f_i \underline{x}^i \in B[[x_1, \dots, x_n]]$ such that $|f_i|_B r^{i_1 + \dots + i_n} \rightarrow 0$ for all $r \in |B|_B$. It is also easy to see that a function $f : L^n \rightarrow B$ belongs to $\text{Hol}_B(\mathbb{A}_L^{n,an} \rightarrow B)$ if, on every bounded subset U of L , f can be obtained as a uniform limit of polynomial functions $f_i \in B[x_1, \dots, x_n]$, $f_i : U \rightarrow B$. These functions are called B -entire (or simply *entire* if the reference to B is understood). The following property is easily checked. Let $(f_i)_{i \geq 0}$ be a sequence of B -entire functions. If for every such r , the sequence $(f_i)_{i \geq 0}$ converges uniformly over $D_L(0, r)^n$, then the limit function $L^n \rightarrow B$ is a B -entire function.

The next result is a simple generalization of the analogue of Liouville's theorem which can be found in Schikhof's [71, Theorems 42.2 and 42.6]. See also [61, Proposition 8].

Proposition 2.11 (B -analogue of Liouville's Theorem). *Assuming that the Banach L -algebra B is countably cartesian, any bounded B -entire function is constant.*

Although the principles of the proof are completely elementary, we prefer to give all the details. Let n be a positive integer and $f : D_L(0, 1)^n \rightarrow B$ a B -analytic function, so that, with $\underline{x} = (x_1, \dots, x_n) \in D_L(0, 1)^n$,

$$f(\underline{x}) = \sum_{\underline{i}} f_{\underline{i}} \underline{x}^{\underline{i}}, \quad f_{\underline{i}} \in B,$$

where $\underline{x} = x_1^{i_1} \cdots x_n^{i_n}$ if $\underline{i} = (i_1, \dots, i_n)$ and $|f_{\underline{i}}|_B \rightarrow 0$ as $\underline{i} \rightarrow \infty$. We set

$$|f|_{B, \text{sup}} := \sup_{\underline{x} \in D_L(0, 1)^n} |f(\underline{x})|_B.$$

We also set $\|f\|_B = \sup\{|f_{\underline{i}}|_B : \underline{i} \in \mathbb{N}^n\} = \max\{|f_{\underline{i}}|_B : \underline{i} \in \mathbb{N}^n\}$.

Lemma 2.12. *We have $|f|_{B, \text{sup}} = \|f\|_B$.*

Proof. There is no loss of generality to suppose that $\|f\|_B = 1$. Indeed, $|B|_B = |L|$ because B is countably cartesian. It is easy to see that $|f|_{B, \text{sup}} \leq \|f\|_B$ and we only need to prove the opposite inequality. We proceed by induction on $n > 0$. Let us write $\underline{x} = (x_1, \underline{x}')$ (concatenation). We note that

$$|f|_{B, \text{sup}} \geq \sup_{x_1 \in D_L(0, 1)} \left(\sup_{\underline{x}' \in D_L(0, 1)^{n-1}} |f(x_1, \underline{x}')|_B \right) = \sup_{x_1} |f(x_1, \cdot)|_{B, \text{sup}} = \sup_{x_1} \|f(x_1, \cdot)\|_B$$

by the induction hypothesis. Let B' be the L -algebra

$$\text{Hol}_B(D_L(0, 1)^{n-1} \rightarrow B)$$

with the norm $|\cdot|_{B, \text{sup}} = \|\cdot\|_B$. It is easy to see that B' is a Banach L -algebra which is countably cartesian. Then, we can identify f with a B' -analytic function $\tilde{f} : D_L(0, 1) \rightarrow B'$, where $\tilde{f} = \sum_{i \geq 0} f_i x_1^i$, $f_i \in B'$, $f_i \rightarrow 0$. We see that $\sup_{x_1} \|f(x_1, \cdot)\|_B = \|\tilde{f}\|_{B'}$, the latter norm equals $\|f\|_B$. Hence $|f|_{B, \text{sup}} = \|f\|_B$. It remains to prove the case $n = 1$ of the Lemma. For this, we follow [71, Lemma 42.1].

Let us therefore consider an element $f \in \text{Hol}_B(D_L(0, 1) \rightarrow B)$ with $f(x) = \sum_{i \geq 0} f_i x^i$, $f_i \rightarrow 0$. Of course $|f|_{B, \text{sup}} \leq \|f\|_B$ and we can again suppose that $\|f\|_B = 1$. If $|f_0|_B = 1$ then $1 = |f(0)|_B \leq |f|_{B, \text{sup}} \leq \|f\|_B = 1$ and we are done. Otherwise, let N be the smallest integer j such that $|f_j|_B = 1$. We have $N > 0$. Let $\epsilon > 0$ be such that $\epsilon < 1 - \max\{|f_i|_B : 0 \leq i < N\}$. Since $|L^\times|$ is dense in $\mathbb{R}_{>0}$ there exists $x \in L^\times$ such that $1 - \epsilon < |x^N| < 1$. We claim that $|f(x)|_B = |x^N| > 1 - \epsilon$. To see this note that $\max\{|f_i|_B : 0 \leq i < N\} < 1 - \epsilon$ so that $|f_0 + \cdots + f_{N-1} x^{N-1}|_B < 1 - \epsilon$. On the other hand the sequence $(|x^i|)_{i \geq N}$ is strictly decreasing so that $|\sum_{i \geq N} f_i x^i|_B = |f_N x^N|_B = |x^N|$. Hence

$$|f(x)|_B = \max \left\{ \left| f_0 + \cdots + f_{N-1} x^{N-1} \right|_B, \left| \sum_{i \geq N} f_i x^i \right|_B \right\} = \left| \sum_{i \geq N} f_i x^i \right|_B > 1 - \epsilon.$$

The claim follows by letting ϵ tend to 0 and the proof of the lemma is complete. \square

Remark 2.13. If B is an algebraically closed field, Lemma 2.12 is contained in the arguments of [15, §5.1.4].

Proof of Proposition 2.11. Let f be B -entire (in n variables). If $r \in |L^\times|$ we can choose $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in (L^\times)^n$ so that $|\alpha_1| = \cdots = |\alpha_n| = r$ and apply Lemma 2.12 to the B -entire function $f(\alpha_1 x_1, \dots, \alpha_n x_n)$. We deduce that

$$\sup_{\underline{x} \in D_L(0, r)^n} |f(\underline{x})|_B = \max_{\underline{i}} |f_{\underline{i}}|_B r^{i_1 + \cdots + i_n}.$$

Assume now that $|f|_B$ is bounded, say, by $M > 0$. Then $\max_{\underline{i}} |f_{\underline{i}}|_B r^{i_1 + \cdots + i_n} \leq M$ for all $r \in |L^\times|$. This means that $|f_{\underline{i}}|_B = 0$ for all $\underline{i} \neq 0$ and f is a constant map $\mathbb{A}_N^{n, \text{an}} \rightarrow B$ that can be identified with its constant term f_0 . \square

2.3. Drinfeld modules and exponential functions. For a more extensive background on Drinfeld modules, lattices and exponential functions we refer to Goss' book [36] and [60, §3]. As in the introduction, we write A for $\mathbb{F}_q[\theta]$, the \mathbb{F}_q -algebra of polynomials in θ . We denote by K its fraction field $\mathbb{F}_q(\theta)$ and by $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$ the local field which is its completion at the infinity place or, which is the same, the completion for $|\cdot|$ the multiplicative valuation of K normalized by $|\theta| = q$. Finally, we denote by \mathbb{C}_∞ the completion of an algebraic closure K^{ac} of K . We recall that the residual field $k_{\mathbb{C}_\infty}$ of \mathbb{C}_∞ is \mathbb{F}_q^{ac} , an algebraic closure of \mathbb{F}_q , that we can view as a subfield of \mathbb{C}_∞ (Lemma 2.1). From now on we set

$$L = \mathbb{C}_\infty$$

and we consider the \mathbb{C}_∞ -algebras

$$\mathbb{T}_\Sigma := \mathbb{T}_{L, \Sigma}, \quad \mathbb{L}_\Sigma := \mathbb{L}_{L, \Sigma}, \quad \mathbb{K}_\Sigma := \mathbb{K}_{L, \Sigma}.$$

In this subsection we collect several tools related to the difference algebras structures on $\mathbb{T}_\Sigma, \mathbb{L}_\Sigma$ and \mathbb{K}_Σ determined by the automorphism $c \mapsto c^q$ of \mathbb{C}_∞ and to the uniformizability of certain $\mathbb{F}_q(t_\Sigma)[\theta]$ -modules associated to Drinfeld A -modules defined over \mathbb{C}_∞ as in [5]. Not all the material illustrated here is used in the body of the paper. However, the tools

we develop can be useful in further investigations in these themes, and the chosen level of generality does not make proofs more complicated.

2.3.1. *On the automorphism τ of $\mathbb{T}_\Sigma, \mathbb{L}_\Sigma, \mathbb{K}_\Sigma$.* The automorphism $c \mapsto c^q$ of \mathbb{C}_∞ (Frobenius) extends in a unique way to an $\mathbb{F}_q[[t_\Sigma]]$ -linear automorphism τ of $\mathbb{C}_\infty[[t_\Sigma]]$ and therefore, to each of the three \mathbb{C}_∞ -algebras $\mathbb{T}_\Sigma \subset \mathbb{L}_\Sigma \subset \mathbb{K}_\Sigma$ defined in §2.1. Recall that $\|\cdot\|$ denotes the unique extension of the Gauss norm to \mathbb{K}_Σ . Recall that by Lemma 2.7, $(\mathbb{K}_\Sigma, \|\cdot\|)$ is a Banach \mathbb{C}_∞ -algebra which is countably cartesian. For all $f \in \mathbb{K}_\Sigma$, we have that $\|\tau(f)\| = \|f\|^q$.

It is also well known that the subring

$$\mathbb{T}_\Sigma^{\tau=1}$$

of the elements $f \in \mathbb{T}_\Sigma$ such that $\tau(f) = f$ is the polynomial subring $\mathbb{F}_q[t_\Sigma]$. Let us recall the proof. Consider the injective morphism of \mathbb{C}_∞ -algebras $\mathbb{T}_\Sigma \rightarrow \mathbb{C}_\infty[[t_\Sigma]]$ defined by (2.2) and notice that $\tau : \mathbb{T}_\Sigma \rightarrow \mathbb{T}_\Sigma$ extends to $\mathbb{C}_\infty[[t_\Sigma]]$ in an unique way to an $\mathbb{F}_q[[t_\Sigma]]$ -linear automorphism. By the uniqueness of the power series expansions of the elements of $\mathbb{C}_\infty[[t_\Sigma]]$ we immediately see that $\mathbb{C}_\infty[[t_\Sigma]]^{\tau=1} = \mathbb{F}_q[[t_\Sigma]]$. Now, it is easily checked, by using (2.2), that $\mathbb{T}_\Sigma \cap \mathbb{F}_q[[t_\Sigma]] = \mathbb{F}_q[t_\Sigma]$.

Furthermore:

Lemma 2.14. *We have the identities of subfields of τ -invariant elements*

$$(2.4) \quad \mathbb{F}_q(t_\Sigma) = \mathbb{L}_\Sigma^{\tau=1} = \mathbb{K}_\Sigma^{\tau=1}.$$

Proof. It suffices to show that $\mathbb{K}_\Sigma^{\tau=1} \subset \mathbb{F}_q(t_\Sigma)$. By Lemma 2.7 with $\Sigma' = \emptyset$, every element $f \in \mathbb{K}_\Sigma$ can be expanded in a unique way as $f = \sum_{i \in I} f_i b_i$, where $(b_i)_{i \in I}$ is an \mathbb{F}_q^{ac} -basis of $\mathbb{F}_q^{ac}(t_\Sigma)$ and $f_i \in \mathbb{C}_\infty$ for $i \in I$, with $f_i \rightarrow 0$. Let f be an element of $\mathbb{K}_\Sigma^{\tau=1}$. All we need to do, is to show that if $\|f\| = 1$, then $f \in \mathbb{F}_q(t_\Sigma)$. Indeed, if $f \neq 0$ we can write $f = \lambda \tilde{f}$ with $\lambda \in \mathbb{C}_\infty^\times$, $\tilde{f} \in \mathbb{K}_\Sigma^\times$ such that $\|\tilde{f}\| = 1$ and $\|f\| = |\lambda|$. Expanding $f = \sum_{i \in I} f_i b_i$ as above, we set $I_1 = \{i \in I : |f_i| = 1\}$ and $I_0 = I \setminus I_1$. We write $f^0 = \sum_{i \in I_0} f_i b_i$ and $f^1 = \sum_{i \in I_1} f_i b_i$ so that $f = f^0 + f^1$. Clearly, I_1 is a finite set and τ induces a permutation of both I_1 and I_0 denoted by σ . There exists $k > 0$ such that σ^k is the identity on I_1 . Since $\tau^k(f) = f$ we have

$$\tau^k(f^1) - f^1 = f^0 - \tau^k(f^0).$$

But $\tau^k(f^1) - f^1 = \sum_{i \in I_1} (f_i^{q^k} - f_i) b_i$ so that if $\tau^k(f^1) - f^1 \neq 0$, then $\|\tau^k(f^1) - f^1\| = 1$. However, $\|f^0 - \tau^k(f^0)\| < 1$ which is impossible. Hence we have $\tau^k(f^1) = f^1$ which means in particular that $f^1 \in \mathbb{F}_{q^k}(t_\Sigma)^\times$ because $f_i^{q^k} = f_i$ for all $i \in I_1$, and $f^0 = \tau^k(f^0)$. Now, $\|\tau^k(f^0)\| = \|f^0\|^{q^k}$ and therefore, $f^0 = 0$. In particular, we have proved that $f = f^1 \in \mathbb{F}_{q^k}(t_\Sigma)^\times$. But it is easily seen that $\mathbb{F}_{q^k}(t_\Sigma)^{\tau=1} = \mathbb{F}_q(t_\Sigma)$ for all $k > 0$. \square

In this text we also consider the non-commutative \mathbb{K}_Σ -algebras $\mathbb{K}_\Sigma[\tau]$ and $\mathbb{K}_\Sigma[[\tau]]$ (the multiplication is defined by the commutation rule $\tau f = \tau(f)\tau$ for $f \in \mathbb{K}_\Sigma$). Similarly, we have the algebras $\mathbb{L}_\Sigma[\tau]$ and $\mathbb{L}_\Sigma[[\tau]]$. We are going to study certain elementary properties

of $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear endomorphisms of \mathbb{K}_Σ and \mathbb{L}_Σ determined by evaluations of operators of $\mathbb{K}_\Sigma[\tau]$ and $\mathbb{L}_\Sigma[\tau]$. The *evaluation* $\mathcal{L}(f)$ of

$$\mathcal{L} = a_0 + a_1\tau + \cdots + a_r\tau^r \in \mathbb{K}_\Sigma[\tau]$$

at $f \in \mathbb{K}_\Sigma$ is by definition the element

$$\mathcal{L}(f) = a_0f + a_1\tau(f) + \cdots + a_r\tau^r(f) \in \mathbb{K}_\Sigma.$$

Analogously, we define the evaluation of an element of $\mathbb{L}_\Sigma[\tau]$ at an element of \mathbb{L}_Σ .

In order to proceed, we can appeal to the next simple lemma that is also used later in the proof of Proposition 2.20. By Lemma 2.7 there exists an \mathbb{F}_q^{ac} -basis $\mathcal{B} = (b_i)_{i \in I}$ of $\mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ determining an orthonormal basis of the Banach \mathbb{C}_∞ -algebra \mathbb{K}_Σ .

Lemma 2.15. *Let \tilde{J} be a non-empty finite subset of I . There exists J finite, with $\tilde{J} \subset J \subset I$ with the following properties: (1) There is a matrix $M_J \in \mathrm{GL}_{|J|}(\mathbb{F}_q^{ac})$ such that, writing \underline{b}_J for the column matrix $(b_i)_{i \in J}$,*

$$\tau(\underline{b}_J) = M_J \underline{b}_J.$$

(2) *There is a vector spaces decomposition*

$$(2.5) \quad \mathbb{F}_q^{ac}(\underline{t}_\Sigma) = \mathrm{Vect}_{\mathbb{F}_q^{ac}}(\underline{b}_J) \oplus \mathrm{Vect}_{\mathbb{F}_q^{ac}}((b_i)_{i \in I \setminus J})$$

which splits the action of τ .

Proof. It follows from an easy study of the orbit under the action of the group $\mathrm{Gal}(\mathbb{F}_q(b_j : j \in \tilde{J})/\mathbb{F}_q(\underline{t}_\Sigma))$. \square

We can now tackle the promised basic properties of $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear endomorphisms of \mathbb{K}_Σ and \mathbb{L}_Σ associated to evaluations as above.

Lemma 2.16. *Let $\mathcal{L} = a_0 + a_1\tau + \cdots + a_r\tau^r \in \mathbb{K}_\Sigma[\tau]$ be such that $a_0a_r \neq 0$. Then the induced $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear evaluation map $\mathcal{L} : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$ is surjective. Similarly, if $\mathcal{L} \in \mathbb{L}_\Sigma[\tau]$, the $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear map $\mathcal{L} : \mathbb{L}_\Sigma \rightarrow \mathbb{L}_\Sigma$ is surjective.*

Proof. First notice that \mathcal{L} , as an $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear endomorphism of \mathbb{K}_Σ (or \mathbb{L}_Σ), is well defined. By the way, we are obviously allowing some (harmless) abuses of notation, because alternatively, \mathcal{L} denotes: an element of $\mathbb{K}_\Sigma[\tau]$ or an element of $\mathbb{L}_\Sigma[\tau]$, and at once alternatively, an endomorphism of \mathbb{K}_Σ or an endomorphism of \mathbb{L}_Σ . Also, we only prove the properties correspondent to the endomorphisms of \mathbb{K}_Σ leaving the rest of the proof to the reader (providing a small hint in a special case). Without loss of generality, we can suppose that $a_0 = 1$. It is easy to see that there exists $\rho \in |\mathbb{C}_\infty^\times|$, $\rho > 0$, such that \mathcal{L} induces an isometric automorphism of $D_{\mathbb{K}_\Sigma}^\circ(0, \rho) := \{f \in \mathbb{K}_\Sigma : \|f\| < \rho\}$. This can be proved with the study of the Newton-Puiseux polygon of the operator \mathcal{L} with respect to $\|\cdot\|$. Let y be in \mathbb{K}_Σ . We can write:

$$y = y^0 + y^1$$

where $y^0 \in D_{\mathbb{K}_\Sigma}^\circ(0, \rho) := \{f \in \mathbb{K}_\Sigma : \|f\| < \rho\}$ and $y^1 \in \bigoplus_{j \in \tilde{J}} \mathbb{C}_\infty b_j$ with \tilde{J} a finite subset of I . There exists $x^0 \in D_{\mathbb{K}_\Sigma}^\circ(0, \rho)$ such that $\mathcal{L}(x^0) = y^0$. It remains to construct $x^1 \in \mathbb{K}_\Sigma$ such that $\mathcal{L}(x^1) = y^1$.

Let J, M_J be respectively the subset of I , and the matrix, both given by Lemma 2.15 and let us write $\underline{b} = \underline{b}_J$ and $M = M_J$. By Lang's theorem [46, Corollary p. 557] there exists $U \in \mathrm{GL}_{|J|}(\mathbb{F}_q^{ac})$ such that $\tau(U) = MU$. We deduce

$$\tau^k(\underline{b}) = \tau^k(U)U^{-1}\underline{b}, \quad \forall k > 0.$$

Note that there exists a unique $\underline{c} \in \mathbb{C}_\infty^{|J| \times 1}$ such that $y^1 = {}^t \underline{c} \cdot \underline{b}$. Since \mathbb{C}_∞ is algebraically closed, there exists $\underline{a} \in \mathbb{C}_\infty^{|J| \times 1}$ such that $\mathcal{L}({}^t \underline{a} U) = {}^t \underline{c} U$. Hence

$$\mathcal{L}({}^t \underline{a} \cdot \underline{b}) = \mathcal{L}({}^t \underline{a} U)U^{-1}\underline{b} = {}^t \underline{c} U U^{-1} \underline{b} = y^1.$$

Setting $x^1 = {}^t \underline{a} \cdot \underline{b}$ yields $\mathcal{L}(x^1) = y^1$ and the proof that $\mathcal{L} : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$ is surjective is complete.

For the case of the endomorphisms of \mathbb{L}_Σ we give some details in a special case only (the reader can easily deduce the general case). We suppose that $\mathcal{L} = 1 - \tau$ (so we partially overlap with the proof of Lemma 2.14). Even though \mathbb{L}_Σ is not complete, it is a filtered union of complete spaces. Let y be an element of \mathbb{L}_Σ . Then there exists $d \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ such that $y \in \widehat{\mathbb{T}_\Sigma[\frac{1}{d}]}$. We can decompose $y = y^0 + y^1$ with $y^0 \in D_{\mathbb{T}_\Sigma[\frac{1}{d}]}^\circ(0, 1)$ (complete space) and $y^1 \in \bigoplus_{j \in \tilde{J}} \mathbb{C}_\infty b_j$ for some finite subset \tilde{J} of I . There exists $x^0 \in D_{\mathbb{T}_\Sigma[\frac{1}{d}]}^\circ(0, 1)$ such that $\tau(x^0) - x^0 = y^0$ and $x^1 \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$ with $\tau(x^1) - x^1 = y^1$. It is easy to show that there exists $d' \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ such that both $\mathbb{T}_\Sigma[\frac{1}{d}]$ and $x^1 \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$ are contained in $\mathbb{T}_\Sigma[\frac{1}{d'}] \subset \mathbb{L}_\Sigma$. \square

The next result we want to study is related with computations of kernels of operators such as \mathcal{L} above. Suppose additionally that $\mathcal{L} = a_0 + \dots + a_r \tau^r \in \mathbb{C}_\infty[\tau]$, again such that $a_0 a_r \neq 0$. It is well known (because \mathbb{C}_∞ is algebraically closed) that the set of the zeroes of the evaluation map $\mathcal{L}|_{\mathbb{C}_\infty} : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is an \mathbb{F}_q -vector space of dimension r :

$$\mathrm{Ker}(\mathcal{L}|_{\mathbb{C}_\infty}) = \mathrm{Vect}_{\mathbb{F}_q}(\beta_1, \dots, \beta_r), \quad \beta_1, \dots, \beta_r \in \mathbb{C}_\infty^\times.$$

We have stressed that \mathcal{L} is restricted to \mathbb{C}_∞ because we can also view \mathcal{L} as an $\mathbb{F}_q(t_\Sigma)$ -linear endomorphism of \mathbb{K}_Σ and \mathbb{L}_Σ . In this case we have:

Lemma 2.17. *With \mathcal{L} as above:*

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{Ker}(\mathcal{L}|_{\mathbb{L}_\Sigma}) = \mathrm{Vect}_{\mathbb{F}_q(t_\Sigma)}(\beta_1, \dots, \beta_r).$$

Proof. Without loss of generality we can suppose that $a_0 = 1$. We first deal with the case of the map $\mathcal{L} : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$. We proceed by induction on r : the result is trivial for $r = 0$. Also, by the right euclidean division [36, Proposition 1.6.2] we can factor $\mathcal{L} \in \mathbb{C}_\infty[\tau]$:

$$\mathcal{L} = (1 - \alpha_r \tau) \cdots (1 - \alpha_1 \tau), \quad \alpha_1, \dots, \alpha_r \in \mathbb{C}_\infty^\times,$$

where we can choose $\alpha_1 = \beta_1^{1-q}$. Note that the factors do not commute in general, and the factorization is not unique. Setting $\mathcal{L}_1 = (1 - \alpha_{r-1} \tau) \cdots (1 - \alpha_1 \tau)$, we have $\mathcal{L} = (1 - \alpha_r \tau) \mathcal{L}_1$.

Note that $\text{Ker}(\mathcal{L}_1)$ is a subvector space of $\text{Ker}(\mathcal{L})$. By induction hypothesis we can choose \mathcal{L}_1 in such a way that

$$\text{Ker}(\mathcal{L}_1) = \text{Vect}_{\mathbb{F}_q(t_\Sigma)}(\beta_1, \dots, \beta_{r-1})$$

Let us consider $\beta \in \text{Ker}(\mathcal{L}) \setminus \text{Ker}(\mathcal{L}_1)$. Then,

$$\mathcal{L}_1(\beta) \in \text{Ker}(1 - \alpha_r \tau),$$

so that $\tau(\mathcal{L}_1(\beta)) = \alpha_r^{-1} \mathcal{L}_1(\beta)$. Using Lemma 2.14 this implies that there exists $x \in \mathbb{C}_\infty^\times$ such that

$$\mathcal{L}_1(\beta) \in \mathbb{F}_q(t_\Sigma)x$$

and there exists $\lambda \in \mathbb{F}_q(t_\Sigma)^\times$ such that $\mathcal{L}_1(\lambda\beta) \in \mathbb{C}_\infty$. By Lemma 2.16, there exists $\tilde{\beta} \in \mathbb{C}_\infty^\times$ such that

$$\mathcal{L}_1(\beta - \lambda^{-1}\tilde{\beta}) = 0.$$

Therefore, by the induction hypothesis, $\beta - \lambda^{-1}\tilde{\beta} \in \text{Vect}_{\mathbb{F}_q(t_\Sigma)}(\beta_1, \dots, \beta_{r-1})$. This implies that $0 = \mathcal{L}(\beta) = \mathcal{L}(\lambda^{-1}\tilde{\beta})$ and $\tilde{\beta} \in \text{Ker}(\mathcal{L}|_{\mathbb{C}_\infty}) = \text{Vect}_{\mathbb{F}_q(t_\Sigma)}(\beta_1, \dots, \beta_r)$ implying our result. The case of the restriction of \mathcal{L} on \mathbb{L}_Σ is similar and left to the reader. \square

Combining with Lemma 2.16, we get the more precise:

Corollary 2.18. *There are exact sequences of $\mathbb{F}_q(t_\Sigma)$ -vector spaces*

$$0 \rightarrow \mathbb{F}_q(t_\Sigma) \rightarrow \mathbb{L}_\Sigma \xrightarrow{1-\tau} \mathbb{L}_\Sigma \rightarrow 0, \quad 0 \rightarrow \mathbb{F}_q(t_\Sigma) \rightarrow \mathbb{K}_\Sigma \xrightarrow{1-\tau} \mathbb{K}_\Sigma \rightarrow 0.$$

2.3.2. The exponential of a Drinfeld module. Let ϕ be a Drinfeld A -module of rank n defined over \mathbb{C}_∞ , let $\exp_\phi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ be its exponential function and $\Lambda_\phi = \text{Ker}(\exp_\phi) \subset \mathbb{C}_\infty$ its lattice period which is a free module of rank n over A discrete for the metric of \mathbb{C}_∞ induced by $|\cdot|$. We recall that \exp_ϕ is an \mathbb{F}_q -linear entire function $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ that can be computed by means of the following everywhere convergent Weierstrass-like product

$$(2.6) \quad \exp_\phi(Z) = Z \prod'_{\lambda \in \Lambda_\phi} \left(1 - \frac{Z}{\lambda}\right), \quad Z \in \mathbb{C}_\infty$$

(the dash ' indicates that the product runs over $\Lambda_\phi \setminus \{0\}$). This product expansion also shows that locally at 0, \exp_ϕ induces an isometric \mathbb{F}_q -linear automorphism. Indeed, if $\rho_\phi := \min_{\lambda \in \Lambda_\phi \setminus \{0\}} |\lambda|$, \exp_ϕ induces an \mathbb{F}_q -linear automorphism of

$$D_{\mathbb{C}_\infty}^\circ(0, \rho_\phi) = \{z \in \mathbb{C}_\infty : |z| < \rho_\phi\}$$

such that for all $z \in D_{\mathbb{C}_\infty}^\circ(0, \rho_\phi)$, $|\exp_\phi(z)| = |z|$. In fact it can be proved that \exp_ϕ induces an isomorphism of \mathbb{C}_∞ -rigid analytic spaces $\mathbb{A}_{\mathbb{C}_\infty}^{1,an}/\Lambda_\phi \cong \mathbb{A}_{\mathbb{C}_\infty}^{1,an}$. With $\phi(\mathbb{C}_\infty)$ the A -module induced by ϕ , there is an exact sequence of A -modules

$$0 \rightarrow \Lambda_\phi \rightarrow \mathbb{C}_\infty \xrightarrow{\exp_\phi} \phi(\mathbb{C}_\infty) \rightarrow 0$$

(\exp_ϕ is uniquely determined by the condition of being an entire A -module morphism with first derivative $\exp'_\phi = 1$). We fix a finite subset $\Sigma \subset \mathbb{N}^*$ and a Drinfeld module ϕ defined over \mathbb{C}_∞ . There is a unique structure of $A \otimes_{\mathbb{F}_q} \mathbb{F}_q(t_\Sigma)$ -module $\phi(\mathbb{K}_\Sigma)$ over \mathbb{K}_Σ which is

defined by extending the operators ϕ_a (of multiplication by $a \in A$ in the A -module $\phi(\mathbb{C}_\infty)$) $\mathbb{F}_q(\underline{t}_\Sigma)$ -linearly to \mathbb{K}_Σ along the extension of the map $(x \mapsto x^q) : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ to the map $\tau : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$. Explicitly, if $\phi_a = (a)_0 + (a)_1\tau + \dots + (a)_r\tau^r$ with $(a)_0, \dots, (a)_r \in \mathbb{C}_\infty$ and $x \in \mathbb{K}_\Sigma$, the action of a on x is defined by the evaluation of ϕ_a at x (compare with [5]). Similarly, we can define the $A \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma)$ -module $\phi(\mathbb{L}_\Sigma)$.

By using the tools developed in §2.3.1 we can easily prove:

Proposition 2.19. *For all $a \in A \setminus \{0\}$ the \mathbb{F}_q -linear map $\phi_a : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ determines a short exact sequence of $\mathbb{F}_q(\underline{t}_\Sigma)$ -vector spaces:*

$$0 \rightarrow \exp_\phi \left(\frac{1}{a} \Lambda_\phi \right) \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma) \rightarrow \mathbb{K}_\Sigma \xrightarrow{\phi_a} \mathbb{K}_\Sigma \rightarrow 0,$$

where the second arrow is the inclusion, and a similar exact sequence holds with \mathbb{K}_Σ replaced by \mathbb{L}_Σ .

Proof. Clearly $\exp_\phi(a^{-1}\Lambda_\phi) \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma)$ is an $\mathbb{F}_q(\underline{t}_\Sigma)$ -subvector space of dimension $r := \deg_\theta(a)n$ of $\text{Ker}(\phi_a)$ where n is the rank of ϕ . Setting $\mathcal{L} = \phi_a$, the computation of the kernel then follows from Lemma 2.16, while the surjectivity of ϕ follows from Lemma 2.17. \square

The case $\Sigma = \{i\}$, $i \in \mathbb{N}^*$ is considered in [33, Theorem 7.1.1, Proposition 8.2.1].

Since \mathbb{K}_Σ is complete we have an $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear map $\exp_\phi : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$, continuous and open, which induces a morphism of $A \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma)$ -modules

$$\mathbb{K}_\Sigma \xrightarrow{\exp_\phi} \phi(\mathbb{K}_\Sigma)$$

such that $\Lambda_\phi \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma) \subset \text{Ker}(\exp_\phi)$. We note that also the map $\mathbb{L}_\Sigma \xrightarrow{\exp_\phi} \phi(\mathbb{L}_\Sigma)$ is well defined, in spite of the fact that \mathbb{L}_Σ is not complete. Indeed, if $x \in \mathbb{L}_\Sigma$, there exists $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that $x \in \widehat{\mathbb{T}_\Sigma[\frac{1}{d}]}$ and this space is complete. It is easy to show that \exp_ϕ induces an isometric $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear automorphism of $D_{\mathbb{K}_\Sigma}^\circ(0, \rho_\phi) = \{f \in \mathbb{K}_\Sigma : \|f\| < \rho_\phi\}$, and a similar property holds with \mathbb{K}_Σ replaced with \mathbb{L}_Σ . Moreover, we have:

Proposition 2.20. *Let ϕ be a Drinfeld A -module with exponential \exp_ϕ . The map \exp_ϕ induces an exact sequence of $A \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma)$ -modules:*

$$(2.7) \quad 0 \rightarrow \Lambda_\phi \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma) \rightarrow \mathbb{K}_\Sigma \xrightarrow{\exp_\phi} \phi(\mathbb{K}_\Sigma) \rightarrow 0.$$

To prove this result we can use the next lemma. Let $\tilde{J} \subset I$ be a finite subset and let J be any finite subset of I as in Lemma 2.15 so that $\tilde{J} \subset J \subset I$.

Lemma 2.21. *The exponential map \exp_ϕ induces a surjective \mathbb{F}_q -linear endomorphism of $\bigoplus_{j \in J} \mathbb{C}_\infty b_j$ with kernel $\Lambda_\phi \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma)^{|J| \times 1}$.*

Proof. Since J is fixed in the proof, let us write more simply $\underline{b} = \underline{b}_J$ and $M = M_J$ where M_J is the matrix given by Lemma 2.15. Also, if X is any matrix with entries in \mathbb{K}_Σ , we set $X^{(i)} = \tau^i(X)$ (coefficient-wise application of τ^i). Note that since $\underline{b}^{(1)} = M\underline{b}$, we have $\underline{b}^{(i)} = M^{(i-1)} \dots M^{(1)} M \cdot \underline{b}$ for all $i \geq 0$. In the proof of Lemma 2.15 we have constructed

a matrix $U \in \mathrm{GL}_{|J|}(\mathbb{F}_q^{ac})$ such that $\tau(U) = MU$. Hence, $U^{(i)} = M^{(i-1)} \dots M^{(1)} MU$ for all $i \geq 0$. We deduce

$$\underline{b}^{(i)} = U^{(i)} U^{-1} \underline{b}, \quad i \geq 0 \quad \text{and} \quad (U^{-1} \underline{b})^{(1)} = U^{-1} \underline{b}.$$

Hence $U^{-1} \underline{b} \in \mathbb{F}_q(\underline{t}_\Sigma)^{|J|}$ by (2.4).

Let us compute, for $\underline{a} \in \mathbb{C}_\infty^{|J| \times 1}$ (column vector), $({}^t \underline{a} \cdot \underline{b})^{(i)}$, $i \geq 0$. We immediately see: $({}^t \underline{a} \cdot \underline{b})^{(i)} = ({}^t \underline{a} \cdot U)^{(i)} U^{-1} \underline{b}$. Transposing we get:

$${}^t ({}^t \underline{a} \cdot \underline{b})^{(i)} = {}^t \underline{b} \cdot {}^t U \cdot ({}^t U \cdot \underline{a})^{(i)},$$

hence, if $f = {}^t \underline{a} \cdot \underline{b} \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$,

$$\exp_\phi(f) = \exp_\phi({}^t \underline{a} \cdot \underline{b}) = {}^t \underline{b} \cdot {}^t (U^{-1}) \exp_\phi({}^t U \cdot \underline{a}) \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j.$$

Since the map $\exp_\phi : \mathbb{C}_\infty^{|J| \times 1} \rightarrow \mathbb{C}_\infty^{|J| \times 1}$ is surjective, $\exp_\phi : \bigoplus_{j \in J} \mathbb{C}_\infty b_j \rightarrow \bigoplus_{j \in J} \mathbb{C}_\infty b_j$ is surjective. Now consider an element $f = {}^t \underline{a} \cdot \underline{b} \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$ such that $\exp_\phi(f) = 0$. By the above computation, this is equivalent to $\exp_\phi({}^t U \cdot \underline{a}) = 0$, so that

$$\underline{a} \in {}^t (U^{-1}) \cdot \Lambda_\phi^{|J| \times 1}.$$

But ${}^t \underline{a} \cdot \underline{b} \in \Lambda_\phi^{1 \times |J|} U^{-1} \cdot \underline{b}$ and we have seen that $U^{-1} \cdot \underline{b} \in \mathbb{F}_q(\underline{t}_\Sigma)^{|J| \times 1}$. The lemma follows. \square

Proof of Proposition 2.20. We first show that \exp_ϕ is surjective. Let us consider $g \in \mathbb{K}_\Sigma$. There exists $J \subset I$ finite with $\underline{b} = (b_j)_{j \in J} = \underline{b}_J$ with $(\underline{b})^{(1)} = M \cdot \underline{b}$ as in Lemma 2.15, and additionally, we can decompose

$$g = g_0 + g_1$$

with $\|g_0\| < \rho_\phi$ and $g_1 \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$. By Lemma 2.21 there exists $f_1 \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$ such that $\exp_\phi(f_1) = g_1$ and since \exp_ϕ induces an isometry over $D_{\mathbb{K}_\Sigma}^\circ(0, \rho_\phi)$, there also exists $f_0 \in D_{\mathbb{K}_\Sigma}^\circ(0, \rho_\phi)$ such that $\exp_\phi(f_0) = g_0$. Setting $f = f_0 + f_1$ we deduce $\exp_\phi(f) = g$.

It remains to compute the kernel of \exp_ϕ over \mathbb{K}_Σ . Let $f \in \mathbb{K}_\Sigma$ be such that $\exp_\phi(f) = 0$. Again, we can write $f = f_0 + f_1$ with $\|f_0\| < \rho_\phi$ and $f_1 \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$. We write $f_0 = f_0^0 \oplus f_0^1$ where f_0^0 belongs to the Banach \mathbb{C}_∞ -sub-vector space of \mathbb{K}_Σ generated by $(b_i)_{i \in I \setminus J}$ and $f_0^1 \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$. By the hypothesis on J we see that $\exp_\phi(f_0^0) = \sum_{i \in I \setminus J} c_i b_i$ while $\exp_\phi(f_0^1 + f_1) \in \bigoplus_{j \in J} \mathbb{C}_\infty b_j$. Hence, again by the fact that \exp_ϕ induces an isometry over $D_{\mathbb{K}_\Sigma}^\circ(0, \rho_\phi)$, we can suppose that $f_0 = 0$. We can conclude by using Lemma 2.21. \square

Let δ be an element of $\mathbb{F}_q(\underline{t}_\Sigma)^\times$. From the proof of Proposition 2.20 one deduces that the exponential function \exp_ϕ of a Drinfeld A -module ϕ also induces an $\mathbb{F}_q[\underline{t}_\Sigma][\delta]$ -linear surjective endomorphism of $\mathbb{T}_\Sigma[\delta]^\wedge \subset \mathbb{K}_\Sigma$, and we deduce the next result (compare with Proposition 2.18 and [3]):

Corollary 2.22. *For any $\delta \in \mathbb{F}_q(\underline{t}_\Sigma)$ the map \exp_ϕ induces an exact sequence of $A[\underline{t}_\Sigma][\delta]$ -modules:*

$$(2.8) \quad 0 \rightarrow \Lambda_\phi \otimes_{\mathbb{F}_q} \mathbb{F}_q[\underline{t}_\Sigma][\delta] \rightarrow \widehat{\mathbb{T}_\Sigma[\delta]} \xrightarrow{\exp_\phi} \phi(\widehat{\mathbb{T}_\Sigma[\delta]}) \rightarrow 0.$$

Hence, we also have an exact sequence of $\mathbb{F}_q(\underline{t}_\Sigma)[\theta]$ -modules:

$$(2.9) \quad 0 \rightarrow \Lambda_\phi \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma) \rightarrow \mathbb{L}_\Sigma \xrightarrow{\exp_\phi} \phi(\mathbb{L}_\Sigma) \rightarrow 0.$$

The proof of the next Lemma, that will be used later, is easy and left to the reader.

Lemma 2.23. *Let ϕ be a Drinfeld module over \mathbb{C}_∞ and \exp_ϕ be its exponential function. Let $f : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$ be a \mathbb{K}_Σ -entire function. Then the composition $\exp_\phi \circ f$ is a \mathbb{K}_Σ -entire function. Additionally, if $f(\mathbb{L}_\Sigma)$ is contained in \mathbb{L}_Σ , then the image of $\exp_\phi \circ f$ is contained in \mathbb{L}_Σ . Finally, if $f(z) = \lambda z$ with $\lambda \in \mathbb{K}_\Sigma$, then $\exp_\phi \circ f$ is a $\mathbb{F}_q(\underline{t}_\Sigma)[\theta]$ -module morphism $\mathbb{K}_\Sigma \rightarrow \phi(\mathbb{K}_\Sigma)$.*

Remark 2.24. In the following, we essentially only use the Drinfeld A -module $\phi = C$, Carlitz's module, defined by the condition that $C_\theta = \theta + \tau$. However, our work has a natural extension to modular forms for the group $\mathrm{GL}_n(A)$ with $n > 2$, where it is important to consider the general case. This motivates the chosen level of generalization so far.

2.4. Some functions associated with the Carlitz module. The functions mentioned in the title of the present subsection, and that will be described here, are required as basic tools to describe the analogues of Fourier series for our modular forms. One of the simplest examples of Drinfeld A -modules is the *Carlitz module* C . In this subsection, after a quick review, in §2.4.1, of well known properties of C and the associated exponential function $\exp_C : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$ (for a more complete treatise, read, for example, Goss' [36, Chapter 3] and [60, §4]), we introduce in §2.4.2 a matrix generalization of the so-called *omega function of Anderson and Thakur* (omega matrices). There is a large amount of references discussing the omega function of Anderson and Thakur, which is a central object in function field arithmetic. Among them we suggest, due to a good compatibility of notations, the paper [3, §2]. It is likely that the first appearance of ω in the literature is in the seminal paper [1] by Anderson and Thakur. In 2.4.3 we introduce a matrix generalization of *Perkins' map* $\chi_t : \mathbb{C}_\infty \rightarrow \mathbb{T}_\Sigma$ [65], another remarkable object in function fields arithmetic. These tools will be heavily used in §3.

2.4.1. Basic notions on Carlitz's module. We recall that the Carlitz module $C(\mathbb{K}_\Sigma)$ over \mathbb{K}_Σ is the $\mathbb{F}_q(\underline{t}_\Sigma)$ -algebra morphism

$$A \otimes_{\mathbb{F}_q} \mathbb{F}_q(\underline{t}_\Sigma) \xrightarrow{C} \mathrm{End}_{\mathbb{F}_q(\underline{t}_\Sigma)}(\mathbb{K}_\Sigma)$$

defined by $C(\theta) = C_\theta = \theta + \tau$, the multiplication by θ . Just like any Drinfeld module ϕ , C can also be viewed as a functor from the category of $\mathbb{C}_\infty(\underline{t}_\Sigma)[\theta][\tau]$ -modules to the category of $\mathbb{F}_q(\underline{t}_\Sigma)[\theta]$ -modules (with appropriate morphisms) so that we can define the modules $C(\mathbb{T}_\Sigma), C(\mathbb{L}_\Sigma), C(A), \dots$ as well. To describe the associated Carlitz exponential, we introduce, following [36, §31 and 3.2], the analogue of the sequence of numbers $q^n!$ in the following way:

$$d_n = \prod_a a,$$

where the product runs over the monic polynomials a of A of degree n . It can be proved (see [36, Proposition 3.1.6]) that

$$(2.10) \quad d_n = (\theta^{q^n} - \theta) \cdots (\theta^{q^n} - \theta^{q^{n-1}}) \in A \setminus \{0\}, \quad n \geq 0.$$

The map $\exp_C : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$ defined by

$$\exp_C(z) = \sum_{i \geq 0} d_i^{-1} \tau^i(z)$$

is the exponential function associated to the Carlitz module, which is a continuous, open $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear endomorphism $\mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$ to which we can apply Proposition 2.20 and Corollary 2.22. In particular, the kernel of \exp_C (over \mathbb{L}_Σ or \mathbb{K}_Σ) is equal to $\tilde{\pi} \mathbb{F}_q(\underline{t}_\Sigma)[\theta]$ where

$$(2.11) \quad \tilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1},$$

which belongs to $K_\infty((-\theta)^{\frac{1}{q-1}}) \setminus K_\infty$ (we make a choice of a $(q-1)$ -th root of $-\theta$, and we note that $(-\theta)^{\frac{1}{q-1}} = \exp_C(\tilde{\pi}\theta^{-1})$). It is rare that, for a given Drinfeld A -module ϕ , we can provide such explicit descriptions of the main characterizing objects \exp_ϕ, Λ_ϕ etc. From this product expansion one immediately sees that $|\tilde{\pi}| = |(-\theta)^{\frac{1}{q-1}}| = |\theta|^{\frac{q}{q-1}}$. It can be proved that $\tilde{\pi}$ is transcendental over K ; there are several ways that lead to this result, using the above product expansion. See [54, 60] for an overview.

We sometimes also use the notation \exp_C for the *Carlitz exponential operator* which is formal series $\sum_{n \geq 0} d_n^{-1} \tau^n \in K[[\tau]] \subset \mathbb{K}_\Sigma[[\tau]]$, unique such that the first term for $n=0$ is $1 = \tau^0$ (normalized), satisfying, for the product rule of $\mathbb{K}_\Sigma[[\tau]]$, $C_\theta \exp_C = \exp_C \theta$.

The inverse of the Carlitz exponential operator $\exp_C \in \mathbb{K}_\Sigma[[\tau]]$ for the composition is the *Carlitz logarithm* defined by the locally convergent series

$$\log_C(z) = \sum_{i \geq 0} l_i^{-1} \tau^i(z),$$

where l_n is equal to $(-1)^n$ times the monic least common multiple of all polynomials of A of degree n . It can be proved (see again [36, Proposition 3.1.6]) that

$$(2.12) \quad l_n = (\theta - \theta^q) \cdots (\theta - \theta^{q^n}).$$

More precisely, \log_C induces an isometric $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear automorphism

$$D_F^\circ(0, |\tilde{\pi}|) \xrightarrow{\log_C} D_F^\circ(0, |\tilde{\pi}|),$$

where $F = \mathbb{L}_\Sigma$ or $F = \mathbb{K}_\Sigma$, and similar properties occur with $F = \mathbb{T}_\Sigma[\delta]$ with $\delta \in \mathbb{F}_q(\underline{t}_\Sigma)$ etc. We also identify, sometimes, \log_C with the *Carlitz logarithm operator* $\sum_{n \geq 0} l_n^{-1} \tau^n \in K[[\tau]]$.

2.4.2. *Omega matrices.* We need certain invertible matrices with entries in \mathbb{L}_Σ similar to Anderson-Thakur omega function. Let

$$A \xrightarrow{\chi} \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$$

be an injective \mathbb{F}_q -algebra morphism. We set

$$\vartheta := \chi(\theta) \in \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}.$$

Let $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ be such that $d\vartheta \in \mathbb{F}_q[\underline{t}_\Sigma]^{n \times n}$. Then, the image of χ lies in $\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d}]^{n \times n}$. We set

$$\omega_\chi := \sum_{i \geq 0} \exp_C \left(\frac{\tilde{\pi}}{\theta^{i+1}} \right) \vartheta^i = \exp_C \left(\tilde{\pi}(\theta I_n - \vartheta)^{-1} \right) \in \widehat{\mathbb{T}_\Sigma[d^{-1}]}^{n \times n} \subset \mathbb{L}_\Sigma^{n \times n},$$

where the map \exp_C is applied coefficientwise and $\mathbb{F}_q(\underline{t}_\Sigma)$ -linearly, on the entries of the matrix $\tilde{\pi}(\theta I_n - \vartheta)^{-1} \in \mathbb{K}_\Sigma^{n \times n}$ (in the same way as in [3, §2.2]). We have, for all $a \in A$, with $C_a \in K[\tau]^{n \times n}$ the multiplication by a over $C(\mathbb{K}_\Sigma)^{n \times n}$:

(2.13)

$$C_a(\omega_\chi) = \exp_C \left(\tilde{\pi}a(\theta I_n - \vartheta)^{-1} \right) = \exp_C \left(\tilde{\pi}(aI_n - \chi(a))(\theta I_n - \vartheta)^{-1} \right) + \chi(a)\omega_\chi = \chi(a)\omega_\chi,$$

because $aI_n - \chi(a) = (\theta I_n - \vartheta)H$ with $H \in A[\vartheta]^{n \times n}$, so that $\tilde{\pi}(aI_n - \chi(a)) \in \text{Ker}(\exp_C)^{n \times n}$.

Lemma 2.25. *We have $\omega_\chi \in \text{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ and ω_χ is solution of the linear τ -difference system*

$$\tau(X) = (\vartheta - \theta I_n)X.$$

Moreover, every solution X in $\mathbb{K}_\Sigma^{n \times 1}$ of this difference system is of the form $X = \omega_\chi m$, with $m \in \mathbb{F}_q(\underline{t}_\Sigma)^{n \times 1}$.

Proof. Observe that

$$\omega_\chi = \exp_C \left(\tilde{\pi}(\theta I_n - \vartheta)^{-1} \right) = \exp_C \left(\tilde{\pi}\theta^{-1}(I_n - \vartheta\theta^{-1})^{-1} \right) = \exp_C(\tilde{\pi}\theta^{-1})I_n + R$$

where $R \in \mathbb{K}_\Sigma^{n \times n}$ is such that $\|R\| < |\theta|^{\frac{1}{q-1}} = |\tilde{\pi}\theta^{-1}| = |\exp_C(\tilde{\pi}\theta^{-1})|$. This proves that $\omega_\chi \in \text{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$. The fact that ω_χ is a matrix solution of the system indicated above follows directly from (2.13) with $a = \theta$. Finally, if X is a column solution of the system above, we have that $\omega_\chi^{-1}X$ has entries in the constant subfield of \mathbb{K}_Σ which is $\mathbb{F}_q(\underline{t}_\Sigma) = \mathbb{F}_q(\underline{t}_\Sigma)$, and this proves the last assertion. \square

We denote by $\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge$ the \mathbb{C}_∞ -algebra generated by all the series

$$(2.14) \quad \sum_{i \geq 0} f_i d^{-i}, \quad f_i \in \mathbb{E}_\Sigma,$$

with the property that $\|f_i\|r^i \rightarrow 0$ for all $r \in |\mathbb{C}_\infty|$. Observe that $\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge$ is not complete for $\|\cdot\|$, unless $\Sigma = \emptyset$; for example, \mathbb{E} is not complete. The completion of $\mathbb{E}_\Sigma[\frac{1}{d}]$ is easily seen to be equal to $\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$ (the completion of \mathbb{E} for the Gauss norm is \mathbb{T}). We therefore adopt this notation that should not lead to confusion. We have the next:

Corollary 2.26. *We have the identity*

$$\omega_\chi = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(I_n - \vartheta \theta^{-q^i} \right)^{-1},$$

up to the choice of an appropriate root $(-\theta)^{\frac{1}{q-1}}$. Hence, $\omega_\chi^{-1} \in \mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge) \cap (\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$.

Note that the factors of the infinite product commute each other.

Proof of Corollary 2.26. First of all note that

$$F := (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(I_n - \vartheta \theta^{-q^i} \right)^{-1} \in (\mathbb{T}_\Sigma[d^{-1}]^\wedge)^{n \times n}$$

is a matrix solution X of the difference system $\tau(X) = (\vartheta - \theta I_n)X$, in $\mathrm{GL}_n(\mathbb{K}_\Sigma)$. Lemma 2.25 applies and there exists a matrix $V \in \mathrm{GL}_n(\mathbb{F}_q[t_\Sigma])$ such that $F = V\omega_\chi$. Now we proceed to prove that $V = I_n$. We recall that $(-\theta)^{\frac{1}{q-1}} = \exp_C(\frac{\tilde{\pi}}{\theta})$ for a unique choice of $(-\theta)^{\frac{1}{q-1}}$. We have seen, in the proof of Lemma 2.25, that $\omega_\chi = \exp_C(\tilde{\pi}\theta^{-1})I_n + R$ where $R \in \mathbb{K}_\Sigma^{n \times n}$ is such that $\|R\| < |\theta|^{\frac{1}{q-1}}$. We also have $F = \exp_C(\tilde{\pi}\theta^{-1})I_n + R'$, $R' \in \mathbb{K}_\Sigma$ such that $\|R'\| < q^{\frac{1}{q-1}}$. Hence $V = I_n$. Additionally, note that $\tilde{\pi}(\theta I_n - \vartheta)^{-1} \in (\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$ so that, by Corollary 2.22, ω_χ has entries in $\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$. Also, F in this case is an element of $\mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$. Writing $\vartheta = d^{-1}\nu$ with $\nu \in \mathbb{F}_q[t_\Sigma]^{n \times n}$, we see that $F = \sum_{i \geq 0} c_i \nu^i d^{-i}$ with $c_i \in \mathbb{C}_\infty$ such that $|c_i| r^i \rightarrow 0$ for all $r \in |\mathbb{C}_\infty|$. But then $c_i \nu^i \in \mathbb{C}_\infty[t_\Sigma]^{n \times n}$ with $\|c_i \nu^i\| r^i \rightarrow 0$ and therefore, the entries of ω_χ^{-1} belong to $\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge$. \square

2.4.3. A class of entire functions: Perkins' maps. We recall that we have set $\vartheta = \chi(\theta) \in \mathbb{F}_q[t_\Sigma]^{n \times n}$ and that $d \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ is such that $d\vartheta \in \mathbb{F}_q[t_\Sigma]$. For $z \in \mathbb{C}_\infty$, we set ⁽⁸⁾:

$$(2.15) \quad \tilde{\chi}(z) := \exp_C \left(\tilde{\pi} z (\theta I_n - \vartheta)^{-1} \right) \omega_\chi^{-1},$$

where $\omega_\chi \in \mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ has been introduced in §2.4.2. By Lemma 2.23, this is an entire function in $\mathrm{Hol}_{\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge}(\mathbb{C}_\infty \rightarrow \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$. We now use the material developed in this section to show the following (compare with [61, Lemmas 15, 17]).

Proposition 2.27. *The function $\tilde{\chi}$ satisfies the following properties:*

- (1) *It has image in $(\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$.*
- (2) *It satisfies $\tilde{\chi}(a) = \chi(a)$ for all $a \in A$.*
- (3) *It satisfies the τ -difference system $\tau(X) = X + \exp_C(\tilde{\pi}z)\omega_\chi^{-1}$.*

Proof. (1) Since

$$\exp_C \left(\tilde{\pi} z (\theta I_n - \vartheta)^{-1} \right) = \sum_{i \geq 0} d_i^{-1} (\tilde{\pi} z)^{q^i} (\theta^{q^i} I_n - \vartheta)^{-1}, \quad z \in \mathbb{C}_\infty$$

⁸Note that the factors commute.

and $\|d_i^{-1}(\theta^{q^i}I_n - \vartheta)^{-1}\| = |\theta|^{(i-1)q^i}$ for all $i \geq 0$, the image of the map $\tilde{\chi}$ is contained in $(\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$ (we recall from Corollary 2.26 that ω_χ^{-1} has entries in $\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge$). (2) Observe that if $a \in A$,

$$\begin{aligned}\tilde{\chi}(a) &= \exp_C \left(\tilde{\pi}a(\theta I_n - \vartheta)^{-1} \right) \omega_\chi^{-1} \\ &= C_a(\omega_\chi) \omega_\chi^{-1} \\ &= \chi(a).\end{aligned}$$

(3) We set $F = \exp_C(\tilde{\pi}z(\theta I_n - \vartheta)^{-1})$. Then,

$$\begin{aligned}\tau(F) &= -\theta F + \exp_C \left(\tilde{\pi}z(\theta I_n - \vartheta + \vartheta)(\theta I_n - \vartheta)^{-1} \right) \\ &= -\theta F + \exp_C \left(\tilde{\pi}(z(\theta I_n - \vartheta)^{-1})\vartheta \right) + \exp_C(\tilde{\pi}z)I_n \\ &= F \cdot (\vartheta - \theta I_n) + \exp_C(\tilde{\pi}z)I_n.\end{aligned}$$

□

From now on, we will denote both maps, $A \xrightarrow{\chi} \mathbb{F}_q(t_\Sigma)^{n \times n}$ and $\mathbb{C}_\infty \xrightarrow{\tilde{\chi}} \mathbb{K}_\Sigma^{n \times n}$, with χ to simplify our notations.

2.4.4. *An example with $n = 1$.* We consider, to illustrate a well known example (the reader familiar with the theory of the function ω can skip this subsection), the above picture in the case when $\chi = \chi_t$, where χ_t is the unique \mathbb{F}_q -algebra map

$$A \xrightarrow{\chi_t} \mathbb{F}_q[t]$$

defined by $\theta \mapsto t$ (therefore, $n = 1$). In this case ω_χ is the *function of Anderson and Thakur* ω . It is likely that this function appeared for the first time in the literature in the paper of Anderson and Thakur [1, Proof of Lemma 2.5.4 p. 177]. We have:

$$\omega(t) = \exp_C \left(\frac{\tilde{\pi}}{\theta - t} \right).$$

Corollary 2.26 allows to recover the well known factorization formula

$$(2.16) \quad \omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}} \right)^{-1} \in \mathbb{T}^\times,$$

for a fixed choice of the $(q-1)$ -th root, and the inverse of ω is an entire function in $\mathbb{E} := \mathbb{E}_\Sigma$ with Σ a singleton. The element ω can also be viewed as a function of the variable $t \in \mathbb{C}_\infty$, because the infinite product converges for all

$$t \in \mathbb{C}_\infty \setminus \{\theta^{q^k}; k \geq 0\}$$

and defines a meromorphic function over the above set, with simple poles at θ^{q^k} , $k \geq 0$. The element ω is a $(\theta - t)$ -torsion point in the Carlitz $A[t]$ -module $C(\mathbb{T})$. In particular, ω

is a generator of the free sub- $\mathbb{F}_q[t]$ -module of rank one of \mathbb{T} , kernel of the evaluation of the operator

$$C_{\theta-t} = \tau + \theta - t \in K[t][\tau],$$

so that ω is a solution of the linear homogeneous τ -difference equation of order 1 (see also [52, Proposition 3.3.6]):

$$(2.17) \quad \tau(\omega)(t) = (t - \theta)\omega(t).$$

All these properties easily follow from Corollary 2.26.

For the function $\chi_t : \mathbb{C}_\infty \rightarrow \mathbb{T}$ the results of §2.4.3 specialize in the foundations of Perkins' theory in [65]. We note that explicitly,

$$\chi_t(z) := \frac{\exp_C\left(\frac{\tilde{\pi}z}{\theta-t}\right)}{\omega(t)}, \quad z \in \mathbb{C}_\infty.$$

We deduce that χ_t defines an entire function $\mathbb{C}_\infty \rightarrow \mathbb{E}$ which satisfies $\chi_t(a) = a(t)$ for all $a \in A$, and the τ -difference equation

$$(2.18) \quad \tau(\chi_t(z)) = \chi_t(z) + \frac{\exp_C(\tilde{\pi}z)}{\tau(\omega)}.$$

To mention an additional property of the entire function χ_t , it can be proved that the function $z \mapsto \frac{\chi_t(z)}{z} \in \mathbb{E}$ is non-constant, entire, without zeroes.

2.4.5. Further commentaries. We already pointed out in the introduction that in this volume, we mainly focus to \mathbb{K}_Σ^N -valued modular forms for $\mathrm{GL}_2(A)$. However, it is natural to compare the theory developed so far with that of scalar modular forms for $\mathrm{GL}_n(A)$ with $n \geq 3$ by Basson, Breuer, Gekeler, Häberli, Hartl, Pink, Yu et al. already cited in the introduction 1. A $\mathrm{GL}_n(A)$ -generalization of our theory is likely to be meaningful but will hopefully be the object of another work. After having read the present volume the reader will attain a more precise intuition: the content of the present section §2.4 can be updated to raise the level of generality of §2.3.1. In the crucial next section 3 we explain in what sense the Carlitz module is involved in the analogues of ‘Fourier series’ of our $\mathrm{GL}_2(A)$ -modular forms. The link is indeed guaranteed by the omega matrices and the functions χ we just described. Higher rank Drinfeld modular forms are involved in the corresponding ‘Fourier series’ of $\mathrm{GL}_n(A)$ -modular forms with $n \geq 3$ and indeed analogues of the elements ω_χ and the functions χ can be associated to Drinfeld modules of any rank. This is briefly outlined, in the special case $\chi = \chi_t$, in the paper [59, §4]. We refrain from giving full details here because this would bring us too far from the purposes we fixed for the present paper.

3. FIELD OF UNIFORMIZERS

The crucial feature of the modular forms we study in the present text is that their entries can be identified with certain formal series generalizing the Fourier series of classical Drinfeld modular forms $f : \Omega \rightarrow \mathbb{C}_\infty$ for Γ . These formal series can be seen as elements of the *field of uniformizers* \mathfrak{K} (Definition 3.33) which provides a natural environment to

do computations and to prove our results. Roughly speaking, if $f : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$ is a modular form of weight w for a representation of the first kind ρ , then the entries of f can be viewed as elements of an algebraically closed field of generalized formal series in the sense of Kedlaya [44], containing the valued field $\mathbb{K}_\Sigma((u))$ with u the uniformiser defined in (1.3). We need to be a bit more precise however, as in practice, these series span a much smaller field and in the sequel, we need to gain a certain control on their expansions. The main results in this section are Propositions 3.16 and 3.32 where the reader can find an explicit description of the elements of \mathfrak{K} as formal Laurent series with coefficients which are *tame series*, certain entire functions defined in §3.2. Similar constructions have also been considered in [59]. We begin with §3.1, where we introduce some algebraic settings.

3.1. Some algebras and fields. In this subsection, we consider an integral commutative A -algebra B with the structure induced by a morphism

$$A \xrightarrow{\iota} B.$$

Additionally, we suppose that B is endowed with an \mathbb{F}_q -algebra endomorphism τ which acts as the map $c \mapsto c^q$ over $\iota(A)$ so that (B, τ) is a difference ring. We set

$$(3.1) \quad \Theta = \iota(\theta).$$

In the paper, we are going to restrict to the case ι injective. In this case, we identify Θ with θ but in the first general discussions, we prefer to keep Θ and θ distinct.

We consider, further, the polynomial B -algebra

$$\mathcal{R} = B[X_i; i \in \mathbb{Z}]$$

in infinitely many variables X_i , and the ideal \mathcal{P} generated by the polynomials

$$X_i^q + \Theta X_i - X_{i-1}, \quad i \in \mathbb{Z}.$$

Then, with \underline{X} the collection $(X_i : i \in \mathbb{Z})$ and $j = \sum_{i \in \mathbb{Z}} j_i q^{-i} \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ expanded in base q (so that only finitely many terms occur), we set

$$\langle \underline{X} \rangle^j = \prod_{i \in \mathbb{Z}} X_i^{j_i} \in \mathcal{R}/\mathcal{P}.$$

The quotient B -algebra \mathcal{R}/\mathcal{P} can be identified with the ring $B\langle \underline{X} \rangle$ whose elements F are formal *finite* sums in the indeterminates X_i , $i \in \mathbb{Z}$:

$$(3.2) \quad F = \sum_{j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} F_j \langle \underline{X} \rangle^j = \sum_{j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} F_j \prod_{k \in \mathbb{Z}} X_k^{j_k}, \quad F_j \in B,$$

where we have expanded the indices $j = \sum_{k \in \mathbb{Z}} j_k q^{-k}$ in base q (the coefficients j_i are almost all zero and belong to $\{0, \dots, q-1\}$).

An expansion (3.2) is uniquely determined. Indeed, supposing the converse, we are led to the existence of elements

$$F = \sum_j F_j \langle \underline{X} \rangle^j = 0$$

in $B\langle \underline{X} \rangle$ with the coefficients F_j not all zero. The B -module $\mathcal{P}(< q)$ with elements the polynomials P in \mathcal{P} such that for all i , $\deg_{X_i}(P) \leq q - 1$, is trivial, while F admits a representative in \mathcal{R} with the degree in X_j which is in $\{0, \dots, q - 1\}$ for some j ; this is impossible.

We observe that a product over $B\langle \underline{X} \rangle$ is well defined in virtue of the rules $X_i^q = X_{i-1} - \Theta X_i$. We have thus identified, after a mild abuse of notation, $B\langle \underline{X} \rangle$ with a complete system of (canonical) representatives of elements of \mathcal{R} modulo \mathcal{P} and we have defined over it a product which makes it isomorphic to the quotient \mathcal{R}/\mathcal{P} . A canonical representative in \mathcal{R} is a polynomial that has degree between $-\infty$ and $q - 1$ in each indeterminate X_i , $i \in \mathbb{Z}$. The reader must carefully distinguish $\langle \underline{X} \rangle^{jq^k}$ and $(\langle \underline{X} \rangle^j)^{q^k}$: these are distinct elements of $B\langle \underline{X} \rangle$!

Examples. If $B = A$ and ι is the identity, since the multiplication by θ of the Carlitz A -module is given by $C_\theta = \theta + \tau$, we have $X_{i-1} = C_\theta(X_i)$ in the A -module $C(A\langle \underline{X} \rangle)$. If $B = \mathbb{C}_\infty$ and ι is the inclusion $A \subset \mathbb{C}_\infty$, the substitution $X_i \mapsto e_C(\frac{z}{\theta^i})$, where e_C is defined by

$$e_C(z) = \exp_C(\tilde{\pi}z),$$

yields a \mathbb{C}_∞ -algebra homomorphism

$$\mathbb{C}_\infty\langle \underline{X} \rangle \rightarrow \text{Map}(K \rightarrow \mathbb{C}_\infty).$$

We come back to the general settings of this §3.1. We define a map

$$B\langle \underline{X} \rangle \xrightarrow{v} \mathbb{Z}[p^{-1}]_{\leq 0} \cup \{\infty\}$$

in the following way. We define $v(0) := \infty$ and we set $v(B \setminus \{0\}) = \{0\}$. Further, for a monomial $\langle \underline{X} \rangle^j = \prod_{i \in \mathbb{Z}} X_i^{j_i}$ (so only finitely many factors satisfy $j_i > 0$), we set $v(\langle \underline{X} \rangle^j) = -j$. Note that distinct monomials $\langle \underline{X} \rangle^j$ correspond to distinct values in $\mathbb{Z}[\frac{1}{p}]_{\leq 0}$ so that v is injective over $\{\langle \underline{X} \rangle^j : j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}\}$. If F is non-zero as in (3.2), then we set

$$v(F) = \inf\{v(\langle \underline{X} \rangle^j) : F_j \neq 0\};$$

the infimum is a minimum.

Lemma 3.1. *With $j, k \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ we have $\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = \langle \underline{X} \rangle^{j+k} + F$ where $F \in B\langle \underline{X} \rangle$ satisfies $v(F) > v(\langle \underline{X} \rangle^{j+k})$.*

Remark 3.2. Note that in general,

$$\langle \underline{X} \rangle^i \langle \underline{X} \rangle^j \neq \langle \underline{X} \rangle^{i+j}, \quad i, j \in \mathbb{Z}[p^{-1}]_{\geq 0}.$$

The equality holds if there is no base- q carry over in the sum $i + j$. For example, the reader can verify the formula:

$$(3.3) \quad \langle \underline{X} \rangle^{(q-1)(\frac{1}{q} + \dots + \frac{1}{q^n})} \langle \underline{X} \rangle^{\frac{1}{q^n}} = \langle \underline{X} \rangle^1 - \Theta \sum_{i=0}^{n-1} \langle \underline{X} \rangle^{(q-1)(\frac{1}{q} + \dots + \frac{1}{q^i})} \langle \underline{X} \rangle^{\frac{1}{q^{i+1}}}, \quad \forall n \geq 1.$$

Proof of Lemma 3.1. The proof is rather long and articulate but elementary. We decided to give all the details so that the reader can better familiarize with these structures. There is no loss of generality if we suppose that $0 < j, k < q$. Indeed we can shift the indexes, i.e. replacing $X_i \mapsto X_{i+l}$ for all i for some $l \in \mathbb{Z}$. This defines a B -linear automorphism of $B\langle \underline{X} \rangle$ that rescales v of a factor q^{-l} because it acts by sending $\langle \underline{X} \rangle^j \mapsto \langle \underline{X} \rangle^{q^{-l}j}$.

For some $r \geq 0$ we can thus suppose that $j = j_0 + j_1q^{-1} + \dots + j_rq^{-r}$, $k = k_0 + k_1q^{-1} + \dots + k_rq^{-r}$ with $0 \leq j_i, k_i \leq q-1$ for all i . We shall show the following properties. (1) There exists $F \in B\langle \underline{X} \rangle$ with $v(F) > -(j+k)$, such that $\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = \langle \underline{X} \rangle^{j+k} + F$. (2) There exists $G \in B\langle \underline{X} \rangle$ with $v(G) > -(j+k+q^{-r})$, such that $\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k X_r = \langle \underline{X} \rangle^{j+k+q^{-r}} + G$. (1) is exactly the statement of the Lemma, but we also need (2) in the proof. Note that F, G need not to belong to $B[X_0, \dots, X_r]$, however one can show that $F, G \in B[\dots, X_0, \dots, X_r]$. We proceed by induction on $r \geq 0$. To see (1) for $r = 0$ we note that there is nothing to prove in case $j+k = j_0+k_0 \leq q-1$. Suppose now that $j_0+k_0 \geq q$. In any case we have $j_0+k_0 \leq 2q-2$ so that $j_0+k_0-q \leq q-2$. Hence we have the identities, in $B\langle \underline{X} \rangle$:

$$\begin{aligned} \langle \underline{X} \rangle^{j+k} &= X_0^{j_0+k_0} = X_0^{j_0+k_0-q} (X_{-1} - \Theta X_0) = \\ &= X_0^{j_0+k_0-q} X_{-1} - \Theta X_0^{j_0+k_0+1-q} = \langle \underline{X} \rangle^{j_0+k_0} - \underbrace{\Theta \langle \underline{X} \rangle^{j_0+k_0+1-q}}_{=: -F}. \end{aligned}$$

To see (2) for $r = 0$ the reasoning is the same and the only case which needs an explanation is when $j_0+k_0+1 \geq q$. Note that in any case, $j_0+k_0+1-q \leq q-1$. Then:

$$\langle \underline{X} \rangle^{j+k} X_r = \langle \underline{X} \rangle^{j+k+1} - \underbrace{\Theta X_0^{j_0+k_0+2-q}}_{=: -G}.$$

If $j_0 = k_0 = q-1$ we have $X_0^{j_0+k_0+2-q} = X_0^q = X_{-1} - \Theta X_0$ (in $B\langle \underline{X} \rangle$) and the v -value is $-1 > -q = -(j_0+k_0+2-q) = v(\langle \underline{X} \rangle^{j_0+k_0+1})$.

We now prove (1) for $r > 0$. If $j_r+k_r \leq q-1$ we can write: $\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = X_r^{j_r+k_r} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'}$ where $j' = j - j_rq^{-r}$ and $k' = k - k_rq^{-r}$ (j', k' have one q -ary digit less so we can apply recursion). By induction hypothesis we get

$$\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = X_r^{j_r+k_r} (\langle \underline{X} \rangle^{j'+k'} + F')$$

where $F' \in B\langle \underline{X} \rangle$ is such that $v(F') > -j' - k'$. Then $\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = \langle \underline{X} \rangle^{j+k} + F$ where $F = X_r^{j_r+k_r} F'$. This proves (1) for the integer r because writing $F' = \sum_{j'} F'_{j'} \langle \underline{X} \rangle^{j'}$ we have $F = \sum_{j'} F'_{j'} \langle \underline{X} \rangle^{j'+q^{-r}j_r-k_r}$ thanks to Remark 3.2. Then, $v(F) = -q^{-j_r-k_r} - v(F') > -q^{-j_r-k_r} - j' - k' = -j - k$.

Let us now suppose that $j_r+k_r \geq q$. Then

$$(3.4) \quad \langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = X_r^{j_r+k_r-q} X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} - \Theta X_r^{j_r+k_r+1-q} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'}.$$

By the induction hypothesis we can apply the properties (1) and (2) for the integer $r-1$ and we deduce that in the first term on the right of (3.4), $X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j'+k'+q^{1-r}} + G'$ with $G' \in B\langle \underline{X} \rangle$ such that $v(G') > -(j'+k'+q^{1-r})$. Since $X_r^{j_r+k_r-q} \langle \underline{X} \rangle^{j'+k'+q^{-r}} = \langle \underline{X} \rangle^{j+k}$ and since writing $G_1 = X_r^{j_r+k_r-q} G'$ we immediately see (Remark 3.2) that $v(G_1) > -(j_r +$

$k_r - q)q^{-r} - (j' + k' + q^{1-r}) = -j - k$ we get $X_r^{j_r+k_r-q} X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j+k} + G_1$ with $v(G_1) > -j - k$. As for the second term on the right of (3.4) we have, always, $j_r + k_r + 1 - q \leq q - 1$. So we can apply Remark 3.2 again to show that there exists $F_1 \in B \langle \underline{X} \rangle$ with $v(F_1) > -(j_r + k_r + 1 - q)q^{-r}$ with the property that $X_r^{j_r+k_r+1-q} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j'+k'+(1-q)q^{-r}} + F_1$ and the whole term $F_2 = \Theta X_r^{j_r+k_r+1-q} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'}$ satisfies $v(F_2) > -j - k - q^{-r}$ so that setting $G := G_1 + F_2$ we get what we want in this case, namely: $\langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = \langle \underline{X} \rangle^{j+k} + G$ with $v(G) > -(j + k)$.

