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ON THE BEHAVIOUR AT THE CUSPS OF DRINFELD MODULAR FORMS

F. PELLARIN

ABSTRACT. In this paper we generalise the notion of Drinfeld modular form for the group $\Gamma := \mathrm{GL}_2(\mathbb{F}_q[\theta])$ to a vector-valued setting, where the target spaces are certain modules over positive characteristic Banach algebras over which are defined what we call the 'representations of the first kind'. Under quite reasonable restrictions, we show that the spaces of such modular forms are finite-dimensional, are endowed with certain generalisations of Hecke operators, with differential operators à la Serre etc. The crucial point of this work is the introduction of a 'field of uniformisers', a valued field in which we can study the expansions at the cusp infinity of our modular forms and which is wildly ramified. Examples of such modular forms are given through the construction of Poincaré and Eisenstein series. After this the paper continues with a more detailed analysis of the special case of modular forms associated to a restricted class of representations ρ_Σ^* of Γ which has more importance in arithmetical applications. More structure results are given in this case, and a harmonic product formula is obtained which allows, with the help of three conjectures on the structure of an \mathbb{F}_p -algebra of A -periodic multiple sums, to produce conjectural formulas for Eisenstein series. Some of these formulas can be proved.

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1. INTRODUCTION

The aim of this text is to revisit the theory of Drinfeld modular forms, initiated by David Goss, in his Ph. D. Thesis (see [19]). We want to extend the notion of *type*, considered by Gekeler in [15], which is a one-dimensional representation of the Drinfeld modular group, to higher dimensional representations.

Let $q = p^e$ be a power of a prime number p with $e > 0$ an integer, let \mathbb{F}_q be the finite field with q elements and characteristic p , and θ an indeterminate over \mathbb{F}_q . All along this text, we denote by A the \mathbb{F}_q -algebra $\mathbb{F}_q[\theta]$. We set $K = \mathbb{F}_q(\theta)$. On K , we consider the absolute value $|\cdot|$ defined by $|a| = q^{\deg_{\theta}(a)}$, a being in K , so that $|\theta| = q$. Let $K_{\infty} := \mathbb{F}_q((1/\theta))$ be the local field which is the completion of K for this absolute value, let K_{∞}^{ac} be an algebraic closure of K_{∞} , let \mathbb{C}_{∞} be the completion of K_{∞}^{ac} for the unique extension of $|\cdot|$ to K_{∞}^{ac} . Then, the field \mathbb{C}_{∞} is at once algebraically closed and complete for $|\cdot|$ with valuation group $q^{\mathbb{Q}}$ and residual field \mathbb{F}_q^{ac} , an algebraic closure of \mathbb{F}_q .

The 'Drinfeld half-plane' $\Omega = \mathbb{C}_{\infty} \setminus K_{\infty}$, with the usual rigid analytic structure in the sense of [14, Definition 4.3.1], carries an action of Γ and $\tilde{\Gamma}$ by homographies: if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}$, and $z \in \Omega$,

$$\gamma(z) := \frac{az + b}{cz + d}.$$

Denote by

$$J_{\begin{pmatrix} * & * \\ c & d \end{pmatrix}}(z) = cz + d$$

the usual factor of automorphy $\Gamma \times \Omega \rightarrow \mathbb{C}_{\infty}^{\times}$. Let us consider $w, m \in \mathbb{Z}$; then, if $w \equiv 2m \pmod{q-1}$, the map $(\gamma, z) \mapsto J_{\gamma}(z)^w \det(\gamma)^{-m}$ defines a factor of automorphy for $\tilde{\Gamma}$. There

is a bijection between these factors of automorphy and the couples $(w, m) \in \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ submitted to the above congruence.

We thus suppose that $w \in \mathbb{Z}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ are such that $w \equiv 2m \pmod{q-1}$. We recall the definition of Drinfeld modular forms (as considered by Gekeler and Goss).

Definition 1.1. A *Drinfeld modular form of weight $w \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$* for the group $\tilde{\Gamma}$ is a rigid analytic function $\Omega \xrightarrow{f} \mathbb{C}_\infty$ such that

$$f(\gamma(z)) = J_\gamma(z)^w \det(\gamma)^{-m} f(z) \quad \forall z \in \Omega, \quad \forall \gamma \in \tilde{\Gamma}$$

and such that additionally, there exists $0 < c < 1$ with the property that if $z \in \Omega$ is such that $|u(z)| \leq c$, where

$$u(z) = \frac{1}{\tilde{\pi}} \sum_{a \in A} \frac{1}{z - a},$$

$\tilde{\pi} \in \mathbb{C}_\infty \setminus K_\infty$ being a fundamental period of Carlitz's module ('our' analogue of $2\pi i$, see (10)), then there is a uniformly convergent series expansion

$$(1) \quad f(z) = \sum_{n \geq 0} f_n u(z)^n, \quad f_n \in \mathbb{C}_\infty.$$

We say that a function f as in (1) is *regular at the infinity cusp*.

The type corresponds to a representation

$$(2) \quad \Gamma \xrightarrow{\det^{-m}} \mathrm{GL}_1(\mathbb{F}_q), \quad m \in \mathbb{Z}/(q-1)\mathbb{Z}.$$

In dimension > 1 it happens that certain representations of Γ naturally have non-trivial analytic deformations, and this makes it natural to consider functions with values in positive-dimensional Tate algebras or in similar ultrametric Banach algebras. We consider $\Sigma \subset \mathbb{N}^*$ a finite subset. Let k_Σ be the field $\mathbb{F}_q(\underline{t}_\Sigma)$ where \underline{t}_Σ is the set of variables $(t_i : i \in \Sigma)$. We choose a representation

$$(3) \quad \Gamma \xrightarrow{\rho} \mathrm{GL}_N(k_\Sigma).$$

Let $w \in \mathbb{Z}$ be such that the map $(\gamma, z) \mapsto J_\gamma(z)^w \rho(\gamma)$ defines a factor of automorphy

$$\tilde{\Gamma} \times \Omega \rightarrow \mathrm{GL}_N(k_\Sigma).$$

This does not happen for any representation of Γ . The necessary and sufficient condition is that

$$(4) \quad \rho(\mu I_2) = \mu^{-w} I_N, \quad \mu \in \mathbb{F}_q^\times,$$

as it comes out after a simple computation.

We consider the field

$$\mathbb{K}_\Sigma = \mathbb{C}_\infty(\underline{t}_\Sigma)^\wedge = \widehat{\mathbb{C}_\infty(\underline{t}_\Sigma)}$$

(the completion for the Gauss norm)¹ so that $\mathrm{GL}_N(k_\Sigma) \subset \mathrm{GL}_N(\mathbb{K}_\Sigma)$. We denote by $\|\cdot\|$ the norm of \mathbb{K}_Σ , extending the norm $|\cdot|$ of \mathbb{C}_∞ . We further extend this norm on matrices

¹Observe the notation $(\cdot)^\wedge$ that will be used when the other notation will lead to a too large hat.

with entries in \mathbb{K}_Σ in the usual way by taking the supremum of the norms of the entries. In §2.2 we discuss the notion of rigid analytic functions with values in \mathbb{K}_Σ . Taking this notion into account:

Definition 1.2. A rigid analytic function

$$\Omega \xrightarrow{f} \mathbb{K}_\Sigma^{N \times 1}$$

such that

$$(5) \quad f(\gamma(z)) = J_\gamma(z)^w \rho(\gamma) f(z) \quad \forall z \in \Omega, \quad \forall \gamma \in \tilde{\Gamma},$$

is called *modular-like of weight w for ρ* . Additionally, we say that such a function $f = {}^t(f_1, \dots, f_N)$ is:

- (1) A *weak modular form of weight w for ρ* if there exists $M \in \mathbb{Z}$ such that $\|u(z)^M f(z)\|$ is bounded as $0 < |u(z)| < c$ for some $c < 1$.
- (2) A *modular form of weight w (for ρ)* if $\|u(z)^M f(z)\|$ is bounded as $0 < |u(z)| < c$ for some $c < 1$.
- (3) A *cuspidal form of weight w* if $\|f(z)\| \rightarrow 0$ as $u(z) \rightarrow 0$.

Let B be a \mathbb{C}_∞ -sub-algebra of \mathbb{K}_Σ . We suppose that ρ as in (3) has image in $\mathrm{GL}_N(B)$. We denote by $M_w^1(\rho; B)$ (resp. $M_w(\rho; B)$, $S_w(\rho; B)$) the B -modules of weak modular forms (resp. modular forms, cuspidal forms) of weight w for ρ such that their images are contained in $B^{N \times 1}$. We have that

$$S_w(\rho; B) \subset M_w(\rho; B) \subset M_w^1(\rho; B).$$

If $B = \mathbb{C}_\infty$, $N = 1$ and $\rho = \det^{-m}$, these \mathbb{C}_∞ -vector spaces coincide with the correspondent spaces of 'classical' Drinfeld modular forms of weight w , type m .

To be of some interest, Definition 1.2 must deliver certain primordial properties such as the finite dimensionality of spaces of modular forms, or their invariance under the action of suitable Hecke operators. We are far from being able to return satisfactory answers in such a level of generality. However, there is a class of representations (called *representations of the first kind*, introduced and discussed in §4.1) which look suitable for our investigation because they contain a variety of arithmetically interesting examples. An explicit example of such representations is, with t a variable, the one which associates to a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the matrix

$$(6) \quad \rho_t(\gamma) = \begin{pmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q[t]),$$

where χ_t is the unique \mathbb{F}_q -algebra morphism $\mathbb{F}_q[\theta] \rightarrow \mathbb{F}_q[t]$ sending θ to t . Another interesting example is $\rho_t^* := {}^t\rho^{-1}$, investigated in [28, 37]; in this last case, we have explicitly described the module structure of $M_w(\rho_t; \widehat{\mathbb{C}_\infty[t]})$ (the values are in a Tate algebra) and proved that these $\widehat{\mathbb{C}_\infty[t]}$ -modules are endowed with endomorphisms given by a natural generalisation of Hecke operators.

Results of the text. The following synthesis summarises our results (more precise statements will be formulated along the text). The key object is called the *field of uniformizers*.

Theorem A. *There exists a valued field \mathfrak{K} , called the field of uniformizers, with valuation v , valuation group $\mathbb{Z}[\frac{1}{p}]$, residual field $\cup_{\Sigma} \mathbb{K}_{\Sigma}$, valuation ring \mathfrak{D} and maximal ideal \mathfrak{M} , satisfying the following properties. Let $\Sigma \subset \mathbb{N}^*$ be a finite subset and $\rho : \Gamma \rightarrow \mathrm{GL}_N(k_{\Sigma})$ be a representation of the first kind, let $w \in \mathbb{Z}$ be such that $(\gamma, z) \mapsto J_{\gamma}(z)^w \rho(\gamma)$ is a factor of automorphy for $\tilde{\Gamma}$. The following properties hold.*

- (1) *There is a natural embedding of \mathbb{K}_{Σ} -vector spaces $M_w^1(\rho; \mathbb{K}_{\Sigma}) \xrightarrow{\iota_{\Sigma}} \mathfrak{K}^{N \times 1}$.*
- (2) *The image by ι_{Σ} of the \mathbb{K}_{Σ} -vector space of modular forms $M_w(\rho; \mathbb{K}_{\Sigma})$ can be identified with $\iota_{\Sigma}(M_w^1(\rho; \mathbb{K}_{\Sigma})) \cap \mathfrak{D}^{N \times 1}$.*
- (3) *The vector space of cusp forms $S_w(\rho; \mathbb{K}_{\Sigma})$ can be identified with the sub-vector space of $M_w(\rho; \mathbb{K}_{\Sigma})$ which is sent to $\mathfrak{M}^{N \times 1}$ by the embedding ι_{Σ} .*
- (4) *We have that $\mathbb{C}_{\infty}((u))$ naturally embeds in \mathfrak{K} and v restricts to the u -adic valuation.*
- (5) *The vector spaces $M_w(\rho; \mathbb{K}_{\Sigma}), S_w(\rho; \mathbb{K}_{\Sigma})$ are endowed with Hecke operators $T_{\mathfrak{a}}$ associated to ideals \mathfrak{a} of A , which provide a totally multiplicative system of endomorphisms reducing, in the case $\Sigma = \emptyset$, to the classical Hecke operators acting on classical scalar Drinfeld modular forms and cusp forms.*
- (6) *We have \mathbb{K}_{Σ} -linear maps $\partial_w^{(n)} : M_w(\rho; \mathbb{K}_{\Sigma}) \rightarrow S_{w+2n}(\rho \det^{-n}; \mathbb{K}_{\Sigma})$, defined for all $n \geq 0$ and generalising Serre's derivatives.*

The corresponding results in the body of the text are more precise and cover a wider spectrum of applications. The field \mathfrak{K} will be constructed explicitly in §3 by taking the fraction field of the *ring of tame series*. The main examples of modular forms (construction of Poincaré series etc.) and the basic results concerning the spaces $M_w(\rho; \mathbb{K}_{\Sigma})$ and $S_w(\rho; \mathbb{K}_{\Sigma})$ are contained in §5. Parts (1), (2), (3) will be proved in Theorem 4.14 and (4) is an obvious consequence of the above (so, when $\rho = \mathbf{1}$ is the trivial representation (sending every element of Γ to $1 \in \mathrm{GL}_1$), our construction specialises to the known setting, and $M = \oplus_w M_w(\mathbf{1}; \mathbb{C}_{\infty})$ is the well known algebra of \mathbb{C}_{∞} -valued Drinfeld modular forms for Γ (of type 0). We will introduce Poincaré series in §5.3 as a first main class of specimens of modular forms. Part (5) is our Theorem 4.19; the proof is very simple, thanks to the flexibility of the use of the field of uniformizers, and we can say the same about part (6), which corresponds to our Theorem 6.7.

A non-complete field \mathbb{L}_{Σ} intermediate between \mathbb{K}_{Σ} and the fraction field of \mathbb{T}_{Σ} will be needed in the next Theorem; it is defined in §2.1.3.

Theorem B. *The following properties hold, for ρ a representation of the first kind.*

- (1) *For all $w \in \mathbb{Z}$, the \mathbb{L}_{Σ} -vector space $M_w(\rho; \mathbb{L}_{\Sigma})$ has finite dimension. The dimension is zero if $w < 0$.*
- (2) *The dimension of the space $M_1(\rho; \mathbb{L}_{\Sigma})$ does not exceed the dimension of the space of common eigenvectors in $\mathbb{L}_{\Sigma}^{N \times 1}$ of all the matrices $\rho(\gamma)$ with γ in the Borel subgroup of Γ .*

Note that (1) of Theorem B only deals with modular forms with values in \mathbb{L}_Σ . One reason for this restriction comes from the fact that we use, in the proof, a specialisation property at roots of unity which is unavailable in the general case of \mathbb{K}_Σ -valued functions. This result corresponds to Theorem 5.4. Part (2), proved in Theorem 5.12 which provides an upper bound for the dimensions of the \mathbb{L}_Σ -vector spaces, and implies a positive answer to the question raised by [37, Problem 1.1] thanks to Theorem C in the following way.

We fix a subset $\Sigma \subset \mathbb{N}^*$ of cardinality s and we consider, for all $i \in \Sigma$,

$$\rho_{t_i}^*(\gamma) = {}^t \begin{pmatrix} a(t_i) & b(t_i) \\ c(t_i) & d(t_i) \end{pmatrix}^{-1},$$

and

$$(7) \quad \rho_\Sigma^* := \bigotimes_{i \in \Sigma} \rho_{t_i}^*.$$

This is indeed a representation of the first kind of degree s where $N = 2^{|\Sigma|}$. Additionally, ρ_Σ^* is an irreducible representation of Γ in $\mathrm{GL}_N(\mathbb{F}_q[\underline{t}_\Sigma])$ (see [31]). An important feature of this class of representations is that it allows to construct certain Eisenstein series in §7.1. If $s \equiv 1 \pmod{q-1}$ we have the Eisenstein series of weight 1:

$$\mathcal{E}(1; \rho_\Sigma^*)(z) := \sum_{(a,b) \in A^2 \setminus \{(0,0)\}} (az + b)^{-1} \bigotimes_{i \in \Sigma} \begin{pmatrix} a(t_i) \\ b(t_i) \end{pmatrix},$$

which is a non-zero holomorphic function $\Omega \rightarrow \mathbb{T}_\Sigma^{N \times 1}$, where $\mathbb{T}_\Sigma = \widehat{\mathbb{C}_\infty[\underline{t}_\Sigma]}$ is the standard Tate algebra in the variables \underline{t}_Σ . We have $\mathcal{E}(1; \rho_\Sigma^*) \in M_1(\rho_\Sigma^*; \mathbb{T}_\Sigma) \setminus S_1(\rho_\Sigma^*; \mathbb{T}_\Sigma)$. Writing $\mathcal{E}(1; \rho_\Sigma^*) = {}^t(\mathcal{E}_1, \dots, \mathcal{E}_N) \in \mathfrak{D}_\Sigma^{N \times 1}$ we can prove that $\mathcal{E}_1, \dots, \mathcal{E}_{N-1} \in \mathfrak{M}$ and $\mathcal{E}_N \in \mathfrak{D} \setminus \mathfrak{M}$ (we recall that \mathfrak{D} and \mathfrak{M} are respectively the valuation ring and the maximal ideal of the field of uniformisers). We also study in depth the v -valuations of the elements \mathcal{E}_i for $i = 1, \dots, N-1$, a problem which, we should say, is not easy. It turns out that

$$\mathcal{E}_N \equiv -\zeta_A(1; \sigma_\Sigma) \pmod{\mathfrak{M}}$$

where

$$(8) \quad \zeta_A(n; \sigma_\Sigma) = \sum_{a \in A^+} a^{-n} \sigma_\Sigma(a), \quad n \in \mathbb{N}^*$$

$\sigma_\Sigma(a) = \prod_{i \in \Sigma} \chi_{t_i}(a)$, are the zeta values in Tate algebras introduced in [28] and studied in [2, 3, 4] as well as in other papers.

Theorem C. *The vector space $M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ is one-dimensional, generated by $\mathcal{E}(1; \rho_\Sigma^*)$.*

This is Theorem 7.6. The proofs of (2) of Theorem B and of Theorem C are easy but use a natural isomorphism between (scalar) Drinfeld modular forms for congruence subgroups of Γ and spaces of *automorphic functions* (harmonic cocycles) over the Bruhat-Tits tree of Ω , and the same specialisation properties in terms of the variables t_i used in the proof of (1). When we do this with the entries of the elements of $M_1(\rho; \mathbb{L}_\Sigma)$ span scalar Drinfeld modular forms of weight one for congruence subgroups of Γ . The proof of this result is thus

based on a crucial earlier remark of Gekeler (which can be found in Cornelissen's paper [11]).

From §7 on the paper focuses on structure properties of modular forms for the representation ρ_Σ^* . We introduce here the notion of *strongly regular modular form* (see Definition 7.12). A strongly regular modular form $f = (f_1, \dots, f_N)$ is a Drinfeld modular form which satisfies certain conditions on the v -valuations of its entries. Theorem 7.15 allows a completely explicit structure description for these modular forms which can be stated as follows (more precise results can be found in the text).

Theorem D. *Every strongly regular modular form associated to the representation ρ_Σ^* can be constructed combining Eisenstein series $\mathcal{E}(1; \rho_{t_i})$ and $\mathcal{E}(q; \rho_{t_i})$ for $i \in \Sigma$ by using the Kronecker product, and scalar Eisenstein series. In particular, the $M \otimes_{\mathbb{C}_\infty} \mathbb{K}_\Sigma$ -module of \mathbb{K}_Σ -valued strongly regular modular forms is free of rank $N = 2^s$ where $s = |\Sigma|$.*

The continuous $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear automorphism τ of \mathbb{K}_Σ extending the automorphism $c \mapsto c^q$ of \mathbb{K}_Σ induces injective maps $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma) \rightarrow M_{qw}(\rho_\Sigma^*; \mathbb{K}_\Sigma)$. We show, in Theorem 7.23 that for every w there exists $k \in \mathbb{N}$ such that $\tau^k(f)$ is *strongly regular* for every $f \in M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$. This allows to deduce:

Theorem E. *The \mathbb{K}_Σ -vector spaces $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ are finitely dimensional.*

Note that the functions of Theorem E have values in $\mathbb{K}_\Sigma^{N \times 1}$, not just in $\mathbb{L}_\Sigma^{N \times 1}$ so that the methods of proof of Theorem B do not apply for Theorem E. After Theorem D for every modular form $f \in M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ there is k such that $\tau^k(f)$ can be constructed combining Eisenstein series, and the coefficients in the construction are in \mathbb{K}_Σ . If f is an Eisenstein series, there seem to emerge another phenomenon: the coefficients are in $\mathbb{F}_p(\theta)$. This is considered in §8 and 9. In §8 we prove a variant of a harmonic product formula for certain A -periodic multiple sums. In §8 we propose conjectures which apply to produce certain conjectural formulas for Eisenstein series $\mathcal{E}(w; \rho_\Sigma^*)$. These formulas are so complicate that are essentially undetectable by numerical experiments.

Acknowledgements. This sub-section will be written once the reviewing process complete.

2. PRELIMINARIES

Most commonly used notation.

- $\mathbb{N} = \{0, 1, \dots\}$ the set of natural integers.
- $\mathbb{N}^* = \{1, \dots\}$ the set of positive natural integers.
- $B^{M \times N}$: M -row, N -column arrays with coefficients in the set B .
- I_r : the $r \times r$ identity matrix.
- \sqcup disjoint union.
- $\text{Diag}(*, \dots, *)$ diagonal matrix.
- $\ell_q(n)$ sum of the digits of the base- q expansion of the positive integer n .
- $A = \mathbb{F}_q[\theta]$
- $\Gamma = \text{GL}_2(A)$
- $\tilde{\Gamma} = \Gamma/\mathbb{F}_q^\times$.

- **1** the trivial representation sending Γ to $1 \in \mathbb{F}_q^\times$.

In this section we collect the basic settings over which we are going to build our theory. We give a very short and self contained account of holomorphic functions with values in Tate algebras or more generally, in Banach algebras, of which we are going to study the first examples, useful for what follows.

2.1. Rings, fields, modules. Let Σ be a finite subset of the set of positive integers \mathbb{N}^* . We denote by \underline{t}_Σ the set of variables $\{t_i : i \in \Sigma\}$. If $\Sigma = \{i\}$ is a singleton, we will often simplify our notations by writing t instead of t_i .

2.1.1. *Tate algebras.* Let

$$\mathbb{C}_\infty[\underline{t}_\Sigma] \xrightarrow{v_\infty} \mathbb{Q} \cup \{\infty\}$$

be the unique extension of the valuation v_∞ over \mathbb{C}_∞ which is trivial over $\mathbb{F}_q[\underline{t}_\Sigma]$ (Gauss valuation). The completion $\widehat{\mathbb{C}_\infty[\underline{t}_\Sigma]}_{v_\infty}$ ($= \mathbb{C}_\infty[\underline{t}_\Sigma]^\wedge$ alternate notation when the hat is too large) of $\mathbb{C}_\infty[\underline{t}_\Sigma]$ with respect to this valuation is the standard *Tate algebra* (of dimension $s = |\Sigma|$) denoted by \mathbb{T}_Σ in all the following. If Σ is a singleton we will prefer the notation \mathbb{T} for this algebra, with variable t . Note that if $\Sigma' \subset \Sigma$ then, the natural embedding $\mathbb{C}_\infty[\underline{t}_{\Sigma'}] \subset \mathbb{C}_\infty[\underline{t}_\Sigma]$ induces an embedding $\mathbb{T}_{\Sigma'} \subset \mathbb{T}_\Sigma$.

It is well known that \mathbb{T}_Σ is a ring which is Noetherian, factorial, of Krull dimension s (see [9] for the general theory of these algebras). The Tate algebra \mathbb{T}_Σ is isomorphic to the sub- \mathbb{C}_∞ -algebra of the formal series

$$f = \sum_{\substack{i_j \geq 0 \forall j \in \Sigma \\ \underline{i} = (i_j : j \in \Sigma)}} f_{\underline{i}} \prod_{j \in \Sigma} t_j^{i_j} \in \mathbb{C}_\infty[[\underline{t}_\Sigma]]$$

which satisfy

$$\lim_{\min\{i_j : j \in \Sigma\} \rightarrow \infty} f_{\underline{i}} = 0.$$

Thus, we have, for f a formal series of \mathbb{T}_Σ expanded as above, and non-zero, that

$$v_\infty(f) = \inf_{\underline{i}} v_\infty(f_{\underline{i}}) = \min_{\underline{i}} v_\infty(f_{\underline{i}}).$$

We also set, for convenience,

$$\|\cdot\| := q^{-v_\infty(\cdot)}$$

and $0 = \|0\| = q^{-\infty}$; this extends the norm $|\cdot|$ of \mathbb{C}_∞ .

2.1.2. *The completion \mathbb{K}_Σ of the fraction field of \mathbb{T}_Σ .* For Σ a finite subset of \mathbb{N}^* , we write \mathbb{K}_Σ for the completion of the fraction field of \mathbb{T}_Σ which is also equal to the completion of $\mathbb{C}_\infty[\underline{t}_\Sigma]$. If $\Sigma' \subset \Sigma$, we have the obvious isometric embedding $\mathbb{K}_{\Sigma'} \subset \mathbb{K}_\Sigma$.

Lemma 2.1. *Let Σ' be a subset of Σ . Let $\mathcal{B} = (b_i)_{i \in I}$ be a basis of the $\mathbb{F}_q^{\text{ac}}(\underline{t}_{\Sigma'})$ -vector space $\mathbb{F}_q^{\text{ac}}(\underline{t}_\Sigma)$. Then, every element f of \mathbb{K}_Σ can be expanded, in an unique way, as a converging series*

$$f = \sum_{i \in I} f_i b_i, \quad f_i \in \mathbb{K}_{\Sigma'}, \quad f_i \rightarrow 0, \quad \|f\| = \max_{i \in I} \|f_i\|.$$

In the above lemma, I is countable, and the limit $f_i \rightarrow 0$ is for the Fréchet filter of I . If $\Sigma' = \emptyset$ this means that the Banach \mathbb{C}_∞ -space \mathbb{K}_Σ is endowed with an *orthonormal basis* providing us with an isometric isomorphism with a Banach \mathbb{C}_∞ -space $c_{\mathbb{C}_\infty}(I)$ in the sense of Serre in [42]. The proof that we present is essentially the same as [42, Lemma 1].

Proof of Lemma 2.1. Let $\mathcal{B} = (b_i)_{i \in I}$ be a basis of the $\mathbb{F}_q^{ac}(\underline{t}_{\Sigma'})$ -vector space $\mathbb{F}_q^{ac}(\underline{t}_\Sigma)$. Let us consider $\alpha \in \mathbb{C}_\infty[\underline{t}_\Sigma] \setminus \{0\}$. We can decompose (in unique way) $\alpha = \alpha_0 + \alpha_1$ with $\alpha_i \in \mathbb{C}_\infty[\underline{t}_\Sigma]$ and $\alpha_1 \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}$, and $\|\alpha_0\| < 1$. For any multi-index $\underline{k} = (k_i : i \in \Sigma) \in \mathbb{N}^\Sigma$ we have, in \mathbb{K}_Σ (with $\underline{t}_\Sigma^{\underline{k}} = \prod_{i \in \Sigma} t_i^{k_i}$):

$$\underline{t}_\Sigma^{\underline{k}} \alpha^{-1} = \frac{\underline{t}_\Sigma^{\underline{k}}}{\alpha_1} \left(1 - \frac{\alpha_0}{\alpha_1} + \frac{\alpha_0^2}{\alpha_1^2} - \dots \right)$$

(the series converges because $\|\alpha_0\| < 1$). For any \underline{k} and $j \geq 0$, $\underline{t}_\Sigma^{\underline{k}} \alpha_1^{-j} \in \mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ can be expanded in the basis \mathcal{B} (in a unique way) We deduce that any element $f = \frac{\beta}{\alpha} \in \mathbb{C}_\infty(\underline{t}_\Sigma)$ can be expanded, in a unique way, as a convergent series:

$$f = \sum_{i \in I} f_i b_i, \quad f_i \rightarrow 0, \quad f_i \in \mathbb{K}_{\Sigma'}, \quad f_i \rightarrow 0.$$

This means that there is an isometric embedding $\mathbb{C}_\infty(\underline{t}_\Sigma) \rightarrow c_{\mathbb{K}_{\Sigma'}}(I)$ where $c_{\mathbb{K}_{\Sigma'}}(I)$ denotes the Banach $\mathbb{K}_{\Sigma'}$ -space of the sequences $(f_i : i \in I) \in \mathbb{K}_{\Sigma'}^I$ with $f_i \rightarrow 0$ and with the max-norm. Completing, we are left with an isometric isomorphism $\mathbb{K}_\Sigma \cong c_{\mathbb{K}_{\Sigma'}}(I)$ which terminates the proof. \square

2.1.3. The non-complete fields \mathbb{L}_Σ . We consider $\delta_1, \dots, \delta_r \in \mathbb{F}_q^{ac}(\underline{t}_\Sigma)$. Then, we have the affinoid \mathbb{C}_∞ -algebra (completion for the Gauss norm)

$$\mathbb{T}_\Sigma[\delta_1, \dots, \delta_r]^\wedge = \mathbb{T}_\Sigma[\widehat{\delta_1, \dots, \delta_r}] = \mathbb{C}_\infty[\underline{t}_\Sigma][\widehat{\delta_1, \dots, \delta_r}]$$

which embeds in \mathbb{K}_Σ (with the Gauss norm). We consider

$$\mathbb{L}_\Sigma = \bigcup_{d \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}} \mathbb{T}_\Sigma[d^{-1}]^\wedge.$$

Lemma 2.2. \mathbb{L}_Σ is a subfield of \mathbb{K}_Σ .

Proof. We note the following elementary properties:

- (1) We can find $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ with $\mathbb{T}_\Sigma[\delta_1, \dots, \delta_r]^\wedge \subset \mathbb{T}_\Sigma[d^{-1}]^\wedge$.
- (2) If $d_1, d_2 \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}$ are relatively prime then $\mathbb{T}_\Sigma[\frac{1}{d_1}]^\wedge \cap \mathbb{T}_\Sigma[\frac{1}{d_2}]^\wedge = \mathbb{T}_\Sigma$.
- (3) If $d_1 \mid d_2$ then $\mathbb{T}_\Sigma[\frac{1}{d_2}]^\wedge \subset \mathbb{T}_\Sigma[\frac{1}{d_1}]^\wedge$.

We deduce that \mathbb{L}_Σ is a \mathbb{C}_∞ -algebra. All it remains to show is that every non-zero element f of \mathbb{L}_Σ is invertible; we follow the same ideas of Lemma 2.1; there is no loss of generality to suppose that $\|f\| = 1$. There exists $\delta \in \mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ such that $f \in \mathbb{T}_\Sigma[\delta]^\wedge$. We can write

$f = \alpha_1 - \alpha_0$ where $\alpha_1 \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma][\delta] \setminus \{0\}$ and where $\alpha_0 \in \mathbb{T}_\Sigma[\delta]$ is such that $\|\alpha_0\| < 1$. Therefore, in \mathbb{K}_Σ :

$$\frac{1}{\alpha} = \frac{1}{\alpha_1} \left(1 - \frac{\alpha_0}{\alpha_1}\right)^{-1} = \frac{1}{\alpha_1} \sum_{i \geq 0} \left(\frac{\alpha_0}{\alpha_1}\right)^i$$

and the series converges in $\mathbb{T}_\Sigma[\tilde{\delta}]^\wedge \subset \mathbb{L}_\Sigma$, for some element $\tilde{\delta} \in \mathbb{F}_q^{ac}(\underline{t}_\Sigma)$. \square

Note that \mathbb{L}_Σ is not complete, and contains the fraction field of \mathbb{T}_Σ . The fields \mathbb{L}_Σ and \mathbb{K}_Σ both have residual field $\mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ and \mathbb{K}_Σ is the completion of \mathbb{L}_Σ for the Gauss norm.

2.1.4. The Carlitz exponential. The automorphism $c \mapsto c^q$ of \mathbb{C}_∞ extends in a unique way to an $\mathbb{F}_q[\underline{t}_\Sigma]$ -linear automorphism τ of $\mathbb{C}_\infty[\underline{t}_\Sigma]$ and therefore, to each of the three \mathbb{C}_∞ -algebras $\mathbb{T}_\Sigma \subset \mathbb{L}_\Sigma \subset \mathbb{K}_\Sigma$, being continuous and open on the first and the third. For all $f \in \mathbb{K}_\Sigma$, we have that $\|\tau(f)\| = \|f\|^q$. The sub-ring $\mathbb{T}_\Sigma^{\tau=1}$ of the elements $f \in \mathbb{T}_\Sigma$ such that $\tau(f) = f$ is the polynomial ring $\mathbb{F}_q[\underline{t}_\Sigma]$, and we have the identities of fixed subfields $k_\Sigma = \mathbb{F}_q(\underline{t}_\Sigma) = \mathbb{L}_\Sigma^{\tau=1} = \mathbb{K}_\Sigma^{\tau=1}$. For all Σ , the \mathbb{C}_∞ -algebra \mathbb{E}_Σ of entire functions $\mathbb{C}_\infty^\Sigma \rightarrow \mathbb{C}_\infty$ further embeds in \mathbb{T}_Σ . We will write \mathbb{E} for the \mathbb{C}_∞ -algebra of entire functions of the variable t (case of Σ a singleton). We can also consider the \mathbb{K}_Σ -algebras $\mathbb{K}_\Sigma[\tau]$ and $\mathbb{K}_\Sigma[[\tau]]$ (the multiplication is defined by the commutation rule $\tau f = \tau(f)\tau$ for $f \in \mathbb{K}_\Sigma$).