It remains to show that the property (2) holds for an integer $r > 0$. We handle the case in which $j_r + k_r + 1 \leq q - 1$ in a way identical to that of the case $j_r + k_r \leq q - 1$ for the property (1) so we omit the details. We suppose that $j_r + k_r + 1 \geq q$. we can write, in $B \langle \underline{X} \rangle$:

$$(3.5) \quad X_r \langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = X_r^{j_r+k_r+1-q} X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} - \Theta X_r^{j_r+k_r+2-q} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'}.$$

We first focus on the first term in (3.5). By the induction hypothesis we have

$$X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j'+k'+q^{1-r}} + G'$$

with $v(G') > -(j' + k' + q^{1-r})$. In any case $j_r + k_r + 1 - q \leq q - 1$ so that applying Remark 3.2, $X_r^{j_r+k_r+1-q} X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j+k+q^{-r}} + G_1$ with $v(G_1) > -(j + k + q^{-r})$. The handling of the second term in (3.5) is slightly more involved because there are two subcases. If (2.a) at least one of the terms j_r, k_r is $< q - 1$ then $j_r + k_r + 2 - q \leq q - 1$ and this term can be handled just as the second term in (3.4) so we omit the details. It remains to consider the subcase (2.b) where $j_r = k_r = q - 1$. We see that

$$G_2 := X_r^{j_r+k_r+2-q} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = X_r^q \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} - \Theta X_r \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'}.$$

By the property (2) for the integer $r - 1$ we can write $X_{r-1} \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j'+k'+q^{1-r}} + G'$ with $v(G') > -(j' + k' + q^{1-r})$. Additionally by Remark 3.2 we have $X_r \langle \underline{X} \rangle^{j'} \langle \underline{X} \rangle^{k'} = \langle \underline{X} \rangle^{j'+k'+q^{-r}}$. Hence $v(G_2) = -(j' + k' + q^{1-r}) > -(j + k + q^{-r}) = v(\langle \underline{X} \rangle^{j+k+q^{-r}})$ (the middle inequality follows from the choice of j_r, k_r). Writing $G := G_1 - \Theta G_2$ we have $v(G) > -(j + k + q^{-r})$, we can write $X_r \langle \underline{X} \rangle^j \langle \underline{X} \rangle^k = \langle \underline{X} \rangle^{j+k+q^{-r}} + G$, and this is what we wanted. \square

As an immediate consequence we record the following important corollary.

Corollary 3.3. *The map v is an additive valuation.*

In other words, $v(FG) = v(F) + v(G)$ if $F, G \in B \langle \underline{X} \rangle$.

Since $B \langle \underline{X} \rangle$ is a valued ring by Corollary 3.3, it is integral and we deduce that \mathcal{P} is a prime ideal. The residual ring of $B \langle \underline{X} \rangle$ is B . Further, defining

$$\tau(X_i) = X_i^q \equiv X_{i-1} - \Theta X_i \pmod{\mathcal{P}}$$

induces an endomorphism of $B \langle \underline{X} \rangle$.

It is easy to show that, for all $F \in B \langle \underline{X} \rangle$, $v(\tau(F)) = qv(F)$. Indeed, for all j , $(\langle \underline{X} \rangle^j)^q = \tau(\langle \underline{X} \rangle^j)$. Since $F \in B \langle \underline{X} \rangle \setminus \{0\}$ is such that $F \in B$ if and only if $v(F) = 0$ (and $F \in B \langle \underline{X} \rangle \setminus B$ if and only if $v(F) < 0$) we immediately see that the subring $B \langle \underline{X} \rangle^{\tau=1}$ of the elements F

such that $\tau(F) = F$ is equal to $B^{\tau=1}$. Note that even in the case of (B, τ) invertive, τ does not extend to an automorphism of $B\langle\underline{X}\rangle$.

3.1.1. *The maps λ, μ, ν .* We present certain auxiliary maps $\lambda : B\langle\underline{X}\rangle \setminus \{0\} \rightarrow \mathbb{N}$ and $\mu, \nu : B\langle\underline{X}\rangle \setminus B \rightarrow \mathbb{N}$ that are necessary to develop the arguments from §3.1.2 on. If $F \in B\langle\underline{X}\rangle$ is non-zero, we call *depth* of F the total degree in the indeterminates $(X_i)_{i \in \mathbb{Z}}$ of the canonical representative in \mathcal{R} of F . We have that $F \in B\langle\underline{X}\rangle \setminus \{0\}$ has depth 0 if and only if $F \in B$. Let $F \in B\langle\underline{X}\rangle \setminus B$ be as in (3.2). We denote by $\mu(F)$ the largest $m \in \mathbb{Z}$ such that the variable X_m occurs in at least one non-zero monomial of F (remember that the elements of $B\langle\underline{X}\rangle$ are polynomials so that $\mu(F)$ is well defined). Similarly, we denote by $\nu(F)$ the smallest $n \in \mathbb{Z}$ such that the variable X_n occurs in at least one non-zero monomial of F . Clearly, the function μ dominates the function ν over $B\langle\underline{X}\rangle \setminus B$ (in the natural ordering of \mathbb{Z}). More precisely, λ, μ, ν are connected by the following inequalities:

$$0 \leq \mu(F) - \nu(F) \leq (q-1)^{-1}\lambda(F), \quad \forall F \in B\langle\underline{X}\rangle \setminus B.$$

The next result collects some properties of the maps λ, μ, ν in relation with the product structure. They are applied in the proof of Proposition 3.11.

Lemma 3.4. *For two monomials $\langle\underline{X}\rangle^i$ and $\langle\underline{X}\rangle^j$ in $B\langle\underline{X}\rangle$ the following properties hold:*

- (1) $\lambda(\langle\underline{X}\rangle^i \langle\underline{X}\rangle^j) \leq \lambda(\langle\underline{X}\rangle^i) + \lambda(\langle\underline{X}\rangle^j)$,
- (2) $\mu(\langle\underline{X}\rangle^i \langle\underline{X}\rangle^j) = \max\{\mu(\langle\underline{X}\rangle^i), \mu(\langle\underline{X}\rangle^j)\}$,
- (3) $\nu(\langle\underline{X}\rangle^i \langle\underline{X}\rangle^j) \in \{\min\{\nu(\langle\underline{X}\rangle^i), \nu(\langle\underline{X}\rangle^j)\}, \min\{\nu(\langle\underline{X}\rangle^i), \nu(\langle\underline{X}\rangle^j)\} - 1\}$.

Proof. (1) We set $\lambda(0) = -\infty$. It is easily seen that any representative in \mathcal{R} of $F \in B\langle\underline{X}\rangle$ modulo \mathcal{P} has its total degree which is larger than $\lambda(F)$. This suffices to justify this property because $\lambda(\langle\underline{X}\rangle^i \langle\underline{X}\rangle^j) \leq \deg(\langle\underline{X}\rangle^i \langle\underline{X}\rangle^j) \leq \deg(\langle\underline{X}\rangle^i) + \deg(\langle\underline{X}\rangle^j) = \lambda(\langle\underline{X}\rangle^i) + \lambda(\langle\underline{X}\rangle^j)$. (2) This property has been already mentioned in the proof of Lemma 3.1 and it is straightforward. We leave the details to the reader. (3) Identifying X_i with $\langle\underline{X}\rangle^{q^{-i}}$ for all i and allowing a harmless abuse of notation, we show by induction on $r \geq 0$ that, for any fixed $j_0, \dots, j_r \in \{0, \dots, 2q-2\}$, $\nu(X_0^{j_0} \cdots X_r^{j_r}) \in \{0, -1\}$ and, if X_{-1} occurs in some term of the expansion of the canonical representative of $X_0^{j_0} \cdots X_r^{j_r}$, it occurs with degree 1. If $r = 0$, this is clear ⁽⁹⁾. Suppose now that the property is proved for the integer $r-1$. Then we have $\nu(X_1^{j_1} \cdots X_r^{j_r}) \in \{0, 1\}$, and if the value is 0 then X_0 occurs in the expansion of the canonical representative with degree 1 by induction hypothesis. The proof is complete by multiplying by $X_0^{j_0}$ (with $j_0 \in \{0, \dots, 2q-2\}$) as we obtain a representative in $B[X_i : i \in \mathbb{N}]$ that has degree $< q$ in all the indeterminates X_i with $i > 0$ and whose degree in X_0 is at most $2q-1$, a property from which it is easy to conclude. \square

3.1.2. *The B -module $B\langle\langle\underline{X}\rangle\rangle$.* We analyze a difference B -module containing $B\langle\underline{X}\rangle$ strictly.

Definition 3.5. We define $B\langle\langle\underline{X}\rangle\rangle$ to be the B -module of formal series as in (3.2), without the condition of finiteness of the sums, and such that the following conditions hold:

⁹It can be proved, more generally, that if $m \geq 1$ and if l is an integer such that $q^l \leq m < q^{l+1}$, then $\nu(X_0^m) \geq -l$. Moreover, $\lambda(X_0^m) \leq \ell_q(m)$, the sum of the digits of m in base q .

- (1) There exists $L \geq 0$ (depending on F) such that if $F_j \neq 0$, then $\ell_q(j) \leq L$, with $\ell_q(j)$ denoting the sum of the digits of j in base q (which means that the length of the base- q expansions of the exponents j involved is $\leq L$).
- (2) If $F_j \neq 0$, then $j < M$ with a constant $M > 0$ depending on F .

It is clear that there is an inclusion of B -modules $B\langle X \rangle \subset B\langle\langle X \rangle\rangle$. The first condition also means that the number of factors X_j of the monomials occurring in $F \in B\langle\langle X \rangle\rangle$ is bounded.

The depth map λ extends to $B\langle\langle X \rangle\rangle \setminus \{0\}$. The second condition can be justified in the following way. If $j \in \mathbb{Z}[\frac{1}{p}]_{>0}$ then we can write, with $r_0 \leq r_1$ integers,

$$j = j_{r_0}q^{-r_0} + j_{r_0+1}q^{-1-r_0} + \cdots + j_{r_1-1}q^{1-r_1} + j_{r_1}q^{-r_1}, \quad j_{r_0}j_{r_1} \neq 0, \quad j_i \in \{0, \dots, q-1\}$$

so that $q^{-r_0} \leq j < q^{1-r_0}$. Hence we have, setting $M = q^{1-r_0}$, that $\langle X \rangle^j$ only contains factors of the form X_i with $i \geq r_0$ if and only if $j < M$. In the above definition, condition (2) is therefore equivalent to the existence, for a series F as in (3.2), of an integer r_0 such that if X_i occurs in a monomial $\langle X \rangle^j$ with $F_j \neq 0$, then $i \geq r_0$. The map ν extends to $B\langle\langle X \rangle\rangle \setminus \{0\}$.

Let $F \in B\langle\langle X \rangle\rangle \setminus \{0\}$ be an element with expansion (3.2). We denote by $\text{Supp}(F)$ the subset

$$\text{Supp}(F) = \{j \in \mathbb{Z}[p^{-1}] : F_j \neq 0\}.$$

We also set:

$$v(F) := \inf\{-j : j \in \text{Supp}(F)\} \in \mathbb{R}, \quad v(0) := \infty.$$

Lemma 3.6. *If $F \in B\langle\langle X \rangle\rangle \setminus \{0\}$, $v(F)$ is a minimum, in $\mathbb{Z}[\frac{1}{p}]_{\leq 0}$.*

Proof. Directly from Definition 3.5 we see that $\text{Supp}(F)$ can be covered by finitely many non-empty subsets S of $\mathbb{Z}[\frac{1}{p}]$ of the form

$$(3.6) \quad S = \left\{ j_0q^{-i_0} + \cdots + j_lq^{-i_l} : i_1, \dots, i_l \in \mathbb{N} \right\}, \quad l \leq \lambda(F), \quad j_0, \dots, j_l \in \{1, \dots, l\}.$$

The lemma follows from the fact that any such subset has the property that every non-empty subset has a maximum. \square

Corollary 3.7. *Let F be an element of $B\langle\langle X \rangle\rangle$. The expansion (3.2) is unique.*

Proof. Otherwise, there would exist an expansion $0 = \sum_j c_j \langle X \rangle^j$ with $c_j \in B$ not all zero and we would have $\infty = -j$ for some $j \in \mathbb{Z}[\frac{1}{p}]$ which is impossible. \square

Let F be an element of $B\langle\langle X \rangle\rangle \setminus \{0\}$. By Lemma 3.6 there is a unique expression

$$(3.7) \quad F = F_{-v(F)} \langle X \rangle^{-v(F)} + F'$$

where $F_{-v(F)} \in B \setminus \{0\}$ and $F' \in B\langle\langle X \rangle\rangle$ is such that $v(F') > v(F)$. The uniqueness follows from Corollary 3.7.

This allows to obtain the following.

Corollary 3.8. *The ring $B\langle\langle X \rangle\rangle$ is complete for the v -metric.*

Proof. Consider a Cauchy sequence $(F_n)_{n \geq 0}$ in $B\langle\langle X \rangle\rangle$. This is equivalent to $v(F_{n+1} - F_n) \rightarrow \infty$ as n tends to infinity. Suppose by contradiction that $F_{n+1} - F_n \neq 0$ for infinitely many integers n . For these integers, by the fact that $F_{n+1} - F_n \in B\langle\langle X \rangle\rangle$, we have $v(F_{n+1} - F_n) \leq 0$ in contradiction with the fact that the sequence is Cauchy. This means that the sequence $(F_n)_{n \geq 0}$ is ultimately constant and therefore, convergent to an element of $B\langle\langle X \rangle\rangle$. \square

The function ν is particularly useful to construct new elements of $B\langle\langle X \rangle\rangle$ out of a given sequence of elements in $B\langle\langle X \rangle\rangle$, by ‘finite sums out of a possibly infinite family of terms.’

Lemma 3.9. *Let $(G_i)_{i \geq 0}$ be a sequence of elements of $B\langle\langle X \rangle\rangle$. Assume that $\nu(G_i) \rightarrow \infty$ and that there exists $L \geq 0$ such that $\lambda(G_i) \leq L$ for all i . Then the infinite sum*

$$\sum_{i \geq 0} G_i$$

defines, canonically, an element $G \in B\langle\langle X \rangle\rangle$ with $\lambda(G) \leq L$.

Proof. Let j be an element of $\mathbb{Z}[\frac{1}{p}]$. Since $\nu(G_i) \rightarrow \infty$, $j \in \text{Supp}(G_i)$ for finitely many indices and we can define

$$G = \sum_j F_j \langle X \rangle^j,$$

where $F_j = \sum_{i \geq 0} G_{i,j}$, having written $G_i = \sum_j G_{i,j} \langle X \rangle^j$ (finite sum). The condition on $\lambda(G)$ is obvious. \square

Remark 3.10. Note that the series defining G in Lemma 3.9 may be divergent for the topology induced by ν . The process of summation defining G is that of finite sums. As an example, let F be in $B\langle\langle X \rangle\rangle$ defined by (3.2). Let us choose a bijection $f : \mathbb{N} \rightarrow \mathbb{Z}[\frac{1}{p}]_{\geq 0}$. Set $G_i := F_{f(i)} \langle X \rangle^{f(i)}$. Then Lemma 3.9 applies and $F = G$ with G the element defined by Lemma 3.9 while the series (3.2) may diverge for the ν -valuation. This process of summation is rather common when one studies wildly ramified extensions of local fields and draws a connection with the so-called *Hahn series*. To illustrate this we recall the famous example by Chevalley constituted by the polynomial $X^p - X - t^{-1} \in \mathbb{F}_p((t))[X]$ that has as a root, the formal series

$$x = \sum_{i \geq 1} t^{-1/p^i},$$

a series that diverges in the complete field $\widehat{\mathbb{F}_p((t))}^{ac}$. For more in this direction, read Kedlaya’s paper [44] and the bibliographical references therein. See also Part (4) of Proposition 3.11.

3.1.3. *Difference algebra structure on $B\langle\langle X \rangle\rangle$.* We show the next result:

Proposition 3.11. *The following properties hold.*

- (1) *The B -module $B\langle\langle X \rangle\rangle$ is endowed with the structure of a difference B -algebra with endomorphism τ , extending that of the difference algebra $(B\langle X \rangle, \tau)$.*

- (2) The B -algebra structure of $B\langle\langle X \rangle\rangle$ is compatible with the map $v : B\langle\langle X \rangle\rangle \setminus \{0\} \rightarrow \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ that therefore defines a valuation extending that on $B\langle X \rangle$. Additionally, for all $F \in B\langle\langle X \rangle\rangle$, $v(\tau(F)) = qv(F)$.
- (3) If (B, τ) is inversive, then $(B\langle\langle X \rangle\rangle, \tau)$ is inversive.
- (4) If (B, τ) is inversive and $F \in B\langle\langle X \rangle\rangle$ has no constant term in its expansion (3.2) then there exists $G \in B\langle\langle X \rangle\rangle$ such that $\tau(G) - G = F$.

Proof. (1) We show that there is a ring product on $B\langle\langle X \rangle\rangle$ that extends the product of $B\langle X \rangle$. We use that $\langle X \rangle^j = X_0^{j_0} \langle X \rangle^{j'}$ if $j = j_0 + j'$ with $0 \leq j_0 \leq q-1$ and $j' \in \mathbb{Z}[\frac{1}{p}]$ is such that $0 \leq j' < 1$. We consider $F, G \in B\langle\langle X \rangle\rangle$ and we proceed by induction on $\lambda(F) + \lambda(G) \geq 0$ to prove that FG is a well defined element of $B\langle\langle X \rangle\rangle$ as we construct the product by using the product of $B\langle X \rangle$ and furthermore, $\lambda(FG) \leq \lambda(F) + \lambda(G)$, $\nu(FG) \geq \min\{\nu(F), \nu(G)\} - 1$, and if $\nu(FG) = \min\{\nu(F), \nu(G)\} - 1$ then the degree of FG in $X_{\min\{\nu(F), \nu(G)\} - 1}$ is equal to one. There is nothing to prove if $\lambda(F) + \lambda(G) = 0$. Let us now suppose that $\lambda(F) + \lambda(G) > 0$. There is no loss of generality if we suppose that $F = X_0^{j_0} F_1$ and $G = X_0^{k_0} G_1$ with $0 \leq j_0, k_0 \leq q-1$, $j_0 + k_0 > 0$ and $\nu(F_1), \nu(G_1) \geq 1$. Clearly, $\lambda(F_1) + \lambda(G_1) < \lambda(F) + \lambda(G)$. Hence the product $F_1 G_1$ is well defined in $B\langle\langle X \rangle\rangle$; let us denote it by H_1 . By Part (3) of Lemma 3.4, $\nu(H_1) \in \mathbb{N}$. If $\nu(H_1) > 0$ we are done, because the product $X_0^{j_0+k_0} H_1$ is trivially well defined. If $\nu(H_1) = 0$ then we know that the canonical representative of H_1 has degree 1 in X_0 . In this case we can write $H_1 = X_0 H_2 + H_3$ where $\nu(H_2), \nu(H_3) > 0$. The products $X_0^{j_0+k_0} H_2, X_0^{j_0+k_0} H_3$ are trivially well defined and $X_0^{j_0+k_0} X_0 H_2 = X_0^{j_0+k_0+1} H_2 = (X_{-1} - \Theta X_0) X_0^{j_0+k_0+1-q} H_2$ is well defined (note that $j_0 + k_0 + 1 \leq 2q - 1$) and $B\langle\langle X \rangle\rangle$ carries a structure of B -algebra, extending the structure of $B\langle X \rangle$. That this is additionally a difference algebra with the extension of τ is clear.

(2) The valuation v of $B\langle X \rangle$ extends to a valuation of $B\langle\langle X \rangle\rangle$. This follows from (3.7). Indeed, clearly, if $F, G \in B\langle\langle X \rangle\rangle$, then $v(FG) = v(F) + v(G)$. A little additional thought allows to also justify that $v(\tau(F)) = qv(F)$ for all $F \in B\langle\langle X \rangle\rangle$.

(3) Assuming now that (B, τ) is inversive, we observe that

$$Y_n := \sum_{i \geq n} (\Theta^{(-1)})^{i-n} X_{i+1} \in B\langle\langle X \rangle\rangle$$

for all $n \in \mathbb{Z}$, with $\Theta^{(-1)} = \tau^{-1}(\iota(\theta))$ in B (this element exists by hypothesis), satisfies $Y_n^{(1)} = X_n$. Indeed,

$$\begin{aligned} Y_n^{(1)} &= \sum_{i \geq n} \Theta^{i-n} X_{i+1}^q \\ &= \sum_{i \geq n} \Theta^{i-n} (X_i - \Theta X_{i+1}) \\ &= X_n. \end{aligned}$$

Therefore, inductively, if we set:

$$(3.8) \quad Y_{n,r} := \sum_{i_1 > \dots > i_r > n} (\Theta^{(-1)})^{i_1 - i_2 - 1} (\Theta^{(-2)})^{i_2 - i_3 - 1} \dots (\Theta^{(-r)})^{i_r - n - 1} X_{i_1} \in B\langle\langle \underline{X} \rangle\rangle,$$

then $Y_{n,r}^{(r)} = X_n$ for all $r \geq 0$ and $n \in \mathbb{Z}$. Note that $\lambda(Y_{n,r}) = 1, \nu(Y_{n,r}) = n + r$ for all n, r and that $Y_{n,r} = X_{n+r} + Y'_{n,r}$ with $Y'_{n,r} \in B\langle\langle \underline{X} \rangle\rangle$ such that $v(Y'_{n,r}) > -q^{-n-r}$. To go a step further and prove that $B\langle\langle \underline{X} \rangle\rangle$ is inversive, let us choose $j \in \mathbb{Z}[\frac{1}{p}]_{>0}$ with q -expansion $\sum_k j_k q^{-k}$ and write

$$\langle \underline{Y} \rangle_r^j := \prod_{k \in \mathbb{Z}} Y_{k,r}^{j_k}.$$

This is a well defined element of $B\langle\langle \underline{X} \rangle\rangle$ and we have $\lambda(\langle \underline{Y} \rangle_r^j) = \lambda(\langle \underline{X} \rangle^j)$ for all r . Moreover,

$$(3.9) \quad \nu(\langle \underline{Y} \rangle_r^j) \geq \nu(\langle \underline{X} \rangle^j) + r + 1 - \lambda(\langle \underline{X} \rangle^j).$$

These properties follow easily from Lemma 3.4 and (3.8). We have that

$$(\langle \underline{Y} \rangle_r^j)^{(r)} = \langle \underline{X} \rangle^j, \quad \forall j, r.$$

Assuming that B is inversive, we are going to prove by induction over $\lambda \geq 0$ the next property. If $F = \sum_{j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} F_j \langle \underline{X} \rangle^j \in B\langle\langle \underline{X} \rangle\rangle$ is such that $\lambda = \lambda(F)$ and $n = \nu(F)$, then, for any $r \geq 0$, there exists $G_r \in B\langle\langle \underline{X} \rangle\rangle$ such that $\tau^r(G_r) = F$ with $\lambda(G) = \lambda, \nu(G) \geq n + r + 1 - \lambda$.

The property is clear for $\lambda = 0$. In fact, (3.8) justifies it also for $\lambda = 1$ but we do not need it. Suppose that $\lambda > 0$. Since r is fixed we write $G = G_r$ for simplicity. Without loss of generality we can suppose that $n = \nu(F) = 0$ and

$$F = X_0^{j_0} F_1$$

with $j_0 \in \{0, \dots, q-1\}$ and $F_1 \in B\langle\langle \underline{X} \rangle\rangle$ is such that $\lambda_1 := \lambda(F_1) < \lambda, \nu(F_1) \geq 1$. Note that $\lambda_1 + j_0 = \lambda$. Indeed, the canonical representative of F_1 does not depend on X_0 . By induction hypothesis there exists $G_1 \in B\langle\langle \underline{X} \rangle\rangle$ such that $\tau^r(G_{1,r}) = F_1$ satisfying $\lambda(G_{1,r}) = \lambda_1, \nu(G_{1,r}) \geq r + 2 - \lambda_1$. Recall that $X_0^{j_0} = \langle \underline{X} \rangle^{j_0}$ and $\langle \underline{Y} \rangle_r^{j_0}$ satisfies $(\langle \underline{Y} \rangle_r^{j_0})^{q^r} = X_0^{j_0}, \lambda(\langle \underline{Y} \rangle_r^{j_0}) = j_0, \nu(\langle \underline{Y} \rangle_r^{j_0}) \geq r + 1 - j_0$ (by (3.9)). Setting $G := \langle \underline{Y} \rangle_r^{j_0} G_{1,r}$ we thus get $\tau^r(G) = F$ with $\lambda(G) = \lambda(F)$. From Part (3) of Lemma 3.4 and (3.7) we deduce

$$\nu(G) \geq \min\{\nu(G_1), \nu(\langle \underline{Y} \rangle_r^{j_0})\} - 1 \geq \min\{r + 1 - \lambda_1, r - j_0\} \geq r + 1 - \lambda_1 - j_0 = r + 1 - \lambda.$$

Hence, if B is inversive, $B\langle \underline{X} \rangle$ is inversive.

(4) It remains to show that if B is inversive then, any equation $\tau(X) - X = F$ is solvable in $B\langle\langle \underline{X} \rangle\rangle$ if $F \in B\langle\langle \underline{X} \rangle\rangle$ has no constant term (note that B needs not to be closed for Artin-Schreier equations). In order to do so, we use the previous Part (3) and its proof. We have seen that for all $r \geq 0$ there exists an element $\tau^{-r}(F) = G_r \in B\langle\langle \underline{X} \rangle\rangle$ such that $\tau^r(\tau^{-r}(F)) = F$, with $\lambda(\tau^{-r}(F)) = \lambda(F)$ and $\nu(\tau^{-r}(F)) \geq \nu(F) + r + 1 - \lambda$. Hence $\nu(\tau^{-r}(F)) \rightarrow \infty$ as $r \rightarrow \infty$ and the hypotheses of Lemma 3.9 are satisfied, so that

$$G := \sum_{r \geq 0} \tau^{-r}(F)$$

is a well defined element of $B\langle\langle\underline{X}\rangle\rangle$ satisfying $\tau(G) - G = F$. \square

Remark 3.12. By using the above proposition and its proof it is easy to deduce the following properties. If L is an inversive field extension of K then $L\langle\langle\underline{X}\rangle\rangle$ is inversive, as well as its fraction field. If additionally L contains all roots of equations $\tau(X) - X = F$ (e.g. $L = \mathbb{K}_\Sigma$), then $L\langle\langle\underline{X}\rangle\rangle$ contains all roots of equations $\tau(X) - X = F$ with $F \in L\langle\langle\underline{X}\rangle\rangle$.

3.1.4. *Depth homogeneity.* We denote by $B\langle\langle\underline{X}\rangle\rangle_s$ the B -submodule of $B\langle\langle\underline{X}\rangle\rangle$ whose elements are the formal series F as in (3.2) such that if $F_j \neq 0$, then $\langle\underline{X}\rangle^j$ has depth equal to s , i.e. $\ell_q(j) = s$. It is easy to see that

$$(3.10) \quad B\langle\langle\underline{X}\rangle\rangle = \bigoplus_{s \geq 0} B\langle\langle\underline{X}\rangle\rangle_s$$

as a B -module. If $F \in B\langle\langle\underline{X}\rangle\rangle$, we can expand in finite sum and in a unique way

$$(3.11) \quad F = \sum_{s \geq 0} F^{[s]},$$

where $F^{[s]} \in B\langle\langle\underline{X}\rangle\rangle_s$.

The next Lemma, not used in the present text, is an aside observation. The proof of which is left to the reader.

Lemma 3.13. *For any $s \geq 0$, τ induces an endomorphism of the B -module $B\langle\langle\underline{X}\rangle\rangle_s$.*

Remark 3.14. The B -algebra $B\langle\langle\underline{X}\rangle\rangle$ is not graded by the depths. Instead, we have that

$$B\langle\langle\underline{X}\rangle\rangle_s B\langle\langle\underline{X}\rangle\rangle_{s'} \subset \bigoplus_{j \geq 0} B\langle\langle\underline{X}\rangle\rangle_{s+s'-j(q-1)},$$

where we set $B\langle\langle\underline{X}\rangle\rangle_s = \{0\}$ if $s < 0$. This property is easy to show and we omit the proof.

3.1.5. *The case of B a difference field.* We keep working under the hypotheses of the previous sections and, although several properties also hold in broader generality, we additionally suppose in this subsection that $B = L$ is a field together with an embedding $A \rightarrow L$ and an endomorphism $\tau : L \rightarrow L$ extending the \mathbb{F}_q -endomorphism $c \mapsto c^q$ in A . We introduce the subvector space of $L\langle\langle\underline{X}\rangle\rangle$:

$$L^\circ\langle\langle\underline{X}\rangle\rangle := \text{Vect}_L(F \in L\langle\langle\underline{X}\rangle\rangle : \nu(F) \geq 1) \oplus L.$$

Only the variables X_1, X_2, \dots occur in the series defining $L^\circ\langle\langle\underline{X}\rangle\rangle$. We have

$$v(L^\circ\langle\langle\underline{X}\rangle\rangle \setminus \{0\}) = \mathbb{Z}[p^{-1}] \cap]-1, 0].$$

In particular, $L^\circ\langle\langle\underline{X}\rangle\rangle$ has no ring structure compatible with v . It has the family $(\langle\underline{X}\rangle^j : j \in \mathbb{Z}[p^{-1}] \cap]-1, 0])$ as an L -basis. Note that $L^\circ\langle\langle\underline{X}\rangle\rangle$ need not to be τ -closed, that is, such that for all $f \in L^\circ\langle\langle\underline{X}\rangle\rangle$, $\tau(f) \in L^\circ\langle\langle\underline{X}\rangle\rangle$. If we suppose that L is inversive, then it follows easily from Part (4) of Proposition 3.11 that τ^{-1} defines an $L^{\tau^{-1}}$ -linear endomorphism of $L^\circ\langle\langle\underline{X}\rangle\rangle$.

We set

$$X := X_0.$$

Lemma 3.15. *For any $F \in L\langle\langle X \rangle\rangle$ there exist n and $f_0, \dots, f_n \in L^\circ\langle\langle X \rangle\rangle$ such that $F = f_0 + f_1X + \dots + f_nX^n$, and this expression is unique.*

Proof. We begin by illustrating a simple claim on polynomials in several variables that holds in broader generality. Let F be an element of $L\langle X \rangle \setminus B$ such that $\mu(F) \leq 0$. By iterate substitution of $X_{-k} \mapsto X_{1-k}^q + \Theta X_{1-k}$ with $k > 0$, defining a map from the set of representatives of $\mathcal{R} = L[X_i : i \in \mathbb{Z}]$ modulo \mathcal{P} , we deduce that there is a unique representative in \mathcal{R} of F which belongs to the subring $L[X]$ (in particular, the degree in X needs not to be between 0 and $q - 1$). This representative is uniquely determined. Let us consider $F \in L\langle\langle X \rangle\rangle \setminus \{0\}$ with expansion (3.2). If $\nu(F) > 0$ there is nothing to prove. Assume now that $\nu(F) \leq 0$. Then we can rearrange the terms in (3.2) in such a way that there are $j_1, \dots, j_h \in \mathbb{N}$ and $G_1, \dots, G_h \in L^\circ\langle\langle X \rangle\rangle$ with

$$F = \sum_{k=1}^h \langle\langle X \rangle\rangle^{j_k} G_k.$$

This expansion is uniquely determined. The above claim now suffices to complete the proof. \square

Hence we have

$$L\langle\langle X \rangle\rangle = \bigoplus_{i \geq 0} L^\circ\langle\langle X \rangle\rangle X^i.$$

We can write, loosely:

$$L\langle\langle X \rangle\rangle = L^\circ\langle\langle X \rangle\rangle[X].$$

We now consider $\widehat{\text{Frac}(L\langle\langle X \rangle\rangle)}_v$, the completion for the valuation v of the fraction field of $L\langle\langle X \rangle\rangle$ (the latter is clearly a domain).

Proposition 3.16. *Every element f of $\widehat{\text{Frac}(L\langle\langle X \rangle\rangle)}_v$ can be expanded in a unique way as a sum*

$$f = \sum_{i \geq i_0} f_i X^{-i}, \quad f_i \in L^\circ\langle\langle X \rangle\rangle.$$

Remark 3.17. Note that in the above expansion the depths of the coefficients f_i may be unbounded in their dependence on i .

We can write $L^\circ\langle\langle X \rangle\rangle((X^{-1}))$ for the L -vector space of the formal series $f = \sum_{i \geq i_0} f_i X^{-i}$ as above, with $f_i \in L^\circ\langle\langle X \rangle\rangle$ for all i , with the warning that this is not a field for the usual Cauchy product rule of formal series, since, as pointed out previously, $L^\circ\langle\langle X \rangle\rangle$ is not a ring but just an L -vector space. The proposition tells us that this set in fact carries a structure of complete field, and equals $\widehat{\text{Frac}(L\langle\langle X \rangle\rangle)}_v$, but the product rule is not the Cauchy's one. To prove the proposition we will need the next two Lemmas. The first one describes the valued ring structure of $L^\circ\langle\langle X \rangle\rangle((X^{-1}))$.

Lemma 3.18. *The set $L^\circ\langle\langle X \rangle\rangle((X^{-1}))$ has a structure of commutative ring with unit, over which the valuation v extends in a unique way from $L^\circ\langle\langle X \rangle\rangle$, and which is complete for it.*

Proof. Recall from Corollary 3.8 that $L\langle\langle\underline{X}\rangle\rangle$ is complete. It is not difficult to deduce that $L^\circ\langle\langle\underline{X}\rangle\rangle$ is also complete. Since $L^\circ\langle\langle\underline{X}\rangle\rangle$ is an L -vector space, in order to show that $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ is a ring, all we need to do is to show that the product of $L\langle\langle\underline{X}\rangle\rangle$ extends to a product structure on $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$. Let $f = \sum_{i \geq i_0} f_i X^{-i}$ and $g = \sum_{j \geq j_0} g_j X^{-j}$ be two elements of $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$. We note that $h_k := \sum_{i+j=k} f_i g_j \in L\langle\langle\underline{X}\rangle\rangle$ has valuation in $] - 2, 0] \cup \{\infty\}$ and we can write $h_k = \alpha_k X + \beta_k$, with $\alpha_k, \beta_k \in L^\circ\langle\langle\underline{X}\rangle\rangle$. We define

$$h = fg = \sum_{k \geq k_0 := i_0 + j_0} X^{-k} h_k = \sum_{k \geq k_0} \alpha_k X^{1-k} + \sum_{k \geq k_0} \beta_k X^{-k} \in L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1})).$$

From this, we obtain the required ring structure. If $f = \sum_{i \geq i_0} f_i X^{-i} \in L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-i}))$ is such that $f_{i_0} \in L^\circ\langle\langle\underline{X}\rangle\rangle \setminus \{0\}$, then we set $v(f) := v(f_{i_0}) + i_0 \in]i_0 - 1, i_0]$ and it is plain that v defines a valuation over the ring $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ and that every such series of $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ converges for this valuation.

Note that $f = \sum_i f_i X^{-i} \in L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ is such that $v(f) > N$ where N is characterised by the following condition: the smallest i_0 such that $f_{i_0} \neq 0$ is such that $i_0 \geq N + 1$. This is meaningful, indeed, if $f_{i_0} \in L^\circ\langle\langle\underline{X}\rangle\rangle \setminus \{0\}$, $v(f_{i_0} X^{-i_0}) \in]i_0 - 1, i_0]$. Thus, if $(F_k)_k$ is a Cauchy sequence of $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$, the sequence $(F_0 - F_k)_k = (\sum_{i=1}^k (F_{i-1} - F_i))_k$ converges to an element of $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ which is then complete. \square

3.1.6. *The rings $L^\bullet\langle\langle\underline{X}\rangle\rangle$.* We introduce the ring:

$$L^\bullet\langle\langle\underline{X}\rangle\rangle := L\langle\langle\underline{X}\rangle\rangle \left[(\langle\langle\underline{X}\rangle\rangle^j)^{-1} : j \in \mathbb{Z}[p^{-1}]_{\geq 0} \right] = L\langle\langle\underline{X}\rangle\rangle [X_i^{-1} : i \in \mathbb{Z}],$$

which contains $L\langle\langle\underline{X}\rangle\rangle$. Every element f of $L^\bullet\langle\langle\underline{X}\rangle\rangle$ has a well defined valuation $v(f)$ in $\mathbb{Z}[\frac{1}{p}]$. To see this we note that for every $g \in L^\bullet\langle\langle\underline{X}\rangle\rangle$, there exists $j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ such that $\langle\langle\underline{X}\rangle\rangle^j g \in L\langle\langle\underline{X}\rangle\rangle$ and this provides the unique extension of the valuation map over $L^\bullet\langle\langle\underline{X}\rangle\rangle$.

Lemma 3.19. *We have $L^\bullet\langle\langle\underline{X}\rangle\rangle \subset L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$.*

Proof. If $n > 0$ we can identify X_{-n}^{-1} with an element in $X^{-q^n}(1 + X^{-1}A[[X^{-1}]])$ and therefore X_{-n} has a multiplicative inverse $X_{-n}^{-1} \in L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ for all $n > 0$. Now, we show that X_i has a multiplicative inverse $X_i^{-1} \in L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$ for all $i \geq 0$. To see this, we need the following useful identity in $L\langle\langle\underline{X}\rangle\rangle$, the proof of which is left to the reader:

$$(3.12) \quad X_i^{q^i} = X_0 - \left(\Theta X_1 + \Theta^q X_2^q + \cdots + \Theta^{q^{i-1}} X_i^{q^{i-1}} \right), \quad n \geq 0.$$

Thanks to the identity (3.12) we can write, in the fraction field of $L\langle\langle\underline{X}\rangle\rangle$:

$$(3.13) \quad \frac{1}{X_i} = \frac{X_i^{q^i-1}}{X_i^{q^i}} = \frac{X_i^{q^i-1}}{X_0 \left(1 - \frac{1}{X_0} (\Theta X_1 + \Theta^q X_2^q + \cdots + \Theta^{q^{i-1}} X_i^{q^{i-1}}) \right)}.$$

Observe that the element of $L^\bullet\langle\langle\underline{X}\rangle\rangle$:

$$(3.14) \quad h = X_0^{-1} (\Theta X_1 + \Theta^q X_2^q + \cdots + \Theta^{q^{i-1}} X_i^{q^{i-1}})$$

has valuation $1 - q^{-1} > 0$. Indeed, one can show that $\Theta X_1 + \Theta^q X_2^q + \cdots + \Theta^{q^{i-1}} X_i^{q^{i-1}} = (\Theta + \Theta^q + \cdots + \Theta^{q^{i-1}})X_1 + \kappa_i$ where $v(\kappa_i) > -q^{-1}$. We have $v(h^j) \rightarrow \infty$ as $j \rightarrow \infty$ so that we obtain a converging series expansion

$$(3.15) \quad X_i^{-1} = \frac{X_i^{q^i-1}}{X_0} \sum_{j \geq 0} h^j \in L^\circ \langle\langle \underline{X} \rangle\rangle ((X^{-1}))$$

(the reader can observe that it is not completely straightforward to determine explicitly the coefficient of X^{-j} for given j , in this series expansion.) More generally, $\frac{1}{\langle \underline{X} \rangle^j} \in L^\circ \langle\langle \underline{X} \rangle\rangle ((X^{-1}))$ for all j , and the lemma follows remembering that $L \langle\langle \underline{X} \rangle\rangle = L^\circ \langle\langle \underline{X} \rangle\rangle [X]$. \square

Proof of Proposition 3.16. It suffices to show that we can embed $\text{Frac}(L \langle\langle \underline{X} \rangle\rangle)$ in $(L^\bullet \langle\langle \underline{X} \rangle\rangle)_v^\wedge$ (completion for v). Taking completions, we get the proposition. To see this property, we only need to show that if $f \in L \langle\langle \underline{X} \rangle\rangle$ is not proportional by an element of L^\times to $\langle \underline{X} \rangle^j$ for some $j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$, then there exists $g \in (L^\bullet \langle\langle \underline{X} \rangle\rangle)_v^\wedge$ such that $fg = 1$. Now, write $f = \alpha \langle \underline{X} \rangle^j - h$ for some j , where $\alpha \in L^\times$ and where $h \in L \langle\langle \underline{X} \rangle\rangle$ is such that $v(h) > -j$. Then, the series $\sum_{i \geq 0} (\frac{h}{\alpha \langle \underline{X} \rangle^j})^i$ converges in $(L^\bullet \langle\langle \underline{X} \rangle\rangle)_v^\wedge$ and we can set

$$(3.16) \quad g = \frac{1}{\alpha \langle \underline{X} \rangle^j} \sum_{i \geq 0} \left(\frac{h}{\alpha \langle \underline{X} \rangle^j} \right)^i \in \widehat{L^\bullet \langle\langle \underline{X} \rangle\rangle}_v.$$

By Lemma 3.19, $(\text{Frac}(L \langle\langle \underline{X} \rangle\rangle))_v^\wedge \subset L^\circ \langle\langle \underline{X} \rangle\rangle ((X^{-1}))$ which is complete. On the other hand, any series $\sum_{i \geq i_0} f_i X^{-i}$ with the coefficients f_i in $L^\circ \langle\langle \underline{X} \rangle\rangle$ converges (for v) and the partial sums are elements of $L^\bullet \langle\langle \underline{X} \rangle\rangle [X^{-1}] \subset (\text{Frac}(L \langle\langle \underline{X} \rangle\rangle))_v^\wedge$ from which we can conclude that $(\text{Frac}(L \langle\langle \underline{X} \rangle\rangle))_v^\wedge = L^\circ \langle\langle \underline{X} \rangle\rangle ((X^{-1}))$ and also, we note that in this way, $L^\circ \langle\langle \underline{X} \rangle\rangle ((X^{-1}))$ carries the structure of a complete valued field (although performing explicitly the product of two formal series in it, or computing the inverse of a non-zero series in it, is in general a difficult matter). \square

Note that the field $L^\circ \langle\langle \underline{X} \rangle\rangle ((X^{-1}))$ has valuation ring

$$L \oplus \bigoplus_{i > 0} L^\circ \langle\langle \underline{X} \rangle\rangle X^{-i}$$

and maximal ideal

$$\bigoplus_{i > 0} L^\circ \langle\langle \underline{X} \rangle\rangle X^{-i}.$$

The residual field is L .

3.1.7. Link with Hahn series. This subsection is not needed in the rest of the paper but it illustrates useful properties of our rings and fields, and comparison with known theories. We suppose here that, in addition to the hypotheses of §3.1.5, $B = L$ is an inersive field containing A such that $\tau(x) = x^q$ for $x \in A$. By Proposition 3.11, $L \langle\langle \underline{X} \rangle\rangle$ is inersive. We give some complements on the structure of $L \langle\langle \underline{X} \rangle\rangle$. We provide here an alternative way to represent the elements of $L^\circ \langle\langle \underline{X} \rangle\rangle$.

Let X be an indeterminate over L . We consider the following set of *generalized formal series*, or *Hahn series*, in the sense of Kedlaya, [44]

$$(3.17) \quad L^\circ\{\{\underline{X}\}\} = \left\{ f = \sum_{i \in \mathbb{Z}[p^{-1}] \cap [0, 1[} f_i X^i : f_i \in L \text{ and there exists } c \geq 0 \text{ such that } f_i \neq 0 \text{ implies } \ell_p(i) \leq c \right\},$$

where $\ell_p(\cdot)$ denotes the sum of the digits in the base- p expansion of an integer. Equivalently, $L^\circ\{\{\underline{X}\}\}$ can be described as the set of all the generalized formal series in the indeterminate $t = X^{-1}$ which are supported by the sets $S_{a,b,c}$ of [44, §3] with $a = 1$, $b = 0$ and $c \geq 0$.

Lemma 3.20. *Identifying the indeterminate X of (3.17) with the element $X = X_0 \in L\langle\langle\underline{X}\rangle\rangle$ gives rise to a canonical isomorphism of L -vector spaces $L^\circ\{\{\underline{X}\}\} \cong L^\circ\langle\langle\underline{X}\rangle\rangle$.*

Proof. First of all, note that, for $S \subset \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ non-empty, $\ell_p(S) \subset \mathbb{N}$ is finite if and only if $\ell_q(S) \subset \mathbb{N}$ is finite. We deduce that every element $f \in L^\circ\{\{\underline{X}\}\}$ can be expanded, in a unique way, as a finite sum of generalized series $f = \sum_{l=0}^{l_0} f_l$ where

$$f_l = \sum_{j_1, \dots, j_l \in \mathbb{N}} f_{\underline{i}, \underline{j}} X^{j_1 q^{-i_1} + \dots + j_l q^{-i_l}}, \quad f_{\underline{i}, \underline{j}} \in L,$$

where $\underline{i} = (i_1, \dots, i_l) \in (\mathbb{N} \setminus \{0\})^l$ and $\underline{j} = (j_1, \dots, j_l) \in \{0, \dots, q-1\}^l$ (the term with $l = 0$ corresponds to the constant term). We have a well defined L -linear map $\phi : L^\circ\{\{\underline{X}\}\} \mapsto L^\circ\langle\langle\underline{X}\rangle\rangle$ defined by $X^{j_1 q^{-i_1} + \dots + j_l q^{-i_l}} \mapsto Y_{0, i_1}^{j_1} \dots Y_{0, i_l}^{j_l}$ where the elements $Y_{n,r}$ are as in (3.8). We note that $Y_{0, i_1}^{j_1} \dots Y_{0, i_l}^{j_l} = X_{i_1}^{j_1} \dots X_{i_l}^{j_l} + F = \langle\underline{X}\rangle^{j_1 q^{-i_1} + \dots + j_l q^{-i_l}} + F$, where $F \in L^\circ\langle\langle\underline{X}\rangle\rangle$ satisfies $v(F) > -(j_1 q^{-i_1} + \dots + j_l q^{-i_l})$. Hence the above mentioned linear map is an isomorphism. \square

The above isomorphism ϕ is canonical in the sense that it is an isometry if we give $L^\circ\{\{\underline{X}\}\}$ the norm induced by the degree in X , and additionally, if $f \in L^\circ\{\{\underline{X}\}\}$ is such that $\tau(f) \in L^\circ\{\{\underline{X}\}\}$, then $\tau(\phi(f)) = \phi(\tau(f))$. If L is inversive, it is easy to see that $\tau^{-1} : L \rightarrow L$ extends in a unique way to an $L^{\tau=1}$ -endomorphism of $L^\circ\{\{\underline{X}\}\}$ and we can identify $L^\circ\{\{\underline{X}\}\}$ and $L^\circ\langle\langle\underline{X}\rangle\rangle$ as $L[\tau]$ -modules.

Corollary 3.21. *If $L \subset \mathbb{C}_\infty$ has no non-trivial Artin-Schreier extensions then the completion of the fraction field of $L\langle\langle\underline{X}\rangle\rangle$ for the valuation v has no non-trivial Artin-Schreier extensions.*

Proof. We consider an Artin-Schreier equation $X^p - X = f$ with f in the completion $\text{Frac}(\widehat{L\langle\langle\underline{X}\rangle\rangle})_v$ of the fraction field of $L\langle\langle\underline{X}\rangle\rangle$. If f lies in the maximal ideal, the result is obvious. Indeed the formal series $f + f^p + f^{p^2} + \dots$ converges in $\text{Frac}(\widehat{L\langle\langle\underline{X}\rangle\rangle})_v$ to a solution g of $X^p - X = f$. Hence, without loss of generality, we can suppose that f belongs to

the complement of the maximal ideal in the completion of the fraction field of $L\langle\langle X \rangle\rangle$. We can decompose, in a unique way, $f = f_0 + f_1$ with $v(f_0) > 0$ (i.e. in the maximal ideal of $\widehat{\text{Frac}(L\langle\langle X \rangle\rangle)}_v$) and $f_1 \in L\langle\langle X \rangle\rangle$ such that $v(f_1) \leq 0$. By the above discussion, there is a solution g_0 in the maximal ideal, of the equation $X^p - X = f_0$.

By Proposition 3.16, the complement of the maximal ideal equals $L\langle\langle X \rangle\rangle$. It remains to solve the equation $X^p - X = f_1$ with $f_1 \in L\langle\langle X \rangle\rangle$. By a simple variant of Remark 3.12, for all $\tilde{f} \in L\langle\langle X \rangle\rangle$ there is $\tilde{g} \in L\langle\langle X \rangle\rangle$ such that $\tilde{g}^q - \tilde{g} = \tilde{f}$. Setting $\tilde{f} = f_1 + f_1^p + \dots + f_1^{p^{e-1}}$ (with $q = p^e$, $e > 0$) we thus get that $\tilde{g}^p - \tilde{g} - f_1$ is solution of $X + X^p + \dots + X^{p^{e-1}} = 0$ hence belonging to the algebraic closure of \mathbb{F}_q in L . There exists $\lambda \in L$ such that $\tilde{g}^p - \tilde{g} = f_1 + \lambda$. Let $\mu \in L$ be such that $\mu^p - \mu = \lambda$ that exists because L has no non-trivial Artin-Schreier extensions. Then $g_1 = \tilde{g} - \mu \in L\langle\langle X \rangle\rangle$ satisfies $g_1^p - g_1 = f_1$ and $g = g_0 + g_1$ is a solution of $X^p - X = f$. Clearly, all the solutions are in $g + \mathbb{F}_p$. \square

Remark 3.22. The reader should compare Proposition 3.16 with [44, Lemma 7]. By Theorem 6 *ibid.*, if L is algebraically closed, the field

$$\bigcup_{n \geq 1} \text{Frac}(\widehat{L\langle\langle X \rangle\rangle}_v((X^{-\frac{1}{n}}))$$

contains an algebraic closure of $L((X^{-1}))$.

3.2. Tame series. The rings $L\langle\langle X \rangle\rangle$ of the previous section, or the completions of their fraction fields, are not enough to study the behavior at ∞ of our modular forms. To do this we need a refined notion and we introduce tame series.

Unless otherwise specified, we shall fix, throughout this subsection, a τ -difference sub- A -algebra B of \mathbb{K}_Σ , for some Σ . We identify θ with $\iota(\theta)$ (so that $\Theta = \theta$ in (3.1)). We denote by $B\langle\langle X \rangle\rangle^b$ the subset of $B\langle\langle X \rangle\rangle$ formed by the series as in (3.2), satisfying $\sup_j \|F_j\| < \infty$ ($(\cdot)^b$ stands for 'bounded'). It is easy to show that it is complete for the v -topology, see the proof of Corollary 3.8. We leave to the reader the proof of the following:

Lemma 3.23. $B\langle\langle X \rangle\rangle^b$ is a difference sub- B -algebra of $B\langle\langle X \rangle\rangle$ containing $B\langle X \rangle$.

We consider the map $B\langle X \rangle \xrightarrow{J} \text{Hol}(\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma)$ defined by $J(X_i) = e_i$, where

$$e_i := e_C \left(\frac{z}{\theta^i} \right) = \exp_C \left(\frac{\tilde{\pi}z}{\theta^i} \right)$$

for all $i \in \mathbb{Z}$. It is easy to see that J is a B -algebra morphism and defines an algebra map from $B\langle\langle X \rangle\rangle^b$ to the maps from \mathbb{C}_∞ to \mathbb{K}_Σ ; this follows from the fact that, for all $z \in \mathbb{C}_\infty$, $|e_i(z)| = \left| \frac{\tilde{\pi}z}{\theta^i} \right|$ for all i sufficiently large (depending on z). We set

$$\underline{e} = (e_i : i \in \mathbb{Z}).$$

We denote by $B\langle\langle \underline{e} \rangle\rangle^b$ the image $J(B\langle\langle X \rangle\rangle^b)$ of J in the \mathbb{K}_Σ -valued maps. We call it the B -algebra of *tame series*. Explicitly, if we set

$$\langle \underline{e} \rangle^j = J(\langle X \rangle^j) = \prod_{i \in \mathbb{Z}} e_i^{j_i}, \quad j = \sum_{i \in \mathbb{Z}} j_i q^{-i} \in \mathbb{Z}[p^{-1}]_{\geq 0}, \quad j_k \in \{0, \dots, q-1\},$$

we can make the next:

Definition 3.24. A *tame series* with coefficients in B is a map $\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma$ which is defined by an everywhere converging series f of the type

$$(3.18) \quad f(z) = \sum_{j \in \mathbb{Z}[p^{-1}]_{\geq 0}} f_j \langle \underline{e} \rangle^j, \quad f_j \in B,$$

satisfying the following properties.

- (1) There exists an integer $L \geq 0$ such that if $f_j \neq 0$, then $\ell_q(j) \leq L$.
- (2) There exists $M > 0$ such that, for all $j \in \mathbb{Z}[p^{-1}]_{\geq 0}$, $f_j \in B$ satisfies $\|f_j\| \leq M$.
- (3) There exists $N \in \mathbb{N}$ such that if $j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ is such that $f_j \neq 0$, then $j < N$.

Proposition 3.25. *The map J extends to a B -algebra morphism*

$$B\langle\langle \underline{X} \rangle\rangle^b \xrightarrow{J} \text{Hol}(\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma)$$

and this is a morphism of τ -difference rings.

Proof. Let us consider a series f as in (3.18). Observe that for all $j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$, the function $z \mapsto \langle \underline{e} \rangle^j$ is \mathbb{K}_Σ -entire. It suffices to show that, for all $R \in |\mathbb{C}_\infty|$, the series defining f converges uniformly over the disk $D_{\mathbb{C}_\infty}(0, R)$. One immediately sees that $f(z)$ is a tame series if and only if $f(\theta^{-1}z)$ is a tame series. Hence, we are reduced to prove the above property in the case $R = 1$. Now, observe that the set $\{j \in \mathbb{Z}[p^{-1}]_{\geq 0} : f_j \neq 0 \text{ and } j \geq 1\}$ is finite (because of the conditions (1) and (3) of Definition 3.24). Hence, we can decompose

$$(3.19) \quad f = \sum_{j \geq 1} f_j \langle \underline{e} \rangle^j + \sum_{0 \leq j < 1} f_j \langle \underline{e} \rangle^j.$$

The first sum is finite and therefore defines an entire function. Note now that if $j = \sum_k j_k q^{-k} < 1$ then we can write

$$\langle \underline{e} \rangle^j = e_{i_1}(z)^{j_1} \cdots e_{i_l}(z)^{j_l}$$

where $\underline{i} = (i_1, \dots, i_l) \in (\mathbb{N}^*)^l$. Then, for $|z| \leq 1$, by the fact that \exp_C is locally an isometry,

$$|\langle \underline{e} \rangle^j| = |e_{i_1}(z)^{j_1} \cdots e_{i_l}(z)^{j_l}| \leq |\tilde{\pi}|^{\ell_q(j)} |\theta|^{-(i_1 j_1 + \cdots + i_l j_l)}.$$

Hence

$$\|f_j \langle \underline{e} \rangle^j\| \leq M |\tilde{\pi}|^L |\theta|^{-(i_1 j_1 + \cdots + i_l j_l)} \rightarrow 0$$

where L, M are as in (1) and (2) of Definition 3.24, the limit being considered for the Fréchet filter over the set of couples $(\underline{i}, \underline{j})$ with $\underline{j} = (j_1, \dots, j_l)$. This means that in the above decomposition (3.19), the second series defines a \mathbb{K}_Σ -entire function and the series defining f converges to a \mathbb{K}_Σ -entire function. \square

3.2.1. *Asymptotic behavior of tame series.* It is the asymptotic behavior of tame series that allows to endow their rings with a weight map, and a valuation. For $j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ we call $\langle \underline{e} \rangle^j$ a *monic tame monomial*. Its *depth* is the integer $\lambda(\langle \underline{e} \rangle^j) = \ell_q(j)$ and its *weight* is j . To fix ideas, the weight of $e_0 = e_C(z)$ is one and the weight of 1 or of a non-zero constant is 0. Distinct tame monomials have distinct weights. The condition of finite depth, jointly with the fact that for any tame series (3.18) e_i does not occur in the series expansion for $i \leq i_0$ for some i_0 (equivalent to Conditions (1) and (3) in Definition 3.24) ensures that the supremum of the weights of the monomials composing a non-zero tame series is a maximum.

In the following, we call *leading tame monomial* of a tame series $f \neq 0$ the unique tame monomial of maximal weight. The weight $w(f)$ of f is by definition equal to the weight of the leading tame monomial. The weight $-\infty$ is assigned to the zero tame series. We now discuss the question on whether, assigning to a non-zero tame series f the weight $w(f)$, we have defined a degree map

$$B\langle\langle \underline{e} \rangle\rangle^b \xrightarrow{w} \mathbb{Z}[p^{-1}]_{\geq 0} \cup \{-\infty\},$$

that is, the opposite of a valuation. Of course, this is related to the uniqueness of the tame expansion of a function such as in (3.18), entire after Proposition 3.25; we are going to focus on these questions now. We recall that $e_C(z) := \exp_C(\tilde{\pi}z) = e_0(z)$.

Lemma 3.26. *We consider a monic tame monomial $f(z) = \langle \underline{e} \rangle^j = e_{i_1}(z)^{j_1} \cdots e_{i_l}(z)^{j_l}$ with $i_1 > \cdots > i_l$ and $j_1, \dots, j_l \in \{0, \dots, q-1\}$. Let $z \in \mathbb{C}_\infty$ be such that $|z| \notin |\theta|^{\mathbb{Z}}$. If $|z| > |\theta|^{i_l}$, we have $|f(z)| = |e_C(z)|^j$.*

Proof. Let $z \in \mathbb{C}_\infty$ be such that $|\theta|^{n-1} < |z| < |\theta|^n$, for $n \in \mathbb{Z}$. Let us suppose that $n \geq 1$. From the Weierstrass product expansion of the function $e_A(z) = \tilde{\pi}^{-1} \exp_C(\tilde{\pi}z)$:

$$(3.20) \quad e_A(z) = z \prod'_{a \in A} \left(1 - \frac{z}{a}\right),$$

we see that

$$|e_A(z)| = |z| \prod_{a \neq 0} \left|1 - \frac{z}{a}\right| = |z| \prod_{0 < |a| < |z|} \left|\frac{z}{a}\right| = |z|^{q^n} \prod_{0 < |a| \leq |\theta|^{n-1}} |a|^{-1}.$$

Therefore

$$\begin{aligned} |e_C(z)| &= |\tilde{\pi}| |z|^{q^n} \prod_{0 < |a| < |\theta|^{n-1}} |a|^{-1}, \\ |e_C(z\theta^{-i})|^{q^i} &= |\tilde{\pi}|^{q^i} \left|\frac{z}{\theta^i}\right|^{q^n} \prod_{0 < |a| < |\theta|^{n-1-i}} |a|^{-q^i}. \end{aligned}$$

One computes easily $\prod_{0 < |a| \leq q^{n-1}} a^{-1} = \frac{l_n}{D_n}$ with D_n and l_n defined in (2.10) and (2.12) and $|\frac{l_n}{D_n}| = |\theta|^{q \frac{q^n - 1}{q-1} - nq^n}$ from which we deduce

$$\frac{\left| e_C \left(\frac{z}{\theta^i} \right)^{q^i} \right|}{|e_C(z)|} = |\tilde{\pi}|^{q^i - 1} |\theta^i|^{-q^n} \left| \frac{l_{n-i}^{q^i} D_n}{D_{n-i}^{q^i} l_n} \right| = 1.$$

To resume, if i is a non-negative integer and $n > i$ (note that $|\frac{z}{\theta^i}| \notin |\theta|^{\mathbb{Z}}$), then

$$|e_i(z)| = \left| e_C \left(\frac{z}{\theta^i} \right) \right| = |e_C(z)|^{\frac{1}{q^i}}.$$

This suffices to complete the proof of the Lemma. \square

Proposition 3.27. *Let us consider a non-zero tame series f as in (3.18) and let $\langle e \rangle^{j_0}$ be its leading tame monomial. Then, for all $z \in \mathbb{C}_\infty$ such that $|z| \notin |\theta|^{\mathbb{Z}}$ and with $|z|$ large enough depending on f , $\|f(z)\| = \|f_{j_0}\| |e_C(z)|^{j_0}$.*

Proof. Let $z \in \mathbb{C}_\infty$ be such that $|\theta|^{n-1} < |z| < |\theta|^n$, for $n \in \mathbb{Z}$. Let us suppose that $n \leq i$. Then, $|z| < |\theta|^i$ and $|z/\theta^i| < 1$. In this case the product expansion (3.20) yields $|e_C(\frac{z}{\theta^i})| = |\tilde{\pi} \frac{z}{\theta^i}|$.

We consider an arbitrary tame monomial $\langle e \rangle^j$, and z as above. Writing $j = j_1 q^{-i_1} + \dots + j_l q^{-i_l}$ with $i_l > \dots > i_1$ and $j_i \in \{0, \dots, q-1\}$, we can set

$$j_{\leq n} = \sum_{\substack{m \text{ such that} \\ n \leq i_m}} j_m q^{-i_m}, \quad j_{> n} = \sum_{\substack{m \text{ such that} \\ n > i_m}} j_m q^{-i_m} \in \mathbb{Z}[p^{-1}]_{\geq 0}$$

so that

$$j = j_{\leq n} + j_{> n}$$

without carrying over in the base- q sum. Then,

$$\langle e \rangle^j = \langle e \rangle^{j_{\leq n}} \langle e \rangle^{j_{> n}}.$$

By Lemma 3.26 we have $|\langle e \rangle^{j_{> n}}| = |e_C(z)|^{j_{> n}}$. On the other hand, writing $j_{\leq n} = j_{k+1} q^{-i_{k+1}} + \dots + j_l q^{-i_l}$ (hence $j_{< n} = j_1 q^{-i_1} + \dots + j_k q^{-i_k}$), we see that

$$|\langle e \rangle^{j_{\leq n}}| = \frac{(\tilde{\pi} z)^{\ell_q(j_{\leq n})}}{|\theta|^{\delta_n}} \leq \frac{|\tilde{\pi}|^{\ell_q(j)}}{|\theta|^{\delta_n}} |z|^{\ell_q(j)},$$

where $\delta_n := i_{k+1} j_{k+1} + \dots + i_l j_l$. Then, we see that

$$|\langle e \rangle^j| \leq |e_C(z)|^j |\theta|^{-\delta_n} |\tilde{\pi} z|^L.$$

Let us choose $\tilde{w} \in \mathbb{Z}[\frac{1}{p}]$, positive. Then, for $|z| \geq R_0$ with $R_0 \in |\mathbb{C}_\infty|$ suitably chosen, depending only on \tilde{w} and L (L as in (1) of Definition 3.24), we have that $|\tilde{\pi} z|^L \leq |e_C(z)|^{\tilde{w}}$, so that

$$|\langle e \rangle^j| \leq |e_C(z)|^{j+\tilde{w}} |\theta|^{-\delta_n}.$$

Now, let us consider a non-zero tame series f that we can write in the following way

$$f = f_{j_0} \langle \underline{e} \rangle^{j_0} + \sum_{j \neq j_0} f_j \langle \underline{e} \rangle^j$$

with $f_{j_0} \neq 0$. There exists $\tilde{w} \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ such that if $j \neq j_0$ is such that $f_j \neq 0$, then $j < j + \tilde{w} < j_0$. Hence:

$$\|f_j \langle \underline{e} \rangle^j\| \leq C_1 |e_C(z)|^{j+\tilde{w}} |\theta|^{-\delta_n}, \quad |z| \geq R_0,$$

where C_1 is an upper bound for the absolute values $\|f_j\|$. Since $\delta_n \rightarrow \infty$, we have that

$$\left\| \sum_{j \neq j_0} f_j \langle \underline{e} \rangle^j \right\| \leq C_2 |e_C(z)|^{w'}, \quad |z| \geq R_0,$$

for $w' \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$, $0 \leq w' < j_0$ and R_0 depending on f . Hence,

$$\|f(z) - f_{j_0} \langle \underline{e} \rangle^{j_0}\| \leq C_3 |e_C(z)|^{w'}$$

and if $|z| \geq R_1$ depending on C_3 and w' , we get

$$\|f(z)\| = \|f_{j_0}\| \cdot |\langle \underline{e} \rangle^{j_0}| = \|f_{j_0}\| \cdot |e_C(z)|^{j_0}$$

(C_2, C_3 are constants depending on f). □

Remark 3.28. We define, for $z \in \mathbb{C}_\infty$,

$$|z|_{\mathfrak{S}} = \inf\{|z - l| : l \in K_\infty\} = \min\{|z - l| : l \in K_\infty\}$$

(see [60, §5]). The statement of Proposition 3.27 holds under the weaker condition that $|z|_{\mathfrak{S}}$ is large enough. We leave the details to the reader.

We have the following important consequence of Proposition 3.27.

Corollary 3.29. *If f is an entire function which belongs to $B\langle\langle \underline{e} \rangle\rangle^b$, then its tame series expansion is unique.*

Proof. It suffices to show that a tame series as in (3.18) cannot vanish identically, if not trivially. But otherwise, such a series would then have a unique leading tame monomial, which would contradict the property of Proposition 3.27. □

Thanks to Corollary 3.29 and Proposition 3.11 (that stipulates that $B\langle\langle \underline{X} \rangle\rangle^b$ is a ring), J is injective, $B\langle\langle \underline{e} \rangle\rangle^b$ has a structure of B -algebra, the map $w \circ J$ is the opposite of the valuation v and the depth $\lambda(f)$ of a tame series f defined as the depth of $g \in B\langle\langle \underline{X} \rangle\rangle^b$ such that $J(g) = f$ becomes a well defined invariant of the entire function it represents. Note that $B\langle\langle \underline{e} \rangle\rangle^b$ is also complete for the v -metric; the proof is identical to that of Corollary 3.8). The map J is continuous.

Remark 3.30. The opposite of the weight is an additive valuation on tame series that we denote by v . While a tame series as in (3.18) in general diverges for the v -valuation, it converges for the inf-valuation associated to any disk $D_{\mathbb{C}_\infty}(0, R)$, $R \in |\mathbb{C}_\infty^\times|$ by the fact that it is an entire function.

3.2.2. *The field of uniformizers.* Several constructions of §3.1 can be reproduced in connection with the B -algebra $B\langle\langle\underline{e}\rangle\rangle^b$, with very little changes. We set $B^\circ\langle\langle\underline{e}\rangle\rangle^b = J(B^\circ\langle\langle\underline{X}\rangle\rangle^b)$. Explicitly, $B^\circ\langle\langle\underline{e}\rangle\rangle^b$ is the B -module of the series satisfying the items (1) to (3) of Definition 3.24 with the additional property that only the functions e_1, e_2, \dots occur, just as for the indeterminates X_1, X_2, \dots in the definition of $L^\circ\langle\langle\underline{X}\rangle\rangle$ at the beginning of §3.1.5. The reader can check, writing

$$e := e_0 = e_C,$$

the next result:

Lemma 3.31. *Every element $f \in B\langle\langle\underline{e}\rangle\rangle^b$ can be expanded, in a unique way, as*

$$f = \sum_{i=0}^r f_i e^i, \quad f_i \in B^\circ\langle\langle\underline{e}\rangle\rangle^b.$$

If $B = L$ is a field, We set

$$\mathfrak{K}_L := \widehat{\text{Frac}(L\langle\langle\underline{e}\rangle\rangle^b)}_v$$

(v -adic completion); we call this the *field of uniformizers over L* . The next proposition provides a simple way to represent the elements of \mathfrak{K}_L .

Proposition 3.32 (u -expansions). *Every element f of \mathfrak{K}_L can be expanded in a unique way as a sum*

$$f = \sum_{i \geq i_0} f_i e^{-i}, \quad f_i \in L^\circ\langle\langle\underline{e}\rangle\rangle^b.$$

Proof. The proof closely follows that of Proposition 3.16. The additional point is that we must take care of condition (2) in Definition 3.24, which is not relevant in Proposition 3.16. But apart from this detail, the proof runs along the same ideas. It suffices to give a deeper look at the proofs in §3.1.6. If we take an element such as h in (3.14) then the series expansion (3.15) belongs to $L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$. Indeed, if we write $x = \Theta X_1 + \Theta^q X_2^q + \dots + \Theta^{q^{i-1}} X_i^{q^{i-1}} \in L^\circ\langle\langle\underline{X}\rangle\rangle^b$, it is rather straightforward, although tedious, to show by induction that for all $n \geq 1$, we can expand in a unique way $x^n \in L^\circ\langle\langle\underline{X}\rangle\rangle^b[X]$ with degree $\leq \max\{0, \log_q(n) - 1\}$ in X . From this we deduce that for all $i \geq 0$, X_i^{-1} can be expanded in a unique way in $L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$ (we know from the proof of Lemma 3.19 that this is already possible in $L^\circ\langle\langle\underline{X}\rangle\rangle((X^{-1}))$). If we set

$$L^\bullet\langle\langle\underline{X}\rangle\rangle^b := L\langle\langle\underline{X}\rangle\rangle^b \left[(\langle\langle\underline{X}\rangle\rangle^j)^{-1} : j \in \mathbb{Z}[p^{-1}]_{\geq 0} \right] = L\langle\langle\underline{X}\rangle\rangle[X_i^{-1} : i \in \mathbb{Z}],$$

we get $L^\bullet\langle\langle\underline{X}\rangle\rangle^b \subset L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$ following the scheme of proof of Lemma 3.19. The other relevant point is justifying that if $h \in L\langle\langle\underline{X}\rangle\rangle^b$, then the series in (3.16) converges in $L^\bullet\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$. Suppose that we have $h \in L\langle\langle\underline{X}\rangle\rangle^b$ with $v(h) > j$, $j \in \mathbb{Z}[\frac{1}{q}]_{\geq 0}$. We have $\frac{h}{\langle\langle\underline{X}\rangle\rangle^j} \in L^\bullet\langle\langle\underline{X}\rangle\rangle^b$. We can expand $\frac{h}{\langle\langle\underline{X}\rangle\rangle^j}$ in $L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$ and therefore, if f is any element of $L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1})) \setminus \{0\}$, thanks to (3.16) f is invertible in $L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$. This justifies that the completion of the fraction field of $L^\circ\langle\langle\underline{X}\rangle\rangle^b$ is $L^\circ\langle\langle\underline{X}\rangle\rangle^b((X^{-1}))$. To recover the corresponding properties of tame series, it suffices to apply the injective map J . \square

We also need to introduce the valuation ring \mathfrak{D}_L and the maximal ideal \mathfrak{M}_L of \mathfrak{K}_L . The residual field is L . We have, as L -vector spaces:

$$\mathfrak{M}_L = \bigoplus_{i>0} \widehat{L^\circ \langle\langle e \rangle\rangle^b} e^{-i}, \quad \mathfrak{D}_L = L \oplus \mathfrak{M}_L.$$

We write, for simplicity, \mathfrak{K}_Σ for $\mathfrak{K}_{\mathbb{K}_\Sigma}$.

Definition 3.33. The *field of uniformizers* is the complete v -valued field

$$\mathfrak{K} = \bigcup_{\Sigma} \widehat{\mathfrak{K}_\Sigma}.$$

We denote by $\mathfrak{D}, \mathfrak{M}$ the valuation ring and the maximal ideal of v .

Remark 3.34. It is easy to see that $f \in \mathfrak{K}$ if and only if we can expand f in series $\sum_{i \geq i_0} f_i e^{-i}$ with $f_i \in \mathbb{K}_{\Sigma_i}^\circ \langle\langle e \rangle\rangle^b$ with, for all i , Σ_i finite subset of \mathbb{N} . This can be easily deduced from the property that, if f is non-zero, then we can write in a unique way in the form

$$f = f_1 \langle e \rangle^w e^{-i} + f_1, \quad f_1 \in \mathbb{K}_\Sigma^\times, \quad w \in \mathbb{Z}[p^{-1}]_{\geq 0}, \quad i \in \mathbb{Z}, \quad f_1 \in \mathfrak{K},$$

(an identity of the same type as (3.7)) where Σ is a finite subset of \mathbb{N} and the weight $w(f_1)$ is strictly smaller than $w - i$.

In the present paper all the elements of \mathfrak{K} that we are going to extract from modular forms are not of such a general form. They can be identified with rigid analytic functions $\Omega \rightarrow \mathbb{K}_\Sigma$ for some Σ finite subset of \mathbb{N} (meromorphic over \mathbb{C}_∞). For example, elements of \mathfrak{K}_\emptyset can be identified with formal series $\sum_{i \geq i_0} f_i e^{-i}$ with $f_i \in \mathbb{C}_\infty^\circ \langle\langle e \rangle\rangle^b$ (the latter are entire functions $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$). The u -expansions of scalar Drinfeld modular forms (1.4) are also of this type (remember that $u = e^{-1}$). In this case the f_i 's are all constant \mathbb{C}_∞ -valued functions. If $\Sigma = \{1\}$ is a singleton, elements of \mathfrak{K}_Σ can be identified with formal series $\sum_{i \geq i_0} f_i e^{-i}$ with $f_i \in \mathbb{K}_\Sigma^\circ \langle\langle e \rangle\rangle^b$.

3.2.3. Some endomorphisms. Hecke operators acting on spaces of our modular forms (see §4.3) are defined through certain affine endomorphisms of \mathfrak{K} that we discuss here. We consider a difference field extension $L \supset \mathbb{C}_\infty$ contained in \mathbb{K}_Σ . Typical examples are \mathbb{L}_Σ and \mathbb{K}_Σ with $\Sigma \subset \mathbb{N}^*$ finite. The rings

$$L \langle\langle \underline{X} \rangle\rangle, L^\circ \langle\langle \underline{X} \rangle\rangle((X^{-1})), L \langle\langle \underline{e} \rangle\rangle^b((e^{-1})), L^\circ \langle\langle \underline{e} \rangle\rangle^b((e^{-1})), \dots$$

also have L -linear endomorphisms coming from affine endomorphisms

$$X \mapsto \alpha X + \beta.$$

We therefore study homotheties and translations in these rings. The main result of this subsection is Proposition 3.35 below. In fact, we are going to only study here a restricted class of endomorphisms of $L \langle\langle \underline{e} \rangle\rangle^b$ and $L^\circ \langle\langle \underline{e} \rangle\rangle^b((e^{-1}))$ but in the course of our proofs we make a partial use of endomorphisms of the other rings as well.

We begin with homotheties. Let y be an indeterminate over L and let us choose $\alpha \in L((y^{-1}))$. Let us write

$$(3.21) \quad \alpha = \sum_{i \geq i_0} \alpha_i y^{-i}, \quad \alpha_i \in L, \quad \alpha_{i_0} \in L^\times.$$

Then we have an L -linear map $\varphi_\alpha : L\langle\langle X \rangle\rangle \rightarrow L\langle\langle X \rangle\rangle$ defined by setting

$$\varphi_\alpha(\langle\langle X \rangle\rangle^j) := \sum_{i \geq i_0} \alpha_i \langle\langle X \rangle\rangle^{jq^{-i}},$$

for all $j \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$. This map is well defined by Lemma 3.9. Indeed note that for all i , $\lambda(\langle\langle X \rangle\rangle^{jq^{-i}}) = \lambda(\langle\langle X \rangle\rangle^j)$. For all $j, k \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ it is easily verified that $\varphi_\alpha(\langle\langle X \rangle\rangle^j \langle\langle X \rangle\rangle^k) = \varphi_\alpha(\langle\langle X \rangle\rangle^j) \varphi_\alpha(\langle\langle X \rangle\rangle^k)$, and φ_α is an L -algebra endomorphism of $L\langle\langle X \rangle\rangle$. Having supposed that the leading term in (3.21) is invertible we recognize that φ_α is in fact an automorphism of inverse $\varphi_{\alpha^{-1}}$. We have thus defined a group homomorphism (injective)

$$L((y^{-1}))^\times \xrightarrow{\varphi: \alpha \mapsto \varphi_\alpha} \text{Aut}_L(L\langle\langle X \rangle\rangle).$$

The above group homomorphism also targets the group $\text{Aut}_L(L^\circ\langle\langle X \rangle\rangle((X^{-1})))$ and we have, for all $f \in L^\circ\langle\langle X \rangle\rangle((X^{-1}))$:

$$v(\varphi_\alpha(f)) = q^{\deg_y(\alpha)} v(f).$$

To see this we recall from Corollary 3.8 that $L\langle\langle X \rangle\rangle$ is complete for v , so that in particular φ_α is v -continuous. It is clear that for $\alpha \in L((y^{-1}))^\times$, φ_α defines a continuous L -algebra automorphism of $L\langle\langle X \rangle\rangle$ such that $v(\varphi_\alpha(f)) = q^{\deg_y(\alpha)} v(f)$ for all $f \in L\langle\langle X \rangle\rangle$ (use (3.7)). We can therefore conclude by using Proposition 3.16 (φ_α extends to the completion of the fraction field of $L\langle\langle X \rangle\rangle$ in a unique way), and the compatibility with respect to v is clear.

Now, we focus on $L\langle\langle \underline{e} \rangle\rangle^b$ and $L^\circ\langle\langle \underline{e} \rangle\rangle^b((e^{-1}))$. Consider again α as in (3.21) with $\alpha_i \in L$, $\alpha_{i_0} \in L^\times$, and additionally suppose that the set $\{\|\alpha_i\| : i \in \mathbb{Z}\}$ is bounded. Then, via the continuous injective map J we already discussed, such that $X_i \mapsto e_i$ for all i , φ_α induces a v -continuous endomorphism of $L\langle\langle \underline{e} \rangle\rangle^b$ satisfying $v(\varphi_\alpha(f)) = q^{\deg_y(\alpha)} v(f)$ for all $f \in L\langle\langle \underline{e} \rangle\rangle^b$. It needs not to be an automorphism this time. This is due to the fact that the above subset of α 's with bounded coefficients in $L((y^{-1}))^\times$ is not a subgroup.

With α as above, φ_α also extends to an endomorphism of $L^\circ\langle\langle \underline{e} \rangle\rangle^b((e^{-1}))$ compatible with v because $L^\circ\langle\langle \underline{e} \rangle\rangle^b((e^{-1}))$ is the completion of the fraction field of $L\langle\langle \underline{e} \rangle\rangle^b$ by Proposition 3.32.

As for the analogues of translations, we shall be briefer. We consider

$$\beta = \sum_{i \geq i_0} \beta_i y^{-i} \in L((y^{-1})),$$

again with the property that the set of positive real numbers $\{\|\beta_i\| : i \in \mathbb{Z}\}$ is bounded. One sees easily that the correspondence

$$e_j \mapsto e_j + \sum_{i \geq i_0} \beta_i e_j(\theta^{-i})$$

defines a continuous automorphism

$$\psi_\beta : L\langle\langle \underline{e} \rangle\rangle^b \rightarrow L\langle\langle \underline{e} \rangle\rangle^b$$

that also extends to a continuous algebra automorphism of $L^\circ\langle\langle \underline{e} \rangle\rangle^b((e^{-1}))$ that is an isometry.

All the properties of tame series that we have discussed so far, and that will be used in the rest of the present paper, are collected for commodity in the next Proposition.

Proposition 3.35. *Let L be a field extension of \mathbb{C}_∞ contained in \mathbb{K}_Σ that is a τ -difference field and let f be in \mathfrak{K}_L^\times .*

(1) *There exists a unique series expansion*

$$f = \sum_{j \geq j_0} f_j e^{-j} \quad f_j \in L^\circ\langle\langle \underline{e} \rangle\rangle^b$$

and $v(f)$ equals $v(f_{j_0}) + j_0$ if $f_{j_0} \neq 0$.

(2) *If α, β are elements of $L((y^{-1}))$ such that the $\|\cdot\|$ -norms of the coefficients of their y^{-1} -expansions are bounded, then $\varphi_\alpha(f), \psi_\beta(f) \in \mathfrak{K}_L^\times$ such that*

$$v(\varphi_\alpha(f)) = q^{\deg_y(\alpha)} v(f), \quad v(\psi_\beta(f)) = v(f).$$

(3) *For all $k \in \mathbb{N}$, $\tau^k(f) \in \mathfrak{K}_L^\times$ is such that $v(\tau^k(f)) = q^k v(f)$.*

3.2.4. *Final remarks on tame series.* There are entire functions $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ which are not tame series. One of them is the identity map $z \mapsto z$. Indeed, one sees easily that for all $w \in \mathbb{Q}$,

$$(3.22) \quad \lim_{|z|_{\mathbb{Q}} \rightarrow \infty} \frac{|z|}{|e_C(z)|^w} \in \{0, \infty\}.$$

Therefore, $(z \mapsto z) \notin \mathbb{C}_\infty\langle\langle \underline{e} \rangle\rangle^b$ as otherwise, we could assign a well defined weight in $\mathbb{Z}[\frac{1}{p}]$ to it.

To define $B\langle\langle \underline{e} \rangle\rangle^b$, we have used formal series with bounded coefficients in B (in Definition 3.24). This seems to be a heavy complication in the theory, but it is necessary. One of the reasons for this choice is that the isomorphism J of Proposition 3.25 is hardly definable over a larger sub-algebra of $B\langle\langle \underline{X} \rangle\rangle$. We illustrate the problem for $B = \mathbb{C}_\infty$.

We set

$$G = \sum_{i \geq 0} \theta^{\frac{i}{q}} X_{i+1} = \varphi_\alpha(X_0) \in \mathbb{C}_\infty^\circ\langle\langle \underline{X} \rangle\rangle,$$

where α is as in (3.21). Then, we have the identities in $\mathbb{C}_\infty\langle\langle X \rangle\rangle$ (we have used a computation similar to what follows to show that $\mathbb{C}_\infty\langle\langle X \rangle\rangle$ is inversive for τ):

$$\begin{aligned}
G^q &= \left(\sum_{i \geq 0} \theta^{\frac{i}{q}} X_{i+1} \right)^q \\
&= \sum_{i \geq 0} \theta^i X_{i+1}^q \\
&= \sum_{i \geq 0} \theta^i (C_\theta(X_{i+1}) - \theta X_{i+1}) \\
&= \sum_{i \geq 0} \theta^i (X_i - \theta X_{i+1}) \\
&\stackrel{!}{=} \sum_{i \geq 0} \theta^i X_i - \sum_{i \geq 0} \theta^{i+1} X_{i+1} \\
&= X_0.
\end{aligned}$$

Note the exclamation point over the next to the last equality. In parallel, let us set

$$g = \sum_{i \geq 0} \theta^{\frac{i}{q}} e_{i+1}.$$

This is not an element of $\mathbb{C}_\infty\langle\langle e \rangle\rangle^b$ because the sequence $(|\theta^{\frac{i}{q}}|)_i$ is not bounded. We claim that g defines an entire function. Indeed, for all $R \in |\mathbb{C}_\infty|$ and all $z \in D(0, R)$, we have, for any i large enough, $|e_{i+1}(z)| = |\tilde{\pi}| |z| |\theta|^{-i-1}$ so that $|\theta^{\frac{i}{q}} e_{i+1}(z)| \leq |\tilde{\pi}| |\theta|^{\frac{i}{q} - i - 1} R \rightarrow 0$ which implies the uniform convergence of the series defining g over any disk $D(0, R)$.

Now, $g^q \neq e$. One way to see this is by observing that $e = \tilde{\pi}z + h^q$, with h an entire function. If $g^q = e$, the identity map $z \mapsto z$ would be equal to the q -th power of an entire function, which is impossible. To compute $g^q - e$ we cannot use the argument we applied to show the identity $G^q = X_0$; this argument breaks at the level of the equality $\stackrel{!}{=}$ because the series of functions $\sum_{i \geq 0} \theta^i e_{i+1}$ is divergent outside 0 although the series $\sum_{i \geq 0} \theta^i X_{i+1}$ defines an element of $\mathbb{C}_\infty\langle\langle X \rangle\rangle$.

To compute g^q we proceed in the following way. We set

$$\phi = e_C \left(\frac{\tilde{\pi}z}{\theta - t} \right) = \sum_{i \geq 0} t^i e_{i+1} \in \mathbb{F}_q[t]^\circ \langle\langle e \rangle\rangle^b,$$

where e_C is defined $\mathbb{F}_q[t]$ -linearly as in §2.3.2. It is easy to see that $\lim_{t \rightarrow \theta} (\theta - t)\phi = \tilde{\pi}z$. But

$$e_C(z) = C_{\theta-t}(\phi) = (\theta - t)\phi + \tau(\phi)$$

so that $e = e_C(z) = \tilde{\pi}z + \lim_{t \rightarrow \theta} \tau(\phi) = \tilde{\pi}z + \sum_{i \geq 0} \theta^i e_{i+1}^q = \tilde{\pi}z + g^q$. We thus obtain:

$$g^q - e = \tilde{\pi}z.$$

From this identity we deduce (1) that $g \notin \mathbb{C}_\infty \langle\langle \underline{e} \rangle\rangle^b$ (because z is not tame) and (2) the map J does not extend to a \mathbb{C}_∞ -algebra map over $\mathbb{C}_\infty \langle\langle \underline{e} \rangle\rangle^b[G]$.

On another hand, the sequence $(X_n^{q^n})_{n \geq 0} \subset L^\circ \langle\langle \underline{X} \rangle\rangle$ diverges for the v -valuation and therefore, also the sequence $(e_n^{q^n})_{n \geq 0}$ of elements of the ring of tame series. But the latter converges to the zero map for the sup-norm associated to the disk $D_{\mathbb{C}_\infty}(0, \rho)$ for all $\rho \in |\mathbb{C}_\infty^\times|$, as the reader can easily verify.

3.2.5. Examples of tame series. To conclude this section, we give examples of tame series of the kind which will be used in the present paper. Following §2.4.3, we consider, in the notations introduced there, a function

$$\chi \in \text{Hol}_{\mathbb{T}_\Sigma[d^{-1}]^\wedge} \left(\mathbb{C}_\infty \rightarrow (\mathbb{E}_\Sigma[d^{-1}]^\wedge)^{n \times n} \right)$$

analytically extending an \mathbb{F}_q -algebra morphism $A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ (see Proposition 2.27, where $\tilde{\chi} = \chi$). We now use the properties of tame series to show the following result.

Proposition 3.36. *The function χ can be identified with an element of $\mathfrak{K}^{n \times n}$ and is the unique entire function $f : \mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma^{n \times n}$ such that $f(a) = \chi(a)$ for all $a \in A$ with $\|\exp_C(\tilde{\pi}z)^{-\frac{1}{q}} f(z)\| \rightarrow 0$ as $\exp_C(\tilde{\pi}z) \rightarrow 0$.*

Proof. We have already seen in Proposition 2.27 that the entire function χ interpolates the map $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$. We now prove the growth estimate. But note that

$$\chi(z) = \exp_C \left(\tilde{\pi}(\theta I_n - \Theta)^{-1} z \right) \omega_\chi^{-1} = \omega_\chi^{-1} \sum_{i \geq 0} e_{i+1} \Theta^{-i} \in \left(\mathbb{T}_\Sigma[d^{-1}]^\wedge \langle\langle \underline{e} \rangle\rangle^b \right)^{n \times n}.$$

We deduce that $w(\chi) = w(e_1) = \frac{1}{q}$. Hence, by Proposition 3.27, we have that the function $\|\exp_C(\tilde{\pi}z)^{-\frac{1}{q}} f(z)\|$ is bounded as $\exp_C(\tilde{\pi}z) \rightarrow 0$.

It remains to show uniqueness. Consider $f \in \text{Hol}_{\mathbb{K}_\Sigma}(\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma^{n \times n})$ such that $f(a) = \chi(a)$ for all $a \in A$. Then the function $g = f - \chi$ is in $\text{Hol}_{\mathbb{K}_\Sigma}(\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma^{n \times n})$ and vanishes on $A \subset \mathbb{C}_\infty$. Therefore $\frac{g(z)}{\exp_C(\tilde{\pi}z)}$ is entire and $\lim_{\exp_C(\tilde{\pi}z) \rightarrow 0} \left\| \frac{g(z)}{\exp_C(\tilde{\pi}z)} \right\| = 0$. By Proposition 2.11, g vanishes identically. \square

4. QUASI-PERIODIC FUNCTIONS, REPRESENTATIONS OF THE FIRST KIND

One of the basic observations in the theory of modular forms for the full modular group $\text{SL}_2(\mathbb{Z})$ is that they are \mathbb{Z} -periodic, so that they have a Fourier series development, also called \mathbf{q} -expansion. There is a very similar property for scalar Drinfeld modular forms for the full modular group $\Gamma = \text{GL}_2(A)$ which are A -periodic, and indeed we have in this case u -expansions, which is the appropriate structure to study their behaviour at the cusp infinity as well as a large part of their theory.

For the vector-valued modular forms in our Definition 1.2 (studied from §4.2.5 on), we note that they behave like *quasi-periodic functions* under the translations $z \mapsto z + a$ with $a \in A$ (Definition 4.1). The first task is to study this behavior for a special class of representations of Γ called *representations of the first kind* introduced below (Definition

4.5). In this section we study quasi-periodic functions, which can also be understood as a kind of generalization of Goss polynomials. The terminology chosen comes from Gekeler's paper [25] ⁽¹⁰⁾. The central result obtained here is Theorem 4.12, which asserts that every modular form in the sense of Definition 1.2 can be expanded as a formal series in the field of uniformizers \mathfrak{K} . We also give an application of these structures in Theorem 4.13, where we show that the spaces of our modular forms and cusp forms are endowed with Hecke endomorphisms, generalizing [63, Proposition 5.12], which deals with the very special case of $N = 2$ and $\rho = \rho_t^*$ (with an ad hoc proof unfortunately very hard to generalize to our more general settings).

4.1. Quasi-periodic functions. Let k be any field, and R a commutative k -algebra. We denote by $B(R)$ the Borel subgroup $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subset \mathrm{GL}_2(R)$ and by $U(R)$ the unit upper-triangular subgroup $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \subset \mathrm{GL}_2(R)$. Let Y be an indeterminate and $E/k(Y)$ be a field extension. Suppose we are given

$$(4.1) \quad \mathrm{GL}_2(k) \xrightarrow{\mu} \mathrm{GL}_N(E) \xleftarrow{\nu} U(k[Y])$$

two representations such that $\mu|_{U(k)} = \nu|_{U(k)}$ and such that for all $\lambda \in k^\times$ and $a \in k[Y]$,

$$(4.2) \quad \mu\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)\nu\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right)\mu\left(\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = \nu\left(\begin{pmatrix} 1 & \lambda a \\ 0 & 1 \end{pmatrix}\right).$$

Then, there is a unique representation $\rho : \mathrm{GL}_2(k[Y]) \rightarrow \mathrm{GL}_N(E)$ which restricts to μ, ν respectively on $\mathrm{GL}_2(k)$ and $U(k[Y])$.

Indeed, see [49, 74], we have that $\mathrm{GL}_2(k[Y])$ is the amalgamated product of $\mathrm{GL}_2(k)$ and $B(k[Y])$ along the common subgroup $B(k)$:

$$\mathrm{GL}_2(k[Y]) = \mathrm{GL}_2(k) *_B(k) B(k[Y]).$$

By Bruhat's decomposition

$$\mathrm{GL}_2(k) = B(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}U(k) \sqcup B(k)$$

this implies that every element $\gamma \in \mathrm{GL}_2(k[Y])$ can be written in a unique way

$$\gamma = A_1 B_1 \cdots A_l B_l$$

for some l , where $A_i \in B(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}U(k)$ and $B_i \in B(k[Y])$. Therefore, the identities

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda a \\ 0 & 1 \end{pmatrix}$$

are the gluing condition for μ, ν giving rise to a unique representation ρ of Γ .

¹⁰See his §2. Gekeler uses what he calls quasi-periodic functions to construct an analogue of the De Rham isomorphism associated to a Drinfeld module (between a 'De Rham module' of classes of biderivations and a 'Betti module'). More precisely, he constructs (in his §4) certain Poincaré series to show that the map is surjective (while injectivity follows essentially from the fact the the logarithm series does not extend to an entire function). These Poincaré series have inspired the construction of Perkins' series and are similar to the quasi-periodic functions we study in the present paper. It is possible to use them to prove an appropriate version of the De Rham isomorphism for the Carlitz functor evaluated on certain difference algebras.

We now take $k = \mathbb{F}_q$ and $Y = \theta$ and we recall that $\Gamma = \mathrm{GL}_2(A)$ with $A = \mathbb{F}_q[\theta]$. We also recall that Ω denotes the rigid analytic space whose underlying set is $\mathbb{C}_\infty \setminus K_\infty$ as defined, for instance, in [24] (see also [60, §5, 6]). We set, for $a \in A$,

$$(4.3) \quad T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(in Γ). Given a representation

$$(4.4) \quad \Gamma \xrightarrow{\rho} \mathrm{GL}_N(B)$$

with $(B, |\cdot|_B)$ a countably cartesian Banach \mathbb{C}_∞ -algebra, we first analyze its restriction to $U(A)$ and the corresponding part in Definition 1.2. This brings us to the next definition.

Definition 4.1. (a) Let ρ be a representation as in (4.4). An analytic function

$$\Omega \xrightarrow{f} B^{N \times 1}$$

such that

$$(4.5) \quad f(z + a) = \rho(T_a)f(z) \quad \forall a \in A,$$

is called a ρ -quasi-periodic function, or more simply, a quasi-periodic function. We say that f is *tempered* if there exists $M \in \mathbb{Z}$ such that

$$\lim_{|z|_{|\mathfrak{S}} \rightarrow \infty} f(z)u(z)^M = \underline{0}$$

where u is defined in (1.3). We further say that f is *regular* if there exists a constant $c > 0$ (depending on f) such that the set $\{|f(z)|_B : |z|_{|\mathfrak{S}} \geq c\}$ is bounded (remember that $|\cdot|_{|\mathfrak{S}}$ has been introduced in Remark 3.28).

(b) Let

$$f : \Omega \rightarrow B^{N \times N}$$

be an analytic matrix function such that its columns are ρ -quasi-periodic in the sense of the point (a) above, so that

$$f(z + a) = \rho(T_a)f(z) \quad \forall a \in A.$$

We say that f is of *type* $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ if for all $\nu \in \mathbb{F}_q^\times$, we have

$$f(\nu z) = \nu^{-l} \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} f(z) \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

(c) We denote by $\mathcal{QP}_l^!(\rho; B)$ the B -module of tempered ρ -quasi-periodic functions

$$\Omega \rightarrow B^{N \times N}$$

of type l , and by $\mathcal{QP}_l(\rho; B)$ the sub-module of quasi-periodic regular functions.

If $n = 1$ and $\rho = \mathbf{1}$ (with $\mathbf{1}$ the trivial map which sends every element of Γ to $1 \in \mathbb{F}_q^\times$), then a quasi-periodic function is a holomorphic function $f : \Omega \rightarrow B$ such that $f(z + a) = f(z)$ for all $a \in A$. Explicit examples are $e_C(z) = \exp_C(\tilde{\pi}z)$ and

$$u(z) = \frac{1}{\tilde{\pi}} \sum_{a \in A} \frac{1}{z - a} = \frac{1}{e_C(z)}.$$

Both functions are obviously tempered. The function $e_C(z)$ is of type -1 and the function $u(z)$ is regular, of type 1 . We record the next Proposition.

Proposition 4.2. *Let $f : \Omega \rightarrow B$ be rigid analytic, such that $f(z+a) = f(z)$ for all $a \in A$. Then, the following properties hold:*

(a) *There is a unique series expansion*

$$(4.6) \quad f = \sum_{n \in \mathbb{Z}} f_n u(z)^n, \quad f_n \in B,$$

convergent if $z \in \Omega$ is such that $|z|_{\mathfrak{S}} > c$ for some $c \in |\mathbb{C}_{\infty}^{\times}|$.

(b) *If $|f(z)|_B$ is bounded for $|u(z)| < c$ for some $c \in |\mathbb{C}_{\infty}^{\times}|$, then $f_n = 0$ for all $n < 0$.*

(c) *If f extends to an entire function over \mathbb{C}_{∞} , and there exists $M \in \mathbb{Z}$ such that*

$$|u(z)^M f(z)|_B \rightarrow 0$$

as $|u(z)| \rightarrow 0$, then $f \in B[u(z)^{-1}]$.

Sketch of proof. This result is basically well known but there is a lack of complete reference in the literature. Let us give some details.

(a) The proof of [60, Proposition 6.1] can be adapted to our setting. We recall from *ibid.* that for an integer n we define

$$\mathcal{B}_n = D_{\mathbb{C}_{\infty}}^{\circ}(0, |\theta|^n) \setminus \bigcup_{a \in A(n)} D_{\mathbb{C}_{\infty}}^{\circ}(a, 1), \quad \mathcal{C}_n = D_{\mathbb{C}_{\infty}}^{\circ}(0, |l_n|) \setminus D_{\mathbb{C}_{\infty}}^{\circ}(0, 1),$$

(recall that $D_{\mathbb{C}_{\infty}}^{\circ}(a, r)$ is the 'open' disk of center a and radius r in \mathbb{C}_{∞}) which are filtered unions of affinoid subsets of \mathbb{C}_{∞} ($A(n)$ denotes the \mathbb{F}_q -vector space of all the elements of A which are of degree $< n$ in θ). One can verify that, for all n ,

$$\mathcal{O}_{\mathcal{C}_n/B}(\mathcal{C}_n) = \left\{ \sum_{k \in \mathbb{Z}} f_k u^k : f_k \in B \text{ for all } k, f_{-k} \rightarrow 0 \text{ as } k \rightarrow \infty, f_k l_n^{(1-\epsilon)k} \rightarrow 0 \right. \\ \left. \text{as } k \rightarrow \infty, \text{ for all } \epsilon > 0 \right\}.$$

This follows from the explicit use of an orthonormal basis of $\mathcal{O}_{\mathcal{C}_n}(\mathcal{C}_n)$ and yields an explicit description of the sheaf $\mathcal{O}_{\mathcal{C}_n/B}$. Similarly, the sub-sheaf of $\mathcal{O}_{\mathcal{B}_n/B}$ whose global sections g are such that $g(z+a) = g(z)$ for all $a \in A(n)$ equals the pull back $\mathcal{E}_n^* \mathcal{O}_{\mathcal{C}_n/B}$ where $\mathcal{E}_n(z) = l_n E_n(z)$, E_n being the n -th Carlitz polynomial (see [60, §4.2]). This follows from an application of Proposition 6.2 of *ibid.* After these observations, the proof of Proposition 6.1 can be slightly modified to yield the existence of the expansion (4.6). Uniqueness follows easily from the connectedness of Ω .

Before considering the point (b) of our proposition, we define, after (4.6):

$$F(u) := \sum_{n \in \mathbb{Z}} f_n u^n, \quad F^-(u) := \sum_{n \leq 0} f_n u^n, \quad F^+(u) := \sum_{n > 0} f_n u^n.$$

We have that F converges for all

$$u \in \dot{D}_{\mathbb{C}_{\infty}}(0, c) := \{z \in \mathbb{C}_{\infty} : 0 < |z| \leq c\}$$

where $c \in |\mathbb{C}_\infty^\times|$, $c < 1$ and $f(z) = F(u(z))$. The series $F^-(u)$ converges for all $0 < |u| \leq c$ and $c < 1$. In other words, $|f_k|_B c^{-k} \rightarrow 0$, which implies that the sequence f_{-k} tends to zero as $k \rightarrow \infty$. In particular, $F^-(u)$ converges for every $u \neq 0$.

(b) Applying (a), $\lim_{u \rightarrow 0} F(u)$ exists and $|F(u)|_B$ is bounded on $\dot{D}_{\mathbb{C}_\infty}(0, c)$. We write $f_n = \sum_{i \in I} f_{n,i} b_i$ with $f_{n,i} \in \mathbb{C}_\infty$, where $(b_i)_{i \in I}$ is an orthonormal basis of B . We note that

$$|f_{n,i}| \max\{r_1^n, r_2^n\} \rightarrow 0 \text{ as } i \rightarrow \infty,$$

for all $r_1, r_2 \in |\mathbb{C}_\infty^\times|$ such that $r_1 < r_2 \leq c$. Therefore we have unconditional series convergence with u in $\dot{D}_{\mathbb{C}_\infty}(0, c)$ for an appropriate choice of c and we can write:

$$F(u) = \sum_{n \in \mathbb{Z}} \left(\sum_{i \in I} f_{n,i} b_i \right) u^n = \sum_{i \in I} \left(\sum_{n \in \mathbb{Z}} f_{n,i} u^n \right) b_i.$$

We get that for all $i \in I$, the limit for $u \rightarrow 0$ of $\sum_n f_{n,i} u^n$ exists. By [7, §3, Theorem (Riemann I)], $f_{n,i} = 0$ for all i , $n < 0$ and $F - F^+ \in B$.

(c) Let $f : \mathbb{C}_\infty \rightarrow B$ be entire, such that $f(z+a) = f(z)$ for all $a \in A$. Then, by (a) of this proposition,

$$f(z) = F(u) = \sum_{k \in \mathbb{Z}} f_k u^k, \text{ with } f_k \in B, \quad \forall k \in \mathbb{Z}.$$

By the above remarks, setting $f^-(z) = F^-(u(z))$ if $z \notin A$ and 0 otherwise, f^- defines a B -entire function. hence $f^+(z) = F^+(u(z)) = f(z) - f^-(z)$ is B -entire and at once, bounded at infinity. By Proposition 2.11, it is constant, hence identically zero; We conclude that $f(z) = f^-(z) = F^-(u(z))$. Now, assume that there exists M such that $|u^M F^-(z)|_B$ is bounded in B as $u \rightarrow 0$ (i.e. as $|z|_{\mathfrak{G}} = |z| \rightarrow \infty$). Then, by (b), we have that $G := u^M F^-$ is such that $G = G^+$ (in the above notations). This suffices to conclude. \square

Proposition 4.2 implies that $\mathcal{QP}_i^!(\mathbf{1}; B)$ can be embedded in $B[[u]][u^{-1}]^{N \times N}$ and for all $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ and a representation ρ as in (4.4), $\mathcal{QP}_i^!(\rho; B)$ is a module over $\mathcal{QP}_0^!(\mathbf{1}; B)^{N \times N}$, and a similar property holds for the regular quasi-periodic functions. Of course, we can specify the target space; the meaning of $\mathcal{QP}_i^!(\rho; \mathbb{L}_\Sigma)$ etc. is therefore understood.

4.1.1. *The series $\Psi_m(\rho)$.* There are three types of quasi-periodic matrix functions that are needed in the present work. They are denoted by $\Psi_m(\rho)$, Ξ_ρ and Φ_ρ . Here we study the first type $\Psi_m(\rho)$. We consider a representation $\rho : \Gamma \rightarrow \text{GL}_N(B)$. We additionally suppose that:

$$(4.7) \quad |a^{-1} \rho(T_a)|_B \rightarrow 0, \quad \text{as } |a| \rightarrow \infty \text{ with } a \in A.$$

Lemma 4.3. *Let m be a positive integer. The function $\Psi_m(\rho)$ defined, for all $z \in \mathbb{C}_\infty \setminus A$, by*

$$\Psi_m(\rho)(z) = \sum_{a \in A} (z - a)^{-m} \rho(T_a),$$

determines a non-zero element of $\mathcal{QP}_m(\rho; B)$.

Proof. It is easy to show that $\Psi_m(\rho)$ converges uniformly for $z \in \mathbb{C}_\infty \setminus (\sqcup_{a \in A} D(a, r))$ with $r \in |\mathbb{C}_\infty^\times|$, $0 < r < 1$. This implies that $\Psi_m(\rho)$ defines a holomorphic function $\Omega \rightarrow B^{N \times N}$, and this function is non-zero because it has, in any disk $D_{\mathbb{C}_\infty}(0, r)$ with $r \in |\mathbb{C}_\infty^\times|$, a meromorphic extension which has poles of order m at every $a \in D(0, r) \cap A$. Moreover, we have, for all $z \in \mathbb{C}_\infty \setminus A$ and $b \in A$, writing Ψ for $\Psi_m(\rho)$:

$$\begin{aligned} \Psi(z - b) &= \sum_{a \in A} (z - a - b)^{-m} \rho(T_a) \\ &= \sum_{a \in A} (z - a - b)^{-m} \rho(T_{a+b}) \rho(T_{-b}) \\ &= \Psi(z) \rho(T_{-b}) = \rho(T_{-b}) \Psi(z). \end{aligned}$$

so that

$$(4.8) \quad \Psi(z + a) = \Psi(z) \rho(T_a) = \rho(T_a) \Psi(z), \quad \forall a \in A.$$

Since

$$T_a = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} T_{\lambda^{-1}a} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall a \in A, \quad \lambda \in \mathbb{F}_q^\times,$$

for all $\lambda \in \mathbb{F}_q^\times$:

$$\begin{aligned} \Psi(\lambda z) &= \sum_{a \in A} (\lambda z - a)^{-m} \rho(T_a) \\ &= \lambda^{-m} \rho \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \Psi(z) \rho \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and the type is m . It remains to show that Ψ is regular. We need to show that there exists $c > 0$ such that if $z \in \mathbb{C}_\infty$ satisfies $|z|_{\mathfrak{S}} \geq c$, then $\|\Psi(z)\| \leq M$ for some M independent on z . But note that if $|z|_{\mathfrak{S}} \geq c > 0$ then $|z - a| \geq c$ for all $a \in A$ and therefore, $|z - a|^{-m} \leq c^{-m}$. Hence $\|\Psi(z)\| \leq c^{-m}$ (because $\|\rho(T_a)\| \leq 1$ for all a , due to the fact that the representation is of the first kind). Hence $\Psi \in \mathcal{QP}_m(\rho; B)$. \square

4.2. Representations of the first kind. We now introduce a class of representations of Γ for which we can construct explicitly entire non-zero quasi-periodic functions in several ways, and these functions turn out to have tame series expansion, or at least, expansion in the field of uniformizers. First of all, we introduce a useful definition.

Definition 4.4. We say that a representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ is of *degree* $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ if for all $\mu \in \mathbb{F}_q^\times$, $\rho(\mu I_2) = \mu^{-l} I_N$. We write $l = \deg(\rho)$.

We recall that after (1.7), $J_\gamma(z)^w \rho(\gamma)$ is a factor of automorphy for $\tilde{\Gamma}$ if and only if ρ is of degree w . For example, \det^{-m} is of degree $2m$ (the double of the type). The identity map over Γ is of degree -1 . All the representations that we consider in this text have a well defined degree.

Definition 4.5. Let $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ be an injective \mathbb{F}_q -algebra morphism, let $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ be such that $d\chi(\theta) \in \mathbb{F}_q[\underline{t}_\Sigma]^{n \times n}$. Then the map

$$\rho_\chi : \Gamma \rightarrow \mathrm{GL}_{2n}(\mathbb{F}_q[\underline{t}_\Sigma][d^{-1}]) \subset \mathrm{GL}_{2n}(\mathbb{F}_q(\underline{t}_\Sigma))$$

defined, with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, by

$$\rho_\chi(\gamma) := \begin{pmatrix} \chi(a) & \chi(b) \\ \chi(c) & \chi(d) \end{pmatrix},$$

is a representation of degree -1 , called the *basic representation* associated to χ . Note also that

$$\det(\rho_\chi(\gamma)) = \det(\chi(ad - bc)) = \det(\gamma)^n.$$

If ρ is a representation, we write

$$\rho^* := {}^t \rho^{-1}$$

for its contragredient (also called dual) representation. If ρ is of degree l , ρ^* is of degree $-l$. Let $\rho : \Gamma \rightarrow \mathrm{GL}_N(\underline{t}_\Sigma)$ be a representation. We say that ρ is a *representation of the first kind* if ρ can be obtained from basic representations by finitely many iterated applications of the following *elementary operations*: $(\cdot)^*$, direct sums \oplus , Kronecker products \otimes , symmetric powers S^m , exterior powers \wedge^m , in such a way that ρ has a well defined degree. For further use, we will call these operations *admissible operations*.

Note that if ρ and ψ are two representations such that ρ has degree l and ψ has degree m , then:

$$\begin{array}{ll} \rho \oplus \psi & \text{has degree } l \text{ (if } l = m) \\ \rho \otimes \psi & l + m, \\ S^r(\rho) & rl, \\ \wedge^r \rho & rl, \\ \rho^* & -l, \end{array}$$

where in the right, $(\cdot)^*$, \oplus , \otimes , S^r and \wedge^r denote respectively the contragredient, direct sum, Kronecker product, r -th symmetric power and the r -th exterior power, of representations.

To define a representation of the first kind, in view of the compatibility conditions (4.1) and (4.2), all we need is: finitely many injective \mathbb{F}_q -algebra maps

$$(4.9) \quad \chi_i : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n_i}, \quad i = 1, \dots, r$$

and a sequence of admissible operations. It is therefore obvious that the set of representations of the first kind is countable. On another side, for any $N > 1$, it is easy to see that the set of equivalence classes of representations of Γ in $\mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ is uncountable. This can be deduced from (4.1) and (4.2) and the fact that $A = \mathbb{F}_q[\theta]$ obviously finitely generated as an \mathbb{F}_q -algebra, is an infinitely dimensional vector space over \mathbb{F}_q . Therefore, there are representations of Γ on $\mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ which are not of the first kind. To see explicit examples, see Remark 4.17.

4.2.1. *The functions Ξ_ρ .* For any representation of the first kind ρ , we can canonically associate a quasi-periodic function Ξ_ρ with entries in the field of uniformizers \mathfrak{K} . This allows to show that, for $L \subset \mathbb{K}_\Sigma$ a field extension of \mathbb{C}_∞ , the $\mathbb{K}_\Sigma((u))^{N \times N}$ -module $\mathcal{QP}_m^1(\rho; \mathbb{K}_\Sigma)$ is free of rank one. Additionally, Ξ_ρ has entries in $(\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^\circ \langle \langle \underline{e} \rangle \rangle^b$. Let us first assume that $\rho = \rho_\chi$ is a basic representation. We denote by χ the function $\tilde{\chi}$ of Proposition 2.27. We know from Proposition 3.36 that it belongs to $\mathfrak{K}^{n \times n}$.

By using Proposition 2.27 and the identity $\chi(z+a) = \chi(z) + \chi(a)$ for $z \in \mathbb{C}_\infty$ and $a \in A$, we see that the function

$$(4.10) \quad \Xi_\rho(z) := \begin{pmatrix} I_n & \chi(z) \\ 0 & I_n \end{pmatrix},$$

belongs to $\mathcal{QP}_0^1(\rho; \mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)$ (with $d\chi(\theta) \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$). In fact, we have more. Indeed, since $\chi(z+a) = \chi(z) + \chi(a) = \chi(z+a) = \chi(a) + \chi(z)$, we have $\Xi_\rho(z) = \rho(T_a)\Xi_\rho(z) = \Xi_\rho(z)\rho(T_a)$ for all $a \in A$.

If now ρ is a representation of the first kind, by definition it can be constructed from basic representations ρ_1, \dots, ρ_m by finitely many iterated applications of direct sums, Kronecker products, exterior and symmetric powers, contragredient, and following the same process, we can combine the functions $\Xi_{\rho_1}, \dots, \Xi_{\rho_m}$ to construct a quasi-periodic matrix function $\Xi_\rho \in \mathcal{QP}_0^1(\rho; \widehat{\mathbb{E}_\Sigma[\frac{1}{d}]}) \cap \mathfrak{K}^{N \times N}$ for some d . More precisely, we set, for ρ, ψ two representations of the first kind:

$$(4.11) \quad \begin{aligned} \Xi_{\rho \oplus \psi} &= \Xi_\rho \oplus \Xi_\psi, \\ \Xi_{\rho \otimes \psi} &= \Xi_\rho \otimes \Xi_\psi, \\ \Xi_{S^r(\rho)} &= S^r(\Xi_\rho), \\ \Xi_{\wedge^r \rho} &= \wedge^r \Xi_\rho, \\ \Xi_{\rho^*} &= (\Xi_\rho)^*. \end{aligned}$$

We thus get:

$$(4.12) \quad \Xi_\rho(z+a) = \rho(T_a)\Xi_\rho(z) = \Xi_\rho(z)\rho(T_a), \quad a \in A.$$

To simplify our notations we write, in the following,

$$(4.13) \quad \mathfrak{E} := \mathbb{E}_\Sigma[d^{-1}]^\wedge,$$

where $\mathbb{E}_\Sigma[d^{-1}]^\wedge$ has been introduced in (2.14).

Proposition 4.6. *If ρ is a representation of the first kind then we have the following properties:*

- (1) $\Xi_\rho \in \mathcal{QP}_0^1(\rho; \mathfrak{E}) \cap (\mathbb{K}_\Sigma \langle \langle \underline{e} \rangle \rangle^b)^{N \times N}$,
- (2) $\Xi_\rho \in \text{GL}_N(\mathbb{K}_\Sigma \langle \langle \underline{e} \rangle \rangle^b)$ and $\Xi_\rho^p = I_N$ and
- (3) $\oplus_m \mathcal{QP}_m^1(\rho; \mathbb{K}_\Sigma) \subset \mathfrak{K}_\Sigma^{N \times N}$ is both a left and a right $\mathbb{K}_\Sigma((u))^{N \times N}$ -module, free of rank one.

Proof. The fact that Ξ_ρ is quasi-periodic is clear from (4.12). Moreover, it is easy to see that Ξ_ρ is of type 0. It suffices to check this for basic representations. For this

note that, for $\nu \in \mathbb{F}_q^\times$, and for any \mathbb{F}_q -algebra morphism $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)$, $\begin{pmatrix} I_n & \chi(\nu z) \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & \nu \chi(z) \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \nu I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & \chi(z) \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \nu^{-1} I_n & 0 \\ 0 & I_n \end{pmatrix}$. But since $\rho = \rho_\chi$, we have $\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a I_n & b I_n \\ c I_n & d I_n \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$, and therefore,

$$(4.14) \quad \Xi_\rho(\nu z) = \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \Xi_\rho(z) \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Additionally, since the entries of the function χ are tame series in virtue of Proposition 3.36, Ξ_ρ is tempered thanks to Proposition 3.27. Now, note that $\det(\Xi_\rho) = 1$ due to the fact that this equality holds true for ρ a basic representation. Hence $\Xi_\rho^{-1} \in (\mathbb{K}_\Sigma \langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$ which confirms (1). For (2), note that $\Xi_\rho \in \mathrm{GL}_N(\mathbb{K}_\Sigma \langle\langle \underline{e} \rangle\rangle^b)$ (with determinant one) and $\Xi_\rho^p = I_N$ for ρ a basic representation, just by construction. The general case follows easily. Finally for (3), note that by (4.12), for all $a \in A$,

$$\Xi_\rho(z+a)^{-1} = \Xi_\rho(z)^{-1} \rho(T_a)^{-1} = \rho(T_a)^{-1} \Xi_\rho(z)^{-1}.$$

Let Φ be an element of $\mathcal{QP}_m^1(\rho; \mathbb{K}_\Sigma)$ for some m . Then $U_1 := \Xi_\rho^{-1} \Phi$ and $U_2 := \Phi \Xi_\rho^{-1}$ are both A -periodic and tempered. By Proposition 4.2 we see that U_1, U_2 belong to $\mathbb{K}_\Sigma((u))^{N \times N}$. Hence $\Phi = \Xi_\rho U_1 = U_2 \Xi_\rho \in \mathfrak{K}_\Sigma^{N \times N}$. A simple computation indicates that U_1, U_2 are both of type m . \square

Along with (4.11) we also define, with $\rho_\chi : \Gamma \rightarrow \mathrm{GL}_{2n}(\mathbb{F}_q(\underline{t}_\Sigma))$ a basic representation and ω_χ as in (2.4.2):

$$(4.15) \quad \omega_{\rho_\chi} = \begin{pmatrix} \omega_\chi & 0_n \\ 0_n & I_n \end{pmatrix}$$

$$(4.16) \quad \begin{aligned} \omega_{\rho \oplus \psi} &= \omega_\rho \oplus \omega_\psi, \\ \omega_{\rho \otimes \psi} &= \omega_\rho \otimes \omega_\psi, \\ \omega_{S^r(\rho)} &= S^r(\omega_\rho), \\ \omega_{\wedge^r \rho} &= \wedge^r \omega_\rho, \\ \omega_{\rho^*} &= (\omega_\rho)^*. \end{aligned}$$

This allows to associate, in a unique way, to every representation of the first kind ρ of dimension N , an element $\omega_\rho \in \mathrm{GL}_N(\mathbb{L}_\Sigma)$. We have:

Lemma 4.7. *If ρ is a representation of the first kind there exist $\vartheta_1, \dots, \vartheta_r \in \mathbb{F}_q(\underline{t}_\Sigma)$ such that $\omega_\rho \Xi_\rho \omega_\rho^{-1} \in (A[\vartheta_1, \dots, \vartheta_r] \langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$.*

Proof. This follows, with $\rho = \rho_\chi$ basic, from $\begin{pmatrix} \omega_\chi & 0_n \\ 0_n & I_n \end{pmatrix} \begin{pmatrix} I_n & \chi(z) \\ 0_n & I_n \end{pmatrix} \begin{pmatrix} \omega_\chi^{-1} & 0_n \\ 0_n & I_n \end{pmatrix} = \begin{pmatrix} I_n & \omega_\chi \chi(z) \\ 0_n & I_n \end{pmatrix}$, and the property that, with $\vartheta = \chi(\theta)$, $\omega_\chi \chi(z) = \exp_C \left(\tilde{\pi} z (\theta I_n - \vartheta)^{-1} \right) = \sum_{i \geq 0} \vartheta^i e_{i+1} \in (\mathbb{F}_q[\vartheta] \langle\langle \underline{e} \rangle\rangle^b)^{n \times n}$ and (2.15). \square

4.2.2. *The functions Φ_ρ .* The third important class of matrix-valued functions is the following one, that we are going to study now:

$$(4.17) \quad \Phi_\rho = e_A \Psi_1(\rho),$$

where we recall that $\Psi_1(\rho) = \sum_{a \in A} (z - a)^{-1} \rho(T_a)$, depending on the choice of a representation of the first kind ρ . For the next proposition, we recall the notation (4.13).

Proposition 4.8. *The following properties hold:*

- (a) *The function Φ_ρ extends to an entire function $\mathbb{C}_\infty \rightarrow \mathfrak{E}^{N \times N}$.*
- (b) *We have that $\Phi_\rho \in \mathcal{QP}_0^1(\rho; \mathfrak{E})$.*
- (c) *There exist, uniquely determined depending on ρ , two matrices*

$$U_1, U_2 \in (\mathfrak{E}[e_C(z)])^{N \times N}$$

of type 0 such that

$$U_i - I_N \in e_C(z) (\mathfrak{E}[e_C(z)])^{N \times N}, \quad i = 1, 2$$

are p -nilpotent, and such that

$$\Phi_\rho = U_1 \Xi_\rho = \Xi_\rho U_2.$$

- (d) *We have $\Phi_\rho \in (\mathfrak{E}^\circ \langle \langle \underline{e} \rangle \rangle^b)^{N \times N}$ and this is the unique element f of $(\mathfrak{E}^\circ \langle \langle \underline{e} \rangle \rangle^b)^{N \times N}$ such that $f(a) = \rho(T_a)$ for all $a \in A$.*

Note that if $\rho = \mathbf{1}$ is the trivial representation, with $N = 1$, then we have $\Xi_\rho = 1$ and $\Phi_\rho = 1$, because $\Psi_1(\rho) = \sum_{b \in A} \frac{1}{z-b}$.

Proof of Proposition 4.8. (a). In any disk $D(0, r)$ with $r \in |\mathbb{C}_\infty^\times|$, the product

$$e_A(z) \Psi_1(\rho)(z)$$

extends to a holomorphic matrix-valued function because of the Weierstrass factorization

$$e_A(z) = z \prod'_{a \in A} \left(1 - \frac{z}{a}\right)$$

(dash indicating the omission of $a = 0$). This immediately implies that Φ_ρ has entire entries, and the target space is easily determined.

(b). Since ρ is a representation of the first kind, Ξ_ρ can be constructed applying finitely many operations as in (4.11) to finitely many functions Ξ_{ρ_i} associated to basic representations ρ_i , which take the elements T_a with $a \in A$ to unipotent matrices (in fact, upper triangular with one on the diagonals). Therefore Ξ_ρ^{-1} defines an entire function $\mathbb{C}_\infty \rightarrow \mathfrak{E}^{N \times N}$. Hence, $U_1(z) := \Phi_\rho(z) \Xi_\rho(z)^{-1}$ has entries which are holomorphic $\Omega \rightarrow \mathfrak{E}^{N \times N}$, and $U_1(z + a) = U_1(z)$ for all $a \in A$, by (4.8). Moreover, since Ξ_ρ is tempered and $\|\Psi_1(\rho)(z)\|$ tends to zero as $|z|_{\mathfrak{S}} = |z| \rightarrow \infty$, there exists $L \in \mathbb{Z}$ such that $u(z)^L U_1(z) \rightarrow \underline{0}$ as $|z|_{\mathfrak{S}} \rightarrow \infty$. By (b) of Proposition 4.2, U_1 can be identified with an element of $\mathfrak{E}[[u]][u^{-1}]^{N \times N}$ and we easily check that $\Phi_\rho \in \mathcal{QP}_0^1(\rho; \mathfrak{E})$.

(c). By (2) of Proposition 4.6 we see that $\Xi_\rho \in \mathrm{GL}_N(\mathfrak{E} \langle \langle \underline{e} \rangle \rangle^b)$ therefore by the arguments of the point (b), we additionally observe that $U_1 = \Phi_\rho(z) \Xi_\rho^{-1} \in \mathfrak{E}[[u]][u^{-1}]^{n \times n}$ extends to an

entire matrix function which, in virtue of (c) of Proposition 4.2, belongs to $\mathfrak{E}[e_C(z)]^{N \times N}$. Note that for all $a \in A \setminus \{0\}$, $U_1(a) = \Phi_\rho(a)\Xi_\rho(a)^{-1} = \Phi_\rho(a)\rho(T_{-a})$. Now, $\Phi_\rho(a) = \lim_{z \rightarrow a} e_A(z)\Psi_1(\rho)(z) = \lim_{z \rightarrow a} e_A(z)(z - a)^{-1}\rho(T_a) = \rho(T_a)$ because $e'_A = 1$. Hence, $U_1(a) = I_N$ and the various properties claimed for U_1 follow. Similar arguments hold for U_2 .

(d). From (c) above, $\Phi_\rho \in (\mathfrak{E}\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$. We denote by $w \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ the supremum of the weights of all the entries of Φ_ρ . Then $\Psi_1(\rho) \in u(\mathfrak{E}\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$ and since we have the obvious limit $\lim_{|z|_{\mathfrak{S}}=|z| \rightarrow \infty} \|\Psi_1(\rho)(z)\| = 0$ we note that $w < 1$ so that $\Phi_\rho \in (\mathfrak{E}^\circ\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$. An element $f \in \mathbb{K}_\Sigma^\circ\langle\langle \underline{e} \rangle\rangle^b$ satisfies $\|u(z)f(z)\| \rightarrow 0$ as $|z|_{\mathfrak{S}} = |z| \rightarrow \infty$. By Proposition 2.11, for any map $g : A \rightarrow \mathbb{K}_\Sigma$ there exists at most one element $f \in \mathfrak{E}^\circ\langle\langle \underline{e} \rangle\rangle^b$ such that $f(a) = g(a)$ for all $a \in A$. Consequently, if f is an element of $(\mathfrak{E}^\circ\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$ such that $f(a) = \rho(T_a)$ for all $a \in A$, then, $\Phi_\rho = f$. \square

We have the next corollary, where ρ is a representation of the first kind.

Corollary 4.9. *The tame series expansion of Φ_ρ is provided by the unique representative in the \mathfrak{E} -module $(\mathfrak{E}^\circ\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$ of the matrix Ξ_ρ in the quotient of $(\mathfrak{E}\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$ by the principal ideal generated by $e_0 I_N$. Moreover, we have $\det(\Phi_\rho) = 1$, $\Phi_\rho - I_N$ is p -nilpotent and $\Phi_\rho^{-1} \in (\mathfrak{E}^\circ\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$. If ω_ρ is the matrix introduced in (4.16), then $\omega_\rho \Phi_\rho \omega_\rho^{-1} \in (A[\vartheta_1, \dots, \vartheta_r]^\circ\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$ for elements $\vartheta_1, \dots, \vartheta_r \in \mathbb{F}_q(\underline{t}_\Sigma)$.*

Proof. The first property follows directly from Proposition 4.8 (c), (d). To show the second property we first note that the matrices $\rho(T_a)$, $a \in A$, can be simultaneously (upper) triangularised over an algebraic closure $\mathbb{F}_q(\underline{t}_\Sigma)^{ac}$ of $\mathbb{F}_q(\underline{t}_\Sigma)$, and the diagonal entries are all equal to one because $T_a^p = I_2$ for all a . Hence, $\Psi_1(\rho)$ is conjugated over $\mathbb{F}_q(\underline{t}_\Sigma)^{ac}$ to an upper triangular matrix having $e_A(z)^{-1}$ as diagonal entries. This implies that Φ_ρ is conjugated over $\mathbb{F}_q(\underline{t}_\Sigma)^{ac}$ to an upper triangular matrix having 1 in the diagonal. Hence, $\det(\Phi_\rho) = 1$, $(\Phi_\rho - I_N)^p = 0$ and $\Phi_\rho^{-1} \in (\mathfrak{E}^\circ\langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$. The last property follows easily from Lemma 4.7. \square

4.2.3. Explicit example. In this part we illustrate an explicit example that governs the quasi-periodic functions associated to basic representations. This covers the representations considered in [60, §9]. Let $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ be an \mathbb{F}_q -algebra morphism and denote by ρ the basic representation $\rho_\chi : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ defined by $\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \chi(a) & \chi(b) \\ \chi(c) & \chi(d) \end{pmatrix}$, with $N = 2n$. For a matrix $f \in \mathfrak{K}_\Sigma^{N \times N}$, $v(f)$ denotes the infimum of the v -valuations of the entries of f (where v is the valuation defined after Proposition 3.32). We have:

Corollary 4.10. *We have $\Phi_\rho = \Xi_\rho$, $v(\Phi_\rho) = -\frac{1}{q}$ and $v(\Phi_\rho - \omega_\chi^{-1} e_1) > -\frac{1}{q}$.*

Proof. By definition, $\Xi_\rho = \begin{pmatrix} I_n & \chi \\ 0 & I_n \end{pmatrix}$ and $\chi(z) = e_C(z(\theta I_n - \vartheta)^{-1})\omega_\chi^{-1}$ (with $\vartheta = \chi(\theta)$) has entries in $\mathbb{K}_\Sigma^\circ\langle\langle \underline{e} \rangle\rangle^b$ so we have already $\Phi_\rho = \Xi_\rho$ by Corollary 4.9. Moreover, the tame series expansion of $e_C(z(\theta I_n - \vartheta)^{-1})$ is $e_C(z(\theta I_n - \vartheta)^{-1}) = e_1 I_n + \text{terms of smaller weight}$, which implies the remaining properties. \square

Note that the proof of the above corollary does not use the injectivity of the maps χ in Definition 4.5.

4.2.4. *Application to column quasi-periodic functions.* We consider, in this subsection, a representation of the first kind $\Gamma \xrightarrow{\rho} \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$. Recall the notation $\mathfrak{K}_\Sigma = \mathfrak{K}_{\mathbb{K}_\Sigma}$ where, for a subfield L of \mathbb{K}_Σ , \mathfrak{K}_L has been defined after Proposition 3.32. We recall that the v -valuation ring is denoted by \mathfrak{D}_Σ , the maximal ideal is denoted by \mathfrak{M}_Σ .

Proposition 4.11. *If $f : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$ is ρ -quasi-periodic and tempered, we can identify it with an element of $\mathfrak{K}_\Sigma^{N \times 1}$. If additionally f is regular, then we can identify it with an element of $\mathfrak{D}_\Sigma^{N \times 1}$. In the latter case, we can expand in a unique way*

$$(4.18) \quad f = f_0 + \sum_{i>0} f_i u^i, \quad f_0 \in \mathbb{K}_\Sigma^{N \times 1}, \quad f_i \in (\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b)^{N \times 1}, \quad i > 0,$$

and the coefficients f_i are \mathbb{K}_Σ -linear combinations of the columns of Φ_ρ . In equivalent terms, we can expand, in unique way,

$$f = \Phi_\rho \sum_{i \geq 0} f'_i u^i, \quad f'_i \in \mathbb{K}_\Sigma^{N \times 1}.$$

Proof. The second type of expansion is clearly equivalent to the first. In the proof of part (c) of Proposition 4.8, we have seen that Φ_ρ can be identified with an element of $\mathrm{GL}_N(\mathbb{K}_\Sigma \langle\langle \underline{e} \rangle\rangle^b)$. Hence, the function $\Phi_\rho^{-1} f : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$ has entries which are all A -periodic and tempered. By part (b) of Proposition 4.2, the entries are thus elements of $\mathbb{K}_\Sigma((e_0^{-1})) = \mathbb{K}_\Sigma((u))$ and the entries of $f = \Phi_\rho \Phi_\rho^{-1} f$ are therefore in $\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b((e_0^{-1}))$ which is equal, by Proposition 3.32, to \mathfrak{K}_Σ . This proves the first part of the proposition.

Since Φ_ρ is a matrix function which is quasi-periodic we have $f = \Phi_\rho g$ where $g \in \mathbb{K}_\Sigma((u))^{N \times 1}$. Corollary 4.9 implies that $\Phi_\rho \in \mathrm{GL}_N(\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b)$. Namely, $\det(\Phi_\rho) = 1$ and $\Phi_\rho^{-1} \in (\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b)^{N \times N}$. Observe that $g = \Phi_\rho^{-1} f$. Since the entries of Φ_ρ^{-1} are in $\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$, for $|z|_\mathfrak{S} \geq c_1$ for some constant $c_1 \in |\mathbb{C}_\infty^\times|$, we have $\|\Phi_\rho^{-1} f\| \leq c_2 |e_A(z)|^w$ by Proposition 3.27, where $w \in \mathbb{Z}[\frac{1}{p}] \cap [0, 1[$, for some $c_2 > 0$. This means that $\|u^w g\| \leq c_2$ as $|z|_\mathfrak{S}$ is large. Let $\alpha > 0$ be such that $p^\alpha w \in \mathbb{Z}$. Then $\|u^{p^\alpha w} g^{p^\alpha}\|$ is bounded at infinity and $u^{p^\alpha w} g^{p^\alpha} \in \mathbb{K}_\Sigma[[u]]^{N \times 1}$. Therefore, $u^w g \in \mathbb{K}_\Sigma[[u^{\frac{1}{p^\alpha}}]]^{N \times 1}$ by Proposition 4.8 (b) and we deduce that, necessarily, $g \in \mathbb{K}_\Sigma[[u]]^{N \times 1}$. Writing $g = \sum_{i \geq 0} g_i u^i$ with $g_i \in \mathbb{K}_\Sigma^{N \times 1}$, by the fact that $f = \Phi_\rho g \in \mathfrak{D}_\Sigma^{N \times 1}$ and since $f_i := \Phi_\rho g_i$ belongs to $(\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b)^{N \times 1}$ by Corollary 4.9, we get the expansion in $\mathfrak{K}_\Sigma^{N \times 1}$

$$f = \sum_{i \geq 0} f_i u^i,$$

from which we also see that $f_0 \in \mathbb{K}_\Sigma^{N \times 1}$. □

4.2.5. *First results on modular forms.* We recall Definition 1.1 and the spaces $S_w(\rho; B) \subset M_w(\rho; B) \subset M_w^1(\rho; B)$. We denote by $\mathcal{M}_w^1(\rho; B)$ the B -module of modular-like forms of weight w for ρ (without the temperedness condition). In the following we also use the term ‘modular-like’ sometimes loosely, to designate the spaces $\mathcal{M}_w^{\text{!!}}(\rho; \mathbb{K}_\Sigma)$ of meromorphic functions $\Omega \xrightarrow{f} \mathbb{K}_\Sigma^{N \times 1}$ (in the obvious sense) satisfying (1.8). Clearly $\mathcal{M}_w^{\text{!!}}(\rho; \mathbb{K}_\Sigma) \supset \mathcal{M}_w^1(\rho; \mathbb{K}_\Sigma)$.

Taking into account Definition 1.2, we deduce parts (1), (2), (3) of Theorem A in the introduction, where the hypothesis that ρ is of the first kind is essential:

Theorem 4.12. *For all $w \in \mathbb{Z}$, there is a natural embedding*

$$M_w^1(\rho; \mathbb{K}_\Sigma) \xrightarrow{\iota_\Sigma} \mathfrak{R}_\Sigma^{N \times 1}$$

such that

$$M_w(\rho; \mathbb{K}_\Sigma) = \iota_\Sigma^{-1} \left(\iota_\Sigma(M_w^1(\rho; \mathbb{K}_\Sigma)) \cap \mathfrak{D}_\Sigma^{N \times 1} \right)$$

and

$$S_w(\rho; \mathbb{K}_\Sigma) = \iota_\Sigma^{-1} \left(\iota_\Sigma(M_w^1(\rho; \mathbb{K}_\Sigma)) \cap \mathfrak{M}_\Sigma^{N \times 1} \right).$$

Proof. Since a weak modular form is also a tempered quasi-periodic (column) function and a modular form is a regular quasi-periodic function, the first part of the result follows directly from Proposition 4.11. To prove the two other parts of the statement, namely the characterisation of the image of $M_w(\rho; \mathbb{K}_\Sigma)$ and $S_w(\rho; \mathbb{K}_\Sigma)$, we combine Proposition 3.32 with Proposition 3.27, which allows to derive, from the fact that f is bounded at infinity (resp. has zero limit at infinity) that valuations of the entries of f are non-negative (resp. positive). \square

4.3. **Hecke operators.** We show here part (5) of Theorem A in the introduction. As an immediate consequence of the above investigations, we will now define Hecke operators acting on the spaces $M_w(\rho; \mathbb{K}_\Sigma)$, $M_w(\rho; \mathbb{L}_\Sigma)$, $S_w(\rho; \mathbb{K}_\Sigma)$ and $S_w(\rho; \mathbb{L}_\Sigma)$, with $w \in \mathbb{Z}$, when $\Gamma \xrightarrow{\rho} \text{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ is a representation of the first kind. Although not explicitly considered in the general purposes of it, Miyake’s book [48] essentially contains everything we need to set up the basis of the present discussion. Following [48, §2.7 and §4.5] we consider the Hecke algebra $\mathcal{R}_A(\Gamma, \Delta)$ where $\Delta = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cap A^{2 \times 2} \cap \text{GL}_2(K)$ is the semigroup generated by the elements of $G = \text{GL}_2(K)$ with entries in A and with the lower left coefficient equal to zero. Explicitly, $\mathcal{R}_A(\Gamma, \Delta)$ is the free A -module generated by the double cosets $\Gamma \delta \Gamma$ with δ in Δ , endowed with the structure of A -algebra induced by *ibid.* (2.7.2), after reduction modulo p of the integral coefficients. It is easy to see, using [48, Theorem 2.7.8], that $\mathcal{R}_A(\Gamma, \Delta)$ is commutative. For $a \in A$, we set $T(a) = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \Gamma \in \mathcal{R}_A(\Gamma, \Delta)$. The proof of *ibid.* Lemma 4.5.7 can be easily modified to show that, if $P \in A$ is irreducible, then

$$T(P)T(P^n) = T(P^{n+1}) + q^{\deg_\theta(P)} T(P, P)T(P^{n-1}), \quad n \geq 1,$$

where $T(P, P) = \Gamma \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \Gamma$ (compare with *ibid.* (4.5.15)). But K has characteristic $p \mid q$ so that $T(P)T(P^n) = T(P^{n+1})$. Similarly, the proof of Lemma 4.5.8 in Miyake’s book implies that if $a, b \in A$ are relatively prime, then $T(a)T(b) = T(ab)$ in $\mathcal{R}_A(\Gamma, \Delta)$. The map $A \mapsto T(a)$ is therefore totally multiplicative. Also, given any right action of Δ on a

B -module \mathcal{M} , $\mathcal{R}(\Gamma, \Delta)$ acts on $\mathcal{M}^\Gamma = \{m \in \mathcal{M} : m|\gamma = m, \forall \gamma \in \Gamma\}$, as described in [48, Lemma 2.7.2], where we denoted by $m|\gamma$ an action of γ on m .

We consider $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ a representation of the first kind. Then, ρ determines in a unique way to a semi-ring map

$$\Delta \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{N \times N}$$

(this even if we drop the injectivity of some \mathbb{F}_q -algebra map in Definition 4.5 and, to a representation $G = \mathrm{GL}_2(K) \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma)^{N \times N})$ if we have injectivity of the \mathbb{F}_q -algebra maps). There exists $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that $\rho(\Delta) \subset \mathbb{F}_q[\underline{t}_\Sigma][d^{-1}]^{N \times N}$. Let w be an integer and B a \mathbb{C}_∞ -algebra contained in \mathbb{K}_Σ such that it contains $\mathbb{T}_\Sigma[d^{-1}]^\wedge$. We set $\mathcal{M}_B := \mathrm{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow B^{N \times 1})$.

Let f be in \mathcal{M}_B . The *Petersson slash operator* $f|_{w, \rho} \gamma$ on f is defined, for any $\gamma \in \mathrm{GL}_2(K)$, by

$$(4.19) \quad (f|_{w, \rho} \gamma)(z) := J_\gamma(z)^{-w} \rho(\gamma)^{-1} f(\gamma(z)).$$

It is easily seen that this gives rise to an action of G over $\mathcal{M}_w(\rho; B)$, the B -module of the modular-like functions of weight w for ρ of Definition 1.2. For instance, the reader can easily check that

$$(f|_{w, \rho} \gamma)|_{w, \rho} \delta = f|_{w, \rho} \gamma \delta$$

for any $\gamma, \delta \in \mathrm{GL}_2(K)$. By the above discussion, we have a well defined $\mathcal{R}_A(\Gamma, \Delta)$ -module structure on $\mathcal{M}_w(\rho; B)$. If $\Gamma \delta \Gamma$ is a double coset in $\mathcal{R}_A(\Gamma, \Delta)$ we can expand in a finite sum $\Gamma \delta \Gamma = \sum_i \Gamma \delta_i$ with $\delta_i \in \Delta$ for all i as described in [48, Lemma 2.7.3] and the action is given by

$$(\Gamma \delta \Gamma, f) \mapsto \sum_i f|_{w, \rho} \delta_i.$$

We also denote by $T_a(f)$ the image of the action of $T(a)$ on f , with $a \in A$. Then, $T_a(T_b(f)) = T_{ab}(f)$ for all $a, b \in A$. For example, since for $P \in A$ irreducible,

$$T(P) = \Gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sqcup \bigsqcup_{\substack{b \in A \\ |b| < |P|}} \Gamma \begin{pmatrix} 1 & b \\ 0 & P \end{pmatrix}$$

(see very similar computations in [48, Lemma 4.5.6]), we have, for $f \in \mathcal{M}_w(\rho; B)$:

$$(4.20) \quad T_P(f)(z) = \rho \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} f(Pz) + P^{-w} \sum_{|b| < |P|} \rho \begin{pmatrix} 1 & b \\ 0 & P \end{pmatrix}^{-1} f\left(\frac{z+b}{P}\right), \quad z \in \Omega.$$

Comparing with [24, (7.1)] we have here a different normalisation for these operators. In the case of $\rho = \mathbf{1}$ so that $N = 1$, denoting by \mathcal{T}_P the weight w operator of *ibid.*, we have $T_P = P^{-w} \mathcal{T}_P$.

The following result holds:

Theorem 4.13. *Assuming that ρ is of the first kind, we have that for all $a \in A$ and $w \in \mathbb{Z}$, T_a defines a B -linear endomorphism of $M_w^1(\rho; B)$ which induces endomorphisms of $M_w(\rho; B)$ and $S_w(\rho; B)$.*

Proof. Thanks to the above observations it suffices to prove the result for $a = P$ irreducible. Theorem 4.12 implies that T_P operates, via the slash operator of weight w associated with ρ , on $\mathfrak{K}_\Sigma^{N \times 1}$ and furthermore, it leaves $\mathfrak{D}_\Sigma^{N \times 1}$ and $\mathfrak{M}_\Sigma^{N \times 1}$ invariant. \square

This generalizes [63, Proposition 5.12] (which deals with the very special case of $N = 2$ and $\rho = \rho_t^*$, with an ad hoc proof hard to generalize to our settings).

Example. Assume that $\rho = \rho_\Sigma^* = {}^t\rho_\Sigma^{-1}$ for a finite subset Σ of \mathbb{N}^* and consider $f = {}^t(f_1, \dots, f_N) \in M_w(\rho; B)$. Then the first entry $(T_P(f))_1$ in (4.20) is

$$(4.21) \quad (T_P(f))_1 = \sigma_\Sigma(P)f_1(Pz) + P^{-w} \sum_{|b| < |P|} f_1\left(\frac{z+b}{P}\right).$$

The last entry is slightly more involved. We have:

$$(4.22) \quad (T_P(f))_N = f_N(Pz) + P^{-w} \sum_{|b| < |P|} \left(\bigotimes_{i \in \Sigma} (\chi_{t_i}(b), \chi_{t_i}(P)) \right) \cdot f\left(\frac{z+b}{P}\right).$$

Note that the whole column vectors $f(\frac{z+b}{P})$ occur in the right-hand side.

4.4. Remarks on representations. We collect here miscellaneous remarks on the settings we choose for this work.

Remark 4.14. For basic representations ρ_1, \dots, ρ_k , any representation of the first kind

$$\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$$

constructed combining them with the admissible operations $\oplus, \otimes, \wedge^r, S^r$ extend to monoid homomorphisms $A^{2 \times 2} \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{N \times N}$. This is used in 4.3.

The operation $(\cdot)^*$ does not satisfy this property. However, the comatrix representation

$$\mathrm{Co}(\rho) := \det(\rho) \otimes \rho^*$$

also extends to a monoid homomorphism and is isomorphic to ρ .

Indeed, in general, if ρ is any representation of Γ , we have that $\det(\rho) = \det^{\deg(\rho)}$, so that

$$(4.23) \quad \rho \cong \det^{\deg(\rho)} \rho^* = \mathrm{Co},$$

(the symbol \cong indicates that the representations ρ and $\det^{\deg(\rho)} \rho^*$ are isomorphic). For all $\gamma \in \Gamma$, writing $\gamma^* := {}^t\gamma^{-1}$ and recalling that S is defined in (4.3),

$$\begin{aligned} \rho^*(\gamma) &= \rho(\gamma^*) \\ &= \rho\left(\det(\gamma)^{-1}(S\gamma S^{-1})\right) \\ &= \det(\gamma)^{\deg(\rho)} \rho(S)\rho(\gamma)\rho(S)^{-1}. \end{aligned}$$

The discussed property is false for GL_n with $n \geq 3$.

Note that, for *any* representation ρ of Γ , there are canonical isomorphisms of vector spaces or modules

$$W_w(\rho^*; B) \cong W_w(\det^{\deg(\rho)} \rho; B), \quad W \in \{S, M, M^!, \mathcal{M}^!, \mathcal{M}^{!!}\}.$$

The isomorphism is $f \mapsto \rho(S)^{-1}f$ in virtue of (4.23).

Remark 4.15. We discuss the choices we made in Definition 4.5. Firstly, we could have decided to set, for representations of the first kind, our target spaces to be vector spaces over $\mathbb{F}_q^{ac}(t_\Sigma)$, but we noticed that this is an unnecessary complication, at least at the stage of the present work, as the most interesting examples are related to representations in $\mathrm{GL}_N(\mathbb{F}_q(t_\Sigma))$. Secondly, and this is perhaps more important to point out, we could have removed the condition of injectivity of χ to define basic representations in Definition 4.5. Several properties we prove for representations of the first kind extend to the larger class of representations satisfying all the conditions in Definition 4.5 but the injectivity of all the algebra maps χ_i in (4.9), used to build them. Many properties still hold. For instance, the maps $\Psi_m(\rho)$ of §4.1.1, or Φ_ρ are defined for any representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(t_\Sigma))$. The reader can also verify that the maps Ξ_ρ of §4.2.1 (which are essential to show that modular forms associated to representations of the first kind have series expansions in the field of uniformizers, see Theorem 4.12) can be constructed also for such representations that are constructed just as in Definition 4.5, starting from \mathbb{F}_q -algebra maps χ_i in (4.9) which are not all injective. But the injectivity condition is important to obtain several other properties. One example is the construction of Hecke operators in §4.3. Another example is the harmonic product formula of §9; see the injectivity in the Data 9.1, which looks essential. Also, if $\rho : \Gamma \rightarrow \mathrm{GL}_2(\mathbb{F}_q(t_\Sigma))$ is an arbitrary representation, even when it is possible to construct Poincaré series as in §5.3 it is hard to show that the functions constructed are non-zero, and in general it does not seem to be possible to construct Eisenstein series in the way we do in §5.4.