For a more complete background on the Carlitz module and the Carlitz exponential, we refer to [21]. See also [34]. To define the Carlitz exponential, we introduce the analogue of the sequence of numbers $q^{n!}$, defined as follows:

$$d_n = \prod_a a,$$

where the product runs over the monic polynomials a of A of degree n . It can be proved (see [21, Proposition 3.1.6]) that

$$d_n = (\theta^{q^n} - \theta) \cdots (\theta^{q^n} - \theta^{q^{n-1}}), \quad n \geq 0.$$

Then, d_n is invertible for all n and the formal series

$$\exp_C := \sum_{n \geq 0} d_n^{-1} \tau^n \in K[[\tau]] \subset \mathbb{K}_\Sigma[[\tau]],$$

well defined, is the unique one such that the first term for $n = 0$ is $1 = \tau^0$ (normalized) and at once,

$$C_\theta \exp_C = \exp_C \theta,$$

where $C_\theta = \theta + \tau$. The inverse of \exp_C for the multiplicative structure of $K[[\tau]]$ is the *Carlitz logarithm operator*

$$\log_C := \sum_{n \geq 0} l_n^{-1} \tau^n \in K[[\tau]],$$

where l_n is equal to $(-1)^n$ times the least common multiple of all polynomials of A of degree n . It can be proved (see [21, Proposition 3.1.6]) that

$$l_n = (\theta - \theta^q) \cdots (\theta - \theta^{q^n}).$$

The formal series \exp_C induces a continuous, open $\mathbb{F}_q(\underline{t}_\Sigma)$ -linear endomorphism

$$\mathbb{K}_\Sigma \xrightarrow{\exp_C} \mathbb{K}_\Sigma,$$

by setting

$$\exp_C(f) = \sum_{i \geq 0} d_i^{-1} \tau^i(f), \quad f \in \mathbb{K}_\Sigma.$$

This induces an isometric \mathbb{F}_q -linear automorphism of the disk

$$D_{\mathbb{K}_\Sigma}^\circ(0, |\tilde{\pi}|) = \{f \in \mathbb{K}_\Sigma : \|f\| < q^{\frac{a}{q-1}}\},$$

of inverse \log_C . Let us first look at the restriction of \exp_C on $\mathbb{C}_\infty \subset \mathbb{K}_\Sigma$. Since it is an entire function $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$, its kernel Λ , a discrete \mathbb{F}_q -vector space, determines \exp_C uniquely, and we have the convergent Weierstrass product expansion

$$(9) \quad \exp_C(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right), \quad z \in \mathbb{C}_\infty.$$

The function \exp_C on \mathbb{C}_∞ , is the unique entire \mathbb{F}_q -linear map $F : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ which satisfies $\frac{dF}{dz} = 1$, and which induces an exact sequence of A -modules:

$$0 \rightarrow \Lambda \rightarrow \mathbb{C}_\infty \xrightarrow{\exp_C} C(\mathbb{C}_\infty) \rightarrow 0,$$

with $C(\mathbb{C}_\infty)$ the *Carlitz module* on \mathbb{C}_∞ ; the unique A -module which has as underlying \mathbb{F}_q -vector space the space \mathbb{C}_∞ , and with multiplication by θ given by $C_\theta = \theta + \tau$. The kernel of the Carlitz exponential function \exp_C is free of rank one over A , generated, as an A -module, by the element

$$(10) \quad \tilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1}.$$

It can be proved that $\tilde{\pi}$ is transcendental over K ; there are several ways that lead to this result, using the above product expansion. See [27] for an overview.

2.1.5. The Carlitz exponential over \mathbb{K}_Σ . We now look more generally at the map $\exp_C : \mathbb{K}_\Sigma \rightarrow \mathbb{K}_\Sigma$. We choose an \mathbb{F}_q^{ac} -basis $\mathcal{B} = (b_i)_{i \in I}$ of $\mathbb{F}_q^{ac}(\underline{t}_\Sigma)$. Let $\tilde{J} \subset I$ be a finite subset. Considering the orbit under the action of the group $\text{Gal}(\mathbb{F}_q(b_j : j \in \tilde{J})/\mathbb{F}_q(\underline{t}_\Sigma))$ we see that there exists J finite, with $\tilde{J} \subset J \subset I$, and a matrix $M_J \in \text{GL}_{|J|}(\mathbb{F}_q^{ac})$ such that, writing b_J for the column matrix $(b_i)_{i \in J}$,

$$\tau(b_J) = M_J b_J.$$

Lemma 2.3. *For any $\tilde{J} \subset I$ and $J \supset \tilde{J}$ as above, the exponential map \exp_C induces a surjective \mathbb{F}_q -linear endomorphism of $\bigoplus_{j \in J} \mathbb{C}_\infty b_j$.*

Proof. By Lang's result [23, Corollary p. 557] there exists $U_J \in \text{GL}_{|J|}(\mathbb{F}_q^{ac})$ such that $\tau(U_J) = M_J U_J$. Hence, $\tau^n(U_J) = M_J U_J \tau(U_J) \cdots \tau^{n-1}(U_J)$ for all n . If b_J is the row

matrix with entries b_j , $j \in J$ and if f_J is the column matrix of $\mathbb{C}_\infty^{|J| \times 1}$ such that $f = b_J f_J$, then

$$\exp_C(f) = b_J U_J^{-1} \exp_C U_J f_J.$$

Since the map $\exp_C : \mathbb{C}_\infty^{|J| \times 1} \rightarrow \mathbb{C}_\infty^{|J| \times 1}$ is surjective, the lemma follows. \square

We denote by $C(\mathbb{K}_\Sigma)$ the k_Σ -vector space \mathbb{K}_Σ with the $k_\Sigma[\theta]$ -module structure determined by the multiplication C_θ by θ given by $C_\theta = \theta + \tau$. This is the *Carlitz module* over \mathbb{K}_Σ . Note that $C(\mathbb{L}_\Sigma), C(A), \dots$ can be defined in analogous way.

We can prove:

Proposition 2.4. *The map \exp_C induces an exact sequence of $\mathbb{F}_q(\underline{t}_\Sigma)[\theta]$ -modules:*

$$(11) \quad 0 \rightarrow \tilde{\pi} \mathbb{F}_q(\underline{t}_\Sigma)[\theta] \rightarrow \mathbb{K}_\Sigma \xrightarrow{\exp_C} C(\mathbb{K}_\Sigma) \rightarrow 0.$$

Proof. We have seen that \exp_C is well defined, and k_Σ -linear. We show surjectivity. Let f be in \mathbb{K}_Σ . By Lemma 2.1 can decompose $f = f_0 + f_1$ where $f_1 \in \mathbb{C}_\infty(\underline{t}_\Sigma)$ and $\|f_0\| < |\tilde{\pi}|$. Since \exp_C is easily seen to induce an isometric isomorphism on $D_{\mathbb{K}_\Sigma}^\circ(0, |\tilde{\pi}|)$ (of inverse the Carlitz logarithm \log_C), there exists $g_0 \in \mathbb{K}_\Sigma$ such that $\exp_C(g_0) = f_0$. We can find $J \subset I$ finite as above, such that $f_1 = \sum_{j \in J} f_j b_j$ and applying Lemma 2.3, there exists $g_1 \in \mathbb{C}_\infty(\underline{t}_\Sigma)$ such that $\exp_C(g_1) = f_1$.

It remains to show that the kernel of \exp_C is $\tilde{\pi} k_\Sigma[\theta]$. It certainly contains it. Now it is easy to modify the standard arguments of [21, Proposition 2.9] to deduce, from the well known structure of the Newton polygon of $\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ that the set of zeroes of \exp_C in $D_{\mathbb{K}_\Sigma}(0, |\tilde{\pi}||\theta|^i)$ is an $\mathbb{F}_q(\underline{t}_\Sigma)$ -vector space V_i of dimension equal to $i + 1$ (see also [34]). This dimension also is the dimension of the space $\text{Vect}_{\mathbb{F}_q(\underline{t}_\Sigma)}(1, \theta, \dots, \theta^i)$ for all $i \geq 0$. Since $\tilde{\pi} \text{Vect}_{\mathbb{F}_q(\underline{t}_\Sigma)}(1, \theta, \dots, \theta^i)$ is contained in V_i , it equals it, and we are done. \square

Let δ be an element of k_Σ^\times . From the proof of Proposition 2.4 one deduces that \exp_C also induces an $\mathbb{F}_q[\underline{t}_\Sigma][\delta]$ -linear surjective endomorphism of $\mathbb{T}_\Sigma[\delta]^\wedge \subset \mathbb{K}_\Sigma$, and we deduce the next result (compare with [3]):

Corollary 2.5. *The map \exp_C induces an exact sequence of $A[\underline{t}_\Sigma][\delta]$ -modules:*

$$(12) \quad 0 \rightarrow \tilde{\pi} A[\underline{t}_\Sigma][\delta] \rightarrow \widehat{\mathbb{T}_\Sigma[\delta]} \xrightarrow{\exp_C} C(\widehat{\mathbb{T}_\Sigma[\delta]}) \rightarrow 0.$$

Hence, we also have an exact sequence of $\mathbb{F}_q(\underline{t}_\Sigma)[\theta]$ -modules:

$$(13) \quad 0 \rightarrow \tilde{\pi} \mathbb{F}_q(\underline{t}_\Sigma)[\theta] \rightarrow \mathbb{L}_\Sigma \xrightarrow{\exp_C} C(\mathbb{L}_\Sigma) \rightarrow 0.$$

2.1.6. *Omega matrices.* Let

$$A \xrightarrow{\chi} k_\Sigma^{n \times n}$$

be an injective \mathbb{F}_q -algebra morphism. We set

$$\vartheta := \chi(\theta) \in \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}.$$

Let $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ be such that $d\vartheta \in \mathbb{F}_q[\underline{t}_\Sigma]^{n \times n}$. Then, the image of χ lies in $\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d}]^{n \times n}$. We set

$$\omega_\chi := \sum_{i \geq 0} \exp_C \left(\frac{\tilde{\pi}}{\vartheta^{i+1}} \right) \vartheta^i = \exp_C \left(\tilde{\pi}(\theta I_n - \vartheta)^{-1} \right) \in \widehat{\mathbb{T}_\Sigma[d^{-1}]}^{n \times n} \subset \mathbb{K}_\Sigma^{n \times n},$$

where the map \exp_C is applied coefficient-wise on the entries of the matrix $\tilde{\pi}(\theta I_n - \vartheta)^{-1} \in \mathbb{K}_\Sigma^{n \times n}$. We have, for all $a \in A$, that

$$C_a(\omega_\chi) = \exp_C(\tilde{\pi}a(\theta I_n - \vartheta)^{-1}) = \exp_C(\tilde{\pi}(aI_n - \chi(a))(\theta I_n - \vartheta)^{-1}) + \chi(a)\omega_\chi = \chi(a)\omega_\chi,$$

because $aI_n - \chi(a) = (\theta I_n - \vartheta)H$ with $H \in A[\vartheta]^{n \times n}$. Recall in the next lemma that $d\vartheta \in \mathbb{F}_q[\underline{t}_\Sigma]^{n \times n}$.

Lemma 2.6. *We have $\omega_\chi \in \mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ and ω_χ is solution of the linear τ -difference system*

$$\tau(X) = (\vartheta - \theta I_n)X.$$

Moreover, every solution X in $\mathbb{K}_\Sigma^{n \times 1}$ of this difference system is of the form $X = \omega_\chi m$, with $m \in k_\Sigma^{n \times 1}$.

Proof. observe that

$$\omega_\chi = \exp_C(\tilde{\pi}(\theta I_n - \vartheta)^{-1}) = \exp_C(\tilde{\pi}\theta^{-1}(I_n - \vartheta\theta^{-1})^{-1}) = \exp_C(\tilde{\pi}\theta^{-1})I_n + R$$

where $\|R\| < q^{\frac{1}{q-1}}$, and that $\exp_C(\tilde{\pi}\theta^{-1}) = |\tilde{\pi}\theta^{-1}| = q^{\frac{1}{q-1}}$. This proves that $\omega_\chi \in \mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$. The fact that ω_χ is a matrix solution of the system indicated above is clear. Finally, if X is a column solution of the system above, we have that $\omega_\chi^{-1}X$ has entries in the constant subfield of \mathbb{K}_Σ which is $k_\Sigma = \mathbb{F}_q(\underline{t}_\Sigma)$, and this proves the last assertion. \square

In the next result, we make a choice of a $(q-1)$ -th root of $-\theta$.

Corollary 2.7. *We have the identity*

$$\omega_\chi = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(I_n - \vartheta\theta^{-q^i} \right)^{-1},$$

up to the choice of an appropriate root $(-\theta)^{\frac{1}{q-1}}$. Moreover, $\omega_\chi^{-1} \in \mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge) \cap (\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$.

Note that the factors of the infinite product commute to each other.

Proof of Corollary 2.7. The methods are standard, but we give the details. First of all note that

$$F := (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(I_n - \vartheta\theta^{-q^i} \right)^{-1} \in (\mathbb{T}_\Sigma[d^{-1}]^\wedge)^{n \times n}$$

is a matrix solution X of the difference system $\tau(X) = (\vartheta - \theta I_n)X$, Lemma 2.6 applies and there exists a matrix $V \in \mathrm{GL}_n(k_\Sigma)$ such that $F = V\omega_\chi$. Now we proceed to prove that $V = I_n$. We recall that $(-\theta)^{\frac{1}{q-1}} = \exp_C(\frac{\tilde{\pi}}{\vartheta})$ for a unique choice of $(-\theta)^{\frac{1}{q-1}}$. We have seen, in the proof of Lemma 2.6, that $\omega_\chi = \exp_C(\tilde{\pi}\theta^{-1})I_n + R$ where $R \in \mathbb{K}_\Sigma^{n \times n}$ and

$\|R\| < q^{\frac{1}{q-1}}$. We also have $F = \exp_C(\tilde{\pi}\theta^{-1})I_n + R'$, $R' \in \mathbb{K}_\Sigma$ such that $\|R'\| < q^{\frac{1}{q-1}}$. Hence $V = I_n$. Additionally, note that $\tilde{\pi}(\theta I_n - \vartheta)^{-1} \in (\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$ so that, by Corollary 2.5, ω_χ has entries in $\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$. Also, F in this case is an element of $\mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ but each factor $(I_n - \vartheta\theta^{-q^i})$ composing the product expansion of its inverse belongs to $K[\underline{t}_\Sigma][\frac{1}{d}]$ so that F^{-1} has the entries which are in $(\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$. \square

An example: the function of Anderson and Thakur. It is likely that this function appeared for the first time in the literature in the paper of Anderson and Thakur [1, Proof of Lemma 2.5.4 p. 177]. It is equal to ω_χ when $\chi = \chi_t$, the map

$$A \xrightarrow{\chi} \mathbb{F}_q[t]$$

defined by $a \in A \mapsto a(t)$ (therefore, $n = 1$), and we denote it by ω for simplicity. We have:

$$\omega(t) = \exp_C \left(\frac{\tilde{\pi}}{\theta - t} \right).$$

Corollary 2.7 implies that

$$(14) \quad \omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}} \right)^{-1} \in \mathbb{T}^\times,$$

for a fixed choice of the $(q-1)$ -th root. This element can be also viewed as a function of the variable $t \in \mathbb{C}_\infty$, because the infinite product converges for all

$$t \in \mathbb{C}_\infty \setminus \{\theta^{q^k}; k \geq 0\}$$

and defines a meromorphic function over the above set, with simple poles at θ^{q^k} , $k \geq 0$. The element ω is a $(\theta - t)$ -torsion point in the Carlitz $A[t]$ -module $C(\mathbb{T})$. In particular, ω is a generator of the free sub- $\mathbb{F}_q[t]$ -module of rank one of \mathbb{T} , kernel of the evaluation of the operator

$$C_{\theta-t} = \tau + \theta - t \in K[t][\tau],$$

so that ω is a solution of the linear homogeneous τ -difference equation of order 1 (see also [26, Proposition 3.3.6]):

$$(15) \quad \tau(\omega)(t) = (t - \theta)\omega(t).$$

All these properties easily follow from Corollary 2.7.

2.2. Analytic functions with values in Banach algebras. In this subsection, B denotes a \mathbb{C}_∞ -Banach algebra with norm $|\cdot|_B$. We identify \mathbb{C}_∞ with a subalgebra of B via $\mathbb{C}_\infty \cdot 1 \subset B$. The norm $|\cdot|_B$ of B extends the norm $|\cdot|$ of \mathbb{C}_∞ , we have $|x + y|_B \leq \sup\{|x|_B, |y|_B\}$, $|xy|_B \leq |x|_B|y|_B$ ($x, y \in B$) and $|cx|_B = |c||x|_B$ ($c \in \mathbb{C}_\infty, x \in B$), and we assume that the valuation group is equal to that of \mathbb{C}_∞ , that is, the set of norms $|f|_B$ for $f \in B$ is equal to the set of norms $|x|$, $x \in \mathbb{C}_\infty$.

Let X/\mathbb{C}_∞ be a rigid analytic space so that it is the datum of (X, T, \mathcal{O}_X) with a G -topology T and a structure sheaf \mathcal{O}_X of \mathbb{C}_∞ -algebras. In all the following, we denote by $\mathcal{O}_{X/B}$ the presheaf of B -algebras defined, for $\mathcal{U} = (U_i)_i$ an affinoid covering of X , by

$$\mathcal{O}_{X/B}(U_i) = \mathcal{O}_X(\widehat{U_i}) \otimes_{\mathbb{C}_\infty} B,$$

the completion being taken for the spectral norm on U_i . Tate's acyclicity theorem [14, Theorem 4.2.2] extends to this framework. If $X = \text{Spm}(\mathcal{A})$ (maximal spectrum) with \mathcal{A} an affinoid \mathbb{C}_∞ -algebra and if M is a finitely generated \mathcal{A} -module, then, we define the pre-sheaf \mathcal{M}_B on X by

$$\mathcal{M}_B(U) = M \otimes_{\mathcal{A}} \mathcal{O}_{X/B}(U).$$

One can see that the covering \mathcal{U} is *acyclic* for \mathcal{M}_B . In particular, $\mathcal{O}_{X/B}$ is a sheaf.

A *holomorphic function*, also called *analytic function*, from X to B is by definition a global section of $\mathcal{O}_{X/B}$, that is, a function $f : X \rightarrow B$ such that for every rational subset $Y \subset X$, the restriction $f|_Y$ is the uniform limit over Y of a sequence of elements of $\mathcal{O}_X(Y) \otimes_{\mathbb{C}_\infty} B$. As an alternative notation, we write $f \in \text{Hol}(X \rightarrow B)$.

We suppose that B has a countable orthonormal basis $\mathcal{B} = (b_i : i \in \mathcal{I})$: for all $i \in \mathcal{I}$ we have $|b_i|_B = 1$ and moreover, for all $g \in B$ there is a unique convergent series expansion

$$g = \sum_{i \in \mathcal{I}} g_i b_i, \quad g_i \in \mathbb{C}_\infty,$$

with $|g_i| \rightarrow 0$ (Fréchet filter). Examples of such algebras are \mathbb{K}_Σ and $\mathbb{T}_\Sigma[\delta]$ for $\delta \in k_\Sigma$. Then, an analytic function

$$X \xrightarrow{f} B$$

can be expanded, in a unique way, as $f = \sum_{i \in \mathcal{I}} f_i b_i$, where $f_i : X \rightarrow \mathbb{C}_\infty$ are analytic functions such that $f_i \rightarrow 0$ for the norm of the uniform convergence, on every rational subset of X .

Let C be a sub-algebra of B (not necessarily complete). We write

$$\text{Hol}_B(X \rightarrow C)$$

for the C -algebra of holomorphic (or analytic) functions from X to B such that the image is contained in C , and we omit the subscript if $B = C$. For instance, we can take $C = \mathbb{L}_\Sigma \subset \mathbb{K}_\Sigma = B$.

2.2.1. Entire functions. We look at B -valued analytic functions on disks. If X is the disk

$$D_{\mathbb{C}_\infty}(0, r) = \{x \in \mathbb{C}_\infty; |x| \leq r\}$$

with $r \in |\mathbb{C}_\infty^\times|$ and with the usual structure sheaf of converging series, then $\text{Hol}(X \rightarrow B)$ equals the ring of series $\sum_{i \geq 0} f_i z^i$ where $f_i \in B$ for all i and $|f_i|_B r^i \rightarrow 0$ as $i \rightarrow \infty$. We thus deduce that the B -algebra $\text{Hol}(\mathbb{C}_\infty \rightarrow B)$ is equal to the B -algebra of the functions $\mathbb{C}_\infty \rightarrow B$ which can be identified with the formal series $\sum_{i \geq 0} f_i z^i \in B[[z]]$ such that $|f_i|_B r^i \rightarrow 0$ for all $r \in |B|_B$. It is easy to see that a function $f : \mathbb{C}_\infty \rightarrow B$ belongs to $\text{Hol}(\mathbb{C}_\infty \rightarrow B)$ if, on every bounded subset U of \mathbb{C}_∞ , f can be obtained as a uniform limit of polynomial functions $f_i \in B[z]$, $f_i : U \rightarrow B$. These functions are called *B-entire* (or

simply *entire* if the reference to B is understood). The following property is easily checked. Let $(f_i)_{i \geq 0}$ be a sequence of B -entire functions. If for every such r , the sequence $(f_i)_{i \geq 0}$ converges uniformly over $D(0, r)$, then the limit function $\mathbb{C}_\infty \rightarrow B$ is an entire function.

The next result is a simple generalization of the analogue of Liouville's theorem which can be found in Schikhof's [41, Theorems 42.2 and 42.6].

Proposition 2.8 (*B*-analogue of Liouville's Theorem). *A bounded B-entire function is constant.*

Proof. See [35, Proposition 8]. The proof uses the fact that the valuation group is dense and that $|B|_B = |\mathbb{C}_\infty|$. \square

To illustrate an application used in this paper we have, with $e_C(z) = \exp_C(\tilde{\pi}z)$:

Corollary 2.9. *For any map $g : A \rightarrow B$, there exists at most one entire function $f : \mathbb{C}_\infty \rightarrow B$ with $|f(z)/e_C(z)|_B \rightarrow 0$ as $|z| \rightarrow \infty$ with $|z| \notin |A|$, such that $f(a) = g(a)$ for all $a \in A$.*

Proof. Let us suppose that f_1, f_2 are entire functions such that $f_1(a) = f_2(a) = g(a)$ for all a , with $|f_i(z)/e_C(z)|_B \rightarrow 0$ as in the statement, for all i . Then, $f = f_1 - f_2$ vanishes on all A . We set $h := f/e_C$; it is an entire function $\mathbb{C}_\infty \rightarrow B$. We consider $r \in |\mathbb{C}_\infty^\times| \setminus |K_\infty^\times|$ and $z \in \mathbb{C}_\infty$ such that $|z| = r$. We can apply Proposition 3.10 and the maximum principle which tells us that $\max_{|z|=r} |h(z)|_B$ tends to 0 as r goes to infinity along a subset of $|\mathbb{C}_\infty^\times| \setminus |K_\infty^\times|$. We conclude that $h = 0$ identically, applying Proposition 2.8. \square

2.2.2. A class of entire functions.

Lemma 2.10. *For any \mathbb{F}_q -algebra morphism $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ there exists $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ and an entire function $\tilde{\chi} : \mathbb{C}_\infty \rightarrow (\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$ such that $\tilde{\chi}(a) = \chi(a)$ for all $a \in A$.*

Proof. We set $\vartheta := \chi(\theta) \in \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ and we choose $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that $d\vartheta \in \mathbb{F}_q[\underline{t}_\Sigma]$. For $z \in \mathbb{C}_\infty$, we set ⁽²⁾:

$$(16) \quad \tilde{\chi}(z) := \exp_C(\tilde{\pi}z(\theta I_n - \vartheta)^{-1}) \omega_\chi^{-1},$$

where $\omega_\chi \in \mathrm{GL}_n(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ has been introduced in §2.1.6. Since

$$\exp_C(\tilde{\pi}z(\theta I_n - \vartheta)^{-1}) = \sum_{i \geq 0} d_i^{-1} (\tilde{\pi}z)^{q^i} (\theta^{q^i} I_n - \vartheta)^{-1}, \quad z \in \mathbb{C}_\infty$$

and $\|d_i^{-1}(\theta^{q^i} I_n - \vartheta)^{-1}\| = q^{(i-1)q^i}$ for all $i \geq 0$, the map $\tilde{\chi}$ defines an entire function $\mathbb{C}_\infty \rightarrow (\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)^{n \times n}$ (we recall from Corollary 2.7 that ω_χ^{-1} has entries in $\mathbb{E}_\Sigma[\frac{1}{d}]^\wedge$). Moreover, observe that if $a \in A$,

$$\begin{aligned} \tilde{\chi}(a) &= \exp_C(\tilde{\pi}a(\theta I_n - \vartheta)^{-1}) \omega_\chi^{-1} \\ &= C_a(\omega_\chi) \omega_\chi^{-1} \\ &= \chi(a). \end{aligned}$$

\square

²Note that the factors commute.

Note that Corollary 2.9 applies to the function $\tilde{\chi}$.

Lemma 2.11. *The matrix function $\tilde{\chi}$ is a solution X of the difference system*

$$\tau(X) = X + e_C(z)\omega_\chi^{-1}.$$

Proof. We set $F = \exp_C(\tilde{\pi}z(\theta I_n - \vartheta)^{-1})$. Then,

$$\begin{aligned} \tau(F) &= -\theta F + e_C(z(\theta I_n - \vartheta + \vartheta)(\theta I_n - \vartheta)^{-1}) \\ &= -\theta F + e_C(z(\theta I_n - \vartheta)^{-1}\vartheta) + e_C(z)I_n \\ &= F \cdot (\vartheta - \theta I_n) + e_C(z)I_n. \end{aligned}$$

□

From now on, we will denote both maps, $A \xrightarrow{\chi} \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ and $\mathbb{C}_\infty \xrightarrow{\tilde{\chi}} \mathbb{K}_\Sigma^{n \times n}$, with χ to simplify our notations.

2.2.3. *An example for the map χ_t .* We define, for $z \in \mathbb{C}_\infty$:

$$\chi_t(z) := \frac{e_C\left(\frac{z}{\theta-t}\right)}{\omega(t)} = \frac{\exp_C\left(\frac{\tilde{\pi}z}{\theta-t}\right)}{\omega(t)}, \quad z \in \mathbb{C}_\infty.$$

We deduce that χ_t defines an entire function $\mathbb{C}_\infty \rightarrow \mathbb{E}$ which satisfies $\chi_t(a) = a(t)$ for all $a \in A$, and the τ -difference equation

$$(17) \quad \tau(\chi_t(z)) = \chi_t(z) + \frac{e_C(z)}{\tau(\omega)}.$$

To mention an additional property of the entire function χ_t , it can be proved that the function $z \mapsto \frac{\chi_t(z)}{z} \in \mathbb{E}$ is non-constant, entire, with no zeroes.

3. FIELD OF UNIFORMISERS

We introduce and discuss the field of uniformisers which provides a natural environment to study the modular forms discussed in the present paper. Similar constructions have also been considered in [33]. We begin with §3.1, where we introduce some algebraic settings. We define *tame series* in §3.2 and we prove the main structural properties of them.

3.1. Some B -algebras. In this subsection, we consider a difference ring (B, τ) with $A \xrightarrow{\iota} B$ an integral, commutative A -algebra and $B \xrightarrow{\tau} B$ an endomorphism which acts as the \mathbb{F}_q -algebra endomorphism $c \mapsto c^q$ over $\iota(A)$. To simplify, we suppose that ι is an inclusion, and we identify θ with $\iota(\theta)$.

We consider, further, the polynomial B -algebra $\mathcal{R} = B[X_i; i \in \mathbb{Z}]$ in infinitely many variables X_i , and the ideal \mathcal{P} generated by the polynomials $X_i^q + \theta X_i - X_{i-1}$ for all $i \in \mathbb{Z}$. Then, with \underline{X} the collection $(X_i : i \in \mathbb{Z})$, the quotient B -algebra \mathcal{R}/\mathcal{P} can be identified

with the ring $B\langle\underline{X}\rangle$ whose elements F are formal finite sums in the indeterminates X_i , $i \in \mathbb{Z}$:

$$(18) \quad F = \sum_{j \in \mathbb{Z}[\frac{1}{p}]} F_j \langle\underline{X}\rangle^j = \sum_{j \in \mathbb{Z}[\frac{1}{p}]} F_j \prod_{k \in \mathbb{Z}} X_k^{j_k}, \quad F_j \in B,$$

where we have expanded the indices $j = \sum_{k \in \mathbb{Z}} j_k q^{-k}$ in base q (the coefficients j_i are almost all zero and belong to $\{0, \dots, q-1\}$). Note that a product over $B\langle\underline{X}\rangle$ is well defined in virtue of the rules $X_i^q = X_{i-1} - \theta X_i$. We have thus identified, after a mild abuse of notation, $B\langle\underline{X}\rangle$ with a complete system of representatives of \mathcal{R} modulo \mathcal{P} and we have defined over it, a product which makes it isomorphic to the quotient \mathcal{R}/\mathcal{P} .

Example. Note that with $C_\theta = \theta + \tau$ the multiplication by θ of the Carlitz A -module, we have $X_{i-1} = C_\theta(X_i)$ in $C(B\langle\underline{X}\rangle)$. Hence, if B is a \mathbb{C}_∞ -algebra, the substitution $X_i \mapsto e_C(\frac{z}{\theta^i})$ yields a natural B -algebra homomorphism

$$B\langle\underline{X}\rangle \rightarrow \text{Map}(K \rightarrow B).$$

We define a map

$$B\langle\underline{X}\rangle \xrightarrow{v} \mathbb{Z}[p^{-1}] \cup \{\infty\}$$

in the following way. We define $v(0) := \infty$ and we set $v(B \setminus \{0\}) = \{0\}$. Further, for a monomial $\langle\underline{X}\rangle^j = \prod_{i \in \mathbb{Z}} X_i^{j_i}$ (so only finitely many factors satisfy $j_i > 0$), we set $v(\langle\underline{X}\rangle^j) = -j$. Note that distinct monomials $\langle\underline{X}\rangle^j$ correspond to distinct values in $\mathbb{Z}[\frac{1}{p}]$ so that v is injective over $\{\langle\underline{X}\rangle^j : j \in \mathbb{Z}[\frac{1}{p}]\}$. If F is non-zero as in (18), then we set $v(F) = \inf\{v(\langle\underline{X}\rangle^j) : F_j \neq 0\}$; the infimum is a minimum. The map v is a valuation: with $j, k \in \mathbb{Z}[\frac{1}{p}]$ and by the definition of the ideal \mathcal{P} , $\langle\underline{X}\rangle^j \langle\underline{X}\rangle^k = \langle\underline{X}\rangle^{j+k} + F$ where $F \in B\langle\underline{X}\rangle$ satisfies $v(F) > v(\langle\underline{X}\rangle^{j+k})$, so that if $F, G \in B\langle\underline{X}\rangle$, $v(FG) = v(F) + v(G)$.

Now, $B\langle\underline{X}\rangle$ being a valued ring, it is integral and we deduce that \mathcal{P} is a prime ideal. The residual ring of $B\langle\underline{X}\rangle$ is B . Further, $\tau(X_i) = X_i^q \equiv X_{i-1} - \theta X_i \pmod{\mathcal{P}}$ induces an endomorphism of $B\langle\underline{X}\rangle$ and the subring $B\langle\underline{X}\rangle^{\tau=1}$ of the elements F such that $\tau(F) = F$ is equal to $B^{\tau=1}$. Note that even in the case of τ inducing an automorphism of B , it does not extend to an automorphism of $B\langle\underline{X}\rangle$.

We analyse now another difference B -algebra containing $B\langle\underline{X}\rangle$ (it will not be complete, but it is *inverse*, that is, τ induces an automorphism).

Definition 3.1. We define $B\langle\langle\underline{X}\rangle\rangle$ to be the B -module of formal series as in (18), without the condition of finiteness of the sums, and such that the following conditions hold:

- (1) There exists $L \geq 0$ (depending on F) such that if $F_j \neq 0$, then $\ell_q(j) \leq L$, with $\ell_q(j)$ denoting the sum of digits of j in base q (which means that the length of the base- q expansions of the exponents j involved is bounded).
- (2) If $F_j \neq 0$, then $j \geq M$ with a constant M depending on F (which means that only the variables X_i with i in a subset of \mathbb{Z} which has a lower bound in \mathbb{Z} occur).

The first condition also means that the number of factors of the monomials occurring in $F \in B\langle\langle\underline{X}\rangle\rangle$ is bounded.

3.1.1. *Product in $B\langle\langle\underline{X}\rangle\rangle$.* Let $F \in B\langle\underline{X}\rangle \setminus \{0\}$ be as in (18). We denote by $\mu(F)$ the largest $m \in \mathbb{Z}$ such that the variable X_m occurs in at least one non-zero monomial of F (remember that the elements of $B\langle\underline{X}\rangle$ are polynomials so that $\mu(F)$ is well defined). Similarly, we denote by $\nu(F)$ the smallest $n \in \mathbb{Z}$ such that the variable X_n occurs in at least one non-zero monomial of F . Clearly, the function μ dominates the function ν over $B\langle\langle\underline{X}\rangle\rangle$ (in the natural ordering of \mathbb{Z}).

Lemma 3.2. *For two monomials $\langle\underline{X}\rangle^i$ and $\langle\underline{X}\rangle^j$ in $B\langle\langle\underline{X}\rangle\rangle$, writing*

$$\langle\underline{X}\rangle^i \langle\underline{X}\rangle^j = \sum_{k \in \mathbb{Z}[\frac{1}{p}]} c_k \langle\underline{X}\rangle^k, \quad c_k \in K,$$

then, if $c_k \neq 0$:

$$(19) \quad \mu(\langle\underline{X}\rangle^k) = \max\{\mu(\langle\underline{X}\rangle^i), \mu(\langle\underline{X}\rangle^j)\},$$

$$(20) \quad \nu(\langle\underline{X}\rangle^k) \in \{\min\{\nu(\langle\underline{X}\rangle^i), \nu(\langle\underline{X}\rangle^j)\}, \min\{\nu(\langle\underline{X}\rangle^i), \nu(\langle\underline{X}\rangle^j)\} - 1\},.$$

The relation (20) says that there are only two possible values for $\nu(\langle\underline{X}\rangle^k)$.

Proof of Lemma 3.2. By using inductively that $X_i^q = X_{i-1} - \theta X_i$, we see that:

$$(21) \quad \langle\underline{X}\rangle^{(q-1)(\frac{1}{q} + \dots + \frac{1}{q^n})} \langle\underline{X}\rangle^{\frac{1}{q^n}} = \langle\underline{X}\rangle^1 - \theta \sum_{i=0}^{n-1} \langle\underline{X}\rangle^{(q-1)(\frac{1}{q} + \dots + \frac{1}{q^i})} \langle\underline{X}\rangle^{\frac{1}{q^{i+1}}}, \quad \forall n \geq 1.$$

From this, (19) and (20) follow easily. It also appears that the minus one in (20) is necessary. \square

Proposition 3.3. *The B -module $B\langle\langle\underline{X}\rangle\rangle$ is endowed with the structure of a difference B -algebra with endomorphism τ , extending that of the difference algebra $(B\langle\underline{X}\rangle, \tau)$. This difference algebra $(B\langle\langle\underline{X}\rangle\rangle, \tau)$ carries a unique extension of the valuation v for which the residual ring is B , and for all $F \in B\langle\langle\underline{X}\rangle\rangle$, $v(\tau(F)) = qv(F)$. Furthermore, if (B, τ) is inversive, then $(B\langle\langle\underline{X}\rangle\rangle, \tau)$ is inversive.*

By this proposition, the fraction field of the integral algebra $K^{\text{per}}\langle\langle\underline{X}\rangle\rangle$, where K^{per} denotes the perfect closure of K , is a perfect field.