Remark 4.16. We briefly explain the chosen terminology: *representations of the first kind*. The reader can notice that Definition 4.5 benefits of quite a long list of good properties making it an excellent starting point in the study of our higher dimensional Drinfeld modular forms. However, a slight generalization can be equally interesting, that can be called *representations of the second kind*. These representations of the second kind will be only marginally discussed in this paper, but they are not difficult to define. Definition 4.5 can be modified allowing the \mathbb{F}_q -algebra map χ defined over A to have, as a target space, $\mathbb{K}_\Sigma^{n \times n}$ for some finite subset $\Sigma \subset \mathbb{N}^*$ and $n \in \mathbb{N}^*$, instead of just $\mathbb{F}_q(t_\Sigma)^{n \times n}$. So, a representation of the second type is one coming from an algebra map χ as above, which is not of the first kind. These representations are also very interesting in that associated non-trivial modular forms in the spirit of the present paper do exist and carry important properties, but they will be the object of another discussion (in general, tame series are not enough to describe their behavior at infinity). To end this remark we point out that it is rather difficult to us to make a comparison with the way one usually classifies complex vector valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$. There is a well known moderate growth condition in vertical strips that allows to expand entries of modular forms in expressions involving

powers of $\log q$ and Puiseux formal expansions, and also a condition of semisimplicity on $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ that, if added, neutralizes the logarithmic components (typical of vector modular forms coming from e.g. quasi-modular forms), that somehow resemble to our expansions in the field of uniformizer (although without phenomenon of wild ramification). This may suggest that these conditions are the analogues of representations of the first and of the second kind respectively. But a notable difference is that we work with analytic families of representations that have no known analogues in the complex theory.

Remark 4.17. Given any representation

$$\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$$

(not necessarily of the first kind), the function

$$\Phi_\rho(z) = e_A(z) \sum_{a \in A} (z - a)^{-1} \rho(T_a)$$

is well defined with entire entries. However, the hypothesis that ρ is of the first kind is crucial in order to obtain that all the entries of Φ_ρ are tame series. This comes from the existence of the exponential function \exp_C , which is not just \mathbb{F}_q -linear, but also a morphism of A -modules. The functions Ξ_ρ can be associated to representations of the first kind only, and occur in the proof of Proposition 4.11 via Proposition 4.8.

We now construct representations

$$\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$$

to which we can associate a quasi periodic function Φ_ρ having entire functions entries, but we also prove that these entries are not all in the field of uniformizers. As a consequence, these representations ρ are explicit examples of representations which are not of the first kind.

To construct such representations ρ we start with a representation of the first kind $\tilde{\rho} : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ and we modify it. Having (4.1) and (4.2) in view, we set ρ to be the unique representation $\Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ such that (a)

$$\rho|_{\mathrm{GL}_2(\mathbb{F}_q)} = \tilde{\rho}|_{\mathrm{GL}_2(\mathbb{F}_q)}$$

(b) $\rho(T_{\theta^i}) = \tilde{\rho}(T_{\theta^i})$ for all $i \in \mathbb{N}$ such that $i \notin S$, where S is a non-empty finite subset of \mathbb{N}^* . We can additionally suppose that (c)

$$G := \sum_{i \in S} (z - a)^{-1} \left(\rho(T_{\theta^i}) - \tilde{\rho}(T_{\theta^i}) \right)$$

is a non-constant matrix function. It is elementary to show that representations like this exist; they even exist when the target space of ρ is \mathbb{F}_q^N and ρ is not of the first kind! We observe that

$$\Phi_\rho = \Phi_{\tilde{\rho}} + G.$$

Assume by contradiction that Φ_ρ can be identified with a matrix with entries in the field of uniformizers. Since also $\Phi_{\tilde{\rho}}$ does, by the fact that $\tilde{\rho}$ is a representation of the first kind,

G is a matrix with entries in the field of uniformizers. But one entry g of G is a non-constant rational function in the variable z . This contradicts (3.22) which, as the reader can verify, implies that $\mathbb{K}_\Sigma(z)$ and \mathfrak{K} are linearly disjoint over \mathbb{K}_Σ . In particular, ρ is not a representation of the first kind.

5. STRUCTURE RESULTS FOR MODULAR FORMS

We consider, in this section, a representation

$$\Gamma \xrightarrow{\rho} \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma)).$$

We recall that $M_w^!(\rho; \mathbb{L}_\Sigma)$, $M_w(\rho; \mathbb{L}_\Sigma)$, $S_w(\rho; \mathbb{L}_\Sigma)$ denote respectively, the \mathbb{L}_Σ -vector spaces of weak modular forms, modular forms, and cusp forms in $\mathrm{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1})$ of weight w for ρ (in the sense of Definition 1.2), so that $S_w(\rho; \mathbb{L}_\Sigma) \subset M_w(\rho; \mathbb{L}_\Sigma) \subset M_w^!(\rho; \mathbb{L}_\Sigma)$. The operator τ induces $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear injective maps

$$M_w(\rho; \mathbb{L}_\Sigma) \xrightarrow{\tau} M_{qw}(\rho; \mathbb{L}_\Sigma),$$

and similarly for $M_w^!(\rho; \mathbb{L}_\Sigma)$, $S_w(\rho; \mathbb{L}_\Sigma)$ etc. Of course, this depends on the choice of Σ . To simplify, we will sometimes also write $M_w(\rho)$ for $M_w(\rho; \mathbb{L}_\Sigma)$ etc. when the reference to the field \mathbb{L}_Σ is clear. The next sub-section also allows to justify this abuse of notation.

5.1. Changing the coefficient field. We have defined the \mathbb{L}_Σ -vector space of modular forms $W_w(\rho; \mathbb{L}_\Sigma)$ and the \mathbb{K}_Σ -vector space of modular forms $W_w(\rho; \mathbb{K}_\Sigma)$ (with W a symbol such that $W \in \{M^!, M, S\}$). Let Σ' be finite such that $\Sigma \subset \Sigma' \subset \mathbb{N}^*$. Then, we also have the spaces $W_w(\rho; \mathbb{L}_{\Sigma'})$ and $W_w(\rho; \mathbb{K}_{\Sigma'})$. The next result allows to compare these spaces for $\Sigma' \supset \Sigma$. It is important in that it confirms that there are bases of these spaces which depend on the representation only. The notation W_w stands for $M_w^!, M_w, S_w$ in all the following.

Proposition 5.1. *We have that (1)*

$$W_w(\rho; \mathbb{K}_{\Sigma'}) = W_w(\rho; \mathbb{K}_\Sigma) \widehat{\otimes}_{\mathbb{K}_\Sigma} \mathbb{K}_{\Sigma'}$$

where $\widehat{\otimes}_{\mathbb{K}_\Sigma}$ means that every element f of $W_w(\rho; \mathbb{K}_{\Sigma'})$ can be expanded as a series $f = \sum_i a_i f_i$ where $a_i \in \mathbb{K}_{\Sigma'}$, $f_i \in W_w(\rho; \mathbb{K}_\Sigma)$ for all i , and $a_i f_i \rightarrow 0$ for the supremum norm of every affinoid subdomain of Ω . (2) If $\dim_{\mathbb{K}_\Sigma}(M_w(\rho; \mathbb{K}_\Sigma)) < \infty$ then

$$W_w(\rho; \mathbb{K}_{\Sigma'}) = W_w(\rho; \mathbb{K}_\Sigma) \otimes_{\mathbb{K}_\Sigma} \mathbb{K}_{\Sigma'}.$$

Moreover (3) if $\dim_{\mathbb{L}_\Sigma}(M_w(\rho; \mathbb{L}_\Sigma)) < \infty$, then

$$M_w(\rho; \mathbb{L}_{\Sigma'}) = M_w(\rho; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{L}_{\Sigma'}, \quad S_w(\rho; \mathbb{L}_{\Sigma'}) = S_w(\rho; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{L}_{\Sigma'}.$$

Proof. Let $(b_i)_{i \in I}$ be a basis of $\mathbb{F}_q^{ac}(\underline{t}_{\Sigma'})$ over $\mathbb{F}_q^{ac}(\underline{t}_\Sigma)$. By Lemma 2.7, $(b_i)_{i \in I}$ is an orthonormal basis. In other words, every element $\kappa \in \mathbb{K}_{\Sigma'}$ can be expanded, in a unique way, as a series $\kappa = \sum_i \kappa_i b_i$ with $\kappa_i \in \mathbb{K}_\Sigma$ such that $\kappa_i \rightarrow 0$. Let us choose a basis $(c_j)_{j \in J}$ of

$\mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ over \mathbb{F}_q^{ac} so that $(b_i c_j)_{i,j}$ is an orthonormal basis of $\mathbb{K}_{\Sigma'}$ over \mathbb{C}_∞ . Now consider $f \in W_w(\rho; \mathbb{K}_{\Sigma'})$. We can expand

$$f(z) = \sum_{i,j} f_{i,j}(z) b_i c_j$$

where $f_{i,j} \in \text{Hol}_{\mathbb{C}_\infty}(\Omega \rightarrow \mathbb{C}_\infty)^{N \times 1}$ for all i, j , with the property that $f_{i,j} \rightarrow 0$ with respect to the supremum norm relative to any choice of an affinoid subdomain of Ω . Observe that

$$f(\gamma(z)) = J_\gamma(z)^w \sum_i \left(\rho(\gamma) \sum_j f_{i,j}(z) c_j \right) b_i,$$

because f is modular-like. Since $\rho(\gamma) \sum_j f_{i,j}(z) c_j \in \mathbb{K}_\Sigma^{N \times 1}$ for $z \in \Omega$ and $(b_i)_i$ is an orthonormal basis of $\mathbb{K}_{\Sigma'}$ over \mathbb{K}_Σ , we deduce that for all $i \in I$, setting $f_i = \sum_j f_{i,j}(z) c_j$,

$$f_i(\gamma(z)) = J_\gamma(z)^w \rho(\gamma) f_i(z),$$

and one sees that $f_i \in W_w(\rho; \mathbb{K}_\Sigma)$. Since $f = \sum_i f_i b_i$ with the above convergence conditions, we get $f \in W_w(\rho; \mathbb{K}_\Sigma) \widehat{\otimes}_{\mathbb{K}_\Sigma} \mathbb{K}_{\Sigma'}$ which proves (1). Now observe, for (2), that the above sum reduces to a finite sum if $\dim_{\mathbb{K}_\Sigma}(M_w(\rho; \mathbb{K}_\Sigma)) < \infty$. The proof of part (3) of the proposition is similar but we restrict to $W_w = M_w, S_w$. First notice that by Lemma 2.10 which can be applied to $B = \mathbb{L}_{\Sigma'}$ (it satisfies the conditions at the beginning of §2.2.1), if $f \in W_w(\rho; \mathbb{L}_{\Sigma'})$ then there exists $d' \in \mathbb{F}_q[\underline{t}_{\Sigma'}] \setminus \{0\}$ such that $f \in W_w(\rho; \mathbb{T}_{\Sigma'}[\frac{1}{d'}]^\wedge)$ by Lemma 2.10. We can even choose d, d' with $d \in \mathbb{F}_q[\underline{t}_{\Sigma'}] \setminus \{0\}$ such that $d \mid d'$ and such that the image of ρ is contained in $\text{GL}_N(\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d}])$. The proof of the first part of the proposition can be modified to obtain that f can be expanded as a series $f = \sum_k a_k f_k$ where $a_k \in \mathbb{T}_{\Sigma'}[\frac{1}{d'}]^\wedge$ and $f_i \in W_w(\rho; \mathbb{T}_\Sigma[\frac{1}{d}])$, and $a_i f_i \rightarrow 0$ for the supremum norm associated to any affinoid subset of Ω . If now $\dim_{\mathbb{L}_\Sigma} M_w(\rho; \mathbb{L}_\Sigma) < \infty$, we deduce the result. \square

Remark 5.2. We have excluded $W_w = M_w^!$ because in general, $\dim_{\mathbb{L}_\Sigma} M_w^!(\rho; \mathbb{L}_\Sigma) = \infty$. However there are some canonical subspaces that are often finite dimensional. For instance, if $v_0 \in \mathbb{R}$ it can be proved (but we omit the details) as a consequence of our Theorem 5.5 that the spaces

$$M_{v_0, w}^!(\rho; \mathbb{L}_\Sigma) := \left\{ f = {}^t(f_1, \dots, f_N) \in M_w^!(\rho; \mathbb{L}_\Sigma) : v(f_1), \dots, v(f_N) \geq v_0 \right\}$$

are finite-dimensional. Then, results similar to the above hold, with similar proofs.

Remark 5.3. In full generality, we do not know if $W_w(\rho; \mathbb{K}_\Sigma) = W_w(\rho; \mathbb{L}_\Sigma) \widehat{\otimes}_{\mathbb{L}_\Sigma} \mathbb{K}_\Sigma$ and we do not know how to compare the dimension over \mathbb{L}_Σ and \mathbb{K}_Σ . Note that the proof above imply the following: when $d \mid d'$ and if $\rho(\Gamma)$ is in $\text{GL}_N(\mathbb{F}_q[\underline{t}_\Sigma][d^{-1}])$, then $M_w(\rho; \mathbb{T}_{\Sigma'}[\frac{1}{d'}]^\wedge) = M_w(\rho; \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge) \otimes_{\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge} \mathbb{T}_{\Sigma'}[\frac{1}{d'}]^\wedge$. In particular, we have

$$M_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge) \otimes_{\mathbb{T}_\Sigma[d^{-1}]^\wedge} \mathbb{L}_\Sigma = M_w(\rho; \mathbb{L}_\Sigma),$$

and similarly for the spaces of cusp forms and the tempered forms.

The next direct consequence of Proposition 5.1 will be used in §8. Compare with [55, Lemma 13].

Corollary 5.4. *With $B = \mathbb{K}_\Sigma, \mathbb{L}_\Sigma, \mathbb{T}_\Sigma[\frac{1}{d}]$ we have, for all $w \in \mathbb{Z}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$,*

$$M_w(\det^{-m}; B) = M_w(\det^{-m}; \mathbb{C}_\infty) \otimes_{\mathbb{C}_\infty} B.$$

In other words, scalar B -valued modular forms are B -linear combinations of Drinfeld modular forms à la Goss-Gekeler (with same weights and types).

5.2. Finiteness results. In this subsection we suppose that the representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ is of the first kind. We also recall that \mathfrak{K}_Σ is the completion of the fraction field of $\mathbb{K}_\Sigma\langle\langle \underline{e} \rangle\rangle^b$ for the valuation v , and that $\mathfrak{D}_\Sigma, \mathfrak{M}_\Sigma$ are respectively the valuation ring and the maximal ideal of v . We have the following results which correspond to part (1) of Theorem B in the introduction:

Theorem 5.5 (Finiteness Theorem). *The \mathbb{L}_Σ -vector space $M_w(\rho; \mathbb{L}_\Sigma)$ has finite dimension $r_\rho(w)$ and we have $r_\rho(w) \leq (1 + \lfloor \frac{w}{q+1} \rfloor)N$ if $q > 2$ and $r_\rho(w) \leq 2(1 + \lfloor \frac{w}{q+1} \rfloor)N$ if $q = 2$.*

In particular, if $w < 0$, then $r_\rho(w) = 0$ and $M_w(\rho; \mathbb{L}_\Sigma) = \{0\}$ but this property will be actually proved separately to obtain the general result. The proof of this theorem makes use of an important feature of our Drinfeld modular forms when they take values in \mathbb{L}_Σ ; the possibility of evaluating the variables t_i ($i \in \Sigma$) at roots of unity. There is not such a property for \mathbb{K}_Σ -valued modular forms. In §5.2.2 we prove Theorem 5.5 by using that the spaces of modular forms of negative weight are trivial. This is a consequence of the fact that classical negative weight (scalar) Drinfeld modular forms for congruence subgroups of Γ are zero. The upper bound for the dimensions in Theorem 5.5 can be slightly refined, but our methods do not allow an explicit computation.

5.2.1. Evaluating at roots of unity. The representation of the first kind ρ is constructed starting from a finite set of basic representations ρ_i associated with injective \mathbb{F}_q -algebra morphisms $\chi_i : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)$ ($i = 1, \dots, r$). If $d_1, \dots, d_r \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ are such that the entries of $d_i \chi_i(\theta)$ are in $\mathbb{F}_q[\underline{t}_\Sigma]$ then the image of ρ is in $\mathrm{GL}_N(\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d_1}, \dots, \frac{1}{d_r}]) \subset \mathrm{GL}_N(\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d}])$ for some $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$. We thus get, after Proposition 4.8, that

$$\Xi_\rho, \Phi_\rho \in \mathrm{Hol}\left(\mathbb{C}_\infty \rightarrow \widehat{\mathbb{E}_\Sigma[d^{-1}]}^{N \times N}\right).$$

Let $\Sigma = U \sqcup V$ be a finite subset of \mathbb{N}^* written as a disjoint union of subsets U, V , with U non-empty. The set

$$\mathcal{V}_U(d) = \{\underline{\zeta} \in (\mathbb{F}_q^{ac})^U : d(\underline{\zeta}) = 0\}$$

is Zariski-dense in $(\mathbb{F}_q^{ac})^U$. Let $\underline{\zeta} = (\zeta_i : i \in U)$ be an element of $(\mathbb{F}_q^{ac})^U \setminus \mathcal{V}_U(d)$.

The *evaluation map*

$$\mathrm{ev}_\zeta : \widehat{\mathbb{T}_\Sigma[d^{-1}]} \rightarrow \widehat{\mathbb{T}_V[\mathrm{ev}_\zeta(d)^{-1}]}$$

is the \mathbb{T}_V -algebra morphism uniquely determined by the correspondence $t_i \mapsto \zeta_i$ for $i \in U$. If there is no possibility of confusion, we write $f(\underline{\zeta})$ in place of $\mathrm{ev}_\zeta(f)$. We extend this map

to matrices with entries in $\mathbb{T}_\Sigma[d^{-1}]^\wedge$. It is easy to see that if X is an analytic space over \mathbb{C}_∞ and $f \in \text{Hol}(X \rightarrow \mathbb{T}_\Sigma[d^{-1}]^\wedge)$, then $\text{ev}_\zeta(f) \in \text{Hol}(X \rightarrow \mathbb{T}_V[d(\zeta)^{-1}]^\wedge)$. Moreover:

Lemma 5.6. *Let X be a rigid analytic space over \mathbb{C}_∞ . If $f \in \text{Hol}(X \rightarrow \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ and if for all $\zeta \in (\mathbb{F}_q^{ac})^U \setminus \mathcal{V}_U(d)$, $\text{ev}_\zeta(f) \in \text{Hol}(X \rightarrow \mathbb{T}_V[\frac{1}{d(\zeta)}]^\wedge)$ is constant, then f is constant.*

Proof. It is enough to prove the result for $X = \text{Spm}(\mathcal{A})$ affinoid. By Lemma 2.5 we can choose an orthonormal basis $(a_i)_{i \in I}$ of the Banach \mathbb{C}_∞ -algebra \mathcal{A} . We can even assume, without loss of generality, that a_{i_0} is the constant function equal to one for an index $i_0 \in I$. Then, for all $i \neq i_0$, a_i is non-constant over X . We can expand every element f of $\text{Hol}(X \rightarrow \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ as $f = \sum_{i \in I} f_i a_i$ with $f_i \in \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$, where the series converges for the supremum norm of X . Hence,

$$\text{ev}_\zeta(f) = \sum_{i \in I} \text{ev}_\zeta(f_i) a_i,$$

and $\text{ev}_\zeta(f_i) = 0$ for all $i \neq i_0$. Since this happens for all $\zeta \in (\mathbb{F}_q^{ac})^U \setminus \mathcal{V}_U(d)$ which is Zariski-dense, we obtain $f_i = 0$ for all $i \neq i_0$ and f is constant over X . \square

Let \mathfrak{n} be a non-zero ideal of A . We denote by $\Gamma(\mathfrak{n})$ the associated principal congruence subgroup of Γ :

$$\Gamma(\mathfrak{n}) = \{\gamma \in \Gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}}\}.$$

We recall that $\rho : \Gamma \rightarrow \text{GL}_N(\mathbb{F}_q[t_\Sigma][d^{-1}])$ is a representation of the first kind.

Lemma 5.7. *Let $\zeta = (\zeta_i : i \in \Sigma)$ be an element of $(\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$. There exists a non-zero ideal \mathfrak{n} of A such that for all $\gamma \in \Gamma(\mathfrak{n})$, $\text{ev}_\zeta(\rho(\gamma)) = I_N$.*

Proof. There exist basic representations $\rho_{\chi_1}, \dots, \rho_{\chi_r}$, associated to \mathbb{F}_q -algebra morphisms $\chi_i : A \rightarrow \mathbb{F}_q(t_\Sigma)^{n_i \times n_i}$ ($i = 1, \dots, r$) such that ρ can be constructed applying admissible operations finitely many times (as in Definition 4.5). We fix $\zeta \in (\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$ where $d \in \mathbb{F}[t_\Sigma] \setminus \{0\}$ is such that $d\chi_i(\theta) \in \mathbb{F}_q[t_\Sigma]^{n_i \times n_i}$. We denote by \mathfrak{n} the ideal generated by $P_1|_{X=\theta}, \dots, P_r|_{X=\theta} \in A \setminus \{0\}$, where $P_i \in \mathbb{F}_q[X]$ is the minimal polynomial of $\eta_i = \chi_i(\theta)|_{t_\Sigma=\zeta}$ (for all i), which are well defined. Then, if $a \in \mathfrak{n}$, we have $\text{ev}_\zeta(\chi_{t_i}(a)) = 0$ for all i so that $\text{ev}_\zeta(\rho(\gamma)) = I_N$ for all $\gamma \in \Gamma(\mathfrak{n})$. \square

We now introduce a slightly more general notion of vector-valued modular form for a congruence subgroup of Γ . Let G be a congruence subgroup of Γ . The quotient space $G \backslash \Omega$ carries a natural structure of analytic curve Y_G with compactification X_G obtained by adding finitely many points to Y_G called *cusps*. We can consider neighbourhoods of a cusp of $G \backslash \Omega$ in Ω in the usual way and therefore, there is a natural notion of modular-like forms $f : \Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ of weight w for ρ , seen as a representation of G by restriction, namely, satisfying the collection of functional equations

$$(5.1) \quad f(\gamma(z)) = J_\gamma(z)^w \rho(\gamma) f(z) \quad \forall z \in \Omega, \quad \forall \gamma \in G.$$

Let c be a cusp of X_G and let us consider $\delta \in \Gamma$ such that $\delta(\infty) = c$. If $f : \Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ is a map and w an integer, we set

$$f^\delta(z) := f|_{w, \rho} \delta = J_\delta(z)^{-w} \rho(\delta)^{-1} f(\delta(z))$$

(Petersson slash operator as in (4.19)). A simple computation shows that if f is modular-like of weight w for the restriction $\rho|_G$ of ρ on G , then $f^\delta : \Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ is modular-like of weight w for $\rho|_{G^\delta}$ where $G^\delta := \delta^{-1}G\delta$ (in particular, if f is modular-like for the group Γ , then $f = f^\delta$).

Definition 5.8. Let w be in \mathbb{Z} . We say that a modular-like function $\Omega \xrightarrow{f} \mathbb{L}_\Sigma^{N \times 1}$ of weight w for $\rho|_G$ is:

- (1) A *weak Drinfeld modular form of weight w for $\rho|_G$* if there exists $H \in \mathbb{Z}$ such that

$$\|u(z)^H f^\delta(z)\| \rightarrow 0$$

as $z \in \Omega$ is such that $|z| = |z|_\mathfrak{S} \rightarrow \infty$, and this, for all $\delta \in \Gamma$.

- (2) A *Drinfeld modular form of weight w for $\rho|_G$* , if $\|f^\delta(z)\|$ is bounded as $|u(z)| < c$ for some constant $c < 1$, for all $\delta \in \Gamma$.
- (3) A *cusp form of weight w for $\rho|_G$* if $\|f^\delta(z)\| \rightarrow 0$ as $z \in \Omega$ is such that $|z| = |z|_\mathfrak{S} \rightarrow \infty$ for all $\delta \in \Gamma$.

We denote by $M_w^!(G; \rho; \mathbb{L}_\Sigma)$ (resp. $M_w(G; \rho; \mathbb{L}_\Sigma), S_w(G; \rho; \mathbb{L}_\Sigma)$) the \mathbb{L}_Σ -vector spaces of weak modular forms (resp. modular forms, cusp forms) of weight w for ρ . More generally, if B is a \mathbb{C}_∞ -subalgebra of \mathbb{K}_Σ , we write $M_w(G; \rho; B)$ for the corresponding B -module of modular forms.

It is easy to see that the \mathbb{C}_∞ -vector space $M_w(G; \mathbf{1}; \mathbb{C}_\infty)$ is equal to the \mathbb{C}_∞ -vector space of the scalar Drinfeld modular forms of weight w for G and a similar property holds for weak modularity and cuspidality of a form. In the next proposition, W_w stands for $M_w^!, M_w, S_w$ (so the proposition is in fact equivalent to three distinct statements).

Proposition 5.9. Let f be in $W_w(\rho; \mathbb{L}_\Sigma)$. Then, there exists $d \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ such that $f \in W_w(\rho; \widehat{\mathbb{T}_\Sigma[\frac{1}{d}]})$ (Lemma 2.10). Let us consider, further, $\zeta \in (\mathbb{F}_q^{\text{ac}})^\Sigma \setminus \mathcal{V}_\Sigma(d)$. We have $\text{ev}_\zeta(f) \in W_w(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}$ where \mathfrak{n} is any ideal as in Lemma 5.7.

Hence, the evaluations of the N entries of $f \in M_w(\rho; \mathbb{L}_\Sigma)$ are scalar Drinfeld modular forms of weight w for $\Gamma(\mathfrak{n})$.

Proof of Proposition 5.9. By Lemma 5.7, for all $\gamma \in \Gamma(\mathfrak{n})$ and $z \in \Omega$, $\text{ev}_\zeta(f)(\gamma(z)) = J_\gamma(z)^w \text{ev}_\zeta(f)(z)$ and also, it is easy to see that $\text{ev}_\zeta(f)$ has rigid analytic entries. It remains to show that the entries of $\text{ev}_\zeta(f)$ have the decay properties of Definition 5.8 which is guaranteed if we show regularity at all cusps of $G \setminus \Omega$. In more detail, if f has image defined

over $\widehat{\mathbb{T}_\Sigma[\frac{1}{d}]}$, we show that the map $\text{ev}_\zeta(\cdot)$ defines maps (\mathbb{C}_∞ -linear maps)

$$(5.2) \quad M_w^!(\rho; \widehat{\mathbb{T}_\Sigma[d^{-1}]}) \rightarrow M_w^!(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1},$$

$$(5.3) \quad M_w(\rho; \widehat{\mathbb{T}_\Sigma[d^{-1}]}) \rightarrow M_w(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1},$$

$$(5.4) \quad S_w(\rho; \widehat{\mathbb{T}_\Sigma[d^{-1}]}) \rightarrow S_w(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}.$$

First of all, a holomorphic function $f : \Omega \rightarrow \mathbb{C}_\infty$ satisfying $f(\gamma(z)) = J_\gamma(z)^w f(z)$ for all $\gamma \in \Gamma(\mathfrak{n})$ is a weak modular form of weight w for $\Gamma(\mathfrak{n})$ if for all $\delta \in \Gamma$, the function $f^\delta(z)$ can be expanded as a series of $\mathbb{C}_\infty((u(\frac{z}{\mathfrak{n}})))$ in the neighborhood of the cusp $\delta(\infty)$, where \mathfrak{n} is a generator of \mathfrak{n} . We deduce that $f^\delta(z)$ is a weak modular form of weight w for the group $\delta^{-1}\Gamma(\mathfrak{n})\delta$. Note indeed that $u(\frac{z}{\mathfrak{n}})$ is a uniformiser at ∞ for the action of $\Gamma(\mathfrak{n})$ over Ω in virtue of the fact that the group $(\frac{1}{0} \ \frac{\mathfrak{n}}{1})$ is contained in $\delta^{-1}\Gamma(\mathfrak{n})\delta$ for all $\delta \in \Gamma$.

Let f be in $M_w^!(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$. Then, $\text{ev}_\zeta(f)$ has all the entries which are \mathfrak{n} -periodic and $\text{ev}_\zeta(f^\delta)$ is tempered for all $\delta \in \Gamma$. This implies that $\text{ev}_\zeta(f) \in M_w^!(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}$ which proves (5.2). Now assume that f is, additionally, a modular form in $M_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$. Then, all the entries b^δ of $\text{ev}_\zeta(f^\delta)$ satisfy $b^\delta \in \mathbb{C}_\infty[[u(\frac{z}{\mathfrak{n}})]]$ for all $\delta \in \Gamma$, which yields (5.3). Similarly, if f is in $S_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$, we see that all the entries of $\text{ev}_\zeta(f)$ vanish at all the cusps of $X(\mathfrak{n})$ hence confirming (5.4) and completing the proof of the Proposition. \square

5.2.2. *Proof of the Finiteness Theorem.* We first study the structure of the space $M_0(\rho; \mathbb{L}_\Sigma)$.

Lemma 5.10. *We have $M_0(\rho; \mathbb{L}_\Sigma) \subset \mathbb{L}_\Sigma^{N \times 1}$.*

Proof. Let f be an element of $M_0(\rho; \mathbb{L}_\Sigma)$. By Lemma 2.10 there exists $d \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ such that the image of f is contained in $\widehat{\mathbb{T}_\Sigma[\frac{1}{d}]}$. By Proposition 5.9, for all $\zeta \in (\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$ there exists a non-zero ideal \mathfrak{n} of A such that $\text{ev}_\zeta(f) \in M_0(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}$. A scalar Drinfeld modular form of weight zero is constant. Hence, for all ζ as above, $\text{ev}_\zeta(f) \in \mathbb{C}_\infty^{N \times 1}$. Therefore, f is a constant map by Lemma 5.6 with $X = \Omega$. \square

We recall from §2.3 the $\mathbb{F}_q(t_\Sigma)$ -linear automorphisms $\tau : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$, $\tau : \mathbb{L}_\Sigma \rightarrow \mathbb{L}_\Sigma$. Since the image of a representation of the first kind ρ lies in $\mathbb{F}_q(t_\Sigma)^{N \times N}$ for some N , we have injective $\mathbb{F}_q(t_\Sigma)$ -linear maps

$$W_w(\rho; \mathbb{K}_\Sigma) \xrightarrow{\tau} W_{qw}(\rho; \mathbb{K}_\Sigma), \quad W_w(\rho; \mathbb{L}_\Sigma) \xrightarrow{\tau} W_{qw}(\rho; \mathbb{L}_\Sigma),$$

where $W_w = M_w^!, M_w, S_w$. With this, we can prove the next corollary to Lemma 5.10.

Corollary 5.11. *If $w < 0$, $M_w(\rho; \mathbb{L}_\Sigma) = \{0\}$.*

Proof. Let f be an element of $M_w(\rho; \mathbb{L}_\Sigma)$ with negative w . For all $k, \alpha, \beta \in \mathbb{N}$ with $\beta > 0$, $\tilde{f} := g^\alpha h^\beta \tau^k(f) \in S_{q^{kw+\alpha(q-1)+\beta(q+1)}}(\rho \det^{-\beta}; \mathbb{L}_\Sigma)$, where g is the normalised Eisenstein series in $M_{q-1}(\mathbf{1}; \mathbb{C}_\infty)$ and h is -1 times the normalised generator of $S_{q+1}(\det^{-1}; \mathbb{C}_\infty)$ (we

are adopting Gekeler's notations in [24], see also §5.3.3). We show that there exist k, α, β with $\beta > 0$ such that

$$(5.5) \quad q^k w + \alpha(q-1) + \beta(q+1) = 0.$$

This is very easy but we give all the details. To find such k, α, β , we first observe that we need $q^k w + \alpha(q-1) + \beta(q+1) \equiv 0 \pmod{q-1}$, and this is guaranteed by $w \equiv -2\beta \pmod{q-1}$. We must have:

$$\begin{aligned} \alpha &= \frac{1}{q-1}(-wq^k - \beta(q+1)) \\ &= \frac{1}{q-1}(-wq^k - 2\beta) + \beta. \end{aligned}$$

Assume first that $p \neq 2$. Then, there exists $\beta \in \{1, \dots, q-1\}$ such that $w \equiv -2\beta \pmod{q-1}$. We can choose k large enough so that $-wq^k - 2\beta$, divisible by $q-1$, is ≥ 0 . Therefore we can choose $\alpha \in \mathbb{N}$ such that, with such β and k , (5.5) holds.

If $p = 2$ we can set $\beta = 1$ and k such that $\alpha = -2^k w - 3 \geq 0$. Since $\beta > 0$ we see that \tilde{f} is a cusp form and Lemma 5.10 now implies that $\tilde{f} = 0$; hence $f = 0$ because τ is injective. \square

Proof of Theorem 5.5. The result is already proved in Lemma 5.10 and Corollary 5.11 if $w \leq 0$. Now assume that $w > 0$ and let f be in $M_w(\rho; \mathbb{L}_\Sigma)$. Again, we can suppose that $f \in M_w(\rho; \widehat{\mathbb{T}_\Sigma[\frac{1}{d}]})$ for some $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$.

We have that $f \in \mathfrak{D}_\Sigma^{N \times 1}$ by Theorem 4.12. In fact, the proof of Proposition 4.11 allows to show that, more precisely, $f \in \mathfrak{D}_{\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge}^{N \times 1}$. Since f is a regular ρ -quasi-periodic function (Definition 4.1), viewing the proof of Proposition 4.11, we obtain that $f = \Phi_\rho g$, where Φ_ρ has been defined in §4.2.1 and studied in Proposition 4.8, and where g is in $\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge[[u]]^{N \times 1}$. We recall that from Corollary 4.9 that $\det(\Phi_\rho) = 1$ and $\Phi_\rho, \Phi_\rho^{-1} \in (\mathfrak{E}^\circ \langle \underline{e} \rangle^b)^{N \times N}$. We now study the association $f \mapsto g$ so that we write g_f to stress the dependence of g on f .

Let ν be in \mathbb{F}_q^\times . We have

$$\rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \Phi_\rho(z) \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}^{-1} g_f(\nu z) = f(\nu z) = \nu^w \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} f(z) = \nu^w \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \Phi_\rho(z) g_f(z), \quad \forall z \in \Omega.$$

Since ρ is of the first kind, $\rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal and we can write:

$$\rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \nu^{-n_1} & & \\ & \ddots & \\ & & \nu^{-n_N} \end{pmatrix}, \quad n_i \in \mathbb{Z}/(q-1)\mathbb{Z}, \quad \nu \in \mathbb{F}_q^\times.$$

Writing additionally $g_f = {}^t(g_1, \dots, g_N)$, we deduce that

$$g_i(\nu z) = \nu^{w-n_i} g_i(z)$$

for all $i = 1, \dots, N$, so that $g_i \in u^{m_i} \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge[[u^{q-1}]]$ where m_i is the unique representative of $n_i - w$ modulo $q-1$ in $\{0, \dots, q-2\}$. This implies that the subspace \mathcal{W}_w of $M_w(\rho; \mathbb{L}_\Sigma)$ spanned by the forms f with g_f having entries of v -valuation in the set $\{0, 1\}$ has dimension

not exceeding N if $q > 2$ and $2N$ if $q = 2$. On the other hand, if $f \in M_w(\rho; \mathbb{L}_\Sigma)$ is such that g_f is not in \mathcal{W}_w , that is, the v -valuations of its entries are ≥ 2 , then, by the fact that $\Phi_\rho^{-1} \in (\mathfrak{E}^\circ \langle \underline{e} \rangle^b)^{N \times N}$, we deduce that the v -valuations of the entries of f are all ≥ 1 and therefore $f \in hM_{w-(q+1)}(\rho \det; \mathbb{L}_\Sigma)$ (where we recall that h is the generator of $S_{q+1}(\det^{-1}; \mathbb{C}_\infty)$ normalised by the coefficient of u in its u -expansion, which is set to -1). We have proved that

$$M_w(\rho; \mathbb{L}_\Sigma) = hM_{w-(q+1)}(\rho \det; \mathbb{L}_\Sigma) \oplus \mathcal{W}_w.$$

This implies

$$\dim_{\mathbb{L}_\Sigma} \left(M_w(\rho; \mathbb{L}_\Sigma) \right) \leq \dim_{\mathbb{L}_\Sigma} \left(M_{w-(q+1)}(\rho \det; \mathbb{L}_\Sigma) \right) + \begin{cases} N & \text{if } q > 2 \\ 2N & \text{if } q = 2 \end{cases}$$

The result follows by induction over w . \square

5.2.3. Modular forms of weight one. We keep working with a representation of the first kind $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ and we set, with L a field extension of $\mathbb{F}_q(\underline{t}_\Sigma)$,

$$H(\rho; L) = \{l \in L^{N \times 1} : \rho(T_a)l = l \text{ for all } a \in A\}.$$

This is equal to the L -vector space generated by the simultaneous eigenvectors of $\rho(T_a)$ in $\mathbb{F}_q(\underline{t}_\Sigma)$, with $a \in A$. Note indeed that for all $a \in A$, $T_a^p = I_2$ so that 1 is the unique eigenvalue of $\rho(T_a)$ for all a . We denote by δ_ρ the dimension of $H(\rho; L)$ (independent on L).

Let us consider $f \in M_w(\rho; L)$ where $L = \mathbb{K}_\Sigma$ or $L = \mathbb{L}_\Sigma$. By Theorem 4.12 we can identify $f = {}^t(f_1, \dots, f_N)$ with an element of $\mathfrak{D}_\Sigma^{N \times 1}$. We denote by \bar{f}_i the image of f_i modulo \mathfrak{M}_Σ for all i . This is an element of \mathbb{K}_Σ and we set $\bar{f} = {}^t(\bar{f}_1, \dots, \bar{f}_N) \in \mathbb{K}_\Sigma^{N \times 1}$. We easily see, by taking the limit for $z \in \Omega$, $|z| = |z|_{\mathfrak{S}} \rightarrow \infty$ that in fact, $\bar{f} \in L^{N \times 1}$. Note that for every $a \in A$, $f|_{w, \rho} T_a = \rho(T_{-a})f(z+a)$ equally belongs to $\mathfrak{D}_\Sigma^{N \times 1}$ (by Lemma 3.35). Therefore $\bar{f} \in H(\rho; L)$. This means that

$$M_w(\rho; L) = S_w(\rho; L) \oplus \mathcal{W}_w,$$

where the map $f \mapsto \bar{f}$ induces an embedding $\mathcal{W}_w \rightarrow H(\rho; L)$ so that δ_ρ is an upper bound for the dimension of \mathcal{W}_w . We can now prove the following result which justifies part (2) of Theorem B in the introduction:

Theorem 5.12. *We have $S_1(\rho; \mathbb{L}_\Sigma) = \{0\}$ and the inequality $\dim_{\mathbb{L}_\Sigma}(M_1(\rho; \mathbb{L}_\Sigma)) \leq \delta_\rho$.*

Proof. It suffices to show that $S_1(\rho; \mathbb{L}_\Sigma) = \{0\}$. Let f be a cusp form of $S_1(\rho; \mathbb{L}_\Sigma)$. In the settings of Proposition 5.9, for $\underline{\zeta} \in (\mathbb{F}_q^{ac})^{\Sigma} \setminus \mathcal{V}_\Sigma(d)$ we get (after this proposition) that the evaluation $\mathrm{ev}_{\underline{\zeta}}(f)$ is well defined and its entries are cusp forms of $S_1(\Gamma(\mathfrak{n}))$. The latter space is zero as it was first noticed by Gekeler (see Cornelissen, in [19, Theorem (1.10)]). Hence, for all $\underline{\zeta}$ as above, $\mathrm{ev}_{\underline{\zeta}}(f) = 0$. By Lemma 5.6, f vanishes identically. \square

A more precise result in a particular case is Theorem 7.5.

5.2.4. *Problems relative to the evaluation at roots of unity.* Consider a representation of the first kind ρ and two elements $f, g \in M_w(\rho; \mathbb{L}_\Sigma)$. By Lemma 2.10 there exists $d \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ such that $f, g \in M_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$ and $\rho(\Gamma) \subset \mathrm{GL}_N(\mathbb{F}_q[t_\Sigma][d^{-1}])$.

Lemma 5.13. *Suppose that $\mathrm{ev}_\zeta(f) = \mathrm{ev}_\zeta(g)$ for all $\zeta \in (\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$. Then either $f = g$ or $f - g \in \mathbb{T}_\Sigma[d^{-1}]^\wedge$ and $w = 0$. If ρ is irreducible, then $f = g$.*

Proof. By Lemma 5.6 there exists $\kappa \in \mathbb{T}_\Sigma[d^{-1}]^{N \times 1}$ such that $f - g = \kappa$. By modularity for all $\gamma \in \Gamma$ and $z \in \Omega$, $\kappa = J_\gamma(z)^w \rho(\gamma) \kappa$. If $\kappa \neq 0$, $w = 0$. Now, again in the case of κ non-zero, $\kappa \mathbb{L}_\Sigma$ is a subvector space of $\mathbb{L}_\Sigma^{N \times 1}$ that is $\rho(\gamma)$ -invariant for all γ . If $N > 1$ and ρ is irreducible, this is impossible. If $N = 1$ then $\rho = \det^{-m}$ for some m . Hence the result follows from Corollary 5.4. \square

If $\zeta \in (\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$ we have a map

$$\mathrm{ev}_\zeta : M_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge) \rightarrow M_w(\rho_\zeta; \mathbb{C}_\infty),$$

where $\rho_\zeta : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q^{ac})$ is the representation obtained by sending $\gamma \in \Gamma$ to $\mathrm{ev}_\zeta(\rho(\gamma))$. We have not yet defined $M_w(\rho_\zeta; \mathbb{C}_\infty)$. Note that the image of ρ_ζ is a finite subgroup and ρ_ζ is not of the first kind, apart from trivial situations. In this case there is an ideal \mathfrak{n} such that $\Gamma(\mathfrak{n}) \subset \mathrm{Ker}(\rho)$. To define $M_w(\rho_\zeta; \mathbb{C}_\infty)$ we require that every entry of an element $f \in M_w(\rho_\zeta; \mathbb{C}_\infty)$ is an element of $M_w(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)$ as suggested by Proposition 5.9. In other words, we require the elements of $M_w(\rho_\zeta; \mathbb{C}_\infty)$ to be regular, in the classical sense, near the cusps of $\Gamma(\mathfrak{n})$ for any \mathfrak{n} such that $\Gamma(\mathfrak{n}) \subset \mathrm{Ker}(\rho)$. From Lemma 5.13 we deduce:

Corollary 5.14. *If ρ is irreducible and $w > 0$, $\bigcap_\zeta \mathrm{Ker}(\mathrm{ev}_\zeta) = (0)$.*

The maps ev_ζ are rarely surjective. We illustrate this with an example, anticipating some tools that are discussed later on. We consider $\rho = \rho_{\chi_t}^*$ the contragredient of the basic representation associated to $\chi_t : A \rightarrow \mathbb{F}_q[t]$ the map $a(\theta) \mapsto a(t)$ of §2.4.4 (irreducible, see Lemma 7.11), so we have $N = 2$ for the rank of the target space. In Theorem 7.5 we shall prove that $M_1(\rho; \mathbb{T})$ is one-dimensional, generated by an Eisenstein series. Now consider $\zeta \in \mathbb{F}_q^{ac}$ and an element $f \in M_1(\rho_\zeta; \mathbb{C}_\infty)$. It is easy to see that if \mathfrak{p} is the prime ideal of $\mathbb{F}_q[t]$ generated by the polynomials that vanish at ζ , then $\Gamma(\mathfrak{p})$ is contained in the kernel of ρ_ζ . By Cornelissen [19, Proposition (1.12)] we have, with $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ (for few lines u will not denote the uniformizer at infinity),

$$f_i(z) = \sum_{u \in S} c_{i,u} F_u(z), \quad c_{i,u} \in \mathbb{C}_\infty$$

where S is any set of representatives of the cusps of $\Gamma(\mathfrak{p})$ (we can make them in a bijective correspondence with the quotient of the set $(A/\mathfrak{p})^2 \setminus \{0\}$ modulo the scalar multiplication by elements of \mathbb{F}_q^\times) and $(F_u)_{u \in S}$ is the set of *restricted Eisenstein series* of weight 1 in the terminology of [19]. In particular, $(F_u)_{u \in S}$ is a basis of $M_1(\Gamma(\mathfrak{p}); \mathbf{1}; \mathbb{C}_\infty)$. The group Γ acts permuting the cusps. If u is a cusp and $\gamma \in \Gamma$, we denote by $u\gamma$ the cusp image of u by γ . It is well known that for all γ , $F_u(\gamma(z)) = J_\gamma(z) F_{u\gamma^{-1}}$. On the other hand we

have $f(\gamma(z)) = J_\gamma(z)\rho_\zeta(\gamma)f(z)$. Hence we get, setting $c_u = \begin{pmatrix} c_{1,u} \\ c_{2,u} \end{pmatrix}$ (the first equality is a rearrangement of sum):

$$\sum_{u \in S} F_u(z)c_{u\gamma} = \sum_{u \in S} F_{u\gamma^{-1}}(z)c_u = \sum_{u \in S} F_u(z)\rho_\zeta(\gamma)(c_u), \quad \forall \gamma \in \Gamma.$$

There exist $(z_v)_{v \in S} \subset \Omega$ such that the matrix $(F_u(z_v))_{u,v}$ is non-singular. This implies that

$$\rho_\zeta(\gamma)(c_u) = c_{u\gamma}, \quad \forall \gamma \in \Gamma, \quad u \in S.$$

Moreover, if $u\gamma = u$ for some u, γ then $\gamma \in \Gamma(\mathfrak{p}) \subset \text{Ker}(\rho_\zeta)$ so that the datum of a vector $c_u \in \mathbb{C}_\infty^2$ determines an element of $M_w(\rho_\zeta; \mathbb{C}_\infty)$ and there is an injective linear map $\mathbb{C}_\infty^2 \rightarrow M_w(\rho_\zeta; \mathbb{C}_\infty)$. This shows that ev_ζ is not surjective. Similarly, if $|\Sigma| > 1$, one can show that the map ev_ζ defined over $M_w(\rho_\zeta^*; \mathbb{T}_\Sigma)$ with $\underline{\zeta}$ a vector of roots of unity is not surjective, apart from obvious trivial cases.

At this point, we would like to ask a question. The next definition prepares it.

Definition 5.15. Let \mathfrak{n} be a non-zero ideal of A , let g be a Drinfeld modular form of weight w for $\Gamma(\mathfrak{n})$. We say that g comes from a modular form for the full modular group if there exist: (1) a representation of the first kind $\rho : \Gamma \rightarrow \text{GL}_N(\mathbb{F}_q(t_\Sigma))$ and $\underline{\zeta} \in (\mathbb{F}_q^{ac})^\Sigma$ such that the evaluations $\text{ev}_\zeta(\rho(\gamma))$ are well defined for every $\gamma \in \Gamma$, and (2) an element $f = {}^t(f_1, \dots, f_N) \in M_w(\rho; \mathbb{L}_\Sigma)$ such that $g = \text{ev}_\zeta(f_i)$ for some $i \in \{1, \dots, N\}$.

Question 5.16. Compute the \mathbb{C}_∞ -span in $M_w(\Gamma(\mathfrak{n}); \mathbf{1}; \mathbb{C}_\infty)$ of the modular forms which come from modular forms for the full modular group. For which \mathfrak{n} and w do we obtain the whole space?

5.3. Poincaré series. We construct explicit examples of modular forms in our generalized setting. We are mainly concerned with a class of matrix-valued Poincaré series.

We consider a representation of the first kind

$$\Gamma \xrightarrow{\rho} \text{GL}_N(\mathbb{F}_q(t_\Sigma)),$$

of degree l . Let w be an integer and, with $L = \mathbb{L}_\Sigma$ or $L = \mathbb{K}_\Sigma$, let $G : \Omega \rightarrow L_\Sigma^{N \times N}$ be a tempered matrix ρ -quasi-periodic matrix function of type m , following Definition 4.1. We shall keep these settings all along §5.3. We set, for $\gamma \in \Gamma$ and $z \in \Omega$:

$$S_\gamma(w, m; G)(z) = \det(\gamma)^m J_\gamma(z)^{-w} \rho(\gamma)^{-1} G(\gamma(z)) \rho \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 5.17. Let $\gamma, \gamma' \in \Gamma$ be in the same left coset modulo $H := \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma$. Then we have the equality $S_\gamma(w, m; G)(z) = S_{\gamma'}(w, m; G)(z)$. Moreover, for all $\delta \in \Gamma$,

$$S_\gamma(w, m; G)(\delta(z)) = \det(\delta)^{-m} J_\delta(z)^w \rho(\delta) S_{\gamma\delta}(w, m; G)(z) \rho \begin{pmatrix} \det(\delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We simplify the notation: $S_\gamma(w, m; G)(z) = S_\gamma(z)$. We prove the first property. Since H is the semidirect product of A by \mathbb{F}_q^\times , it suffices to show that: (1) for all $a \in A$, $S_{T_a\gamma}(z) = S_\gamma(z)$ and (2) for all $\nu \in \mathbb{F}_q^\times$, $S_{\delta\gamma}(z) = S_\gamma(z)$ if $\delta = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$. For (1), we observe,

by the properties of G , that

$$\begin{aligned} S_{T_a\gamma}(z) &= \det(T_a\gamma)^m J_{T_a\gamma}(z)^{-w} \rho(T_a\gamma)^{-1} G(T_a(\gamma(z))) \rho\left(\begin{smallmatrix} \det(T_a\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \det(\gamma)^m J_\gamma(z)^{-w} \rho(\gamma)^{-1} \rho(T_a)^{-1} \rho(T_a) G(\gamma(z)) \rho\left(\begin{smallmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= S_\gamma(z). \end{aligned}$$

For (2), we see, similarly, with $\delta = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$ (here we use that G has type m):

$$\begin{aligned} S_{\delta\gamma}(z) &= \det(\delta\gamma)^m J_{\delta\gamma}(z)^{-w} \rho(\delta\gamma)^{-1} G(\delta(\gamma(z))) \rho\left(\begin{smallmatrix} \det(\delta\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \det(\gamma)^m J_\gamma(z)^{-w} \rho(\gamma)^{-1} \rho(\delta)^{-1} \det(\delta)^m \det(\delta)^{-m} \rho(\delta) G(\gamma(z)) \rho(\delta)^{-1} \rho\left(\begin{smallmatrix} \det(\delta\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= S_\gamma(z). \end{aligned}$$

This completes the proof of the first part of the Lemma. For the second, observe, if $\gamma' = \gamma\delta$ with $\delta \in \Gamma$:

$$\begin{aligned} S_\gamma(\delta(z)) &= \det(\gamma)^m J_\gamma(\delta(z))^{-w} \rho(\gamma)^{-1} G(\gamma(\delta(z))) \rho\left(\begin{smallmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= \det(\delta)^{-m} \det(\gamma')^m J_\delta(z)^w J_{\gamma'}(z)^{-w} \rho(\gamma'\delta^{-1})^{-1} G(\gamma'(z)) \rho\left(\begin{smallmatrix} \det(\gamma') & 0 \\ 0 & 1 \end{smallmatrix}\right) \rho\left(\begin{smallmatrix} \det(\delta) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \\ &= \det(\delta)^{-m} J_\delta(z)^w \rho(\delta) S_{\gamma'}(z) \rho\left(\begin{smallmatrix} \det(\delta) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}. \end{aligned}$$

□

We consider the formal series (Poincaré series):

$$(5.6) \quad \mathcal{P}_w(G)(z) := \sum_{\gamma} S_\gamma(w, m; G)(z),$$

running over a complete set of representatives of $H \backslash \Gamma$. Note that, if well defined, this is a matrix function. Compare this with Bruinier's definition of Poincaré series in [11, §1.2, 1.3].

Two elements $\gamma, \gamma' \in \Gamma$ are in the same left coset modulo H if and only if they have the same second row and there exists a couple $(\alpha, \beta) \in A^{2 \times 1}$ with α, β relatively prime, such that

$$[\gamma']_1 = [(\alpha, \beta) \cdot \gamma]_1,$$

where $[\cdot]_1$ here denotes the first row. In particular, we can run the series (5.6), if convergent, over a set of representatives contained in $\mathrm{SL}_2(A)$, so that, with such a choice, the sum becomes:

$$\mathcal{P}_w(G)(z) = \sum_{\gamma} J_\gamma(z)^{-w} \rho(\gamma)^{-1} G(\gamma(z)).$$

We have the next property.

Proposition 5.18. *If the series $\mathcal{P}_w(G)(z)$ converges to an element of $\mathrm{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow L^{N \times N})$ then it satisfies, for all $z \in \Omega$ and $\gamma \in \Gamma$:*

$$\mathcal{P}_w(G)(\gamma(z)) = \det(\gamma)^{-m} J_\gamma(z)^w \rho(\gamma) \mathcal{P}_w(G)(z) \rho\left(\begin{smallmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}.$$

For each column f of $\mathcal{P}_w(G)$ there exists $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ such that

$$f(\delta(z)) = \det(\delta)^{i-m} J_\delta(z)^w \rho(\delta) f(z), \quad \forall z \in \Omega, \quad \delta \in \Gamma.$$

Proof. We assume that the series converges, giving rise to an element of $\text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow L^{N \times N})$. We note that for $\nu \in \mathbb{F}_q^\times$, $\rho\left(\begin{smallmatrix} \nu & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is diagonal in $\text{GL}_N(\mathbb{F}_q)$ and there are integers n_i with $i = 0, \dots, q-2$ such that $\sum_i n_i = N$ so that we can decompose

$$(5.7) \quad \mathcal{P}_w(G) = \bigoplus_{i=0}^{q-2} \mathcal{P}_w^{[i]}(G),$$

where $\mathcal{P}_w^{[i]}(G) : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times n_i}$ for all i , and

$$\mathcal{P}_w^{[i]}(G)(\delta(z)) = \det(\delta)^{i-m} J_\delta(z)^w \rho(\delta) \mathcal{P}_w^{[i]}(G), \quad \forall z \in \Omega, \quad \delta \in \Gamma, \quad i = 0, \dots, q-2.$$

□

In full generality (for any quasi-periodic function G), we do not have a good criterion of convergence for the series $\mathcal{P}_w(G)$. We discuss these series for two choices of G .

We will need the next Lemma in the book [32] of Gerritzen and van der Put.

Lemma 5.19. *There exists a complete set of representatives $\gamma_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \text{SL}_2(A)$ of $H \backslash \Gamma$ in which each matrix is of one of the following three types:*

- (1) $\gamma_{0,\mu} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$ with $\mu \in \mathbb{F}_q^\times$,
- (2) $\gamma_{\mu,\nu} = \begin{pmatrix} 0 & -\mu^{-1} \\ \mu & \nu \end{pmatrix}$ with $\mu \in \mathbb{F}_q^\times$ and $\nu \in \mathbb{F}_q$,
- (3) $\gamma_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in A$ such that $ad - bc = 1$, $|cd| > 1$, $|a| < |c|$, $|b| < |d|$.

We note that the first two sets are finite. Let us look at the corresponding extracted series in the series (5.6) defining $\mathcal{P}_w(G)$; we denote them by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ (in agreement with the order of the types in the above set of representatives), so that $\mathfrak{A}, \mathfrak{B}$ correspond to finite sums while \mathfrak{C} is an infinite sum. We set, with l the degree of ρ :

$$(5.8) \quad \epsilon(\rho) := \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} \rho\left(\begin{smallmatrix} \mu^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{F}_q^{N \times N},$$

a matrix that has a natural block decomposition induced by the way ρ is constructed in terms of basic representations. Note that this is also a diagonal matrix with entries in $\{-1, 0\}$. For the first sub-sum we have, in virtue of the fact that G is of type m (second equality) and that ρ is of degree l (third equality):

$$(5.9) \quad \begin{aligned} \mathfrak{A} &:= \sum_{\mu \in \mathbb{F}_q^\times} S_{\gamma_{0,\mu}}(z) = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{-w} \rho\left(\begin{smallmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{smallmatrix}\right)^{-1} G(\mu^{-2}z) \\ &= \sum_{\mu \in \mathbb{F}_q^\times} \mu^{-w} \rho\left(\begin{smallmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{smallmatrix}\right)^{-1} \mu^{2m} \rho\left(\begin{smallmatrix} \mu^{-2} & 0 \\ 0 & 1 \end{smallmatrix}\right) G(z) \rho\left(\begin{smallmatrix} \mu^{-2} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \\ &= G(z) \epsilon(\rho). \end{aligned}$$

For the second sub-sum we have, similarly:

$$(5.10) \quad \mathfrak{B} := \sum_{\substack{\mu \in \mathbb{F}_q^\times \\ \nu \in \mathbb{F}_q}} S_{\gamma_{\mu, \nu}}(z) =$$

$$(5.11) \quad = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} \sum_{\nu \in \mathbb{F}_q} \left(z + \frac{\nu}{\mu} \right)^{-w} \rho \left(\begin{smallmatrix} \frac{\nu}{\mu} & -1 \\ 1 & 0 \end{smallmatrix} \right) G \left(\frac{-1}{z + \frac{\nu}{\mu}} \right) \rho \left(\begin{smallmatrix} \mu^2 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ = (-1)^m \left(\sum_{\beta \in \mathbb{F}_q} (z + \beta)^{-w} \rho \left(\begin{smallmatrix} -\beta & 1 \\ 1 & 0 \end{smallmatrix} \right) G \left(\frac{1}{z + \beta} \right) \rho \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \epsilon(\rho).$$

We easily deduce that $\mathfrak{A} + \mathfrak{B} \in \text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow L^{N \times N})$. We now make explicit choices for G .

5.3.1. The case of G entire. We suppose that $G \in \mathcal{QP}_m^1(\rho; \mathbb{K}_\Sigma)$ extends to an entire function $\mathbb{C}_\infty \rightarrow L^{N \times N}$, where L is a field extension of \mathbb{C}_∞ contained in \mathbb{K}_Σ . In this part we study $\mathcal{P}_w(G)$ with $w \in \mathbb{Z}$, $w > 0$. Let $M \geq 0$ be such that $\|G(z)u(z)^M\|$ is bounded for $|u(z)| < c$ for some $c < 1$ (it exists as G is tempered).

Lemma 5.20. *There are three constants $c_1, c_2, c_3 \in |\mathbb{C}_\infty^\times|$ such that $c_1 \geq 1$ and $\eta \in \mathbb{Z}[\frac{1}{p}] \cap [0, M + 1[$ such that if $|z|_{\mathfrak{S}} \geq c_1$ then $\|G(z)\| \leq c_2 |e_C(z)|^\eta$ and if $|z|_{\mathfrak{S}} \leq c_1$ then $\|G(z)\| \leq c_3$.*

Proof. We recall that Φ_ρ , introduced at the end of §4.1, is entire (Proposition 4.8 (a)), ρ -quasi-periodic of type 0 (same proposition (b)) and that Φ_ρ^{-1} is entire (Corollary 4.9) and has its entries which are at once tame series of degrees in $[0, 1[\cap \mathbb{Z}[\frac{1}{p}] \cup \{-\infty\}]$. Then $G\Phi_\rho^{-1}$ is also entire and A -periodic. Therefore, by Proposition 4.2 (c), $G\Phi_\rho^{-1} \in L[e_C(z)]^{N \times N}$ and the degrees in e_C of the entries of this matrix function, well defined, are $\leq M$ while the matrix function itself is of type m . We deduce that

$$G \in L[e_C(z)]^{N \times N} \Phi_\rho.$$

By Proposition 3.27, there exist constants $c_1 \geq 1$ and c_2 with $c_1, c_2 \in |\mathbb{C}_\infty^\times|$, $\eta \in \mathbb{Z}[\frac{1}{p}] \cap [0, M + 1[$ such that if $|z|_{\mathfrak{S}} \geq c_1$, then $\|G(z)\| \leq c_2 |e_C(z)|^\eta$. Suppose now that $|z|_{\mathfrak{S}} \leq c_1$. There exists $\lambda \in K_\infty$ such that $|z - \lambda| = |z|_{\mathfrak{S}} \leq c_1$. We can write $\lambda = a + m$ with $a \in A$ and $m \in \frac{1}{\theta} \mathbb{F}_q[[\frac{1}{\theta}]]$. Then $|z - a| = |z - \lambda + m| \leq \max\{|z - \lambda|, |m|\} \leq c_1$ because $|m| < 1 \leq c_1$. Now, since $G(z)$ is ρ -quasi-periodic, $\|G(z)\| \leq \|G(z - a)\| \leq c_3$ for some constant $c_3 \in |\mathbb{C}_\infty^\times|$, because the entries of G are entire functions, hence bounded in the disk $D_{\mathbb{C}_\infty}(0, c_1)$. \square

Proposition 5.21. *Let w be a positive integer. If G is an entire tempered ρ -quasi-periodic function of type m the series defining $\mathcal{P}_w(G)$ converges to an element of $\text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow L^{N \times N})$ and the matrix functions $\mathcal{P}_w^{[i]}(G)$ defined in (5.7) are elements of $M_w^1(\rho \det^{i-m}; L)^{1 \times n_i}$ for i varying in $\mathbb{Z}/(q-1)\mathbb{Z}$. If the i -th block of $\epsilon(\rho)$ is non-zero, then the columns of $\mathcal{P}_w^{[i]}(G)$ are non-zero. Moreover, the matrix functions $h^{M+1} \mathcal{P}_w^{[i]}(G)$ are elements of $S_{w+(M+1)(q+1)}(\rho \det^{i-m-M-1}; L)^{1 \times n_i}$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in Γ , such that $c \neq 0$ and let us consider $z \in \Omega$. Then:

$$(5.12) \quad \gamma(z) = \frac{a}{c} - \frac{\det(\gamma)}{c(cz + d)}.$$

We consider $c_1 \in |\mathbb{C}_\infty^\times|$ such that $c_1 > 1$ and we consider $z \in \Omega$ such that $c_1^{-1} \leq |z|_{\mathfrak{S}} \leq |z| \leq c_1$. We note that if γ is of type (2) or (3) as in Lemma 5.19, then $|\gamma(z)| \leq c_1$. Since G has entire entries, we therefore get that the series defining $\mathcal{P}_w(G)$ converges uniformly over all the affinoid subdomains of Ω of the type $\{z \in \Omega : c_3 \leq |z|_{\mathfrak{S}} \leq |z| \leq c_4\}$ with $c_3, c_4 \in |\mathbb{C}_\infty^\times|$ hence defining an element of $\text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow \mathbb{L}_\Sigma^{N \times N})$. Now observe that if $|z|_{\mathfrak{S}} = |z| \rightarrow \infty$ and γ is of type (2) or (3), then $|\gamma(z)| \rightarrow 0$ uniformly on the set of representatives γ of $H \backslash \Gamma$ and therefore, the sum $\mathfrak{B} + \mathfrak{C}$, as a function of the variable z , is bounded as $|z|_{\mathfrak{S}} = |z| \rightarrow \infty$. By Lemma 5.20 and the expression we found for \mathfrak{A} , we therefore have that $\mathcal{P}_w(G)$ is tempered, because for $|z|_{\mathfrak{S}}$ large enough, $\|\mathcal{P}_w(G)(z)\| = \|G(z)\epsilon(\rho)\|$. More precisely, $|e_C(z)|^{-\eta} \|G(z)\|$ is bounded as $|z|_{\mathfrak{S}} = |z| \rightarrow \infty$ where η is given in Lemma 5.20. Thanks to Proposition 5.18, this suffices to show that $\mathcal{P}_w(G)$ has its columns in $M_w^1(\rho \det^{i-m}; L)$. If $\epsilon(\rho)$ does not vanish identically, looking at the blocks which are not zero we deduce the properties regarding $\mathcal{P}_w^{[i]}(G)$. The last assertion of the proposition is verified by noticing that $\|G(z)u(z)^\eta\| \rightarrow 0$ as $|z|_{\mathfrak{S}} = |z| \rightarrow \infty$, and $0 \leq \eta < M + 1$. Therefore $h^{M+1}\mathcal{P}_w(G)$ vanishes at infinity because $v(h) = 1$. \square

Corollary 5.22. *If $G = \Phi_\rho$ and $\epsilon(\rho) \neq 0$ then there exists i such that*

$$h\mathcal{P}_w^{[i]}(G) \in \left(S_{w+q+1}(\rho \det^{i-1}; L) \setminus \{0\} \right)^{1 \times n_i}.$$

5.3.2. *The case $G = \Psi_m(\rho)$.* With $m \geq 1$, we study $\mathcal{P}_m(G)$ where:

$$(5.13) \quad G = \Psi_m(\rho) = \sum_{a \in A} \frac{1}{(z-a)^m} \rho(T_a).$$

The functions $\Psi_m(\rho)$ have been introduced in §4.1.1. By Lemma 4.3 we have $\Psi_m(\rho) \in \mathcal{QP}_m(\rho; \mathbb{L}_\Sigma)$. If $\rho = \mathbf{1} : \Gamma \rightarrow \{1\}$ we recover the (scalar) sums $S_{m,\Lambda}$ for the lattice $\Lambda = A$ (see [35, §6] and [24, §3]). In particular, for any $m \geq 1$ there exists a polynomial $G_m \in K[X]$ (called the *Goss' polynomial* of order m) such that

$$(5.14) \quad G = \tilde{\pi}^m G_m(u).$$

The Goss' polynomials G_m can be computed inductively by using the generating series:

$$(5.15) \quad \sum_{m \geq 1} G_m(u) X^m = \frac{uX}{1 - u \exp_C(X)}.$$

See [24, (3.6)], [26], and [53, Theorem 3.2], [31] for more recent results on these polynomials. See also our Lemma 6.5.

The next result holds:

Proposition 5.23. *Let us consider $w, m \in \mathbb{N}^*$. If $G = \Psi_m(\rho)$, the columns of $\mathcal{P}_w(G)$ are in $S_w(\rho \det^{-j}; \mathbb{L}_\Sigma)$ with j varying in $\mathbb{Z}/(q-1)\mathbb{Z}$.*

Proof. It suffices to show that the sum defining $\mathcal{P}_w(G)$ is uniformly convergent on affinoid subdomains of Ω of the type $\mathcal{C} := \{z \in \Omega : c_1^{-1} \leq |z|_{\mathfrak{S}} \leq |z| \leq c_1\}$ with $c_1 \in |\mathbb{C}_{\infty}^{\times}|$ such that $c_1 > 1$. For this, we use the decomposition $\mathcal{P}_w(G) = \mathfrak{A} + \mathfrak{B} + \mathfrak{C}$. We need to show that the series \mathfrak{C} converges uniformly over \mathcal{C} . We note that if $\gamma = \gamma_{c,d}$ is of type (3) as in Lemma 5.19, then if $z \in \mathcal{C}$ we get $|\gamma(z)| \leq c_1$. In fact, we have $\gamma(z) \rightarrow 0$ by (5.12) as γ varies in the chosen representative set of $H \backslash \Gamma$ and $\gamma(\mathcal{C}) \subset D_{\mathbb{C}_{\infty}}(0, |\theta|^{-1}) \cap \Omega$ for all but finitely many γ . If we denote by \mathcal{E} the set of such homographies, we get $\|G(\gamma(z))\| \leq |z|^{-m}$ for all $z \in \mathcal{C}$ and for all $\gamma \in \mathcal{E}$. Therefore we can decompose $\mathfrak{C} = \mathfrak{C}_0 + \mathfrak{C}_1$ where \mathfrak{C}_1 is a finite sum of holomorphic functions and $\mathfrak{C}_0 = \sum_{\gamma \in \mathcal{E}} S_{\gamma}(G)(z)$ which converges uniformly on \mathcal{C} in virtue of the fact that $w > 0$. We deduce that $\mathcal{P}_w(G)$ defines a holomorphic function over Ω , with values in $\mathbb{L}_{\Sigma}^{N \times N}$. Since moreover, $\|G(z)\| \rightarrow 0$ as $|z| = |z|_{\mathfrak{S}} \rightarrow \infty$, we see that the columns of $\mathcal{P}_w(G)$ are cusp forms. \square

Giving sufficiently general conditions for the non-vanishing of $\mathcal{P}_w(G)$ is more difficult in the case $G = \Psi_m(\rho)$. We have the next proposition:

Proposition 5.24. *Assuming that m, w are two positive integers such that $w > 2m$, if $G = \Psi_m(\rho)$ and $\epsilon(\rho) \neq 0$, then $\mathcal{P}_w(G)$ has a non-zero column in $S_w(\rho \det^{-i}; \mathbb{L}_{\Sigma})$ for some i .*

Proof. We need to analyze the various subsums $\mathfrak{A}, \mathfrak{B}$ and \mathfrak{C} of $\mathcal{P}_w(G)$ that we know being convergent series, by Proposition 5.23. We begin by studying the subsum \mathfrak{A} . Note that $\rho(T_a) - I_N$ is a nilpotent matrix having zeroes in the diagonal for all $a \in A$. The diagonal of $G = \Psi_m(\rho)$ is equal to $I_N \sum_{a \in A} (z - a)^{-m}$ and the hypothesis on $\epsilon(\rho)$ implies that $G\epsilon(\rho)$ has some non-zero coefficients on the diagonal of valuation $|\cdot|$ equal to $|\Psi_m(\mathbf{1})|$. By Lemma 5.26 (proof postponed to §5.3.5) there exists

$$\kappa_1 \in]1, |\theta|[\cap |\mathbb{C}_{\infty}^{\times}|$$

and a non-negative integer ω_2 such that if $\kappa_1 < |z| < |\theta|$, then $\Psi_m(z) = |\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2}$. We deduce that

$$(5.16) \quad \|\mathfrak{A}(z)\| = |\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2}, \quad \kappa_2 < z < |\theta|.$$

We now study the subsum \mathfrak{B} . To do this, we assume that $|z| > 1$. By (5.11) and the definition of G :

$$(5.17) \quad \mathfrak{B} = \underbrace{\left(\sum_{\beta \in \mathbb{F}_q} (z + \beta)^{m-w} \rho \begin{pmatrix} -\beta & 1 \\ -1 & 0 \end{pmatrix} \right)}_{\mathfrak{B}_0} + \underbrace{\sum'_{a \in A} \sum_{\beta \in \mathbb{F}_q} \frac{(z + \beta)^{m-w}}{(1 - a(z + \beta))^m} \rho \begin{pmatrix} -\beta & 1 \\ 1 & 0 \end{pmatrix} \rho(T_a) \rho \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathfrak{B}_1} \epsilon(\rho),$$

where the sum is split in two pieces, the first sum corresponding to $a = 0$, while the dash ' on the second sum designates the term corresponding to $a = 0$ omitted. If $a \neq 0$ we

get $|1 - a(z + \beta)| = |a||z| \geq |z| = |z + \beta|$ and therefore, $\|\mathfrak{B}_1(z)\| \leq |z|^{-w}$ for $1 < |z|$. As for \mathfrak{B}_0 , we see that

$$\mathfrak{B}_0 = \sum_{\beta \in \mathbb{F}_q} (z + \beta)^{m-w} \rho \begin{pmatrix} -\beta & 1 \\ -1 & 0 \end{pmatrix} \epsilon(\rho).$$

Hence, $\|\mathfrak{B}_0(z)\| \leq |z|^{m-w}$ again for $1 < |z|$. Thus,

$$(5.18) \quad \|\mathfrak{B}(z)\| \leq |z|^{m-w}, \quad 1 < |z|.$$

It remains to handle the subsum \mathfrak{C} and we consider, for this purpose, $z \in \Omega$ such that $1 < |z|$ and $|z| \notin |\theta|^{\mathbb{Z}}$. Suppose that $\gamma = \gamma_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is of type (3) as in Lemma 5.19. We notice that $|az+b| < |cz+d|$. This follows easily from the conditions on a, b, c, d determining the type (3) and the fact that $|az+b| = \max\{|az|, |b|\}$ and $|cz+d| = \max\{|cz|, |d|\}$ because $|z| \notin |\theta|^{\mathbb{Z}}$.

Then

$$S_\gamma(G) = \tilde{\pi}^{-m} J_\gamma(z)^{-w} \rho(\gamma) \sum_{\tilde{b} \in A} (\gamma(z) - \tilde{b})^{-m} \rho(T_{\tilde{b}}) \rho \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix}.$$

One sees easily that

$$(\gamma(z) - \tilde{b})^{-m} = \frac{J_\gamma(z)^m}{(az + b - \tilde{b}J_\gamma(z))^m}.$$

Note that $az + b - \tilde{b}J_\gamma(z) = z(a - \tilde{b}c) + b - \tilde{b}d$ so that, if $\tilde{b} \neq 0$, $|az + b - \tilde{b}J_\gamma(z)| = \max\{|z||a - \tilde{b}c|, |b - \tilde{b}d|\} = \max\{|z||c|, |d|\}|\tilde{b}| = |\tilde{b}||J_\gamma(z)|$. Hence $\tilde{b} \neq 0$ implies that $|(\gamma(z) - \tilde{b})^{-m}| \leq 1$. If $\tilde{b} = 0$, since $|az + b| < |cz + d|$, we get $|\gamma(z)|^{-m} \leq |J_\gamma(z)|^m$. Therefore, we deduce that $\|S_\gamma(G)\| \leq |J_\gamma(z)|^{m-w}$ for γ of type (3) and we can conclude that

$$(5.19) \quad \|\mathfrak{C}(z)\| \leq |z|^{m-w}, \quad \text{if } 1 < |z|, \quad |z| \notin |\theta|^{\mathbb{Z}}.$$

Assuming by contradiction that $\mathcal{P}_w(G)$ vanishes identically, we have that $\mathfrak{A} = -(\mathfrak{B} + \mathfrak{C})$. Looking at Lemma 5.26 we observe that $|z| > |\theta|^{\frac{\omega_2+m}{\omega_2+w-m}}$ if and only if $|z|^{\omega_2+w-m} > |\theta|^{\omega_2+m}$, equivalent to $|\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2} > |z|^{m-w}$. But

$$\frac{\omega_2 + m}{\omega_2 + w - m} = 1 - \frac{w - 2m}{\omega_2 + w - m}$$

and the hypothesis $w > 2m$ ensures that there exists $\kappa_2 \in]1, |\theta|[|\cap|\mathbb{C}_\infty^\times|$ such that for all $z \in \Omega$ such that $\kappa_2 < |z| < |\theta|$,

$$\|\mathfrak{A}(z)\| \geq |\Psi_m(z)| = |\Psi_m^\geq(z)| = |\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2} > |z|^{m-w} \geq \|\mathfrak{B}(z) + \mathfrak{C}(z)\|,$$

by (5.16), (5.18) and (5.19) (more precisely, a non-zero column of \mathfrak{A} has an entry which has $\|\cdot\|$ equal to $|\Psi_m(z)|$). This is impossible. Hence $\mathcal{P}_w(G)$ does not vanish identically. \square

5.3.3. *Example: Poincaré series in a class introduced by Gekeler.* We consider the case $N = 1$, $\rho = \mathbf{1}$, we choose $G(z) = G_m(u) = \tilde{\pi}^{-m} \Psi_m(\mathbf{1})$ the Goss' polynomial (in u) of order m with $m > 0$. Then, we see that $\epsilon(\rho) = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w}$ which is non-zero if and only if $w \equiv 2m \pmod{q-1}$. We therefore reach the next result.

Corollary 5.25. *If $w \equiv 2m \pmod{q-1}$ and $w > 2m$ then, with $G(z) = G_m(u)$, the Poincaré series $\mathcal{P}_w(G)$ determines a non-zero element of $S_w(\det^{-m}; \mathbb{C}_\infty)$.*

Note that in [68, Remark 4.1] the condition on $w \equiv 2m \pmod{q-1}$ is stronger: $w \geq (q+1)m$. See also [32, pp. 304-307] and [24, §(5.11)]. However, we do not get new scalar Poincaré forms. For instance, if $w \equiv 2m \pmod{q-1}$ and $w > 2m$ the minimal data is given by $m = 1$ and $w = q+1$. If we take $w > 2m$, $m \in \{1, \dots, q\}$, $w \equiv 2m \pmod{q-1}$ and $G = u^m$, we see that

$$\mathcal{P}_w(G) = \sum_{\gamma \in H \backslash \Gamma} \det(\gamma)^m J_\gamma^{-w} u(\gamma(z)) = P_{w,m} \in S_w(\det^{-m}; \mathbb{C}_\infty)$$

in the notations of Gekeler, [24, (5.11)]. We recall that if $w = q+1$ and $m = 1$, then $h = P_{q+1,1}$.

5.3.4. *Example: Poincaré series of weight three.* We consider $\rho = \rho_\Sigma^*$ which is of degree $s = |\Sigma|$, where $\Sigma \subset \mathbb{N}^*$. We suppose that $s \equiv 1 \pmod{q-1}$. A simple computation shows that

$$\rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} = \text{Diag}(\nu^{-s}, \dots, \nu^{-n_1}, \nu^{-n_0})$$

where the integer sequence $(n_i)_{i \geq 0}$ does not depend on s and coincides with the so-called one-counting sequence, that is, the sequence $(n_i)_{i \geq 0}$ which gives the number of one's in the binary expansion of i . The degree of ρ_Σ^* is s . We also consider integers $w, m > 0$ such that $w > 2m$ and we set $r = 2m - w + s$. If $s = |\Sigma| \equiv 1 \pmod{q-1}$ then the smallest parameters allowable in the above construction of a non-zero Poincaré series as above are $w = 3$ and $m = 1$. We note that the last column of $\epsilon(\rho)$, defined in (5.8), is the opposite of the last element of the canonical basis of $\mathbb{F}_q^{N \times 1}$. Hence the last column of $\mathcal{P}_3(G)$ where $G = \Psi_1(\rho_\Sigma^*)$ is an element of $S_3(\rho_\Sigma^* \det^{-1}; \mathbb{L}_\Sigma) \setminus \{0\}$. Explicitly we have, with a choice of representatives of $H \backslash \Gamma$ in $\text{SL}_2(A)$:

$$(5.20) \quad \mathcal{P}_3(G) = \sum_{\gamma} J_\gamma(z)^3 ({}^t \rho_\Sigma)(\gamma) G(\gamma(z)), \quad G(z) = \sum_{a \in A} (z-a)^{-1} \rho_\Sigma^*(T_a).$$

Note that if $\Sigma = \emptyset$ then $N = 1$, $\rho = \mathbf{1}$ and $G = \tilde{\pi}u$. we have $|\Sigma| \equiv 1 \pmod{q-1}$ if and only if $q = 2$. In this case

$$\mathcal{P}_3(G) = \tilde{\pi} \sum_{\gamma} J_\gamma(z)^{-3} u(\gamma(z)) = \tilde{\pi}h,$$

that is, a non-zero multiple of Gekeler's function h that has weight 3 in this case.

5.3.5. *Growth in annuli.* This section is mainly devoted to the proof of Lemma 5.26, only used in the proof of Proposition 5.24. We suppose that $B \subset \mathbb{K}_\Sigma$. We study the series $\Psi_m(\rho)$ in the annuli $\mathcal{C}_0 = \{z \in \mathbb{C}_\infty : 0 < |z| < 1\}$ and $\mathcal{C}_n = \{z \in \mathbb{C}_\infty : |\theta|^{n-1} < |z| < |\theta|^n\}$. The representation ρ being fixed, we now write Ψ_m or $\Psi_m(z)$, instead of $\Psi_m(\rho)$. We also write:

$$\begin{aligned}\Psi_m^<(z) &= I_N z^{-m} \quad \text{if } n = 0, \\ &= \sum_{\substack{a \in A \\ |a| < |\theta|^n}} (z - a)^{-m} \rho(T_a) \quad \text{if } n > 0, \\ \Psi_m^{\geq}(z) &= \sum_{\substack{a \in A \\ |a| \geq |\theta|^n}} (z - a)^{-m} \rho(T_a) \quad \text{for all } n.\end{aligned}$$

Note that $\Psi_m^< \in \mathbb{K}_\Sigma(z)^{N \times N}$ and that $\Psi_m = \Psi_m^< + \Psi_m^{\geq}$. Also, if \mathcal{D}_k denotes the higher divided derivative of order k in the variable z applied coefficientwise (the operator defined by $\mathcal{D}_k(z^m) = \binom{m}{k} z^{m-k}$), we have

$$(5.21) \quad \Psi_m = (-1)^{m-1} \mathcal{D}_{m-1}(\Psi_1) = (-1)^{m-1} \mathcal{D}_{m-1}(\Psi_1^<) + (-1)^{m-1} \mathcal{D}_{m-1}(\Psi_1^{\geq}).$$

We note that if $a \in A$ is such that $|a| < |\theta|^n$ then $|\frac{a}{z}| < 1$ and

$$\frac{1}{z - a} = \frac{1}{z} \frac{1}{1 - \frac{a}{z}} = z^{-1} \left(1 + \sum_{i \geq 0} \left(\frac{a}{z} \right)^i \right).$$

Hence, we get

$$(5.22) \quad \Psi_1^<(z) = z^{-1} \left(\sum_{i \geq 0} H_{-i}(\rho) z^{-i} \right), \quad |z| > |\theta|^{n-1},$$

where

$$H_{-i}(\rho) = \sum_{\substack{a \in A \\ |a| < |\theta|^n}} a^i \rho(T_a), \quad i \geq 0,$$

where we adopt the convention $a^0 = 1$ including when $a = 0$, so that $H_0(\rho) = \sum_{|a| < |\theta|^n} \rho(T_a)$.

Similarly, if $|a| \geq |\theta|^n$ then $|\frac{z}{a}| < 1$ and

$$\frac{1}{z - a} = -\frac{1}{a} \frac{1}{1 - \frac{z}{a}} = -\frac{1}{a} \sum_{i \geq 0} \left(\frac{z}{a} \right)^i = -z^{-1} \sum_{j \geq 1} \left(\frac{z}{a} \right)^j,$$

and we derive the expansion

$$(5.23) \quad \Psi_1^{\geq}(z) = -z^{-1} \sum_{j \geq 1} H_j(\rho) z^j, \quad |z| < |\theta|^n,$$

where

$$H_j(\rho) = - \sum_{\substack{a \in A \\ |a| \geq |\theta|^n}} a^{-j} \rho(T_a), \quad j \geq 1.$$

We deduce that

$$\Psi_1(z) = z^{-1} \sum_{i \in \mathbb{Z}} H_i(\rho) z^i, \quad z \in \mathcal{C}_n.$$

We suppose now that $\rho = \mathbf{1}$ so that $N = 1$ (this is the only case we need in the study of Poincaré series). We have the next technical result:

Lemma 5.26. *For any $m \geq 1$ there exists $\kappa_1 \in \mathbb{R}$ such that $1 < \kappa_1 < q$ two non-negative integers ω_1, ω_2 such that if $1 < |z| < \kappa_1$, then $|\Psi_m(z)| = |z|^{1-q-m-\omega_1}$ and if $\kappa_1 < |z| < |\theta|$, then $|\Psi_m(z)| = |\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2}$.*

Proof. Writing

$$S_d(i) = \sum_{a \in A^+(d)} a^{-i} \in K$$

,

$$S_{<d}(i) = \sum_{0 \leq k < d} S_k(i) \in K$$

and

$$\zeta_A(i) = \sum_{a \in A^+} a^{-i} \in K_\infty,$$

we get $H_i(\mathbf{1}) = 0$ if $q-1 \nmid i$, and if $q-1 \mid i$ then $H_i(\mathbf{1}) = -S_{<n}(-i)$ if $i < 0$, $H_0(\mathbf{1}) = 1$ if $n = 0$ and $H_0(\mathbf{1}) = 0$ if $n > 0$, and $H_i(\mathbf{1}) = \zeta_A(i) - S_{<n}(i) \in K_\infty$. If $n = 0$ we get $H_i(\mathbf{1}) = 0$ for all $i < 0$ and therefore we conclude with the identity:

$$(5.24) \quad \Psi_1(z) = \frac{1}{z} \left(1 + \sum_{\substack{j>0 \\ q-1 \mid j}} \zeta_A(j) z^j \right), \quad 0 < |z| < 1.$$

If we choose $n = 1$ and $z \in \mathcal{C}_1$ (i.e. $1 < |z| < |\theta|$) we get:

$$\Psi_1^<(z) = \sum_{\lambda \in \mathbb{F}_q} \frac{1}{z - \lambda} = \frac{-1}{z^q - z} = - \sum_{\substack{i \geq q \\ q-1 \mid i}} z^{-i}, \quad |z| > 1.$$

Similarly, we compute

$$\Psi_1^>(z) = z^{-1} \sum_{\substack{i \geq 1 \\ q-1 \mid i}} (\zeta_A(i) - 1) z^i, \quad |z| < |\theta|.$$

In other words, to construct the formal series which represents Ψ_1 on \mathcal{C}_1 it suffices to compute

$$\tilde{\Psi} = z^{-1} \sum_{\substack{j \in \mathbb{Z} \\ q-1 \mid j}} z^j,$$

where $\tilde{\Psi}$ is the formal series (5.24) which represents Ψ_1 on \mathcal{C}_0 (note that the second series is nowhere converging).

Since $\zeta_A(i) - 1 \equiv \theta^{-i} \pmod{\mathcal{M}_\infty^{i+1}}$ for all $i > 0$ where $\mathcal{M}_\infty = \frac{1}{\theta}\mathbb{F}_q[[\frac{1}{\theta}]]$ is the maximal ideal of K_∞ , we observe that the ∞ -adic Newton polygon of Ψ_1 over \mathcal{C}_1 has three slopes. If $z \in \mathcal{C}_1$, we have $|\Psi_1^<(z)| = |z|^{-q}$ and $|\Psi_1^>(z)| = |z|^{q-2}|\theta|^{1-q}$. We therefore have that $|\Psi_1^<(z)| = |\Psi_1^>(z)|$ if and only if $|z| = |\theta|^{\frac{1}{2}}$ and if $1 < |z| < |\theta|^{\frac{1}{2}}$ we have $|\Psi_1(z)| = |\Psi_1^<(z)| = |z|^{-q}$ while if $|\theta|^{\frac{1}{2}} < |z| < |\theta|$ we have $|\Psi_1(z)| = |\Psi_1^>(z)| = |z|^{q-2}|\theta|^{1-q}$. This yields the lemma in the case $m = 1$.

Using (5.21) and the fact that $(-1)^{m-1}\mathcal{D}_{m-1}(\Psi_1^<) = \Psi_m^<$ and $(-1)^{m-1}\mathcal{D}_{m-1}(\Psi_1^>) = \Psi_m^>$ we deduce that if $z \in \mathcal{C}_1$:

$$(5.25) \quad |\Psi_m^<(z)| = |z|^{1-q-m-\omega_1}, \quad |\Psi_m^>(z)| = |\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2},$$

where $-1 + q + m + \omega_1$ is the order of $\Psi_m^<(z)$ in z^{-1} with $\omega_1 \geq 0$, and $\omega_2 \geq 0$ is the order of $\Psi_m^>(z)$ in z . Indeed, the reader can easily verify that $\theta^m \mathcal{D}_{m-1}(\Psi_1^>) = \sum_k \alpha_k z^k$ where $|\alpha_k| = |\theta|^{-k}$ for all k . The computation of ω_1 and ω_2 and their dependence in m is a combinatorial problem which goes beyond our scopes but fortunately, irrelevant here. We see that the ∞ -adic Newton polygon has three slopes in this case too. Note that $|\Psi_m^<(z)| = |\Psi_m^>(z)|$ if and only if $|z|^{1-q-m-\omega_1} = |\theta|^{-m} \left| \frac{z}{\theta} \right|^{\omega_2}$ which is equivalent to $|z| = |\theta|^{\frac{m+\omega_2}{\omega_1+\omega_2+m+q-1}}$. Now,

$$\frac{m + \omega_2}{\omega_1 + \omega_2 + m + q - 1} = 1 - \frac{\omega_1 + q - 1}{\omega_1 + \omega_2 + m + q - 1} \in]0, 1[.$$

□

5.4. Eisenstein series. The process that leads to the construction of Eisenstein series is different from that of Poincaré series and delivers, in general, vector-valued modular forms rather than matrix-valued modular forms. We describe it in our particular setting but the discussion that follows easily generalizes to e.g. the case of vector-valued modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$ etc. Let ρ be a representation

$$\Gamma \xrightarrow{\rho} \mathrm{GL}_N(B),$$

with $(B, |\cdot|_B)$ a countably cartesian Banach \mathbb{C}_∞ -algebra. Suppose that there is a map

$$(5.26) \quad A^{1 \times 2} \xrightarrow{\mu} B^{N \times 1}$$

such that for all $\gamma \in \Gamma$, if $(a, b)\gamma = (a', b')$ in $A^{1 \times 2}$, then

$${}^t \rho(\gamma) \mu(a, b) = \mu(a', b').$$

Assume further that the image of μ is bounded, that is, there is $c_1 > 0$ such that $|\mu(a, b)|_B \leq c_1$ for all $a, b \in A$. Then, for all $w > 0$, the series

$$\mathcal{E} = \sum'_{a, b \in A} (az + b)^{-w} \mu(a, b)$$

(where the dash ' indicates that the term corresponding to $a = b = 0$ is omitted) converges to a rigid analytic map

$$\Omega \rightarrow B^{N \times 1}$$

and moreover:

Lemma 5.27. *We have that $\mathcal{E} \in M_w(\rho^*; B)$. If $\sum_{b \in A \setminus \{0\}} b^{-w} \mu(0, b)$ is non-zero, then \mathcal{E} does not vanish identically.*

Proof. We consider $\gamma \in \Gamma$. Then:

$$\begin{aligned} \mathcal{E}_w(\rho; \mu) &= J_\gamma(z)^w \sum'_{a, b \in A} \left((a, b) \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^{-w} \mu(a, b) \\ &= J_\gamma(z)^w \sum'_{a', b' \in A} \left((a', b') \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^{-w} \rho^*(\gamma) \mu(a', b') \\ &= J_\gamma(z)^w \rho^*(\gamma) \mathcal{E}_w(\rho; \mu). \end{aligned}$$

Since $|(az + b)^{-w} \mu(a, b)|_B$ tends to zero, we easily conclude that $\mathcal{E} \in M_w(\rho^*; B)$ and the non-vanishing condition is clear. \square

Definition 5.28. We call the function \mathcal{E} of Lemma 5.27 the *Eisenstein series of weight w associated with the data (ρ^*, μ)* and we denote it by $\mathcal{E}_w(\rho^*; \mu)$ or more simply $\mathcal{E}_w(\rho^*)$ when the reference to μ is understood.

Although we can always associate Poincaré series to representations of the first kind ρ (it follows from Proposition 5.24 that for any representation of the first kind ρ there exists $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $w > 0$ such that a column of a Poincaré series constructed there defines a non-zero element of $M_w(\rho)$) not every representation ρ can be enriched by a map μ as above. The reader can check that if ρ is a representation of the first kind that can be constructed by starting from basic representations by using only the elementary operations $\oplus, \otimes, S^m, \wedge^m$ (so the operation $(\cdot)^*$ is omitted) then maps like μ exist which are not zero and Lemma 5.27 can be applied to construct non-zero Eisenstein series in $M_w(\rho^*)$ for certain $w > 0$. In this paper Eisenstein series will be studied in depth for specific choices of ρ only. Namely, we will study, in §7, Eisenstein series associated to the representation ρ_Σ^* with Σ a finite subset of \mathbb{N}^* .

6. DIFFERENTIAL OPERATORS ON MODULAR FORMS, PERKINS' SERIES

A classic feature of modular forms for the group $SL_2(\mathbb{Z})$ is the existence of differential operators acting homogeneously on them (sending families of modular forms to modular forms). For instance, one can mention the so-called Serre's derivatives, Rankin-Cohen brackets etc. For scalar Drinfeld modular forms associated to the characters \det^{-m} , similar structures exist and have been investigated (see [13, 14, 53]). Here we describe the natural extension of Serre's derivatives over the Drinfeld modular forms for a representation of the first kind. In order to justify the existence of such operators, we need to first show that divided derivatives leave the fields of uniformizers invariant.

In this section (see §6.4) we will also apply our results on quasi-periodic functions and higher derivatives to determine, in Theorems 6.12 and 6.16, the v -valuations of certain series introduced by Perkins in his Thesis [64], which turn out to be related to tame series. Perkins noticed that these series play a singular role in series expansions of Eisenstein series (see §7).

All along this section, we consider the divided higher derivatives:

$$\mathcal{D}_m(z^n) = \binom{n}{m} z^{n-m}, \quad n, m \in \mathbb{N}.$$

We choose $(B, |\cdot|_B)$ a Banach L -algebra which is countably cartesian in the sense of Definition 2.4. For all $n \geq 0$, \mathcal{D}_n defines a B -linear endomorphism of $\mathcal{O}_{\mathbb{A}_{\mathbb{C}_\infty}^1/B}$. Note that these operators satisfy Leibnitz rule

$$\mathcal{D}_n(fg) = \sum_{i+j=n} \mathcal{D}_i(f)\mathcal{D}_j(g),$$

for f, g analytic functions. To handle divided derivatives it is convenient to introduce the following map, where x is an indeterminate and where $\underline{\mathcal{D}}$ denotes the family of operators $(\mathcal{D}_n)_{n \geq 0}$ (Taylor's map):

$$\mathcal{T}_{\underline{\mathcal{D}},x} : \mathcal{O}_{\mathbb{A}_{\mathbb{C}_\infty}^1/B} \rightarrow \mathcal{O}_{\mathbb{A}_{\mathbb{C}_\infty}^1/B}[[x]], \quad \mathcal{T}_{\underline{\mathcal{D}},x}(f) = \sum_{i \geq 0} \mathcal{D}_i(f)x^i.$$

Then, $\mathcal{T}_{\underline{\mathcal{D}},x}$ induces B -algebra morphisms at the level of the sections, and Leibnitz rule is equivalent to the multiplicativity $\mathcal{T}_{\underline{\mathcal{D}},x}(fg) = \mathcal{T}_{\underline{\mathcal{D}},x}(f)\mathcal{T}_{\underline{\mathcal{D}},x}(g)$. Let Y be an affinoid subdomain of $\mathbb{A}_{\mathbb{C}_\infty}^1/B$, $z \in Y$ and $x_0 \in \mathbb{C}_\infty$ such that $z + x_0 \in Y$. If $f \in \mathcal{O}_{\mathbb{A}_{\mathbb{C}_\infty}^1/B}$ then $\mathcal{T}_{\underline{\mathcal{D}},x}(f)_{x=x_0} = f(z + x_0)$. If x, y are two indeterminates, we therefore have

$$\mathcal{T}_{\underline{\mathcal{D}},x}(\mathcal{T}_{\underline{\mathcal{D}},y}(f)) = \mathcal{T}_{\underline{\mathcal{D}},x+y}(f).$$

This implies that the family of higher derivatives $\underline{\mathcal{D}}$ is *iterative*:

$$\mathcal{D}_{m+n} = \binom{m+n}{m} \mathcal{D}_m \circ \mathcal{D}_n = \binom{m+n}{n} \mathcal{D}_n \circ \mathcal{D}_m,$$

for all $m, n \geq 0$. By an application of Lucas' formula, if $n = n_0 + n_1q + \dots + n_rq^r \in \mathbb{N}$ with $n_0, \dots, n_r \in \{0, \dots, q-1\}$, we have the identity

$$\mathcal{D}_n = \mathcal{D}_{n_0} \circ \mathcal{D}_{n_1q} \circ \dots \circ \mathcal{D}_{n_rq^r},$$

and the operators $\mathcal{D}_{n_iq^i}$ mutually commute, for $i = 0, \dots, r$.

6.1. Higher derivatives on tame series. We show that tame series are closed under higher derivations. The main result of this subsection is Proposition 6.2 but we also present some auxiliary properties that can be of interest for the reader willing to do computations. Let Σ be a finite subset of \mathbb{N}^* with s elements. Let $m \geq 0$ be the unique integer such that $(m-1)(q-1) + 1 \leq s \leq m(q-1)$. If $s = 0$ then $m = 0$. Let l be the unique integer with $s = (m-1)(q-1) + l$ (so that $1 \leq l \leq q-1$ and if $s = m = 0$, then $l = q-1$). We set:

$$(6.1) \quad M_s = e_1^{q-1} \dots e_{m-1}^{q-1} e_m^l \in \mathbb{F}_q^\circ \langle\langle e \rangle\rangle^b$$

(note that we can define the B -module $B^\circ\langle\langle e \rangle\rangle^b$ for any \mathbb{F}_q -algebra B). We clearly have, by the fact that $s = (m-1)(q-1) + l$:

$$(6.2) \quad \lambda(M_s) = s, \quad w(M_s) = (q-1) \sum_{i=1}^{m-1} \frac{1}{q^i} + \frac{l}{q^m} =: w_{\max}(s).$$

We set, for B as in §3.2, $B^\circ\langle\langle e \rangle\rangle_s^b = B^\circ\langle\langle e \rangle\rangle^b \cap B\langle\langle e \rangle\rangle_s^b$ (recall the graduation by depths (3.10) and λ in Definition 3.5). We have the direct sum of B -modules $B^\circ\langle\langle e \rangle\rangle^b = \bigoplus_{s \geq 0} B^\circ\langle\langle e \rangle\rangle_s^b$. We call M_s the *maximal tame monomial*, a terminology which is motivated by the following result which tells us that in the homogeneous module $B^\circ\langle\langle e \rangle\rangle_s^b$, M_s has maximal weight (the proof is easy and left to the reader).

Lemma 6.1. *For all $f \in \mathbb{K}_\Sigma^\circ\langle\langle e \rangle\rangle_s^b$ we have $w(f) \leq w_{\max}(s)$.*

We have the next rather straightforward result, where w_{\max} has been defined in (6.2) (recall that if $f \in B\langle\langle e \rangle\rangle^b$ then $f^{[i]}$ is the projection of f on $B\langle\langle e \rangle\rangle_i^b$ of (3.11)), and where we suppose that $K(\tilde{\pi}) \subset B$:

Proposition 6.2. *The following properties hold. (1) The operators $(\mathcal{D}_i)_{i \geq 0}$ induce B -linear endomorphisms of $B^\circ\langle\langle e \rangle\rangle^b, B\langle\langle e \rangle\rangle^b, B\langle\langle e \rangle\rangle^b[[u]]$. (2) If $f = \sum_i f^{[i]} \in B\langle\langle e \rangle\rangle^b$ is of depth $\leq L$ we have, for all $n \geq 1$:*

$$\mathcal{D}_n(f) = \sum_{L \geq i \geq \ell_q(n)} \mathcal{D}_n(f^{[i]}).$$

(3) For all $n \geq 0$ and for all $f \in B^\circ\langle\langle e \rangle\rangle^b$ of depth $\leq s$, $\mathcal{D}_n(f) \in B^\circ\langle\langle e \rangle\rangle^b$ is of depth $\leq s - \ell_q(n)$ and of weight $\leq w_{\max}(s - \ell_q(n))$. (4) We have the commutation rules

$$(6.3) \quad \mathcal{D}_n \tau = \begin{cases} 0 & \text{if } q \nmid n \\ \tau \mathcal{D}_{\frac{n}{q}} & \text{if } q \mid n, \end{cases} \quad n \geq 1.$$

Sketch of proof. If $M \in B\langle\langle e \rangle\rangle_s^b$ is a tame monomial of depth s (as in §3.2.1), then $\mathcal{D}_n(M)$ is a tame polynomial, and

$$\mathcal{D}_n(M) \in \bigoplus_{i \geq 0} B\langle\langle e \rangle\rangle_{s - \ell_q(n) - i(q-1)}^b.$$

To see this consider more generally, for $i \in U$ with U a finite subset of \mathbb{N}^* of cardinality s , \mathbb{F}_q -linear functions $f_i \in \text{Hol}(\mathbb{C}_\infty \rightarrow B)$, so that we can write

$$f_i = \sum_{j \geq 0} f_{i,q^j} z^{q^j}, \quad f_{i,q^j} \in B, \quad i \in U.$$

By Leibnitz rule we have for $n \geq 0$:

$$\mathcal{D}_n \left(\prod_{i \in U} f_i \right) = \sum_{i_1 + \dots + i_s = n} \prod_{k \in U} \mathcal{D}_{i_k}(f_k).$$

By \mathbb{F}_q -linearity we have that $\mathcal{D}_k(f_i) = f_i$ if $k = 0$, $\mathcal{D}_k(f_i) = f_{i,q^j}$ if $k = q^j$ with $j \in \mathbb{N}$, and 0 otherwise. Hence, setting $f_{i,0} := f_i$, we can write:

$$(6.4) \quad \mathcal{D}_n \left(\prod_{i \in U} f_i \right) = \sum_{\substack{i_1 + \dots + i_s = n \\ i_k \in \{0\} \cup q^{\mathbb{N}}; \forall k}} \prod_{k \in U} f_{k,i_k},$$

if the subset of indices is non-empty, and 0 otherwise, by the usual conventions on empty sums. Coming back to our elements of $B\langle\langle \underline{e} \rangle\rangle^b$, since for all i , e_i is \mathbb{F}_q -linear, we deduce that for all $n \geq 0$, \mathcal{D}_n sends tame monomials on tame polynomials and therefore the operators \mathcal{D}_i induce B -linear endomorphisms of $B\langle\langle \underline{e} \rangle\rangle^b$ as expected and the property corresponding to $B^\circ\langle\langle \underline{e} \rangle\rangle^b$ follows easily. Now, it is easy to see that the operators \mathcal{D}_i extend to B -linear endomorphisms of $B[u^{-1}][[u]]$ so that we can also deduce the expected property for $B\langle\langle \underline{e} \rangle\rangle^b[[u]]$ and this suffices to justify (1). For (2), let n be in \mathbb{N}^* and let us consider the set of decompositions of length $r \geq 1$

$$n = \sum_{i=1}^r n_i q^i, \quad r \in \mathbb{N}, \quad n_i \in \mathbb{N}^*.$$

Then, the q -ary expansion of n (the unique one which has the coefficients $n_i \in \{0, \dots, q-1\}$) minimises the length $r = \ell_q(n)$. The reader can complete the verifications of the remaining properties of the proposition. \square

Remark 6.3. The behavior of v with respect to the action of the operator τ is multiplicative. On the other hand, it is difficult to make the interaction between v and the collection of operators $\underline{\mathcal{D}}$ explicit which introduces a difficulty in handling our modular forms.

6.2. Divided higher derivatives of ρ -quasi-periodic functions. We discuss here the problem of the computation of higher divided derivatives of the entries of the matrix functions Φ_ρ and $\Psi_m(\rho)$ for $m \geq 1$. We added this section to allow readers to perform explicit computations of higher derivatives of our modular forms. Indeed, the latter are all ρ -quasi-periodic and Proposition 4.11 tells us that in order to explicitly compute higher derivatives of ρ -quasi-periodic functions, it suffices to explicitly compute higher derivatives of u and Φ_ρ .

For this purpose it is convenient to choose a different normalisation for the higher divided derivatives. We set

$$D_n = (-\tilde{\pi})^{-n} \mathcal{D}_n$$

for all $n \geq 0$ and we write $\underline{D} = (D_i)_{i \geq 0}$. The formalism of the function $\mathcal{T}_{\underline{D},x}$ extends to \underline{D} and matrix functions. Additionally, we record the next straightforward corollary of Proposition 6.2:

Corollary 6.4. *The operators of the family \underline{D} determine K -linear endomorphisms of the K -vector spaces*

$$K(\underline{t}_\Sigma)^\circ\langle\langle \underline{e} \rangle\rangle^b, \quad K(\underline{t}_\Sigma)\langle\langle \underline{e} \rangle\rangle^b, \quad K(\underline{t}_\Sigma)^\circ\langle\langle \underline{e} \rangle\rangle^b[[u]], \quad K(\underline{t}_\Sigma)^\circ\langle\langle \underline{e} \rangle\rangle^b[[u]][u^{-1}].$$

We set, for f an analytic function $\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$,

$$\mathcal{T}_{\underline{D},x}(f) = \sum_{i \geq 0} D_i(f)x^i.$$

This defines, with $\mathbb{H} := \text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N})$ a $\mathbb{K}_\Sigma^{N \times N}$ -algebra morphism

$$\mathbb{H} \xrightarrow{\mathcal{T}_{\underline{D},x}} \mathbb{H}[[x]].$$

We also set

$$(6.5) \quad G_m(\rho) = \tilde{\pi}^{-m} \sum_{a \in A} (z - a)^{-m} \rho(T_a) = D_{m-1}(G_1(\rho)), \quad m \geq 1,$$

and $G_0(\rho) = 0$. The generating series of these functions is

$$(6.6) \quad \mathbf{G}(\rho) := \sum_{i \geq 0} G_i(\rho)x^i = x\mathcal{T}_{\underline{D},x}(G_1(\rho)).$$

We have the next lemma where we recall that $\exp_C(x) = \sum_{i \geq 0} d_i^{-1}x^i$ is Carlitz's exponential in x (see §2.4).

Lemma 6.5. *The following formula holds:*

$$\mathbf{G}(\rho) = \frac{ux}{1 - u \exp_C(x)} \mathcal{T}_{\underline{D},x}(\Phi_\rho).$$

Proof. It suffices to compute $\mathcal{T}_{\underline{D},x}(G_1(\rho))$. Since $G_1(\rho) = u\Phi_\rho$ the formula is obvious if we prove that

$$\mathcal{T}_{\underline{D},x}(G_1(\mathbf{1})) = \frac{u}{1 - u \exp_C(x)}.$$

This is well known, see Gekeler [24, (3.6)]. Nevertheless, we recall the proof here. From $1 = uu^{-1}$ we see that $1 = \mathcal{T}_{\underline{D},x}(u)\mathcal{T}_{\underline{D},x}(u^{-1})$.

Note that

$$u^{-1} = \exp_C(\tilde{\pi}z) = \sum_{i \geq 0} d_i^{-1}(\tilde{\pi}z)^i$$

so that $D_0(u^{-1}) = u^{-1}$ and, for $n > 0$, $D_n(u^{-1}) = 0$ if n is not a power of q and for $i \geq 0$, $D_{q^i}(u^{-1}) = D_{q^i}(d_i^{-1}\tilde{\pi}^{q^i}z^{q^i}) = -d_i^{-1}$. Hence

$$\mathcal{T}_{\underline{D},x}(u^{-1}) = u^{-1} - \sum_{i \geq 0} d_i^{-1}x^{q^i}$$

and

$$\mathcal{T}_{\underline{D},x}(u) = \frac{1}{u^{-1} - \sum_{i \geq 0} d_i^{-1}x^{q^i}} = \frac{1}{u^{-1} - \exp_C(x)} = \frac{u}{1 - u \exp_C(x)}.$$

□

If $\rho = \mathbf{1}$ then the formula of Lemma 6.5 reduces to [24, (3.6)] because in this case $\Phi_\rho = 1$. In general, the next Lemma can be helpful in determining some properties of $\mathcal{T}_{\underline{D},x}(\Phi_\rho)$.

We recall the matrix ω_ρ defined in (4.16).

Lemma 6.6. *There exist $\vartheta_1, \dots, \vartheta_r \in \mathbb{F}_q(\underline{t}_\Sigma)$ and a matrix*

$$M \in K[\vartheta_1, \dots, \vartheta_r][[x]]^{N \times N}$$

such that

$$\mathcal{T}_{\underline{D}, x}(\omega_\rho \Phi_\rho \omega_\rho^{-1}) = \omega_\rho \Phi_\rho \omega_\rho^{-1} M.$$

Proof. We recall that $\Phi_\rho \in \mathcal{QP}_0^1(\rho; \mathfrak{E})$ (Proposition 4.8) and $\omega_\rho \Phi_\rho \omega_\rho^{-1} \in A[\vartheta]^\circ \langle \langle \underline{e} \rangle \rangle^b)^{N \times N}$ (Corollary 4.9, where the elements ϑ_i are also introduced). $\Phi_\rho \in \mathcal{QP}_0^1(\rho; \mathfrak{E})$ implies that for all n , the columns of $D_n(\Phi_\rho)$ are ρ -quasi-periodic. By Proposition 6.2 for all $n \geq 0$, $D_n(\omega_\rho \Phi_\rho \omega_\rho^{-1})$ belongs to $(K[\vartheta_1, \dots, \vartheta_r]^\circ \langle \langle \underline{e} \rangle \rangle^b)^{N \times N}$. Note that the coefficients do not necessarily belong to $A[\vartheta_1, \dots, \vartheta_r]$. This comes from the fact that $D_n(e_i) \in K$ for $n > 0$ and these coefficients are not, in general, in A . By the proof of Proposition 4.11 the columns of $D_n(\Phi_\rho)$ being ρ -quasi-periodic, they are linear combinations with coefficients in \mathbb{K}_Σ of the columns of Φ_ρ . This means that for all $n \geq 0$,

$$\omega_\rho D_n(\Phi_\rho) \omega_\rho^{-1} = \omega_\rho \Phi_\rho \omega_\rho^{-1} M_n$$

for some $M_n \in \mathbb{K}_\Sigma^{N \times N}$. From Corollary 4.9 we deduce easily that $\omega_\rho \Phi_\rho \omega_\rho^{-1}$ belongs to $\text{GL}_N(A[\vartheta_1, \dots, \vartheta_r] \langle \langle \underline{e} \rangle \rangle^b)$. Hence

$$M_n = \omega_\rho D_n(\Phi_\rho) \Phi_\rho^{-1} \omega_\rho^{-1} \in \mathbb{K}_\Sigma^{N \times N} \cap (K[\vartheta_1, \dots, \vartheta_r] \langle \langle \underline{e} \rangle \rangle^b)^{N \times N} = K[\vartheta_1, \dots, \vartheta_r]^{N \times N}.$$

Hence

$$\mathcal{T}_{\underline{D}, x}(\omega_\rho \Phi_\rho \omega_\rho^{-1}) = \omega_\rho \Phi_\rho \omega_\rho^{-1} \sum_{n \geq 0} M_n x^n = \omega_\rho \Phi_\rho \omega_\rho^{-1} M,$$

with $M = \sum_n M_n x^n \in K[\vartheta_1, \dots, \vartheta_r]^{N \times N}[[x]]$. \square

Corollary 6.7. *There exists $\vartheta_1, \dots, \vartheta_r \in \mathbb{F}_q(\underline{t}_\Sigma)$ such that*

$$\omega_\rho \mathbf{G}(\rho) \omega_\rho^{-1} \in \mathfrak{M}_\Sigma^{N \times N} \cap \left(K[\vartheta_1, \dots, \vartheta_r] \langle \langle \underline{e} \rangle \rangle^b [u][[x]] \right)^{N \times N}.$$

Proof. We have $\frac{u}{1 - u \exp_C(x)} \in K[u][[x]]$ and

$$\mathcal{T}_{\underline{D}, x}(\omega_\rho \Phi_\rho \omega_\rho^{-1}) \in \left(K[\vartheta_1, \dots, \vartheta_r]^\circ \langle \langle \underline{e} \rangle \rangle^b [[x]] \right)^{N \times N}$$

by Corollary 4.9. The result follows applying Lemma 6.5. \square

For example, if $\rho = \rho_\chi$ is basic, we have seen in Corollary 4.10 that $\Phi_\rho = \Xi_\rho = \begin{pmatrix} I_n & \chi(z) \\ 0_n & I_n \end{pmatrix}$, with $N = 2n$. By (2.15) $D_{q^k}(\chi) = D_k^{-1}(\vartheta - \theta^{q^k} I_n)^{-1} \omega_\chi^{-1}$ for $k \geq 0$, and $D_j(\chi) = 0$ if $j > 0$ is not a q -power. Hence in this case the matrix M of Lemma 6.6 is:

$$M = \begin{pmatrix} I_n & \chi(z) \\ 0_n & I_n \end{pmatrix} + \omega_\chi^{-1} \exp_C \left((\vartheta - \theta I_n)^{-1} x \right) \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix},$$

with $\tau(x) = x^q$.

6.3. Serre’s derivatives. In this subsection we prove part (6) of our Theorem A. We discuss variants of *Serre’s higher derivatives* introduced in [14, §1.2.3]. Following this reference, we set, for $n, w \in \mathbb{N}$ and $f \in \text{Hol}(\Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1})$:

$$(6.7) \quad \partial_n^{(w)}(f) := D_n(f) + \sum_{i=1}^n (-1)^i \binom{w+n-1}{i} D_{i-1}(E) D_{n-i}(f),$$

where E is the normalised false Eisenstein series of weight 2 and type 1 of Gekeler, defined in [24, §8]. We recall the definition here, for convenience of the reader. We can define E by using the conditionally convergent lattice sum

$$(6.8) \quad E(z) = \tilde{\pi}^{-1} \sum_{a \in A^+} \sum_{b \in A} \frac{a}{az + b}.$$

This defines a rigid analytic function $E : \Omega \rightarrow \mathbb{C}_\infty$ which satisfies

$$E(\gamma(z)) = J_\gamma(z)^2 \det(\gamma)^{-1} \left(E(z) - \tilde{\pi} \frac{c}{cz + d} \right), \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$$

(a *Drinfeld quasi-modular form of weight 2, type 1* and depth 1 in the terminology of [13]). We also recall the u -expansion, with $u_a = e_C(az)^{-1}$:

$$E = \sum_{a \in A^+} au_a.$$

Another property of E is that it can be computed as a logarithmic derivative $E = \frac{D_1(\Delta)}{\Delta}$ of Δ the cusp form of weight $q^2 - 1$ defined in [24, §(6.4)]. See also §7.6.4. Coming back to our modular forms, note that the case $n = 1$ of (6.7) yields the operator $\partial_1^{(w)} = D_1 - wEI_N$. This is the analogue of Ramanujan’s derivative introduced by Gekeler in [24, (8.5)].

Theorem 6.8. *Let $\rho : \Gamma \rightarrow \text{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ be a representation of the first kind. The operator $\partial_n^{(w)}$ determines a \mathbb{K}_Σ -linear map $M_w(\rho; \mathbb{K}_\Sigma) \rightarrow S_{w+2n}(\rho \det^{-n}; \mathbb{K}_\Sigma)$ and an \mathbb{L}_Σ -linear map $M_w(\rho; \mathbb{L}_\Sigma) \rightarrow S_{w+2n}(\rho \det^{-n}; \mathbb{L}_\Sigma)$.*

Proof. If $f \in M_w(\rho; \mathbb{K}_\Sigma)$ then f can be identified with an element of $\mathfrak{D}_\Sigma^{N \times 1}$ (Theorem 4.12) which is $\partial_n^{(w)}$ -stable for all n, w . The same arguments of the proof of [14, Theorem 4.1] (which holds in a wider context of *Drinfeld quasi-modular forms*) imply that $\partial_n^{(w)}(f) \in M_{w+2n}(\rho \det^{-n}; \mathbb{K}_\Sigma)$. Further, it is easy to see that $\partial_n^{(w)}(f)$ has entries in \mathfrak{M}_Σ so it is a cusp form. \square

6.4. Application to Perkins’ series. In this subsection we present the series indicated in the title, originally introduced by Perkins in his Ph. D. Thesis [64], as generating series for certain zeta values in Tate algebras introduced by the author in [55]. These series define elements of \mathfrak{D}_Σ and the problem of computing their v -valuations (or equivalently, weights) arises. This is quite an intricate problem that we partially solve here. One of the difficulties is that the matrix formalism of the preceding sections does not seem suitable to extract this kind of information.

Let U be a finite subset of \mathbb{N}^* . We set

$$\sigma_U = \prod_{i \in U} \chi_{t_i}.$$

Explicitly, $\sigma_U(a) = \prod_{i \in U} \chi_{t_i}(a) \in \mathbb{F}_q[t_U]$ for all $a \in A$.

For further use, with Σ a given finite subset of \mathbb{N}^* :

Definition 6.9. A *semicharacter* is a map $\sigma : A \rightarrow \mathbb{F}_q[t_\Sigma]$ defined, for $a \in A$, by $\sigma(a) = \prod_{i \in \Sigma} \chi_{t_i}(a)^{\alpha_i}$ for integers $\alpha_i \geq 0$.

We are interested in the following class of function.

Definition 6.10. Let U be a finite subset of \mathbb{N}^* . The *Perkins series of order $n \geq 1$* associated to σ_U is the series:

$$\psi(n; \sigma_U) = \sum_{a \in A} (z - a)^{-n} \sigma_U(a).$$

For any U and n as above, the series converges for $z \in \mathbb{C}_\infty \setminus A$ (with respect to the norm $\|\cdot\|$ of \mathbb{K}_Σ , Σ being a finite subset of \mathbb{N}^* containing U) and $z \mapsto e_A(z)^n \psi(n; \sigma_U)(z)$ define entire functions $\mathbb{C}_\infty \rightarrow \mathbb{E}_\Sigma$, as it is easily seen. If $U = \emptyset$ we have $\sigma_\emptyset = \mathbf{1}$ the *trivial semi-character*, and Perkins' generating series are related to Goss' polynomials associated to the lattice $A \subset \mathbb{C}_\infty$ as in [35, §6] and [24, §3]. Indeed,

$$(6.9) \quad \psi(n; \mathbf{1}) = S_{n,A} = \sum_{b \in A} \frac{1}{(z - b)^n} = G_{n,A}(S_{1,A}),$$

for polynomials $G_{n,A} \in K_\infty[X]$ (in the notations of [24].) The functions $\psi(n; \sigma_U)$ with $U \subset \Sigma$ occur in the entries of $\Psi_n(\rho_\Sigma)$, where ρ_Σ is the representation of the first kind

$$\rho_\Sigma = \bigotimes_{i \in \Sigma} \rho_{t_i},$$

where $\rho_{t_i} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a(t_i) & b(t_i) \\ c(t_i) & d(t_i) \end{pmatrix}$ (or alternatively, one can also use $\rho = \rho_\Sigma^*$). Since $\Psi_n(\rho_\Sigma) \in \mathcal{QP}_n^1(\rho_\Sigma; \mathbb{E}_\Sigma)$ by Proposition 4.6, Lemma 6.5 implies:

Lemma 6.11. *For all $U \subset \Sigma$ and $n \geq 1$ we have $\psi(n; \sigma_U) \in \mathfrak{K}_\Sigma$. Additionally, $\phi(1; \sigma_U) := e_0 \psi(1; \sigma_U) \in \mathbb{E}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$.*

6.4.1. *Perkins' series of order $n = 1$.* We focus now on $\phi(1; \sigma_\Sigma) \in \mathbb{E}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$. The next question is the computation of its weight. We set, for Σ non-empty with $s = |\Sigma| = (m - 1)(q - 1) + l$ with $m \geq 1$ and $l \in \{1, \dots, q - 1\}$:

$$(6.10) \quad \kappa(\Sigma) := q^{-m}(q - l) \in]0, 1[\cap \mathbb{Z}[p^{-1}].$$

For $\Sigma = \emptyset$, we extend the definition to $\kappa(\emptyset) := 1$. Note that $\kappa(\Sigma)$ defines a strictly decreasing function $|\Sigma| \mapsto \kappa(\Sigma)$, and $\lim_{|\Sigma| \rightarrow \infty} \kappa(\Sigma) = 0$. We prove:

Theorem 6.12. *The function $\phi(1; \sigma_\Sigma) \in \mathbb{E}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$ has weight*

$$(6.11) \quad w(\phi(1; \sigma_\Sigma)) = 1 - \kappa(\Sigma) = 1 - q^{1-m} + lq^{-m} = w_{\max}(s).$$

Proof. The identities connecting κ and w_{\max} are easily verified. If $\Sigma = \emptyset$, it is clear that $\phi(1; \sigma_\Sigma)$ has weight 0 (it is in this case a constant function). We suppose that Σ is non-empty. We consider the unique representative $g_\Sigma \in \mathbb{E}_\Sigma^\circ \langle\langle e \rangle\rangle^b$ of $\prod_{i \in \Sigma} \chi_{t_i}(z)$ (see §2.4.4 for the definition of $\chi_t(z)$) modulo the ideal of $\mathbb{E}_\Sigma \langle\langle e \rangle\rangle^b$ generated by e_0 . By Corollary 4.9, we have $\phi(1; \sigma_\Sigma) = g_\Sigma$. We can write $g_\Sigma = \sum_{i=0}^s g_\Sigma^{[i]}$ with $g_\Sigma^{[i]} \in \mathbb{E}_\Sigma^\circ \langle\langle e \rangle\rangle_i^b$ (see (3.11)) ⁽¹¹⁾. We note that

$$(6.12) \quad g_\Sigma^{[s]} = \left[\prod_{i \in \Sigma} \chi_{t_i}(z) \right]^{[s]} = \underbrace{e_1^{q-1} \cdots e_{m-1}^{q-1} e_m^l}_{\text{Tame monomial } M_s} \mathcal{P}_\Sigma + \Phi,$$

with $w(M_s) = w_{\max}(s)$, $\Phi \in \mathbb{E}_\Sigma \langle\langle e \rangle\rangle^b$, with $w(\Phi) < w_{\max}(s)$, and where

$$\mathcal{P}_\Sigma := \sum_{\substack{I_0 \sqcup I_1 \sqcup \cdots \sqcup I_m = \Sigma \\ |I_0| = \cdots = |I_{m-1}| = q-1 \\ |I_m| = l}} \left(\prod_{i_1 \in I_1} t_{i_1} \right) \cdots \left(\prod_{i_m \in I_m} t_{i_m}^{m-1} \right) \in \mathbb{F}_p[t_\Sigma].$$

This polynomial is non-zero as it is easily verified by tracking the contribution coming from a subset $\tilde{I} \subset \Sigma$ such that $|\Sigma \setminus \tilde{I}| = l$. Substituting t_i by 1 if $i \in \Sigma \setminus \tilde{I}$ and by 0 if $i \in \tilde{I}$, we get the value 1. By Lemma 6.1:

$$w\left(g_\Sigma^{[s]} - \mathcal{P}_\Sigma M_s\right) < w_{\max}(s).$$

This implies the theorem because the map $s \mapsto w_{\max}(s)$ is a strictly increasing function ($s > 0$) so that

$$w(\phi(1; \sigma_\Sigma)) = w(g_\Sigma) = w(g_\Sigma^{[s]}) = w_{\max}(s).$$

□

For all $\Sigma \subset \mathbb{N}^*$ a finite subset, the above proof yields the next corollary:

Corollary 6.13. *We have*

$$\lim_{|z|_{\mathbb{Q}} \rightarrow \infty} e_A(z)^{\kappa(\Sigma)} \psi(1; \sigma_\Sigma) = \mathcal{P}_\Sigma.$$

Example. If Σ is a singleton we can work with one variable t and we have the explicit formula, due to Perkins, a simple proof of which can be found in [61] (combine (3) and Theorem 1):

$$(6.13) \quad \psi(1; \chi) = \tilde{\pi} u(z) \chi_t(z).$$

¹¹In fact, one sees that if $i \not\equiv s \pmod{q-1}$, then $g_\Sigma^{[i]} = 0$.

Let Σ be a subset of \mathbb{N}^* of cardinality q . Developing the product $\prod_{k \in \Sigma} e_C \left(\frac{z}{\theta - t_k} \right)$ we get, after elimination of the q -th powers:

$$\begin{aligned} \prod_{k \in \Sigma} e_C \left(\frac{z}{\theta - t_k} \right) &= e_0 - \sum_{j \geq 0} \left(\theta \prod_{i \in \Sigma} t_i^j - \prod_{i \in \Sigma} t_i^{j+1} \right) e_{j+1} + \\ &+ \sum_{\substack{0 \leq i_1 \leq \dots \leq i_q \\ i_k \text{ not all equal}}} e_{i_1+1} \cdots e_{i_q+1} \sum_{\substack{\underline{\alpha} = (\alpha_i; i \in \Sigma) \in \mathbb{N}^{|\Sigma|} \\ |\underline{\alpha}| = q^{i_1} + \dots + q^{i_q}}} \prod_{k \in \Sigma} t_k^{\alpha_k}. \end{aligned}$$

from this tame series expansion (of depth q) we deduce that the leading tame monomial of the tame series $\prod_{i \in \Sigma} \chi_{t_i}(z)$ is e_0 . Hence, $\prod_{k \in \Sigma} e_C \left(\frac{z}{\theta - t_k} \right) - e_0 \in A[t_\Sigma]^\circ \langle\langle \underline{e} \rangle\rangle^b$ and we get an explicit computation of $\phi(1; \sigma_\Sigma)$ for this choice of Σ , first obtained by Perkins.

6.4.2. *Perkins' series of higher order.* In this part we are interested in the following question:

Question 6.14. *Compute the valuation $v(\psi(n; \sigma_\Sigma)) \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ explicitly in terms of l, m, n .*

The case $\Sigma = \emptyset$, where $N = 1$ was partially settled by Gekeler in [26]. The complete solution is now available in Gekeler's manuscript [31]. In Theorem 6.16 we give a partial answer in the several variables case. We suppose that $s = |\Sigma| \neq 0$. We recall that by Proposition 6.2, \mathcal{D}_n induces \mathbb{K}_Σ -linear endomorphisms of $\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$ and $\mathbb{K}_\Sigma \langle\langle \underline{e} \rangle\rangle^b$ for all n . We also recall that w denotes the opposite of the valuation v (degree).

Proposition 6.15. *Let i be a non-negative integer, let $r \geq 0$ be such that $\mathcal{D}_{(i+1)q^r-1}(f_\Sigma) \neq 0$. Then,*

$$w(\psi(1+i; \sigma_\Sigma)) = \frac{1}{q^r} w(\mathcal{D}_{(i+1)q^r-1}(f_\Sigma)) - \frac{1}{q^r} \in \left[-\frac{1}{q^r}, 0 \right].$$

Proof. We observe that $\tau^r(\psi(i+1; \sigma_\Sigma)) \in \mathfrak{K}_\Sigma$. Further, we have:

$$(6.14) \quad \tau^r(\mathcal{D}_i(\psi(1; \sigma_\Sigma))) = (-1)^i \tau^r(\psi(i+1; \sigma_\Sigma)) = \mathcal{D}_{q^r(i+1)-1}(\psi(1; \sigma_\Sigma)), \quad i, r \geq 0.$$

We are interested in the computation of the weight of $\tau^r(\psi(i+1; \sigma_\Sigma))$ (it is equal to q^r times the weight of $\psi(i+1; \sigma_\Sigma)$, which is the quantity we ultimately want to compute). We set $f_i = \mathcal{D}_i(e_C(z)\psi(1; \sigma_\Sigma)) \in \mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$. In particular, $f_0 = f_\Sigma = e_C(z)\psi(1; \sigma_\Sigma) \in \mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$. By Leibnitz rule, we have

$$(6.15) \quad f_i = e_C(z) \mathcal{D}_i(\psi(1; \sigma_\Sigma)) + \underbrace{\sum_{\substack{\alpha + \beta = i \\ \alpha > 0}} \mathcal{D}_\alpha(e_C(z)) \mathcal{D}_\beta(\psi(1; \sigma_\Sigma))}_{=:\Xi}.$$

All terms of the above sum are in \mathfrak{K}_Σ . Since the higher derivatives of positive order of $e_C(z)$ are constant and all the functions $\psi(1+\beta; \sigma_\Sigma)$ for $\beta \geq 0$ tend to zero as $|z| = |z|_{\mathfrak{S}} \rightarrow \infty$,

the weight of the above defined term Ξ is < 0 . We apply the operator τ^r . We get, by (6.14):

$$(6.16) \quad \tau^r(f_i) = e_C(z)^{q^r} \mathcal{D}_{(i+1)q^r-1}(\psi(1; \sigma_\Sigma)) + \tau^r(\Xi).$$

We have that $\tau^r(\Xi) \in \mathfrak{K}$ and the weight is ≤ 0 ; we also set $n = (i+1)q^r - 1$. Then,

$$\begin{aligned} \mathcal{D}_n(\psi(1; \sigma_\Sigma)) &= \mathcal{D}_n(uf_\Sigma) \\ &= u\mathcal{D}_n(f_\Sigma) + \underbrace{\sum_{\substack{\alpha+\beta=n \\ \alpha>0}} \mathcal{D}_\alpha(u)\mathcal{D}_\beta(f_\Sigma)}_{=: \Upsilon}. \end{aligned}$$

If $\alpha > 0$, $\mathcal{D}_\alpha(u) \in \mathbb{C}_\infty[u] \subset \mathfrak{K}$ which is of weight ≤ -2 as the reader can easily check. Since $f_\Sigma \in \mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$, the weights of all its higher derivatives are in $\{-\infty\} \cup [0, 1[$ and thus, the weight of the term Υ above defined is < -1 . Let us suppose that $\mathcal{D}_n(f_\Sigma)$ is non-zero. Then, its weight belongs to $[0, 1[$ and the weight of $u\mathcal{D}_n(f_\Sigma)$ belongs to $[-1, 0[$. We deduce that, under this hypothesis of non-vanishing, the weight of $\mathcal{D}_n(\psi(1; \sigma_\Sigma))$ is equal to the weight of $u\mathcal{D}_n(f_\Sigma)$, belonging to the interval $[-1, 0[$. Coming back to the identity (6.16) and recalling that $\tau^r(\Xi)$ has negative weight, we deduce that $\tau^r(f_i)$ and $e_C(z)^{q^r-1}\mathcal{D}_n(f_\Sigma)$ have the same weight, belonging to the interval $[q^r - 1, q^r[$, and the weight of f_i satisfies:

$$(6.17) \quad w(f_i) = 1 + \frac{1}{q^r} w(\mathcal{D}_{q^r(i+1)-1}(f)) - \frac{1}{q^r} \in \left[1 - \frac{1}{q^r}, 1 \right], \quad r \geq 0, \quad i \geq 1.$$

Coming back to (6.15), we have noticed that the term Ξ has weight < 0 . But f_i has non-negative weight by (6.17). Hence, the weight of the first term in the right-hand side of (6.15) has the same weight as f_i and the result follows. \square

We recall that if $s = |\Sigma| = (m-1)(q-1) + l$ with $m \geq 1$ and $l \in \{1, \dots, q-1\}$, then $w(\psi(1; \sigma_\Sigma)) = lq^{-m} - q^{1-m}$ (see Theorem 6.12). We want to compute the weight of $\psi(1+n; \sigma_\Sigma)$ for $n \geq 0$ and this allows to compute the v -valuation of these elements. The following Theorem generalizes Theorem 6.12:

Theorem 6.16. *Let Σ , s, m, l as above and let n be ≥ 0 such that $\ell_q(n) \leq l$. Then,*

$$v(\psi(1+n; \sigma_\Sigma)) = q^{1-m} - (l - \ell_q(n))q^{-m}.$$

Proof. We choose $i = n$ and $r = 0$ in Proposition 6.15 (note that in this case $n = (i+1)q^r - 1$). We show that $\mathcal{D}_n(f_\Sigma) \neq 0$ and we compute its depth. To construct f_Σ , we have applied the rule $e_{i-1} = C_\theta(e_i)$ to the product $\prod_{i \in \Sigma} e_C\left(\frac{z}{\theta - t_i}\right)$ which implies that

$$f_\Sigma = f_\Sigma^{[s]} + f_\Sigma^{[s-q+1]} + \dots.$$

We recall that we have already seen that $f_\Sigma^{[s]}$ is equal to $\chi(\sigma_\Sigma)^{[s]}$ and has the monic maximal tame monomial M_s as a non-zero term of its tame expansion. Further, $w(f_\Sigma^{[s-j(q-1)]}) <$

$w_{\max}(s)$ for all $j > 0$. Hence, we can write $f_{\Sigma} = M_s + g$ with $w(g) < w_{\max}(s)$ and $\mathcal{D}_n(g)$ has weight strictly less than

$$w_{\max}(s - \ell_q(n)) = 1 - q^{1-m} + (l - \ell_q(n))q^{-m}.$$

We now claim that $w(\mathcal{D}_n(f_{\Sigma})) = w_{\max}(s - \ell_q(n))$. If this is true, we deduce, from Proposition 6.15, the formula $w(\psi(1+n; \sigma_{\Sigma})) = (l - \ell_q(n))q^{-m} - q^{1-m}$ hence completing the proof of the Theorem. \square

The claim is the object of the next Lemma, where M_s is defined in (6.1):

Lemma 6.17. *With $s = |\Sigma|$ equal to $(m-1)(q-1) + l$, $m \geq 1$ and $1 \leq l \leq q-1$, let $n \in \mathbb{N}$ be such that $\ell_q(n) \leq l$. Then,*

$$\mathcal{D}_n(M_s) = \kappa_n M_{s-\ell_q(n)} + h,$$

with

$$\kappa_n := \left(\frac{\tilde{\pi}}{\theta^m} \right)^{n_0} \left(\frac{\tilde{\pi}^q}{\theta^{mq} d_1} \right)^{n_1} \cdots \left(\frac{\tilde{\pi}^{q^r}}{\theta^{mq^r} d_r} \right)^{n_r} \in \mathbb{C}_{\infty}^{\times},$$

where $n = n_0 + n_1 q + \cdots + n_r q^r$ is the base- q expansion of n , and with $h \in \mathcal{T}^{\circ}(\mathbb{C}_{\infty})$ of weight $< w_{\max}(s - \ell_q(n))$.

Proof. We write $M_s = FG$ with $F = (e_1 \cdots e_{m-1})^{q-1}$ and $G = e_m^l$. By Leibnitz rule $\mathcal{D}_n(M_s) = \sum_{\alpha+\beta=n} \mathcal{D}_{\alpha}(F) \mathcal{D}_{\beta}(G)$. If $\alpha > 0$, then $w(\mathcal{D}_{\alpha}(F) \mathcal{D}_{\beta}(G))$ is strictly smaller than $w_{\max}(s - \ell_q(n))$. Now, we consider the term with $\alpha = 0$. Note, by the formula (6.4) applied to the product of \mathbb{F}_q -linear maps $G = e_m^l$, that

$$\mathcal{D}_n(G) = (\mathcal{D}_{n_0} \circ \mathcal{D}_{n_1 q} \circ \cdots \circ \mathcal{D}_{n_r q^r})(e_m^l) = \kappa_n e_m^{l-\ell_q(n)}.$$

The result follows. \square

7. EISENSTEIN SERIES FOR ρ_{Σ}^*

This section contains the proofs of the various items of Theorem C in the introduction. We present several aspects of Eisenstein series for the representation $\rho = \rho_{\Sigma}^*$, with $N = 2^s$. These functions provide important examples of the modular forms we consider (see also [57]). We set, for $w \in \mathbb{N}^*$:

$$\mathcal{E}(w; \rho_{\Sigma}^*) := \sum'_{(a,b) \in A} (az+b)^{-w} \bigotimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(a) \\ \chi_{t_i}(b) \end{pmatrix},$$

where the sum runs over the $a, b \in A$ which are not both zero. This series corresponds to the choice

$$\mu(a, b) = \bigotimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(a) \\ \chi_{t_i}(b) \end{pmatrix}$$

in (5.26) (this is the transposition of the first line of $\rho_{\Sigma} \begin{pmatrix} a & b \\ * & * \end{pmatrix}$) so that by Lemma 5.27 $\mathcal{E}_w(\rho_{\Sigma}^*) \in M_w(\rho_{\Sigma}^*) \setminus \{0\}$ if $s = |\Sigma| \equiv w \pmod{q-1}$ (see also [57, §5]). Note also that this

series defines a holomorphic function $\Omega \rightarrow \mathbb{E}_\Sigma^{N \times 1}$. We call $\mathcal{E}(w; \rho_\Sigma^*)$ the *Eisenstein series of weight w associated to ρ_Σ^** .

Here is the plan of this section. In §7.2, Corollary 7.4, we compute the v -valuation of the entries of $\mathcal{E}(1; \rho_\Sigma^*)$. The computation uses results of §6 on Perkins' series. The general problem of the computation of the v -valuation of the entries of $\mathcal{E}(m; \rho_\Sigma^*)$ for $m > 0$ is likely to be a difficult problem. Some partial results can be obtained applying Theorem 6.16. In §7.3 we use the Eisenstein series $\mathcal{E}(1; \rho_\Sigma^*)$ to show that the dimension of $M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ equals one if $|\Sigma| \equiv 1 \pmod{q-1}$. This is one of the very few spaces of non-scalar Drinfeld modular forms that we are able to fully characterize. As a corollary, the series $\mathcal{E}(1; \rho_\Sigma^*)$ are Hecke eigenforms. In §7.4, Theorem 7.8 we describe integrality properties of the u -expansions (in the sense of Proposition 3.32) of the entries of $\mathcal{E}(m; \rho_\Sigma^*)$. Naturally, these series expansions are much more complicated and less explicit than those obtained by Gekeler in [24] for the scalar Eisenstein series. In §7.6 we show how certain results of Petrov [67] on A -expansions can be generalized to show that series such as

$$\sum_{a \in A^+} a^l G_m(u_a) \in K[[u]]$$

with $l, m > 0$ such that $l \equiv m \pmod{q-1}$ give rise to u -expansions of *quasi-modular forms* in the sense of [13]. These series occur as special values of an entry of the Eisenstein series $\mathcal{E}(m; \rho_\Sigma^*)$ hence confirming a prediction of D. Goss on a link between Petrov's A -expansions and Eisenstein series; see Theorem 7.15. In §7.6.5 we present, succinctly, some applications to \mathfrak{v} -adic modular forms.

7.1. Link between Eisenstein series and Poincaré series. The next lemma provides a connection with Poincaré series.

Lemma 7.1. $\mathcal{E}(w; \rho_\Sigma^*) = \zeta_A(w; \sigma_\Sigma) \mathcal{P}_w^{(0)}(\Phi_{\rho_\Sigma^*})$.

Here $\mathcal{P}_w^{(0)}(\Phi_{\rho_\Sigma^*})$ denotes the last column of the matrix valued Poincaré series $\mathcal{P}_w(\Phi_{\rho_\Sigma^*})$ defined in (5.6), with $G = \Phi_{\rho_\Sigma^*}$ and, as in (1.17),

$$\zeta_A(w; \sigma_\Sigma) = \sum_{a \in A^+} \frac{\sigma_\Sigma(a)}{a^w}.$$

Proof of Lemma 7.1. We consider a matrix $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. We note that the last column of $\Phi_{\rho_\Sigma^*}(\gamma(z))$ is the last entry of the canonical basis of the vector space $\mathbb{F}_q^{N \times 1}$. Indeed, $\Phi_{\rho_\Sigma^*}(z)$ itself is a matrix function which is lower triangular with 1 on the diagonal. Moreover, the last column of $\rho_\Sigma^*(\gamma)^{-1} = {}^t \rho_\Sigma(\gamma)$ is $\otimes_{i \in \Sigma} \begin{pmatrix} \chi^{t_i(c)} \\ \chi^{t_i(d)} \end{pmatrix}$, which is therefore also equal to the last column of $\rho_\Sigma^*(\gamma)^{-1} \Phi_{\rho_\Sigma^*}(\gamma(z))$ and to the last column of

$$\rho_\Sigma^*(\gamma)^{-1} \Phi_{\rho_\Sigma^*}(\gamma(z)) \rho_\Sigma^* \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the last column $\mathcal{P}_w^{(0)}(\Phi_{\rho_\Sigma^*})$ of $\mathcal{P}_w(\Phi_{\rho_\Sigma^*})$ is

$$\sum_{\substack{\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \\ c, d \in A \\ \text{relatively prime}}} (cz + d)^{-w} \bigotimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(c) \\ \chi_{t_i}(d) \end{pmatrix},$$

independent on the choice of the representatives modulo the subgroup H of Γ . Observe that the index set of the sum defining the series $\mathcal{E}(w; \rho_\Sigma^*)$, $A^2 \setminus \{(0, 0)\}$, is equal to $\mathcal{I}A^+$, where \mathcal{I} is the set of couples $(c, d) \in A^2$ with c, d relatively prime. This means that

$$\mathcal{E}(w; \rho_\Sigma^*) = \sum_{a \in A^+} \frac{\sigma_\Sigma(a)}{a^w} \sum_{(c, d) \in \mathcal{I}} (cz + d)^{-w} \bigotimes_{i \in \Sigma} (\chi_{t_i}(c), \chi_{t_i}(d)) = \zeta_A(w; \sigma_\Sigma) \mathcal{P}_w^{(0)}(\Phi_{\rho_\Sigma^*}).$$

□

7.2. The v -valuation of Eisenstein series. We expand the entries of our vector-valued Eisenstein series along the principles of Theorem 4.12 and we compute their v -valuations in certain cases.

If $|\Sigma| = s > 0$ and $N = 2^s$, the ordering on Σ induces a bijection $\Sigma \xrightarrow{\varepsilon} \{0, \dots, s-1\}$. This in turn defines a bijection between subsets $J \subset \Sigma$ and integers $0 \leq n \leq N-1$. If $n = n_0 + n_1 2 + \dots + n_{s-1} 2^{s-1}$ is the base-2 expansion of n , the image of n is the subset $J = \{j \in \Sigma : n_j \neq 0\} \subset \Sigma$. We can write $|J|_\Sigma := n$. For example, $|\emptyset|_\Sigma = 0$. Then, we can describe in two ways an N -tuple of objects parametrized by the subsets of $\{1, \dots, 2^s\}$:

$$f = (f^J)_{J \subset \Sigma} = (f_i)_{1 \leq i \leq N},$$

by using that the latter is $(f_{|J|_\Sigma+1})_{J \subset \Sigma}$ (note how we distinguish the \mathbb{N}^* -indexing from the Σ -indexing). Note that the first entry is

$$f_0 = f^\emptyset.$$

The Perkins series $\psi(w; \sigma_U)$ defined in (6.10) are elements of the ring of integers of the field of uniformizers \mathfrak{D}_Σ , if $U \subset \Sigma$. We set

$$\psi_a(w; \sigma_\Sigma) := \psi(w; \sigma_\Sigma)(za),$$

functions which also belong to \mathfrak{D}_Σ . Their valuations v are positive and we have, for all $a \in A^+$,

$$v(\psi_a(w; \sigma_\Sigma)) = |a|v(\psi(w; \sigma_\Sigma))$$

by Proposition 3.35. We set

$$(7.1) \quad V(w; \rho_\Sigma^*) := \frac{1}{\tilde{\pi}^w} \sum_{b \in A} \frac{1}{(z+b)^w} (\sigma_J(b))_{J \subset \Sigma}.$$

We denote by $V(w; \rho_\Sigma^*)_a$ the function of the variable z in $\mathfrak{D}_\Sigma^{N \times 1}$ obtained by rescaling $z \mapsto az$. We also set

$$(7.2) \quad \mathcal{Z}(w; \rho_\Sigma^*) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_A(w; \sigma_\Sigma) \end{pmatrix}.$$

The next Proposition generalizes [63, Proposition 3.7] to the case of $\rho = \rho_\Sigma^*$.

Proposition 7.2. *If $s = |\Sigma| \equiv w \pmod{q-1}$ and $w > 0$, then:*

$$(7.3) \quad \mathcal{E}(w; \rho_\Sigma^*) = -\mathcal{Z}(w; \rho_\Sigma^*) - \tilde{\pi}^w \sum_{a \in A^+} \rho_\Sigma \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} V(w; \rho_\Sigma^*)_a.$$

Writing $\mathcal{E}(w; \rho_\Sigma^*) = (\mathcal{E}^J)_{I \sqcup J = \Sigma}$, we have, more explicitly:

$$(7.4) \quad \mathcal{E}^J = -(-1)^{|J|} \sum_{a \in A^+} \sigma_I(a) \psi_a(w; \sigma_J), \quad J \neq \Sigma,$$

$$(7.5) \quad \mathcal{E}^\Sigma = -\zeta_A(w; \sigma_\Sigma) - (-1)^{|\Sigma|} \sum_{a \in A^+} \psi_a(w; \sigma_\Sigma).$$

In particular, if $J = \emptyset \neq \Sigma$, we have

$$(7.6) \quad \mathcal{E}^\emptyset = -\tilde{\pi}^w \sum_{a \in A^+} \sigma_\Sigma(a) G_w(u_a(z)) \in \mathbb{K}_\Sigma[[u]].$$

Moreover, if $\Sigma = \emptyset$, we have, for $q-1 \mid n$:

$$(7.7) \quad \mathcal{E}(w; \mathbf{1}) = -\zeta_A(w) - \tilde{\pi}^n \sum_{a \in A^+} G_w(u_a(z)).$$

In all cases, we can identify $\mathcal{E}(w; \rho_\Sigma^*)$ with an element of $\mathfrak{D}_\Sigma^{N \times 1}$.