Proof of Proposition 3.3. It is clear that there is an inclusion of B -modules $B\langle\underline{X}\rangle \subset B\langle\langle\underline{X}\rangle\rangle$. Let us consider $k \in \mathbb{Z}[\frac{1}{p}]$, and let us write

$$k = k_r q^r + \dots + k_{r+s} q^{r+s}, \quad k_i \in \{0, \dots, q-1\}, \quad k_r k_{r+s} \neq 0.$$

Let L be a positive integer and let us consider the set (depending on k):

$$\mathcal{S} = \{(i, j) \in \mathbb{Z}[p^{-1}]^2 : \ell_q(i), \ell_q(j) \leq L \text{ and } \langle\underline{X}\rangle^k \text{ occurs in a summand of } \langle\underline{X}\rangle^i \langle\underline{X}\rangle^j\}.$$

Let (i, j) be in \mathcal{S} . By Lemma 3.2, we have

$$r \in \{\min\{\nu(\langle\underline{X}\rangle^i), \nu(\langle\underline{X}\rangle^j)\}, \min\{\nu(\langle\underline{X}\rangle^i), \nu(\langle\underline{X}\rangle^j)\} - 1\},$$

and

$$r + s = \max\{\mu(\langle\underline{X}\rangle^i), \mu(\langle\underline{X}\rangle^j)\}.$$

This implies that \mathcal{S} is a finite set and therefore the product of $B\langle\langle\underline{X}\rangle\rangle$ is well defined.

The property involving the endomorphism τ can be checked similarly, and also the fact that the valuation v extends to $B\langle\langle\underline{X}\rangle\rangle$ is easily verified. We now prove the last part of the proposition, in the case of (B, τ) inversive. Observe that

$$(22) \quad Y_j := \sum_{i \geq j} \theta^{\frac{i}{q}} X_{i+1} \in B\langle\langle\underline{X}\rangle\rangle, \forall j \in \mathbb{Z},$$

with $\theta^{\frac{1}{q}}$ the q -th root of θ in B (which is inversive by hypothesis, and therefore contains $\{x^{\frac{1}{p}} : x \in \iota(A)\}$). By $X_{i+1}^q = X_i - \theta X_{i+1}$ for $i \geq 0$, we deduce that $\tau(Y_j) = X_j$ for all j . Now consider $f = \sum_{j \in \mathbb{Z}[\frac{1}{p}]} F_j \langle\underline{X}\rangle^j \in B\langle\langle\underline{X}\rangle\rangle$, with $\ell_q(j) \leq L$ for all j such that $F_j \neq 0$. Let j be such that $F_j \neq 0$ and write $\langle\underline{X}\rangle^j = \prod_{k \in \mathbb{Z}} Y_k^{j_k}$. We set

$$\mathcal{Y}^{(j)} := \prod_{k \in \mathbb{Z}} Y_k^{j_k} \in B\langle\langle\underline{X}\rangle\rangle.$$

Note that $\nu(\mathcal{Y}^{(j)}) \geq \nu(\langle\underline{X}\rangle^j) + 1$. Moreover, writing $\mathcal{Y}^{(j)} = \sum_{k \in \mathbb{Z}[\frac{1}{p}]} \mathcal{Y}_k^{(j)} \langle\underline{X}\rangle^k$, if $\mathcal{Y}_k^{(j)} \neq 0$, then $\ell_q(k) \leq M$ for some M . Let k be an element of $\mathbb{Z}[\frac{1}{p}]$. In the same way as before, one shows that there are only finitely many elements $j \in \mathbb{Z}[\frac{1}{p}]$ such that $\mathcal{Y}_k^{(j)} \neq 0$ in the above expansion. Hence, the formal series

$$g = \sum_{j \in \mathbb{Z}[\frac{1}{p}]} \tau^{-1}(F_j) \mathcal{Y}^{(j)}$$

defines an element of $B\langle\langle\underline{X}\rangle\rangle$ and $\tau(g) = f$. \square

3.2. Tame series. We shall fix, throughout this subsection, a τ -difference sub- A -algebra B of \mathbb{K}_Σ , for some Σ . We denote by $B\langle\langle\underline{X}\rangle\rangle^b$ the sub- B -algebra of $B\langle\langle\underline{X}\rangle\rangle$ formed by the series as in (18), satisfying $\sup_j \|F_j\| < \infty$ ($(\cdot)^b$ stands for 'bounded'). We leave to the reader the proof of the following:

Lemma 3.4. *$B\langle\langle\underline{X}\rangle\rangle^b$ is a difference sub- B -algebra of $B\langle\langle\underline{X}\rangle\rangle$ containing $B\langle\underline{X}\rangle$.*

We consider the map $B\langle\underline{X}\rangle \xrightarrow{J} \text{Hol}(\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma)$ defined by $J(X_i) = e_i$, where

$$e_i := e_C \left(\frac{z}{\theta^i} \right)$$

for all $i \in \mathbb{Z}$. It is easy to see that J is a B -algebra morphism. By the fact that \exp_C defines, locally at zero, an \mathbb{F}_q -linear isometry of \mathbb{C}_∞ , it is easy to see that J defines an algebra map from $B\langle\langle\underline{X}\rangle\rangle^b$ to the maps from \mathbb{C}_∞ to \mathbb{K}_Σ ; this follows from the fact that, for all $z \in \mathbb{C}_\infty$, $|e_i(z)| = |\frac{\pi z}{\theta^i}|$ for all but finitely many i (depending on z).

We denote by $\mathcal{T}(B)$ the image of J in the \mathbb{K}_Σ -valued maps. We call it the B -algebra of *tame series*. Explicitly:

Definition 3.5. A *tame series* with coefficients in B is a map $\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma$ defined by an everywhere converging series f of the type

$$(23) \quad f(z) = \sum_{l=0}^{\infty} \sum_{\substack{\underline{i}=(i_1, \dots, i_l) \in \mathbb{Z}^l \\ i_1 < \dots < i_l \\ \underline{j} \in \{1, \dots, q-1\}^l}} f_{\underline{i}, \underline{j}} e_{i_1}^{j_1} \cdots e_{i_l}^{j_l}, \quad f_{\underline{i}, \underline{j}} \in B,$$

satisfying the following properties.

- (1) There exists an integer $L \geq 0$ such that if $f_{\underline{i}, \underline{j}} \neq 0$, then $|\underline{j}| = \sum_k j_k \leq L$.
- (2) There exists $M > 0$ such that, for all $\underline{i}, \underline{j}$, $f_{\underline{i}, \underline{j}} \in B$ satisfies $\|f_{\underline{i}, \underline{j}}\| \leq M$.
- (3) There exists $N \in \mathbb{Z}$ such that if $\underline{i} = (i_1, \dots, i_l)$ is such that $f_{\underline{i}, \underline{j}} \neq 0$, then $i_1 \geq N$.

Proposition 3.6. *The map J extends to a τ -difference B -algebra morphism*

$$B\langle\langle \underline{X} \rangle\rangle^b \xrightarrow{J} \text{Hol}(\mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma).$$

Proof. Let us consider a series f such as in (23). Observe that for all $\underline{i}, \underline{j}$, the function $z \mapsto f_{\underline{i}, \underline{j}} e_{i_1}^{j_1} \cdots e_{i_l}^{j_l}$ is \mathbb{K}_Σ -entire. It suffices to show that, for all $R \in |\mathbb{C}_\infty|$, the series defining f converges uniformly over $D(0, R)$.

Note that $f(z)$ is a tame series if and only if $f(\theta^{-1}z)$ is a tame series. Hence, we are reduced to prove the above property in the case $R = 1$. Now, observe that the set $\mathcal{S} = \{(\underline{i}, \underline{j}) : f_{\underline{i}, \underline{j}} \neq 0 \text{ and } \exists r \in \{1, \dots, l\} \text{ such that } i_r \leq 0\}$ is finite (in the definition of \mathcal{S} , $\underline{i} = (i_1, \dots, i_l)$ and $\underline{j} = (j_1, \dots, j_l)$ and l varies between 0 and L). Hence, we can decompose

$$f = \sum_{(\underline{i}, \underline{j}) \in \mathcal{S}} f_{\underline{i}, \underline{j}} e_{i_1}^{j_1} \cdots e_{i_l}^{j_l} + \sum_{l=0}^{\infty} \sum_{\substack{\underline{i}=(i_1, \dots, i_l) \in (\mathbb{N}^*)^l \\ i_1 < \dots < i_l \\ \underline{j} \in \{1, \dots, q-1\}^l}} f_{\underline{i}, \underline{j}} e_{i_1}^{j_1} \cdots e_{i_l}^{j_l}.$$

The first sum is finite and therefore defines an entire function. Note now that if $\underline{i} = (i_1, \dots, i_l) \in (\mathbb{N}^*)^l$, then, for $|z| \leq 1$,

$$|e_{i_1}(z)^{j_1} \cdots e_{i_l}(z)^{j_l}| \leq |\tilde{\pi}|^{j_1 + \dots + j_l} |\theta|^{-(i_1 j_1 + \dots + i_l j_l)}.$$

Hence

$$\|f_{\underline{i}, \underline{j}} e_{i_1}(z)^{j_1} \cdots e_{i_l}(z)^{j_l}\| \leq M |\tilde{\pi}|^{L(q-1)} |\theta|^{-(i_1 j_1 + \dots + i_l j_l)} \rightarrow 0$$

the limit considered being along the Fréchet filter over the set of couples $(\underline{i}, \underline{j})$. This means that in the above decomposition, the second series defines a B -entire function and the series defining f converges to a \mathbb{K}_Σ -entire function. \square

3.2.1. Examples of tame series. We give examples of such functions in the setting of $B = \mathbb{F}_q(\underline{t}_\Sigma)[\theta] \subset \mathbb{K}_\Sigma$, which carries the Gauss norm.

Lemma 3.7. *Let a be in $\mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1}))^\times \subset \mathbb{K}_\Sigma^\times$. Then, the entire function $z \mapsto e_C(az) : \mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma$ defines an element of $\mathcal{T}(\mathbb{F}_q(\underline{t}_\Sigma)[\theta])$.*

Proof. We write:

$$a = \sum_{i \geq i_0} \frac{a_i}{\theta^i}, \quad a_i \in \mathbb{F}_q(\underline{t}_\Sigma),$$

and we immediately get that

$$e_C(az) = \sum_{i \geq i_0} a_i e_i.$$

□

In particular, $e_C(\frac{z}{a}) \in \mathcal{T}(A)$ for all $a \in A \setminus \{0\}$ and $e_C(\frac{z}{\theta - t_i}) \in \mathcal{T}(\mathbb{F}_q[\underline{t}_\Sigma][\theta])$, for all $i \in \Sigma$. Let $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)$ be an injective \mathbb{F}_q -algebra morphism. Then, there exists $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that the image is in $\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d}]$. We consider its entire extension $\chi : \mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma^{n \times n}$, defined in §2.2.2. Note that, for all $z \in \mathbb{C}_\infty$:

$$\chi(z) = \omega_\chi^{-1} e_C(z(\theta I_n - \vartheta)^{-1}) = \omega_\chi^{-1} \sum_{i \geq 0} e_{i+1} \vartheta^i,$$

where $\vartheta = \chi(\theta)$, with the convergence for the Gauss norm of \mathbb{K}_Σ . The last sum defines a matrix of tame series of $\mathcal{T}(\mathbb{F}_q(\underline{t}_\Sigma)[\theta])^{n \times n}$. Hence, $\chi \in \mathcal{T}(\mathbb{K}_\Sigma)^{n \times n}$, and in fact:

Lemma 3.8. *The entire matrix functions χ define elements of $\mathcal{T}(\widehat{\mathbb{E}_\Sigma[\frac{1}{d}]})^{n \times n}$. The entries of χ , when non-zero, have depth one.*

Another important example is given by an apparently different class of functions related to the previous ones thanks to work of Perkins (see [39]). For simplicity, we confine our description to some very particular examples, but see §4 for more general *quasi-periodic functions*.

Define the *Perkins series* (of order 1 associated to the semicharacter χ_t):

$$\psi(1; \chi_t) = \sum_{a \in A} \frac{\chi_t(a)}{z - a}.$$

Note that for all $z \in \mathbb{C}_\infty \setminus (A \setminus \{0\})$, this series converges. Then, the function $z \mapsto e_C(z)\psi(1; \chi_t)$ is \mathbb{E} -entire and in fact belongs to $\mathcal{T}(\mathbb{E})$. Indeed Perkins (see [39, Theorem 1.1]) proved the identity of \mathbb{E} -entire functions

$$(24) \quad e_A(z)\psi(1; \chi_t) = \chi_t(z),$$

where $e_A(z) = z \prod'_{a \in A} (1 - \frac{z}{a})$.

3.2.2. Asymptotic behavior of tame series. We suppose that $l > 0$ and we set:

$$(25) \quad M = M_{\underline{i}, \underline{j}} = e_{i_1}^{j_1} \cdots e_{i_l}^{j_l} = J(X_{i_1}^{j_1} \cdots X_{i_l}^{j_l}) = J(\langle \underline{X} \rangle^{\underline{j}}) \in \mathcal{T}(A),$$

so that $j_k \in \{1, \dots, q-1\}$ for all $k = 1, \dots, l$ and $i_1 < \dots < i_l$, for some $l > 0$. We call this a *monic tame monomial*. Its *depth* is the integer $d(M) = |\underline{j}| = \sum_k j_k$. We define its *weight* to be the rational number

$$w(M) := \frac{j_1}{q^{i_1}} + \cdots + \frac{j_l}{q^{i_l}} \in \mathbb{Z}[p^{-1}] = j = -v(\langle \underline{X} \rangle^{\underline{j}}) \cap \mathbb{R}_{\geq 0}.$$

In particular, the weight of $e_0 = e_C(z)$ is one and the weight of 1 or of a non-zero constant is 0. The fact that the exponents j_k are all $\leq q - 1$ ensures that distinct tame monomials have distinct weights. The condition of finite depth ensures that the supremum of the weights of the monomials composing a tame series is a maximum. In the following, we call the unique tame monomial of maximal weight in a non-zero tame series f , the *leading tame monomial*. The weight $w(f)$ of f is by definition equal to the weight of the leading tame monomial. The weight $-\infty$ is assigned to the zero tame series. We now discuss the question on whether, assigning to a non-zero tame series f the weight $w(f)$, we have defined a degree map

$$\mathcal{T}(B) \xrightarrow{w} \mathbb{Z}[p^{-1}] \cup \{-\infty\},$$

that is, the opposite of a valuation. Of course, this is related to the uniqueness of the tame expansion of a function such as in (23), entire after Lemma 3.6; we are going to focus on these questions now.

Lemma 3.9. *Let M be a monic tame monomial of weight w as in (25), of depth $l \leq L$. Let $z \in \mathbb{C}_\infty$ such that $|z| \notin q^{\mathbb{Z}}$. If $|z| > q^i$, we have $|M(z)| = |e_C(z)|^w$.*

Proof. We recall that $|\cdot|$ is normalised so that $|\mathbb{C}_\infty| = q^{\mathbb{Q}}$. Let $z \in \mathbb{C}_\infty$ be such that $q^{n-1} < |z| < q^n$, for $n \in \mathbb{Z}$. Then, there exists a rational number $\epsilon \in]0, 1[$ such that $|z| = q^{n-1+\epsilon}$. Let us suppose that $n \geq 1$. From the Weierstrass product expansion of the function $e_A(z) = \tilde{\pi}^{-1} \exp_C(\tilde{\pi}z)$:

$$(26) \quad e_A(z) = z \prod_{a \in A \setminus \{0\}} \left(1 - \frac{z}{a}\right),$$

we see that

$$|e_A(z)| = |z| \prod_{a \neq 0} \left|1 - \frac{z}{a}\right| = |z| \prod_{0 < |a| < |z|} \left|\frac{z}{a}\right| = |z|^{q^n} \prod_{0 < |a| \leq q^{n-1}} |a|^{-1}.$$

One computes easily $\prod_{0 < |a| \leq q^{n-1}} |a|^{-1} = q^{q \frac{q^n-1}{q-1} - nq^n}$ so that $|e_A(z)| = q^{q^n \left(\epsilon - 1 + \frac{q}{q-1}\right) - \frac{q}{q-1}}$ and, by $e_C(z) = \exp_C(\tilde{\pi}z)$, we deduce that

$$\log_q |e_C(z)| = q^n \left(\frac{1}{q-1} - \epsilon\right)$$

(where \log_q denotes the logarithm in base q of a positive real number). Let i be a non-negative integer. If $n > i$ (note that $|\frac{z}{\theta^i}| \notin q^{\mathbb{Z}}$)

$$|e_i(z)| = \left|e_C\left(\frac{z}{\theta^i}\right)\right| = q^{q^{n-i} \left(\frac{1}{q-1} + \epsilon\right)} = |e_C(z)|^{\frac{1}{q^i}}.$$

This suffices to complete the proof of the Lemma. \square

Proposition 3.10. *Let us consider a non-zero tame series f as in (23) and let M_{i_0, j_0} be its leading tame monomial. Then, for all $z \in \mathbb{C}_\infty$ such that $|z| \notin q^{\mathbb{Z}}$ and with $|z|$ large enough depending on f , $\|f(z)\| = \|f_{i_0, j_0}\| |e_C(z)|^{w(f)}$.*

Proof. Let $z \in \mathbb{C}_\infty$ be such that $q^{n-1} < |z| < q^n$, for $n \in \mathbb{Z}$. Then, as observed above, there exists a rational number $\epsilon \in]0, 1[$ such that $|z| = q^{n-1+\epsilon}$. Let us suppose that $n \leq i$. Then, $|z| < q^i$ and $|z/\theta^i| = q^{n-i-1+\epsilon} < 1$. In this case the product expansion (26) tells us that $|e_C(\frac{z}{\theta^i})| = |\tilde{\pi} \frac{z}{\theta^i}|$. We now consider a tame monomial of weight w , as in (25), and z as above. Then, we can write (note that what follows is a well defined tame monomial):

$$M = \underbrace{e_{i_1}^{j_1} \cdots e_{i_k}^{j_k}}_{[M]_{>n}} e_{i_{k+1}}^{j_{k_1}} \cdots e_{i_l}^{j_l}, \quad \underbrace{\hspace{10em}}_{[M]_{\leq n}}$$

with $i_1 < \cdots < i_l$, where

$$[M]_{\leq n} := \prod_{m \text{ such that } n \leq i_m} e_{i_m}^{j_m},$$

and $[M]_{>n}$ is defined analogously. Both $[M]_{\leq n}$ and $[M]_{>n}$ are tame monomials, and $w' := w([M]_{>n}) = \sum_{m \leq k} j_m q^{-i_m} \leq w(M)$. There is equality only if $M = [M]_{>n}$. We have $||[M]_{>n}(z)|| = |e_C(z)|^{w'}$ (Lemma 3.9) and

$$|[M]_{\leq n}(z)| = \frac{(\tilde{\pi}z)^{j_{k+1}+\cdots+j_l}}{|\theta|^{i_{k+1}j_{k+1}+\cdots+i_lj_l}} \leq \frac{|\tilde{\pi}|^{L(q-1)}}{|\theta|^{i_{k+1}j_{k+1}+\cdots+i_lj_l}} |z|^{L(q-1)}.$$

To ease our notation, we set $\delta_M(z) := i_{k+1}j_{k+1} + \cdots + i_lj_l$. Then, we see that

$$|M(z)| \leq |e_C(z)|^w |\theta|^{-\delta_M(z)} |\tilde{\pi}|^{L(q-1)} |z|^{L(q-1)}.$$

Let us choose $\tilde{w} \in \mathbb{Z}[\frac{1}{p}] \cap \mathbb{R}_{>0}$. Then, for $|z| \geq R_0$ with $R_0 \in |\mathbb{C}_\infty|$ large enough, depending only on \tilde{w} and L , we have that $|\tilde{\pi}|^{L(q-1)} |z|^{L(q-1)} \leq |e_C(z)|^{\tilde{w}}$, so that

$$|M(z)| \leq |e_C(z)|^{w+\tilde{w}} |\theta|^{-\delta_M(z)}.$$

Now, let us consider a non-zero tame series f that we can write in the following way

$$f = f_{\underline{i}_0, \underline{j}_0} M_{\underline{i}_0, \underline{j}_0} + \sum_{(\underline{i}, \underline{j}) \neq (\underline{i}_0, \underline{j}_0)} f_{\underline{i}, \underline{j}} M_{\underline{i}, \underline{j}}$$

with $f_{\underline{i}_0, \underline{j}_0} \neq 0$ and $w(M_{\underline{i}_0, \underline{j}_0}) = w(f)$ (we have seen that $(\underline{i}_0, \underline{j}_0)$ is uniquely determined with this property). From now on, we always suppose that $z \notin q^{\mathbb{Z}}$. There exists $\tilde{w} \in \mathbb{Z}[\frac{1}{p}] \cap \mathbb{R}_{>0}$ such that if $(\underline{i}, \underline{j}) \neq (\underline{i}_0, \underline{j}_0)$ is such that $f_{\underline{i}, \underline{j}} \neq 0$, then $w(M_{\underline{i}, \underline{j}}) + \tilde{w} < w(f)$. Hence, for all $(\underline{i}, \underline{j}) \neq (\underline{i}_0, \underline{j}_0)$ such that $f_{\underline{i}, \underline{j}} \neq 0$,

$$\|f_{\underline{i}, \underline{j}} M_{\underline{i}, \underline{j}}\| \leq C |e_C(z)|^{w'} |\theta|^{-\delta_{M_{\underline{i}, \underline{j}}}(z)}, \quad |z| \geq R_0,$$

where C is an upper bound for all the absolute values $\|f_{\underline{i}, \underline{j}}\|$. Since $\delta_M(z) \rightarrow \infty$ for M running in the set of monic tame monomials (25) of bounded depth such that $i_1 \geq \alpha$ for some fixed α ,

$$\left\| \sum_{(\underline{i}, \underline{j}) \neq (\underline{i}_0, \underline{j}_0)} f_{\underline{i}, \underline{j}} M_{\underline{i}, \underline{j}} \right\| \leq C |e_C(z)|^{w'}, \quad |z| \geq R_0,$$

for $w' \in \mathbb{Z}[\frac{1}{p}]$, $0 \leq w' < w(f)$ and R_0 depending on \tilde{w} . Hence,

$$\left\| f(z) - f_{\underline{i}_0, \underline{j}_0} M_{\underline{i}_0, \underline{j}_0} \right\| \leq C |e_C(z)|^{w'}$$

and if $|z| \geq R_1$ depending on C and w' , we get

$$\|f(z)\| = \|f_{\underline{i}_0, \underline{j}_0}\| \|M_{\underline{i}_0, \underline{j}_0}(z)\| = \|f_{\underline{i}_0, \underline{j}_0}\| |e_C(z)|^{w(f)}.$$

□

Remark 3.11. We define, for $z \in \mathbb{C}_\infty$, $|z|_{\mathfrak{S}} = \inf\{|z-l| : l \in K_\infty\} = \min\{|z-l| : l \in K_\infty\}$. The statement of Proposition 3.10 holds under the weaker condition that $|z|_{\mathfrak{S}}$ is large enough. We leave the details to the reader.

We have the following important consequence of Proposition 3.10.

Corollary 3.12. *If $f \in \mathcal{T}(B)$, then its tame series expansion is unique.*

Proof. It suffices to show that a tame series as in (23) cannot vanish identically, if not trivially. But otherwise, such a series would then have a unique leading tame monomial, which would contradict the property of Proposition 3.10. □

Thanks to the above Corollary, J is injective, the map $w \circ J$ is the opposite of the valuation v and the depth $d(f)$ of a tame series f defined as the depth of $g \in B\langle\langle \underline{X} \rangle\rangle^b$ such that $J(g) = f$ becomes a well defined invariant of the entire function it represents. Let f be in $\mathcal{T}(B)$ as in (23). We denote by $\mathcal{S}(f)$ the subset of \mathbb{Z} whose elements are the i_k 's which occur in $(\underline{i}, \underline{j})$ such that $f_{\underline{i}, \underline{j}} \neq 0$. We set $\nu(f) := \inf(\mathcal{S}(f))$ and $\mu(f) := \sup(\mathcal{S}(f))$. Then, $\mu(f) \geq \nu(f)$ and $\nu(f) \in \mathbb{Z}$ if and only if $f \neq 0$ and $\mu(f) \in \mathbb{Z}$ if and only if f is a polynomial in e_i , $i \in \mathbb{Z}$. Note also that for any $\nu_0 \leq \mu_0$ integers, the set of monic tame monomials as in (25) such that

$$\nu_0 \leq \nu(M) \leq \mu(M) \leq \mu_0$$

is finite.

We denote by $\mathcal{T}_s(B)$ the B -submodule of $\mathcal{T}(B)$ whose elements are the tame series f as in (23) such that if $f_{\underline{i}, \underline{j}} \neq 0$, then $M_{\underline{i}, \underline{j}}$ has depth equal to s (homogeneous tame series of depth s). It is easy to see that

$$(27) \quad \mathcal{T}(B) = \bigoplus_{s \geq 0} \mathcal{T}_s(B).$$

If $f \in \mathcal{T}(B)$, we can expand in finite sum and in a unique way

$$(28) \quad f = \sum_{s \geq 0} f^{[s]},$$

where $f^{[s]} \in \mathcal{T}_s(B)$. Moreover, we have the next Lemma, the proof of which is left to the reader.

Lemma 3.13. *For any $s \geq 0$, τ induces an endomorphism of the B -module $\mathcal{T}_s(B)$.*

Unfortunately, the B -algebra $\mathcal{T}(B)$ is not graded by the depths. Instead, we have that

$$\mathcal{T}_s(B)\mathcal{T}_{s'}(B) \subset \bigoplus_{j \geq 0} \mathcal{T}_{s+s'-j(q-1)}(B),$$

where we set $\mathcal{T}_s(B) = \{0\}$ if $s < 0$.

3.2.3. Some endomorphisms of $\mathcal{T}(B)$. We consider the B -linear endomorphism Φ of $B[e_i : i \in \mathbb{Z}]$ determined by

$$e_i \mapsto \sum_{j=0}^d c_j e_{i-j},$$

for fixed $c_0, \dots, c_d \in B$ not all zero.

Lemma 3.14. *The endomorphism Φ extends to an endomorphism of $\mathcal{T}(B)$.*

Proof. Let M be a monic tame monomial. It is easy to see that if $d(M) \leq c_0$, then $d(\Phi(M)) \leq c_1$, where c_1 is a constant depending on c_0 and Φ . Further, $\mu(\Phi(M)) \leq \mu(M)$ and $\nu(\Phi(M)) \geq \nu(M) - c_3$. Hence, for any monic tame monomial N , the set of monic tame monomials M such that N occurs in the expansion of $\Phi(M)$, is finite. This implies that if $f \in \mathcal{T}(B)$. Expanding f as in (23), we see that the formal series

$$\Phi(f) := \sum_{\underline{i}, \underline{j}} f_{\underline{i}, \underline{j}} \Phi(M_{\underline{i}, \underline{j}}) = \sum_{\underline{i}, \underline{j}} \tilde{f}_{\underline{i}, \underline{j}} M_{\underline{i}, \underline{j}}$$

has the property that $\tilde{f}_{\underline{i}, \underline{j}} \neq 0$ implies $d(M_{\underline{i}, \underline{j}}) \leq c_2$ (for some c_2) and $\nu(M_{\underline{i}, \underline{j}}) \geq \nu(f) - c_3$. Moreover, it is plain that $\|\tilde{f}_{\underline{i}, \underline{j}}\| \leq c_4$ (for some c_4) so that $\Phi(f)$ is a tame series. \square

Lemma 3.15. *Let f be an element of $\mathcal{T}(\mathbb{K}_\Sigma)$ and $\alpha \in \mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1}))^\times$, $\beta \in \mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1}))$. Then, the function $h(z) := f(\alpha z + \beta)$ belongs to $\mathcal{T}(\mathbb{K}_\Sigma)$ and we have $d(h) \leq d(f)$ and $w(h) = |\alpha|w(f)$.*

Proof. We first show the lemma in the case $\beta = 0$. For f an entire function over \mathbb{C}_∞ , f_α denotes the function $f(\alpha z)$. We recall that we have seen, in Lemma 3.7, that $(e_i)_\alpha \in \mathcal{T}^\circ(\mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1})))$ (of depth ≤ 1 , and the weight is $|\alpha|q^{-i}$). Hence, if M is a tame monomial, then $M_\alpha \in \mathcal{T}(\mathbb{K}_\Sigma)$ (because $\mathcal{T}(\mathbb{K}_\Sigma)$ is a \mathbb{K}_Σ -algebra), and $d(M_\alpha) \leq d(M)$, $w(M_\alpha) = |\alpha|w(M)$. Expanding $f = \sum_M f_M M$ ($f_M \in \mathbb{K}_\Sigma$ for all M tame monomial) then, given a tame monomial N , the set $\{M : N \text{ occurs in the tame expansion of } M_\alpha\}$ is finite. Hence, $f_\alpha = \sum_M f_M M_\alpha$ defines a tame series and we have the claimed properties of the depth and the weight. We now assume that $\alpha = 1$ and $\beta \in \mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1}))$. Since $e_i(z + \beta) = e_i + e_i(\beta)$, we then see that $f(z + \beta) = f + g$ where $g \in \mathcal{T}(\mathbb{K}_\Sigma)$ is such that $d(g) \leq d(f)$ and $w(g) < w(f)$. \square

Note that the map $f \mapsto f_\theta$ is an automorphism of $\mathcal{T}(\mathbb{K}_\Sigma)$ (recall that $f_\theta(z) = f(\theta z)$).

3.2.4. *Tame series of weight < 1 .* We denote by $\mathcal{T}^\circ(B)$ the B -submodule of $\mathcal{T}(B)$ of the tame series f which have weight $w(f) < 1$: note that $\mathcal{T}^\circ(B)$ is not a ring. Also, if a tame monomial as in (25) is in $\mathcal{T}^\circ(B)$, then $i_1 > 0$. Indeed, $\mathbb{Z}[\frac{1}{p}] \cap [0, 1[$ contains only elements of the form $j_{i_1} q^{-i_1} + \dots$ with $i_1 > 0$ if $j_{i_1} \neq 0$.

Lemma 3.16. *Every element $f \in \mathcal{T}(B)$ can be expanded, in a unique way, as*

$$f = \sum_{i=0}^r f_i e_0^i, \quad f_i \in \mathcal{T}^\circ(B).$$

Proof. Since any monic tame monomial involving e_i with $i > 0$ always has weight in $[0, 1[\cap\mathbb{Z}[\frac{1}{p}]$, this follows from the fact that we have an isomorphism of B -modules $\mathcal{T}^\circ(B) \cong \mathcal{T}(B)/(e_0)$, the quotient of $\mathcal{T}(B)$ by the principal ideal generated by e_0 . \square

It is suggestive to write, after this lemma, $\mathcal{T}(B) = \mathcal{T}^\circ(B)[e_0]$, with the warning that $\mathcal{T}^\circ(B)$ is not a ring, so that the multiplication of $\mathcal{T}^\circ(B)[e_0]$ is not the Cauchy one.

3.2.5. *Some remarks.* There are entire functions $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ which do not come from evaluation of tame series. One of them is the identity map $z \mapsto z$. Indeed, one sees easily that for all $w \in \mathbb{Z}[\frac{1}{p}]$,

$$\lim_{|z|_{\mathbb{C}} \rightarrow \infty} \frac{|z|}{|e_C(z)|^w} \in \{0, \infty\}.$$

therefore, $(z \mapsto z) \notin \mathcal{T}(\mathbb{C}_\infty)$ as otherwise, we could assign a well defined weight in $\mathbb{Z}[\frac{1}{p}] \cap \mathbb{R}_{>0}$ to it.

To define $\mathcal{T}(B)$, we have used formal series with bounded coefficients in B . One of the reasons for this choice is that the isomorphism J of Proposition 3.6 is likely not to extend to a larger sub-algebra of $B\langle\langle X \rangle\rangle$. We illustrate the problem for $B = \mathbb{C}_\infty$.

We set

$$G = \sum_{i \geq 0} \theta^{\frac{i}{q}} X_{i+1} \in \mathbb{C}_\infty\langle\langle X \rangle\rangle.$$

Then, we have the identities in $\mathbb{C}_\infty\langle\langle X \rangle\rangle$ (we have used the following computation to show that $\mathbb{C}_\infty\langle\langle X \rangle\rangle$ is inversive for τ):

$$\begin{aligned}
G^q &= \left(\sum_{i \geq 0} \theta^{\frac{i}{q}} X_{i+1} \right)^q \\
&= \sum_{i \geq 0} \theta^i X_{i+1}^q \\
&= \sum_{i \geq 0} \theta^i (C_\theta(X_{i+1}) - \theta X_{i+1}) \\
&= \sum_{i \geq 0} \theta^i (X_i - \theta X_{i+1}) \\
&\stackrel{!}{=} \sum_{i \geq 0} \theta^i X_i - \sum_{i \geq 0} \theta^{i+1} X_{i+1} \\
&= X_0.
\end{aligned}$$

Note the exclamation mark over the next to the last equality. In parallel, let us set

$$g = \sum_{i \geq 0} \theta^{\frac{i}{q}} e_{i+1}.$$

We claim that this defines an entire function. Indeed, for all $R \in |\mathbb{C}_\infty|$ and all $z \in D(0, R)$, we have, for any i large enough, $|e_{i+1}(z)| = |\tilde{\pi}||z||\theta|^{-i-1}$ so that $|\theta^{\frac{i}{q}} e_{i+1}(z)| \leq |\tilde{\pi}||\theta|^{\frac{i}{q}-i-1} R \rightarrow 0$ which implies the uniform convergence of the series defining g over any disk $D(0, R)$.

Now, $g^q \neq e_0$. One way to see this is observing that $e_0 = \tilde{\pi}z + h^q$, with h an entire function. If $g^q = e_0$ we would have the identity map $z \mapsto z$ equal to the q -th power of an entire function, which is false. However, to compute $g^q - e_0$ we cannot use the same argument we applied to show the identity $G^q = X_0$; where the argument breaks at the level of the equality $\stackrel{!}{=}$ because the series of functions $\sum_{i \geq 0} \theta^i e_{i+1}$ is divergent outside 0 although the series $\sum_{i \geq 0} \theta^i X_{i+1}$ defines an element of $\mathbb{C}_\infty\langle\langle X \rangle\rangle$.

To compute g^q we proceed in the following way. We set $\phi = e_C(\frac{z}{\theta-t}) = \sum_{i \geq 0} t^i e_{i+1} \in \mathcal{T}(\mathbb{E})$. It is easy to see that $\lim_{t \rightarrow \theta} (\theta - t)\phi = \tilde{\pi}z$. But

$$e_C(z) = C_{\theta-t}(\phi) = (\theta - t)\phi + \tau(\phi)$$

so that $e_C(z) = \tilde{\pi}z + \lim_{t \rightarrow \theta} \tau(\phi) = \tilde{\pi}z + \sum_{i \geq 0} \theta^i e_{i+1}^q = \tilde{\pi}z + g^q$. We thus obtain:

$$g^q - e_0 = \tilde{\pi}z.$$

From this identity we deduce (1) that $g \notin \mathcal{T}(\mathbb{C}_\infty)$ (because z is not tame) and (2) the map J does not extend to a \mathbb{C}_∞ -algebra map over $\mathbb{C}_\infty\langle\langle X \rangle\rangle^b[G]$.

Also, note that the condition of finite depth in the definition of $\mathcal{T}(\mathbb{C}_\infty)$ is necessary. It is not difficult to show that there is a uniformly convergent series expansion (in any bounded

subset of \mathbb{C}_∞)

$$\tilde{\pi}z = \sum_{i \geq 0} c_i e_i^q,$$

with $c_0 = 1$ and $c_i \in K_\infty$ such that $c_i \rightarrow 0$ so that the sequence $(c_i)_{i \geq 0}$ is bounded. The reader can compute the coefficients c_i inductively.

3.2.6. The field of uniformisers. let $L \subset \mathbb{K}_\Sigma$ be a field extension of K with an extension $\|\cdot\|$ of the norm $|\cdot|$ and with an endomorphism τ extending the map $c \mapsto c^q$ of K , such that for all $x \in L$, $\|\tau(x)\| = \|x\|^q$. We set

$$\mathfrak{K}_L = \widehat{\text{Frac}(\mathcal{T}(L))}_v$$

(v -adic completion); we call this the *field of uniformisers over L* . The next proposition provides a simple way to represent the elements of \mathfrak{K}_L .

Proposition 3.17. *Every element f of \mathfrak{K}_L can be expanded in a unique way as a sum*

$$f = \sum_{i \geq i_0} f_i e_0^{-i}, \quad f_i \in \mathcal{T}^\circ(L).$$

We can write $\mathcal{T}^\circ(L)((e_0^{-1}))$ for the set of the formal series $f = \sum_{i \geq i_0} f_i e_0^{-i}$ as above, with $f_i \in \mathcal{T}^\circ(L)$ for all i , with the warning that this needs not to be a field for the usual Cauchy product rule of formal series, since, as pointed out previously, $\mathcal{T}^\circ(L)$ is not a ring but just an L -vector space. The proposition tells that this set in fact carries a structure of complete field, and equals \mathfrak{K}_L , but the product rule is not the Cauchy's one. To prove the proposition we will need the next two Lemmas. The first one describes the valued ring structure of $\mathcal{T}^\circ(L)((e_0^{-1}))$.

Lemma 3.18. *The set $\mathcal{T}^\circ(L)((e_0^{-1}))$ has a natural structure of commutative ring with unit, over which the valuation v extends from $\mathcal{T}^\circ(L)$, and which is complete for it.*

Proof. Since $\mathcal{T}^\circ(L)$ is an L -vector space, in order to show that $\mathcal{T}^\circ(L)((e_0^{-1}))$ is a ring, all we need to do is to describe the product structure. Let $f = \sum_{i \geq i_0} f_i e_0^{-i}$ and $g = \sum_{j \geq j_0} g_j e_0^{-j}$ be two elements of $\mathcal{T}^\circ(L)((e_0^{-1}))$. We note that $h_k := \sum_{i+j=k} f_i g_j \in \mathcal{T}(L)$ has weight in $\{-\infty\} \cup [0, 2[$ so that we can write, by Lemma 3.16, $h_k = \alpha_k e_0 + \beta_k$, with $\alpha_k, \beta_k \in \mathcal{T}^\circ(L)$. We define

$$h = fg = \sum_{k \geq k_0 := i_0 + j_0} e_0^{-k} h_k = \sum_{k \geq k_0} \alpha_k e_0^{1-k} + \sum_{k \geq k_0} \beta_k e_0^{-k} \in \mathcal{T}^\circ(L)((e_0^{-1})).$$

From this, the ring structure follows easily. If $f = \sum_i f_i e_0^{-i} \in \mathcal{T}^\circ(L)((e_0^{-1}))$ is such that $f_{i_0} \neq 0$, then we set $v(f) := v(f_{i_0}) + i_0 \in]i_0 - 1, i_0]$ and it is plain that v defines a valuation over the ring $\mathcal{T}^\circ(L)((e_0^{-1}))$ and that every series of $\mathcal{T}^\circ(L)((e_0^{-1}))$ converges for this valuation.

Note that $f = \sum_i f_i e_0^{-i} \in \mathcal{T}^\circ(L)((e_0^{-1}))$ is such that $v(f) > N$ if the smallest i_0 such that $f_{i_0} \neq 0$ is such that $i_0 \geq N + 1$ (indeed, if $f_{i_0} \in \mathcal{T}^\circ(L) \setminus \{0\}$, $v(f_{i_0} e_0^{-i_0}) \in]i_0 - 1, i_0]$). Thus, if

$(F_k)_k$ is a Cauchy sequence of $\mathcal{T}^\circ(L)((e_0^{-1}))$, the sequence $(F_0 - F_k)_k = (\sum_{i=1}^k (F_{i-1} - F_i))_k$ converges to an element of $\mathcal{T}^\circ(L)((e_0^{-1}))$ which is then complete. \square

We now introduce the ring:

$$\mathcal{T}^\bullet(L) := \mathcal{T}(L)[M^{-1} : M \text{ monic tame monomial}] = \mathcal{T}(L)[e_i^{-1} : i \in \mathbb{Z}],$$

which contains $\mathcal{T}(L)$. Every element f of $\mathcal{T}^\bullet(L)$ has a well defined weight $w(f)$ in $\mathbb{Z}[\frac{1}{p}]$. Indeed, for every $g \in \mathcal{T}^\bullet(L)$, there exists a monic tame monomial M such that $Mg \in \mathcal{T}(L)$ and this provides the unique extension of the weight map over $\mathcal{T}^\bullet(L)$. If L is a complete valued field extension of \mathbb{C}_∞ , It is easy to see that $\mathcal{T}^\bullet(L)$ is an L -algebra which embeds in the L -algebra $\text{Hol}(\Omega \rightarrow L)$.

Lemma 3.19. *We have $\mathcal{T}^\bullet(L) \subset \mathcal{T}^\circ(L)((e_0^{-1}))$.*

Proof. If $n > 0$ we have $e_{-n}^{-1} \in e_0^{-q^n} (1 + e_0^{-1} A[[e_0^{-1}]])$ (recall that $C_a(e_C(z)) = e_C(az)$ for all $a \in A$) and therefore, $e_{-n}^{-1} \in \mathcal{T}^\circ(L)((e_0^{-1}))$ for all $n > 0$. Now, we show that $e_i^{-1} \in \mathcal{T}^\circ(L)((e_0^{-1}))$ for all $i \geq 0$. Since this is clear for $i = 0$, let us assume (induction) that $e_0^{-1}, \dots, e_{i-1}^{-1} \in \mathcal{T}^\circ(L)((e_0^{-1}))$. We observe, in the fraction field of $\mathcal{T}(L)$:

$$\frac{1}{e_i} = \frac{e_i^{q-1}}{e_i^q} = \frac{e_i^{q-1}}{e_{i-1} - \theta e_i} = \frac{e_i^{q-1}}{e_{i-1} \left(1 - \frac{\theta e_i}{e_{i-1}}\right)}.$$

Since $v(\theta e_i/e_{i-1}) > 0$, the series $\sum_{j \geq 0} (\frac{\theta e_{i-1}}{e_i})^j$ converges in $(\mathcal{T}(L)[e_{i-1}^{-1}])_v^\wedge$ to an element h such that $h(1 - \frac{\theta e_i}{e_{i-1}}) = 1$. Now, we have $\mathcal{T}(L)[e_{i-1}^{-1}]_v^\wedge \subset \mathcal{T}^\circ(L)((e_0^{-1}))$ by our induction hypothesis. Since $\mathcal{T}^\circ(L)((e_0^{-1}))$ is a ring, $\frac{1}{e_i} = e_i^{q-1} \cdot \frac{1}{e_{i-1}} \cdot h \in \mathcal{T}^\circ(L)((e_0^{-1}))$, and more generally, $\frac{1}{M} \in \mathcal{T}^\circ(L)((e_0^{-1}))$, and the lemma follows remembering Lemma 3.16 which says that $\mathcal{T}(L) = \mathcal{T}^\circ(L)[e_0]$. \square

Proof of Proposition 3.17. The uniqueness of the series expansion of an element of the L -vector space $\mathcal{T}^\circ(L)((e_0^{-1}))$ follows from Lemma 3.18. We show that $\text{Frac}(\mathcal{T}(L))$ embeds in $(\mathcal{T}^\bullet(L))_v^\wedge$. To see this, we only need to show that if $f \in \mathcal{T}(L)$ is not proportional by an element of L^\times to a tame monomial, then there exists $g \in (\mathcal{T}^\bullet(L))_v^\wedge$ such that $fg = 1$. Now, write $f = \alpha M - h$ where M is the leading monomial and where $h \in \mathcal{T}(L)$ is such that $v(h) > v(M)$. Then, the series $\sum_{i \geq 0} (\frac{h}{\alpha M})^i$ converges in $(\mathcal{T}^\bullet(L))_v^\wedge$ and we can set

$$g = \frac{1}{\alpha M} \sum_{i \geq 0} \left(\frac{h}{\alpha M}\right)^i \in \widehat{\mathcal{T}^\bullet(L)}_v.$$

By Lemma 3.19, $(\text{Frac}(\mathcal{T}(L)))_v^\wedge \subset \mathcal{T}^\circ(L)((e_0^{-1}))$ which is complete. On the other hand, any series $\sum_{i \geq i_0} f_i e_0^{-i}$ with the coefficients f_i in $\mathcal{T}^\circ(L)$ converges (for v) and the partial sums are elements of $\mathcal{T}^\bullet(L)[e_0^{-1}] \subset \text{Frac}(\mathcal{T}(L))_v^\wedge$ from which we can conclude that $(\text{Frac}(\mathcal{T}(L)))_v^\wedge = \mathcal{T}^\circ(L)((e_0^{-1}))$ and also, we note that $\mathcal{T}^\circ(L)((e_0^{-1}))$ carries the structure of a complete, valued field. \square

We also introduce the corresponding valuation ring \mathfrak{D}_L together with its maximal ideal \mathfrak{M}_L . The residual field is L . Denote, for simplicity, \mathfrak{K}_Σ for $\mathfrak{K}_{\mathbb{K}_\Sigma}$ etc.

Definition 3.20. The *field of uniformisers* is the complete v -valued field

$$\mathfrak{K} = \widehat{\bigcup_{\Sigma} \mathfrak{K}_{\Sigma}}.$$

We denote by $\mathfrak{D}, \mathfrak{M}$ the valuation ring and the maximal ideal of v .

3.2.7. Galoisian interpretation. Let $\Lambda \subset \mathbb{C}_\infty$ be the set of torsion points of the Carlitz module. We consider the field $F := \mathbb{F}_q(\Lambda)(x)$ with x indeterminate; note that it contains K . Let F^{sep} be a separable closure of F , set $x_0 := x$, and choose, inductively, $x_i \in F^{\text{sep}}$ with $C_\theta(x_i) = x_{i-1}$. The field $H := K(\Lambda)(x_0, x_1, \dots)$ is a Galois extension of F_0 contained in F^{sep} . Note that the field $L = K(\Lambda)(e_0, e_1, \dots) \subset \mathfrak{K}_{K(\Lambda)}$ (subfield of the field of meromorphic functions over \mathbb{C}_∞), wildly ramified at v (the valuation which is trivial on K^{sep} and such that $v(x_0) = -1$), is isomorphic to H . the translations $z \mapsto z + a$ with $a \in A$ induce Galois automorphisms of $L/K(\Lambda)(e_0)$ and therefore, of H/F .

4. QUASI-PERIODIC MATRIX FUNCTIONS

In the next lemma, we temporarily go to a more general setting. Let k be any field, and R a commutative k -algebra. We denote by $B(R)$ the Borel subgroup $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \text{GL}_2(R)$ and by $U(R)$ the unit-triangular subgroup $\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \text{GL}_2(R)$. Let E/k be a field extension.

Lemma 4.1. *Suppose we are given*

$$\text{GL}_2(k) \xrightarrow{\mu} \text{GL}_N(E) \xleftarrow{\nu} U(k[\theta])$$

two representations such that $\mu|_{U(k)} = \nu|_{U(k)}$ and such that for all $\lambda \in k^\times$ and $a \in k[\theta]$,

$$\mu \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \nu \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mu \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \nu \begin{pmatrix} 1 & \lambda a \\ 0 & 1 \end{pmatrix}.$$

Then, there is a unique representation $\rho : \text{GL}_2(k[\theta]) \rightarrow \text{GL}_N(E)$ which restricts to μ, ν respectively on $\text{GL}_2(k)$ and $U(k[\theta])$.

Proof. We recall, from Nagao's paper [25], that

$$\text{GL}_2(k[\theta]) = \text{GL}_2(k) *_{B(k)} B(k[\theta]),$$

which means that $\text{GL}_2(k[\theta])$ is the amalgamated product of $\text{GL}_2(k)$ and $B(k[\theta])$ along the common subgroup $B(k)$. This implies that every element $\gamma \in \text{GL}_2(k[\theta])$ can be written, in a unique way

$$\gamma = A_1 B_1 \cdots A_l B_l$$

for some l , where $A_i \in B(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(k)$ and $B_i \in B(k[\theta])$ (because of Bruhat's decomposition $\text{GL}_2(k) = B(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(k) \sqcup B(k)$). Therefore, the identities

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda a \\ 0 & 1 \end{pmatrix}$$

are the gluing condition for μ, ν to be extended to a unique representation ρ . \square

We come back to our initial setting $k = \mathbb{F}_q$. We recall that $\Gamma = \mathrm{GL}_2(A)$. We also recall that we have denoted by Ω the rigid analytic space whose underlying set is $\mathbb{C}_\infty \setminus K_\infty$ as defined, for instance, in [15]. We fix a representation

$$\Gamma \xrightarrow{\rho} \mathrm{GL}_N(k_\Sigma).$$

We set, for $a \in A$, $T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (in Γ).

Then we see that, in order to study modular forms over Ω for a representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(k_\Sigma)$, we can start by separately studying their behaviour when we apply homographies in $U(\mathbb{F}_q[\theta])$. This suggests the next definition.

Definition 4.2. Let $f : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$ be a holomorphic matrix function. We say that it is ρ -quasi-periodic if it is such that, for all $z \in \Omega$ and $a \in A$

$$(29) \quad f(z+a) = \rho(T_a)f(z).$$

We say that f is of type $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ if for all $\nu \in \mathbb{F}_q^\times$, we have

$$f(\nu z) = \nu^{-l} \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} f(z) \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

We say that f is *tempered at ∞* if there exists $L \in \mathbb{Z}$ such that $u(z)^L f(z) \rightarrow \underline{0}$ as $z \in \Omega$ is such that $|u(z)| \rightarrow 0$ and that f is *regular* if $\|f\|$ is bounded as $|u(z)| \rightarrow 0$.

If $\rho = \mathbf{1}$ (with $\mathbf{1}$ the trivial map which sends every element of Γ to $1 \in \mathbb{F}_q^\times$), then a quasi-periodic function is a holomorphic function $f : \Omega \rightarrow \mathbb{K}_\Sigma$ such that $f(z+a) = f(z)$ for all $a \in A$. Examples are $e_C(z)$ and

$$u(z) = \frac{1}{\pi} \sum_{a \in A} \frac{1}{z-a} = \frac{1}{e_C(z)}.$$

Both functions are obviously tempered. The function $e_A(z)$ is of type -1 and the function $u(z)$ is of type 1 . For further use, we record the next Proposition whose proof runs along standard arguments (see [34]).

Proposition 4.3. *Let $f : \Omega \rightarrow \mathbb{K}_\Sigma$ be holomorphic, such that $f(z+a) = f(z)$ for all $a \in A$. Then, the following properties hold:*

- (a) *There is a unique series expansion*

$$f = \sum_{n \in \mathbb{Z}} f_n u(z)^n, \quad f_n \in \mathbb{K}_\Sigma,$$

convergent if $z \in \Omega$ is such that $|z|_{\mathfrak{S}} > c$ for some $c \in |\mathbb{C}_\infty^\times|$.

- (b) *If $\lim_{|z|_{\mathfrak{S}} \rightarrow \infty} f(z) = 0$, then $f_n = 0$ for all $n \leq 0$.*
(c) *If f extends to an entire function over \mathbb{C}_∞ , and there exists $L \in \mathbb{Z}$ such that $u(z)^L f(z) \rightarrow 0$ as $|z|_{\mathfrak{S}} \rightarrow \infty$, then $f \in \mathbb{K}_\Sigma[u(z)^{-1}]$.*

We denote by $\mathcal{QP}_i^1(\rho; \mathbb{K}_\Sigma)$ the \mathbb{K}_Σ -vector space of tempered ρ -quasi-periodic functions

$$\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$$

of type l , and by $\mathcal{QP}_l(\rho; \mathbb{K}_\Sigma)$ the sub-vector space of quasi-periodic regular functions. Proposition 4.3 implies that $\mathcal{QP}_l^1(\mathbf{1}; \mathbb{K}_\Sigma)$ can be embedded in the field $\mathbb{K}_\Sigma((u))$ and that for all Σ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$, $\mathcal{QP}_l^1(\rho; \mathbb{K}_\Sigma)$ is a module over $\mathcal{QP}_0^1(\mathbf{1}; \mathbb{K}_\Sigma)^{N \times N}$, and a similar property holds for the regular quasi-periodic functions. Of course, we can reduce the target space; the meaning of $\mathcal{QP}_l^1(\rho; \mathbb{L}_\Sigma)$ etc. is therefore understood.

Lemma 4.4. *Let l be a positive integer. The function $\Psi(\rho; l)$ defined, for all $z \in \mathbb{C}_\infty \setminus A$, by*

$$\Psi(\rho; l)(z) = \sum_{a \in A} (z - a)^{-l} \rho(T_a),$$

determines a non-zero element of $\mathcal{QP}_l(\rho; \mathbb{K}_\Sigma)$.

Proof. It is easy to show that $\Psi(\rho; l) = \Psi(\rho; l)(z)$ converges uniformly for $z \in \mathbb{C}_\infty \setminus (\sqcup_{a \in A} D(a, r))$ with $r \in |\mathbb{C}_\infty^\times|$, $0 < r < 1$. This shows that $\Psi(\rho; l)$ defines a holomorphic function $\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$, and this function is non-zero because it has, in any disk $D(0, r)$ with $r \in |\mathbb{C}_\infty^\times|$, a meromorphic extension which has poles of order l at any $a \in D(0, r) \cap A$. Moreover, we have, for all $z \in \mathbb{C}_\infty \setminus A$ and $b \in A$:

$$\begin{aligned} \Psi_\rho(z - b) &= \sum_{a \in A} (z - a - b)^{-1} \rho(T_a) \\ &= \sum_{a \in A} (z - a - b)^{-1} \rho(T_{a+b}) \rho(T_{-b}) \\ &= \Psi_\rho(z) \rho(T_{-b}) = \rho(T_{-b}) \Psi_\rho(z). \end{aligned}$$

so that

$$(30) \quad \Psi_\rho(z + a) = \Psi_\rho(z) \rho(T_a) = \rho(T_a) \Psi_\rho(z), \quad \forall a \in A.$$

Since

$$T_a = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} T_{\lambda^{-1}a} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall a \in A, \quad \lambda \in \mathbb{F}_q^\times,$$

for all $\lambda \in \mathbb{F}_q^\times$:

$$\begin{aligned} \Psi(\rho; l)(\lambda z) &= \sum_{a \in A} (\lambda z - a)^{-l} \rho(T_a) \\ &= \lambda^{-l} \rho \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \Psi(\rho; l)(z) \rho \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and the type is l . Now, as $|z|_{\mathfrak{S}} \rightarrow \infty$, we get $\Psi(\rho; l)(z) \rightarrow 0$ so that $\Psi(\rho; l) \in \mathcal{QP}_l(\rho; \mathbb{K}_\Sigma)$. \square

Remark 4.5. We consider a representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{K}_\Sigma)$. We additionally suppose that:

$$(31) \quad \|a^{-1} \rho(T_a)\| \rightarrow 0, \quad \text{as } a \text{ runs in } A.$$

Then, the functions $\Psi(\rho; l)$ can be defined for all $l > 0$ and the statement of Lemma 4.4 holds in this more general situation.

In the following, we set $\Psi_\rho := \Psi(1; \rho)$ and $\Phi_\rho := e_A \Psi(1; \rho)$.

4.1. Representations of the first kind. We now introduce a class of representations of Γ for which we can construct explicitly entire non-zero quasi-periodic functions. First of all, we introduce a useful technical definition.

Definition 4.6. We say that a representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(k_\Sigma)$ is of *degree* $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ if for all $\mu \in \mathbb{F}_q^\times$, $\rho(\mu I_2) = \mu^{-l} I_N$.

We recall that after (4), $J_\gamma(z)^w \rho(\gamma)$ is a factor of automorphy if and only if ρ is of degree w . For example, \det^{-m} is of degree $2m$ (the double of the type). The identity map over Γ is of degree -1 . All the representations that we consider in this text necessarily have a well defined degree (otherwise, 0 will be the only associated modular form).

Definition 4.7. Let $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ be an injective \mathbb{F}_q -algebra morphism, let $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ be such that $d\chi(\theta) \in \mathbb{F}_q[\underline{t}_\Sigma]^{n \times n}$. Then the map

$$\rho_\chi : \Gamma \rightarrow \mathrm{GL}_{2n}(\mathbb{F}_q[\underline{t}_\Sigma][d^{-1}]) \subset \mathrm{GL}_{2n}(\mathbb{F}_q(\underline{t}_\Sigma))$$

defined, with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, by

$$\rho_\chi(\gamma) := \begin{pmatrix} \chi(a) & \chi(b) \\ \chi(c) & \chi(d) \end{pmatrix},$$

is a representation of degree -1 , called the *basic representation* associated to χ . Note also that

$$\det(\rho_\chi(\gamma)) = \det(\chi(ad - bc)) = \det(\gamma)^n.$$

If ρ is a representation, we write $\rho^* := {}^t \rho^{-1}$; it is its contragredient representation. If ρ is of degree l , ρ^* is of degree $-l$. Let $\rho : \Gamma \rightarrow \mathrm{GL}_N(\underline{t}_\Sigma)$ be a representation. We say that ρ is a *representation of the first kind* if ρ can be obtained from basic representations by finitely many iterated applications of $(\cdot)^*$, direct sums \oplus , Kronecker products \otimes , symmetric powers S^n , exterior powers \wedge^n , in such a way that ρ has a well defined degree.

Note that if ρ has degree l and if ψ has degree m , then:

$$\begin{array}{ll} \rho \oplus \psi & \text{has degree } l \text{ (if } l = m) \\ \rho \otimes \psi & l + m, \\ S^r(\rho) & rl, \\ \wedge^r \rho & rl, \\ \rho^* & -l, \end{array}$$

where in the right, $(\cdot)^*$, \oplus , \otimes , S^r and \wedge^r denote respectively the contragredient, direct sum, Kronecker product, r -th symmetric power and the r -th exterior power, of representations.

4.1.1. *Quasi-periodic functions for representations of the first kind.* For any representation of the first kind ρ , we can canonically associate two quasi-periodic functions Ξ_ρ and Φ_ρ . Let us first assume that $\rho = \rho_\chi$ is a basic representation. We denote by χ the function $\widetilde{\chi}$ of Lemma 2.10.

By using Lemma 3.8, we see that the function

$$\Xi_\rho(z) = \begin{pmatrix} 1 & \chi(z) \\ 0 & 1 \end{pmatrix},$$

belongs to $\mathcal{QP}_0^!(\rho; \mathbb{E}_\Sigma[\frac{1}{d}]^\wedge)$ (with $d\chi(\theta) \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$). If now ρ is a representation of the first kind, it can be constructed from basic representations ρ_1, \dots, ρ_m by finitely many iterated applications of direct sums, Kronecker products, exterior and symmetric powers, and the star operation, and following the same process, we can combine the functions $\Xi_{\rho_1}, \dots, \Xi_{\rho_m}$ to construct a quasi-periodic matrix function $\Xi_\rho \in \mathcal{QP}_0^!(\rho; \widehat{\mathbb{E}_\Sigma[\frac{1}{d}]})$ for some d . More precisely, we set, for ρ, ψ two representations of the first kind:

$$(32) \quad \begin{aligned} \Xi_{\rho \oplus \psi} &= \Xi_\rho \oplus \Xi_\psi, \\ \Xi_{\rho \otimes \psi} &= \Xi_\rho \otimes \Xi_\psi, \\ \Xi_{S^r(\rho)} &= S^r(\Xi_\rho), \\ \Xi_{\wedge^r \rho} &= \wedge^r \Xi_\rho, \\ \Xi_{\rho^*} &= (\Xi_\rho)^*. \end{aligned}$$

To simplify our notations we write, in the following,

$$\mathfrak{E} := \widehat{\mathbb{E}_\Sigma[d^{-1}]}$$

Lemma 4.8. *If ρ is a representation of the first kind, the function Ξ_ρ has all its entries in $\mathcal{T}(\mathfrak{E})$ and is a tempered ρ -quasi-periodic function of type 0.*

Proof. The fact that Ξ_ρ is quasi-periodic is clear from the properties of the functions $\chi : \mathbb{C}_\infty \rightarrow \mathbb{K}_\Sigma^{n \times n}$. Moreover, it is easy to see that it is of type 0. It suffices to check this for basic representations. For this note that, for $\nu \in \mathbb{F}_q^\times$, and for any \mathbb{F}_q -algebra morphism $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)$, $\begin{pmatrix} I_n & \chi(\nu z) \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & \nu \chi(z) \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \nu I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & \chi(z) \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \nu^{-1} I_n & 0 \\ 0 & I_n \end{pmatrix}$. But since $\rho = \rho_\chi$, we have $\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a I_n & b I_n \\ c I_n & d I_n \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q)$, and therefore,

$$(33) \quad \Xi_\rho(\nu z) = \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \Xi_\rho(z) \rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Additionally, since the entries of the function χ are tame series, they are tempered. \square

Proposition 4.9. *The following properties hold:*

- (a) *The function Φ_ρ extends to an entire function $\mathbb{C}_\infty \rightarrow \mathfrak{E}^{N \times N}$.*
- (b) *We have that $\Phi_\rho, \Psi_\rho \in \mathcal{QP}_1^!(\rho; \mathfrak{E})$.*
- (c) *There exist matrices $U_1, \dots, U_k \in \mathfrak{E}^{N \times N}$, uniquely determined and depending on ρ , such that*

$$\Phi_\rho(z) = (I_N + U_1 e_C(z) + \dots + U_k e_C(z)^k) \Xi_\rho(z).$$

- (d) We have $\Phi_\rho \in \mathcal{T}^\circ(\mathfrak{E})^{N \times N}$ and this is the unique element f of $\mathcal{T}^\circ(\mathfrak{E})^{N \times N}$ such that $f(a) = \rho(T_a)$ for all $a \in A$.

Note that if $\rho = \mathbf{1}$ is the trivial representation, then we have $\Xi_\rho = 1 = \Phi_\rho$, $\Psi_\rho = \sum_{b \in A} \frac{1}{z-b}$.

Proof of Proposition 4.9. (a). In any disk $D(0, r)$ with $r \in |\mathbb{C}_\infty^\times|$, the product $e_A(z)\Psi_\rho(z)$ extends to a holomorphic function because of the Weierstrass factorization

$$e_A(z) = z \prod_{a \in A \setminus \{0\}} \left(1 - \frac{z}{a}\right).$$

This immediately implies that Φ_ρ is entire.

(b). Since ρ is a representation of the first kind, Ξ_ρ can be constructed applying finitely many operations as in (32) to finitely many functions Ξ_{ρ_i} associated to basic representations ρ_i , which are unipotent (in fact, triangular with one on the diagonals). Therefore Ξ_ρ^{-1} defines an entire function $\mathbb{C}_\infty \rightarrow \mathfrak{E}^{N \times N}$. Hence, $M(z) := \Psi_\rho(z)\Xi_\rho(z)^{-1}$ has entries which are holomorphic $\Omega \rightarrow \mathfrak{E}^{n \times n}$, and $M(z+a) = M(z)$ for all $a \in A$, by (30). Moreover, since Ξ_ρ is tempered, there exists $L \in \mathbb{Z}$ such that $u(z)^L M(z) \rightarrow \mathbf{0}$ as $|z|_{\mathfrak{S}} \rightarrow \infty$. By (b) of Proposition 4.3, M can be identified with an element of $\mathfrak{E}((u))^{N \times N}$ and we easily check that $\Psi_\rho \in \mathcal{QP}_1^1(\rho; \mathfrak{E})$.

(c). We have seen that Ξ_ρ defines an element of $\mathcal{T}(\mathfrak{E})^{N \times N}$ therefore by the arguments at the point (b), we additionally observe that $\Xi_\rho \in \mathrm{GL}_N(\mathcal{T}(\mathfrak{E}))$ and therefore, $\Phi_\rho(z)\Xi_\rho^{-1} \in \mathfrak{E}((u))^{n \times n}$ extends to an entire matrix function which, in virtue of (c) of Proposition 4.3, belongs to $\mathfrak{E}[e_C(z)]^{n \times n}$.

Thus, we have proved the existence of matrices $U_0, U_1, \dots, U_k \in \mathfrak{E}^{n \times n}$ such that

$$\Phi_\rho(z)\Xi_\rho(z)^{-1} = U_0 + U_1 e_C(z) + \dots + U_k e_C(z)^k.$$

To determine U_0 , we replace $z = a \in A \setminus \{0\}$; we get $U_0 = \Phi_\rho(a)\Xi_\rho(a)^{-1} = \Phi_\rho(a)\rho(T_{-a})$. Now, $\Phi_\rho(a) = \lim_{z \rightarrow a} e_A(z)\Psi_\rho(z) = \lim_{z \rightarrow a} e_A(z)(z-a)^{-1}\rho(T_a) = \rho(T_a)$ because $e'_A = 1$. Hence, $U_0 = I_n$.

(d). From (c) above, $\Phi_\rho \in \mathcal{T}(\mathfrak{E})^{N \times N}$. We denote by $w \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}$ the supremum of the weights of all the entries of Φ_ρ . Note that $w < 1$ so that $\Phi_\rho \in \mathcal{T}^\circ(\mathfrak{E})^{N \times N}$. An element of $\mathcal{T}^\circ(\mathfrak{E})$ clearly satisfies the decay condition of Corollary 2.9. Hence, for any map $g : A \rightarrow B$, there exists at most one element $f \in \mathcal{T}^\circ(\mathfrak{E})$ such that $f(a) = g(a)$ for all $a \in A$. Consequently, if f is element of $\mathcal{T}^\circ(\mathfrak{E})^{N \times N}$ such that $f(a) = \rho(T_a)$ for all $a \in A$, then, $\Phi_\rho = f$. The other properties are obvious. \square

We have the next corollary, where ρ is a representation of the first kind.

Corollary 4.10. *The tame series expansion of Φ_ρ is provided by the unique representative in the \mathfrak{E} -module $\mathcal{T}^\circ(\mathfrak{E})^{N \times N}$ of the matrix Ξ_ρ in the quotient of $\mathcal{T}(\mathfrak{E})^{N \times N}$ by the principal ideal generated by $e_0 I_N$. Moreover, we have $\det(\Phi_\rho) = 1$ and $\Phi_\rho \in \mathcal{QP}_0^1(\rho; \mathfrak{E})$.*

Proof. The first property follows directly from Proposition 4.9 (c), (d). To show the second property we first note that the matrices $\rho(T_a)$, $a \in A$, can be simultaneously (upper) triangularised over $\mathbb{F}_q(\underline{t}_\Sigma)^{ac}$, and the diagonal entries are all equal to one because $T_a^p = I_2$ for all a . Hence, Ψ_ρ is conjugated over $\mathbb{F}_q(\underline{t}_\Sigma)^{ac}$ to a upper triangular matrix having $e_A(z)^{-1}$ as diagonal entries. This implies that Φ_ρ is conjugated over $\mathbb{F}_q(\underline{t}_\Sigma)^{ac}$ to a upper triangular matrix having 1 in the diagonal. Hence, $\det(\Phi_\rho) = 1$ while it is clear that it is quasi-periodic of type 0. \square

Let $\chi : A \rightarrow \mathbb{F}_q(\underline{t}_\Sigma)^{n \times n}$ be an \mathbb{F}_q -algebra morphism and denote by ρ the basic representation $\rho_\chi : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q(\underline{t}_\Sigma))$ defined by $\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \chi(a) & \chi(b) \\ \chi(c) & \chi(d) \end{pmatrix}$, with $N = 2n$. For a matrix $f \in \mathfrak{K}_\Sigma^{N \times N}$, $v(f)$ denotes the infimum of the valuations of the entries of f .

Corollary 4.11. *We have $\Phi_\rho = \Xi_\rho$, $v(\Phi_\rho) = -\frac{1}{q}$ and $v(\Phi_\rho - \omega_\chi^{-1}e_1) > -\frac{1}{q}$.*

Proof. By definition, $\Xi_\rho = \begin{pmatrix} I_n & \chi \\ 0 & I_n \end{pmatrix}$ and $\chi(z) = \omega_\chi^{-1}e_C(z(\theta I_n - \vartheta)^{-1})$ has entries in $\mathcal{T}^\circ(\mathbb{K}_\Sigma)$ so we have already $\Phi_\rho = \Xi_\rho$ by Corollary 4.10. Moreover, the tame series expansion of $e_C(z(\theta I_n - \vartheta)^{-1})$ is $e_C(z(\theta I_n - \vartheta)^{-1}) = e_1 I_n + \text{terms of smaller weight}$, which implies the remaining properties. \square

4.1.2. *Application to column quasi-periodic functions.* We consider, in this subsection, a representation $\Gamma \xrightarrow{\rho} \mathrm{GL}_N(k_\Sigma)$.