We deduce, in yet another way, that $\mathcal{E}(w; \rho_\Sigma^*) \in M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$. Additionally, we see that it does not belong to $S_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ because of the non-vanishing of $\zeta_A(w; \sigma_\Sigma)$ in (7.5). Note that writing $\mathcal{E}(w; \rho_\Sigma^*) = {}^t(\mathcal{E}_1, \dots, \mathcal{E}_{N-1}, \mathcal{E}_N)$, we have $v(\mathcal{E}_i) > 0$ for $i = 1, \dots, N-1$ and $v(\mathcal{E}_N) = 0$.

More precisely, in the case $\mathcal{E} = (\mathcal{E}^I)_I = \mathcal{E}(1; \rho_\Sigma^*)$ with $|\Sigma| \equiv 1 \pmod{q-1}$, recalling that the map κ is defined in (6.10) and combining with Theorem 6.12:

Corollary 7.3. *Assume that $\Sigma \neq \emptyset$. We have that $v(\mathcal{E}^\Sigma) = 0, v(\mathcal{E}^\emptyset) = 1$ and, for $\emptyset \subsetneq I \subsetneq \Sigma$, $v(\mathcal{E}^I) = \kappa(I)$. Explicitly, if $|I| = (\mu-1)(q-1) + \lambda$ with $\mu \geq 1$ and $1 \leq \lambda \leq q-1$,*

$$v(\mathcal{E}^I) = q^{-\mu}(q-\lambda) \in]0, 1[\cap \mathbb{Z}[p^{-1}].$$

Note that the above valuations do not depend on Σ .

Proof of Proposition 7.2. The sum defining $\mathcal{E}(w; \rho_\Sigma^*)$ splits in two pieces, a sum over the couples $(a, b) \in A \times A$ with $a \neq 0$ and a sum over the couples $(0, b)$ with $b \neq 0$. While the second sum is easily seen to be equal to $-\mathcal{Z}(w; \rho_\Sigma^*)$, for the first sum we have

$$\begin{aligned}
& \sum_{\substack{a \in A \setminus \{0\} \\ b \in A}} \frac{1}{(az + b)^w} \bigotimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(a) \\ \chi_{t_i}(b) \end{pmatrix} = \\
&= \sum_{a \in A \setminus \{0\}} \rho_\Sigma \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \sum_{b \in A} \frac{1}{(az + b)^w} \bigotimes_{i \in \Sigma} \begin{pmatrix} 1 \\ \chi_{t_i}(b) \end{pmatrix} \\
&= \sum_{a' \in A^+} \rho_\Sigma \begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix} \left(\sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{-w} \rho_\Sigma \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \cdot \rho_\Sigma \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \right) \sum_{b' \in A} \frac{1}{(a'z + b')^w} (\sigma_J(b'))_{J \subset \Sigma} \\
&= -\tilde{\pi}^w \sum_{a' \in A^+} \rho_\Sigma \begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix} V(w; \rho_\Sigma^*)_{a'},
\end{aligned}$$

where we made the change of variables $a = \lambda a'$ (with $\lambda \in \mathbb{F}_q^\times$), $b = \lambda b'$ in the summation, and used that $|\Sigma| \equiv w \pmod{q-1}$ because $\sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{|\Sigma|-w} = -1$. Now note that

$$V(w; \rho_\Sigma^*) = \frac{1}{\tilde{\pi}^w} \left((-1)^{|J|} \psi(w; \sigma_J) \right)_{J \subset \Sigma}.$$

The identity concerning the case $J = \emptyset \neq \Sigma$ is clear, and the last identity, concerning the scalar Eisenstein series, is well known; see, for instance, [24, (6.3)]. The last assertion of the proposition is a direct consequence of the fact that $\psi_a(w; \sigma_\Sigma) \in \mathfrak{D}_\Sigma$ for all $a \in A$ and $w \in \mathbb{N}^*$ and the fact that $v(\psi_a(w; \sigma_\Sigma)) = |a|v(\psi_a(w; \sigma_\Sigma)) \rightarrow \infty$ as a runs in A^+ (Proposition 3.35). \square

Thanks to Theorem 6.12 we can compute the v -valuations of the entries of $\mathcal{E}(1; \rho_\Sigma^*)$ (recall that κ has been introduced in (6.10)). The corresponding problem for $\mathcal{E}(w; \rho_\Sigma^*)$ for general w is at the moment unsolved but the reader can apply Theorem 6.16 to some specific cases.

Corollary 7.4. *If $|\Sigma| \equiv 1 \pmod{q-1}$ and $\mathcal{E}(1; \rho_\Sigma^*) = (\mathcal{E}^J)_{J \subset \Sigma}$, we have $v(\mathcal{E}^J) = \kappa(J)$ if $J \subsetneq \Sigma$ and $v(\mathcal{E}^\Sigma) = 0$.*

7.3. Application to modular forms of weight one for ρ_Σ^* . In this subsection we prove Theorem D of the introduction. We recall that $N = 2^s$. We have:

Theorem 7.5. *Assuming that $|\Sigma| \equiv 1 \pmod{q-1}$, $M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ is of dimension one over \mathbb{L}_Σ , generated by the Eisenstein series $\mathcal{E}(1; \rho_\Sigma^*)$.*

Proof. We note that in the case $\rho = \rho_\Sigma^*$ we have the following identity for the space $H(\rho; \mathbb{K}_\Sigma)$ defined in §5.2.3:

$$(7.8) \quad H(\rho; \mathbb{K}_\Sigma) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbb{K}_\Sigma \end{pmatrix}.$$

We claim that if $f = (f_1, \dots, f_N)$ is a modular form for ρ_Σ^* , we can identify f_1, \dots, f_{N-1} with elements of \mathfrak{M}_Σ and f_N with an element of \mathfrak{D}_Σ . Indeed we know already that $f \in \mathfrak{D}_\Sigma^{N \times 1}$. In particular, there exists $\underline{\alpha} \in \mathbb{K}_\Sigma^{N \times 1}$ such that $f \equiv \underline{\alpha} \pmod{\mathfrak{M}^{N \times 1}}$. But note that for all $a \in A$, $f(z+a) = \rho_\Sigma^*(T_a)f(z)$ for all $z \in \Omega$ so that $\underline{\alpha} = \rho_\Sigma^*(T_a)\underline{\alpha}$ for all $a \in A$. Identity (7.8) allows to deduce the claim.

We conclude by observing that $\mathcal{E}(1; \rho_\Sigma^*) \in M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma) \setminus S_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ and applying Theorem 5.12 knowing that in this case, $\delta_\rho = 1$. \square

This yields a positive answer to [63, Problem 1.1]. By Theorem 4.13, $\mathcal{E}(1; \rho_\Sigma^*)$ is an eigenform for all the Hecke operators defined in §4.3. We deduce:

Corollary 7.6. *For all $a \in A \setminus \{0\}$ we have $T_a(\mathcal{E}(1; \rho_\Sigma^*)) = \mathcal{E}(1; \rho_\Sigma^*)$.*

Proof. By Theorem 7.5 $M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ is one-dimensional generated by $\mathcal{E}(1; \rho_\Sigma^*)$ and we have $T_a(\mathcal{E}(1; \rho_\Sigma^*)) = \lambda_a \mathcal{E}(1; \rho_\Sigma^*)$ for all $a \in A \setminus \{0\}$ for elements $\lambda_a \in \mathbb{L}_\Sigma$. It suffices to show that $\lambda_P = 1$ for every irreducible element $P \in A$ by using the Hecke operators T_P described in (4.20). We set $f = \mathcal{E}(1; \rho_\Sigma^*)$. In (4.20), $g := \sum_{|b| < |P|} \rho \begin{pmatrix} 1 & b \\ 0 & P \end{pmatrix}^{-1} f \left(\frac{z+b}{P} \right) \in \mathfrak{M}_\Sigma^{N \times 1}$. Indeed, let f_1, \dots, f_N be the entries of f . We have $f_1, \dots, f_{N-1} \in \mathfrak{M}_\Sigma$ and $f_N \in \mathfrak{D}_\Sigma$. This implies that the first $N-1$ coefficients of g are in \mathfrak{M}_Σ and by (4.22) the last coefficient of g is

$$P^{-1} \sigma_\Sigma(P) \sum_{|b| < |P|} f_N \left(\frac{z+b}{P} \right)$$

so it is an element of $\mathbb{L}_\Sigma^\circ \langle \langle \underline{e} \rangle \rangle^b$ with zero constant term. Hence λ_P equals the lower right coefficient of $\rho_\Sigma^* \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ which is equal to 1. \square

7.3.1. Digression: another class of Eisenstein series. One of the main motivations for the introduction of the Eisenstein series $\mathcal{E}(w; \rho_\Sigma^*)$, for which they have been initially considered in [55], is that the non-zero entry (which is the last one, in the prescribed ordering) tends to $-\zeta_A(w; \sigma_\Sigma)$ (the zeta values defined in (1.17)) as $z \in \Omega$ approaches the cusp infinity or, in other words, it is congruent to $-\zeta_A(w; \sigma_\Sigma)$ modulo \mathfrak{M}_Σ . These are not the only Eisenstein series which enjoy this property. Another example is discussed in this remark; further investigations will lead to a better understanding of these examples. We consider

the \mathbb{F}_q -algebra morphism $\chi : A \rightarrow \mathbb{F}_q[t_\Sigma]^{s \times s}$ (with $s = |\Sigma|$) defined by

$$\vartheta = \chi(\theta) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -P_0 & -P_1 & \cdots & -P_{s-1} \end{pmatrix},$$

where $P_0, \dots, P_{s-1} \in \mathbb{F}_q[t_\Sigma]$ are defined by $\prod_{i \in \Sigma} (X - t_i) = X^s + P_{s-1}X^{s-1} + \dots + P_0$. Then, for all $a \in A$, $\det(\chi(a)) = \sigma_\Sigma(a)$ (see [57, §2.1]). We consider the representation of the first kind $\varphi_\Sigma^* = \wedge^s \rho_\chi^*$, of dimension $N := \binom{2s}{s}$. We suppose that $w \equiv s \pmod{q-1}$ and $w > 0$. Just as in Lemma 7.1, the last column of the Poincaré series $\mathcal{P}_w(\Phi_{\varphi_\Sigma^*})$ multiplied by $\zeta_A(w; \sigma_\Sigma)$ equals

$$\mathcal{E}(w; \varphi_\Sigma^*) := \sum_{(a,b) \in A \setminus \{(0,0)\}} (az + b)^{-w} \wedge^s \begin{pmatrix} \chi(a) \\ \chi(b) \end{pmatrix}.$$

This defines an element of $\text{Hol}(\Omega \rightarrow \mathbb{E}_\Sigma^{N \times 1})$ and a modular form in $M_w(\varphi_\Sigma^*; \mathbb{K}_\Sigma) \setminus S_w(\varphi_\Sigma^*; \mathbb{K}_\Sigma)$. Moreover, the only entry \mathcal{E}_N of $\mathcal{E}(w; \varphi_\Sigma^*)$ which does not vanish at infinity, which is the last one, satisfies

$$\mathcal{E}_N \equiv -\zeta_A(w; \sigma_\Sigma) \pmod{\mathfrak{M}_\Sigma}.$$

In other words, $-\zeta_A(w; \sigma_\Sigma)$ is the 'constant term' of the last entry of $\mathcal{E}(w; \varphi_\Sigma^*)$. It is evident that $\mathcal{E}(w; \varphi_\Sigma^*) \neq \mathcal{E}(w; \rho_\Sigma^*)$.

7.4. Rationality and integrality of coefficients. We investigate rationality and integrality properties of coefficients of Eisenstein series. Our main result in this subsection is Theorem 7.8, in the same vein as classical results of Gekeler [24, §5]. We will also obtain, with an alternative proof, a weaker version of [2, Theorem 1] and a generalization of the principles of [56, Theorem 8], namely, a 'modular proof' of Theorem 7.9.

Definition 7.7. An element $f \in M_w^1(\rho; \mathbb{K}_\Sigma)$ is said to be *rationally definable* if there exists a matrix $M \in \text{GL}_N(\mathbb{K}_\Sigma)$ such that the image of Mf by the embedding ι_Σ of Theorem 4.12 is an element of $K(t_\Sigma)^\circ \langle \langle \underline{e} \rangle \rangle^b ((u))^{N \times 1}$. It is *integrally definable* if this image lies in $A[t_\Sigma]^\circ \langle \langle \underline{e} \rangle \rangle^b [u^{-1}][[u]]^{N \times 1}$. If $\mathfrak{v} : K(t_\Sigma) \rightarrow \mathbb{Z} \cup \{\infty\}$ is a valuation of $K(t_\Sigma)$ we say that a rationally defined element $f \in M_w^1(\rho; \mathbb{K}_\Sigma)$ is *\mathfrak{v} -integrally definable* if, writing f_i for the i -th entry of Mf with M the above mentioned matrix and expanding it as a formal series $f_i = \sum_{j \geq j_0} f_{i,j} u^j$ with $f_{i,j} \in K(t_\Sigma)^\circ \langle \langle \underline{e} \rangle \rangle^b$, which can be done in a unique way after Proposition 3.32, we have $\mathfrak{v}(f_{i,j}) \geq 0$ for all i, j .

Note that if $N = 1$ and $\Sigma = \emptyset$, this coincides, up to multiplication by a proportionality factor, with the scalar modular forms having u -expansions in $K((u))$ and $A[[u]]$, or \mathfrak{v} -integral respectively.

We borrow from Proposition 7.2 the notation \mathcal{E}^I that designates the I -th entry of $\mathcal{E} = \mathcal{E}(m; \rho_\Sigma^*)$ with $I \subset \Sigma$, $|\Sigma| \equiv m \pmod{q-1}$. Also, we recall that $\omega_I = \prod_{i \in I} \omega(t_i) \in \mathbb{T}_\Sigma^\times$. We have:

Theorem 7.8. *For all $I \subsetneq \Sigma$ we have*

$$\mathcal{E}_I \in \tilde{\pi}^m \omega_I^{-1} \left(K(\underline{t}_\Sigma)^\circ \langle \underline{e} \rangle^b((u)) \cap \mathfrak{M}_\Sigma \right)$$

and

$$\mathcal{E}_\Sigma \in -\zeta_A(m; \sigma_\Sigma) + \tilde{\pi}^m \omega_\Sigma^{-1} \left(K(\underline{t}_\Sigma)^\circ \langle \underline{e} \rangle^b((u)) \cap \mathfrak{M}_\Sigma \right).$$

Moreover, $\mathcal{E}(m; \rho_\Sigma^*)$ is \mathfrak{v} -integrally definable for the valuations of $K(\underline{t}_\Sigma)$ associated with a non-zero prime ideal \mathfrak{p} of A , and this for all but finitely many \mathfrak{p} .

The proof of Theorem 7.8 is easy if we introduce another class of matrix-valued functions. As seen in §5.3, Poincaré series naturally occur as square matrix functions. On the other hand, Eisenstein series, following our constructions in §5.4, are defined as vector functions. The following matrix function is very useful in studying Eisenstein series for the representation of the first kind ρ_Σ^* :

$$(7.9) \quad \mathcal{E}(m; \rho_\Sigma^*) := \sum'_{c \in A} \rho_\Sigma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \Psi_m(cz) + E_\Sigma \sum'_{d \in A} d^{-m} \rho_\Sigma^*(T_{-d}),$$

where $m > 0$, E_Σ denotes, with $N = 2^s$, $s = |\Sigma|$, the $N \times N$ -matrix with zero coefficients, except the bottom-right coefficient which is equal to 1, $\Psi_m(z) = \Psi_m(\rho_\Sigma^*)(z)$ (as defined in §4.1.1) and the sums over c, d run in $A \setminus \{0\}$. We have, as it is easily seen,

$$\mathcal{E}(m; \rho_\Sigma^*) \in \text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow \mathbb{E}_\Sigma^{N \times N}).$$

There is a bijection between the columns of $\mathcal{E}(m; \rho_\Sigma^*)$ and the subsets I of Σ . We use the ordering described at the beginning of §7.2 and we denote by \mathcal{E}_I the I -th column in such a way that the first column corresponds to $I = \Sigma$ and the last one to $I = \emptyset$. It is easy to show that

$$\mathcal{E}_I = \mathcal{E}(m; \rho_I^*) \otimes \bigotimes_{j \in \Sigma \setminus I} I_2 \in M_m \left(\rho_\Sigma^* \otimes \bigotimes_{j \in \Sigma \setminus I} \mathbf{1}_2; \mathbb{K}_\Sigma \right),$$

where $\mathbf{1}_2$ is the representation (of the first kind) $\gamma \in \Gamma \mapsto I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that the first column \mathcal{E}_Σ equals $\mathcal{E}(m; \rho_\Sigma^*)$ (compare with (7.3)).

Proof of Theorem 7.8. By (6.5) we see $(G_m(\rho)_c := G_m(\rho)_{z \mapsto cz})$ that

$$\mathcal{E}(m; \rho_\Sigma^*) = \underbrace{\tilde{\pi}^m \sum'_{c \in A} \rho_\Sigma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} G_m(\rho_\Sigma^*)_c}_{=:\mathcal{E}} + E_\Sigma \sum'_{d \in A} d^{-m} \rho_\Sigma^*(T_{-d}).$$

In virtue of Corollary 6.7 we have (recall the definitions (4.16)) that

$$\mathcal{E} \in \mathfrak{M}_\Sigma^{N \times N} \cap \tilde{\pi}^m \omega_{\rho_\Sigma^*}^{-1} K(\underline{t}_\Sigma)^\circ \langle \underline{e} \rangle^b[[u]]^{N \times N} \omega_{\rho_\Sigma^*}.$$

Additionally we see that the coefficients are \mathfrak{v} -integral for \mathfrak{v} as expected in the statement of the theorem. From this it is very easy to conclude the proof. \square

7.5. Negative weight modular forms and functional identities. From the theory described in the present work we can deduce a weaker form of the functional zeta-identities for the *zeta values in Tate algebras* of [2, Theorem 1] (the zeta values defined in (1.17)). We shall give a ‘modular’ proof of the following:

Theorem 7.9. *Let Σ be a finite subset of \mathbb{N}^* and $w > 0$ an integer such that $w \equiv |\Sigma| \pmod{q-1}$. If q is larger than a constant depending on $|\Sigma|$ and w then*

$$\zeta_A(w; \sigma_\Sigma) = \lambda_{w, \Sigma} \frac{\tilde{\pi}^w}{\omega_\Sigma},$$

where $\lambda_{w, \Sigma}$ is an element of $K(\underline{t}_\Sigma)^\times$.

This result is weaker than the original result of loc. cit. In that reference, there is no condition on q and denominators of $\lambda_{w, \Sigma}$ are explicitly computed. The interest in presenting this in our work in this part exclusively relies in the nature of the proof. Indeed, we will derive the formula of Theorem 7.9 from a comparison between the constant term and the positive terms of the tame series expansion of our Eisenstein series, in a way which is not completely different from Serre’s [73].

Previously, ‘modular proofs’ of such identities were only known in the two-dimensional case [55, Theorem 8] and in the slightly, and partially, more general setting of [60, Theorem 4.9.9]. Both proofs rest in fact on a duality principle between on one side modular forms of weight w in our settings and on the other side, weak modular forms of weight -1 , for suitable choices of representations (a representation and its contragredient). Functional identities occur in the comparison of rational structures arising from duality. For the sake of simplicity, we will only illustrate the case $w = 1$ here (this case contains the main principles so it is the most relevant). As the zeta values (1.17) are also involved in certain variants of *Taelman’s class number formula* in [76], see [5], it is certainly desirable to extend Taelman’s theory to the settings of the present paper. We are going to prove that, in order to reach our conclusion, it suffices that

$$q > 2(m-1)^{m-1}$$

if $|\Sigma| = m(q-1) + 1$ with $m \geq 1$ ⁽¹²⁾.

The original proof of Theorem 7.9 in [2] notably relies on certain arithmetic properties of Gauss-Thakur sums. There are other available proofs for similar results. In [4, Theorem 5.7], a class of even more general functional identities is proved ⁽¹³⁾, as an application of a generalized variant of *Anderson log-algebraicity theorem*, to certain ‘Dirichlet-like values in Tate algebras’. Other proofs of variants, or similar or more general results are equally available in the literature.

¹²The condition on q can be improved to the price of a more complicated proof which is not suitable for the present work. We know that the result is true for any choice of q , but we do not know how to make the present proof unconditional.

¹³That is, relative to a ‘base ring’ $A = H^0(X \setminus \{\infty\}, \mathcal{O}_X)$ where X is a projective, smooth, geometrically irreducible curve over \mathbb{F}_q and ∞ a point of $X(\mathbb{F}_q)$.

7.5.1. *Duality.* We begin with two lemmas. In the first lemma we consider two subfields $F \subset L$ of \mathbb{K}_Σ and a representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(F)$. The lemma is applied with the choices $F = \mathbb{F}_q(\underline{t}_\Sigma)$ and $L = \mathbb{L}_\Sigma$ or $L = \mathbb{K}_\Sigma$.

Lemma 7.10. *Assume that ρ is irreducible and let f be an element in $\mathcal{M}_w^{\mathrm{ll}}(\rho; L)$. If the entries of f are linearly dependent over L , then f vanishes identically.*

Proof. Straightforward, but we prefer to give full details. Let V be the L -subspace of $L^{1 \times N}$ the elements of which are the v 's such that $v \cdot f = 0$. Assume that $V \neq \{0\}$ and let us consider $\gamma \in \Gamma$. Then

$$0 = v \cdot f(\gamma^{-1}(z)) = J_{\gamma^{-1}}(z)^w v \cdot \rho(\gamma^{-1})f(z).$$

Hence $v \cdot \rho(\gamma^{-1}) \in V$ and this, for all $\gamma \in \Gamma$. This means that ρ^* has the invariant space $W = {}^tV$ that is, for all $\gamma \in \Gamma$, $\rho^*(\gamma)W \subset W$ with $W \neq \{0\}$. But ρ is irreducible if and only if ρ^* is irreducible. \square

In the next lemma we choose $L = \mathbb{K}_\Sigma$ and $F = \mathbb{F}_q(\underline{t}_\Sigma)$. We give explicit examples of irreducible representations of the first kind.

Lemma 7.11. *For all Σ finite subset of \mathbb{N}^* the representations*

$$\rho_\Sigma, \rho_\Sigma^* : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{K}_\Sigma)$$

are irreducible.

Proof. Since $\mathbb{F}_q(\underline{t}_\Sigma)$ is contained in the residual field $\mathbb{F}_q^{\mathrm{ac}}(\underline{t}_\Sigma)$ of \mathbb{K}_Σ , if the statement of the lemma were false there would exist a non-trivial subvector space $\{0\} \subsetneq U \subsetneq \mathbb{F}_q^{\mathrm{ac}}(\underline{t}_\Sigma)^{N \times 1}$ such that $\rho_\Sigma(\gamma)U \subset U$ for all $\gamma \in \Gamma$. This would be, however, in contradiction with [57, Theorem 14]. \square

In particular, the representations $\rho_\Sigma, \rho_\Sigma^*$ are irreducible for Σ a finite subset of \mathbb{N}^* .

We set

$$\mathcal{E} := \mathcal{E}(1; \rho_\Sigma^*),$$

where $\mathcal{E}(1; \rho_\Sigma^*)$ is the Eisenstein series of weight 1 defined in §7. The main result in the present subsection is the following, where $|\Sigma|, m \geq 1$ are as above:

Proposition 7.12. *If $m = 1, 2$ or if $m \geq 3$ and $q > 2(m-1)^{m-1}$, there exists a non-zero element*

$$(7.10) \quad \mathcal{F} \in M_{-q^m}^1(\rho_\Sigma; \mathbb{L}_\Sigma) \cap \tilde{\pi}^{-1} \bigotimes_{j \in \Sigma} \begin{pmatrix} 1 & 0 \\ 0 & \omega(t_j) \end{pmatrix} \left(K(\underline{t}_\Sigma)^\circ \langle \underline{e} \rangle^b((u)) \right)^{N \times 1}$$

such that

$$(7.11) \quad {}^t\mathcal{E} \cdot \mathcal{F} = 0.$$

Before proving it, we show how this result implies Theorem 7.9.

Proof of Theorem 7.9. By Lemma 7.11, given $\mathcal{F} = (\mathcal{F}^I)_{I \subset \Sigma}$ as in the statement of Proposition 7.12, we have \mathcal{F}^Σ non-zero. Hence we have an identity of non-zero elements of \mathfrak{K}_Σ :

$$\mathcal{E}^\Sigma \mathcal{F}^\Sigma = - \sum_{I \subsetneq \Sigma} \mathcal{E}^I \mathcal{F}^I.$$

By Theorem 7.8 and (7.10) the right-hand side is an element of $K(\underline{t}_\Sigma)^\circ \langle \langle \underline{e} \rangle \rangle^b((u))$. Moreover, the left-hand side is the sum of a non-zero element of $-\zeta(1; \sigma_\Sigma) \omega_\Sigma \tilde{\pi}^{-1} K(\underline{t}_\Sigma)^\circ \langle \langle \underline{e} \rangle \rangle^b((u))$ and an element of $K(\underline{t}_\Sigma)^\circ \langle \langle \underline{e} \rangle \rangle^b((u))$. The theorem follows. \square

The reader can easily deduce, from a small variation of these arguments, and a simple explicit computation (case $m = 0$), that

$$(7.12) \quad \zeta_A(1; \chi_t) = \frac{\tilde{\pi}}{(\theta - t)\omega(t)}$$

which holds in \mathbb{T} . In this case, the proof is very similar to that given in [55], where this formula was first noticed.

Proof of Proposition 7.12. We shall use certain weak modular forms of weight -1 associated to $\rho_{\chi_{t_i}}$ that have been originally introduced in [55]. In the classical theory of complex-valued vectorial modular forms there seem to be no analogue of these forms yet. By [60, Theorem 4.9.3],

$$\mathcal{F}_i(z) := \exp_{\Lambda_z} \left(\begin{pmatrix} z \\ 1 \end{pmatrix} \frac{1}{\theta - t_i} \right) \in \text{Hol}(\Omega \rightarrow \mathbb{L}_\Sigma)^{2 \times 1}$$

where $\exp_{\Lambda_z} : \mathbb{L}_\Sigma \rightarrow \mathbb{L}_\Sigma$ is the exponential function associated to the A -lattice of rank two $\Lambda_z = Az \oplus A$ with $z \in \Omega$ studied in §2.3.2, is an element of $M_{-1}^1(\rho_{\chi_{t_i}}; \mathbb{L}_\Sigma)$. If we write

$$\mathcal{F}_i = \begin{pmatrix} \mathcal{F}_{i,1} \\ \mathcal{F}_{i,2} \end{pmatrix},$$

then the proof of loc. cit. yields that

$$(7.13) \quad v(\mathcal{F}_{i,1}) = -\frac{1}{q}, \quad v(\mathcal{F}_{i,2}) = 0, \quad i \in \Sigma.$$

An application of [60, Lemma 4.9.4] that we leave to the reader ensures that

$$\mathcal{F}_i \in \tilde{\pi}^{-1} \left(\begin{pmatrix} 1 & 0 \\ 0 & \omega(t_i) \end{pmatrix} K(\underline{t}_\Sigma)^\circ \langle \langle \underline{e} \rangle \rangle^b[[u]]^{2 \times 1} \right).$$

We are given with $m > 0$ and $\Sigma \subset \mathbb{N}^*$ with $|\Sigma| = m(q - 1) + 1$. We now proceed to construct a class of weak modular forms of weight $-q^m$. We consider the set $\mathbb{U}(\Sigma)$ whose elements are the ordered m -tuples

$$\underline{U} = (U_1, \dots, U_m) \subset (\Sigma, \dots, \Sigma)$$

with $\Sigma = U_1 \sqcup \dots \sqcup U_m$ and $|U_1| = \dots = |U_{m-1}| = q - 1$ (hence $|U_m| = q$). For instance, if $|\Sigma| = q$, $m = 0$ and $\mathbb{U}(\Sigma) = \{(\Sigma)\}$.

We choose $\underline{U} \in \mathbb{U}(\Sigma)$ and we define

$$\mathcal{F}_{\underline{U}} = \bigotimes_{i=1}^m \bigotimes_{j \in U_i} \mathcal{F}_j^{(m-i)}.$$

The condition on \underline{U} and [60, Lemma 4.9.4] imply that

$$\mathcal{F}_{\underline{U}} \in M_{-q^m}^!(\rho_{\Sigma}; \mathbb{L}_{\Sigma}) \setminus \{0\}.$$

The rationality property of the tame series expansion of the forms \mathcal{F}_j along with the functional identity (2.17) implies that

$$\mathcal{F}_{\underline{U}} \in \tilde{\pi}^{-1} \left(\bigotimes_{j \in \Sigma} \begin{pmatrix} 1 & 0 \\ 0 & \omega(t_i) \end{pmatrix} \right) K(t_{\Sigma})^{\circ} \langle \langle \underline{e} \rangle \rangle^b ((u))^{N \times 1}.$$

We have lower bounds for the v -valuations of the entries of $\mathcal{F}_{\underline{U}}$. By (7.13) and by the choice of \underline{U} we have, writing $\mathcal{F}_{\underline{U}} = (\mathcal{F}_{\underline{U}}^I)_{I \subset \Sigma}$,

$$(7.14) \quad v(\mathcal{F}_{\underline{U}}^I) = - \sum_{i=1}^m q^{m-i} |U_i \setminus I| \geq -q^{m-1}, \quad I \subset \Sigma.$$

In the lower bound in the right, we have equality if and only if $I = \emptyset$, and if $I = \Sigma$ we get $v(\mathcal{F}_{\underline{U}}^{\Sigma}) = 0$.

Recall that $\mathcal{E} \in M_1(\rho_{\Sigma}^*; \mathbb{L}_{\Sigma})$. For all the matrices $\gamma \in \Gamma$ it is clear that ${}^t(\rho_{\Sigma}^*(\gamma)) \cdot \rho_{\Sigma}(\gamma)$ is the identity matrix of size N . Hence

$${}^t\mathcal{E} \cdot \mathcal{F}_{\underline{U}} \in M_{1-q^m}^!(\mathbf{1}; \mathbb{L}_{\Sigma}) = M_{1-q^m}^!(\mathbf{1}; \mathbb{C}_{\infty}) \otimes_{\mathbb{C}_{\infty}} \mathbb{L}_{\Sigma},$$

and ${}^t\mathcal{E} \cdot \mathcal{F}_{\underline{U}} \in \mathbb{K}_{\Sigma}((u^{q-1}))$. By (7.14) and the comments that follow it, and by the fact that $v(\mathcal{E}^{\emptyset}) = 1$, $v(\mathcal{E}^I) > 0$ for $i \neq \emptyset, \Sigma$ and $v(\mathcal{E}^{\Sigma}) = 0$ (see Corollary 7.3),

$$v({}^t\mathcal{E} \cdot \mathcal{F}_{\underline{U}}) \geq 1 - q^m.$$

If $m = 1$ we see that, setting $\mathcal{F} = \mathcal{F}_{\underline{U}}$ with \underline{U} the unique element of $\mathbb{U}(1)$, ${}^t\mathcal{E} \cdot \mathcal{F} \in M_{1-q}(\mathbf{1}; \mathbb{L}_{\Sigma}) = (0)$ and the proposition is proved in this case (unconditionally on q).

We now suppose that $m \geq 2$. We consider

$$\mathcal{F} \in \text{Vect}_{K(t_{\Sigma})} \left(\mathcal{F}_{\underline{U}} : \underline{U} \in \mathbb{U}(m) \right).$$

We can expand

$$\begin{aligned} {}^t\mathcal{E} \cdot \mathcal{F} &= c_{\mu} u^{-\mu(q-1)} + c_{\mu-1} u^{-(\mu-1)(q-1)} + \dots + c_1 u^{-(q-1)} + \\ &\quad + \left(\text{element of } \mathbb{L}_{\Sigma}((u^{q-1})) \right), \quad c_1, \dots, c_{\mu} \in \mathbb{L}_{\Sigma}, \end{aligned}$$

where

$$\mu := \mu(m) := 1 + q + \dots + q^{m-2}$$

(we can also set $\mu(1) := 0$). Since $v(\mathcal{F}_{\underline{U}}^{\Sigma}) = 0$, we can also expand

$${}^t\mathcal{E} \cdot \mathcal{F} \in K(t_{\Sigma})((u^{q-1})) + \zeta_A(1; \sigma_{\Sigma}) \tilde{\pi}^{-1} \omega_{\Sigma} K(t_{\Sigma})[[u^{q-1}]].$$

In particular, we have $c_1, \dots, c_\mu \in K(\underline{t}_\Sigma)$ for any choice of \mathcal{F} as above (note that a priori, a coefficient of the u -expansion of such an \mathcal{F} needs not to be in $K(\underline{t}_\Sigma)$, only the coefficients corresponding to negative powers have this property). We set

$$\nu(m) := \dim_{K(\underline{t}_\Sigma)} \left(\text{Vect}_{K(\underline{t}_\Sigma)} (\mathcal{F}_{\underline{U}} : \underline{U} \in \mathbb{U}(m)) \right).$$

To show the proposition we can use the next lemma, with $\nu := \nu(m)$.

Lemma 7.13. *We have $\nu(2) > \mu(2)$ for all q and $\nu(m) > \mu(m)$ if $m \geq 3$ and $q > 2(m-1)^{m-1}$.*

Proof. It suffices to find $I \subset \Sigma$ such that the set

$$\{v(\mathcal{F}_{\underline{U}}^I) : \underline{U} \in \mathbb{U}(\Sigma)\}$$

has cardinality $> \mu$. Note that $\mu(2) = 1$. In this case, it suffices to choose $I = \{i\} \subset \Sigma$ a singleton as there certainly are at least two elements in the above set, regardless of the choice of q , so that $\nu(2) \geq 2$.

We now assume that $m \geq 3$. Here we need a few combinatorial additional remarks. The number of m -tuples (i_1, \dots, i_m) with $i_k \in \mathbb{N}$ for all k and $i_1 + \dots + i_m = i$, with i given, is $\binom{i-1}{m-1}$. Suppose $i = q+1$ (assuming that $q \geq m-1$). The number of m -tuples as above with $i_k \leq q-1$ that sum to $q+1$ is therefore

$$\nu_0 = \binom{q}{m-1} - m - \binom{m}{2}$$

(we subtract two terms to exclude m -tuples of the type $qe_k + e_h$ with $(e_r)_r$ the canonical basis). The integer ν_0 is a lower bound for $\nu(m)$ because given any subset $I \subset \Sigma$ of cardinality $q+1$, and given any (i_1, \dots, i_m) as above, there are at least ν_0 m -tuples $\underline{U} \in \mathbb{U}(m)$ such that $|U_j \setminus I| = i_j$. The valuations $v(\mathcal{F}_{\underline{U}}^I)$ for such \underline{U} are all distinct (because by (7.14) they correspond to integers that have distinct q -ary expansions). It is elementary to verify that if $q \geq 2m$ then $\nu_0 \geq \frac{1}{2} \binom{q}{m-1}$. Now, $\binom{q}{m-1} \geq \left(\frac{q}{m-1}\right)^{m-1}$ so that if additionally $q > 2(m-1)^{m-1}$, we get the desired lower bound $\nu \geq \nu_0 > \mu$. Note that we have used, in a crucial way, that the u -expansion of ${}^t\mathcal{E} \cdot \mathcal{F}$ is in $\mathbb{L}_\Sigma((u^{q-1}))$. Having solved linear equations corresponding to initial parts of formal series in $\mathbb{L}_\Sigma((u))$ would have lead to too many equations and we would have needed supplementary arguments to achieve this proof. \square

Applying Lemma 7.13 we can construct $\mathcal{F} \in M_{-q}^1(\rho_\Sigma; \mathbb{L}_\Sigma)$ non-zero with the property that ${}^t\mathcal{E} \cdot \mathcal{F} \in M_{1-q}(\mathbf{1}; \mathbb{L}_\Sigma)$. The latter space is zero and ${}^t\mathcal{E} \cdot \mathcal{F} = 0$. \square

Remark 7.14. There is no apparent reason, in the above proof, that a modular form as in Proposition 7.12 is unique up to scalar multiplication. We did not compute the dimension $\nu(m)$ and we do not know if the condition on q , when $m \geq 3$, is really necessary. Note that the set $\mathbb{U}(\Sigma)$ can be enlarged; see §10.2.1. It would be interesting to construct explicitly a basis for the space of modular forms satisfying the properties illustrated in Proposition 7.12. In fact, the isomorphism (4.14) could allow to bypass these constructions of modular forms in negative weight. The principle is simple. Choose, for example, two Eisenstein series \mathcal{E} and \mathcal{E}' for ρ^* of weights w and w' respectively. If $\tilde{\mathcal{E}}'$ denotes the image of \mathcal{E} via

the isomorphism (4.14), so it is a modular form of weight w' for $\det^{\deg(\rho)} \rho$. Then the inner product $f := {}^t\mathcal{E} \cdot \widetilde{\mathcal{E}}'$ is in $M_{w+w'}(\det^{\deg(\rho)}; \mathbb{L}_\Sigma)$ and one can compute explicitly the 'first' coefficients of the u -expansion. By using that this is a finite-dimensional \mathbb{L}_Σ -vector space, this in turn leads in explicit non-trivial linear dependence relations involving elements of \mathbb{T}_Σ that explicitly contain $\zeta_A(w; \sigma_\Sigma)$ and $\zeta_A(w'; \sigma_\Sigma)$. The author tried several examples, some of which are complicated already in small weight. What we did above seems to be fairly simple, but the reader must be aware that this is not the only way to approach results such as Theorem 7.9.

7.6. Some applications to quasi-modular and \mathfrak{v} -adic modular forms. In this subsection we illustrate how constructions of Drinfeld modular forms defined over Ω with values in \mathbb{C}_∞ having ' A -expansions' as considered by Petrov in [67] can be naturally carried out as evaluations of our Eisenstein series $\mathcal{E}(m; \rho_\Sigma^*)$ at certain specific points. This also leads to some properties of \mathfrak{v} -adic modular forms with \mathfrak{v} a valuation of $K(t_\Sigma)$ that will be sketched at the end of the present subsection to illustrate further directions of research.

Consider a finite subset $\Sigma \subset \mathbb{N}^*$ of cardinality s and, for $i \in \Sigma$, integers $k_i \in \mathbb{N}$. With $\underline{k} = (k_i)_{i \in \Sigma} \in \mathbb{N}^\Sigma$, set $\text{ev} = \text{ev}_{\vartheta^{q\underline{k}}}$ the evaluation map that sends an element f of $\mathbb{E}_\Sigma^{M \times N}$ for integers M, N to

$$\text{ev}(f) = (f)_{t_i = \vartheta^{q^{k_i}}, \forall i \in \Sigma} \in \mathbb{C}_\infty^{M \times N}.$$

The family $\underline{k} \in \mathbb{N}^\Sigma$ is fixed all along the subsection.

The Eisenstein series $\mathcal{E}(m; \rho_\Sigma^*)$ defines a non-zero rigid analytic function $\Omega \rightarrow \mathbb{E}_\Sigma^{N \times 1}$ with $N = 2^s$. Hence the evaluation $\text{ev}(\mathcal{E}(m; \rho_\Sigma^*))$ can be viewed as a rigid analytic function $\Omega \rightarrow \mathbb{C}_\infty^{N \times 1}$. We recall, from [67], the series

$$(7.15) \quad f_{k,m} = \sum_{a \in A^+} a^{k-m} G_m(u_a) \in K[[u]],$$

running over the monic polynomials in A . This series converges in $K[[u]]$ (for the u -adic valuation) for every $m > 0$ and $k \in \mathbb{Z}$.

We show the following result, where we use the notion of *Drinfeld quasi-modular form* introduced in [13, Definition 2.1], answering a question that D. Goss addressed to A. Petrov [37] on the general nature of the A -series defined in (7.15).

The \mathbb{C}_∞ -algebra

$$\widetilde{M} := \mathbb{C}_\infty[E, g, h],$$

where g, h are the already discussed normalized modular forms, respectively in $M_{q-1}(\mathbf{1}; \mathbb{C}_\infty)$ and $M_{q+1}(\det^{-1}; \mathbb{C}_\infty)$ (Gekeler's notations) and E is the false Eisenstein series (6.8), has dimension 3 (this is not difficult to see) and is $\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ -graded by weights and types, and filtered by depths, where weights and types of E, g, h are respectively $(2, 1), (q-1, 0), (q+1, 1)$, and the depth is just the degree in E . We recall that a polynomial f of \widetilde{M} that is homogeneous of weight w , type m , and has depth $\leq l$ is by definition a Drinfeld quasi-modular form of same weight, type, and depth. For example, E is a quasi-modular form of weight 2, type 1 and depth ≤ 1 which is not a modular form, and the

\mathbb{C}_∞ -algebra M of Drinfeld modular forms can be identified with the sub-algebra of \widetilde{M} of the quasi-modular forms that have depth ≤ 0 .

We suppose that $s = |\Sigma|$ and the integer $m > 0$ are chosen so that $s \equiv m \pmod{q-1}$. We also set

$$l := \sum_{i \in \Sigma} q^{k_i} \in \mathbb{N},$$

so that $l \equiv s \pmod{q-1}$.

Theorem 7.15. *The first entry of $\text{ev}(\mathcal{E}(m; \rho_\Sigma^*))$ equals $-\widetilde{\pi}^m f_{l+m, m}$ and is a non-zero quasi-modular form of weight $l+m$ type m and depth $\leq l$.*

7.6.1. *Preliminaries, Hypothesis H.* We choose a representation of the first kind $\rho : \Gamma \rightarrow \text{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ satisfying the next:

Hypothesis H. *We suppose that ρ is constructed starting from the basic representations ρ_{t_i} with $i \in \Sigma$ applying the usual elementary operations $\oplus, \otimes, S^\alpha, \wedge^\beta, (\cdot)^*$.*

Assuming the Hypothesis H amounts to make an initial restriction on the basic representations used to define ρ . This condition can be relaxed but is convenient for our exposition.

We note that the matrix functions Ξ_ρ, Φ_ρ introduced in §4.2.1 and §4.2.2 belong to $\text{GL}_N(\mathbb{E}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b)$ for $N \geq 1$ so that

$$\widetilde{\Phi}_\rho := \text{ev}(\Phi_\rho), \quad \widetilde{\Xi}_\rho := \text{ev}(\Xi_\rho)$$

define entire functions $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty^{N \times N}$.

Lemma 7.16. *Assuming the Hypothesis H we have $\widetilde{\Phi}_\rho, \widetilde{\Xi}_\rho \in \text{GL}_N(\mathbb{C}_\infty[z])$.*

Proof. We begin by proving the property for $\widetilde{\Phi}_\rho$. The Hypothesis H implies that every entry of Φ_ρ ‘comes from Perkins’ series’ in that they are of the type

$$\sum_{a \in A} (z - a)^{-1} \Theta(a)$$

where $\Theta : A \rightarrow \mathbb{F}_q[\underline{t}_\Sigma]$ is a map such that there exists a polynomial $P \in \mathbb{F}_p[X_1, \dots, X_r]$ (for some r) and semi-characters $\sigma_1, \dots, \sigma_r : A \rightarrow \mathbb{F}_q[\underline{t}_\Sigma]$ (see Definition 6.9) such that $\Theta(a) = P(\sigma_1(a), \dots, \sigma_r(a))$ for all $a \in A$. Hence, to prove the lemma, it suffices to show that, with $f = e_0 \psi(1; \sigma_\Sigma)$ (ψ is a Perkins series, see Definition 6.10). We have

$$\widetilde{f} := \text{ev}(f) \in \mathbb{C}_\infty[z].$$

To justify this we note that after [61, Theorem 2],

$$f(z) = \frac{\widetilde{\pi} \prod_{i \in \Sigma} \exp_C \left(\frac{\widetilde{\pi} z}{\theta - t_i} \right)}{\prod_{i \in \Sigma} \omega(t_i)} + e_0 g(z)$$

where $g : \mathbb{C}_\infty \rightarrow \mathbb{E}_\Sigma$ is an entire function which vanishes identically after evaluation ev at $t_\Sigma = \theta^{q^k}$. Recall that

$$\chi_{t_i}(z) = \omega(t_i)^{-1} \exp_C \left(\frac{\tilde{\pi} z}{\theta - t_i} \right),$$

where \exp_C is the exponential map associated to Carlitz's module as discussed in §2.3.2, for all $i \in \Sigma$ and for all $z \in \mathbb{C}_\infty$. It is therefore easily seen that

$$\text{ev}(\chi_{t_i}(z)) = z^{q^{k_i}}.$$

Hence the claimed property of \tilde{f} follows, and together with it, that of $\tilde{\Phi}_\rho$.

To show that $\tilde{\Xi}_\rho \in \text{GL}_N(\mathbb{C}_\infty[z])$ it suffices to verify it for $\rho = \rho_{t_i}$ with $i \in \Sigma$ so we assume now $\Sigma = \{i\}$ and $k = k \geq 0$. In this case however, $\tilde{\Phi}_\rho = \tilde{\Xi}_\rho = \begin{pmatrix} 1 & z^{q^k} \\ 0 & 1 \end{pmatrix}$, and thanks to the Hypothesis H,

$$(7.16) \quad \tilde{\Xi}_\rho = \text{ev} \left(\rho(T_\theta) \right)_{\theta \mapsto z} \in \text{GL}_N(\mathbb{F}_q[z]).$$

The proof of the lemma is complete. \square

We can now prove:

Lemma 7.17. *Under the Hypothesis H we have $\tilde{\Phi}_\rho = \tilde{\Xi}_\rho$.*

Proof. By Proposition 4.8 (c) we have $\Xi_\rho = \Phi_\rho(I_N + \mathcal{N}_1)$ with \mathcal{N}_1 a function belonging to $e_0 \mathbb{E}_\Sigma[e_0]^{N \times N}$. evaluating we get

$$\tilde{\Xi}_\rho = \tilde{\Phi}_\rho(I_N + \mathcal{N}_2)$$

for $\mathcal{N}_2 \in e_0 \mathbb{C}_\infty[e_0]^{N \times N}$. By Lemma 7.16 we see that $I_N + \mathcal{N}_2 \in \text{GL}_N(\mathbb{C}_\infty[z])$ and this shows that $\mathcal{N}_2 = 0_N$ because the functions $z \mapsto z$ and $z \mapsto e_0(z)$ are algebraically independent over \mathbb{C}_∞ (easy to check). \square

We now choose an integer $n > 0$ and we study $\text{ev}(G_m(\rho))$ where $G_m(\rho)$ has been defined in (6.5). We recall that $G_1(\rho) = \tilde{\pi}^{-1} \Psi_1(\rho) = u \Phi_\rho$. It is easy to see (we leave the verification to the reader) that for all ρ satisfying the Hypothesis H, Φ_ρ can be expanded into an $N \times N$ matrix of entire functions of the variables z and t_Σ ($|\Sigma| + 1$ variables). It follows that for all $m \geq 1$,

$$D_{m-1}(\tilde{\Xi}_\rho) = D_{m-1}(\tilde{\Phi}_\rho) = \text{ev}(D_{m-1}(\Phi_\rho)),$$

so we have:

$$(7.17) \quad \text{ev} \left(G_m(\rho) \right) = D_{m-1} \left(u \tilde{\Xi}_\rho \right).$$

7.6.2. Matrix functions and proof of Theorem 7.15. From now on we suppose that $\rho = \rho_\Sigma^*$ and that $|\Sigma| \equiv m \pmod{q-1}$ with $m > 0$. Recalling the matrix functions \mathcal{E} of §7.4, from (7.17) we obtain the series expansion:

$$(7.18) \quad \tilde{\pi}^{-m} \text{ev} \left(\mathcal{E}(m; \rho_\Sigma^*) \right) = \sum'_{c \in A} \text{ev} \left(\rho_\Sigma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \right) D_{m-1} \left(u \tilde{\Xi}_{\rho_\Sigma^*} \right)_c + \tilde{\pi}^{-m} E_\Sigma \text{ev} \left(\sum'_{d \in A} d^{-m} \rho_\Sigma^*(T_{-d}) \right),$$

where $(\cdot)_c$ indicates that we have applied the substitution $z \mapsto cz$. We rewrite the identity (7.18) at the level of the first columns in a more convenient way. Our next task is to show the subsequent identities (7.19) and (7.21). We note that the first column of

$$E_\Sigma \sum'_{d \in A} d^{-m} \rho_\Sigma^*(T-d)$$

equals

$$- \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathcal{Z} \end{pmatrix}$$

where $\mathcal{Z} = \zeta_A(m; \sigma_\Sigma)$ is the ζ -value (1.17), and we get

$$\text{ev}(\mathcal{Z}) = \zeta_A(m-l) = \sum_{d \geq 0} \sum_{a \in A^+(d)} a^{l-m}$$

(sum over the polynomials of A which are monic of degree n), a special value of the *Goss zeta function* associated to A , see [36, §8.6]. The Goss' zeta values

$$\zeta_A(k) = \sum_{d \geq 0} \sum_{a \in A^+(d)} a^{-k}$$

are well defined elements of K_∞ for all $k \in \mathbb{Z}$. We recall that we have the following properties: (a) if $k > 0$ is such that $q-1 \mid k$, then $\zeta_A(k) \in K^\times \tilde{\pi}^k$, (b) $\zeta_A(k) \in A$ for $k \leq 0$, and (c) $\zeta_A(k) = 0$ if and only if $k < 0$, $q-1 \mid k$ (see Goss' book [36] for an introduction to the theory of these functions). In particular \mathcal{Z} is zero if and only if $l > m$ and $l \equiv m \pmod{q-1}$.

We resume the computation as follows (the index 1 indicates that we are extracting the first column):

$$(7.19) \quad \tilde{\pi}^{-m} \left(E_\Sigma \sum'_{d \in A} d^{-m} \rho_\Sigma^*(T-d) \right)_1 = - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{\pi}^{-m} \zeta_A(m-l) \end{pmatrix}.$$

We now compute $D_{m-1}(u \tilde{\Xi}_{\rho_\Sigma^*})$. For this, set, with $I \subset \Sigma$,

$$l_I = \sum_{i \in I} q^{k_i}$$

so that $I = I_\Sigma$ (note that $I \subset J$ implies $l_I \leq l_J$ and if $I \sqcup J = \Sigma$, $l_I + l_J = l$). In place of (7.16) we have the explicit formula

$$\tilde{\Xi}_{\rho_\Sigma^*} = \bigotimes_{i \in \Sigma} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix},$$

and, by Leibnitz rule,

$$D_j(\tilde{\Xi}\rho_\Sigma^*) = \sum_{k \in \Sigma} \bigotimes_{i \in J} D_{i_k} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix}$$

where the sum runs over the families $(i_k : k \in \Sigma) \subset \mathbb{N}$ such that $\sum_k i_k = j$. Since $D_i(z^{q^k}) = -\tilde{\pi}^{q^k}, z^{q^k}, 0$ depending on whether $i = q^k, 0$ or another value distinct from the previous ones, we see that

$$D_j(\tilde{\Xi}\rho_\Sigma^*) = \sum_{\substack{I \sqcup J = \Sigma \\ l_I = j}} \bigotimes_{i \in J} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix}.$$

By $D_{m-1-l_I}(u) = G_{m-l_I}(u)$ we deduce the formula

$$(7.20) \quad D_{m-1}(u\tilde{\Xi}\rho_\Sigma^*) = \sum_{\substack{I \sqcup J = \Sigma \\ l_I < m}} G_{m-l_I}(u) \bigotimes_{i \in J} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix}.$$

Note that, with $c \in A$,

$$\begin{aligned} \text{ev} \left(\rho_\Sigma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \right) & \left(\bigotimes_{i \in J} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix} \right) = \\ & = \bigotimes_{i \in J} \left(\begin{pmatrix} c^{q^{k_i}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix} \right) \otimes \bigotimes_{h \in I} \left(\begin{pmatrix} c^{q^{k_h}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix} \right) = \\ & = \left(\bigotimes_{i \in J} \begin{pmatrix} c^{q^{k_i}} & 0 \\ -(cz)^{q^{k_i}} & 1 \end{pmatrix} \right) \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix}. \end{aligned}$$

Considering (7.20) we get (remember that the index $(\cdot)_1$ means that we are extracting the first column):

$$\begin{aligned} & \left(\sum'_{c \in A} \text{ev} \left(\rho_\Sigma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \right) D_{m-1}(u\tilde{\Xi}\rho_\Sigma^*) \right)_1 = \\ & = \sum_{\substack{I \sqcup J = \Sigma \\ l_I < m}} \left(\sum'_{c \in A} c^{l_J} G_{m-l_I}(u_c) \right) \left(\bigotimes_{i \in J} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix} \right)_1. \end{aligned}$$

Note that

$$\sum_{c \in A \setminus \{0\}} c^{l_J} G_{m-l_I}(u_c) = -f_{m-l_I+l_J, m-l_I}$$

(see (7.15)) if $l = l_\Sigma \equiv m \pmod{q-1}$, and it equals zero otherwise (because $l = l_I + l_J$). But $|\Sigma| \equiv l \pmod{q-1}$. Hence, writing F_I for $f_{m-l_I+l_J, m-l_I}$ for simplicity:

$$(7.21) \quad \left(\sum'_{c \in A} \text{ev} \left(\rho_\Sigma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \right) D_{m-1}(u\tilde{\Xi}\rho_\Sigma^*) \right)_1 = - \sum_{\substack{I \sqcup J = \Sigma \\ l_I < m}} F_I \left(\bigotimes_{i \in J} \begin{pmatrix} 1 & 0 \\ -z^{q^{k_i}} & 1 \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 & 0 \\ (-\tilde{\pi})^{-q^{k_h}} & 0 \end{pmatrix} \right).$$

Proof of Theorem 7.15. We study the first column of $\tilde{\pi}^{-m} \text{ev}(\mathcal{E}(m; \rho_\Sigma^*))$. Gathering together (7.19) and (7.21) we find

$$(7.22) \quad \tilde{\mathcal{E}} := \text{ev} \left(\mathcal{E}(m; \rho_\Sigma^*) \right)_1 = -\tilde{\pi}^m \sum_{\substack{I \sqcup J = \Sigma \\ l_I < m}} F_I \left(\bigotimes_{i \in J} \begin{pmatrix} 1 \\ -z^{q^{k_i}} \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 \\ (-\tilde{\pi})^{-q^{k_h}} \end{pmatrix} \right) - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_A(m-l) \end{pmatrix}.$$

Observe, from the modularity of $\mathcal{E} := \mathcal{E}(m; \rho_\Sigma^*)$, the following identity, where $(v)_1$ now denotes the first entry of an element $v \in R^{N \times 1}$ for some ring R , and where $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$:

$$\left(\mathcal{E}(\gamma(z)) \right)_1 = J_\gamma(z)^m \left(\rho_\Sigma^*(\gamma) \mathcal{E}(z) \right)_1 = \det(\gamma)^{-m} J_\gamma(z)^m \bigotimes_{i \in \Sigma} \left(\chi_{t_i}(d), -\chi_{t_i}(c) \right) \mathcal{E}(z).$$

This is obtained by noticing that $\det(\gamma)^{-|\Sigma|} \otimes_{i \in \Sigma} (\chi_{t_i}(d), -\chi_{t_i}(c))$ is the first row of $\rho_\Sigma^*(\gamma)$, and $|\Sigma| \equiv m \pmod{q-1}$. Evaluating at $t_i = \theta^{q^{k_i}}$ for all $i \in \Sigma$ this becomes $\det(\gamma)^{-m} \otimes_{i \in \Sigma} (d^{q^{k_i}}, -c^{q^{k_i}})$. Observe that, for any $x \in \mathbb{C}_\infty^\times$, with the dot \cdot being the standard scalar product,

$$\bigotimes_{i \in \Sigma} (d^{q^{k_i}}, -c^{q^{k_i}}) \cdot \left(\bigotimes_{i \in J} \begin{pmatrix} 1 \\ -z^{q^{k_i}} \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 \\ -x^{-q^{k_h}} \end{pmatrix} \right) = J_\gamma(z)^{l_J} (cx)^{l_I} = J_\gamma^l L_\gamma(z)^{l_j},$$

where $L_\gamma(z) := -\frac{c}{cz+d}$.

Moreover,

$$\bigotimes_{i \in \Sigma} (d^{q^{k_i}}, -c^{q^{k_i}}) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_A(m-l) \end{pmatrix} = (-c)^l \zeta_A(m-l).$$

Using (7.22) yields the identity for $f := (\tilde{\mathcal{E}})_1$ (first entry):

$$(7.23) \quad f(\gamma(z)) = \det(\gamma)^{-m} J_\gamma(z)^m \bigotimes_{i \in \Sigma} (d^{q^{k_i}}, -c^{q^{k_i}}) \cdot \left(-\tilde{\pi}^m \sum_{\substack{I \sqcup J = \Sigma \\ l_I < m}} F_I \left(\bigotimes_{i \in J} \begin{pmatrix} 1 \\ -z^{q^{k_i}} \end{pmatrix} \otimes \bigotimes_{h \in I} \begin{pmatrix} 0 \\ (-\tilde{\pi})^{-q^{k_h}} \end{pmatrix} \right) - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_A(m-l) \end{pmatrix} \right) = \\ = \det(\gamma)^{-m} J_\gamma(z)^{m+l} \left(- \sum_{\substack{I \sqcup J = \Sigma \\ l_I < m}} \tilde{\pi}^{m-l_I} F_I L_\gamma(z)^{l_I} - \zeta_A(m-l) L_\gamma(z)^l \right).$$

This implies that f is a *Drinfeld quasi-modular form of weight $l+m$ type m and depth $\leq l$* in the sense of [13, Definition 2.1], which is equivalent to our definition of quasi-modular

form in virtue of loc. cit., Theorem 1. With γ the identity we have $L_\gamma = 0$ and $f = -\tilde{\pi}^m F_\emptyset$ and the proof of our theorem is complete. \square

If $l \leq m$, the quasi-modular form has depth l because of the non-vanishing of $\zeta_A(m-l)$. If $l > m$ the depth is $< l$ and these results can be compared with with Petrov's work [67]. In his Theorem 1.3 Petrov shows that $l > m$, $l \equiv m \pmod{q-1}$ and if

$$m \leq p^{-v_p(l)}$$

where v_p is the p -adic valuation of \mathbb{Z} , then $f_{l+m,m}$ is the u -expansion of a Drinfeld cusp form in $S_{l+m}(\det^{-m}; \mathbb{C}_\infty)$, a Drinfeld cusp form of weight $l+m$ and type m in the terminology of [24] and therefore a quasi-modular form of depth zero.

The reader can easily deduce the following result which is however slightly weaker than Petrov's (note that $p^{v_p(l)} = q^{v_q(l)} p^{v_p(\ell_q(l))}$, with v_q denoting the order of divisibility by q and ℓ_q denoting the sum of the digits in the q -ary expansion).

Corollary 7.18. *If $l > m$ with $l \equiv m \pmod{q-1}$ and $m \leq q^{v_q(l)}$ then $f_{l+m,m}$ is the u -expansion of a modular form in $S_l(\det^{-m}; \mathbb{C}_\infty)$.*

Proof. Indeed with this hypothesis on the order of divisibility by q in the sums in (7.23) there is no I such that $l_I < m$, unless $I = \emptyset$. Moreover, $\zeta_A(m-l) = 0$ (trivial zero) and the depth of f is zero. \square

7.6.3. An example of Hecke eigenform. Consider Σ such that $s = |\Sigma| \equiv 1 \pmod{q-1}$ and set $m = 1$. Both Corollary 7.18 and Petrov's [67, Theorem 3.1] imply that $f := f_{l+1,1}$ is the u -expansion of an element of $S_{l+1}(\det^{-1}; \mathbb{C}_\infty) \setminus \{0\}$. It is proportional to an entry of $\text{ev}(\mathcal{E}(1; \rho_\Sigma^*))$. It is easy to see that this cusp form is not doubly cuspidal (that is, with multiplicity ≥ 2 at ∞). It is also well known that f is the u -expansion of an Hecke eigenform. We can deduce this property from the fact that $\mathcal{E} := \mathcal{E}(1; \rho_\Sigma^*)$ is a Hecke eigenform. We come back to (4.21). We have, for all $P \in A^+$ irreducible,

$$\left(T_P(\mathcal{E})\right)_1 = \sigma_\Sigma(P) \left(\mathcal{E}(Pz)\right)_1 + P^{-1} \sum_{|b| < |P|} \left(\mathcal{E}\left(\frac{z+b}{P}\right)\right)_1$$

and this equals $(\mathcal{E})_1$ by Corollary 7.6. Evaluating at $t_i = \theta^{q^{k_i}}$ for all $i \in \Sigma$ implies the identity

$$f(Pz) + P^{-1-l} \sum_{|b| < |P|} f\left(\frac{z+b}{P}\right) = P^{-l} f(z)$$

which tells us that f is a Hecke eigenform for all the Hecke operators \mathcal{T}_P , with eigenvalue $P \in A^+$ irreducible (the operators \mathcal{T}_P are those of [24], we use the normalisation of [24] to allow an easier comparison with existing results).

7.6.4. *Examples of quasi-modular forms.* The content of this subsection is also related to the sequence of *extremal quasi-modular forms* $(x_k)_{k \geq 0}$ introduced in [14], where the initial explicit elements are $x_0 = -E$, $x_1 = -Eg - h$, in the notations of [24], and where we recall that E is the normalized false Eisenstein series of weight 2 (6.8). From Theorem 7.15 we deduce that $\mathcal{E}(q^n; \rho_t^*)_{t=\theta} = -\tilde{\pi}^{q^n} f_{q^n+1, q^n}$ for all $n \geq 0$ and $x_n = -f_{q^n+1, q^n}$. If $n = 0$, we deduce Gekeler's series expansion [24, p. 686]:

$$(7.24) \quad E = \sum_{a \in A^+} au_a.$$

Taking $\mathcal{E}(1; \rho_t^*)_{t=\theta q^n}$ for $n \geq 1$ we get, up to a proportionality factor, Petrov's sequence of Hecke eigenforms

$$F_n = \sum_{a \in A^+} a^{q^n} u_a$$

of weight $q^n + 1$ and type 1, notably the initial values $F_1 = h$ and $F_2 = hg^q$ (see [67, §3.2] and the proof of Theorem 3.6 *ibid.*).

7.6.5. *v-adic modular forms from Eisenstein series.* In this short subsection we quickly introduce further desirable directions of investigation, with few details to preserve the flow of the main topics of the present work. Consider an element

$$f \in K(\underline{t}_\Sigma) + uK(\underline{t}_\Sigma)^\circ \langle\langle \underline{e} \rangle\rangle^b[[u]].$$

We say that f is an entry of a rational Drinfeld modular form if there exist $w \in \mathbb{Z}$, $\rho : \Gamma \rightarrow \mathrm{GL}_N(K(\underline{t}_\Sigma))$ a representation of the first kind, $F \in M_w(\rho; \mathbb{K}_\Sigma)$ and a linear map $\lambda : \mathbb{K}_\Sigma^N \rightarrow \mathbb{K}_\Sigma$ such that $f = \lambda(F)$. We denote by \mathcal{X} the set of all entries of rational Drinfeld modular forms.

Write

$$f = f_0 + \sum_{i>0} f_i u^i$$

with $f_0 \in K(\underline{t}_\Sigma)$ and $f_i \in K(\underline{t}_\Sigma)^\circ \langle\langle \underline{e} \rangle\rangle^b$ for $i > 0$. This expansion exists and is unique (see Proposition 3.32). Let $\mathfrak{v} : K(\underline{t}_\Sigma) \rightarrow \mathbb{Z} \cup \{\infty\}$ be an additive valuation. We say that f is *v-integral* if $f_i \in \mathcal{O}_{\mathfrak{v}}^\circ \langle\langle \underline{e} \rangle\rangle^b$, where $\mathcal{O}_{\mathfrak{v}}$ is the subring of $K(\underline{t}_\Sigma)$ of elements with non-negative \mathfrak{v} -valuation, i. e. $f \in \mathcal{O}_{\mathfrak{v}} + u\mathcal{O}_{\mathfrak{v}}^\circ \langle\langle \underline{e} \rangle\rangle^b[[u]]$. Over the ring $\mathcal{O}_{\mathfrak{v}} + u\mathcal{O}_{\mathfrak{v}}^\circ \langle\langle \underline{e} \rangle\rangle^b[[u]]$ of \mathfrak{v} -integral series we have the infimum \mathfrak{v} -valuation (relative to the series expansion $f = \sum_i f_i u^i$) and we denote by $\mathcal{X}_{\mathfrak{v}}$ the metric space of all entries of rational Drinfeld modular forms which are \mathfrak{v} -integral (compare with Definition 7.7).

Definition 7.19. *A v-adic Drinfeld modular form is an element of the completed space $\widehat{\mathcal{X}}_{\mathfrak{v}}$.*

Following the ideas of Goss in [39] the reader can verify the following explicit example. Consider $\Sigma = \Sigma' \sqcup \{1\}$ with $s' = |\Sigma'|$ and set \mathfrak{v} to be the $\chi_{t_1}(\mathfrak{p})$ -adic valuation of $K(\underline{t}_\Sigma)$ with $\mathfrak{p} = (P)$ a prime ideal of A of degree d (and P monic). We choose $m > 0$. We consider a sequence of positive integers $(k_i)_{i \geq 0}$ with $k_i = r + \alpha_i(q^d - 1)$, with $r \in \{0, \dots, q^d - 2\}$

with $k_i \rightarrow \infty$ as $i \rightarrow \infty$ and with α_i converging p -adically. We also suppose that for all i , $k_i + s' \equiv m \pmod{q - 1}$. Then, as $i \rightarrow \infty$, the sequence of series

$$\sum_{\substack{a \in A^+ \\ P|a}} \chi_{t_1}(a)^{k_i} \sigma_{\Sigma'}(a) G_m(u_a) \in K(t_\Sigma)[[u]],$$

all \mathfrak{v} -integral, defines a \mathfrak{v} -adic Drinfeld modular form which is non-zero. Of course, it is related to an Eisenstein series $\mathcal{E}(m; \rho_{\Sigma''}^*)$, for a suitable Σ'' , after an appropriate evaluation.

A remark. It is an interesting problem to determine an appropriate complete topological group of weights for \mathfrak{v} -adic modular forms in the sense of our Definition 7.19. We note indeed that the union

$$\bigcup_{w, \Sigma, \rho} M_w(\rho; \mathbb{K}_\Sigma), \quad w > 0, \quad \Sigma \subset \mathbb{N}^*, \quad \rho \text{ of the first kind,}$$

Σ being finite, generates an algebra over $\cup_\Sigma \mathbb{K}_\Sigma$ with multiplication \otimes . It is not difficult to show that this algebra is graded over the monoid $(\mathbb{Z}, +) \oplus (\{\rho : \text{of the first kind}\}, \otimes)$. To define his ∞ -adic and \mathfrak{v} -adic zeta and L -functions, Goss introduced several complete topological spaces containing \mathbb{Z} , see [36, Chapter 8]. For instance, the complete topological group \mathbb{S} projective limit of the groups $\mathbb{Z}/((q^d - 1)p^n)\mathbb{Z}$ as $n \rightarrow \infty$ with $d = \deg_\theta(P)$ and $q = p^e$, isomorphic to $\mathbb{Z}/(q^d - 1)\mathbb{Z} \times \mathbb{Z}_p$, contains the weights of the \mathfrak{p} -adic modular forms of [38], with \mathfrak{p} the ideal of A generated by P irreducible. The same question arises when one wants to define a topological space over which interpolate the L -series of [56], see [38]. At the time being, there is no complete topological group containing $(\{\rho : \text{of the first kind}\}, \otimes)$ behaving as nicely as \mathbb{S} , allowing to give rise to a nice space of weights for our \mathfrak{v} -adic modular forms. A similar question has been addressed in connection with multiple zeta values in Tate algebras, see [34, Remark 3.1.2].

8. MODULAR FORMS FOR THE REPRESENTATIONS ρ_Σ^*

In this section we consider modular forms associated to representations of the first kind, with values in vector spaces over \mathbb{K}_Σ rather than vector spaces over \mathbb{L}_Σ as we did in the previous sections. To classify them we cannot use the techniques of specialization at roots of unity of §5.2.1. We are therefore led to introduce other techniques which, however, are harder to apply in the general setting of all the representations of the first kind. At least, they lead to proofs of Theorems E, F in the introduction. We will focus on the representations

$$\rho = \rho_\Sigma^* \det^{-m}$$

only, as they seem to have a larger spectrum of applications. We are going to determine the complete structure of the spaces $M_w^1(\rho; \mathbb{K}_\Sigma)$ in Theorem 8.1. An important tool introduced in this section (see §8.2) is the notion of *strongly regular modular form*. The v -valuations of the entries of a strongly regular modular form are submitted to certain sharp lower bounds making them into a module over the scalar modular forms, the structure of which can be easily computed, see Theorem 8.7. If $|\Sigma| \leq q - 1$, the notions of modular form and strongly regular modular form agree (Corollary 8.12). If $|\Sigma| \geq q$, this is no longer true but

in Theorem 8.9 we show that twisting an element of $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ by a large enough power of the operator τ defined in the corresponding section (the exponent depending on Σ) yields a strongly regular modular form. Besides these properties, the precise structure of the \mathbb{K}_Σ -vector spaces $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ for a general choice of Σ subset of \mathbb{N}^* remains presently unknown.

8.1. Structure of weak modular forms. We consider a finite non-empty subset $\Sigma \subset \mathbb{N}^*$. The structure of the \mathbb{K}_Σ -vector space $M_w^!(\rho; \mathbb{K}_\Sigma)$ is quite simple to describe.

We recall that, after Corollary 5.4, for all m , with $W \in \{S, M\}$, $B \in \{\mathbb{L}_\Sigma, \mathbb{K}_\Sigma\}$,

$$W_w(\det^m; B) = W_w(\det^m; \mathbb{C}_\infty) \otimes_{\mathbb{C}_\infty} B.$$

In particular, $M_w(\det^m; \mathbb{K}_\Sigma)$ is finite-dimensional. The main result of this subsection is the following.

Theorem 8.1. *Assuming that $\rho = \rho_\Sigma^* \det^{-m}$, we have:*

$$M_w^!(\rho; \mathbb{K}_\Sigma) = M_{w-1}^!(\rho; \mathbb{K}_\Sigma) \otimes \mathcal{E}(1; \rho_{t_k}^*) + M_{w-q}^!(\rho; \mathbb{K}_\Sigma) \otimes \mathcal{E}(q; \rho_{t_k}^*).$$

Note that $\mathcal{E}(q; \chi_{t_k}) = \tau(\mathcal{E}(1; \chi_{t_k}))$ where τ is defined in (1.18). We choose $k \in \Sigma$. We set $\Sigma' = \Sigma \setminus \{k\}$. We denote by $\rho_{\Sigma'}^*$ the Kronecker factor of the representation ρ_Σ^* . Hence:

$$(8.1) \quad \rho_\Sigma^* = \rho_{\Sigma'}^* \otimes \rho_{t_k}^*.$$

We can suppose, without loss of generality, that $k = \min(\Sigma)$. The natural ordering of $\Sigma \subset \mathbb{N}^*$ is considered in the Kronecker product (it is non-commutative).