Definition 4.12. A holomorphic column function

$$\Omega \xrightarrow{f} \mathbb{K}_\Sigma^{N \times 1}$$

such that

$$f(z+a) = \rho(T_a)f(z), \quad \forall a \in A,$$

is called a ρ -quasi-periodic function. We say that f is *tempered* if there exists $L \in \mathbb{Z}$ such that $\lim_{|z|=|z|_{\mathfrak{S}} \rightarrow \infty} f(z)u(z)^L = {}^t(0, \dots, 0)$. We further say that f is *regular* if there exists $c > 0$ such that the set $\{\|f(z)\| : |z|_{\mathfrak{S}} \geq c\}$ is bounded.

Note that in §4, we discussed quasi-periodic $N \times N$ matrix functions. Their columns are quasi-periodic in the above sense. We now suppose that the representation ρ is of the first kind. Recall the notation $\mathfrak{K}_\Sigma = \mathfrak{K}_{\mathbb{K}_\Sigma}$ where, for a subfield L of \mathbb{K}_Σ , \mathfrak{K}_L has been defined in §3.2.6. The v -valuation ring is denoted by \mathfrak{D}_Σ (and the maximal ideal is denoted by \mathfrak{M}_Σ).

Proposition 4.13. *Assuming that ρ is of the first kind, if f is ρ -quasi-periodic and tempered, we can identify it with an element of $\mathfrak{K}_\Sigma^{N \times 1}$. If additionally f is regular, then we can identify it with an element of $\mathfrak{D}_\Sigma^{N \times 1}$.*

Proof. In the proof of part (c) of Proposition 4.9, we have seen that Ξ_ρ can be identified with an element of $\mathrm{GL}_N(\mathcal{T}(\mathbb{K}_\Sigma))$. Hence, the function $\Xi_\rho^{-1}f : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$ has entries which are all A -periodic and tempered. By part (b) of Proposition 4.3, the entries are thus elements of $\mathbb{K}_\Sigma((e_0^{-1}))$ and the entries of $f = \Xi_\rho \Xi_\rho^{-1}f$ are therefore in $\mathcal{T}^\circ(\mathbb{K}_\Sigma)((e_0^{-1}))$ which is equal, by Proposition 3.17, to \mathfrak{K}_Σ . This proves the first part of the proposition.

Since Φ_ρ is quasi-periodic we have $f = \Phi_\rho g$ where $g \in \mathbb{K}_\Sigma((u))^{N \times 1}$. Corollary 4.10 implies that $\Phi_\rho \in \mathrm{GL}_N(\mathcal{T}^\circ(\mathbb{K}_\Sigma))$. Observe that $g = \Phi_\rho^{-1} f$. Since the entries of Φ_ρ^{-1} are in $\mathcal{T}^\circ(\mathbb{K}_\Sigma)$, for $|z|_{\mathfrak{S}} \geq c_1$ for some constant $c_1 \in |\mathbb{C}_\infty^\times|$, we have $\|\Phi_\rho^{-1} f\| \leq c_2 |e_A(z)|^w$, where $w \in \mathbb{Z}[\frac{1}{p}] \cap [0, 1[$, for some $c_2 > 0$. This means that $\|u^w g\| \leq c_2$ as $|z|_{\mathfrak{S}}$ is large. Let $\alpha > 0$ be such that $p^\alpha w \in \mathbb{Z}$. Then $\|u^{p^\alpha w} g^{p^\alpha}\|$ is bounded at infinity and $u^{p^\alpha w} g^{p^\alpha} \in \mathbb{K}_\Sigma((u))^{N \times 1}$. Therefore, $u^w g \in \mathbb{K}_\Sigma[[u^{\frac{1}{p^\alpha}}]]^{N \times 1}$ and we deduce that, necessarily, $g \in \mathbb{K}_\Sigma[[u]]^{N \times 1}$. Hence, $f = \Phi_\rho g \in \mathfrak{D}_\Sigma^{N \times 1}$. \square

Now viewing Definition 1.2, we deduce parts (1), (2), (3) of Theorem A in the introduction:

Theorem 4.14. *For all $w \in \mathbb{Z}$, there is a natural embedding $M_w^!(\rho; \mathbb{K}_\Sigma) \xrightarrow{\iota_\Sigma} \mathfrak{K}_\Sigma^{N \times 1}$ such that $M_w(\rho; \mathbb{K}_\Sigma) = \iota_\Sigma^{-1}(\iota_\Sigma(M_w^!(\rho; \mathbb{K}_\Sigma)) \cap \mathfrak{D}_\Sigma^{N \times 1})$ and such that $S_w(\rho; \mathbb{K}_\Sigma) = \iota_\Sigma^{-1}(\iota_\Sigma(M_w^!(\rho; \mathbb{K}_\Sigma)) \cap \mathfrak{M}_\Sigma^{N \times 1})$.*

Proof. Since a weak modular form is a tempered quasi-periodic function and a modular form is a regular quasi-periodic function, the first part of the result follows directly from Proposition 4.13, where the hypothesis that ρ is of the first kind is essential. To prove the two other parts of the statement, namely the characterisation of the image of $M_w(\rho; \mathbb{K}_\Sigma)$ and $S_w(\rho; \mathbb{K}_\Sigma)$, we combine Proposition 3.17 with Proposition 3.10, which allows to derive, from the fact that f is bounded at infinity (resp. has zero limit at infinity) that valuations of the entries of f are non-negative (resp. positive). \square

4.1.3. *Application to Perkins' series.* Let U be a finite subset of \mathbb{N}^* , let

$$\sigma_U = \prod_{i \in U} \chi_{t_i}$$

be the corresponding semi-character $A \rightarrow \mathbb{F}_q[t_U]$.

Definition 4.15. The *Perkins series* of order $n \geq 1$ associated to σ_U is:

$$\psi(n; \sigma_U) = \sum_{a \in A} (z - a)^{-n} \sigma_U(a).$$

All these series converge for $z \in \mathbb{C}_\infty \setminus A$ and the functions $z \mapsto e_A(z)^n \psi(n; \sigma_U)(z)$ converge to entire functions $\mathbb{C}_\infty \rightarrow \mathbb{E}_\Sigma$, as it is easily seen. If $\Sigma = \emptyset$ we have $\sigma_\emptyset = \mathbf{1}$ the trivial semi-character, and Perkins' generating series are related to Goss' polynomials associated to the lattice $A \subset \mathbb{C}_\infty$ as in [19, §6] and [15, §3]. Indeed,

$$(34) \quad \psi(n; \mathbf{1}) = S_{n,A} = \sum_{b \in A} \frac{1}{(z - b)^n} = G_{n,A}(S_{1,A}),$$

for polynomials $G_{n,A} \in K_\infty[X]$ (we adopt the notations of *ibid.*) These functions have been considered by Perkins in his Thesis [38]. We study Perkins' series of order 1: we show that they are in \mathfrak{K}_Σ and we compute their weight in the case $n = 1$.

We need to introduce a few tools. Let Σ be a finite subset of \mathbb{N}^* with s elements. Let $m \geq 0$ be the unique integer such that $(m - 1)(q - 1) + 1 \leq s \leq m(q - 1)$. If $s = 0$, then

$m = 0$. Let l be the unique integer with $s = (m - 1)(q - 1) + l$ (so that $1 \leq l \leq q - 1$ and if $s = m = 0$, then $l = q - 1$). We set:

$$(35) \quad M_s = e_1^{q-1} \cdots e_{m-1}^{q-1} e_m^l \in \mathcal{T}^\circ(\mathbb{F}_q)$$

(note that we can define the B -module $\mathcal{T}^\circ(B)$ for any \mathbb{F}_q -algebra B). We call M_s the *maximal* tame monomial, a terminology which is motivated by the following result. We clearly have, by the fact that $s = (m - 1)(q - 1) + l$:

$$(36) \quad d(M_s) = s, \quad w(M_s) = (q - 1) \sum_{i=1}^{m-1} \frac{1}{q^i} + \frac{l}{q^m} =: w_{\max}(s).$$

We set $\mathcal{T}_s^\circ(B) = \mathcal{T}^\circ(B) \cap \mathcal{T}_s(B)$ (recall the grading (27)). We have $\mathcal{T}^\circ(B) = \bigoplus_{s \geq 0} \mathcal{T}_s^\circ(B)$. Then, the next lemma holds (the proof is easy and left to the reader), which tells us that in the homogeneous piece \mathcal{T}_s° , M_s has maximal weight.

Lemma 4.16. *For all $f \in \mathcal{T}_s^\circ(\mathbb{K}_\Sigma)$ there exists $f_s \in \mathbb{K}_\Sigma$ unique such that $w(f - f_s M_s) < w(f) \leq w_{\max}(s)$.*

We come back to Perkins' series. All the functions $\psi(1; \sigma_U)$ with $U \subset \Sigma$ occur in the entries of Ψ_{ρ_Σ} , where ρ_Σ is the representation of the first kind

$$\rho_\Sigma = \bigotimes_{i \in \Sigma} \rho_{t_i},$$

where $\rho_{t_i} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a(t_i) & b(t_i) \\ c(t_i) & d(t_i) \end{pmatrix}$. Since $\Psi_{\rho_\Sigma} \in \mathcal{QP}_1^1(\rho_\Sigma; \mathbb{E}_\Sigma)$ from (d) of Proposition 4.9 we deduce that

$$\phi(1; \sigma_U) := e_0 \psi(1; \sigma_U) \in \mathcal{T}^\circ(\mathbb{E}_\Sigma), \quad U \subset \Sigma.$$

Without loss of generality, we focus now on $\phi(1; \sigma_\Sigma) \in \mathcal{T}^\circ(\mathbb{E}_\Sigma)$. The next question is the computation of its weight. We set, for Σ non-empty with l, m as above:

$$(37) \quad \kappa(\Sigma) := q^{-m}(q - l) \in]0, 1[\cap \mathbb{Z}[p^{-1}].$$

For $\Sigma = \emptyset$, we extend the definition to $\kappa(\emptyset) := 1$. Note that $\kappa(\Sigma)$ defines a strictly decreasing function $|\Sigma| \mapsto \kappa(\Sigma)$, and $\lim_{|\Sigma| \rightarrow \infty} \kappa(\Sigma) = 0$. We prove:

Theorem 4.17. *The function $\phi(1; \sigma_\Sigma) \in \mathcal{T}^\circ(\mathbb{E}_\Sigma)$ has weight*

$$(38) \quad w(\phi(1; \sigma_\Sigma)) := 1 - \kappa(\Sigma) = 1 - q^{1-m} + lq^{-m}.$$

Proof. If $\Sigma = \emptyset$, it is clear that $\phi(1; \sigma_\Sigma)$ has weight 0 (it is in this case a constant function). We suppose that Σ is non-empty. We consider the unique representative $g_\Sigma \in \mathcal{T}^\circ(\mathbb{E}_\Sigma)$ of $\prod_{i \in \Sigma} \chi_{t_i}(z)$ (see §2.2.3 for the definition of $\chi_t(z)$) modulo the ideal of $\mathcal{T}(\mathbb{E}_\Sigma)$ generated by e_0 . By Corollary 4.10, we have $\phi(1; \sigma_\Sigma) = g_\Sigma$. We can write $g_\Sigma = \sum_{i=0}^s g_\Sigma^{[i]}$ with

$g_\Sigma^{[i]} \in \mathcal{T}_i^\circ(\mathbb{E}_\Sigma)$ (see (28)) ⁽³⁾. We note that

$$(39) \quad g_\Sigma^{[s]} = \left[\prod_{i \in \Sigma} \chi_{t_i}(z) \right]^{[s]} = \underbrace{e_1^{q-1} \cdots e_{m-1}^{q-1} e_m^l}_{\text{Tame monomial } M_s} \mathcal{P}_\Sigma + \Phi,$$

with $w(M_s) = w_{\max}(s)$, $\Phi \in \mathcal{T}(\mathbb{E}_\Sigma)$, with $w(\Phi) < w_{\max}(s)$, and where

$$\mathcal{P}_\Sigma := \sum_{\substack{I_0 \sqcup I_1 \sqcup \cdots \sqcup I_m = \Sigma \\ |I_0| = \cdots = |I_{m-1}| = q-1 \\ |I_m| = l}} \left(\prod_{i_1 \in I_1} t_{i_1} \right) \cdots \left(\prod_{i_m \in I_m} t_{i_m}^{m-1} \right) \in \mathbb{F}_p[\underline{t}_\Sigma],$$

a polynomial which is easily seen to be non-zero. By Lemma 4.16:

$$w(g_\Sigma^{[s]} - \mathcal{P}_\Sigma M_s) < w_{\max}(s).$$

This implies the theorem because the map $s \mapsto w_{\max}(s)$ is a strictly increasing function ($s > 0$) so that

$$w(\phi(1; \sigma_\Sigma)) = w(g_\Sigma) = w(g_\Sigma^{[s]}) = w_{\max}(s).$$

□

For all $\Sigma \subset \mathbb{N}^*$ a finite subset, the above proof yields the next corollary:

Corollary 4.18. *We have*

$$\lim_{|z|_{\mathbb{G}} \rightarrow \infty} e_A(z)^{\kappa(\Sigma)} \psi(1; \sigma_\Sigma) = \mathcal{P}_\Sigma.$$

Example. Let Σ be a subset of \mathbb{N}^* of cardinality q . Developing the product $\prod_{k \in \Sigma} e_C\left(\frac{z}{\theta - t_k}\right)$ we get, after elimination of the q -th powers:

$$\begin{aligned} \prod_{k \in \Sigma} e_C\left(\frac{z}{\theta - t_k}\right) &= e_C(z) - \sum_{j \geq 0} \left(\theta \prod_{i \in \Sigma} t_i^j - \prod_{i \in \Sigma} t_i^{j+1} \right) e_C\left(\frac{z}{\theta^{j+1}}\right) + \\ &+ \sum_{\substack{0 \leq i_1 \leq \cdots \leq i_q \\ i_k \text{ not all equal}}} e_C\left(\frac{z}{\theta^{i_1+1}}\right) \cdots e_C\left(\frac{z}{\theta^{i_q+1}}\right) \sum_{\substack{\underline{\alpha} = (\alpha_i; i \in \Sigma) \in \mathbb{N}^{|\Sigma|} \\ |\underline{\alpha}| = q^{i_1} + \cdots + q^{i_q}}} \prod_{k \in \Sigma} t_k^{\alpha_k}. \end{aligned}$$

from this tame series expansion (of depth q) we deduce that the leading tame monomial of $\prod_{i \in \Sigma} \chi_{t_i}(z)$ is $e_C(z)$. Hence, $\prod_{k \in \Sigma} e_C\left(\frac{z}{\theta - t_k}\right) - e_C(z) \in \mathcal{T}^\circ(A[\underline{t}_\Sigma])$ and equals $\phi(1; \sigma_\Sigma)$ for this choice of Σ .

³In fact, one sees that if $i \not\equiv s \pmod{q-1}$, then $g_\Sigma^{[i]} = 0$.

4.2. Hecke operators. As simple consequences of the above investigations, we will now define Hecke operators acting on the spaces $W_w(\rho; B)$ with symbols $W \in \{M, S\}$ and $B \in \{\mathbb{L}_\Sigma, \mathbb{K}_\Sigma\}$. Let us consider a representation $\Gamma \xrightarrow{\rho} \mathrm{GL}_N(k_\Sigma)$ and let \mathfrak{a} be a non-zero ideal of A . The group Γ acts (on the left) on $M_{\mathfrak{a}} = \{\gamma \in A^{2 \times 2} : \det(\gamma) \text{ is a generator of } \mathfrak{a}\}$. Let \mathcal{G} be an element of $\mathrm{Hol}(\Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1})$ and let us consider $k \in \mathbb{Z}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. We define, for $\gamma \in M_{\mathfrak{a}}$, the Petersson slash operator $\mathcal{G}|_{k,m,\rho}\gamma$ on \mathcal{G} by

$$(40) \quad (\mathcal{G}|_{k,m,\rho}\gamma)(z) := J_\gamma(z)^{-k} \det(\gamma)^m \rho(\gamma)^{-1} \mathcal{G}(\gamma(z)),$$

and we set, choosing a complete set of representatives of $\Gamma \backslash M_{\mathfrak{a}}$:

$$T_{\mathfrak{a}}^{k,m,\rho}(\mathcal{G})(z) := \mathfrak{a}^{k-m} \sum_{\gamma \in \Gamma \backslash M_{\mathfrak{a}}} \mathcal{G}|_{k,m,\rho}\gamma.$$

Let \mathcal{G} be such that

$$(41) \quad \mathcal{G}(z) = J_\gamma(z)^k \det(\gamma)^{-1} \rho(\gamma) \mathcal{G}(z), \quad \forall z \in \Omega, \quad \forall \gamma \in \Gamma.$$

It is a standard computation to show that the function $\mathcal{H} := T_{\mathfrak{a}}(\mathcal{G}) = T_{\mathfrak{a}}^{k,m,\rho}(\mathcal{G})$, regardless of the choice of the representatives involved in the sum defining it, also satisfies the collection of functional equations (41). Moreover, $T_{\mathfrak{a}}T_{\mathfrak{b}} = T_{\mathfrak{a}\mathfrak{b}}$ for all $\mathfrak{a}, \mathfrak{b}$ non-zero ideals of A . We have the next result, where $T_{\mathfrak{a}} = T_{\mathfrak{a}}^{w,0,\rho}$ for all ideals \mathfrak{a} :

Theorem 4.19. *Assuming that ρ is of the first kind, we have that for all \mathfrak{a} , $T_{\mathfrak{a}}$ defines a B -linear endomorphism of $M_w^!(\rho; B)$ which induces endomorphisms of $M_w(\rho; B)$ and $S_w(\rho; B)$.*

Proof. If $\mathfrak{a} = \mathfrak{p} = (P)$ is a prime ideal, then we can choose $M_{\mathfrak{p}} = \left\{ \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & P \end{pmatrix} : b \in A, |b| < |P| \right\}$. Thanks to Lemma 3.15, $T_{\mathfrak{p}}$ operates on $\mathfrak{K}_\Sigma^{N \times 1}$ and furthermore, leaves $\mathfrak{D}_\Sigma^{N \times 1}$ and $\mathfrak{M}_\Sigma^{N \times 1}$ invariant. The total multiplicativity of the operators $T_{\mathfrak{a}}$ completes the proof. \square

This generalises [37, Proposition 5.12] (which deals the very special case of $N = 2$ and $\rho = \rho_t^*$, with an ad hoc proof very hard to generalise to our settings).

5. STRUCTURE RESULTS FOR MODULAR FORMS

We consider, in this section, a representation

$$\Gamma \xrightarrow{\rho} \mathrm{GL}_N(k_\Sigma).$$

We recall that $M_w^!(\rho; \mathbb{L}_\Sigma)$, $M_w(\rho; \mathbb{L}_\Sigma)$, $S_w(\rho; \mathbb{L}_\Sigma)$ denote respectively, the \mathbb{L}_Σ -vector spaces of weak modular forms, modular forms, and cusp forms in $\mathrm{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1})$ of weight w for ρ (in the sense of Definition 1.2), so that $S_w(\rho; \mathbb{L}_\Sigma) \subset M_w(\rho; \mathbb{L}_\Sigma) \subset M_w^!(\rho; \mathbb{L}_\Sigma)$. The operator τ induces k_Σ -linear injective maps

$$M_w(\rho; \mathbb{L}_\Sigma) \xrightarrow{\tau} M_{qw}(\rho; \mathbb{L}_\Sigma),$$

and similarly for $M_w^!(\rho; \mathbb{L}_\Sigma)$, $S_w(\rho; \mathbb{L}_\Sigma)$ etc. Of course, this depends on the choice of Σ . To simplify, we will sometimes also write $M_w(\rho)$ for $M_w(\rho; \mathbb{L}_\Sigma)$ etc. when the reference to the field \mathbb{L}_Σ is clear. The next sub-section also allows to justify this abuse of notation.

5.1. Changing the coefficient field. We have defined, for $\rho : \Gamma \rightarrow \mathrm{GL}_N(k_\Sigma)$ a representation, the \mathbb{L}_Σ -vector space of modular forms $W_w(\rho; \mathbb{L}_\Sigma)$ (with W a symbol such that $W \in \{M^!, M, S\}$). Let Σ' be finite such that $\Sigma \subset \Sigma' \subset \mathbb{N}^*$. Then, we also have the spaces $W_w(\rho; \mathbb{L}_{\Sigma'})$. The next result allows to compare the various spaces $M_w(\rho; \mathbb{L}_{\Sigma'})$ for $\Sigma' \supset \Sigma$ under the hypothesis of finite dimensionality.

Proposition 5.1. *If $\dim_{\mathbb{L}_\Sigma}(M_w(\rho; \mathbb{L}_\Sigma)) < \infty$ and ρ is of the first kind, then*

$$M_w(\rho; \mathbb{L}_{\Sigma'}) = M_w(\rho; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{L}_{\Sigma'}.$$

To prove this proposition we need the next two Lemmas.

Lemma 5.2. *Let X be a rigid analytic curve. We have*

$$\mathrm{Hol}_{\mathbb{K}_\Sigma}(X \rightarrow \mathbb{L}_\Sigma) = \bigcup_{d \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}} \mathrm{Hol}(X \rightarrow \widehat{\mathbb{T}_\Sigma[d^{-1}]})$$

Proof. We first show the lemma when $X = \mathrm{Spm}(\mathcal{A})$ where \mathcal{A} is an integral \mathbb{C}_∞ -affinoid algebra. Remember that in this case, if $f : X \rightarrow \mathbb{C}_\infty$ is holomorphic with infinitely many zeroes, then it is identically zero. Now, let $f : X \rightarrow \mathbb{K}_\Sigma$ be a global section of $\mathcal{O}_{X/\mathbb{K}_\Sigma}$ such that the image lies in \mathbb{L}_Σ . For all $x \in X$ there exists $d \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}$ such that $f(x) \in \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$. We thus have a map

$$X \xrightarrow{\Phi} \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\},$$

defined by sending $x \in X$ to any d such that $f(x) \in \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$.

Since X is, as a set, uncountable, while the target set is countable, there exists an infinite subset $X_0 \subset X$ and d , such that $\Phi(x) = d$ for all $x \in X_0$. We can complete an orthonormal basis $\mathcal{B}_d \subset \mathbb{F}_q^{ac}[\underline{t}_\Sigma][\frac{1}{d}]$ of $\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$ to an orthonormal basis $\mathcal{B} = \mathcal{B}_d \sqcup \mathcal{B}' \subset \mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ of $\mathbb{K}_\Sigma/\mathbb{C}_\infty$. Then, we can expand f in a uniformly convergent series over X :

$$f(x) = \sum_{b \in \mathcal{B}} f_b(x)b, \quad x \in X$$

where $f_b : X \rightarrow \mathbb{C}_\infty$ are holomorphic functions (explicitly, f_b tends to zero for the spectral norm over X with respect to the Fréchet filter). If $b \in \mathcal{B}'$, then f_b vanishes over X_0 infinite and therefore vanishes identically. Hence, $f \in \mathrm{Hol}_{\mathbb{T}_\Sigma[d^{-1}]^\wedge}(X \rightarrow \mathbb{T}_\Sigma[d^{-1}]^\wedge)$, or in other words, we have a uniformly convergent series expansion

$$(42) \quad f(x) = \sum_{b \in \mathcal{B}_d} f_b(x)b, \quad x \in X$$

with $f_b \in \mathcal{O}_X$ for all $b \in \mathcal{B}_d$. Now, suppose that $X \subsetneq Y$ are two affinoids, let f be in $\mathrm{Hol}_{\mathbb{K}_\Sigma}(Y \rightarrow \mathbb{L}_\Sigma)$. Consider an orthonormal basis \mathcal{B}_d of $\mathbb{T}_\Sigma[d^{-1}]^\wedge$ and complete it to an orthonormal basis $\mathcal{B}_{d'} = \mathcal{B}_d \sqcup \mathcal{C}$ of $\mathbb{T}_\Sigma[d'^{-1}]^\wedge$ assuming that $\mathbb{T}_\Sigma[d^{-1}]^\wedge \subset \mathbb{T}_\Sigma[d'^{-1}]^\wedge$ and that at once, $f \in \mathrm{Hol}_{\mathbb{T}_\Sigma[d'^{-1}]^\wedge}(Y \rightarrow \mathbb{T}_\Sigma[d'^{-1}]^\wedge)$ and $f \in \mathrm{Hol}_{\mathbb{T}_\Sigma[d^{-1}]^\wedge}(X \rightarrow \mathbb{T}_\Sigma[d^{-1}]^\wedge)$. Then, by (42), we have

$$f(x) = \sum_{b \in \mathcal{B}_{d'}} f_b(x)b, \quad x \in Y$$

and f_b vanishes identically for all $b \in \mathcal{C}$. This implies that $\mathbb{T}_\Sigma[d'^{-1}] = \mathbb{T}_\Sigma[d^{-1}]$. The lemma follows easily working on an admissible covering of a rigid curve by affinoids. \square

Let $\Sigma' = \Sigma \sqcup U$ be a finite subset of \mathbb{N}^* decomposed as a disjoint union of two subsets.

Lemma 5.3. *For all $d' \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}$ there exists $S \subset D_{\mathbb{C}_\infty}(0, 1)^U$ affinoid, with $S = \text{Spm}(R)$ (maximal spectrum), R affinoid \mathbb{C}_∞ -algebra of dimension $|U|$, and $d \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma] \setminus \{0\}$, such that $\mathbb{T}_{\Sigma'}[\frac{1}{d'}]^\wedge \subset \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge \widehat{\otimes}_{\mathbb{C}_\infty} R$.*

Proof. There exists $S = \text{Spm}(R)$ as in the statement of the lemma such that $d' = du$ with $d \in \mathbb{F}_q^{ac}[\underline{t}_\Sigma]$ and $u \in R^\times$. Then, we can identify g with a convergent series $\sum_{i \in \mathcal{I}} g_i b_i$ with $g_i \in \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$ and $(b_i)_{i \in \mathcal{I}}$ an orthonormal basis of R over \mathbb{C}_∞ . Hence $g \in \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge \widehat{\otimes}_{\mathbb{C}_\infty} R$. \square

Proof of Proposition 5.1. In the statement, the space on the right-hand side is obviously contained in the space on the left-hand side so we prove now the opposite inclusion. Since $\rho : \Gamma \rightarrow \text{GL}_N(k_\Sigma)$ is of the first kind, there exists $d_1 \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that the image of ρ lies in $\mathbb{F}_q[\underline{t}_\Sigma][\frac{1}{d_1}]^{N \times N}$. We also write $\Sigma' = \Sigma \sqcup U$. Let f be an element of $M_w(\rho; \mathbb{L}_{\Sigma'})$. Since $f : \Omega \rightarrow \mathbb{L}_{\Sigma'}^{N \times 1}$ is analytic, by Lemma 5.2 with $X = \Omega$, there exists $d_2 \in \mathbb{F}_q[\underline{t}_{\Sigma'}] \setminus \{0\}$ such that the image has entries in the completion, for the Gauss norm, of $\mathbb{T}_{\Sigma'}[\frac{1}{d_2}]$. By Lemma 5.3, we can find d_3 and R an affinoid \mathbb{C}_∞ -algebra such that the image of f is in

$$(\mathbb{T}_\Sigma[\widehat{d_3^{-1}}] \widehat{\otimes}_{\mathbb{C}_\infty} R)^{N \times 1},$$

with R of Krull dimension $|U|$, $S = \text{Spm}(R)$ with $S \subset D_{\mathbb{C}_\infty}(0, 1)^U$. In the above, we can in fact make a choice so that $d_1 = d_2 = d_3 =: d$.

Since by hypothesis $\dim_{\mathbb{L}_\Sigma}(M_w(\rho; \mathbb{L}_\Sigma)) < \infty$, we can choose a basis (g_1, \dots, g_r) , writing

$$g_i = \begin{pmatrix} g_i^{[1]} \\ \vdots \\ g_i^{[N]} \end{pmatrix}, \quad i = 1, \dots, r, \quad f = \begin{pmatrix} f^{[1]} \\ \vdots \\ f^{[N]} \end{pmatrix}.$$

We can view f as an analytic map

$$\Omega \times S \xrightarrow{(z,s) \mapsto f(z,s)} \mathbb{T}_\Sigma[\widehat{d^{-1}}]^{N \times 1}$$

and adjusting again the choice of d , the maps g_i take their values in the module on this right-hand side. If we fix $s \in S$, the resulting map $f_s = f(\cdot, s)$, which is analytic $\Omega \rightarrow \mathfrak{T}^{N \times 1}$, belongs to $M_w(\rho; \mathfrak{T}) \subset M_w(\rho; \mathbb{L}_\Sigma)$, where we have written

$$\mathfrak{T} := \mathbb{T}_\Sigma[\widehat{d^{-1}}].$$

Indeed, note that for all $s \in S$, $\|f_s\|$ is bounded on $\{z \in \mathbb{C}_\infty : |z|_{\mathfrak{T}} \geq M\}$, for some $M \in |\mathbb{C}_\infty^\times|$ with $M > 1$. Hence, for any $s \in S$:

$$f_s(z) = \sum_{j=1}^r \phi_j(s) g_j(z),$$

with coefficients $\phi_j(s) \in \mathbb{L}_\Sigma$, or more precisely, in \mathfrak{T} , for an appropriate choice of d compatible with all the above. Now, we choose $z_1, \dots, z_l \in \Omega$. Then, we can write, for all $k = 1, \dots, l$:

$$(43) \quad (f_s^{[1]}(z_k), \dots, f_s^{[N]}(z_k)) = (\phi_1(s), \dots, \phi_r(s)) \underbrace{\begin{pmatrix} g_1^{[1]}(z_k) & \cdots & g_1^{[N]}(z_k) \\ \vdots & & \vdots \\ g_r^{[1]}(z_k) & \cdots & g_r^{[N]}(z_k) \end{pmatrix}}_{=:\mathcal{M}_k}.$$

Now, since g_1, \dots, g_r are linearly independent over \mathbb{L}_Σ , we can find appropriate l and z_1, \dots, z_l such that the matrix $\mathcal{M} = (\mathcal{M}_1, \dots, \mathcal{M}_l)$ has rank r . We can thus extract an $r \times r$ submatrix $\widetilde{\mathcal{M}} \in \mathrm{GL}_r(\mathbb{L}_\Sigma)$. Then, for a vector \widetilde{f}_s of length r extracted from

$$\underbrace{(f_s^{[1]}(z_1), \dots, f_s^{[N]}(z_l))}_{\text{length } Nl}$$

which represents an analytic function

$$S \rightarrow \mathbb{L}_\Sigma^r,$$

we have

$$(44) \quad \widetilde{f}_s = (\phi_1(s), \dots, \phi_r(s)) \widetilde{\mathcal{M}}$$

and therefore

$$\phi = \phi(s) = (\phi_1(s), \dots, \phi_r(s)) = \widetilde{f}_s \mathcal{M}^{-1}$$

is an analytic function $S \rightarrow \mathfrak{T}^{1 \times r}$ (of course for a good choice of d invisible in our notation) and therefore an element of

$$(\mathfrak{T} \widehat{\otimes}_{\mathbb{C}_\infty} R)^{r \times 1} = (\mathbb{T}_\Sigma[\widehat{d^{-1}}] \widehat{\otimes}_{\mathbb{C}_\infty} R)^{r \times 1},$$

(for this good choice of d). Therefore,

$$f \in M_w(\rho; \mathfrak{T}) \otimes_{\mathfrak{T}} (\mathfrak{T} \widehat{\otimes}_{\mathbb{C}_\infty} R).$$

We still need to justify that $f \in M_w(\rho; \mathbb{L}_\Sigma) \otimes_{\mathbb{L}_\Sigma} \mathbb{L}_{\Sigma'}$. For this, note that the identity (43) has the left-hand side in $\mathbb{L}_{\Sigma'}^{1 \times N}$ hence (44) has the left-hand side in $\mathbb{L}_{\Sigma'}^{1 \times r}$. Therefore ϕ_1, \dots, ϕ_r extend, as elements of $\mathrm{Hol}_{\mathbb{K}_\Sigma}(S \rightarrow \mathbb{L}_\Sigma)$ to elements of $\mathbb{L}_{\Sigma'}$ and we are done. \square

5.2. Finiteness results. We suppose, along this subsection, that the representation $\rho : \Gamma \rightarrow \mathrm{GL}_N(k_\Sigma)$ is of the first kind. We also recall that \mathfrak{K}_Σ is the completion of the fraction field of $\mathcal{T}(\mathbb{K}_\Sigma)$ for the valuation v , and that $\mathfrak{D}_\Sigma, \mathfrak{M}_\Sigma$ are respectively the valuation ring and the maximal ideal of v . We have the following results which corresponds to part (1) of Theorem B in the introduction:

Theorem 5.4 (Finiteness Theorem). *The \mathbb{L}_Σ -vector space $M_w(\rho; \mathbb{L}_\Sigma)$ has finite dimension $r_\rho(w)$ not exceeding $(1 + \lfloor \frac{w}{q+1} \rfloor)N$.*

In particular, if $w < 0$, then $r_\rho(w) = 0$ and $M_w(\rho; \mathbb{L}_\Sigma) = \{0\}$ but this property will be actually proved separately to obtain the general result. The proof of this theorem makes use of an important feature of our Drinfeld modular forms when they take values in \mathbb{L}_Σ ; the possibility of evaluating the variables t_i ($i \in \Sigma$) at roots of unity. This will be the subject of the next subsection. In the subsequent subsection, we will prove Theorem 5.4 by using that the spaces of modular forms of negative weight are trivial, which comes from a corresponding result of classical (scalar) Drinfeld modular forms for congruence subgroups of Γ .

5.2.1. Evaluating at roots of unity. The representation of the first kind ρ is constructed starting from a finite set of basic representations which are themselves associated with injective \mathbb{F}_q -algebra morphisms $\chi_i : A \rightarrow k_\Sigma$ ($i = 1, \dots, r$). If $d_1, \dots, d_r \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$ are such that the entries of $d_i \chi_i(\theta)$ are in $\mathbb{F}_q[t_\Sigma]$ then the image of ρ is in $\mathrm{GL}_N(\mathbb{F}_q[t_\Sigma][\frac{1}{d_1}, \dots, \frac{1}{d_r}]) \subset \mathrm{GL}_N(\mathbb{F}_q[t_\Sigma][\frac{1}{d}])$ for some $d \in \mathbb{F}_q[t_\Sigma] \setminus \{0\}$. We thus get, after Proposition 4.9, that

$$\Xi_\rho, \Phi_\rho \in \mathrm{Hol}(\mathbb{C}_\infty \rightarrow \widehat{\mathbb{E}_\Sigma[d^{-1}]}^{N \times N}).$$

Let $\Sigma = U \sqcup V$ be a finite subset of \mathbb{N}^* written as a disjoint union of sets U, V . The set

$$\mathcal{V}_U(d) = \{\underline{\zeta} \in (\mathbb{F}_q^{ac})^U : d(\underline{\zeta}) = 0\}$$

is contained in a proper hypersurface of $(\mathbb{F}_q^{ac})^U$. Let $\underline{\zeta} = (\zeta_i : i \in U)$ be an element of $(\mathbb{F}_q^{ac})^U$ over which d does not vanish.