Proof of Theorem 8.1. We recall that:

$$\mathcal{E}(1; \rho_t^*) = \sum'_{a,b \in A} (az + b)^{-1} \begin{pmatrix} \chi_t(a) \\ \chi_t(b) \end{pmatrix}.$$

We denote by \mathcal{E} the transposition (row function) of $\mathcal{E}(1; \rho_t^*)$. It satisfies:

$$\mathcal{E}(\gamma(z)) = J_\gamma(z) \mathcal{E}(z) \rho_t(\gamma)^{-1}, \quad \gamma \in \Gamma.$$

We also consider the matrix function

$$\mathfrak{E} = \begin{pmatrix} \mathcal{E} \\ \tau(\mathcal{E}) \end{pmatrix} \in \text{Hol}(\Omega \rightarrow \mathbb{K}_\Sigma^{2 \times 2}),$$

satisfying

$$\mathfrak{E}(\gamma(z)) = \begin{pmatrix} J_\gamma(z) & 0 \\ 0 & J_\gamma(z)^q \end{pmatrix} \mathfrak{E}(z) \rho_t(\gamma)^{-1}, \quad \gamma \in \Gamma.$$

Note that $\tau(\mathcal{E}) = {}^t \mathcal{E}(q; \rho_t^*)$. Let $h = -u + o(u)$ be as in §5.3.3. By [63, Theorem 3.9]:

$$\det(\mathfrak{E}) = -\tilde{\pi} \zeta_A(q; \chi_t) h(z),$$

which is also equal to

$$-\frac{\tilde{\pi}^{q+1} h(z)}{(\theta^q - t)(\theta - t)\omega(t)}$$

by the formula (7.12), after application of τ . The function h does not vanish on Ω and $v(h) = 1$. Since the function $\det(\mathfrak{E})$ can vanish identically for certain values of t with $|t| > 1$, the matrix function $\mathfrak{E}(z)^{-1}$ belongs to $\text{Hol}(\Omega \rightarrow \mathbb{T}^{2 \times 2})$ but not to $\text{Hol}(\Omega \rightarrow \mathbb{E}^{2 \times 2})$, and satisfies

$$\mathfrak{E}(\gamma(z))^{-1} = \rho(\gamma)\mathfrak{E}(\gamma(z))^{-1} \begin{pmatrix} J_\gamma(z)^{-1} & 0 \\ 0 & J_\gamma(z)^{-q} \end{pmatrix}, \quad \gamma \in \Gamma.$$

In any case, $\tau^2(\omega)^{-1}\mathfrak{E}(z)^{-1}$ defines a function of $\text{Hol}(\Omega \rightarrow \mathbb{E}^{2 \times 2})$. We are going to generalize some aspects of the proof of [63, Theorem 3.9]. We set

$$\mathbf{E} = \tau^2(\omega(t))^{-1}\mathfrak{E}^{-1},$$

and \mathbf{E}_{t_k} the same function in the variable t_k instead of t . Note that the function

$$(8.2) \quad \mathbf{F} := I_{N'} \otimes \mathbf{E}_{t_k} \in \text{Hol}(\Omega \rightarrow \mathbb{E}_{\{k\}}^{N \times N}),$$

with $N = 2^s$, $s' = s - 1$, and $N' = 2^{s'}$, satisfies:

$$\mathbf{F}(\gamma(z)) = (\mathbf{1}_{N'} \otimes \rho_{t_k}(\gamma))\mathbf{F}(z) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{-1} & 0 \\ 0 & J_\gamma(z)^{-q} \end{pmatrix} \right), \quad \gamma \in \Gamma.$$

Let \mathcal{G} be an element of $M_w^!(\rho; \mathbb{K}_\Sigma)$. Then by definition, for all $\gamma \in \Gamma$ and $z \in \Omega$, we have

$$\mathcal{G}(\gamma(z)) = J_\gamma(z)^w \det(\gamma)^{-m} \rho_\Sigma^*(\gamma)\mathcal{G}(z).$$

Now setting $\mathbf{G} = {}^t\mathcal{G}$ and, denoting with \mathbf{H} the row function $\mathbf{G}\mathbf{F}$, with values in $\mathbb{K}_\Sigma^{1 \times N}$, we have, for $\gamma \in \Gamma$:

$$\begin{aligned} \mathbf{H}(\gamma(z)) &= \\ &= J_\gamma(z)^w \det(\gamma)^{-m} \mathbf{G}(z) \rho_{\Sigma'}^{-1}(\gamma) (\mathbf{1}_{N'} \otimes \rho_{t_k}(\gamma)) (\mathbf{1}_{N'} \otimes \mathbf{E}_{t_k}(z)^{-1}) \times \\ &\quad \times \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{-1} & 0 \\ 0 & J_\gamma(z)^{-q} \end{pmatrix} \right) \\ &= \det(\gamma)^{-m} \mathbf{G}(z) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) (\mathbf{1}_{N'} \otimes \mathbf{E}_{t_k}(z)^{-1}) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right) \\ &= \det(\gamma)^{-m} \mathbf{G}(z) (\mathbf{1}_{N'} \otimes \mathbf{E}_{t_k}(z)^{-1}) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right) \\ &= \det(\gamma)^{-m} \mathbf{H}(z) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right). \end{aligned}$$

In the above computation, we have observed the distributive property of the mixed product $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ (for matrices A, B, C, D). This identity that we have found,

$$\mathbf{H}(\gamma(z)) = \det(\gamma)^{-m} \mathbf{H}(z) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right), \quad \gamma \in \Gamma$$

can be interpreted in the following way. The column holomorphic function $\mathcal{H} := {}^t\mathbf{H}$, with values in $\mathbb{K}_\Sigma^{N \times 1}$ can be written as $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ with both \mathcal{H}_1 and \mathcal{H}_2 columns of size

$N' = 2^{\Sigma'}$, where the symbol \odot is defined, if $a = {}^t(a_1, \dots, a_{N'})$ and $b = {}^t(b_1, \dots, b_{N'})$, by $a \odot b = (a_1, b_1, a_2, b_2, \dots, a_{N'}, b_{N'})$. Then, both $\mathcal{H}_1, \mathcal{H}_2$ are separately weak modular forms for $\rho_{\Sigma}^* \det^{-m}$, with values in \mathbb{K}_{Σ} and the weights are respectively $w - 1$ and $w - q$. \square

We have, with ρ as above:

Theorem 8.2. *The following equality of \mathbb{K}_{Σ} -vector spaces holds, for any $w \in \mathbb{Z}$, $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ and finite $\Sigma \subset \mathbb{N}^*$:*

$$(8.3) \quad M_w^!(\rho; \mathbb{K}_{\Sigma}) = \bigoplus_{I \sqcup J = \Sigma} \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right) M_{w-|I|-q|J|}^!(\det^{-m}; \mathbb{K}_{\Sigma}).$$

We denote by $M^!(\det^{\bullet}; \mathbb{K}_{\Sigma})$ the $\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ -graded B -algebra of scalar weak \mathbb{K}_{Σ} -valued Drinfeld modular forms for Γ of any weight and type, and we set $M^!(\rho_{\Sigma}^* \det^{\bullet}; \mathbb{K}_{\Sigma}) = \bigoplus_{w,m} M_w^!(\rho_{\Sigma}^* \det^{-m}; \mathbb{K}_{\Sigma})$, which is a graded module over $M^!(\mathbf{1}; \mathbb{K}_{\Sigma})$.

Also, we denote by $M^!(\rho_{\Sigma}^*; \mathbb{K}_{\Sigma}) = \bigoplus_{w \in \mathbb{Z}} M_w^!(\rho_{\Sigma}^*; \mathbb{K}_{\Sigma})$ the sub-module of $M^!(\det^{\bullet}; \mathbb{K}_{\Sigma})$ of weak modular forms for ρ_{Σ}^* and $M^!(\mathbf{1}; \mathbb{K}_{\Sigma}) = \bigoplus_w M_w^!(\mathbf{1}; \mathbb{K}_{\Sigma})$. We have the next corollary:

Corollary 8.3. (a) *The \mathbb{K}_{Σ} -vector space $M^!(\rho_{\Sigma}^* \det^{\bullet}; \mathbb{K}_{\Sigma})$ is a graded free $M^!(\det^{\bullet}; \mathbb{K}_{\Sigma})$ -module of rank $N = 2^s$. (b) *The \mathbb{K}_{Σ} -vector space $M^!(\rho_{\Sigma}^*; \mathbb{K}_{\Sigma})$ is a graded free $M^!(\mathbf{1}; \mathbb{K}_{\Sigma})$ -module of rank N .**

Observe that further, the $N = 2^s$ generators of these modules are explicitly described in Theorem 8.2 and are the elements

$$\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \otimes \bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \in M_{|I|+q|J|}^!(\rho_{\Sigma}^*; \mathbb{T}_{\Sigma}), \quad I \sqcup J = \Sigma.$$

Proof of Theorem 8.2. We deduce from Theorem 8.1, by induction on $|\Sigma|$, that a weaker version of (8.3) holds, with \sum in place of \bigoplus . It remains to show that the sum is a direct sum. For this, it suffices to show that the $N = 2^s$ functions $\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \otimes \bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*)$, for $I \sqcup J = \Sigma$, which define elements of $\mathfrak{D}_{\Sigma}^{N \times 1}$, are linearly independent over the field $\mathbb{K}_{\Sigma}((u))$. Note indeed that $M_{w-|I|-q|J|}^!(\det^{-m}; \mathbb{K}_{\Sigma}) \hookrightarrow \mathbb{K}_{\Sigma}((u))$ because all the elements of the space on the left are A -periodic and tempered.

Let a, b be two elements of \mathfrak{K}_{Σ} . We write $a \approx b$ if $v(a) = v(b)$ (note that if $a = 0$ and $a \approx b$ then $b = 0$) and we extend the definition to vectors and matrices whose entries are in \mathfrak{K} by saying that $(a_{i,j}) \approx (b_{i,j})$ if for all i, j , $v(a_{i,j}) = v(b_{i,j})$. Then by Proposition 7.2, we have $\mathcal{E}(1; \rho_{t_i}^*) \approx \begin{pmatrix} u \\ 1 \end{pmatrix}$ and $\mathcal{E}(q; \rho_{t_j}^*) \approx \begin{pmatrix} u^q \\ 1 \end{pmatrix}$. Hence, up to permutation of rows and columns, we have the \approx -equivalence of $N \times N$ -matrices in $\mathfrak{D}_{\Sigma}^{N \times N}$:

$$\mathcal{N} := \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right)_{I \sqcup J = \Sigma} \approx \begin{pmatrix} u^q & u \\ 1 & 1 \end{pmatrix}^{\otimes s}.$$

The anti-diagonal of the matrix on the right is equal to $(1, u)^{\otimes s}$ (up to reordering). This corresponds to a unique monomial which minimizes the v -valuation in the series expansion

of the determinant of \mathcal{N} . We deduce that $\det(\mathcal{N}) \approx u^{a_s}$, where $(a_s)_{s \geq 1}$ is the sequence defined, inductively, by $a_1 = 1$ and

$$a_s = 2a_{s-1} + 2^{s-1}$$

for $s > 1$. The matrix \mathcal{N} is therefore non-singular, and the functions $\otimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \otimes \otimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*)$ for $I \sqcup J = \Sigma$ are linearly independent over $\mathbb{K}_\Sigma((u))$, from which the result follows. \square

8.2. Strongly regular modular forms. We keep considering a finite non-empty subset $\Sigma \subset \mathbb{N}^*$ of cardinality s , the representation $\rho = \rho_\Sigma^* \det^{-m}$, $k := \max(\Sigma)$. We discuss quite a restricted but useful class of modular forms which have a particularly simple behaviour at infinity.

Definition 8.4. A tempered ρ_Σ^* -quasi-periodic holomorphic function

$$\mathcal{G} : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$$

is called *strongly regular at infinity* if

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes s} \mathcal{G}(z) \in \mathfrak{D}_\Sigma^{N \times 1}.$$

Note, with Diag denoting a diagonal matrix, that

$$\begin{aligned} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes 2} &= \text{Diag}(u^{-2}, u^{-1}, u^{-1}, 1) \\ \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes 3} &= \text{Diag}(u^{-3}, u^{-2}, u^{-2}, u^{-1}, u^{-2}, u^{-1}, u^{-1}, 1). \end{aligned}$$

Writing

$$(8.4) \quad \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes s} = \text{Diag}(u^{-s}, \dots, u^{-n_1}, u^{-n_0}),$$

and letting s tend to infinity, an integer sequence $(n_i)_{i \geq 0}$ is defined and coincides with the one's-counting sequence (compare with the sequence $(a_i)_i$ in the proof of Theorem 8.2; we used it also in §5.3.4). We need the next Lemma, where we use the sequence introduced in (8.4) and the notation \odot introduced in the course of the proof of Theorem 8.1.

Lemma 8.5. *We have $(n_i)_{i \geq 0} = (n_{2i})_{i \geq 0} \odot (n_{2i+1} + 1)_{i \geq 0}$.*

Proof. Straightforward computation of the carry over in binary addition when we add one to an integer. \square

The above serves to make the next definition.

Definition 8.6. A weak modular form $\mathcal{G} \in M_w^1(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ is said *strongly regular* (of weight w) if it is strongly regular at infinity after definition 8.4. Taking (8.4) into account and writing $\mathcal{G} = {}^t(\mathcal{G}_1, \dots, \mathcal{G}_N)$, this is equivalent to $v(\mathcal{G}_{N-i}) \geq n_i$ for $i = 0, \dots, N - 1$.

The \mathbb{K}_Σ -vector subspaces of the spaces $M_w(\rho; \mathbb{K}_\Sigma)$ of strongly regular modular forms have quite a simple structure which can be described essentially by adapting the proof of Theorem 8.1; The main result is Theorem 8.7 below. Also, regarding the Definition 8.4 of strongly regular functions, if we want to use the indexation of the components of \mathcal{G} , $\mathcal{G} = (\mathcal{G}^J)_{I \sqcup J = \Sigma}$ (so that the first entry $\mathcal{G}^\emptyset = \mathcal{G}_1$ has a u -expansion) we then get that the above condition is equivalent to

$$(8.5) \quad \mathcal{G}^J u^{-|I|} \in \mathfrak{D}_\Sigma, \quad \forall I, J \text{ such that } I \sqcup J = \Sigma.$$

We denote by $M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ the \mathbb{K}_Σ -sub-vector space of $M_w^1(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ generated by the strongly regular modular forms of weight w for $\rho_\Sigma^* \det^{-m}$ (with values in \mathbb{K}_Σ).

Examples. Any scalar Drinfeld modular form is strongly regular. In fact, we have

$$M_w^\dagger(\det^{-m}; \mathbb{K}_\Sigma) = M_w(\det^{-m}; \mathbb{K}_\Sigma) = M_w(\det^{-m}; \mathbb{C}_\infty) \otimes_{\mathbb{C}_\infty} \mathbb{K}_\Sigma$$

for all w, m by Corollary 5.4. From Proposition 7.2 we immediately see that $\mathcal{E}(1; \rho_t^*) \in M_1^\dagger(\rho_t^*; \mathbb{L})$ and $\mathcal{E}(q; \rho_t^*) \in M_q^\dagger(\rho_t^*; \mathbb{L})$ (recall that we write $\mathbb{L} = \mathbb{L}_\Sigma$ when Σ is a singleton, with variable t); these functions are strongly regular. In particular, after Theorem 8.2 and Corollary 8.3, the generators of the module $M^1(\rho_\Sigma^* \det^\bullet; \mathbb{K}_\Sigma)$ described in the statements are all strongly regular modular forms.

8.2.1. *Structure of strongly regular modular forms.* We shall prove the next result where B can be taken to be equal to \mathbb{L}_Σ or \mathbb{K}_Σ :

Theorem 8.7. *The following equality of B -vector spaces holds, for any $w \in \mathbb{Z}$, $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, finite $\Sigma \subset \mathbb{N}^*$:*

$$(8.6) \quad M_w^\dagger(\rho_\Sigma^* \det^{-m}; B) = \bigoplus_{I \sqcup J = \Sigma} \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right) M_{w-i-qj}(\det^{-m}; B).$$

The direct sum $M^\dagger(\rho_\Sigma^* \det^\bullet; B) := \bigoplus_{w, m} M_w^\dagger(\rho_\Sigma^* \det^{-m}; B)$ is a graded module over the graded algebra $M(\det^\bullet; B)$ of scalar Drinfeld modular forms $\Omega \rightarrow \mathbb{K}_\Sigma$ for any power of the determinant character. Similarly, we have, recalling that $M(\mathbf{1}; B)$ is the graded algebra of (scalar) Drinfeld modular forms for Γ (it is equal to the graded algebra $B[g, \Delta]$) and $M^\dagger(\rho_\Sigma^*; B)$ the $M(\mathbf{1}; B)$ -module of strongly regular modular forms for ρ_Σ^* , we immediately deduce, with the same settings and $s = |\Sigma|$:

Corollary 8.8. *(a) The $M(\det^\bullet; B)$ -module $M^\dagger(\rho_\Sigma^* \det^\bullet; B)$ is free of rank $N = 2^s$. (b) The graded $M(\mathbf{1}; B)$ -module $M^\dagger(\rho_\Sigma^*; B)$ is free of rank $N = 2^s$. (c) Both modules are generated by the N modular forms $(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*)) \otimes (\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*))$, for $I, J \subset \Sigma$ such that $I \sqcup J = \Sigma$.*

The fact that the rank is N is not a surprise in view of the papers of Marks and Mason [47] and of Bantay and Gannon [6], in the settings of complex vector-valued modular forms. These authors prove that vector spaces of vector valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$ associated to an indecomposable finite dimensional complex representation of this group

(and satisfying some additional conditions we do not want to discuss here) all are free of dimension, the dimension of the representation. However, note that the vector-valued modular forms that we study in the present work are not simple variants of the above complex valued ones.

Proof of Theorem 8.7. It is easily seen that the left-hand side of (8.6) is contained in the right-hand side and we have to prove the reverse inclusion. Corollary 8.2 ensures the equality of the corresponding B -vector spaces of weak modular forms (“when \dagger is replaced with $!$ ”). This means that if $\mathcal{G} \in M_w^\dagger(\rho_\Sigma^* \det^{-m}; B)$, then

$$\mathcal{G} \in M_w^!(\rho_\Sigma^* \det^{-m}; B) = \bigoplus_{I \sqcup J = \Sigma} \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right) M_{w-i-qj}^!(\det^{-m}; B).$$

All we need to prove is that the coefficients occurring in the various spaces of scalar weak modular forms $M_{w-i-qj}^!(\det^{-m}; B)$ are in fact Drinfeld modular forms (regular at infinity). To see this it suffices to show that

$$\mathcal{G} \in M_{w-1}(\rho_{\Sigma'}^*, \det^{-m}; B) \otimes \mathcal{E}(1; \rho_{t_k}^*) + M_{w-q}(\rho_{\Sigma'}^*, \det^{-m}; B) \otimes \mathcal{E}(q; \rho_{t_k}^*),$$

where k is an integer such that $k < \min(\Sigma')$ with $\Sigma = \Sigma' \sqcup \{k\}$. A simple induction will then allow to complete the proof.

Lemma 8.5 implies that for all $s \geq 1$, writing

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes s} = \text{Diag}(U_s),$$

then

$$(8.7) \quad U_s = u^{-1} U_{s-1} \odot U_{s-1}.$$

Now, we set $\mathbf{G} = \mathbf{G}_1 \odot \mathbf{G}_2$ with $\mathbf{G} = {}^t \mathcal{G}$ an element of $M_w^\dagger(\rho_\Sigma^* \det^{-m}; B)$, hence we also write $\mathbf{G}_i = {}^t \mathcal{G}_i$. We know by the proof of Theorem 8.1 that

$$\mathbf{H} = \mathbf{H}_1 \odot \mathbf{H}_2 = \mathbf{G}\mathbf{F}$$

(with \mathbf{F} as in (8.2)) is such that

$$\mathcal{H}_1 = {}^t \mathbf{H}_1 \in M_{w-1}^!(\rho_{\Sigma'}^*, \det^{-m}; B), \quad \text{and } \mathcal{H}_2 = {}^t \mathbf{H}_2 \in M_{w-q}^!(\rho_{\Sigma'}^*, \det^{-m}; B).$$

It remains to prove that \mathcal{H}_1 and \mathcal{H}_2 are both strongly regular. We have to show that

$$\mathbf{H}_j(z) \text{Diag}(U_{s-1}) \in \mathfrak{D}_\Sigma^{1 \times N'}, \quad j = 1, 2.$$

By hypothesis, we know that the entries of $\mathbf{G}(z) \text{Diag}(U_s)$ are in \mathfrak{D}_Σ . Explicitly, the entries of $u(z)^{-1} \mathbf{G}_1(z) \text{Diag}(U_{s-1})$ and of $\mathbf{G}_2(z) \text{Diag}(U_{s-1})$ are in \mathfrak{D}_Σ . We recall the relation $a \approx b$, for elements of $\mathfrak{K}_\Sigma^\times$, and its extension to matrices. We note that $\mathbf{H}_1, \mathbf{H}_2$ are given, explicitly, by the formulas:

$$\mathbf{H}_1 = \frac{-\mathbf{G}_1 \tau(e_2) + \mathbf{G}_2 \tau(e_1)}{\tilde{\pi} \zeta_A(q; \chi_{t_k}) h}, \quad \mathbf{H}_2 = \frac{\mathbf{G}_1 e_2 - \mathbf{G}_2 e_1}{\tilde{\pi} \zeta_A(q; \chi_{t_k}) h},$$

where $\mathcal{E} = (e_1, e_2)$ ⁽¹⁴⁾. By the well-known u -expansion $h \in -u + u\mathbb{C}_\infty[[u^{q-1}]]$ (which tells us that $v(h) = 1$ and $h \approx u$), we thus have

$$u\mathbf{H}_1 \approx -\mathbf{G}_1\tau(e_2) + \mathbf{G}_2\tau(e_1), \quad u\mathbf{H}_2 \approx \mathbf{G}_1e_2 - \mathbf{G}_2e_1.$$

We first study \mathcal{H}_1 . We have:

$$\begin{aligned} \mathbf{H}_1 \text{Diag}(U_{s-1}) &\approx u^{-1}(-\mathbf{G}_1\tau(e_2) + \mathbf{G}_2\tau(e_1)) \text{Diag}(U_{s-1}) \\ &\approx -u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})\tau(e_2) + \mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}\tau(e_1). \end{aligned}$$

Now, by hypothesis $u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1}) \in \mathfrak{D}_\Sigma^{1 \times N'}$, while $v(\tau(e_2)) = 0$, from which we deduce that $u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})\tau(e_2) \in \mathfrak{D}_\Sigma^{1 \times N'}$. On the other hand, we have that $\tau(e_1) \approx u^q$, hence, we have that $\mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}\tau(e_1) \approx \mathbf{G}_2 \text{Diag}(U_{s-1})u^{q-1} \in \mathfrak{M}_\Sigma^{1 \times N'} \subset \mathfrak{D}_\Sigma^{1 \times N'}$. Therefore all entries of $\mathbf{H}_1 \text{Diag}(U_{s-1})$ are in \mathfrak{D}_Σ and \mathcal{H}_1 is strongly regular.

Let us now deal with \mathcal{H}_2 . Similarly, we have that

$$\begin{aligned} \mathbf{H}_2 \text{Diag}(U_{s-1}) &\approx u^{-1}(\mathbf{G}_1e_2 - \mathbf{G}_2e_1) \text{Diag}(U_{s-1}) \\ &\approx u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})e_2 - \mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}e_1. \end{aligned}$$

Since $v(e_2) = 0$, we have that the term $u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})e_2$ has all the entries in \mathfrak{D}_Σ . Moreover, $e_1 \approx u$ so that all the entries of $\mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}e_1$ are in \mathfrak{D}_Σ by the hypothesis on \mathcal{G}_2 . Hence, $\mathbf{H}_2 \text{Diag}(U_{s-1}) \in \mathfrak{D}_\Sigma^{1 \times N'}$ and \mathcal{H}_2 is strongly regular. This completes the proof of the Theorem. \square

8.3. More structure properties. In contrast with that of strongly regular modular forms, the structure of the vector spaces $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ is more difficult to describe. In this subsection, we report on some properties in this direction. Let $r \geq 0$ be the unique integer such that $r(q-1) + 1 \leq s := |\Sigma| \leq (r+1)(q-1)$. We recall that the map $\tau^r : M_w(\rho; \mathbb{K}_\Sigma) \rightarrow M_{q^r w}(\rho; \mathbb{K}_\Sigma)$ is defined in (1.18). We also write, sometimes, $f^{(k)}$ instead of $\tau^k(f)$, with f a modular form. Note that the definition also makes sense if $k < 0$. However, in this case, the resulting function over Ω needs not to be analytic. We want to show:

Theorem 8.9. *Let $f \in M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$. Then, $\tau^r(f) \in M_{wq^r}^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$.*

To prove this result we need preliminary tools to handle the representations ρ_Σ and ρ_Σ^* . We order, for $\gamma \in \Gamma$, the columns of $\rho_\Sigma(\gamma)$ from \emptyset to Σ along the total order described in §7.2, and we order the rows from Σ to \emptyset along the opposite of this order. In parallel, we also label rows and columns with integers $1, \dots, N = 2^{|\Sigma|}$ in the usual way (this will serve to transpose matrices), so we have two orderings. Let $M = (M_{I,J})_{I,J \subset \Sigma} \in B^{N \times N}$ be a matrix with entries in some ring B , with rows and columns indexed as above (the first index always indicates rows). Since the order opposite of the inclusion order on the subsets of Σ is obtained by computing complementaries $I \mapsto I^c := \Sigma \setminus I$, we have the following transposition rule:

$$(8.8) \quad {}^t M = (M_{J^c, I^c})_{I, J \subset \Sigma} \in B^{N \times N}.$$

¹⁴The reader will not mix these functions with the functions e_i of §3.

Note that the transposition is relative to the ordering by integers in $\{1, \dots, N\}$. In relation to the ordering by subsets of Σ , this is anti-transposition (that is, transposition with respect to the anti-diagonal). Now we write with $a \in A$:

$$\rho_\Sigma(T_a) = (\rho_{I,J}(T_a))_{I,J \subset \Sigma} \in \mathbb{F}_q(\underline{t}_\Sigma)^{N \times N},$$

and similarly, we write $\rho_\Sigma^*(T_a) = (\rho_{I,J}^*(T_a))_{I,J \subset \Sigma}$. For $U \subset \Sigma$, we recall the map (semi-character) $\sigma_U : A \rightarrow \mathbb{F}_q[\underline{t}_U]$ defined by

$$\sigma_U(a) = \prod_{i \in U} \chi_{t_i}(a).$$

An elementary computation, the fact that the inverse of $\rho_{t_i}(T_a)$ is $\rho_{t_i}(T_{-a})$, and an application of (8.8), lead to:

Lemma 8.10. *For $I, J \subset \Sigma$, we have:*

$$\rho_{I,J}(T_a) = \begin{cases} 0 & \text{if } I \cup J \subsetneq \Sigma \\ \sigma_{I \cap J}(a) & \text{if } I \cup J = \Sigma \end{cases}, \quad \rho_{I,J}^*(T_a) = \begin{cases} 0 & \text{if } J \cap I \neq \emptyset \\ (-1)^{|(J \cup I)^c|} \sigma_{(J \cup I)^c}(a) & \text{if } J \cap I = \emptyset \end{cases}.$$

Note that $\rho_\Sigma(T_a)$ is symmetric with respect to the anti-diagonal (in the ordering by $\{1, \dots, N\}$; this is the diagonal in the ordering by subsets $I, J \subset \Sigma$ because we can swap I, J). Note also that the entries in the diagonal (in the ordering over $\{1, \dots, N\}$) are all equal to 1 because these are the entries indexed by I, J with $I \sqcup J = \Sigma$. The coefficient of $\rho_\Sigma(T_a)$ in the upper-right corner is equal to $\sigma_\Sigma(a) = \prod_{i \in \Sigma} \chi_{t_i}(a)$. We deduce the explicit expression of the coefficients of $\Phi_{\rho_\Sigma^*} = (\Phi_{I,J})_{I,J}$ (defined in §4.2.2) in term of Perkins' series. In particular, since the function κ in (6.10) is strictly decreasing (with respect to inclusion), we deduce from Theorem 6.12 the following property. If $I, J \subset \Sigma$ with $I \cap J = \emptyset$ and $I \cup J \neq \Sigma$ (not corresponding to a diagonal coefficient), then

$$(8.9) \quad v(\Phi_{I,J}) \geq \kappa(I) - 1.$$

We set $\rho = \rho_\Sigma^* \det^{-m}$. The above properties can be used to prove:

Lemma 8.11. *Let $f = (f^I)_I$ be a ρ -quasi-periodic function in \mathfrak{D}_Σ . Then, if $I \subsetneq \Sigma$, $v(f^I) \geq \kappa(I)$.*

Proof. By Proposition 4.11, we have

$$f = \Phi_\rho g$$

where $g = (g^I)_I \in \mathbb{K}_\Sigma[[u]]^{N \times 1}$. Since the entries of Φ_ρ are in $\mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b$ (v -valuations in $] - 1, 0] \cup \{\infty\}$) we see, inductively, that $g^I \in u\mathbb{K}_\Sigma[[u]]$ if $I \subsetneq \Sigma$ (while $g^\Sigma \in \mathbb{K}_\Sigma[[u]]$) and (8.9) allows to conclude. \square

This generalizes Corollary 7.4. Theorem 8.9 now follows easily. Thanks to the alternative condition for strong regularity (8.5) and Lemma 8.11, the property of the Theorem is verified taking into account that if $I \subsetneq \Sigma$ then $q^r \kappa(I) \geq |I|$, which is easily seen.

Note that if $s = 1$ (case in which we denote by t the unique variable), every Drinfeld modular form for ρ_t^* is strongly regular, which is a restatement of Theorem 3.9 of [63]. We have

$$M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) \subset M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) \subset M_w^1(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma),$$

and the inclusions are in general strict. Moreover, as an immediate consequence of Theorem 8.9, we have:

Corollary 8.12. *If $s = |\Sigma| < q$, then $M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$. For any s , $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ is of finite dimension over \mathbb{K}_Σ .*

In particular, one can easily check that, in the above hypotheses,

$$(8.10) \quad \mathcal{E}(s; \rho_\Sigma^*) = (-1)^s \bigotimes_{i \in \Sigma} \mathcal{E}(1; \rho_{t_i}^*).$$

In fact, the formula (8.10) can be proved also for $s = q$ by using the methods of §9. This implies and generalizes [21, Theorem 4.4] (see the identity at the level of the first coefficients).

We also deduce the next result which asserts, in particular, that there are no non-zero \mathbb{K}_Σ -valued modular forms of negative weight:

Corollary 8.13. *We have $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \{0\}$ for $w < 0$, for $w = 0$ and $m \neq 0$, or for $w = 0$ and $\Sigma \neq \emptyset$.*

Proof. Note that $M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \{0\}$ if $w < 0$. Hence we obtain the first assertion, combining with Theorem 8.9. The other properties are easy. \square

Corollary 8.13 is also a consequence of the main result of the next subsection.

8.3.1. *Another consequence of Theorem 8.9.* We shall show:

Theorem 8.14. *For all $\Sigma \subset \mathbb{N}^*$ finite and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ we have that*

$$M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = M_w(\rho_\Sigma^* \det^{-m}; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{K}_\Sigma.$$

We point out that we do not know if the dimensions, of $M_w(\rho; \mathbb{K}_\Sigma)$ over \mathbb{K}_Σ and of $M_w(\rho; \mathbb{L}_\Sigma)$ over \mathbb{L}_Σ , agree. The proof of Theorem 8.14 rests also on a notion of *analytic part of non-analytic modular form*. The main result regarding this notion is Proposition 8.16. Recall that $h = -u + o(u)$ is Gekeler's cusp form in $S_{q+1}(\det^{-1}; \mathbb{C}_\infty)$ as in [24, (5.11), (9.3)]. We need the following technical refinement of Proposition 3.32 where, for $k \geq 0$, $\mathfrak{K}_\Sigma^{(k)}$ denotes the image of \mathfrak{K}_Σ by τ^k in \mathfrak{K}_Σ and $\mathfrak{K}_\Sigma^{(-k)}$ denotes the τ -difference field generated by the elements $f^{(k)}$ with $f \in \mathfrak{K}_\Sigma$. This field can be embedded in the perfect closure $\mathfrak{K}_\Sigma^{\text{perf}}$ of \mathfrak{K}_Σ . Note that $h^{1/q^k} = h^{(-k)}$ is a well defined element of $\mathbb{C}_\infty((u))^{\text{perf}}$.

Proposition 8.15. (1) *For $k \geq 0$ the field \mathfrak{K}_Σ is a $\mathfrak{K}_\Sigma^{(k)}$ -vector space of dimension q^k with basis $(1, h, \dots, h^{q^k-1})$. (2) *Every element f of $\mathfrak{K}_\Sigma^{(-k)}$ can be expanded, in a unique way, as**

$$f = \sum_{i \geq i_0} f_i h^{-i/q^k}, \quad f_i \in \mathbb{K}_\Sigma^\circ \langle\langle \underline{e} \rangle\rangle^b.$$

Proof. (1) We note that

$$\mathbb{K}_\Sigma[[u]] = \bigoplus_{i=0}^{q^k-1} \mathbb{K}_\Sigma[[u^{q^k}]]u^i.$$

We claim that

$$\mathfrak{K}_\Sigma = \bigoplus_{i=0}^{q^k-1} \mathfrak{K}_\Sigma^{(k)}u^i.$$

Indeed otherwise, there would exist a non-trivial linear dependence relation

$$(8.11) \quad \sum_{i=0}^{q^k-1} \alpha_i e_0^i = 0, \quad \alpha_i \in \mathfrak{K}_\Sigma^{(k)}.$$

We know from Corollary 6.4 that the higher divided derivatives \underline{D} stabilize the field \mathfrak{K}_Σ . In particular, $\mathfrak{K}_\Sigma^{(k)}$ is contained in the subfield of constants of each one of the operators D_i , with $i = 1, \dots, q^k - 1$. In (8.11), consider the integer $i \in \{0, \dots, q^k - 1\}$ maximal with the property that $\alpha_i \neq 0$. Since

$$e_0^j = \sum_{k_1, \dots, k_j \geq 0} \frac{(\tilde{\pi}z)^{q^{k_1} + \dots + q^{k_j}}}{d_{k_1} \cdots d_{k_j}}, \quad 0 \leq j \leq q^k - 1,$$

we have that $D_i(e_0^j) = 0$ if $j < i$ and $D_i(e_0^i) = (-1)^i$. Applying D_i on both sides we get $\alpha_i = 0$ in contradiction with our assumptions.

Note that h is a uniformizer of $\mathbb{C}_\infty[[u]]$. By the u -expansion

$$h \in -u + u\mathbb{C}_\infty[[u^{q-1}]] \subset \mathfrak{M}_\Sigma$$

[24, (9.3)] there is an h -expansion:

$$u = -h + \sum_{i \geq 1} c_i h^i, \quad c_i \in A$$

and a canonical isomorphism $\mathbb{K}_\Sigma[[h]] \cong \mathbb{K}_\Sigma[[u]]$ (consider a compositional inverse). We identify these two rings. Then

$$\mathfrak{K}_\Sigma = \bigoplus_{i=0}^{q^k-1} \mathfrak{K}_\Sigma^{(k)}h^i.$$

(2) From the identities $e_i^q + \theta e_i = e_{i-1}$, equivalent to

$$(8.12) \quad e_i^{\frac{1}{q}} = \frac{1}{\theta^{1/q}} \left(e_{i-1}^{\frac{1}{q}} - e_i \right), \quad i \in \mathbb{Z}$$

we deduce, with K^{perf} the perfect closure of K , that

$$(\langle e \rangle^j)^{(-1)} \in \left(K^{\text{perf}}(\underline{t}_\Sigma) \langle e \rangle \cap \mathbb{K}_\Sigma^\circ \langle \langle e \rangle \rangle^b \right) [e_0^{\frac{1}{q}}], \quad j \in \mathbb{Z}[p^{-1}]_{\geq 0}$$

so that $\mathfrak{K}_\Sigma^{(-1)} = \mathfrak{K}_\Sigma[u^{\frac{1}{q}}]$, which implies, identifying $\mathbb{K}_\Sigma[[u^{1/q^k}]]$ and $\mathbb{K}_\Sigma[[h^{1/q^k}]]$ (u^{1/q^k} and h^{1/q^k} are two uniformizers)

$$\mathfrak{K}_\Sigma^{(-k)} = \mathfrak{K}_\Sigma[u^{\frac{1}{q^k}}] = \mathfrak{K}_\Sigma[h^{\frac{1}{q^k}}] = \bigoplus_{i=0}^{q^k-1} \mathfrak{K}_\Sigma h^{\frac{i}{q^k}}, \quad k \geq 0.$$

Combining with Proposition 3.32 we see that

$$\mathfrak{K}_\Sigma^{(-k)} = \mathbb{K}_\Sigma^\circ \langle\langle \varepsilon \rangle\rangle^b((u))[u^{\frac{1}{q^k}}] = \mathbb{K}_\Sigma^\circ \langle\langle \varepsilon \rangle\rangle^b((h))[h^{\frac{1}{q^k}}] = \mathbb{K}_\Sigma^\circ \langle\langle \varepsilon \rangle\rangle^b((h^{\frac{1}{q^k}})).$$

□

Let f be an element of $\mathfrak{K}_\Sigma^{N \times 1}$. We can expand in a unique way (depending on the choice of the modular form h)

$$(8.13) \quad f = \sum_{i=0}^{q^k-1} f_i h^i, \quad f_i \in (\mathfrak{K}_\Sigma^{(k)})^{N \times 1}.$$

With $\rho : \Gamma \rightarrow \mathbb{K}_\Sigma^{N \times 1}$ a representation of the first kind and w an integer, we have:

Proposition 8.16. *Assume that f is an element of $M_w^1(\rho; B)$ with $B = \mathbb{L}_\Sigma$ or $B = \mathbb{K}_\Sigma$. Then, for all i , f_i equals $g_i^{(k)}$ for some analytic function $g_i : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$ and*

$$f_i \in M_{w-i(q+1)}^1(\rho \det^i; B) \cap (\mathfrak{K}_\Sigma^{(k)})^{N \times 1}, \quad i = 0, \dots, q^k - 1.$$

Proof. In the statement we understand that for all i there exists $\tilde{f}_i \in M_w^1(\rho; B)^{(k)}$, uniquely determined, such that it can be identified, in $(\mathfrak{K}_\Sigma)^{N \times 1}$, with f_i (and we identify the two objects). Consider a weak modular form f of weight w . We consider the decomposition (8.13) determined by Proposition 8.15. Denote by $\text{Mero}(\Omega \rightarrow B^{N \times 1})$ the B -vector space of meromorphic functions $\Omega \rightarrow B^{N \times 1}$ (in the obvious sense, extending our notion of analytic functions $\Omega \rightarrow B^{N \times 1}$). Then

$$(8.14) \quad \text{Mero}(\Omega \rightarrow B^{N \times 1}) = \bigoplus_{i=0}^{q^k-1} \text{Mero}(\Omega \rightarrow B^{N \times 1})^{(k)} h^i.$$

Since f is meromorphic, we have an expansion

$$(8.15) \quad f = \sum_{i=0}^{q^k-1} \tilde{f}_i h^i, \quad \tilde{f}_i \in \text{Mero}(\Omega \rightarrow B^{N \times 1})^{(k)}.$$

If $\gamma \in \Gamma$, applying the Petersson slash operator (4.19) we obtain, by using the modularity of h :

$$\sum_{i=0}^{q^k-1} (\tilde{f}_i|_{w-i(q+1), \rho \det^i \gamma}) h^i = f|_{w, \rho \gamma} = f = \sum_{i=0}^{q^k-1} \tilde{f}_i h^i.$$

By the uniqueness of the decomposition (8.14), we deduce that

$$\tilde{f}_i \in \mathcal{M}_{w^{-i(q+1)}}^!(\rho \det^i; B), \quad i = 0, \dots, q^k - 1.$$

We now proceed to show that all the functions $\tilde{f}_i^{(-k)}$ are analytic. Let z_0 be an element of Ω . All we need to show is that for all i , \tilde{f}_i is locally analytic at z_0 . Being $\text{ord}_{z_0} : \text{Mero}(\Omega \rightarrow B) \rightarrow \mathbb{Z}$ the map that associates to a meromorphic function its order at z_0 we note that, by the fact that $\text{ord}_{z_0}(h) = 0$, locally at z_0 ,

$$f(z) = \sum_{i \geq 0} g_i (h(z) - h(z_0))^i, \quad g_i \in B^{N \times 1}.$$

Now, again locally at z_0 ,

$$\begin{aligned} f(z) &= \sum_{i=0}^{q^k-1} (h(z) - h(z_0))^i \sum_{j \geq 0} g_{i+jq^k} (h(z) - h(z_0))^{jq^k} \\ &= \sum_{i=0}^{q^k-1} \tilde{g}_i(z) (h(z) - h(z_0))^i \\ &= \sum_{i=0}^{q^k-1} h^i \tilde{f}_i(z), \end{aligned}$$

where $\tilde{g}_i := \sum_{j \geq 0} g_{i+jq^k} (h(z) - h(z_0))^{jq^k}$ is locally analytic at z_0 and the last step follows, after expansion of the powers $(h(z) - h(z_0))^i$, by uniqueness in (8.15). This means that \tilde{f}_i is locally analytic at all $z_0 \in \Omega$. Being modular-like, it is ρ -quasiperiodic. Combining the proofs of Propositions 4.2 and 4.11 we see that it is also tempered. By uniqueness, we can identify \tilde{f}_i with its tame expansion $f_i \in (\mathfrak{K}_B^{(k)})^{N \times 1}$ and $f_i \in M_{w^{-i(q+1)}}^!(\rho \det^i; B)$, each function being the k -th twist of an analytic function over Ω . \square

With B as above we have:

Corollary 8.17. *Consider $f \in M_{q^k w}^!(\rho; B)$. Then*

$$[f]_k := f_0^{(-k)}$$

is in $M_w^!(\rho; B)$.

Proof. By Proposition 8.16 we have that $f_0 \in M_{wq^k}^!(\rho; B)$ and f is the k -th twist of an analytic function. Hence, writing $g := [f]_k$,

$$g|_{w, \rho \gamma} = \left(g^{(k)}|_{wq^k, \rho} \right)^{(-k)} = g, \quad \gamma \in \Gamma.$$

\square

Given $f \in M_{q^k w}^!(\rho; B)$ we call $[f]_k \in M_w^!(\rho; B)$ the *analytic part* of $f^{(-k)}$. The latter needs not to be analytic. Note that if $f = \tau^k(g)$ with g analytic, then $[f]_k = g$.

Proof of Theorem 8.14. Clearly $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{K}_\Sigma \subset M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ so we need to prove the opposite inclusion; this is a consequence of the fact that for all w' , $M_{w'}^\dagger(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ is of finite dimension over \mathbb{K}_Σ , by Corollary 8.8.

We set once again $\rho = \rho_\Sigma^* \det^{-m}$ (for our choice of Σ and m). By Theorem 8.9 there exists $k \geq 0$ such that

$$\tau^k \left(M_w(\rho; \mathbb{K}_\Sigma) \right) \subset M_{q^k w}^\dagger(\rho; \mathbb{K}_\Sigma)$$

and by Corollary 8.8, the space on the right is finitely dimensional over \mathbb{K}_Σ and generated by modular forms that take values in \mathbb{L}_Σ and are defined over \mathbb{L}_Σ , with a basis (g_1, \dots, g_r) constituted by modular forms that are also elements of

$$M_{q^k w}(\rho; \mathbb{L}_\Sigma) \cap \left(\mathbb{L}_\Sigma^\circ \langle \underline{e} \rangle^b((u)) \right)^{N \times 1}.$$

This property on the tame series expansion follows easily from the properties of the functions \mathcal{F}_i mentioned and used in the proof of Proposition 7.12 and in §8.2.

Let f be an element of $M_w(\rho; \mathbb{K}_\Sigma)$. Since $\tau^k(f) \in M_{q^k w}^\dagger(\rho; \mathbb{K}_\Sigma)$ we can write, in a unique way,

$$f = \sum_{i=1}^r \alpha_i g_i^{(-k)}, \quad \alpha_1, \dots, \alpha_r \in \mathbb{K}_\Sigma.$$

By Corollary 8.17,

$$f = \sum_{i=1}^r \alpha_i [g_i]_k \in M_w^\dagger(\rho; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{K}_\Sigma \cap \left(\mathbb{L}_\Sigma^\circ \langle \underline{e} \rangle^b((u)) \otimes_{\mathbb{L}_\Sigma} \mathbb{K}_\Sigma \right)^{N \times 1}.$$

Note that $[g_i]_k \in M_w^\dagger(\rho; \mathbb{L}_\Sigma)$ for all i , but we do not have, in general $[g_i]_k \in M_w(\rho; \mathbb{L}_\Sigma)$. We still need to show that $f \in M_w(\rho; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{K}_\Sigma$, that is, that f is a linear combination of regular \mathbb{L}_Σ -valued modular forms. We already know that there exist $h_1, \dots, h_r \in M_w^\dagger(\rho; \mathbb{L}_\Sigma)$ such that $f \in \text{Vect}_{\mathbb{K}_\Sigma}(h_1, \dots, h_r)$. We can even choose h_1, \dots, h_r linearly independent over \mathbb{L}_Σ with $r = r_0$ minimal with the property that the given element f is in their \mathbb{K}_Σ -span. Also, there is no loss of generality if we suppose that $\mu := \min\{v(h_i) : i \leq r\}$ is maximal, where $v(h_i)$ stands for the infimum of the v -valuations of the entries of the weak modular form h_i . A last reduction that we can assume, without losing generality, is that the set \mathcal{I} of indices i such that $v(h_i) = \mu$ is non-empty and minimal; note that it cannot be a singleton. We select one such minimal family (h_1, \dots, h_r) of \mathbb{L}_Σ -linearly independent weak modular forms.

Assume by contradiction that $\mu < 0$. We know that there exist $\alpha_1, \dots, \alpha_r \in \mathbb{K}_\Sigma$, not all zero, such that $v(\sum_i \alpha_i h_i) \geq 0$ (because f belongs to the span). By Proposition 3.32, we can expand in a unique way $h_i = \sum_j h_{i,j} u^j$ with $h_{i,j} \in (\mathbb{L}_\Sigma^\circ \langle \underline{e} \rangle^b)^{N \times 1}$. Then,

$$(8.16) \quad \sum_i \alpha_i h_{i,j} = 0, \quad j < 0, \quad \sum_i \alpha_i h_{i,0} \in \mathbb{K}_\Sigma^{N \times 1}.$$

We claim that given elements $c_1, \dots, c_r \in (\mathbb{L}_\Sigma^\circ \langle \underline{e} \rangle^b)^{N \times 1}$ that are \mathbb{K}_Σ -linearly dependent, then they also are \mathbb{L}_Σ -linearly dependent. To see the latter property, note that a \mathbb{K}_Σ -linear

dependence relation connecting the r elements c_1, \dots, c_r of $(\mathbb{L}_\Sigma^\circ \langle\langle e \rangle\rangle^b)^{N \times 1}$ is equivalent to a relation

$$\sum_{l=1}^r \alpha_l \sum_{j \in \mathbb{Z}[1/p]_{\geq 0}} c_{l,j} \langle e \rangle^j = 0$$

for $\alpha_1, \dots, \alpha_h \in \mathbb{K}_\Sigma$ and $c_{l,j} \in \mathbb{L}_\Sigma^{N \times 1}$ (with Gauss' norm of entries that are uniformly bounded over $\mathbb{Z}[\frac{1}{p}]_{\geq 0}$). Since the elements $\langle e \rangle^j$ are linearly independent over \mathbb{K}_Σ , the above is equivalent to

$$\sum_{l=1}^r \alpha_l c_{l,j} = 0, \quad j \in \mathbb{Z}[1/p]_{\geq 0}.$$

If $\alpha_1, \dots, \alpha_r$ are not all zero, then there exist $\alpha'_1, \dots, \alpha'_r \in \mathbb{L}_\Sigma$, not all zero, such that $\sum_i \alpha'_i c_i = 0$ hence proving the claim.

Returning to the proof of our Theorem, let us suppose that $j = j_0 < 0$ is the smallest integer such that in (8.16), the linear dependence relation is non-trivial, among not all zero elements h_{i,j_0} . There exist $\alpha'_1, \dots, \alpha'_r \in \mathbb{L}_\Sigma$, not all zero, such that $\sum_i \alpha'_i h_{i,j_0} = 0$ so that

$$v\left(\sum_{i \in \mathcal{I}} \alpha'_i h_i\right) > \mu.$$

If $i_0 \in \mathcal{I}$ is such that $\alpha'_{i_0} \neq 0$ (it exists), we can set $h'_{i_0} := \sum_{i \in \mathcal{I}} \alpha'_i h_i$ and $h'_i = h_i$ if $i \neq i_0$. Then (h'_1, \dots, h'_r) is a basis of $\text{Vect}_{\mathbb{K}_\Sigma}(h_1, \dots, h_r)$ hence containing f so that by maximality of μ , $\min\{v(h'_i) : i \leq r\} = \mu$. Now, by construction, $\{i : v(h'_i) = \mu\} \subsetneq \mathcal{I}$ contradicting the minimality of \mathcal{I} . Hence $\mu \geq 0$ and the proof of the theorem is complete in this case. The handling of the case $j = j_0 = 0$ in (8.16) is slightly different but similar in spirit (note that the v -valuation of a tame series is in $] - 1, 0[$). We leave it to the reader. \square

9. HARMONIC PRODUCT AND EISENSTEIN SERIES

In this section we study another aspect of the Eisenstein series of §7 associated to representations of the form ρ_Σ^* with Σ a finite subset of \mathbb{N}^* that connects to the *multiple zeta values* as introduced and discussed by Thakur in [77]. The first entries of these Eisenstein series are proportional, by Proposition 7.2, to combinations of series such as

$$(9.1) \quad \sum_{a \in A^+} \sigma_\Sigma(a) G_m(u_a) \in A[t_\Sigma][[u]]$$

where σ_Σ is the semi-character $a \mapsto \prod_{i \in \Sigma} \chi_{t_i}(a)$ and G_m the m -th Goss polynomial associated to the lattice $\tilde{\pi}A \subset \mathbb{C}_\infty$. In [58, 34] an \mathbb{F}_p -algebra structure is described, over the set of *multiple zeta series in the Tate algebras* \mathbb{T}_Σ (in fact, these are functions in $\mathbb{E}_\Sigma \subset \mathbb{T}_\Sigma$) generalizing Thakur's multiple zeta values (see for example [77, 1, 79]). We will see, in this section, that this algebra structure determines a multiplication rule for the series 9.1 and can be viewed as a source of explicit relations connecting Eisenstein series.

The results of the present section cover various aspects of an *harmonic product formula* (Theorem 9.4 and complements) generalizing [58, Theorems 2.3, 3.1]. We present now the basic tools.

We recall that, as usual in this text, Σ denotes a finite subset of \mathbb{N}^* of cardinality s (the empty set is allowed). Let L/\mathbb{F}_q be a field extension.

Data 9.1. *Let us suppose we are given with:*

- (1) Injective \mathbb{F}_q -linear maps $\delta_i : A \rightarrow L$, for $i \in \Sigma$.
- (2) For $\alpha_{i,j} \in \mathbb{N}$ ($i \in \Sigma$ and $j = 1, \dots, r$), maps $\sigma_j : A \rightarrow L$ defined by $\sigma_j(a) := \prod_{i \in \Sigma} \delta_i(a)^{\alpha_{i,j}}$. We call *semi-characters* such maps $A \rightarrow L$ ⁽¹⁵⁾.
- (3) Injective \mathbb{F}_q -linear map $\gamma : A \rightarrow L$ (we adopt the notation γ_a for the evaluation of γ in $a \in A$).

We consider a semi-character $\sigma = \prod_{i \in \Sigma} \delta_i^{\alpha_i}$ with linear maps δ_i as above, $i \in \Sigma$ (empty products are allowed). The map $\mathbf{1}$ sending A to $1 \in L$ is the trivial semi-character.

Together with the objects that we have introduced so far, we consider, for integers $n_i \in \mathbb{N}^*$ with $i = 1, \dots, r$ *composition arrays*, that is, tables of the form:

$$(9.2) \quad \mathcal{C} := \begin{pmatrix} \sigma_1 & \cdots & \sigma_r \\ n_1 & \cdots & n_r \end{pmatrix}.$$

When $r = 1$, we may sometimes write $(n; \sigma)$ instead of $\begin{pmatrix} \sigma \\ n \end{pmatrix}$. If $\mathcal{C} = \begin{pmatrix} \mathbf{1} & \cdots & \mathbf{1} \\ n_1 & \cdots & n_r \end{pmatrix}$ we simplify it to $\mathcal{C} = (n_1, \dots, n_r)$. The *degree* of \mathcal{C} is $\begin{pmatrix} \sigma \\ n \end{pmatrix}$ where $\sigma = \sigma_1 \cdots \sigma_r$ and $n = \sum_i n_i$. The *weight* is n and the *type* is σ . If $\sigma = \mathbf{1}$ we say that the type is *trivial*. For a composition array as in (9.2), we introduce the *twisted power sum*

$$S_d(\mathcal{C}) := \sum_{\substack{d_1 > \cdots > d_r \geq 0 \\ a_1, \dots, a_r \in A^+ \\ \deg_{\theta}(a_i) = d_i, \forall i = 1, \dots, r}} \frac{\sigma_1(a_1) \cdots \sigma_r(a_r)}{\gamma_{a_1}^{n_1} \cdots \gamma_{a_r}^{n_r}} \in L.$$

These twisted power sums generalize the classical power sums of Thakur in [77], as well as the twisted power sums of [62]. We shall show the following generalization of [58, Theorem 3.1]:

Theorem 9.2. *Let σ, ψ be two semi-characters and m, n two positive integers. For any α, β semi-characters and $i, j \in \mathbb{N}^*$ there is an element $f_{\alpha, \beta, i, j} \in \mathbb{F}_p$ such that, for all $d \geq 0$,*

$$S_d \begin{pmatrix} \sigma \\ m \end{pmatrix} S_d \begin{pmatrix} \psi \\ n \end{pmatrix} - S_d \begin{pmatrix} \sigma\psi \\ m+n \end{pmatrix} = \sum_{\substack{\alpha\beta = \sigma\psi \\ i+j = m+n}} f_{\alpha, \beta, i, j} S_d \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix}.$$

In the theorem, the sum is on the couples of semi-characters (α, β) such that $\alpha\beta = \sigma\psi$, and over the decompositions $n + m = i + j$, so there are only finitely many terms in it. In order to proceed further, we need additional data of conditions:

Data 9.3. Let us assume that:

- (1) L is endowed with a valuation $\nu : L \rightarrow \mathbb{Q} \cup \{\infty\}$ and it is complete for this valuation
- (2) $\nu(\delta_{i,j}(a)) \in \{0, \infty\}$ for all i, j and $a \in A$

¹⁵Note that they generalize the semi-characters that we have discussed so far.

(3) $\gamma_a^{-1} \rightarrow 0$ as a runs in A (for the valuation ν).

Assuming the Data 9.3 it is easy to see that the series

$$(9.3) \quad f_A(\mathcal{C}) := \sum_{d \geq 0} S_d(\mathcal{C})$$

converges in L for any composition array \mathcal{C} as in (9.2). Let n be a positive integer, and let $\sigma : A \rightarrow L$ be a semi-character such that ν is trivial over its image. We denote by \mathcal{F}_n^σ the \mathbb{F}_p -sub-vector space of L generated by the elements $f_A(\frac{\sigma_1}{n_1} \cdots \frac{\sigma_r}{n_r})$ with $r > 0$, $\prod_i \sigma_i = \sigma$, $\sum_i n_i = n$ (with $n_i > 0$ for all i). We also set $\mathcal{F}_0^1 := \mathbb{F}_p$ and $\mathcal{F}_0^\sigma := (0)$ if $\sigma \neq \mathbf{1}$. We consider the sum $\mathcal{F} := \sum_{n, \sigma} \mathcal{F}_n^\sigma$. The above result can be used, in a lengthy but straightforward way very similar to that of [58], to prove the next result.

Theorem 9.4. *For all $m, n > 0$ and σ, ψ semi-characters, We have that $\mathcal{F}_m^\sigma \mathcal{F}_n^\psi \subset \mathcal{F}_{m+n}^{\sigma\psi}$, and the \mathbb{F}_p -vector space \mathcal{F} is an \mathbb{F}_p -algebra.*

9.1. Existence of the harmonic product. In this section we prove Theorem 9.2. We will use the methods of [58, §3.1.2 and §3.1.3] which deeply borrow from Thakur in [79]. The following result can be found in [58].

Proposition 9.5. *Let Σ be a finite subset of \mathbb{N}^* . Consider U, V such that $U \sqcup V = \Sigma$. Let L/\mathbb{F}_q be a field extension and let us suppose that x_i ($i \in \Sigma$) are elements of L and let z be an element of $L \setminus \mathbb{F}_q$. Then, the following formula holds:*

$$\sum_{\mu, \nu \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} (x_i + \mu) \prod_{j \in V} (x_j + \nu)}{(z + \mu)(z + \nu)} = - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{k \in I} (x_k + \mu)}{(z + \mu)}.$$

With appropriate choices of the set Σ , of the subsets U, V , of the elements x_i and z and applying a power of an endomorphism of L which is $\mathbb{F}_q(x_i : i \in \Sigma)$ -linear and which sends z to z^q , and specialization of some x_i to z , we deduce:

Corollary 9.6. *Considering a finite set $\Sigma \subset \mathbb{N}^*$, an ordered partition $\Sigma = U \sqcup V$, a positive integer N and two integers α, β such that $N = \alpha + \beta$, for all $1 \leq k \leq N$ and $I \subset \Sigma$, there exists $c_{I,k} \in \mathbb{F}_p$ such that*

$$\sum_{\mu, \nu \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} (x_i + \mu) \prod_{j \in V} (x_j + \nu)}{(z + \mu)^\alpha (z + \nu)^\beta} = \sum_{\substack{k=1, \dots, N \\ I \subset \Sigma}} c_{I,k} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{i \in I} (x_i + \mu)}{(z + \mu)^k}.$$

In the above formula, Δ denotes the diagonal subset. We can now prove Theorem 9.2. We recall that we have denoted by $A^+(d)$ the set of monic polynomials of degree d in A . We also denote by $A^+(\leq d)$ the set of monic polynomials of A which have degree $\leq d$. For $n \in A^+(d)$ and $m \in A^+(\leq d)$, we write

$$S_{m,n} = \{(n + \mu m, n + \nu m); \mu, \nu \in \mathbb{F}_q, \mu \neq \nu\} \subset (A^+(d) \times A^+(\leq d)) \setminus \Delta,$$

where Δ is the diagonal of $A^+(d) \times A^+(d)$. Similarly, we define for $n \in A^+(d)$ and $m \in A^+(< d)$:

$$S'_{m,n} = \{(n + \mu m, m); \mu \in \mathbb{F}_q\} \subset A^+(d) \times A^+(< d).$$

From [58, Lemmas 3.10 and 3.11] and following the original ideas of Thakur in [79], we deduce that the sets $S_{m,n}$ determine a partition of $A^+(d) \times A^+(d) \setminus \Delta$ and the sets $S'_{m,n}$ determine a partition of $A^+(d) \times A^+(< d)$. Moreover, $S'_{m,n} = S'_{m',n'}$ if and only if $S_{m,n} = S_{m',n'}$.

Now, let us choose $d > 0$. We write $\sigma\psi = \prod_{i \in \Sigma} \delta_i$ with δ_i an injective \mathbb{F}_q -linear map $A \rightarrow L$ for all $i \in \Sigma$ (there can be multiple occurrences of such maps), and $\sigma = \prod_{i \in U} \delta_i$, $\psi = \prod_{i \in V} \delta_i$ with $U \sqcup V = \Sigma$. We have, with \mathcal{U} a set of representatives of the above-mentioned partition:

$$\begin{aligned} S_d \begin{pmatrix} \sigma \\ \alpha \end{pmatrix} S_d \begin{pmatrix} \psi \\ \beta \end{pmatrix} - S_d \begin{pmatrix} \sigma\psi \\ N \end{pmatrix} &= \sum_{(a,b) \in A^+(d) \times A^+(d) \setminus \Delta} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta} = \\ &= \sum_{(m,n) \in \mathcal{U}} \sum_{(a,b) \in S_{m,n}} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta}. \end{aligned}$$

We focus on the sub-sum corresponding to the choice of a set $S_{m,n}$. We want now to compute:

$$\begin{aligned} &\sum_{(a,b) \in S_{m,n}} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta} = \\ &= \sum_{(\mu,\nu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\sigma(n + \mu m)\psi(n + \nu m)}{\gamma_{n+\mu m}^\alpha \gamma_{n+\nu m}^\beta} \\ &= \sum_{(\mu,\nu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} \delta_i(n + \mu m) \prod_{j \in V} \delta_j(n + \nu m)}{(\gamma_n + \mu \gamma_m)^\alpha (\gamma_n + \nu \gamma_m)^\beta} \\ &= \frac{\sigma(m)\psi(m)}{\gamma_m^N} \sum_{(\mu,\nu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} \left(\frac{\delta_i(n)}{\delta_i(m)} + \mu \right) \prod_{j \in V} \left(\frac{\delta_j(n)}{\delta_j(m)} + \nu \right)}{\left(\frac{\gamma_n}{\gamma_m} + \mu \right)^\alpha \left(\frac{\gamma_n}{\gamma_m} + \nu \right)^\beta}. \end{aligned}$$

Note that we have used the \mathbb{F}_q -linearity of δ_i for all $i \in \Sigma$ so that $\delta_i(n + \mu m) = \delta_i(n) + \mu \delta_i(m)$ and the hypothesis of injectivity, to divide by $\delta_i(m)$ which needs to be non-zero. Similarly, we have used the \mathbb{F}_q -linearity of the map $a \mapsto \gamma_a$ and the fact that $\gamma_n + \lambda \gamma_m$ does not vanish, because n, m , in the above computation, have distinct degrees. Applying Corollary 9.6 with $x_i = \frac{\delta_i(n)}{\delta_i(m)}$ for $i \in \Sigma$ and $z = \frac{\gamma_n}{\gamma_m}$ which does not belong to \mathbb{F}_q , we obtain the

identity:

$$\begin{aligned}
 (9.4) \quad \sum_{(a,b) \in S_{m,n}} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta} &= \\
 &= \sigma(m)\psi(m)\gamma_m^{-N} \sum_{\substack{I \subset \Sigma \\ k=1, \dots, N}} c_{I,k} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{i \in I} \left(\frac{\delta_i(n)}{\delta_i(m)} + \mu \right)}{\left(\frac{\gamma_n}{\gamma_m} + \mu \right)^k} = \\
 &= \sum_{\substack{I \sqcup J = \Sigma \\ k=1, \dots, N}} c_{I,k} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{i \in I} \delta_i(n + \mu m) \prod_{j \in J} \delta_j(m)}{\gamma_{n+\mu m}^k \gamma_m^{N-k}}.
 \end{aligned}$$

The latter is a sum over $S'_{m,n}$. In view of our previous observations, this concludes the proof of our Theorem. The deduction of Theorem 9.4 from Theorem 9.2 is standard and we omit it. If we choose $\delta_i = \chi_{t_i}$ for $i \in \Sigma$ and $\gamma_a = e_C(az)$, and we follow closely the above proof of Theorem 9.4 in conjunction with [58, Theorem 3.1], we deduce the following explicit result that will be used later, with $\sigma_\Sigma = \prod_{i \in \Sigma} \chi_{t_i}$ and $\gamma_a = e_C(az)$ for $a \in A \setminus \{0\}$.

Theorem 9.7. *The following formula holds, for all $\Sigma \subset \mathbb{N}^*$ and $U \sqcup V = \Sigma$:*

$$\begin{aligned}
 f_A \begin{pmatrix} \sigma_U \\ 1 \end{pmatrix} f_A \begin{pmatrix} \sigma_V \\ 1 \end{pmatrix} - f_A \begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix} &= \\
 f_A \begin{pmatrix} \sigma_U & \sigma_V \\ 1 & 1 \end{pmatrix} + f_A \begin{pmatrix} \sigma_V & \sigma_U \\ 1 & 1 \end{pmatrix} - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} f_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix}.
 \end{aligned}$$

In the next three short subsections we give the three main sets of Data 9.1 that are considered in this paper (we will mainly consider the second one, described in §9.1.2).

9.1.1. Multiple zeta values. We choose the Data 9.1 in the following way. We consider variables $t_\Sigma = \{t_i : i \in \Sigma\}$ and the field $L = \mathbb{K}_\Sigma := \widehat{K(t_\Sigma)}_{v_\infty}$ obtained by completing $K(t_\Sigma)$ with respect to the Gauss' valuation ν extending the valuation v_∞ of K . We consider further the injective \mathbb{F}_q -algebra morphisms $\delta_i(a) := \chi_{t_i}(a)$ for all $i \in \Sigma$ to build our semi-characters. As we did previously, we write, for U a finite subset of \mathbb{N}^* , $\sigma_U(a) := \prod_{i \in U} \chi_{t_i}(a)$. More generally, we can also consider elements in the *monoid of degrees* of [34, §2.1] in place of U ; this amounts in considering semi-characters σ defined by

$$(9.5) \quad \sigma(a) = \prod_{i \in \Sigma} \chi_{t_i}(a)^{n_i}$$

with $\Sigma \subset \mathbb{N}^*$ finite and $n_i \geq 0$. Finally, we choose γ the identity map, so that for all $a \in A$, $\gamma_a = a \in L$. Then we also have the Data 9.3 and we are in the settings of [58]. In the notations of *ibid.*, we have $\zeta_A(\mathcal{C}) = f_A(\mathcal{C})$ for any \mathcal{C} as in (9.2) and we can speak about degree, weight and type of $\zeta_A(\mathcal{C})$. One proves (see [34, Corollary 3.3]) that the $K[t_\Sigma]$ -algebra they generate is graded by the degrees. Note also that for any such element

there exists a finite subset Σ of \mathbb{N}^* such that it belongs to $\mathbb{E}_\Sigma \subset \mathbb{K}_\Sigma$. If we consider the particular case of composition arrays \mathcal{C} as in (9.2) such that the semi-characters σ_i are all equal to the trivial semi-character $\mathbf{1}$ (trivial type), then it is easy to see that the series $\zeta_A(\mathcal{C}) \in K_\infty$ are the *multiple zeta values* of Thakur (the reader can find more in the papers [1, 79] and the survey [80] also provides a wider set of references).

9.1.2. *A-periodic multiple sums.* These are closely related to first entries of Eisenstein series for ρ_Σ^* . We choose, for the Data 9.1:

$$\gamma_a := e_C(az), \quad a \in A \setminus \{0\}.$$

This choice leads us to work with the same semi-characters as in §9.1.1, and in the field $L = K(\underline{t}_\Sigma)((u))$ which is complete for the valuation $\nu = v$, giving the order at $u = 0$ of a formal power series of u . We are also in the settings of Data 9.3. In this case, for \mathcal{C} as in (9.2), we set $\varphi_A(\mathcal{C}) = f_A(\mathcal{C})$ and we can continue to speak about degree, weight and type of such a sum. Explicitly:

$$\varphi_A(\mathcal{C}) = \sum_{\substack{d_1 > \dots > d_r \geq 0 \\ a_1, \dots, a_r \in A^+ \\ \deg_\theta(a_i) = d_i, \\ \forall i=1, \dots, r}} \sigma_1(a_1) \cdots \sigma_r(a_r) u_{a_1}^{n_1} \cdots u_{a_r}^{n_r} \in L,$$

(with $u_a = \exp_C(\tilde{\pi}az)^{-1}$). These series define formal series of $K(\underline{t}_\Sigma)[[u]]$ and each of them is also converging for u in a non-empty disk of \mathbb{C}_∞ of radius $\leq c$ for some $c \in |\mathbb{C}_\infty| \cap]0, 1[$, containing 0. From Theorem 9.4 we deduce:

Corollary 9.8. *The \mathbb{F}_p -vector space spanned by 1 and the series $\varphi_A(\mathcal{C})$ with \mathcal{C} as in (9.2) is an \mathbb{F}_p -algebra. The multiplication rule is compatible with the filtration induced by the semigroup of the elements (w, σ) with $w \in \mathbb{Z}$ and σ semi-characters as in §9.1.1.*

Again with \mathcal{C} as in (9.2), we consider a variant of the above sums based on Goss' polynomials:

$$(9.6) \quad \phi_A(\mathcal{C}) = \sum_{|a_1| > \dots > |a_r| > 0} \sigma_1(a_1) \cdots \sigma_r(a_r) G_{n_1}(u_{a_1}) \cdots G_{n_r}(u_{a_r}),$$

with the sum running over elements $a_1, \dots, a_r \in A^+$. These sums are more closely related to the first entries of our Eisenstein series. We have the next result.

Corollary 9.9. *The K -vector space spanned by 1 and the series $\phi_A(\mathcal{C})$ with \mathcal{C} as in (9.2) is a K -algebra and equals the K -vector space spanned by the series $\varphi_A(\mathcal{C})$.*

Proof. We claim that the family $(G_m(X))_{m>0}$ is a K -basis of $XK[X]$. First of all, these polynomials are linearly independent over K because the functions $z \mapsto G_m(u(z))$, meromorphic over \mathbb{C}_∞ , have poles of distinct orders at the elements $a \in A \subset \mathbb{C}_\infty$. To show that these polynomials span $XK[X]$ it suffices to prove that for $k > 0$, u^k belongs to the K -span V of the polynomials $G_m(u)$ with $m > 0$. This is clear for $k = 1$. Now assuming that u^{k-1} belongs to V , by the fact that $u^k = uu^{k-1}$, it suffices to show that $uG_m(u) \in V$

for all m , but this easily follows from [24, Proposition (3.4) (ii)] and induction on m hence proving the claim. The result now follows from Corollary 9.8. \square

9.1.3. *Remark on product rule.* The product rule of Corollary 9.9 does not seem to be compatible with a filtration involving the composition arrays in a simple way, unlike Corollary 9.8. Note however the following formula, which is homogeneous in the orders of the Goss' polynomials:

$$\sum_{m+n=k} G_m(X)G_n(X) = \left(\binom{k}{1} - 1 \right) G_k(X), \quad k \geq 0.$$

To prove this formula we use (6.6) and Lemma 6.5, and

$$\mathbf{G}(\mathbf{1}) = x\mathcal{T}_{D,x}(G_1(u)) = \frac{ux}{1 - u \exp_C(x)}.$$

Hence we obtain the following Riccati-like differential equation from which the above identities can be derived:

$$\mathbf{G}(\mathbf{1})^2 = x \frac{\partial}{\partial x} (\mathbf{G}(\mathbf{1})) - \mathbf{G}(\mathbf{1}).$$

9.1.4. *Multiple sums in \mathfrak{K}_Σ .* This is the third important type of multiple sums that is determined by making the following choice of Data 9.1, but it will be only studied in §10.1. We consider $L = \mathfrak{K}$ the field of uniformizers with the valuation $\nu = v$. As in §9.1.2 we use $\gamma_a := e_C(az)$ for $a \in A \setminus \{0\}$. Instead of the semi-characters of §9.1.1, we use, for $i \in \mathbb{N}^*$, $\delta_i : A \rightarrow L$ defined by

$$\delta_i(a) = \chi_{t_i}(az) = \frac{\exp_C\left(\frac{\tilde{\pi}z}{\theta-t}\right)}{\omega(t)},$$

seen as a tame series in $\mathbb{L}^\circ \langle\langle \underline{e} \rangle\rangle^b$. These maps are clearly \mathbb{F}_q -linear and injective, and they give rise to semi-characters

$$\tilde{\sigma}_U(a) := \prod_{i \in U} \chi_{t_i}(az)$$

with U a finite subset of \mathbb{N}^* ⁽¹⁶⁾. With them we can construct the formal series

$$(9.7) \quad \tilde{\varphi}_A \left(\begin{matrix} \tilde{\sigma}_1 & \dots & \tilde{\sigma}_r \\ n_1 & \dots & n_r \end{matrix} \right) = \sum_{|a_1| > \dots > |a_r| > 0} \tilde{\sigma}_1(a_1) \cdots \tilde{\sigma}_r(a_r) u_{a_1}^{n_1} \cdots u_{a_r}^{n_r},$$

where the semi-characters $\tilde{\sigma}_i$ are of the above form, where n_1, \dots, n_r are positive integers, and with the sum running over elements $a_1, \dots, a_r \in A^+$. This time however, we do not have a consistent set of Data 9.3 but not for this reason, this case is less interesting. Condition (2) does not hold in general. We cannot guarantee the convergence of the series in (9.7) for the v -valuation. However, when these series converge for the v -valuation (this can happen), they give rise to well defined elements of L .

We have:

¹⁶Or more generally, an element of the monoid of degrees as in §9.1.1.