The *evaluation map*

$$\mathrm{ev}_\zeta : \widehat{\mathbb{T}_\Sigma[d^{-1}]} \rightarrow \widehat{\mathbb{T}_V[\mathrm{ev}_\zeta(d)^{-1}]}$$

is the \mathbb{T}_V -algebra morphism uniquely determined by the assignment $t_i \mapsto \zeta_i$ for $i \in U$. We extend this map to matrices with entries in $\mathbb{T}_\Sigma[d^{-1}]^\wedge$. If M is a matrix with entries in $\mathbb{T}_\Sigma[d^{-1}]^\wedge$ we also write $M(\underline{\zeta}) = \mathrm{ev}_\zeta(M)$. It is easy to see that if X is an analytic space and $f \in \mathrm{Hol}(X \rightarrow \mathbb{T}_\Sigma[d^{-1}]^\wedge)$, then $\mathrm{ev}_\zeta(f) \in \mathrm{Hol}(X \rightarrow \mathbb{T}_V[d(\underline{\zeta})^{-1}]^\wedge)$. Moreover:

Lemma 5.5. *Let X be a rigid analytic space over \mathbb{C}_∞ . If $f \in \mathrm{Hol}(X \rightarrow \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$ and if for all $\underline{\zeta} \in (\mathbb{F}_q^{ac})^U \setminus \mathcal{V}_U(d)$, $\mathrm{ev}_\zeta(f) \in \mathrm{Hol}(X \rightarrow \mathbb{T}_V[\frac{1}{d(\underline{\zeta})}]^\wedge)$ is constant, then f is constant.*

Proof. This is obvious if X is the polydisk $D_{\mathbb{C}_\infty}(0, 1)^n$. Indeed, if $f \in \mathrm{Hol}(X \rightarrow \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge)$, then we can write in a unique way $f(z) = \sum_{\underline{i} \in \mathbb{N}^n} f_{\underline{i}} z^{\underline{i}}$ (multi-indices notation) with $f_{\underline{i}} \in \mathbb{T}_\Sigma[\frac{1}{d}]^\wedge$, $f_{\underline{i}} \rightarrow 0$. Evaluating, we get $f(\underline{\zeta})(z) = \sum_{\underline{i}} f_{\underline{i}}(\underline{\zeta}) z^{\underline{i}}$ with $f_{\underline{i}}(\underline{\zeta}) \rightarrow 0$ in \mathbb{K}_V . But such a function is constant only if all the coefficients are zero except, possibly, the constant term. The hypotheses imply $f_{\underline{i}}(\underline{\zeta}) = 0$ for all $\underline{i} \in \mathbb{N}^n \setminus \{0\}$ for all $\underline{\zeta} \in (\mathbb{F}_q^{ac})^U \setminus \mathcal{V}_U(d)$ which is Zariski-dense in $(\mathbb{F}_q^{ac})^U$ so that $f_{\underline{i}} = 0$ for all such \underline{i} . From this we also deduce the statement for X affinoid (use quotients of standard Tate algebras). This suffices to prove the lemma. \square

Let \mathfrak{n} be an element of A^+ . We denote by $\Gamma(\mathfrak{n})$ the principal congruence subgroup of Γ :

$$\Gamma(\mathfrak{n}) = \{\gamma \in \Gamma : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}}\}.$$

We recall that $\rho : \Gamma \rightarrow \mathrm{GL}_N(\mathbb{F}_q[t_\Sigma][d^{-1}])$ is a representation of the first kind.

Lemma 5.6. *Let $\underline{\zeta} = (\zeta_i : i \in \Sigma)$ be an element of $(\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$. There exists $\mathbf{n} \in A^+$ such that for all $\gamma \in \Gamma(\mathbf{n})$, $\rho(\gamma)(\underline{\zeta}) = I_N$.*

Proof. By definition, ρ is build on a finite set of basic representations $\rho_{\chi_1}, \dots, \rho_{\chi_r}$. The evaluations $\eta_i := \chi_i(\theta)(\underline{\zeta})$ are well defined in $(\mathbb{F}_q^{ac})^{n_i \times n_i}$. We consider the minimal polynomials $P_i \in \mathbb{F}_q[X]$ of η_i (for all i) and we denote by $\mathbf{n}_i \in A^+$ the element $(P_i)_{X=\theta}$. If $\gamma \in \Gamma(\mathbf{n}_i)$, then $\rho_{\chi_i}(\gamma) = I_{n_i}$. Since the operations which construct all the representations of the first kind send m -tuples of identity matrices (of various sizes) to identity matrices, considering \mathbf{n} the least common multiple of the elements \mathbf{n}_i , we see that $\rho(\gamma)(\underline{\zeta}) = I_N$ for all $\gamma \in \Gamma(\mathbf{n})$. \square

Let G be a congruence subgroup of Γ . The quotient space $G \backslash \Omega$ carries a natural structure of analytic curve Y_G with compactification X_G obtained by adding finitely many cusps to Y_G . We can consider neighbourhoods of a cusp of $G \backslash \Omega$ in Ω in the usual way and therefore, there is a natural notion of modular-like forms $f : \Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ for ρ , seen as a representation of G (restriction), namely, satisfying the collection of functional equations

$$(45) \quad f(\gamma(z)) = J_\gamma(z)^w \rho(\gamma) f(z) \quad \forall z \in \Omega, \quad \forall \gamma \in G.$$

Let c be a cusp of X_G and let us consider $\delta \in \Gamma$ such that $\delta(\infty) = c$. If $f : \Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ is a map and w an integer, we set

$$f^\delta(z) := f|_{w,0,\rho}\delta = J_\delta(z)^{-w} \rho(\delta)^{-1} f(\delta(z))$$

(Peterson slash operator as in (40)). A simple computation shows that if f is modular-like of weight w for G and the restriction ρ on G , then $f^\delta : \Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ is modular-like of weight w for $G^\delta := \delta^{-1}G\delta$ and the restriction ρ on G^δ (in particular, if f is modular-like for the group Γ , then $f = f^\delta$).

Definition 5.7. *Let w be in \mathbb{Z} . We say that a modular-like form $\Omega \xrightarrow{f} \mathbb{L}_\Sigma^{N \times 1}$ of weight w for the restriction of ρ over G is:*

- (1) A *weak Drinfeld modular form of weight w for ρ* if there exists $H \in \mathbb{Z}$ such that

$$\|u(z)^H f^\delta(z)\| \rightarrow 0$$

for $z \in \Omega$ such that $|u(z)| \neq 0$ is smaller than a constant < 1 , for all $\delta \in \Gamma$.

- (2) A *Drinfeld modular form of weight w for ρ* , if $\|f^\delta(z)\|$ is bounded as z is such that $|u(z)| \neq 0$ is smaller than a constant < 1 , for all $\delta \in \Gamma$.
- (3) A *cusp form of weight w for ρ* is a modular form f of weight w such that $\|f(z)\| \rightarrow 0$ as z approaches any cusp of $G \backslash \Omega$.

We denote by $M_w^!(G; \rho; \mathbb{L}_\Sigma)$ (resp. $M_w(G; \rho; \mathbb{L}_\Sigma), S_w(G; \rho; \mathbb{L}_\Sigma)$) the \mathbb{L}_Σ -vector spaces of weak modular forms (resp. modular forms, cusp forms) of weight w for ρ . More generally, if B is a Banach \mathbb{C}_∞ -subalgebra of \mathbb{K}_Σ , we will write $M_w(G; \rho; B)$ for the corresponding B -module of modular forms.

It is easy to see that the \mathbb{C}_∞ -vector space $M_w(G; \mathbf{1}; \mathbb{C}_\infty)$ is equal to the \mathbb{C}_∞ -vector space of the classical (scalar) Drinfeld modular forms of weight w for G and a similar property holds for weak modularity and cuspidality of a form. In the next proposition, W_w stands for $M_w^!, M_w, S_w$ (so the proposition is in fact equivalent to three distinct statements).

Proposition 5.8. *Let f be in $W_w(\rho; \mathbb{L}_\Sigma)$. Then, there exists $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that $f \in W_w(\rho; \widehat{\mathbb{T}_\Sigma[\frac{1}{d}]})$. Let us consider, further, $\underline{\zeta} \in (\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$. We have $\text{ev}_{\underline{\zeta}}(f) \in W_w(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}$ where \mathbf{n} is any element as in Lemma 5.6.*

Hence, the evaluations of the entries of $f \in M_w(\rho; \mathbb{L}_\Sigma)$ are scalar Drinfeld modular forms of weight w for $\Gamma(\mathbf{n})$.

Proof of Proposition 5.8. By Lemma 5.6, for all $\gamma \in \Gamma(\mathbf{n})$ and $z \in \Omega$, $f(\underline{\zeta})(\gamma(z)) = J_\gamma(z)^w f(\underline{\zeta})(z)$ and also, it is easy to see that $f(\underline{\zeta})$ has holomorphic entries. It remains to show that the entries of $f(\underline{\zeta}) = \text{ev}_{\underline{\zeta}}(f)$ have the decay properties of Definition 5.7 which is guaranteed if we show regularity at all cusps of $G \setminus \Omega$. In more detail, if f has image defined over $\widehat{\mathbb{T}_\Sigma[\frac{1}{d}]}$, we show that the map $\text{ev}_{\underline{\zeta}}(\cdot)$ defines maps (\mathbb{C}_∞ -linear maps)

$$(46) \quad M_w^!(\rho; \widehat{\mathbb{T}_\Sigma[d^{-1}]}) \rightarrow M_w^!(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1},$$

$$(47) \quad M_w(\rho; \widehat{\mathbb{T}_\Sigma[d^{-1}]}) \rightarrow M_w(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1},$$

$$(48) \quad S_w(\rho; \widehat{\mathbb{T}_\Sigma[d^{-1}]}) \rightarrow S_w(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}.$$

First of all, a holomorphic function $f : \Omega \rightarrow \mathbb{C}_\infty$ satisfying $f(\gamma(z)) = J_\gamma(z)^w f(z)$ for all $\gamma \in \Gamma(\mathbf{n})$ is a weak modular form of weight w for $\Gamma(\mathbf{n})$ if for all $\delta \in \Gamma$, the function $f^\delta(z)$ can be expanded as a series of $\mathbb{C}_\infty((u(\frac{z}{\mathbf{n}})))$ in the neighborhood of the cusp $\delta(\infty)$. We deduce that $f^\delta(z)$ is weak modular form of weight w for the group $\delta^{-1}\Gamma(\mathbf{n})\delta$. Note indeed that $u(\frac{z}{\mathbf{n}})$ is a uniformizer at ∞ for the action of $\Gamma(\mathbf{n})$ over Ω in virtue of the fact that the group $(\begin{smallmatrix} 1 & \mathbf{n} \\ 0 & 1 \end{smallmatrix})$ is contained in $\delta^{-1}\Gamma(\mathbf{n})\delta$ for all $\delta \in \Gamma$.

Let F be in $M_w^!(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$. Then, $\text{ev}_{\underline{\zeta}}(F)$ has all the entries which are nA -periodic and $\text{ev}_{\underline{\zeta}}(F^\delta)$ is tempered for all $\delta \in \Gamma$. This implies that $\text{ev}_{\underline{\zeta}}(F) \in M_w^!(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}$ which proves (46). Now assume that F is, additionally, a modular form in $M_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$. Then, all the entries b^δ of $\text{ev}_{\underline{\zeta}}(F^\delta)$ satisfy $b^\delta \in \mathbb{C}_\infty[[u(\frac{z}{\mathbf{n}})]]$ for all $\delta \in \Gamma$, which yields (47). Similarly, if F is in $S_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$, we see that all the entries of $\text{ev}_{\underline{\zeta}}(F)$ vanish at all the cusps of $X(\mathbf{n})$ hence confirming (48) and completing the proof of the Proposition. \square

Question 5.9. *Compute the span in $M_w(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)$ of all the modular forms which occur as an entry of the evaluation at some element of $(\mathbb{F}_q^{ac})^\Sigma$ of some element of $M_w(\rho; \mathbb{T}_\Sigma[d^{-1}]^\wedge)$ for some Σ and for some representation of the first kind ρ . For which \mathbf{n} do we obtain the whole space?*

5.2.2. *Proof of the Finiteness Theorem.* We first study the structure of the space $M_0(\rho; \mathbb{L}_\Sigma)$.

Lemma 5.10. *We have $M_0(\rho; \mathbb{L}_\Sigma) \subset \mathbb{L}_\Sigma^{N \times 1}$.*

Proof. Let f be an element of $M_0(\rho; \mathbb{L}_\Sigma)$. Then, there exists $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$ such that the image of f is in $\widehat{\mathbb{T}_\Sigma[\frac{1}{d}]}$. By Proposition 5.8, for all $\underline{\zeta} \in (\mathbb{F}_q^{ac})^\Sigma \setminus \mathcal{V}_\Sigma(d)$ there exists $\mathbf{n} \in A^+$ such that $f(\underline{\zeta}) \in M_0(\Gamma(\mathbf{n}); \mathbf{1}; \mathbb{C}_\infty)^{N \times 1}$. A scalar Drinfeld modular form of weight zero is constant. Hence, for all $\underline{\zeta}$ as above, $f(\underline{\zeta}) \in \mathbb{C}_\infty^{N \times 1}$. Therefore, f is constant by Lemma 5.5 with $X = \Omega$. \square

Corollary 5.11. *If $w < 0$, $M_w(\rho; \mathbb{L}_\Sigma) = \{0\}$.*

Proof. Let f be an element of $M_w(\rho; \mathbb{L}_\Sigma)$ with negative w . For all $k, \alpha, \beta \in \mathbb{N}$ with $\beta > 0$, $\tilde{f} := g^\alpha h^\beta \tau^k(f) \in S_{q^k w + \alpha(q-1) + \beta(q+1)}(\rho \det^{-\beta}; \mathbb{L}_\Sigma)$, where g is the normalised Eisenstein series in $M_{q-1}(\mathbf{1}; \mathbb{C}_\infty)$ and h is -1 times the normalised generator of $S_{q+1}(\det^{-1}; \mathbb{C}_\infty)$ (we are adopting Gekeler's notations in [15], see also §5.3.1). We show that there exist k, α, β with $\beta > 0$ such that

$$(49) \quad q^k w + \alpha(q-1) + \beta(q+1) = 0.$$

This is very easy but we give all the details. To find such k, α, β , we first observe that we need $q^k w + \alpha(q-1) + \beta(q+1) \equiv 0 \pmod{q-1}$, and this is guaranteed by $w \equiv -2\beta \pmod{q-1}$. We must have:

$$\begin{aligned} \alpha &= \frac{1}{q-1}(-wq^k - \beta(q+1)) \\ &= \frac{1}{q-1}(-wq^k - 2\beta) + \beta. \end{aligned}$$

Assume first that $p \neq 2$. Then, there exists $\beta \in \{1, \dots, q-1\}$ such that $w \equiv -2\beta \pmod{q-1}$. We can choose k large enough so that $-wq^k - 2\beta$, divisible by $q-1$, is ≥ 0 . Therefore we can choose $\alpha \in \mathbb{N}$ such that, with such β and k , (49) holds.

If $p = 2$ we can set $\beta = 1$ and k such that $\alpha = -2^k w - 3 \geq 0$. Since $\beta > 0$ we see that \tilde{f} is a cusp form and Lemma 5.10 now implies that $\tilde{f} = 0$; hence $f = 0$ because τ is injective. \square

Proof of Theorem 5.4. The result is already proved in Lemma 5.10 and Corollary 5.11 if $w \leq 0$. Now assume that $w > 0$ and let f be in $M_w(\rho; \mathbb{L}_\Sigma)$. Again, we can suppose that $f \in M_w(\rho; \widehat{\mathbb{T}_\Sigma[\frac{1}{d}]})$ for some $d \in \mathbb{F}_q[\underline{t}_\Sigma] \setminus \{0\}$.

Then, $f = \Phi_\rho g$, where Φ_ρ has been defined in §4.1.1 and studied in Proposition 4.9, and where g is in $\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge[u^{-1}][[u]]^{N \times 1}$ (by using the same methods of proof of (b) of Proposition 4.9). Recall from Proposition 4.9 and Corollary 4.10 that $\Phi_\rho \in \mathrm{GL}_N(\mathcal{T}^\circ(\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge))$ (this means that it is a matrix with entries in $\mathcal{T}^\circ(\dots)$ which is invertible, and the entries of the inverse are in $\mathcal{T}^\circ(\dots)$). Also, by the fact that f is quasi-periodic and regular, we have $f \in \mathfrak{D}_{\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge}^{N \times 1}$ by Theorem 4.14. Since the v -valuations of the entries of Φ_ρ^{-1} are either ∞ or in $\mathbb{Z}[\frac{1}{p}] \cap [0, 1[$, we get, from $g = \Phi_\rho^{-1} f$, $g \in (\mathbb{T}_\Sigma[\frac{1}{d}]^\wedge[[u]])^{N \times 1}$.

Let ν be in \mathbb{F}_q^\times . We have

$$\rho\left(\begin{smallmatrix} \nu & 0 \\ 0 & 1 \end{smallmatrix}\right) \Phi_\rho(z) \rho\left(\begin{smallmatrix} \nu & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} g(\nu z) = f(\nu z) = \nu^w \rho\left(\begin{smallmatrix} \nu & 0 \\ 0 & 1 \end{smallmatrix}\right) f(z) = \nu^w \rho\left(\begin{smallmatrix} \nu & 0 \\ 0 & 1 \end{smallmatrix}\right) \Phi_\rho(z) g(z), \quad \forall z \in \Omega.$$

Since ρ is of the first kind, $\rho\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal and we can write:

$$\rho\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \nu^{-n_1} & & \\ & \ddots & \\ & & \nu^{-n_N} \end{pmatrix}, \quad n_i \in \mathbb{Z}/(q-1)\mathbb{Z}, \quad \nu \in \mathbb{F}_q^\times.$$

Writing additionally $g = {}^t(g_1, \dots, g_N)$, we deduce that

$$g_i(\nu z) = \nu^{w-n_i} g_i(z)$$

for all $i = 1, \dots, N$, so that $g_i \in u^{m_i} \mathbb{T}_\Sigma[\frac{1}{q}]^\wedge[[u^{q-1}]]$ where m_i is the unique representative of $n_i - w$ modulo $q - 1$ in $\{0, \dots, q - 2\}$. We deduce that there is a vector space W_w over \mathbb{L}_Σ of dimension at most N such that

$$M_w(\rho; \mathbb{L}_\Sigma) = hM_{w-(q+1)}(\rho \det; \mathbb{L}_\Sigma) \oplus W_w.$$

By induction over w , the result follows. \square

5.2.3. Modular forms of weight one. Since $T_a^p = I_2$ we deduce that for all $a \in A$ there exists $N_a \in k^{N \times N}$:

$$(50) \quad \rho(T_a) = I_N + N_a,$$

with $N_a N_b = N_b N_a$ and $N_{a+b} = N_a + N_b + N_a N_b$ (and N_a is nilpotent of exponent p). On the other hand, $\rho(S)$ is involutory, and diagonalisable if $p > 2$.

By Lemma 5.10 and by (50), we see that $M_0(\rho; \mathbb{L}_\Sigma)$ is contained in the \mathbb{K}_Σ -span $H(\rho)$ in $\mathbb{K}_\Sigma^{N \times 1}$ of the common eigenvectors in $\mathbb{K}_\Sigma^{N \times 1}$ of eigenvalue in \mathbb{F}_q^\times for the left multiplication by the matrices $\rho(\gamma)$, $\gamma \in B(A)$, the Borel subgroup of A . The eigenvalue is one for $\gamma = T_a$. In the following, we denote by $\delta_\rho(0)$ the dimension of this space.

We can now prove the following result which justifies part (2) of Theorem B in the introduction:

Theorem 5.12. *We have the inequality $\dim_{\mathbb{L}_\Sigma}(M_1(\rho; \mathbb{L}_\Sigma)) \leq \delta_\rho(0)$.*

Proof. Let $f = f(z)$ be in $\mathfrak{D}_\Sigma^{N \times 1}$. Then, for all $a \in A$ and $\lambda \in \mathbb{F}_q^\times$ we have that the function $f(\lambda z + a)$ is well defined in $\mathfrak{R}_\Sigma^{N \times 1}$ and again belongs to $\mathfrak{D}_\Sigma^{N \times 1}$ (this follows from Lemma 3.15). Similarly, if now $f \in \mathfrak{M}_\Sigma^{N \times 1}$, then again $f(\lambda z + a) \in \mathfrak{M}_\Sigma^{N \times 1}$. If $b \in \mathfrak{D}_\Sigma$, we denote by $\bar{b}_v \in \mathbb{L}_\Sigma$ its reduction modulo \mathfrak{M}_Σ (constant term of the tame series expansion) and we extend this to matrices in the obvious way. Then, for any $f \in M_w(\rho; \mathbb{L}_\Sigma)$ (temporarily for any $w \in \mathbb{Z}$) we have that the well defined element $\bar{f}_v \in \mathbb{L}_\Sigma^{N \times 1}$ belongs to the above defined space $H(\rho)$ (observe that the entries are in \mathbb{L}_Σ , not just in \mathbb{K}_Σ). Assume by contradiction that $\dim_{\mathbb{L}_\Sigma} M_1(\rho; \mathbb{L}_\Sigma) > \delta_\rho(0)$. then, there is a non-zero cusp form in $S_1(\rho; \mathbb{L}_\Sigma)$.

Let $\underline{\zeta} = (\zeta_i; i \in \Sigma)$ and \mathfrak{n} as in Proposition 5.8, let f be a non-zero cusp form of $S_1(\rho; \mathbb{L}_\Sigma)$. this proposition implies that for all such $\underline{\zeta}$, the evaluation $\text{ev}_{\underline{\zeta}}(f)$ is well defined and its entries are cusp forms of $S_1(\Gamma(\mathfrak{n}))$. The latter space is zero as it was first noticed by Gekeler (see Cornelissen, in [11, Theorem (1.10)]). Hence, for all $\underline{\zeta}$ as above, $\text{ev}_{\underline{\zeta}}(f) = 0$. By Lemma 5.5, f vanishes identically, contradicting our assumptions. \square

A more precise result in a particular case is Theorem 7.6.

5.3. Poincaré series. In the last part of this section we construct modular forms in our generalised setting. We are mainly concerned with a class of matrix-valued Poincaré series.

We keep considering a representation

$$\Gamma \xrightarrow{\rho} \mathrm{GL}_N(k_\Sigma).$$

Let w be an integer and let $G : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$ be a tempered ρ -quasi-periodic matrix function of type m , following definition 4.2. We set, for $\gamma \in \Gamma$ and $z \in \Omega$:

$$S_\gamma(w, m; G)(z) = \det(\gamma)^m J_\gamma(z)^{-w} \rho(\gamma)^{-1} G(\gamma(z)) \rho \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 5.13. *Let $\gamma, \gamma' \in \Gamma$ be in the same left coset modulo $H := \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma$. Then we have the equality $S_\gamma(w, m; G)(z) = S_{\gamma'}(w, m; G)(z)$. Moreover, for all $\delta \in \Gamma$,*

$$S_\gamma(w, m; G)(\delta(z)) = \det(\delta)^{-m} J_\delta(z)^w \rho(\delta) S_{\gamma\delta}(w, m; G)(z) \rho \begin{pmatrix} \det(\delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We simplify the notation: $S_\gamma(w, m; G)(z) = S_\gamma(z)$. We prove the first property. Since H is the semidirect product of A by \mathbb{F}_q^\times , it suffices to show that: (1) for all $a \in A$, $S_{T_a\gamma}(z) = S_\gamma(z)$ and (2) for all $\nu \in \mathbb{F}_q^\times$, $S_{\delta\gamma}(z) = S_\gamma(z)$ if $\delta = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$. For (1), we observe, by the properties of G , that

$$\begin{aligned} S_{T_a\gamma}(z) &= \det(T_a\gamma)^m J_{T_a\gamma}(z)^{-w} \rho(T_a\gamma)^{-1} G(T_a(\gamma(z))) \rho \begin{pmatrix} \det(T_a\gamma) & 0 \\ 0 & 1 \end{pmatrix} \\ &= \det(\gamma)^m J_\gamma(z)^{-w} \rho(\gamma)^{-1} \rho(T_a)^{-1} \rho(T_a) G(\gamma(z)) \rho \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix} \\ &= S_\gamma(z). \end{aligned}$$

For (2), we see, similarly, with $\delta = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{aligned} S_{\delta\gamma}(z) &= \det(\delta\gamma)^m J_{\delta\gamma}(z)^{-w} \rho(\delta\gamma)^{-1} G(\delta(\gamma(z))) \rho \begin{pmatrix} \det(\delta\gamma) & 0 \\ 0 & 1 \end{pmatrix} \\ &= \det(\gamma)^m J_\gamma(z)^{-w} \rho(\gamma)^{-1} \rho(\delta)^{-1} \det(\delta)^m \det(\delta)^{-m} \rho(\delta) G(\gamma(z)) \rho(\delta)^{-1} \rho \begin{pmatrix} \det(\delta\gamma) & 0 \\ 0 & 1 \end{pmatrix} \\ &= S_\gamma(z). \end{aligned}$$

This completes the proof of the first part of the Lemma. For the second, observe, if $\gamma' = \gamma\delta$ with $\delta \in \Gamma$:

$$\begin{aligned} S_\gamma(\delta(z)) &= \det(\gamma)^m J_\gamma(\delta(z))^{-w} \rho(\gamma)^{-1} G(\gamma(\delta(z))) \rho \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix} \\ &= \det(\delta)^{-m} \det(\gamma')^m J_\delta(z)^w J_{\gamma'}(z)^{-w} \rho(\gamma'\delta^{-1})^{-1} G(\gamma'(z)) \rho \begin{pmatrix} \det(\gamma') & 0 \\ 0 & 1 \end{pmatrix} \rho \begin{pmatrix} \det(\delta) & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \det(\delta)^{-m} J_\delta(z)^w \rho(\delta) S_{\gamma'}(z) \rho \begin{pmatrix} \det(\delta) & 0 \\ 0 & 1 \end{pmatrix}^{-1}. \end{aligned}$$

□

From now on, we suppose that ρ is of the first kind, of degree l and that G is of type m . We consider the formal series (Poincaré series):

$$(51) \quad \mathcal{P}_{w,m}(G)(z) := \sum_{\gamma} S_\gamma(w, m; G)(z),$$

where the sum runs over a complete set of representatives of $H \backslash \Gamma$. Note that this is a matrix function.

Remark 5.14. Our definition of Poincaré series is inspired by the one that can be found in the book of Bruinier [6, §1.2, 1.3].

We have the next result.

Proposition 5.15. *If the series $\mathcal{P}_{w,m}(G)(z)$ converges to a holomorphic function $\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$, then it satisfies, for all $z \in \Omega$ and $\gamma \in \Gamma$:*

$$\mathcal{P}_{w,m}(G)(\gamma(z)) = \det(\gamma)^{-m} J_\gamma(z)^w \rho(\gamma) \mathcal{P}_{w,m}(G)(z) \rho \left(\begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right).$$

For each column f of $\mathcal{P}_{w,m}(G)$ there exists $i \in \mathbb{Z}/(q-1)\mathbb{Z}$ such that

$$f(\delta(z)) = \det(\delta)^{i-m} J_\delta(z)^w \rho(\delta) f(z), \quad \forall z \in \Omega, \quad \delta \in \Gamma.$$

Proof. The first part of Lemma 5.13 implies that the sum $\mathcal{P}_{w,m}(G)$, if convergent, is well defined, and the second part describes the functional behaviour. We note that $\rho \left(\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \right)$ is diagonal in $\mathrm{GL}_N(\mathbb{F}_q)$ and we can decompose, up to permutation of the columns of $\mathcal{P}_{w,m}(G)$:

$$\mathcal{P}_{w,m}(G) = \bigoplus_{i=0}^{q-1} \mathcal{P}_{w,m}^{(i)}(G),$$

where $\mathcal{P}_{w,m}^{(i)}(G) : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times n_i}$ for integers n_i such that $\sum_i n_i = N$. Then, if the series $\mathcal{P}_{w,m}(G)$ converges, we have the modular-like behaviour

$$\mathcal{P}_{w,m}^{(i)}(G)(\delta(z)) = \det(\delta)^{i-m} J_\delta(z)^w \rho(\delta) \mathcal{P}_{w,m}^{(i)}(G), \quad \forall z \in \Omega, \quad \delta \in \Gamma, \quad i = 0, \dots, q-1.$$

□

In full generality (for any quasi-periodic function G), we do not have a good criterion of convergence for the series $\mathcal{P}_{w,m}(G)$. We discuss these series for two choices of G .

We will need the next Lemma; see [17].

Lemma 5.16. *There exists a complete set of representatives $\gamma_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ of $H \backslash \Gamma$ in which each matrix belongs to one of the following three types:*

- (1) $\gamma_{0,\mu} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$ with $\mu \in \mathbb{F}_q^\times$,
- (2) $\gamma_{\mu,\nu} = \begin{pmatrix} 0 & -\mu^{-1} \\ \mu & \nu \end{pmatrix}$ with $\mu \in \mathbb{F}_q^\times$ and $\nu \in \mathbb{F}_q$,
- (3) $\gamma_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in A$ such that $ad - bc = 1$, $|cd| > 1$, $|a| < |c|$, $|b| < |d|$.

We note that the first two sets are finite. Let us look at the corresponding sub-sums in the series (51) defining $\mathcal{P}_{w,m}(G)$. For the first sub-sum we have, in virtue of the fact that

G is of type m and that ρ is of degree l :

$$\begin{aligned}
\mathfrak{A} &:= \sum_{\mu \in \mathbb{F}_q^\times} S_{\gamma_{0,\mu}}(z) = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{-w} \rho(\gamma_{0,\mu})^{-1} G(\mu^{-2}z) \rho \begin{pmatrix} \det(\gamma_{0,\mu}) & 0 \\ 0 & 1 \end{pmatrix} \\
&= \sum_{\mu \in \mathbb{F}_q^\times} \mu^{-w} \rho \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}^{-1} G(\mu^{-2}z) \\
&= \sum_{\mu \in \mathbb{F}_q^\times} \mu^{-w} \rho \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}^{-1} \mu^{2m} \rho \begin{pmatrix} \mu^{-2} & 0 \\ 0 & 1 \end{pmatrix} G(z) \rho \begin{pmatrix} \mu^{-2} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\
&= \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w} \rho \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} G(z) \rho \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix} \\
&= G(z) \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} \rho \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

For the second sub-sum we have:

$$\begin{aligned}
\mathfrak{B} &:= \sum_{\substack{\mu \in \mathbb{F}_q^\times \\ \nu \in \mathbb{F}_q}} S_{\gamma_{\mu,\nu}}(z) = \\
&= \sum_{\substack{\mu \in \mathbb{F}_q^\times \\ \nu \in \mathbb{F}_q}} (\mu z + \nu)^{-w} \rho(\gamma_{\mu,\nu})^{-1} G \left(\frac{-\mu^{-1}}{\mu z + \nu} \right) \\
&= \sum_{\substack{\mu \in \mathbb{F}_q^\times \\ \nu \in \mathbb{F}_q}} \mu^{-w} \left(z + \frac{\nu}{\mu} \right)^{-w} \rho \begin{pmatrix} \nu & \mu^{-1} \\ -\mu & 0 \end{pmatrix} (-\mu^2)^m \rho \begin{pmatrix} -\mu^{-2} & 0 \\ 0 & 1 \end{pmatrix} G \left(\frac{1}{z + \frac{\nu}{\mu}} \right) \rho \begin{pmatrix} -\mu^2 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (-1)^m \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} \sum_{\nu \in \mathbb{F}_q} \left(z + \frac{\nu}{\mu} \right)^{-w} \rho \begin{pmatrix} -\frac{\nu}{\mu} & 1 \\ 1 & 0 \end{pmatrix} G \left(\frac{1}{z + \frac{\nu}{\mu}} \right) \rho \begin{pmatrix} -\mu^2 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (-1)^m \sum_{\beta \in \mathbb{F}_q} (z + \beta)^{-w} \rho \begin{pmatrix} -\beta & 1 \\ 1 & 0 \end{pmatrix} G \left(\frac{1}{z + \beta} \right) \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} \rho \begin{pmatrix} -\mu^2 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

It is easy to see that these sums define holomorphic functions over Ω (the sums are finite).

We denote the third (infinite) sum by \mathfrak{C} so that $\mathcal{P}_{w,m}(G) = \mathfrak{A} + \mathfrak{B} + \mathfrak{C}$.

We study the case $G = \Phi_\rho$.

Proposition 5.17. *If $G = \Phi_\rho$ and $w > 0$, the columns of $\mathcal{P}_{w,m}(G)$ are elements of $M_w^l(\rho \det^l; \mathbb{L}_\Sigma)$ for i varying in $\mathbb{Z}/(q-1)\mathbb{Z}$. If \mathfrak{A} is non-zero, the matrix function $\mathcal{P}_{w,m}(G)$ is not identically zero.*

Proof. The functional properties of the series, under the hypotheses of its convergence, are guaranteed by Proposition 5.15. After Corollary 4.10, this function can be identified with

an element of $\mathrm{GL}_N(\mathcal{T}^\circ(\mathbb{K}_\Sigma))$ and therefore all the entries are entire functions, and there exists $c > 0$ such that for all $z \in \Omega$, $\|G(z)\| \leq \max\{c, |e_C(z)|^j\}$, for some $j \in \mathbb{Z}[\frac{1}{p}] \cap [0, 1[$.

Let us prove that $\mathcal{P}_{w,m}(G)$ converges to an analytic function $\Omega \rightarrow \mathbb{K}_\Sigma^{N \times N}$. To see this, it suffices to check that the series defining \mathfrak{C} converges uniformly on affinoids defined by inequalities $|z| \leq c_1$, $|z|_{\mathfrak{S}} \geq c_2$, where c_1, c_2 are in $|\mathbb{C}_\infty^\times|$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in Γ , such that $c \neq 0$ and let us consider $z \in \Omega$. Then:

$$(52) \quad \gamma(z) = \frac{a}{c} - \frac{\det(\gamma)}{c(cz + d)}.$$

Note that if $z \in \Omega$, then $|z - a| \geq |z|_{\mathfrak{S}}$ for all $a \in A$. If γ is one of the matrices of the family (3) of Lemma 5.16, then $|cz + d| \geq |c||z|_{\mathfrak{S}} \geq |c|c_2$ and $|\frac{a}{c}| \leq q^{-1}$. This means that the series \mathfrak{C} is uniformly convergent over any affinoid as above, depending on the choice of the constants c_1, c_2 . This proves that $\mathcal{P}_{w,m}(G)$ defines an analytic function over Ω . Also, note that by the above description of the sums $\mathfrak{B}, \mathfrak{C}$, we have that $\|\mathfrak{B}\|, \|\mathfrak{C}\|$ are bounded for $|z|_{\mathfrak{S}}$ bounded from below by a non-zero constant, and tend to zero as $|z|_{\mathfrak{S}} \rightarrow \infty$. Since G is tempered, \mathfrak{A} is tempered, and so is the function $\mathcal{P}_{w,m}(G)$. From this, we derive that if the function \mathfrak{A} is not identically zero, then $\mathcal{P}_{w,m}(G)$ does not vanish identically. It is easy to see that the image is \mathbb{L}_Σ -valued, because this is true for Φ_ρ ; this follows from the definition of our Poincaré series. \square

With $m \geq 1$, we set:

$$(53) \quad \mathbf{S}(m; \rho) := \tilde{\pi}^{-m} \sum_{a \in A} \frac{1}{(z - a)^m} \rho(T_a).$$

If $\rho = \mathbf{1} : \Gamma \rightarrow \{1\}$ we recover the (scalar) sums $S_{m,\Lambda}$ for the lattice $\Lambda = \tilde{\pi}A$ (see [19, §6] and [15, §3]). In particular, for any $m \geq 1$ there exists a polynomial $G_m \in K[X]$ (called the *Goss' polynomial* of order m) such that

$$(54) \quad \mathbf{S}(m; \mathbf{1}) = G_m(u).$$

The polynomials G_m can be computed inductively by using the generating series:

$$(55) \quad \sum_{m \geq 1} G_m(u) X^m = \frac{uX}{1 - u \exp_C(X)}$$

(see [15, (3.6)]). If $\rho \neq \mathbf{1}$, we do not know a satisfactory generalisation of (55).

We now study Poincaré series in the case $G = \mathbf{S}(m; \rho)$, with $m > 0$, defined in (53). We have:

Proposition 5.18. *If $w > 2m$ and if $G = \mathbf{S}(m; \rho)$, the columns of $\mathcal{P}_{w,m}(G)$ are in $S_w(\rho \det^{-j}; \mathbb{L}_\Sigma)$ with j varying in $\mathbb{Z}/(q-1)\mathbb{Z}$.*

Proof. We need again to show that the sum defining $\mathcal{P}_{w,m}(G)$ is uniformly convergent on any affinoid in Ω defined by $|z| \leq c_1$ and $|z|_{\mathfrak{S}} \geq c_2 > 0$, with $c_1, c_2 \in |\mathbb{C}_\infty^\times|$. For all $z \in \Omega$, $\|G(z)\| \leq |z|_{\mathfrak{S}}^{-m}$. Hence, for all $\gamma \in \Gamma$, $\|G(\gamma(z))\| \leq |J_\gamma(z)|^{2m} / |\gamma(z)|_{\mathfrak{S}}^m$ and therefore, $\|S_\gamma(w, m; G)\| \leq |J_\gamma(z)|^{2m-w}$. The fact that $w > 2m$ ensures the uniform convergence and the fact that the function is holomorphic over Ω . Since moreover, $\|G(z)\| \rightarrow 0$ as

$|z| = |z|_{\mathfrak{S}} \rightarrow \infty$, we see that the columns of $\mathcal{P}_{w,m}(G)$ are cusp forms. Moreover, these functions similarly take values in \mathbb{L}_{Σ} . \square

Giving necessary conditions for the non-vanishing of $\mathcal{P}_{w,m}(G)$ is more difficult in the case $G = \mathbf{S}(m; \rho)$, but there are cases in which it is possible to check that the series converge to non-zero functions without too much work. In the case $m = 1$ we have:

Proposition 5.19. *Let us suppose that $w \geq 3$, that ρ is of degree l , that $\rho(T_a)$ is triangular for all $a \in A$ ⁽⁴⁾ and that*

$$\sum_{\mu \in \mathbb{F}_q^\times} \mu^{2-w+l} \rho \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix} \neq 0.$$

Then, if $G = \mathbf{S}(1; \rho)$, $\mathcal{P}_{w,1}(G)$ does not vanish identically.

Proof. In the beginning of this proof, we suppose that G is of type m . By the fact that $\rho(T_a)$ is triangular for $a \in A$, G is triangular. The diagonal of G is given by a vector with entries equal to $\tilde{\pi}^{-m} \sum_{a \in A} \frac{1}{(z-a)^m}$ and since for $\mu \in \mathbb{F}_q^\times$, $\rho \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal we have that some diagonal coefficients of \mathfrak{A} are proportional to the Goss polynomial $G_m(u)$ of order m in u (see [15, §3]), with non-zero factor of proportionality.

In particular, if $m = 1$ the Goss polynomial is just the function u and there exists an entry of the diagonal of \mathfrak{A} which belongs to $\mathbb{F}_q^\times u(z)$. Now, we suppose that z is such that $1 < |z| < |\tilde{\pi}|$. Then, there is a diagonal entry of \mathfrak{A} which has absolute value $|\tilde{\pi}z|^{-1}$. Let us consider $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the family (2) or in the family (3). Then, by using (52), we see that $|\gamma(z)| < 1$. Hence, if $e \in A \setminus \{0\}$, we have $|\gamma(z) - e| \leq 1$. This means, writing $\xi = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2-w+l} \rho \begin{pmatrix} -\mu^2 & 0 \\ 0 & 1 \end{pmatrix}$, that

$$\begin{aligned} \mathfrak{B} &= \\ &= (-1)^m \tilde{\pi}^{-1} \left(\sum_{\beta \in \mathbb{F}_q} (z + \beta)^{-w} \underbrace{\rho \begin{pmatrix} -\beta & 1 \\ 1 & 0 \end{pmatrix} \sum_{b \in A \setminus \{0\}} \rho(T_b) \frac{1}{z+\beta} - b}_{\|\cdot\| \leq 1} + \sum_{\beta \in \mathbb{F}_q} (z + \beta)^{1-w} \rho \begin{pmatrix} -\beta & 1 \\ 1 & 0 \end{pmatrix} \right) \xi. \end{aligned}$$

For all $\beta \in \mathbb{F}_q$, $|z + \beta| = |z|$, and therefore, $\|\mathfrak{B}\| \leq |\tilde{\pi}^{-1}| |z|^{1-w}$ (with z in the annulus above).

Similarly, considering now the sum over $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the family (3) of Lemma 5.16, we can decompose

$$\mathfrak{C} = \tilde{\pi}^{-1} \left(\mathfrak{C}_0 + \sum_{\gamma \text{ of type (3)}} J_\gamma(z)^{-w} \rho(\gamma)^{-1} I_N \gamma(z)^{-1} \right) \tilde{\xi},$$

for some matrix $\tilde{\xi}$ with entries in \mathbb{F}_q , and with $\|\mathfrak{C}_0\| \leq |cz|^{1-w} \leq |z|^{-w}$. Now, observe that $U = J_\gamma(z)^{-w} \gamma(z)^{-1} = (cz + d)^{1-w} (az + b)^{-1}$. If $a \neq 0$ then $|az + b|^{-1} \leq |az|^{-1}$. If

⁴Lower or upper triangular.

$a = 0$ then $|az + b|^{-1} \leq 1$. Hence, if $a \neq 0$ we have $|U| \leq \frac{|c|}{|a|} |c|^{-w} |z|^{-w} \leq |z|^{-w}$. If $a = 0$, $|U| \leq |z|^{1-w}$. All together, we see that $\|\mathfrak{B} + \mathfrak{C}\| \leq \|\tilde{\pi}^{-1}\| |z|^{1-w}$. Since, as seen previously, \mathfrak{A} has coefficients on the diagonal which have absolute value equal to $|\tilde{\pi}^{-1}| |z|^{-1}$, $\mathcal{P}_{w,1}(G)(z)$ does not vanish identically. \square

5.3.1. *Example: a Poincaré series in a class introduced by Gekeler.* We consider the case $N = 1$, $\rho = \mathbf{1}$, $G(z) = u(z)$ and we take $w = q + 1 \geq 3$. In this case, we have the scalar function $\mathcal{P}_{q+1,1}(G)$ which is a non-zero cusp form of weight $q + 1$ and type 1 (that is, a non-zero element of $S_{q+1}(\det^{-1}; \mathbb{C}_\infty)$) and was studied by Gekeler in [15, (9.1)] as a first example of a class of Poincaré series that he defined, and is proportional to the cusp form h .

5.3.2. *Example: cusp forms of weight 3.* We consider $\rho = \rho_\Sigma^*$ which is of degree $s = |\Sigma|$ and $\Sigma \subset \mathbb{N}^*$. Note that

$$\rho \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} = \text{Diag}(\nu^{-s}, \dots, \nu^{-n_1}, \nu^{-n_0})$$

Hence, if $s \equiv 1 \pmod{q-1}$, $w = 3$, $m = 1$, the last entry of the diagonal of

$$\xi = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} \rho \begin{pmatrix} \mu^2 & 0 \\ 0 & 1 \end{pmatrix}$$

is $\sum_{\mu \in \mathbb{F}_q^\times} \mu^{2m-w+l} = \sum_{\mu \in \mathbb{F}_q^\times} \mu^{2-3+s} = -1$. This means that the last column $\mathcal{P}_{3,1}^{(0)}$ of $\mathcal{P}_{3,1}(G)$ with $G = \mathbf{S}(1; \rho)$ is non-zero and $\mathcal{P}_{3,1}(G)$ does not vanish identically. We deduce:

Lemma 5.20. *If $s = |\Sigma| \equiv 1 \pmod{q-1}$, then $\mathcal{P}_{3,1}^{(0)} \in S_3(\rho_\Sigma^* \det^{-1}; \mathbb{K}_\Sigma) \setminus \{0\}$ where $\mathcal{P}_{3,1}^{(0)}$ is the last column of the matrix function $\mathcal{P}_{3,1}(G)$.*

If $q = 2$, $\Sigma = \emptyset$ and $G = u$ we are back to Gekeler's example. In general, there are several non-zero columns in $\mathcal{P}_{3,1}(G)$.

6. DIFFERENTIAL OPERATORS ON MODULAR FORMS

A classical feature of modular forms for the group $\text{SL}_2(\mathbb{Z})$ is the existence of a large variety of differential operators acting homogeneously on them (that is, sending sets of modular forms to modular forms). For instance, one can mention the so-called Serre's derivatives, the Rankin-Cohen brackets etc. For scalar Drinfeld modular forms associated to the characters \det^{-m} , similar structures exist and have been partially investigated (see [7, 8]). Here we describe the natural extension of Serre's derivatives over the Drinfeld modular forms for a representation of the first kind. In order to justify the existence of such operators, we need first to show that higher derivatives leave the fields of uniformisers invariant. All along this section, we set

$$\mathcal{D}_m(z^n) = \binom{n}{m} z^{n-m}, \quad n, m \in \mathbb{N},$$

an operator which gives rise to B -linear endomorphisms of the B -algebra $\text{Hol}(X \rightarrow B)$ if $X \subset \mathbb{C}_\infty$ is an admissible open subset (e.g. $X = \Omega$), where B is a Banach \mathbb{C}_∞ -algebra as in §3.2. Note that these operators satisfy Leibnitz's identity

$$\mathcal{D}_n(fg) = \sum_{i+j=n} \mathcal{D}_i(f)\mathcal{D}_j(g),$$

for f, g analytic functions.

6.1. Higher derivatives on tame series. We study the stability of $\mathcal{T}(B) \subset \text{Hol}(\mathbb{C}_\infty \rightarrow B)$, with respect to the collection of operators $\mathcal{D} = (\mathcal{D}_n)_{n \geq 0}$. We state and prove three preliminary lemmas.

Lemma 6.1. *Let $n = n_0 + n_1q + \dots + n_vq^v \in \mathbb{N}$ with $n_0, \dots, n_v \in \{0, \dots, q-1\}$. Then, as linear operators over the entire functions $\mathbb{C}_\infty \rightarrow B$ we have the identity*

$$\mathcal{D}_n = \mathcal{D}_{n_0} \circ \mathcal{D}_{n_1q} \circ \dots \circ \mathcal{D}_{n_vq^v},$$

where the operators $\mathcal{D}_{n_iq^i}$ mutually commute, for $i = 0, \dots, v$.

Proof. The family of higher derivatives \mathcal{D} is *iterative* in the sense that

$$\mathcal{D}_{m+n} = \binom{m+n}{m} \mathcal{D}_m \circ \mathcal{D}_n = \binom{m+n}{n} \mathcal{D}_n \circ \mathcal{D}_m,$$

for all $m, n \geq 0$. The lemma follows easily by an application of Lucas' formula. \square

In the next lemma we recall that $\mathcal{T}_s(B)$ is the homogeneous subspace coming from the decomposition in direct sum (27).

Lemma 6.2. *If $M \in \mathcal{T}_s(B)$ is a tame monomial of depth s , then $\mathcal{D}_n(M)$ is a tame polynomial, and*

$$\mathcal{D}_n(M) \in \bigoplus_{i \geq 0} \mathcal{T}_{s-\ell_q(n)-i(q-1)}(B).$$

Proof. For $i \in U$, U being a finite subset of \mathbb{N}^* of cardinality s , we consider \mathbb{F}_q -linear functions $f_i \in \text{Hol}(\mathbb{C}_\infty \rightarrow B)$, so that we can write

$$f_i = \sum_{j \geq 0} f_{i,q^j} z^{q^j}, \quad f_{i,q^j} \in B, \quad i \in U.$$

By Leibnitz's formula we have for $n \geq 0$:

$$\mathcal{D}_n \left(\prod_{i \in U} f_i \right) = \sum_{i_1 + \dots + i_s = n} \prod_{k \in U} \mathcal{D}_{i_k}(f_k).$$

By \mathbb{F}_q -linearity we have that $\mathcal{D}_k(f_i) = f_i$ if $k = 0$, f_{i,q^j} if $k = q^j$ with $j \in \mathbb{N}$, and 0 otherwise. Hence, setting $f_{i,0} := f_i$, we can write:

$$(56) \quad \mathcal{D}_n \left(\prod_{i \in U} f_i \right) = \sum_{\substack{i_1 + \dots + i_s = n \\ i_k \in \{0\} \cup q^{\mathbb{N}}; \forall k}} \prod_{k \in U} f_{k,i_k},$$

if the subset of indices is non-empty, and 0 otherwise, by the usual conventions on empty sums. Since for all i , e_i is \mathbb{F}_q -linear, we deduce that for all $n \geq 0$, \mathcal{D}_n sends tame monomials on tame polynomials. Let n be in \mathbb{N}^* and consider the set of decompositions of length $r \geq 1$

$$n = \sum_{i=1}^r n_i q^i, \quad r \in \mathbb{N}, \quad n_i \in \mathbb{N}^*.$$

Then, the q -ary expansion of n (the unique one which has the coefficients $n_i \in \{0, \dots, q-1\}$) minimises the length $r = \ell_q(n)$. The lemma follows. \square

We deduce the next result, the proof of which is left to the reader, where w_{\max} has been defined in (36) (recall that if $f \in \mathcal{T}(B)$ then $f^{[i]}$ is the projection of f on $\mathcal{T}_i(B)$ of (28)):

Proposition 6.3. *With $f = \sum_i f^{[i]} \in \mathcal{T}(B)$ of depth $\leq L$ we have, for all $n \geq 1$:*

$$\mathcal{D}_n(f) = \sum_{L \geq i \geq \ell_q(n)} \mathcal{D}_n(f^{[i]}).$$

For all $n \geq 0$ and for all $f \in \mathcal{T}^\circ(B)$ of depth $\leq s$, $\mathcal{D}_n(f) \in \mathcal{T}^\circ(B)$ is of depth $\leq s - \ell_q(n)$ and of weight $\leq w_{\max}(s - \ell_q(n))$. The algebra $\mathcal{T}(B)$ is endowed with the collection of operators \mathcal{D} and with the operator τ . These operators satisfy the commutation rules

$$(57) \quad \mathcal{D}_n \tau = \begin{cases} 0 & \text{if } q \nmid n \\ \tau \mathcal{D}_{\frac{n}{q}} & \text{if } q \mid n, \end{cases} \quad n \geq 1,$$

Note that the commutation rules (57) hold on $\text{Hol}(\mathbb{C}_\infty \rightarrow B)$ and easily follow from Lemma 6.1.

We recall that the operator τ is *multiplicative*, in the sense that $v(\tau(f)) = qv(f)$ for all $f \in \mathcal{T}(B)$.

Definition 6.4. The properties illustrated in the above Proposition will be synthesised by saying that the quadruple $(\mathcal{T}(B), \tau, \mathcal{D}, v)$ is a *compatible $B[\tau, \mathcal{D}]$ -algebra*. Alternatively, it might be more convenient to consider a norm $\|\cdot\|_v$ associated to v and speak about a compatible $B[\tau, \mathcal{D}]$ -algebra $(\mathcal{T}(B), \tau, \mathcal{D}, \|\cdot\|_v)$.

Remark 6.5. The behavior of v with respect to the action of the operator τ is multiplicative. On the other hand, it is difficult to make the interaction between v and the collection of operators \mathcal{D} explicit.

As a simple consequence of Propositions 6.3 and 3.17 we deduce, with the terminology of Definition 6.4, the next corollary in which we collect the simplest properties that we need for \mathfrak{K}_L where L/K is a τ -difference field extension:

Corollary 6.6. *The field \mathfrak{K}_L is endowed with a unique extension of the operator τ and of the collection of operators \mathcal{D} , and the quadruple $(\mathfrak{K}_L, \tau, \mathcal{D}, v)$ is $L[\tau, \mathcal{D}]$ -compatible. Further, \mathfrak{K}_L carries an action of the group $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, with $\alpha \in \mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1}))^\times$ and $\beta \in \mathbb{F}_q(\underline{t}_\Sigma)((\theta^{-1}))$, given by $(\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}, f) \mapsto f(\alpha z + \beta)$, which satisfies $v(f(\alpha z + \beta)) = |\alpha|v(f)$.*

Proof. This is clear. The last property on the group action follows from Lemma 3.15. \square

6.2. Serre's derivatives. We are ready to discuss variants of *Serre's higher derivatives* introduced in [8, §1.2.3]. Following this reference, we set, for $n, w \in \mathbb{N}$ and $f \in \text{Hol}(\Omega \rightarrow \mathbb{K}_\Sigma)$:

$$\partial_n^{(w)}(f) := D_n(f) + \sum_{i=1}^n (-1)^i \binom{w+n-1}{i} D_{n-i}(f) D_i(E),$$

where $D_n = (-\tilde{\pi})^n \mathcal{D}_n$ and E is the normalized false Eisenstein series of weight 2 and type 1 of Gekeler, defined in [15, §8]. For instance, note that $\partial_1^{(w)} = D_1 - wE$ is the analogue of Ramanujan's derivative introduced by Gekeler in [15, (8.5)].

Theorem 6.7. *Let $\rho : \Gamma \rightarrow \text{GL}_N(k_\Sigma)$ be of the first kind. The operator $\partial_n^{(w)}$ induces a \mathbb{K}_Σ -linear map $M_w(\rho; \mathbb{K}_\Sigma) \rightarrow S_{w+2n}(\rho \det^{-n}; \mathbb{K}_\Sigma)$.*

Proof. If $f \in M_w(\rho; \mathbb{K}_\Sigma)$ then f can be identified with an element of $\mathfrak{A}_\Sigma^{1 \times N}$ which is, thanks to Corollary 6.6, $\partial_n^{(w)}$ -stable for all n, w . The same arguments of the proof of [8, Theorem 4.1] (which holds in a wider context of *Drinfeld quasi-modular forms*) imply that $\partial_n^{(w)}(f) \in M_{w+2n}(\rho \det^{-n}; \mathbb{K}_\Sigma)$. Further, it is easy to see that $\partial_n^{(w)}(f)$ has entries in \mathfrak{M}_Σ so it is a cusp form. \square

Note that we also have \mathbb{L}_Σ -linear maps $\partial_n^{(w)} : M_w(\rho; \mathbb{L}_\Sigma) \rightarrow S_{w+2n}(\rho \det^{-n}; \mathbb{L}_\Sigma)$.

7. MODULAR FORMS FOR THE REPRESENTATIONS ρ_Σ^*

In this section we give a closer look at modular forms associated with the representations (1). We especially focus on modular forms of weight one, of which examples are given by certain Eisenstein series which we introduce below. In Theorem 7.6 we show that they are, up to a scalar factor, the only modular forms of weight one for ρ_Σ^* . We also give some structural properties of the vector spaces $M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$. Since we consider now modular forms with values in vector spaces over \mathbb{K}_Σ , we cannot use the techniques of specialisation at roots of unity of §5.2.1. We are therefore forced to introduce other techniques which, however, are hard to apply in the more general setting of representations of the first kind.

7.1. Eisenstein series. We describe Eisenstein series for the representation $\rho = \rho_\Sigma^*$. These functions provide important examples of the modular forms we consider (see also [31]). We set, for $w \in \mathbb{N}^*$:

$$\mathcal{E}(w; \rho_\Sigma^*) := \sum'_{(a,b) \in A} (az+b)^{-w} \bigotimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(a) \\ \chi_{t_i}(b) \end{pmatrix},$$

where the sum runs over the $a, b \in A$ which are not both zero. We have that $\mathcal{E}(w; \rho_\Sigma^*)$ is not identically zero if and only if $w \equiv s \pmod{q-1}$ (see [31, §5]). It is easy to see that in this case, this series defines a holomorphic function $\Omega \rightarrow \mathbb{E}_\Sigma^{N \times 1}$, where $N = 2^s$. Further, it is a simple exercise to show that (5) holds so that $\mathcal{E}(w; \rho_\Sigma^*)$ is modular-like of weight w for ρ_Σ^* . We call $\mathcal{E}(w; \rho_\Sigma^*)$ the *Eisenstein series of weight w associated to ρ_Σ^** . The next lemma provides a connection with Poincaré's series.

Lemma 7.1. $\mathcal{E}(w; \rho_\Sigma^*) = \zeta_A(w; \sigma_\Sigma) \mathcal{P}_{w,0}^{(0)}(\Phi_{\rho_\Sigma^*})$.

Here $\mathcal{P}_{w,0}^{(0)}(\Phi_{\rho_\Sigma^*})$ denotes the last column of the matrix valued Poincaré series $\mathcal{P}_{w,0}(\Phi_{\rho_\Sigma^*})$ defined in (51), with $G = \Phi_{\rho_\Sigma^*}$ and:

$$\zeta_A(w; \sigma_\Sigma) := \prod_P \left(1 - \frac{\sigma_\Sigma(P)}{P^w} \right)^{-1} \in \mathbb{T}_\Sigma^\times \cap \mathbb{E}_\Sigma,$$

the product running over the irreducible monic polynomials of A .

Proof of Lemma 7.1. We observe that for all $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ the last column of

$$\rho_\Sigma^*(\gamma)^{-1} \Phi_{\rho_\Sigma^*}(\gamma(z)) \begin{pmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{pmatrix}$$

is equal to the last column of ${}^t \rho_\Sigma(\gamma) \Phi_{\rho_\Sigma^*}(\gamma(z))$ which is equal to $\otimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(c) \\ \chi_{t_i}(d) \end{pmatrix}$. Therefore, the last column of $\mathcal{P}_{w,0}(\Phi_{\rho_\Sigma^*})$ is

$$\sum_{\substack{\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \\ c, d \in A \\ \text{relatively prime}}} (cz + d)^{-w} \bigotimes_{i \in \Sigma} \begin{pmatrix} \chi_{t_i}(c) \\ \chi_{t_i}(d) \end{pmatrix},$$

and does not depend on the choice of the representatives modulo H . Observe that the index set of the sum defining the series $\mathcal{E}(w; \rho_\Sigma^*)$, $A^2 \setminus \{(0, 0)\}$, is equal to $\mathcal{I}A^+$, where \mathcal{I} is the set of couples $(c, d) \in A^2$ with c, d relatively prime. This means that

$$\mathcal{E}(w; \rho_\Sigma^*) = \sum_{a \in A^+} \frac{\sigma_\Sigma(a)}{a^w} \sum_{(c,d) \in \mathcal{I}} (cz + d)^{-w} \bigotimes_{i \in \Sigma} (\chi_{t_i}(c), \chi_{t_i}(d)) = \zeta_A(w; \sigma_\Sigma) \mathcal{P}_{w,0}^{(0)}(\Phi_{\rho_\Sigma^*}).$$

□

7.1.1. The expansion at infinity of Eisenstein series. We expand the entries of our vector-valued Eisenstein series along the principles of Theorem 4.14.

If $|\Sigma| = s > 0$ and $N = 2^s$, the ordering on Σ induces a bijection $\Sigma \xrightarrow{\varepsilon} \{0, \dots, s-1\}$. This in turn defines a bijection between subsets $J \subset \Sigma$ and integers $0 \leq n \leq N-1$. If $n = n_0 + n_1 2 + \dots + n_{s-1} 2^{s-1}$ is the base-2 expansion of n , the image of n is the subset $J = \{j \in \Sigma : n_j \neq 0\} \subset \Sigma$. We can write $|J|_\Sigma := n$. For example, $|\emptyset|_\Sigma = 0$. Then, we can describe in two ways an N -tuple of objects parametrized by the subsets of $\{1, \dots, 2^s\}$:

$$f = (f^J)_{J \subset \Sigma} = (f_i)_{1 \leq i \leq N},$$

by using that the latter is $(f_{|J|_\Sigma+1})_{J \subset \Sigma}$ (note how we distinguish the \mathbb{N}^* -indexing from the Σ -indexing). Note that the first entry is

$$f_0 = f^\emptyset.$$

The Perkins series $\psi(w; \sigma_U)$ defined in (4.15) are elements of \mathfrak{D}_Σ , if $U \subset \Sigma$. We set

$$\psi_a(w; \sigma_\Sigma) := \psi(w; \sigma_\Sigma)(za),$$

functions which also belong to \mathfrak{D}_Σ (use Corollary 6.6). Their valuations v are positive and we we have, for all $a \in A^+$,

$$v(\psi_a(w; \sigma_\Sigma)) = |a|v(\psi(w; \sigma_\Sigma)).$$

The next Proposition generalizes [37, Proposition 3.7] to the case of $\rho = \rho_\Sigma^*$.

Proposition 7.2. *If $s = |\Sigma| \equiv w \pmod{q-1}$ and $w > 0$, then, writing $\mathcal{E}(w; \rho_\Sigma^*) = (\mathcal{E}^J)_{I \sqcup J = \Sigma}$, we have:*

$$(58) \quad \mathcal{E}^J = -(-1)^{|J|} \sum_{a \in A^+} \sigma_I(a) \psi_a(w; \sigma_J), \quad J \neq \Sigma,$$

$$(59) \quad \mathcal{E}^\Sigma = -\zeta_A(w; \sigma_\Sigma) - (-1)^{|\Sigma|} \sum_{a \in A^+} \psi_a(w; \sigma_\Sigma).$$

In particular, if $J = \emptyset \neq \Sigma$, we have that

$$(60) \quad \mathcal{E}^\emptyset = -\tilde{\pi}^w \sum_{a \in A^+} \sigma_\Sigma(a) G_w(u_a(z)) \in \mathbb{K}_\Sigma[[u]].$$

Moreover, if $\Sigma = \emptyset$, we have, for $q-1 \mid n$:

$$(61) \quad \mathcal{E}(w; \mathbf{1}) = -\zeta_A(w) - \tilde{\pi}^n \sum_{a \in A^+} G_w(u_a(z)).$$

In all cases, we can identify $\mathcal{E}(w; \rho_\Sigma^*)$ with an element of $\mathfrak{D}_\Sigma^{N \times 1}$.

We deduce, in yet another way, that $\mathcal{E}(w; \rho_\Sigma^*) \in M_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$. Additionally, we see that it does not belong to $S_w(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ because of the non-vanishing of $\zeta_A(w; \sigma_\Sigma)$ in (59). Note that writing $\mathcal{E}(w; \rho_\Sigma^*) = {}^t(\mathcal{E}_1, \dots, \mathcal{E}_{N-1}, \mathcal{E}_N)$, we have $v(\mathcal{E}_i) > 0$ for $i = 1, \dots, N-1$ and $v(\mathcal{E}_N) = 0$.

Proof of Proposition 7.2. We compute, with $I \sqcup J = \Sigma$ and $J \neq \Sigma$:

$$\begin{aligned} \mathcal{E}^J &= \sum_{a \in A \setminus \{0\}} \sigma_I(a) \sum_{b \in A} \sigma_J(b) (az + b)^{-w} \\ &= \sum_{a \in A^+} \sum_{\lambda \in \mathbb{F}_q^\times} \sigma_I(\lambda a) \sum_{b \in A} \sigma_J(b) (\lambda az + b)^{-w} \\ &= \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{|I|-n} \sum_{a \in A^+} \sigma_I(a) \sum_{b \in A} \sigma_J(b) \left(az + \frac{b}{\lambda} \right)^{-w} \\ &= \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{|I|+|J|-n} \sum_{a \in A^+} \sigma_I(a) \sum_{b \in A} \sigma_J \left(\frac{b}{\lambda} \right) \left(az + \frac{b}{\lambda} \right)^{-w} \\ &= - \sum_{a \in A^+} \sigma_I(a) \sum_{c \in A} \sigma_J(c) (az + c)^{-w}, \end{aligned}$$

because $\sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{|I|+|J|-w} = \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{|\Sigma|-w} = -1$. The result now follows in the case $J \neq \Sigma$ from the fact that

$$\psi(n, \sigma_J)(az) = \sum_{c \in A} \sigma_J(-c)(az+c)^{-w} = (-1)^J \sum_{c \in A} \sigma_J(c)(az+c)^{-w}.$$

In the case $J = \Sigma$, the argument is similar, but with an additional sum:

$$\mathcal{E}^\Sigma = \sum_{a \in A \setminus \{0\}} \sum_{b \in A} \sigma_\Sigma(b)(az+b)^{-w} + \sum_{b \in A \setminus \{0\}} \sigma_\Sigma(b)b^{-w}.$$

The first sum can be handled as above, while the second one gives:

$$\sum_{b \in A \setminus \{0\}} \sigma_\Sigma(b)b^{-w} = \sum_{b \in A^+} \sigma_\Sigma(b)b^{-w} \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{|\Sigma|-w} = -\zeta_A(w; \sigma_\Sigma).$$

The identity concerning the case $J = \emptyset \neq \Sigma$ is clear, and the last identity, concerning the scalar Eisenstein series, is well known; see, for instance, [15, (6.3)]. The last assertion of the proposition is a direct consequence of Corollary 6.6 by the fact that $\psi_a(w; \sigma_\Sigma) \in \mathfrak{D}_\Sigma$ for all $a \in A$ and $w \in \mathbb{N}^*$ and the fact that $v(\psi_a(w; \sigma_\Sigma)) = |a|v(\psi_a(w; \sigma_\Sigma)) \rightarrow \infty$ as a runs in A^+ . \square

Thanks to Theorem 4.17 we can compute the v -valuations of the entries of $\mathcal{E}(1; \rho_\Sigma^*)$ (recall that κ has been introduced in (37)). The corresponding problem for $\mathcal{E}(w; \rho_\Sigma^*)$ for general w is at the moment unsolved.

Corollary 7.3. *If $|\Sigma| \equiv 1 \pmod{q-1}$ and $\mathcal{E}(1; \rho_\Sigma^*) = (\mathcal{E}^J)_{J \subset \Sigma}$, we have $v(\mathcal{E}^J) = \kappa(J)$ if $J \subsetneq \Sigma$ and $v(\mathcal{E}^\Sigma) = 0$.*

For the next definition, we recall that $\mathfrak{K}_\Sigma = \mathcal{T}^\circ(\mathbb{K}_\Sigma)((u))$.

Definition 7.4. An element $f \in M_w^1(\rho; \mathbb{K}_\Sigma)$ is said *rational*, if the image of f by the embedding ι_Σ of Theorem 4.14 is an element of $\mathcal{T}^\circ(K(\underline{t}_\Sigma))((u))^{N \times 1}$, and *integral* if this image lies in $\mathcal{T}^\circ(A[\underline{t}_\Sigma])[u^{-1}][[u]]^{N \times 1}$.

Note that if $N = 1$ and $\Sigma = \emptyset$, this coincides with the scalar modular forms having u -expansions in $K((u))$ and $A[[u]]$ respectively, studied in [15]. We have the simple but important result:

Theorem 7.5. *If $w \equiv |\Sigma| \pmod{q-1}$ then $\zeta_A(w; \sigma_\Sigma)^{-1} \mathcal{E}(w; \rho_\Sigma^*)$ is integral.*

Proof. It is a direct consequence of [2, Theorem 1] and Proposition 7.2. \square

7.1.2. *Modular forms of weight one for ρ_Σ^* .* We recall that $N = 2^s$. We have:

Theorem 7.6. *Assuming that $|\Sigma| \equiv 1 \pmod{q-1}$, $M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma)$ is of dimension one over \mathbb{L}_Σ , generated by the Eisenstein series $\mathcal{E}(1; \rho_\Sigma^*)$.*

Proof. We note that in the case $\rho = \rho_\Sigma^*$ we have the following identity for the space $H(\rho)$ defined in §5.2.3:

$$H(\rho) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbb{K}_\Sigma \end{pmatrix}.$$

We conclude by observing that $\mathcal{E}(1; \rho_\Sigma^*) \in M_1(\rho_\Sigma^*; \mathbb{L}_\Sigma) \setminus \{0\}$ and applying Theorem 5.12. \square

This yields a positive answer to [37, Problem 1.1]. By Theorem 4.19, $\mathcal{E}(1; \rho_\Sigma^*)$ is an eigenform for all the Hecke operators defined in §4.2.