Corollary 9.10. *There is a multiplication rule on the series (9.7) that are convergent for the v -valuation. Choosing a correspondence $\chi_{t_i} \leftrightarrow \delta_i$ identifies, if all the terms are well defined, the multiplication rule with that of Corollary 9.8 and that of §9.1.1.*

Example. We have the following formulas expressing the same harmonic product rule in the three different settings of §9.1.1, 9.1.2 and 9.1.4. We use δ the semi-character defined by $\delta(a) = \chi_t(az)$ ⁽¹⁷⁾:

$$(9.8) \quad \zeta_A(1; \chi_t) \zeta_A(q-1) = \zeta_A(q; \chi_t) + \zeta_A \left(\begin{smallmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{smallmatrix} \right),$$

$$\varphi_A(1; \chi_t) \varphi_A(q-1) = \varphi_A(q; \chi_t) + \varphi_A \left(\begin{smallmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{smallmatrix} \right),$$

$$\tilde{\varphi}_A(1; \delta) \tilde{\varphi}_A(q-1) = \tilde{\varphi}_A(q; \delta) + \tilde{\varphi}_A \left(\begin{smallmatrix} \delta & \mathbf{1} \\ 1 & q-1 \end{smallmatrix} \right).$$

It is not difficult to verify that all the multiple series involved in the third formula converge for the v -valuation. To prove the identities (9.8) one observes that the first identity follows from identities on multiple power sums (see [34, §7.2]), then uses that the product rules of §9.1.1, 9.1.2, 9.1.4 are the same upon choice of the appropriate correspondence between the semi-characters.

Remark 9.11. In the settings of §9.1.1 we mention that Khac Nhuan Le [45] gave a direct and completely explicit proof of Theorem 9.2. The method he uses is a generalization of Chen's method in [18]. It would be nice to see if his method extends to our general setting of Data 9.1.

9.2. Explicit formulas. The harmonic product can be applied to obtain identities for certain modular forms, notably Eisenstein series. We give three examples. In §9.2.1 an identity for Eisenstein series of weight $q+1$ for $\rho_{\{1,2\}}^*$, in §9.2.2 an identity for Eisenstein series of weight 2 for ρ_Σ^* with $|\Sigma| \equiv 2 \pmod{q-1}$ and in §9.2.3 we present a question on Serre's derivatives of Eisenstein series of weight 1 and their possible relation with Poincaré series of weight 3.

We are going to use Lemma 7.10 by means of the following consequence: if an element of $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ has vanishing first entry, then it vanishes identically. This can be applied to prove (8.10) for $s \leq q$. To see this we choose $k \in \Sigma$ and we write $\Sigma' := \Sigma \setminus \{k\}$, with Σ non-empty finite subset of \mathbb{N}^* of cardinality $s \leq q$. The harmonic product formula of Theorem 9.7 yields inductively

$$\varphi_A(s-1, \sigma_{\Sigma'}) \varphi_A(1, \chi_{t_k}) = \varphi_A(s, \sigma_\Sigma).$$

This formula can also be written more explicitly in the following way:

$$\prod_{i \in \Sigma} \left(\sum_{a \in A^+} \chi_{t_i}(a) u_a \right) = \sum_{a \in A^+} \sigma_\Sigma(a) u_a^s.$$

¹⁷Note that for coherence with other references, we render in different ways the multiple series of 'depth' $r = 1$ or with trivial semi-character (scalar). In particular, we sometimes write, for the arguments of multiple sums, $(n; \sigma)$ instead of $\binom{\sigma}{n}$, and (n_1, \dots, n_r) instead of $\begin{pmatrix} 1 & \dots & 1 \\ n_1 & \dots & n_r \end{pmatrix}$.

This implies (8.10); we leave the details to the reader.

9.2.1. *An identity for Eisenstein series of weight $q + 1$.* We use $\Sigma = \{1, 2\}$ and we suppose that $q > 2$. We denote by g the (scalar) normalized Eisenstein series of weight $q - 1$ for $\mathbf{1}$ (following Gekeler's notations in [24]).

Proposition 9.12. *The following identity holds when $q > 2$:*

$$\mathcal{E}(q + 1; \rho_\Sigma^*) = -\mathcal{E}(1; \rho_{t_1}^*) \otimes \mathcal{E}(q; \rho_{t_2}^*) - \mathcal{E}(q; \rho_{t_1}^*) \otimes \mathcal{E}(1; \rho_{t_2}^*) - (\theta^q - \theta)^{-1} g \mathcal{E}(1, \rho_{t_1}^*) \otimes \mathcal{E}(1, \rho_{t_2}^*).$$

To prove it, we use the next Lemma in the settings of Theorem 9.7.

Lemma 9.13. *The following formula holds:*

$$(9.9) \quad \begin{aligned} \varphi_A(q + 1, \sigma_\Sigma) &= \\ &= \varphi_A(1, \chi_{t_1})\varphi_A(q, \chi_{t_2}) + \varphi_A(q, \chi_{t_1})\varphi_A(1, \chi_{t_2}) - \varphi_A(q - 1)\varphi_A(1, \chi_{t_1})\varphi_A(1, \chi_{t_2}). \end{aligned}$$

Proof. We have the following formulas where we also observe, with $\Sigma = \{1, 2\}$, the formula $\varphi_A(1; \chi_{t_1})\varphi_A(1; \chi_{t_2}) = \varphi_A(2; \sigma_\Sigma)$:

$$\begin{aligned} \varphi_A(1, \chi_{t_1})\varphi_A(q, \chi_{t_2}) &= \varphi_A\left(\begin{matrix} \sigma_\Sigma \\ q + 1 \end{matrix}\right) + \varphi_A\left(\begin{matrix} \sigma_\Sigma & \mathbf{1} \\ 2 & q - 1 \end{matrix}\right) + \varphi_A\left(\begin{matrix} \chi_{t_2} & \chi_{t_1} \\ 2 & q - 1 \end{matrix}\right) \\ \varphi_A(1, \chi_{t_2})\varphi_A(q, \chi_{t_1}) &= \varphi_A\left(\begin{matrix} \sigma_\Sigma \\ q + 1 \end{matrix}\right) + \varphi_A\left(\begin{matrix} \sigma_\Sigma & \mathbf{1} \\ 2 & q - 1 \end{matrix}\right) + \varphi_A\left(\begin{matrix} \chi_{t_1} & \chi_{t_2} \\ 2 & q - 1 \end{matrix}\right) \\ \varphi_A(q - 1)\varphi_A(2, \sigma_\Sigma) &= \varphi_A\left(\begin{matrix} \sigma_\Sigma \\ q + 1 \end{matrix}\right) + 2\varphi_A\left(\begin{matrix} \sigma_\Sigma & \mathbf{1} \\ 2 & q - 1 \end{matrix}\right) - \varphi_A\left(\begin{matrix} \chi_{t_2} & \chi_{t_1} \\ 2 & q - 1 \end{matrix}\right) - \\ &\quad - \varphi_A\left(\begin{matrix} \chi_{t_1} & \chi_{t_2} \\ 2 & q - 1 \end{matrix}\right). \end{aligned}$$

The formula (9.9) follows easily; it also holds for $q = 2$. □

Proof of Proposition 9.12. We note that since $q > 2$, $\mathcal{E}(2, \rho_\Sigma^*) = \mathcal{E}(1, \rho_{t_1}^*) \otimes \mathcal{E}(1, \rho_{t_2}^*)$ by (8.10). The first coordinates of the modular forms

$$\mathcal{E}(q + 1; \rho_\Sigma^*), \quad \mathcal{E}(1; \rho_{\chi_{t_1}}^*) \otimes \mathcal{E}(q; \rho_{\chi_{t_2}}^*), \quad \mathcal{E}(q; \rho_{\chi_{t_1}}^*) \otimes \mathcal{E}(1; \rho_{\chi_{t_2}}^*), \quad g\mathcal{E}(2, \rho_\Sigma^*)$$

are proportional to the following A -expansions (where we recall once again that $G_n(X)$ denotes the n -th Goss polynomial [24, §(3.4)]):

$$\begin{aligned} \mathcal{X} &:= \sum_{a \in A^+} \sigma_\Sigma(a) G_{q+1}(u_a), \\ \mathcal{Y}_1 &:= \left(\sum_{a \in A^+} \chi_{t_1}(a) u_a \right) \left(\sum_{b \in A^+} \chi_{t_2}(b) u_b^q \right) = \varphi_A(1, \chi_{t_1}) \varphi_A(q, \chi_{t_2}), \\ \mathcal{Y}_2 &:= \left(\sum_{a \in A^+} \chi_{t_2}(a) u_a \right) \left(\sum_{b \in A^+} \chi_{t_1}(b) u_b^q \right) = \varphi_A(q, \chi_{t_1}) \varphi_A(q, \chi_{t_2}), \\ \mathcal{Z} &:= \left(1 - (\theta^q - \theta) \sum_{a \in A^+} u_a^{q-1} \right) \left(\sum_{a \in A^+} \sigma_\Sigma(a) u_a^2 \right) = (1 - (\theta^q - \theta)) \varphi_A(q-1) \varphi_A(2, \sigma_\Sigma). \end{aligned}$$

Note that $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{F}_{q+1}^{\sigma_\Sigma}$. A simple computation yields $G_{q+1}(X) = X^{q+1} + (\theta^q - \theta)^{-1} X^2$. Hence

$$\mathcal{X} = (\theta^q - \theta)^{-1} \varphi_A(2; \sigma_\Sigma) + \varphi_A(q+1; \sigma_\Sigma).$$

By using Lemma 9.13 with $f_A = \varphi_A$, the first entry of the modular form given by the difference of both sides of the identity of our statements vanishes identically so this modular form vanishes identically by Lemmas 7.10 and 7.11. \square

9.2.2. *An identity for Eisenstein series of weight 2.* We prove here a more complicated identity involving Eisenstein series of weights 1 and 2 in the case of q odd. We suppose that $|\Sigma| \equiv 2 \pmod{q-1}$ and we write $s = |\Sigma| = \alpha(q-1) + 2$, $\alpha \in \mathbb{N}$. We have:

Proposition 9.14. *If q is odd the following formula holds:*

$$\sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1} \\ |V| \equiv 1 \pmod{q-1}}} \mathcal{E}(1; \rho_U^*) \otimes \mathcal{E}(1; \rho_V^*) = 2\mathcal{E}(2; \rho_\Sigma^*).$$

Proof. This is a simple combination of Lemmas 7.10 and 7.11 and the next Lemma 9.15. \square

Lemma 9.15. *The following formula holds:*

$$(9.10) \quad \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1} \\ |V| \equiv 1 \pmod{q-1}}} \varphi_A \begin{pmatrix} \sigma_U \\ 1 \end{pmatrix} \varphi_A \begin{pmatrix} \sigma_V \\ 1 \end{pmatrix} = 2\varphi_A \begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix}.$$

Proof. We set $m = \alpha(q - 1) + 2$ and $n = \alpha(q - 1) + 1$, for $\alpha \geq 0$. We claim that

$$(9.11) \quad \sum_{\substack{k \equiv 1 \pmod{q-1} \\ 0 \leq k \leq m}} \binom{m}{k} \equiv 2 \pmod{p},$$

$$(9.12) \quad \sum_{\substack{k \equiv 1 \pmod{q-1} \\ 0 < k \leq n}} \binom{n}{k} \equiv 0 \pmod{p}.$$

To see this we consider more generally $N \in \mathbb{N}$ and we write $N = \alpha(q - 1) + l$ with $\alpha \geq 0$ and $0 \leq l \leq q - 2$. Let λ, μ be in \mathbb{F}_q with $\lambda + \mu \neq 0$. Then,

$$1 = (\lambda + \mu)^l = (\lambda + \mu)^N = \sum_{r=0}^N \binom{N}{r} \lambda^r \mu^{N-r} = \sum_{r'=0}^{q-2} \lambda^{r'} \mu^{\nu(r')} \underbrace{\sum_{\substack{r \equiv r' \pmod{q-1} \\ 0 \leq r \leq N}} \binom{N}{r}}_{=:\beta_{r'}},$$

where $\nu(r')$ is the unique integer in $\{0, \dots, q - 2\}$ such that $l - r' \equiv \nu(r') \pmod{q - 1}$. Setting further $\mu = 1$, we have the polynomial

$$P(X) = \sum_{r'=0}^{q-2} \beta_{r'} X^{r'} - (X + 1)^l \in \mathbb{F}_p[X],$$

which vanishes identically over the set $\mathbb{F}_q \setminus \{-1\}$ with $q - 1$ elements, and has degree $\leq q - 2$. This implies that it is identically zero; in other words, $\beta_{r'} = \binom{l}{r'}$ for $r' = 0, \dots, q - 2$. Taking $N = m = \alpha(q - 1) + 2$ we have $l = 2$ and computing the coefficient of X in P , we deduce (9.11). Taking $N = n = \alpha(q - 1) + 1$ and computing the constant term of P , we deduce (9.12). This shows the claim. We can complete the proof of formula 9.10. We use Theorem 9.7, which tells us that if $U \sqcup V = \Sigma$ with $|U| \equiv |V| \equiv 1 \pmod{q - 1}$,

$$\begin{aligned} \varphi_A \begin{pmatrix} \sigma_U \\ 1 \end{pmatrix} \varphi_A \begin{pmatrix} \sigma_V \\ 1 \end{pmatrix} - \varphi_A \begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix} = \\ \varphi_A \begin{pmatrix} \sigma_U & \sigma_V \\ 1 & 1 \end{pmatrix} + \varphi_A \begin{pmatrix} \sigma_V & \sigma_U \\ 1 & 1 \end{pmatrix} - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} \varphi_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

We sum these identities over all such partitions $\Sigma = U \sqcup V$. First of all, the number of such partitions is equal to

$$\sum_{k=0}^{\alpha} \binom{|\Sigma|}{k(q - 1) + 1}$$

which is congruent to 2 modulo p by (9.11). Let

$$f : \mathcal{P}(\Sigma)^2 \rightarrow L$$

be any map with values in a field L of characteristic p , where $\mathcal{P}(\Sigma)$ is the set of subsets of Σ . Then,

$$\begin{aligned} \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1}}} \left(\sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} f(I, J) - f(U, V) - f(V, U) \right) &= \\ &= \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1}}} \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} f(I, J) = \\ &= \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1}}} f(I, J) \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1} \\ U \supseteq J \text{ or } V \supseteq J}} 1, \end{aligned}$$

which vanishes by (9.12). Observing that we can choose $f(I, J) = f_A(\begin{smallmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{smallmatrix})$ terminates the proof. \square

As a complement of Proposition 9.14 we propose the following question, to be compared with Cornelissen, [19, Proposition (1.15)]. We assume that $|\Sigma| \equiv 2 \pmod{q-1}$.

Question 9.16. *Do the forms $\mathcal{E}(1; \rho_U^*) \otimes \mathcal{E}(1; \rho_V^*)$, for $U \sqcup V = \Sigma$ and $|U| \equiv |V| \equiv 1 \pmod{q-1}$ generate the module $M_2(\rho_\Sigma^*; \mathbb{K}_\Sigma)$?*

9.2.3. *Serre's derivatives of Eisenstein series.* The last type of explicit formulas we want to discuss in this volume is related to Serre's derivatives of Eisenstein series. We are going to see that they are closely related to the harmonic product. We return to the operators $\partial_n^{(w)}(f)$ introduced in §6.3. We suppose that $\Sigma \subset \mathbb{N}^*$ is such that $s = |\Sigma| \equiv 1 \pmod{q-1}$ and we study the u -expansion of the first entry (the one which is indexed by \emptyset) of

$$\partial_1^{(1)}(\mathcal{E}(1; \rho_\Sigma^*)) \in S_3(\rho_\Sigma^* \det^{-1}; \mathbb{K}_\Sigma).$$

By Proposition 7.2, the first entry of $\mathcal{E}(1; \rho_\Sigma^*)$ is equal to $-\tilde{\pi} \varphi_A(1; \sigma_\Sigma)$. We compute, by setting $\Sigma' = \Sigma \sqcup \{0\}$:

$$\begin{aligned} \partial_1^{(1)}(\varphi_A(1; \sigma_\Sigma)) &= \\ &= \sum_{a \in A^+} \sigma_\Sigma(a) a u_a^2 - \sum_{a \in A^+} a u_a \sum_{b \in A^+} \sigma_\Sigma(b) u_b \\ &= -[\varphi_A(1; \chi_{t_0}) \varphi_A(1; \sigma_\Sigma) - \varphi_A(2; \sigma_{\Sigma'})]_{t_0 = \theta}. \end{aligned}$$

This implies that the image of $\partial_1^{(1)}$ at $\varphi_A(1; \sigma_\Sigma)$ is the evaluation of a linear combinations of multiple sums as in §9.1.2 and we transcribe Serre's derivatives in terms of specializations of the harmonic relations of §9:

Lemma 9.17. *We have the formula:*

$$(9.13) \quad \partial_1^{(1)}(\varphi_A(1; \sigma_\Sigma)) = \left[\sum_{\substack{I \sqcup J = \Sigma' \\ |J| \equiv 1 \pmod{q-1} \\ J \subsetneq \Sigma}} \varphi_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix} \right]_{t_0 = \theta}.$$

Proof. This follows directly from Theorem 9.7. □

Viewing Lemma 9.17, it is natural to ask for non-vanishing properties of Serre’s derivatives of our Eisenstein series. In many cases indeed, we can prove that these are non-zero. For instance, we have the next lemma:

Lemma 9.18. *If $q > 2$, $m \geq 1$ and $p \nmid m - 1$, and if Σ is a finite subset of \mathbb{N}^* of cardinality $m(q - 1) + 1$, then $\partial_1^{(1)}(\mathcal{E}(1; \rho_\Sigma^*)) \neq 0$.*

Sketch of proof. In view of Lemmas 7.10, 7.11 and 9.17, it suffices to show that the right-hand side of (9.13) does not vanish identically if our hypotheses are verified. Note that for $I \sqcup J = \Sigma'$ as in the sum in (9.13),

$$\varphi_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix} = u \sum_{\lambda \in \mathbb{F}_q} \sigma_I(\theta + \lambda) u_{\theta + \lambda} + (\text{elements in } u^{q+2} K(\underline{t}_\Sigma)[[u]]),$$

and note that for all λ , $u u_{\theta + \lambda} \equiv u^{q+1} \pmod{(u^{q+2})}$. This follows from

$$u_{\theta + \lambda} = u^q (1 + (\theta + \lambda) u^{q-1})^{-1}, \quad \lambda \in \mathbb{F}_q$$

a consequence of the formula $(e_0)_\theta = C_\theta(e_0)$. Therefore

$$\varphi_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix} \equiv u^{q+1} \sum_{\lambda \in \mathbb{F}_q} \sigma_I(\theta + \lambda) \pmod{(u^{q+2})}.$$

If I occurs in the sum, $|I| \equiv 1 \pmod{q - 1}$, $0 \in I$ and I is not a singleton. The next step is to move outside the formalism of power sums and to study algebraic relations between elementary symmetric polynomials (modulo p). There is an explicit formula that can be proved by elementary combinatorial way (but we omit the details of the proof) which is:

$$(9.14) \quad \mathcal{Q}_I := \sum_{\lambda \in \mathbb{F}_q} \sigma_I(\theta + \lambda) = - \sum_{j=0}^{i-1} e_{j(q-1)+1}(\underline{t}_I), \quad |I| = i(q - 1) + 1, \quad i \geq 0, \quad q > 2,$$

where $e_k(\underline{t}_I)$ is the k -th elementary symmetric polynomial in the variables \underline{t}_I . It is precisely here that we suppose $q > 2$. There is a similar formula in the case $q = 2$ but since it is not identical we disregard this case to avoid a too lengthy proof (but things proceed in a sensibly close way).

By the formula (9.14) the homogeneous term of \mathcal{Q}_I of highest total degree in $\underline{t}_{\Sigma'}$ is equal to $-\mathbf{e}_{(i-1)(q-1)+1}(\underline{t}_I)$. Now we consider:

$$\begin{aligned} \mathcal{R}_{\Sigma} &:= \sum_{\substack{I \sqcup J = \Sigma' \\ |J| \equiv 1(q-1) \\ J \subsetneq \Sigma}} \sum_{\lambda \in \mathbb{F}_q} \sigma_I(\theta + \lambda) \\ &= - \sum_{\substack{I \sqcup J = \Sigma' \\ |J| \equiv 1(q-1) \\ J \subsetneq \Sigma}} \sum_{j=0}^{\frac{|I|-1}{q-1}-1} \mathbf{e}_{j(q-1)+1}(\underline{t}_I) \\ &= - \sum_{j=0}^{m-1} \sum_{\substack{|I|=(j+1)(q-1)+1 \\ I \sqcup J = \Sigma' \\ 0 \in I}} \mathbf{e}_{j(q-1)+1}(\underline{t}_I). \end{aligned}$$

We have $\partial_1^{(1)}(\varphi_A(1; \sigma_{\Sigma})) \neq 0$ if and only if $\mathcal{R}_{\Sigma} \neq 0$. There is a linear dependence relation (in any polynomial ring of characteristic zero):

$$\sum_{\substack{|I|=m(q-1)+1 \\ 0 \in I}} \mathbf{e}_{(m-1)(q-1)+1}(\underline{t}_I) = (m(q-1)+1) \mathbf{e}_{(m-1)(q-1)+1}(\underline{t}_{\Sigma'}).$$

Clearly, if $p \nmid m-1$, the linear form in elementary symmetric polynomials in the left-hand side does not vanish and therefore the homogeneous part of highest total degree in \mathcal{R}_{Σ} is non-zero hence proving that \mathcal{R}_{Σ} itself is non-zero. This completes the proof: $\partial_1^{(1)}(\varphi_A(1; \sigma_{\Sigma})) \neq 0$. \square

We avoided the case $|\Sigma| = 1$ in the above lemma because, on the opposite side:

Lemma 9.19. *If $s = |\Sigma| \leq q-1$, then $\partial_1^{(s)}(\mathcal{E}(s; \rho_{\Sigma}^*)) = 0$.*

Proof. The result follows from the case $s = 1$ because of (8.10) and Leibnitz rule for ∂_1^{\bullet} (relative to \otimes). In the remaining case of one variable $t = t_i$ observe that if $\mathcal{F} = \mathcal{F}_i$ is as in §7.5.1 (we recall that this is an element of $M_{-1}^1(\rho_{\chi_t}; \mathbb{L}_{\Sigma})$), then $h\mathcal{F}^{(1)} \in M_1(\det^{-1} \rho_{\chi_t}; \mathbb{L}_{\Sigma}) \setminus \{0\}$ thanks to (7.13). By (4.23) we see that

$$h\rho_{\chi_t}(S)\mathcal{F}^{(1)} \in M_1(\rho_{\chi_t}^*; \mathbb{L}_{\Sigma}) \setminus \{0\}.$$

Since $M_1(\rho_{\chi_t}^*; \mathbb{L}_{\Sigma})$ is one-dimensional generated by $\mathcal{E}(1; \rho_{\chi_t}^*)$ we deduce that, for some $\lambda \in \mathbb{L}_{\Sigma}^{\times}$,

$$\mathcal{E}(1; \rho_{\chi_t}^*) = \lambda h\rho_{\chi_t}(S)\mathcal{F}^{(1)}.$$

But then,

$$\partial_1^{(1)}(\mathcal{E}(1; \rho_{\chi_t}^*)) = \lambda \partial_1^{(q+1)}(h)\rho_{\chi_t}(S)\mathcal{F}^{(1)} = 0.$$

The identity $\partial_1^{(q+1)}(h) = 0$ is well known, see [24, Proposition (8.8)]. \square

We propose the next question with $s = |\Sigma| \equiv 1 \pmod{q-1}$, $s > 1$, where we recall that $\mathcal{P}_3(G)$ is the Poincaré series defined in (5.20):

Question 9.20. *In the case $|\Sigma| \equiv 1 \pmod{q-1}$ and $|\Sigma| > 1$ when are the form $\partial_1^{(1)}(\mathcal{E}(1; \sigma_\Sigma))$ and the last column of $\mathcal{P}_3(G)$ proportional with a proportionality factor in \mathbb{L}_Σ^\times ?*

In the above question, G is as in Proposition 5.24 with $m = 1$. The question is suggested by the fact that $\partial_1^{(q-1)}(g) = h$ [24, Theorem (9.1)], and in the case $\Sigma = \emptyset$ and $q = 2$, this responds positively to one (and only one) particular case. But if $|\Sigma| = 1$, from our results one sees that the question has negative answer. Indeed $\mathcal{P}_3(G) \neq 0$ and $\partial_1^{(1)}(\mathcal{E}(1; \rho_{\chi_t}^*)) = 0$.

9.3. A conjecture on multiple sums. We write $\mathcal{Z}_\zeta = \bigoplus_{n,\sigma} \mathcal{Z}_{n,\sigma}$ for the graded \mathbb{F}_p -algebra of the *multiple zeta values in Tate algebras* where $\mathcal{Z}_{n,\sigma}$ is the \mathbb{F}_p -subvector space of $\mathbb{F}_p[t_i : i \in \mathbb{N}^*][[\frac{1}{\theta}]]$ generated by the corresponding multiple sums of §9.1.1. Hence, this space is endowed with the Gauss norm $\|\cdot\|$ extending $|\cdot|$ and is generated by the sums $f_A(\mathcal{C})$ of (9.3) in the settings of §9.1.1 and the semi-characters σ involved in the compositions arrays (9.2) are maps from A to $\mathbb{F}_q[t_i : i \in \mathbb{N}^*]$ defined by

$$(9.15) \quad \sigma(a) = \prod_{i \in \mathbb{N}} \chi_{t_i}(a)^{n_i}, \quad a \in A,$$

with $n_i \in \mathbb{N}$ and $n_i = 0$ for all but finitely many $i \in \mathbb{N}^*$. Here we prefer to write $\zeta_A(\mathcal{C})$ instead of $f_A(\mathcal{C})$.

Similarly, we write \mathcal{Z}_φ for the \mathbb{F}_p -algebra $\mathcal{F} = \sum_{n,\sigma} \mathcal{F}_{n,\sigma}$ where $\mathcal{F}_{n,\sigma}$ is this time the \mathbb{F}_p -subvector space of $\mathbb{F}_p[\theta][t_i : i \in \mathbb{N}^*][[u]]$ (with the v -valuation) generated by the sums $\varphi_A(\mathcal{C})$ of (9.1.2). Theorem 9.4 implies that \mathcal{Z}_ζ and \mathcal{Z}_φ are \mathbb{F}_p -algebras. However, we do not know if \mathcal{Z}_φ is graded by the degrees like \mathcal{Z}_ζ . The algebra \mathcal{Z}_φ is the algebra of *A-periodic multiple sums*. We propose:

Conjecture 9.21. *The correspondence $\zeta_A(\mathcal{C}) \leftrightarrow \varphi_A(\mathcal{C})$ induces an isomorphism of \mathbb{F}_p -algebras $\mathcal{Z}_\zeta \cong \mathcal{Z}_\varphi$.*

Conjecture 9.21 implies that \mathcal{Z}_φ is graded by the degrees. Moreover, all the identities for multiple zeta values in \mathcal{Z}_ζ correspond to identities for multiple A -periodic sums, many of which can be proved directly (e.g. Lemmas 9.13 and 9.15). For example, note that in the proof of Proposition 9.12, \mathcal{X} and \mathcal{Z} are not homogeneous. By Conjecture 9.21, any linear dependence relation between $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ and \mathcal{Z} must come from two homogeneous ones, one in $\mathcal{F}_{q+1}^{\sigma_\Sigma}$ and another one in $\mathcal{F}_2^{\sigma_\Sigma}$, both defined over \mathbb{F}_p . We see that these relations exist and are indeed derived from (9.9) and the identity $\varphi_A(2; \sigma_\Sigma) = \varphi_A(1; \chi_{t_1})\varphi_A(1; \chi_{t_2})$.

10. PERSPECTIVES ON ALGEBRAIC PROPERTIES OF EISENSTEIN SERIES

We give here further conjectures which allow to produce examples of relations which can be in certain cases verified by explicit computations. This section provides perspectives suggested by experimental investigations we did for modular forms associated to the representations ρ_Σ^* . Conjecture 10.4 using the notion of *multiple Eisenstein series*, and

Conjectures 10.6, 10.7 and 9.21 together provide a collection of identities between our Eisenstein series. Although yet hypothetical, some special cases of these identities can be verified by explicit computation. Therefore these conjectures can be viewed as challenging problems to pursue researches in this domain.

10.1. Multiple Eisenstein series. In [18], Chen introduces a function field variant of *Eisenstein* and *double Eisenstein series* as initially defined by Gangl, Kaneko and Zagier in [23]. We propose here a generalization of her viewpoint. We begin with a description of the required settings, introducing a vector-valued generalization of multiple Eisenstein series. We state Conjecture 10.4 suggesting correspondences between multiple zeta values and multiple Eisenstein series.

We consider ρ_1, \dots, ρ_r representations of the first kind which are constructed starting from basic representations by using the operations $\oplus, \otimes, \wedge^\alpha, S^\beta$ as well as the ‘comatrix operation’ Co , defined through the comatrix map. All these representations extend to monoid maps defined over $A^{2 \times 2}$, with its standard matrix product. Before going on we need some notation: we need to work with composition arrays having the first line composed by representations of the first kind.

We consider positive integers n_1, \dots, n_r and composition arrays (with $(\cdot)^*$ contragredient)

$$\widehat{\mathcal{C}} = \begin{pmatrix} \rho_1 & \cdots & \rho_r \\ n_1 & \cdots & n_r \end{pmatrix}, \quad \widehat{\mathcal{C}}^* = \begin{pmatrix} \rho_1^* & \cdots & \rho_r^* \\ n_1 & \cdots & n_r \end{pmatrix}.$$

We also set, for $j \in \{0, \dots, r\}$

$$\widehat{\mathcal{C}}_{\leq j} = \begin{pmatrix} \rho_1 & \cdots & \rho_j \\ n_1 & \cdots & n_j \end{pmatrix}, \quad \widehat{\mathcal{C}}_{> j} = \begin{pmatrix} \rho_{j+1} & \cdots & \rho_r \\ n_{j+1} & \cdots & n_r \end{pmatrix},$$

so that $\widehat{\mathcal{C}}_{\leq r} = \widehat{\mathcal{C}}$ and we set $\widehat{\mathcal{C}}_{\leq 0} = \emptyset$. We now define:

$$\widetilde{\Phi}(\widehat{\mathcal{C}}^*) = \sum_{|a_1| > \cdots > |a_r| > 0} \rho_1 \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \rho_r \begin{pmatrix} a_r & 0 \\ 0 & 1 \end{pmatrix} \cdot \Psi_{n_1}(\rho_1^*)_{a_1} \otimes \cdots \otimes \Psi_{n_r}(\rho_r^*)_{a_r},$$

with the sum running over elements $a_1, \dots, a_r \in A^+$. The dot \cdot is the usual matrix product, and the index $(\cdot)_a$ with $a \in A$ designates the the matrix function (with entries in \mathfrak{K}_Σ for appropriate $\Sigma \subset \mathbb{N}^*$) obtained after substitution $z \mapsto az$. The matrices $\begin{pmatrix} a_j & 0 \\ 0 & 1 \end{pmatrix}$ do not belong to Γ but all the terms of the series are well defined thanks to the hypothesis on ρ_1, \dots, ρ_r . This series converges to a rigid analytic map $\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$ for appropriate $\Sigma \subset \mathbb{N}^*$ and $N > 0$ and to an element of $\mathfrak{M}_\Sigma^{N \times N}$.

In the case $\rho_i = \rho_{U_i}$ with, for $i = 1, \dots, r$, U_i finite subsets of \mathbb{N}^* , the case that interests us the most in the present paper, we have:

$$\left[\widetilde{\Phi}(\widehat{\mathcal{C}}^*) \right]_1 = \widetilde{\pi}^{\sum_i n_i} \sum_{\substack{a_1, \dots, a_r \in A^+ \\ |a_1| > \cdots > |a_r|}} \rho_{U_1} \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \rho_{U_r} \begin{pmatrix} a_r & 0 \\ 0 & 1 \end{pmatrix} \cdot V(n_1; \rho_{U_1}^*)_{a_1} \otimes \cdots \otimes V(n_r; \rho_{U_r}^*)_{a_r}$$

where $[\cdot]_1$ denotes the first column of a matrix and $V(n; \rho_U^*)$ is defined in (7.1).

We also set

$$\mathcal{Z}(\widehat{\mathcal{C}}^*) := (\rho_1 \otimes \cdots \otimes \rho_r) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \sum_{\substack{a_1, \dots, a_r \in A^+ \\ |a_1| > \cdots > |a_r| > 0}} a_1^{-n_1} \cdots a_r^{-n_r} \rho_1^*(T_{-a_1}) \otimes \cdots \otimes \rho_r^*(T_{-a_r}),$$

a series which converges in $\mathbb{F}_q(\!(t_\Sigma)\!)((\theta^{-1}))^{N \times N}$ (we follow the same conventions used in the definition of $\widetilde{\Phi}(\widehat{\mathcal{C}}^*)$). In the case $\rho_i = \rho_{U_i}$ for all i , it is easy to see that

$$\left[\mathcal{Z}(\widehat{\mathcal{C}}^*) \right]_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \zeta_A \begin{pmatrix} \sigma_{U_1} & \cdots & \sigma_{U_r} \\ n_1 & \cdots & n_r \end{pmatrix} \end{pmatrix} \in \mathbb{T}_\Sigma^{N \times 1}$$

where $\Sigma = \cup_i U_i$ and N is the product of the dimensions of the representations ρ_i , in agreement with (7.2).

Definition 10.1. The *multiple Eisenstein series* associated with the composition array $\widehat{\mathcal{C}}^*$ is the series

$$\mathcal{E}_A(\widehat{\mathcal{C}}^*) = \left[\widetilde{\Phi}(\widehat{\mathcal{C}}^*) + \widetilde{\Phi}(\widehat{\mathcal{C}}^*_{\leq r-1}) \otimes \mathcal{Z}(\widehat{\mathcal{C}}^*_{> r-1}) + \cdots + \mathcal{Z}(\widehat{\mathcal{C}}^*) \right]_1 \in \mathfrak{D}_\Sigma^{N \times 1}.$$

We say that $\mathcal{E}_A(\widehat{\mathcal{C}}^*)$ is of *degree* $\binom{\rho_1^* \otimes \cdots \otimes \rho_r^*}{n_1 + \cdots + n_r}$.

It is easy to verify that, if the representations ρ_i are all equal to $\mathbf{1}$ (case in which $N = 1$ and $\Sigma = \emptyset$) and $r = 1, 2$, this coincides with [18, Definition 3.2]. The function $E_k(z)$ defined in *ibid.* coincides with our $\mathcal{E}_A(k) = \mathcal{E}_A\left(\frac{\mathbf{1}}{k}\right)$ (for $k > 0$) and similarly, $E_{r',s'}(z)$ of *ibid.* coincides with our $\mathcal{E}_A(r', s') = \mathcal{E}_A\left(\frac{\mathbf{1}}{r'} \frac{\mathbf{1}}{s'}\right)$. The case of depth $r = 1$ can be resumed in the next formula which follows easily from (7.9) and Proposition 7.2, where $m > 0$ and $\Sigma \subset \mathbb{N}^*$ a finite subset such that $|\Sigma| \equiv m \pmod{q-1}$:

$$(10.1) \quad \mathcal{E}_A\left(\frac{\rho_\Sigma^*}{m}\right) = \left[\widetilde{\Phi}\left(\frac{\rho_\Sigma^*}{m}\right) + \mathcal{Z}\left(\frac{\rho_\Sigma^*}{m}\right) \right]_1 = -\mathcal{E}(m; \rho_\Sigma^*).$$

It is also easy to verify the next lemma, where $\rho_1^*, \dots, \rho_r^*$ are representations of the form $\rho_{U_j}^*$ with $U_j \subset \Sigma$ for some finite subset Σ of \mathbb{N}^* , and where $\sigma_1, \dots, \sigma_r$ denote the projections of ρ_1, \dots, ρ_r on their upper-right coefficients (these are semi-characters). We recall the multiple sums ϕ_A defined in (9.6).

Lemma 10.2. *Writing*

$$\widehat{\mathcal{C}}^* = \begin{pmatrix} \rho_1^* & \cdots & \rho_r^* & \mathbf{1} & \cdots & \mathbf{1} \\ n_1 & \cdots & n_r & m_1 & \cdots & m_s \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \sigma_1 & \cdots & \sigma_r & \mathbf{1} & \cdots & \mathbf{1} \\ n_1 & \cdots & n_r & m_1 & \cdots & m_s \end{pmatrix},$$

for $r > 0, s \geq 0$, the first entry \mathcal{E}_1 of $\mathcal{E} = \mathcal{E}_A(\widehat{\mathcal{C}}^*)$ satisfies, with $n = \sum_i n_i$,

$$\begin{aligned} \mathcal{E}_1 = & \widetilde{\pi}^n \left(\widetilde{\pi}^{\sum_{j \leq s} m_j} \phi_A \left(\begin{pmatrix} \sigma_1 & \cdots & \sigma_r & \mathbf{1} & \cdots & \mathbf{1} \\ n_1 & \cdots & n_r & m_1 & \cdots & m_s \end{pmatrix} \right) + \widetilde{\pi}^{\sum_{j \leq s-1} m_j} \phi_A \left(\begin{pmatrix} \sigma_1 & \cdots & \sigma_r & \mathbf{1} & \cdots & \mathbf{1} \\ n_1 & \cdots & n_r & m_1 & \cdots & m_{s-1} \end{pmatrix} \right) \zeta_A(m_s) + \cdots \\ & \cdots + \widetilde{\pi}^{m_1} \phi_A \left(\begin{pmatrix} \sigma_1 & \cdots & \sigma_r & \mathbf{1} \\ n_1 & \cdots & n_r & m_1 \end{pmatrix} \right) \zeta_A(m_2, \dots, m_s) + \phi_A \left(\begin{pmatrix} \sigma_1 & \cdots & \sigma_r \\ n_1 & \cdots & n_r \end{pmatrix} \right) \zeta_A(m_1, \dots, m_s) \end{aligned}$$

and the last entry $\mathcal{E}_N \in \mathfrak{D}_\Sigma$ of $\mathcal{E} = \mathcal{E}_A(\widehat{\mathcal{C}}^*)$ satisfies

$$\mathcal{E}_N - \zeta_A(\mathcal{C}) \in \mathfrak{M}_\Sigma.$$

10.1.1. *Eulerian multiple zeta values.* We consider semi-characters $\sigma_1, \dots, \sigma_r$ defined as in (9.5) and positive integers n_1, \dots, n_r . We write $\sigma = \prod_i \sigma_i = \prod_j \chi_{t_j}^{\nu_j}$ for the type and $n = \sum_i n_i$ for the weight of the multiple zeta value $\zeta_A(\frac{\sigma_1}{n_1} \cdots \frac{\sigma_r}{n_r})$. In this subsection we return to the settings of §9.1.1 to make the following definition.

Definition 10.3. Let Z be a K -linear combination of multiple zeta values of degree $\binom{\sigma}{n}$. We say that Z is *Eulerian* if

$$Z \in K(\underline{t}_\Sigma) \frac{\widetilde{\pi}^n}{\prod_j \omega(t_j)^{\nu_j}}.$$

This agrees with the notion of eulerian multiple zeta value of Thakur as in [77, Definition 5.10.8] because in the case of trivial type the product involving the Anderson-Thakur function is equal to one. See [17] for deep properties of Eulerian multiple zeta values in the case of trivial type. Examples of Eulerian combinations of multiple zeta values in our settings are given by the elements $\zeta_A(n; \sigma_\Sigma)$ with $|\Sigma| \equiv n \pmod{q-1}$. By using [34, (39)] we see that the elements $\zeta_A(\begin{smallmatrix} \chi_t & \mathbf{1} & \cdots & \mathbf{1} \\ 1 & q-1 & \cdots & q^{r-1}(q-1) \end{smallmatrix})$ are eulerian for all $r \geq 0$.

10.1.2. *A conjecture for multiple Eisenstein series.* We denote by $\mathcal{W}_n^{\rho^*}$ the \mathbb{F}_p -vector space of multiple Eisenstein series of degree $\binom{\rho^*}{n}$, with $n > 0$ where ρ^* is a product of representations of the type $\rho_{U_i}^*$. Writing $\rho^* = \otimes_j (\rho_{t_j}^*)^{\otimes \nu_j}$, we set $\sigma = \prod_j \chi_{t_j}^{\nu_j}$. We consider $\mathcal{C}, \widehat{\mathcal{C}}^*$ as in Lemma 10.2. We address the following:

Conjecture 10.4. *The following properties hold:*

- (1) *We have inclusions $\mathcal{W}_m^{\rho^*} \otimes \mathcal{W}_n^{\psi^*} \subset \mathcal{W}_{m+n}^{\rho^* \otimes \psi^*}$.*
- (2) *The correspondence $\zeta_A(\mathcal{C}) \mapsto \mathcal{E}_A(\widehat{\mathcal{C}}^*)$ defines an isomorphism η of \mathbb{F}_p -vector spaces between the space \mathcal{Z}_n^σ of multiple zeta values of degree $\binom{\sigma}{n}$ and $\mathcal{W}_n^{\rho^*}$ which is compatible with the multiplication rules in such a way that the sum $\mathcal{W} := \sum_{n, \rho^*} \mathcal{W}_n^{\rho^*}$ is graded, and endowed with a structure of \mathbb{F}_p -algebra with multiplication \otimes , isomorphic to the algebra $\bigoplus_{n, \sigma} \mathcal{Z}_n^\sigma$.*
- (3) *An element $f \in \mathcal{Z}_n^\sigma$ is eulerian if and only if $\eta(f)$ is a modular form in $M_n(\rho^*; \mathbb{L}_\Sigma)$.*

The next result describes a depth two identity which illustrates the pertinence of the above conjecture in a special case, interesting because lying outside the case of Eisenstein series. The reader will notice that the proof given is quite ad hoc and not easily generalizable. While the first item of the conjecture is likely to be at reach by an appropriate generalization of the harmonic product of §9, the equivalence between eulerianity of constant terms and modularity of vector functions may require deeper arithmetic/geometric tools.

Proposition 10.5. *The following identity holds:*

$$\mathcal{E}(1; \rho_t^*) \otimes \mathcal{E}(q-1; \mathbf{1}) + \mathcal{E}(q; \rho_t^*) = \mathcal{E}_A\left(\begin{smallmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{smallmatrix}\right).$$

Proof. We claim that

$$(10.2) \quad \mathcal{E}(1; \rho_t^*) = \tilde{\pi} \begin{pmatrix} -\varphi_A \begin{pmatrix} \chi_t \\ 1 \end{pmatrix} \\ \tilde{\varphi}_A \begin{pmatrix} \delta \\ 1 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} 0 \\ \zeta_A(1; \chi_t) \end{pmatrix},$$

where δ is the semi-character $a \mapsto \chi_t(az)$ and $\tilde{\varphi}_A$ has been introduced in §9.1.4. This follows easily from (10.1), Perkins' identity (6.13) and Proposition 7.2. In a similar way, we can easily prove the identity

$$(10.3) \quad \mathcal{E}(q; \rho_1^*) = \tilde{\pi}^q \begin{pmatrix} -\varphi_A \begin{pmatrix} \chi_t \\ q \end{pmatrix} \\ \tilde{\varphi}_A \begin{pmatrix} \delta \\ q \end{pmatrix} \end{pmatrix} + \frac{\tilde{\pi}^q}{(t - \theta)\omega(t)} \begin{pmatrix} 0 \\ \tilde{\varphi}_A(q - 1) \end{pmatrix} - \begin{pmatrix} 0 \\ \zeta_A(q; \chi_t) \end{pmatrix}.$$

To see this, note that $\tau(\mathcal{E}(1; \rho_1^*)) = \mathcal{E}(q; \rho_1^*)$ and use (2.18); all the series involved in these formulas are convergent for the v -valuation. We deduce from (10.2) and (10.3) that

$$\begin{aligned} \mathcal{E}(1; \rho_t^*) \otimes \mathcal{E}(q - 1; \mathbf{1}) + \mathcal{E}(q; \rho_t^*) &= \\ &= \tilde{\pi}^q \begin{pmatrix} \varphi_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix} \\ -\tilde{\varphi}_A \begin{pmatrix} \delta & \mathbf{1} \\ 1 & q-1 \end{pmatrix} \end{pmatrix} + \tilde{\pi} \begin{pmatrix} \varphi_A \begin{pmatrix} \chi_t \\ 1 \end{pmatrix} \\ -\tilde{\varphi}_A \begin{pmatrix} \delta \\ 1 \end{pmatrix} \end{pmatrix} \zeta_A(q - 1) + \begin{pmatrix} 0 \\ \zeta_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

This identity is reached applying the second and the third identities in (9.8) and the formula (7.12). But a direct computation shows that

$$\mathcal{E}_A \begin{pmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{pmatrix} = \left[\tilde{\mathfrak{F}} \begin{pmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{pmatrix} + \tilde{\mathfrak{F}} \begin{pmatrix} \rho_t^* \\ 1 \end{pmatrix} \otimes \mathfrak{Z} \begin{pmatrix} \mathbf{1} \\ q-1 \end{pmatrix} + \mathfrak{Z} \begin{pmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{pmatrix} \right]_1$$

equals the right-hand side of the above identity. \square

We deduce that $\mathcal{E}_A \begin{pmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{pmatrix}$ is in $M_q(\rho_t^*; \mathbb{L})$. One further proves that it is non-zero and is not a cusp form. In fact we have that

$$\mathcal{E}_A \begin{pmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{pmatrix} - \begin{pmatrix} 0 \\ \zeta_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix} \end{pmatrix} \in \mathfrak{M}_\Sigma^{2 \times 1},$$

with \mathfrak{M}_Σ the maximal ideal of the valuation v , and Σ a singleton, so that $\eta(\zeta_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix}) = \mathcal{E}_A \begin{pmatrix} \rho_t^* & \mathbf{1} \\ 1 & q-1 \end{pmatrix}$ and we see, by [34, Lemma 6.12] that $\zeta_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix}$ is Eulerian. One proves easily that $\zeta_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix} = \frac{\theta-t}{\theta^q-\theta} \zeta_A \begin{pmatrix} \chi_t \\ q \end{pmatrix}$. However, the cusp form $\mathcal{E}_A \begin{pmatrix} \chi_t & \mathbf{1} \\ 1 & q-1 \end{pmatrix} - \frac{\theta-t}{\theta^q-\theta} \mathcal{E}_A \begin{pmatrix} \chi_t \\ q \end{pmatrix}$ does not vanish identically by Corollary 8.12. Hence the item (3) of Conjecture 10.4 does not extend to K -linear combinations of multiple Eisenstein series.

10.2. A conjecture for zeta values in Tate algebras. We now focus on zeta values in Tate algebras (1.17). Recall from the introduction that $q = p^e$ with $e > 0$. Hence $\tau = \mu^e$ where μ is the \mathbb{F}_p -linear automorphism of \mathbb{C}_∞ given by $c \mapsto c^p$ for $c \in \mathbb{C}_\infty$, which can be extended $\mathbb{F}_p(\underline{t}_\Sigma)$ -linearly to \mathbb{K}_Σ for any finite set Σ . We introduce the following \mathbb{F}_p -algebra

$$\mathbb{I} := \mathbb{F}_p \left[\mu^m(\zeta_A(1, \chi_{t_i})) : \begin{matrix} i \in \mathbb{N}^* \\ m \in \mathbb{Z} \end{matrix} \right] \subset \bigcup_{k \geq 0} \mathbb{F}_p[t_i : i \in \mathbb{N}^*][[\theta^{-\frac{1}{p^k}}]].$$

We set $\zeta_A(0) := 1$. The \mathbb{F}_p -algebra \mathbb{I} is thus generated by all the μ -twists (negative or positive) of the functions $\zeta_A(1, \chi_{t_i})$ for $i \in \Sigma$. It is very important to allow negative values for m , and for this reason this \mathbb{F}_p -algebra carries a structure of inversive μ -difference algebra. We address the following

Conjecture 10.6. *For all $n \in \mathbb{N}^*$ and $\Sigma \subset \mathbb{N}^*$ such that $|\Sigma| \equiv n \pmod{q-1}$ we have a unique expansion*

$$(10.4) \quad \zeta_A(n; \sigma_\Sigma) = \sum_{\substack{0 \leq k \leq n \\ k \equiv 0 \pmod{q-1}}} \zeta_A(k) \eta_k, \quad \eta_k \in \mathbb{I}.$$

Recall that in our conventions, $\zeta_A(k) = \zeta_A(k; \mathbf{1})$ are the usual Carlitz zeta values. We are going to give some examples of relations along the predictions of this conjecture. Note that the factors η_k need not to lie in $\mathbb{F}_p[t_i : i \in \mathbb{N}^*](\frac{1}{\theta})$. However, there exists $l \in \mathbb{N}$ such that $\mu^l(\eta_k) \in \mathbb{F}_p[t_i : i \in \mathbb{N}^*](\frac{1}{\theta})$ for all $k \equiv 0 \pmod{q-1}$ in the range $0 \leq k \leq n$ and all the terms involved are products of zeta values. Since $\mu^l(\zeta_A(k; \sigma_\Sigma)) = \zeta_A(kp^l; \sigma_\Sigma)$, the identity (10.4) is equivalent to an algebraic identity of zeta values as in (1.17) defined over \mathbb{F}_p . We recall from Thakur conjectures in [79, §5.3] that the only \mathbb{F}_p -relations among his multiple zeta values in K_∞ are those which come from the harmonic product.

Conjecture 10.7. *The only \mathbb{F}_p -algebraic relations in \mathbb{I} are those coming from the harmonic product.*

After Conjecture 10.7, all the algebraic relations defined over \mathbb{F}_p between the elements $\zeta_A(n; \sigma_\Sigma)$ with $n \equiv |\Sigma| \pmod{q-1}$ can be derived from the harmonic product and for each zeta value $\zeta_A(n; \sigma_\Sigma)$ it should be possible to derive explicit formulas like in (10.4) by using the harmonic product of Theorem 9.7 (or in [58]). However, carrying this program might be very difficult in practice due to the combinatorial computations involved. The challenge is to introduce other techniques to tackle it. This was accomplished by Hung Le and Ngo Dac in [43], where they proved a particular case of this conjecture hence proving a conjectural formula of the author of the present text. Their result is reviewed in the following §10.2.1.

10.2.1. *Some evidences.* We focus on the case $n = 1$ in Conjecture 10.7 so that we can now suppose that $|\Sigma| = m(q-1) + 1$ with $m \geq 0$. We know from [2, 5] that

$$(10.5) \quad \zeta_A(1; \sigma_\Sigma) = \frac{(-1)^m \tilde{\pi} \mathbb{B}_\Sigma}{\omega_\Sigma}, \quad |\Sigma| \equiv 1 \pmod{q-1}, \quad |\Sigma| > 1,$$

where $\mathbb{B}_\Sigma \in A[t_\Sigma]$ ⁽¹⁸⁾ is a monic polynomial in θ of degree $m-1$ when $m \geq 1$ and $\omega_\Sigma = \prod_{i \in \Sigma} \omega(t_i) \in \mathbb{T}_\Sigma^\times$. If $m = 0$, the conjecture is clearly verified thanks to the formula (7.12). If $m = 1$ then $\mathbb{B}_\Sigma = 1$ by [5, Corollary 7.3] so that

$$\zeta_A(1, \sigma_\Sigma) = \tau^{-1} \left(\prod_{i \in \Sigma} \zeta_A(1, \chi_{t_i}) \right) \in \mathbb{I}$$

¹⁸ \mathbb{B} stands for 'Bernoulli'.

confirming Conjecture 10.6 also in this case.

To describe the case $m = 2$ (so that $|\Sigma| = 2q - 1$) we shall introduce the notation

$$\mathcal{L}_U^{(m)} := \tau^m \left(\prod_{i \in U} \zeta_A(1, \chi_{t_i}) \right),$$

for $U \subset \Sigma$. Then it is possible to show the following explicit formula:

$$\zeta_A(1, \sigma_\Sigma) = \sum_{\substack{\Sigma = U_1 \sqcup U_2 \\ |U_1| = q-1 \\ |U_2| = q}} \mathcal{L}_{U_1}^{(-1)} \mathcal{L}_{U_2}^{(-2)},$$

where \sqcup denotes disjoint union. Now, recall that the right-hand side is equal to $\frac{\tilde{\pi} \mathbb{B}_\Sigma^*}{\omega_\Sigma}$, with

$$\mathbb{B}_\Sigma^* = - \sum_{\substack{U_2 \subset \Sigma \\ |U_2| = q}} \prod_{i \in U_2} \left(t_i - \theta^{\frac{1}{q}} \right),$$

while the left-hand side is easily seen to be equal to $\frac{\tilde{\pi} \mathbb{B}_\Sigma}{\omega_\Sigma}$, with

$$\mathbb{B}_\Sigma = \theta - \sum_{\substack{V \subset \Sigma \\ |V| = q}} \prod_{i \in V} t_i = -e_q \left(t_i - \theta^{\frac{1}{q}} : i \in \Sigma \right)$$

(with e_n denoting here the n -th elementary symmetric polynomial), and it is easy to see that $\mathbb{B}_\Sigma = \mathbb{B}_\Sigma^*$ (all the terms defined over $\mathbb{F}_p[\theta^{\frac{1}{q}}]$, but not all over $\mathbb{F}_p[\theta]$, cancel. More generally we have the next result (see [43, Theorem 1.3]) ⁽¹⁹⁾:

Theorem 10.8 (Hung Le and Ngo Dac). *For all $m \geq 0$ and for all $q > m$ the following formula holds:*

$$(10.6) \quad \zeta_A(1; \sigma_\Sigma) = \sum_{\substack{U_1 \sqcup \dots \sqcup U_m = \Sigma \\ q^{-1}|U_1| + \dots + q^{-m}|U_m| = 1}} \mathcal{L}_{U_1}^{(-1)} \dots \mathcal{L}_{U_m}^{(-m)}.$$

Although Conjecture 10.7 predicts that such formulas can all be derived from the harmonic product the method of Hung Le and Ngo Dac does not use it, and introduces new tools which do not reduce it to a mere computational verification, the latter being most likely out of reach.

¹⁹The formula (10.6) has been conjectured by the author of the present paper and incorporated in an earlier version of it. The work [43] has been anticipated by the verification of the cases $m = 1, 2, 3, 4$ by Ngo Dac in [20].

10.2.2. *More about Theorem 10.8.* It is not hard to show that Hung Le and Ngo Dac's Theorem is equivalent to the following corollaries ⁽²⁰⁾:

Corollary 10.9. *Assuming that $m \geq 2$ and that $q > m$, we have the formula*

$$\zeta_A(1, \sigma_\Sigma) = \sum_{r=0}^{m-2} \sum_{\substack{U \sqcup V \sqcup \Sigma' = \Sigma \\ |V|=q-r-1 \\ |U|=rq}} \tau^{-1}(\zeta_A(1, \sigma_{\Sigma'})) \mathcal{L}_U^{(-2)} \mathcal{L}_V^{(-1)}.$$

The interest of Corollary 10.9 is that it can be considered in parallel with similar (but not analogue) classical formulas by Euler. We recall that the well-known Riccati's differential equation $f' = -1 - f^2$ satisfied by the cotangent function $f(x) = \cot(x)$ implies, via the formula $-\frac{\pi x}{2} \cot(\pi x) = \sum_{i \geq 0} \zeta(2i)x^{2i}$:

$$\left(n + \frac{1}{2}\right) \zeta(2n) = \sum_{i=1}^{n-1} \zeta(2i)\zeta(2n - 2i), \quad n > 1.$$

Note that the coefficients in the quadratic expression on the right-hand side are all equal to 1.

Theorem 10.8 implies nice formulas for the polynomials $\mathbb{B}_\Sigma \in A[\underline{t}_\Sigma]$ (when $|\Sigma| > q$. Indeed, observe that for all $m \geq 1$,

$$(10.7) \quad \tau^{-m}((t - \theta)\omega)^{-1} = \left(t - \theta^{\frac{1}{q^{m-1}}}\right) \cdots \left(t - \theta^{\frac{1}{q}}\right) \omega^{-1}.$$

Hence,

$$\tau^{-m}\left(\zeta_A(1, \chi_t)\right) = -\frac{\tilde{\pi}^{\frac{1}{q^m}} \left(t - \theta^{\frac{1}{q^{m-1}}}\right) \cdots \left(t - \theta^{\frac{1}{q}}\right)}{\omega}, \quad m \geq 1.$$

Setting $b_m^* := \left(t - \theta^{\frac{1}{q^{m-1}}}\right) \cdots \left(t - \theta^{\frac{1}{q}}\right)$ (again for $m \geq 1$) and $B_m^*(\underline{t}_\Sigma) = \prod_{i \in \Sigma} b_m^*(t_i)$, we thus have:

Corollary 10.10. *The following formula holds, when $q > m$.*

$$\mathbb{B}_\Sigma = (-1)^{m-1} \sum_{\substack{U_1 \sqcup \cdots \sqcup U_m = \Sigma \\ q^{-1}|U_1| + \cdots + q^{-m}|U_m| = 1}} B_1^*(\underline{t}_{U_1}) \cdots B_m^*(\underline{t}_{U_m}).$$

Similarly, Corollary 10.9 is equivalent to:

Corollary 10.11. *The following formula holds, for $|\Sigma| = m(q - 1) + 1$ with $q > m \geq 2$.*

$$\mathbb{B}_\Sigma = \sum_{r=0}^{m-2} (-1)^{r+1} \sum_{\substack{U \sqcup V \sqcup \Sigma' = \Sigma \\ |U|=qr \\ |V|=q-r-1 \\ |\Sigma'|=(m-r-1)(q-1)+1}} \tau^{-1}(\mathbb{B}_{\Sigma'}) \prod_{i \in U \sqcup \Sigma'} \left(t_i - \theta^{\frac{1}{q}}\right).$$

²⁰They were stated as Conjectures in the earlier versions of the present manuscript.

10.3. A modular analogue. We end this work with a conjectural formula which can be derived from Theorem 10.8. We set, with $U \subset \mathbb{N}^*$ a finite subset and $j \in \mathbb{Z}$:

$$\mathcal{E}_U^{(j)} := \tau^j \left(\bigotimes_{i \in U} \mathcal{E}(1; \rho_{t_i}^*) \right),$$

for $U \subset \Sigma$. Note that this needs not to represent an analytic function $\Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ for $N \geq 1$ if $j < 0$.

Conjecture 10.12. *For all $m \geq 0$, $|\Sigma| = s = m(q-1) + 1$ and for all $q > m$, the following formula holds:*

$$(10.8) \quad \mathcal{E}(1; \rho_\Sigma^*) = \sum_{\substack{U_1 \sqcup \dots \sqcup U_m = \Sigma \\ q^{-1}|U_1| + \dots + q^{-m}|U_m| = 1}} \mathcal{E}_{U_1}^{(-1)} \otimes \dots \otimes \mathcal{E}_{U_m}^{(-m)}.$$

We note that (10.8) expresses the analytic function $\mathcal{E}(1; \rho_\Sigma^*)$ as a linear combination of non-analytic functions if $s \geq 2q - 1$. Clearly, Theorem 10.8 and Conjecture 9.21, or Conjecture 10.4 imply Conjecture 10.12 (and the latter implies Theorem 10.8). The cases $s = 1, q$ are obviously verified, see (8.10). The case $s = 2q - 1$ is at the moment unsolved. The author was only able to verify, numerically, for few values of q a prime number, that the u -expansions of the \emptyset -coordinates of both sides in (10.8) agree up to a certain order. But this is not enough to conclude.

ADDENDUM

The reader may notice that some results in [34] depend on results written here, and some results of loc. cit. are cited in our §9. However, there is no loop in the chains of deductions.

REFERENCES

- [1] G. ANDERSON & D. THAKUR. *Tensor powers of the Carlitz module and zeta values.* Ann. of Math. **132** (1990), 159-191.
- [2] B. ANGLÈS & F. PELLARIN. *Functional identities for L-series values in positive characteristic.* J. Number Theory **142** (2014), 223-251.
- [3] B. ANGLÈS, F. PELLARIN. *Universal Gauss-Thakur sums and L-series.* Invent. Math. 200 (2), 653-669, (2015). 28, 2015.
- [4] B. ANGLÈS, T. NGO DAC & F. TAVARES RIBEIRO. *Special functions and twisted L-series.* Journal de théorie des nombres de Bordeaux, 29 (2017) 931-961.
- [5] B. ANGLÈS, F. PELLARIN & F. TAVARES RIBEIRO. *Arithmetic of positive characteristic L-series values in Tate algebras.* Comp. Math. **152** (2016), 1-61.
- [6] P. BANTAY & T. GANNON. *Vector-valued modular functions for the modular group and the hypergeometric equation.* Communications in number theory and physics, Vol. 1, 651-680, 2007.
- [7] W. BARTENWERFER. *Der erste Riemannsche Hebbarkeitssatz im nichtarchimedischen Fall.* J. reine angew. Math. 20, (1976), 144-163.
- [8] D. BASSON, F. BREUER & R. PINK. *Drinfeld modular forms of arbitrary rank.* To appear in Memoirs of the AMS, 2022.

- [9] G. BÖCKLE. *An Eichler-Shimura isomorphism over function fields between Drinfeld modular forms and cohomology classes of crystals*. Manuscript, currently available on the webpage of the author.
- [10] G. BÖCKLE. *Hecke characters associated to Drinfeld modular forms*. With an appendix by the author and T. Centeleghe. *Compositio Math.* 151 (2015), 2006-2058.
- [11] JAN H. BRUINIER. *Borchers Products on $O(2, l)$ and Chern Classes of Heegner Divisors* Springer Lecture Notes in Mathematics 1780.
- [12] S. BOSCH. *Lectures on rigid and formal geometry*. Lect. Notes in Math. 2105, Springer-Verlag, (2014).
- [13] V. BOSSER & F. PELLARIN. *Hyperdifferential properties of Drinfeld quasi-modular forms*, *Int. Math. Res. Not. IMRN* **11** (2008), Art. ID rnn032, 56 pp.
- [14] V. BOSSER & F. PELLARIN. *On certain families of Drinfeld quasi-modular forms*. *Journal of Number Theory* 129 (2009) 2952-2990.
- [15] S. BOSCH, U. GÜNTZER & R. REMMERT. *Non-Archimedean Analysis*. Grundlehren der mathematischen Wissenschaften, Vol. 261.
- [16] J. W. S. CASSELS. *Local fields*. Cambridge University Press, (1986).
- [17] C.-Y. CHANG, M. PAPANIKOLAS & J. YU. *An effective criterion for Eulerian multizeta values in positive characteristic*. *J. Of the EMS*. Volume 21, Issue 2, 2019, 405-440.
- [18] H.-J. CHEN. *On shuffle of double Eisenstein series in positive characteristic*. *J. de Théorie des Nombres de Bordeaux*, 29, 815-825, (2017).
- [19] G. CORNELISSEN. *Drinfeld Modular Forms of Weight One*. *J. of Number Theory*, 67, 215-228 (1997).
- [20] T. NGO DAC. *Letter to the author, May, 10, 2017*.
- [21] A. EL GUINDY & A. PETROV. *On symmetric powers of τ -recurrent sequences and deformations of Eisenstein series*. *Prof. of the Amer. Math. Soc.* Volume 143, Number 8, August 2015, 3303-3318.
- [22] J. FRESNEL, & M. VAN DER PUT. *Rigid Analytic Geometry and its Applications*. Birkhäuser, Boston (2004).
- [23] H. GANGL, M. KANEKO & D. ZAGIER. *Double zeta values and modular forms*. In *Automorphic forms and zeta functions*. Tokyo, Japan, World Scientific (2006).
- [24] E.-U. GEKELER. *On the coefficients of Drinfeld modular forms*. *Invent. Math.* 93, No.3, 667-700 (1988).
- [25] E.-U. GEKELER. *On the de Rham isomorphism for Drinfeld modules*. *J. reine angew Math.* 401, 188-208 (1989).
- [26] E.-U. GEKELER. *On the zeroes of Goss polynomials*. *Trans. of the Amer. Math. Soc.* (2012).
- [27] E.-U. GEKELER. *On Drinfeld modular forms of higher rank*. *Journal de Théorie des Nombres de Bordeaux* Vol. 29 (2017), 875-902.
- [28] E.-U. GEKELER. *On Drinfeld modular forms of higher rank II*. *J. Number Theory*, (2019).
- [29] E.-U. GEKELER. *On Drinfeld modular forms of higher rank III: The analogue of the $k/12$ -formula*. *J. Number Theory* Vol. 192, (2018), 293-306.
- [30] E.-U. GEKELER. *On Drinfeld modular forms of higher rank IV: Modular forms with level*. *J. Number Theory*, (2019).
- [31] E.-U. GEKELER. *Goss polynomials, q -adic expansions, and Sheats compositions*. *J. Number Theory* 240 (2022) 196-253.
- [32] L. GERRITZEN & M. VAN DER PUT. *Schottky Groups and Mumford Curves*. LNM 817. Springer Berlin Heidelberg, 1980.
- [33] O. GEZMIŞ & M. A. PAPANIKOLAS. *The de Rham isomorphism for Drinfeld modules over Tate algebras*. *J. Algebra* 525 (2019), 454-496.
- [34] O. GEZMIŞ & F. PELLARIN. *Trivial multiple zeta values in Tate algebras*. *IMRN* (2021) Art. ID rnab104, 65 pp.
- [35] D. GOSS. *π -adic Eisenstein series for Function Fields*. *Compositio Math.* 41, 3-38 (1980).
- [36] D. GOSS. *Basic Structures of Function Field Arithmetic*. Springer Verlag, Berlin, (1996).
- [37] D. GOSS. *Letter to A. Petrov, February 20, 2013*.
- [38] D. GOSS. *On the L -series of F. Pellarin*. *J. of Number Theory* 133 (2013) 955-962.
- [39] D. GOSS. *A construction of \mathfrak{v} -adic modular forms*. *J. Number Theory* 136 (2014) 330-338.

- [40] S. HÄBERLI. *Satake compactification of analytic Drinfeld modular varieties*. J. Number Theory 219, (2021), 1-92.
- [41] U. HARTL & CHIA-FU YU. *Arithmetic Satake compactifications and algebraic Drinfeld modular forms*. Preprint [arXiv:2009.13934](https://arxiv.org/abs/2009.13934) (2020).
- [42] S. HATTORI. *P-adic continuous families of Drinfeld eigenforms of finite slope*. Adv. Math. 380 (2021), 107594.
- [43] H. HUNG LE & T. NGO DAC. *On identities for zeta values in Tate algebras*. Trans. Amer. Math. Soc. 374 (2021), 5623-5650 <https://doi.org/10.1090/tran/8357>.
- [44] K. KEDLAYA. *The algebraic closure of the power series field in positive characteristic*. Proc. Amer. Math. Soc. 129 (2001), 3461-3470.
- [45] K. NHUAN LE. *Letter to the author, August 10, 2022*.
- [46] S. LANG. *Algebraic groups over finite fields*. Amer. J. Math. 76 (1956), 555-563.
- [47] C. MARKS & G. MASON. *Structure of the module of vector-valued modular forms*. Journal of the London Math. Soc., Vol. 82, 2010, 32-48, <https://doi.org/10.1112/jlms/jdq020>
- [48] T. MIYAKE. *Modular forms*. Springer-Verlag, Berlin, New York, (1989).
- [49] H. NAGAO. *On $GL(2, K[x])$* . J. Inst. Polytechn., Osaka City Univ., Ser. A. 10: (1959) 117-121.
- [50] M.-H. NICOLE & G. ROSSO. *Familles de formes modulaires de Drinfeld pour le groupe général linéaire*. Trans. AMS. 374, 2021, 4227-4266.
- [51] M.-H. NICOLE & G. ROSSO. *Perfectoid Drinfeld Modular Forms* J. de Th. des Nombres de Bordeaux 33, 2021, 1045-1067.
- [52] M. A. PAPANIKOLAS. *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms*, Invent. Math. 171, 123-174 (2008).
- [53] M. PAPANIKOLAS & G. ZENG. *Theta operators, Goss polynomials, and v-adic modular forms*. J. Théor. Nombres Bordeaux. 3, (2017), p. 729-753.
- [54] F. PELLARIN. *Aspects de l'indépendance algébrique en caractéristique non nulle*. Bourbaki seminar. Volume 2006/2007. Exposés 967–981. Paris: Société Mathématique de France. Astérisque 317, 205-242 (2008).
- [55] F. PELLARIN. *Values of certain L-series in positive characteristic*. Ann. of Math. 176 (2012), 2055-2093 <http://dx.doi.org/10.4007/annals.2012.176.3.13>
- [56] F. PELLARIN. *On the generalized Carlitz module*. Journal of Number Theory. Vol. 133, 2013, 1663-1692.
- [57] F. PELLARIN. *A note on certain representations in characteristic p and associated functions*. J. Number Theory, 176, (2017), 420-438 <http://dx.doi.org/10.1016/j.jnt.2016.12.001>.
- [58] F. PELLARIN. *A sum-shuffle formula for zeta values in Tate algebras*. J. de Théorie des Nombres de Bordeaux, 29 (2017), 1025-1048.
- [59] F. PELLARIN. *On a variant of Schanuel conjecture for the Carlitz exponential*. J. de Théorie des Nombres de Bordeaux, 29 (2017), 845-873.
- [60] F. PELLARIN. *From the Carlitz exponential to Drinfeld modular forms*. In *Arithmetic and Geometry over Local Fields*. Springer LNM 2275, edited by B. Anglès and T. Ngo Dac. (2020). 93-175.
- [61] F. PELLARIN, & R. B. PERKINS. *On certain generating functions in positive characteristic*. Monat. Math. (2016) 180, pp 123-144. doi:10.1007/s00605-016-0880-6
- [62] F. PELLARIN, & R. B. PERKINS. *On twisted A-harmonic sums and Carlitz finite zeta values*. J. of Number Theory 232, March 2022, 355-378
- [63] F. PELLARIN, & R. B. PERKINS. *On vectorial Drinfeld modular forms over Tate algebras*. Int. J. of Number Theory 14, 1729-1783 (2018)
- [64] R. B. PERKINS. *On Special Values of Pellarin's L-Series*. Ph. D. thesis. The Ohio State University (2013).
- [65] R. B. PERKINS. *Explicit formulae for L-values in positive characteristic*. Math. Z. (2014), 248, 279-299. doi:10.1007/s00209-014-1315-5.
- [66] A. PETROV. *On A-expansions of Drinfeld modular forms*. Ph.D. thesis, The University of Arizona, (2012).

- [67] A. PETROV. *A-expansions of Drinfeld modular forms*. Journal of Number Theory 133 (2013) 2247–2266.
- [68] A. PETROV. *On hyperderivatives of single-cuspidal Drinfeld modular forms with A-expansion*. J. Number Theory, 149, (2015), 153-165.
- [69] R. PINK. *Compactification of Drinfeld modular varieties and Drinfeld modular forms of arbitrary rank*. Manuscripta Math. 140 (2013), 333-361.
- [70] M. ROSEN. *Number Theory in Function Fields*. Graduate Texts in Mathematics 210, Springer Verlag (2002).
- [71] W. H. SCHIKHOF. *Ultrametric Calculus: An Introduction to p -Adic Analysis*. Cambridge University Press, (1984).
- [72] J.-P. SERRE. *Endomorphismes complètement continus des espaces de Banach p -adiques*. Publ. math. I.H.E.S. 12, (1962), 69-85.
- [73] J.-P. SERRE. *Formes modulaires et fonctions zêta p -adiques*. In: Kuijk, W., Serre, JP. (eds) Modular Functions of One Variable III. Lecture Notes in Mathematics, vol 350. Springer, Berlin, Heidelberg (1973).
- [74] J.-P. SERRE. *Trees*. Berlin, New York: Springer-Verlag, (1980).
- [75] J.-P. SERRE. *Local fields*. Berlin, New York: Springer-Verlag, (1980).
- [76] L. TAELEMAN. *Special L -values of Drinfeld modules*. Ann. of Math. 175-1, (2012), 369-391.
- [77] D. THAKUR. *Function field arithmetic*. World Scientific Publishing, River Edge NJ, 2004.
- [78] D. THAKUR. *Power sums of polynomials over finite fields and applications: A survey*. Finite Fields Appl. **32** (2015) 171-191.
- [79] D. THAKUR. *Shuffle Relations for Function Field Multizeta Values*. Int. Math. Res. Not., (2010), No. 11, 1973-1980. doi:10.1093/imrn/rnp202.
- [80] D. THAKUR. *Multizeta in function field arithmetic*. In t-motives: Hodge structures, transcendence and other motivic aspects, 441-452, Ed. by G. Böckle, D. Goss, U. Hartl, M. Papanikolas, European Math. Soc. pub. May 2020.

FEDERICO PELLARIN, ISTITUTO GUIDO CASTELNUOVO, UNIVERSITÀ SAPIENZA, PIAZZALE ALDO MORO 5,, 00185 ROME, ITALY