Remark 7.7. One of the main motivations for the introduction of the Eisenstein series $\mathcal{E}(w; \rho_\Sigma^*)$, for which they have been initially considered in [28], is that the non-zero entry (which is the last one, in the prescribed ordering) tends to $-\zeta_A(w; \sigma_\Sigma)$ as $z \in \Omega$ approaches the cusp infinity or, in other words, it is congruent to $-\zeta_A(w; \sigma_\Sigma)$ modulo \mathfrak{M}_Σ . These are not the only Eisenstein series which enjoy this property. Another example is discussed in this remark. We consider the \mathbb{F}_q -algebra morphism $\chi : A \rightarrow \mathbb{F}_q[\underline{t}_\Sigma]^{s \times s}$ (with $s = |\Sigma|$) defined by

$$\chi(\theta) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -P_0 & -P_1 & \cdots & -P_{s-1} \end{pmatrix},$$

where $P_0, \dots, P_{s-1} \in \mathbb{F}_q[\underline{t}_\Sigma]$ are defined by $\prod_{i \in \Sigma} (X - t_i) = X^s + P_{s-1}X^{s-1} + \cdots + P_0$. Then, for all $a \in A$, $\det(\chi(a)) = \sigma_\Sigma(a)$ (see [31, §2.1]). We consider the representation of the first kind $\varphi_\Sigma^* = \wedge^s \rho_\chi^*$, of dimension $N := \binom{2s}{s}$. We suppose that $w \equiv s \pmod{q-1}$ and $w > 0$. The last column of the Poincaré series $\mathcal{P}_{w,0}(\Phi_{\varphi_\Sigma^*})$ multiplied by $\zeta_A(w; \sigma_\Sigma)$ equals

$$\mathcal{E}(w; \varphi_\Sigma^*) := \sum_{(a,b) \in A \setminus \{(0,0)\}} (az + b)^{-w} \bigwedge^s \begin{pmatrix} \chi(a) \\ \chi(b) \end{pmatrix}.$$

This defines an element of $\text{Hol}(\Omega \rightarrow \mathbb{E}_\Sigma^{1 \times N})$ and a modular form in $M_w(\varphi_\Sigma^*; \mathbb{K}_\Sigma) \setminus S_w(\varphi_\Sigma^*; \mathbb{K}_\Sigma)$. Moreover, the only entry \mathcal{E}_N of $\mathcal{E}(w; \varphi_\Sigma^*)$ which does not vanish at infinity, which is the last one, satisfies

$$\mathcal{E}_N \equiv -\zeta_A(w; \sigma_\Sigma) \pmod{\mathfrak{M}_\Sigma}.$$

In other words, $-\zeta_A(w; \sigma_\Sigma)$ is the 'constant term' of the last entry of $\mathcal{E}(w; \varphi_\Sigma^*)$.

7.2. Structure of weak modular forms. We consider a finite non-empty subset $\Sigma \subset \mathbb{N}^*$ and $k \in \Sigma$. We set $\Sigma' = \Sigma \setminus \{k\}$. We denote by $\rho_{\Sigma'}^*$ for the tensor factor of the representation ρ_Σ^* . Hence:

$$(62) \quad \rho_\Sigma^* = \rho_{\Sigma'}^* \otimes \rho_{t_k}^*.$$

We can suppose, without loss of generality, that $k = \min(\Sigma)$. The natural ordering of $\Sigma \subset \mathbb{N}^*$ has to be considered to write the tensor product. We set $\rho = \rho_\Sigma^* \det^{-m}$. The structure of the \mathbb{K}_Σ -vector space $M_w^!(\rho; \mathbb{K}_\Sigma)$ is quite simple to describe. The main result of this subsection is the following.

Theorem 7.8. *Assuming that $\rho = \rho_\Sigma^* \det^{-m}$, we have:*

$$M_w^!(\rho; \mathbb{K}_\Sigma) = M_{w-1}^!(\rho; \mathbb{K}_\Sigma) \otimes \mathcal{E}(1; \rho_{t_k}^*) + M_{w-q}^!(\rho; \mathbb{K}_\Sigma) \otimes \mathcal{E}(q; \rho_{t_k}^*).$$

Proof. We denote by $\mathcal{E} := \mathcal{E}(1; \rho_t^*)$ the row vectorial Eisenstein series of weight 1 associated with the representation ρ_t^* . Explicitly, we have

$$\mathcal{E}(1; \rho_t^*) = \sum'_{a,b \in A} (az + b)^{-1} \begin{pmatrix} \chi_t(a) \\ \chi_t(b) \end{pmatrix}, \quad n > 0.$$

We also set

$$\mathfrak{E} = \begin{pmatrix} \mathcal{E} \\ \tau(\mathcal{E}) \end{pmatrix} \in \text{Hol}(\Omega \rightarrow \mathbb{K}_\Sigma^{2 \times 2}).$$

Note that $\tau(\mathcal{E}) = \mathcal{E}(q; \rho_t^*)$. Let $h = -u + o(u)$ be as in §5.3.1. By [37, Theorem 3.9]:

$$\det(\mathfrak{E}) = -\tilde{\pi} \zeta_A(q; \chi_t) h(z),$$

which is also equal to

$$-\frac{\tilde{\pi}^{q+1} h(z)}{(\theta^q - t)(\theta - t)\omega(t)}$$

by the formula

$$(63) \quad \zeta_A(1; \chi_t) = \frac{\tilde{\pi}}{(\theta - t)\omega(t)}$$

which holds in \mathbb{T} and can be found in [28], after application of τ . The function h does not vanish on Ω and $v(h) = 1$. Since the function $\det(\mathfrak{E})$ can vanish identically for certain values of t with $|t| > 1$, the matrix function $\mathfrak{E}(z)^{-1}$ belongs to $\text{Hol}(\Omega \rightarrow \mathbb{T}^{2 \times 2})$ but not to $\text{Hol}(\Omega \rightarrow \mathbb{E}^{2 \times 2})$. Note that $\tau^2(\omega)^{-1} \mathfrak{E}(z)^{-1}$ defines a function of $\text{Hol}(\Omega \rightarrow \mathbb{E}^{2 \times 2})$. We are going to generalize some aspects of the proof of [37, Theorem 3.9]. Let \mathcal{G} be an element of $M_w^!(\rho; \mathbb{K}_\Sigma)$. Then by definition for all $\gamma \in \Gamma$ and $z \in \Omega$, we have

$$\mathcal{G}(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m} \rho_\Sigma^*(\gamma) \mathcal{G}(z).$$

We now set $\mathbf{E} = \tau^2(\omega(t))^{-1} \mathfrak{E}^*$, \mathbf{E}_{t_k} the same function in the variable t_k instead of t , and

$$(64) \quad \mathbf{F} := I_{N'} \otimes \mathbf{E}_{t_k} \in \text{Hol}(\Omega \rightarrow \mathbb{E}_{\{k\}}^{N \times N}),$$

with $N = 2^s$, $s' = s - 1$, and $N' = 2^{s'}$. We have:

$$\mathbf{F}(\gamma(z)) = (\mathbf{1}_{N'} \otimes \rho_{t_k}(\gamma)) \mathbf{F}(z) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{-1} & 0 \\ 0 & J_\gamma(z)^{-q} \end{pmatrix} \right).$$

Now setting $\mathbf{G} = {}^t\mathcal{G}$ and, denoting with \mathbf{H} the row function $\mathbf{G}\mathbf{F}$, with values in $\mathbb{K}_\Sigma^{1 \times N}$, we have:

$$\begin{aligned}
\mathbf{H}(\gamma(z)) &= \\
&= J_\gamma(z)^w \det(\gamma)^{-m} \mathbf{G}(z) \rho_{\Sigma'}^{-1}(\gamma) (\mathbf{1}_{N'} \otimes \rho_{t_k}(\gamma)) (\mathbf{1}_{N'} \otimes \mathbf{E}_{t_k}(z)^{-1}) \times \\
&\quad \times \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{-1} & 0 \\ 0 & J_\gamma(z)^{-q} \end{pmatrix} \right) \\
&= \det(\gamma)^{-m} \mathbf{G}(z) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) (\mathbf{1}_{N'} \otimes \mathbf{E}_{t_k}(z)^{-1}) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right) \\
&= \det(\gamma)^{-m} \mathbf{G}(z) (\mathbf{1}_{N'} \otimes \mathbf{E}_{t_k}(z)^{-1}) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right) \\
&= \det(\gamma)^{-m} \mathbf{H}(z) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right).
\end{aligned}$$

In the above computation, we have observed the distributive property of the mixed product $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ (for matrices A, B, C, D). This identity that we have found,

$$\mathbf{H}(\gamma(z)) = \det(\gamma)^{-m} \mathbf{H}(z) (\rho_{\Sigma'}^{-1}(\gamma) \otimes \mathbf{1}_2) \left(\mathbf{1}_{N'} \otimes \begin{pmatrix} J_\gamma(z)^{w-1} & 0 \\ 0 & J_\gamma(z)^{w-q} \end{pmatrix} \right)$$

means the following. The column holomorphic function $\mathcal{H} := {}^t\mathbf{H}$, with values in $\mathbb{K}_\Sigma^{N \times 1}$ can be written as $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ with both \mathcal{H}_1 and \mathcal{H}_2 columns of size $N' = 2^{\Sigma'}$, where the symbol \odot is defined, if $a = {}^t(a_1, \dots, a_{N'})$ and $b = {}^t(b_1, \dots, b_{N'})$, by $a \odot b = (a_1, b_1, a_2, b_2, \dots, a_{N'}, b_{N'})$. Then, both $\mathcal{H}_1, \mathcal{H}_2$ are separately weak modular forms for $\rho_{\Sigma'}^* \det^{-m}$, with values in \mathbb{K}_Σ and the weights are respectively $w-1$ and $w-q$. \square

We have:

Theorem 7.9. *The following equality of \mathbb{K}_Σ -vector spaces holds, for any $w \in \mathbb{Z}$, $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ and finite $\Sigma \subset \mathbb{N}^*$:*

$$(65) \quad M_w^!(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \bigoplus_{I \sqcup J = \Sigma} \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right) M_{w-|I|-q|J|}^!(\det^{-m}; \mathbb{K}_\Sigma).$$

Denoting by $M^!(\det^\bullet; \mathbb{K}_\Sigma)$ the $\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ -graded B -algebra of scalar weak \mathbb{K}_Σ -valued Drinfeld modular forms for Γ of any weight and type, and setting $M^!(\rho_\Sigma^* \det^\bullet; \mathbb{K}_\Sigma) = \bigoplus_{w,m} M_w^!(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$, which is a graded module over $M^!(\rho_\emptyset; \mathbb{K}_\Sigma)$, we obtain:

Corollary 7.10. *The \mathbb{K}_Σ -vector space $M^!(\rho_\Sigma^* \det^\bullet; \mathbb{K}_\Sigma)$ is a graded free $M^!(\det^\bullet; \mathbb{K}_\Sigma)$ -module of rank $N = 2^s$.*

Observe that further, the generators of this module are explicitly described in Theorem 7.9. Denoting by $M^!(\rho_\Sigma^*; \mathbb{K}_\Sigma) = \bigoplus_{w \in \mathbb{Z}} M_w^!(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ the sub-module of $M^!(\det^\bullet; \mathbb{K}_\Sigma)$ of

weak modular forms for ρ_Σ^* and setting $M^1(\rho_\emptyset; \mathbb{K}_\Sigma) = \oplus_w M_w^1(\rho_\emptyset; \mathbb{K}_\Sigma)$, We also deduce the following corollary:

Corollary 7.11. *The \mathbb{K}_Σ -vector space $M^1(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ is a graded free $M^1(\rho_\emptyset; \mathbb{K}_\Sigma)$ -module of rank N .*

Proof of Theorem 7.9. We deduce from Theorem 7.8, by induction on $|\Sigma|$, that a weaker version of (65) holds, with \sum in place of \bigoplus . It remains to show that the sum is a direct sum. For this, it suffices to show that the $N = 2^s$ functions $\otimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \otimes \otimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*)$, for $I \sqcup J = \Sigma$, which define elements of $\mathfrak{D}_\Sigma^{N \times 1}$ are linearly independent over the field $\mathbb{K}_\Sigma((u))$. Note indeed that $M_{w-|I|-q|J|}^1(\det^{-m}; \mathbb{K}_\Sigma) \hookrightarrow \mathbb{K}_\Sigma((u))$ because all the elements of the space on the left are A -periodic and tempered.

Let a, b be two elements of \mathfrak{K}_Σ . We write $a \approx b$ if $v(a) = v(b)$ (note that if $a = 0$ and $a \approx b$ then $b = 0$) and we extend the definition to vectors and matrices whose entries are all in \mathfrak{K}^\times by saying that $(a_{i,j}) \approx (b_{i,j})$ if for all i, j , $v(a_{i,j}) = v(b_{i,j})$. Then by Proposition 7.2, we have $\mathcal{E}(1; \rho_{t_i}^*) \approx \begin{pmatrix} u \\ 1 \end{pmatrix}$ and $\mathcal{E}(q; \rho_{t_j}^*) \approx \begin{pmatrix} u^q \\ 1 \end{pmatrix}$. Hence, up to permutation of rows and columns, we have the \approx -equivalence of $N \times N$ -matrices in $\mathfrak{D}_\Sigma^{N \times N}$:

$$\mathcal{N} := \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right)_{I \sqcup J = \Sigma} \approx \begin{pmatrix} u^q & u \\ 1 & 1 \end{pmatrix}^{\otimes s}.$$

The anti-diagonal of the matrix on the right is equal to $(1, u)^{\otimes s}$ (up to reordering). This corresponds to a unique monomial which minimises the v -valuation in the series expansion of the determinant of \mathcal{N} . We deduce that $\det(\mathcal{N}) \approx u^{a_s}$, where $(a_s)_{s \geq 1}$ is the sequence defined, inductively, by $a_1 = 1$ and $a_s = 2a_{s-1} + 2^{s-1}$ for $s > 1$. The matrix \mathcal{N} is therefore non-singular, and the functions $\otimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \otimes \otimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*)$ for $I \sqcup J = \Sigma$ are linearly independent over $\mathbb{K}_\Sigma((u))$, from which the result follows. \square

7.3. Strongly regular modular forms. We keep considering a finite non-empty subset $\Sigma \subset \mathbb{N}^*$ of cardinality s , the representation $\rho = \rho_\Sigma^*$, $k := \max(\Sigma)$. We will discuss quite a restricted but useful class of modular forms which have a particularly simple behaviour at infinity.

Definition 7.12. A tempered ρ_Σ^* -quasi-periodic holomorphic function

$$\mathcal{G} : \Omega \rightarrow \mathbb{K}_\Sigma^{N \times 1}$$

is called *strongly regular at infinity* if

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes s} \mathcal{G}(z) \in \mathfrak{D}_\Sigma^{N \times 1}.$$

Note, with Diag denoting a diagonal matrix, that

$$\begin{aligned} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes 2} &= \text{Diag}(u^{-2}, u^{-1}, u^{-1}, 1) \\ \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes 3} &= \text{Diag}(u^{-3}, u^{-2}, u^{-2}, u^{-1}, u^{-2}, u^{-1}, u^{-1}, 1). \end{aligned}$$

Note also that writing

$$(66) \quad \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes s} = \text{Diag}(u^{-s}, \dots, u^{-n_1}, u^{-n_0}),$$

and letting s tend to infinity, an integer sequence $(n_i)_{i \geq 0}$ is defined and coincides with the so-called one's-counting sequence, that is, the sequence which gives the number of one's in the binary expansion of i . We need the next Lemma, where we use the sequence introduced in (66) and the notation \odot introduced in the course of the proof of Theorem 7.8.

Lemma 7.13. *We have $(n_i)_{i \geq 0} = (n_{2i})_{i \geq 0} \odot (n_{2i+1} + 1)_{i \geq 0}$.*

Proof. Straightforward computation of the carry over in binary addition when we add one to an integer. \square

The above serves to make the next definition.

Definition 7.14. A weak modular form $\mathcal{G} \in M_w^{\dagger}(\rho_{\Sigma}^* \det^{-m}; \mathbb{K}_{\Sigma})$ is said *strongly regular* (of weight w) if it is strongly regular at infinity after definition 7.12.

The \mathbb{K}_{Σ} -vector spaces of strongly regular modular forms have quite a simple structure which can be described essentially by adapting the proof of Theorem 7.8; see Theorem 7.15. Also, regarding the Definition 7.12 of strongly regular functions, if we want to use the indexation of the components of \mathcal{G} , $\mathcal{G} = (\mathcal{G}^J)_{I \sqcup J = \Sigma}$ (so that the first entry \mathcal{G}^{\emptyset} has a u -expansion) we then get that the above condition is equivalent to

$$(67) \quad \mathcal{G}^J(z)u^{-|I|} \in \mathfrak{D}_{\Sigma}, \quad \forall I, J \text{ such that } I \sqcup J = \Sigma.$$

We denote by $M_w^{\dagger}(\rho_{\Sigma}^* \det^{-m}; \mathbb{K}_{\Sigma})$ the \mathbb{K}_{Σ} -sub-vector space of $M_w^{\dagger}(\rho_{\Sigma}^* \det^{-m}; \mathbb{K}_{\Sigma})$ generated by the strongly regular modular forms of weight w for $\rho_{\Sigma}^* \det^{-m}$ (with values in \mathbb{K}_{Σ}).

Examples of strongly regular modular forms. Any scalar Drinfeld modular form is strongly regular. In fact, we have $M_w^{\dagger}(\det^{-m}; \mathbb{K}_{\Sigma}) = M_w(\det^{-m}; \mathbb{K}_{\Sigma})$ for all w, m . From Proposition 7.2 we immediately see that $\mathcal{E}(1; \rho_i^*) \in M_1^{\dagger}(\rho_i^*; \mathbb{K})$ and $\mathcal{E}(q; \rho_i^*) \in M_q^{\dagger}(\rho_i^*; \mathbb{K})$. In particular, after Theorem 7.9 and Corollary 7.10, the generators of the module $M^{\dagger}(\rho_{\Sigma}^* \det^{\bullet}; \mathbb{K}_{\Sigma})$ described in the statements are all strongly regular modular forms.

7.3.1. *Structure of strongly regular modular forms.* We shall prove:

Theorem 7.15. *The following equality of \mathbb{K}_Σ -vector spaces holds, for any $w \in \mathbb{Z}$, $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, finite $\Sigma \subset \mathbb{N}^*$:*

$$(68) \quad M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \bigoplus_{I \sqcup J = \Sigma} \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right) M_{w-i-qj}(\det^{-m}; \mathbb{K}_\Sigma).$$

The direct sum $M^\dagger(\rho_\Sigma^* \det^\bullet; \mathbb{K}_\Sigma) := \bigoplus_{w,m} M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ is a graded module over the graded algebra $M(\det^\bullet; \mathbb{K}_\Sigma)$ of scalar Drinfeld modular forms $\Omega \rightarrow \mathbb{K}_\Sigma$ of any power of the determinant character. We immediately deduce:

Corollary 7.16. *the $M(\det^\bullet; \mathbb{K}_\Sigma)$ -module $M^\dagger(\rho_\Sigma^* \det^\bullet; \mathbb{K}_\Sigma)$ is free of rank N generated by the functions $(\otimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*)) \otimes (\otimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*))$, for $I, J \subset \Sigma$ such that $I \sqcup J = \Sigma$.*

After the work of Marks and Mason [24] and Bantay and Gannon [5] in the setting of complex vector-valued modular forms, this is expected. These authors prove that vector spaces of vector valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$ associated to an indecomposable finite dimensional complex representation of this group (and satisfying some additional mild technical conditions) all are free of the dimension that of the representation.

Similarly, we have, writing $M(\mathbf{1}; \mathbb{K}_\Sigma)$ for the graded algebra of scalar Drinfeld modular forms for Γ (it is equal to the graded algebra $\mathbb{K}_\Sigma[g, \Delta]$ see [15, Corollary (6.5)]) and $M^\dagger(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ the $M(\mathbf{1}; \mathbb{K}_\Sigma)$ -module of strongly regular modular forms for ρ_Σ^* :

Corollary 7.17. *The graded $M(\mathbf{1}; \mathbb{K}_\Sigma)$ -module $M^\dagger(\rho_\Sigma^*; \mathbb{K}_\Sigma)$ is free of rank $N = 2^s$.*

We can take in the above result the generators of Corollary 7.16.

Proof of Theorem 7.15. It is easily seen that the left-hand side of (68) is contained in the right-hand side and we have to prove the reverse inclusion. Corollary 7.9 ensures the equality of the corresponding \mathbb{K}_Σ -vector spaces of weak modular forms (“when \dagger is replaced with $!$ ”). This means that if $\mathcal{G} \in M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$, then

$$\mathcal{G} \in M_w^!(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \bigoplus_{I \sqcup J = \Sigma} \left(\bigotimes_{i \in I} \mathcal{E}(1; \rho_{t_i}^*) \right) \otimes \left(\bigotimes_{j \in J} \mathcal{E}(q; \rho_{t_j}^*) \right) M_{w-i-qj}^!(\det^{-m}; \mathbb{K}_\Sigma).$$

All we need to prove is that the coefficients occurring in the various spaces of scalar weak modular forms $M_{w-i-qj}^!(\det^{-m}; \mathbb{K}_\Sigma)$ are in fact Drinfeld modular forms (regular at infinity). To see this it suffices to show that

$$\mathcal{G} \in M_{w-1}(\rho_{\Sigma'}^*, \det^{-m}; \mathbb{K}_\Sigma) \otimes \mathcal{E}(1; \rho_{t_k}^*) + M_{w-q}(\rho_{\Sigma'}^*, \det^{-m}; \mathbb{K}_\Sigma) \otimes \mathcal{E}(q; \rho_{t_k}^*),$$

where k is an integer such that $k < \min(\Sigma')$ with $\Sigma = \Sigma' \sqcup \{k\}$. A simple induction will then allow to complete the proof.

Lemma 7.13 implies that for all $s \geq 1$, writing

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{\otimes s} = \mathrm{Diag}(U_s),$$

then

$$(69) \quad U_s = u^{-1}U_{s-1} \odot U_{s-1}.$$

Now, we set $\mathbf{G} = \mathbf{G}_1 \odot \mathbf{G}_2$ with $\mathbf{G} = {}^t\mathcal{G}$ an element of $M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$. We know by the proof of Theorem 7.8 that

$$\mathbf{H} = \mathbf{H}_1 \odot \mathbf{H}_2 = \mathbf{G}\mathbf{F}$$

(with \mathbf{F} as in (64)) is such that

$$\mathcal{H}_1 = {}^t\mathbf{H}_1 \in M_{w-1}^\dagger(\rho_\Sigma^*, \det^{-m}; \mathbb{K}_\Sigma), \quad \text{and } \mathcal{H}_2 = {}^t\mathbf{H}_2 \in M_{w-q}^\dagger(\rho_\Sigma^*, \det^{-m}; \mathbb{K}_\Sigma).$$

It remains to prove that \mathcal{H}_1 and \mathcal{H}_2 are both strongly regular. We have to show that

$$\mathbf{H}_j(z) \text{Diag}(U_{s-1}) \in \mathfrak{D}_\Sigma^{1 \times N'}, \quad j = 1, 2.$$

By hypothesis, we know that the entries of $\mathbf{G}(z) \text{Diag}(U_s)$ are in \mathfrak{D}_Σ . Explicitly, the entries of $u(z)^{-1}\mathbf{G}_1(z) \text{Diag}(U_{s-1})$ and of $\mathbf{G}_2(z) \text{Diag}(U_{s-1})$ are in \mathfrak{D}_Σ . We recall the relation $a \approx b$, for elements of $\mathfrak{K}_\Sigma^\times$, and its extension to matrices with non-zero entries. We note that $\mathbf{H}_1, \mathbf{H}_2$ are given, explicitly, by the formulas:

$$\mathbf{H}_1 = \frac{-\mathbf{G}_1\tau(e_2) + \mathbf{G}_2\tau(e_1)}{\tilde{\pi}\zeta_A(q; \chi_{t_k})h}, \quad \mathbf{H}_2 = \frac{\mathbf{G}_1e_2 - \mathbf{G}_2e_1}{\tilde{\pi}\zeta_A(q; \chi_{t_k})h},$$

where $\mathcal{E} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ ⁽⁵⁾. By the well-known u -expansion $h = -u + o(u)$ (which tells us that $v(h) = 1$ and $h \approx u$), we thus have

$$u\mathbf{H}_1 \approx -\mathbf{G}_1\tau(e_2) + \mathbf{G}_2\tau(e_1), \quad u\mathbf{H}_2 \approx \mathbf{G}_1e_2 - \mathbf{G}_2e_1.$$

We first study \mathcal{H}_1 . We have:

$$\begin{aligned} \mathbf{H}_1 \text{Diag}(U_{s-1}) &\approx u^{-1}(-\mathbf{G}_1\tau(e_2) + \mathbf{G}_2\tau(e_1)) \text{Diag}(U_{s-1}) \\ &\approx -u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})\tau(e_2) + \mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}\tau(e_1). \end{aligned}$$

Now, by hypothesis $u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1}) \in \mathfrak{D}_\Sigma^{1 \times N'}$, while $v(\tau(e_2)) = 0$, from which we deduce that $u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})\tau(e_2) \in \mathfrak{D}_\Sigma^{1 \times N'}$. On the other hand, we have that $\tau(e_1) \approx u^q$. hence, we have that $\mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}\tau(e_1) \approx \mathbf{G}_2 \text{Diag}(U_{s-1})u^{q-1} \in \mathfrak{M}_\Sigma^{1 \times N'} \subset \mathfrak{D}_\Sigma^{1 \times N'}$. Therefore all entries of $\mathbf{H}_1 \text{Diag}(U_{s-1})$ are in \mathfrak{D}_Σ and \mathcal{H}_1 is strongly regular.

Let us now deal with \mathcal{H}_2 . Similarly, we have that

$$\begin{aligned} \mathbf{H}_2 \text{Diag}(U_{s-1}) &\approx u^{-1}(\mathbf{G}_1e_2 - \mathbf{G}_2e_1) \text{Diag}(U_{s-1}) \\ &\approx u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})e_2 - \mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}e_1. \end{aligned}$$

Since $v(e_2) = 0$, we have that the term $u^{-1}\mathbf{G}_1 \text{Diag}(U_{s-1})e_2$ has all the entries in \mathfrak{D}_Σ . Moreover, $e_1 \approx u$ so that all the entries of $\mathbf{G}_2 \text{Diag}(U_{s-1})u^{-1}e_1$ are in \mathfrak{D}_Σ by the hypothesis on \mathcal{G}_2 . Hence, $\mathbf{H}_2 \text{Diag}(U_{s-1}) \in \mathfrak{D}_\Sigma^{1 \times N'}$ and \mathcal{H}_2 is strongly regular. This completes the proof of the Theorem. \square

⁵The reader will not mix these functions with the functions e_i of §3.

7.4. Some structural properties of modular forms. In contrast with that of strongly regular modular forms, the structure of the vector spaces $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ is more difficult to describe. In this subsection, we give some properties of them. The first result we want to discuss is Lemma 7.19. To prove it we need the next result (where $N = 2^s$ and $s = |\Sigma|$).

Lemma 7.18. *Let $\underline{\alpha} = {}^t(\alpha_1, \dots, \alpha_N) \in \mathbb{K}_\Sigma^{N \times 1}$ be such that $\rho_\Sigma^*(T_a)\underline{\alpha} = \underline{\alpha}$ for all $a \in A$. Then, $\alpha_1 = \dots = \alpha_{N-1} = 0$.*

In particular, we have $\rho_\delta(0) = 1$ in Theorem 5.12.

Proof of Lemma 7.18. We proceed by induction on $s \geq 0$. If $s = 0$, case of $\Sigma = \emptyset$, there is nothing to prove. Assume now that $s > 0$ and write $\Sigma = \{k\} \sqcup \Sigma'$ with $k < \min(\Sigma')$ and $s' = |\Sigma'| \geq 0$. We have

$$\rho_\Sigma^* = \rho_{\Sigma'}^* \otimes \rho_{t_k}^* = \begin{pmatrix} \rho_{\Sigma'}^* & 0 \\ -\chi_{t_k}(\cdot)\rho_{\Sigma'}^* & \rho_{\Sigma'}^* \end{pmatrix}.$$

We also decompose $\underline{\alpha} = \underline{\beta} \oplus \underline{\gamma}$, with $\underline{\beta}, \underline{\gamma} \in \mathbb{K}_\Sigma^{N' \times 1}$ (and $N' = 2^{s'}$). Then, we have two families of relations. The first one is

$$\rho_{\Sigma'}^*(T_a)\underline{\beta} = \underline{\beta}, \quad \forall a \in A,$$

and the second one is

$$\rho_{\Sigma'}^*(T_a)\underline{\gamma} - \chi_{t_k}(a)\rho_{\Sigma'}^*(T_a)\underline{\beta} = \underline{\gamma}, \quad \forall a \in A.$$

By induction, we have two cases: either (1) $\underline{\beta} = \underline{0}$, or (2) $\underline{\beta} = {}^t(0, \dots, 0, \beta)$ with $\beta \neq 0$. If we are in the first case, we deduce from the second family of relations and induction that $\underline{\gamma} = {}^t(0, \dots, 0, *)$ and the lemma is proved. It remains to show that the second case cannot hold. But let us assume by contradiction that for all $a \in A$,

$$(\rho_{\Sigma'}^* - I_{N'})\underline{\gamma} = \chi_{t_k}(a){}^t(0, \dots, 0, \beta), \quad \forall a \in A.$$

We denote by $\rho_1^a, \dots, \rho_{N'}^a$ the entries of the last row of $\rho_{\Sigma'}^*(a)$. These are monomials in the elements $\chi_{t_i}(a)$ with $i \in \Sigma'$. We also write $\gamma_1, \dots, \gamma_{N'}$ for the entries of $\underline{\gamma}$. At the level of the bottom entry, we have:

$$\rho_1^a \gamma_1 + \dots + \rho_{N'-1}^a \gamma_{N'-1} = \chi_{t_k}(a)\beta, \quad \forall a \in A.$$

Since we have supposed that $\beta \neq 0$, we have $\max_i \{\|\gamma_i\|\} = \|\beta\| \neq 0$. Rescaling, we can suppose that $\max_i \{\|\gamma_i\|\} = \|\beta\| = 1$. Reducing modulo the maximal ideal of the valuation we see that $\bar{\gamma}_1, \dots, \bar{\gamma}_{N'-1}, \bar{\beta} \in \mathbb{F}_q^{ac}(\underline{t}_\Sigma)$ are such that

$$\rho_1^a \bar{\gamma}_1 + \dots + \rho_{N'-1}^a \bar{\gamma}_{N'-1} = \chi_{t_k}(a)\bar{\beta}, \quad \forall a \in A,$$

with the right-hand side which does not vanish identically if $a \neq 0$. Let $\underline{\zeta} = (\zeta_i : i \in \Sigma') \in (\mathbb{F}_q^{ac})^{s'}$ be such that the following is well defined:

$$\delta := \beta|_{t_i = \zeta_i, \forall i \in \Sigma'} \in \mathbb{F}_q^{ac}(t_k)^\times.$$

such a vector of elements $\underline{\zeta}$ exists, by the hypothesis that β is non-zero. There exists $a \in A^+$ such that $a(\zeta_i) = 0$ for all $i \in \Sigma'$ and therefore $\chi_{t_i}(a)_{t_i=\zeta_i} = 0$ and $\rho_j^a|_{t_i=\zeta_i, \forall i \in \Sigma'} = 0$ for all j . This means that $\chi_{t_k}(a)\delta = 0$ but this is in contradiction with the injectivity of χ_{t_k} . Hence, $\underline{\beta} = \underline{0}$ is the only possible case and we are done. \square

We deduce:

Lemma 7.19. *If $f = (f_1, \dots, f_N)$ is a modular form for ρ_Σ^* , then we can identify f_1, \dots, f_{N-1} with elements of \mathfrak{M}_Σ and f_N with an element of \mathfrak{D}_Σ .*

Proof. We have $f \in \mathfrak{D}_\Sigma^{N \times 1}$. In particular, there exists $\underline{\alpha} \in \mathbb{K}_\Sigma^{N \times 1}$ such that $f \equiv \underline{\alpha} \pmod{\mathfrak{M}^{N \times 1}}$. But note that for all $a \in A$, $f(z+a) = \rho_\Sigma^* f(z)$ for all $z \in \Omega$ so that $\underline{\alpha} = \rho_\Sigma^*(T_a)\underline{\alpha}$ for all $a \in A$. Lemma 7.18 allows to conclude. \square

7.4.1. Link between regular and strongly regular forms. We begin with some tools to handle the representations ρ_Σ and ρ_Σ^* . We order the columns of $\rho_\Sigma(\gamma)$ from \emptyset to Σ along the total order described in §7.1.1, and we order the rows from Σ to \emptyset along the opposite of this order. Let $M = (M_{I,J})_{I,J \subset \Sigma} \in B^{N \times N}$ be a matrix with entries in some ring B , with rows and columns indexed as above (the first index always refers to rows). Since the opposite order of the inclusion order on the subsets of Σ is obtained by computing complementaries $I \mapsto I^c := \Sigma \setminus I$, we have the following transposition rule:

$$(70) \quad {}^t M = (M_{J^c, I^c})_{I, J \subset \Sigma} \in B^{N \times N}.$$

Now we write with $a \in A$:

$$\rho_\Sigma(T_a) = (\rho_{I,J}(T_a))_{I, J \subset \Sigma} \in \mathbb{F}_q(\underline{t}_\Sigma)^{N \times N},$$

and we do similarly for $\rho_\Sigma^*(T_a) = (\rho_{I,J}^*(T_a))_{I, J \subset \Sigma}$. For $U \subset \Sigma$, we recall the semi-character $\sigma_U = \prod_{i \in U} \chi_{t_i}$. An elementary computation, the fact that the inverse of $\rho_{t_i}(T_a)$ is $\rho_{t_i}(T_{-a})$, and an application of (70), lead to:

Lemma 7.20. *For $I, J \subset \Sigma$, we have:*

$$\rho_{I,J}(T_a) = \begin{cases} 0 & \text{if } I \cup J \subsetneq \Sigma \\ \sigma_{I \cap J}(a) & \text{if } I \cup J = \Sigma \end{cases}, \quad \rho_{I,J}^*(T_a) = \begin{cases} 0 & \text{if } J \cap I \neq \emptyset \\ (-1)^{|(J \cup I)^c|} \sigma_{(J \cup I)^c}(a) & \text{if } J \cap I = \emptyset \end{cases}.$$

Note that $\rho_\Sigma(T_a)$ is symmetric with respect to the anti-diagonal (we can switch I, J) and that the entries in the diagonal are all equal to 1 because these are the entries indexed by I, J with $I \sqcup J = \Sigma$. The coefficient of $\rho_\Sigma(T_a)$ in the upper-right corner is equal to $\sigma_\Sigma(a) = \prod_{i \in \Sigma} \chi_{t_i}(a)$. We deduce the explicit expression of the coefficients of $\Phi_{\rho_\Sigma^*} = (\Phi_{I,J})_{I,J}$ in term of Perkins' series. In particular, since the function κ is strictly decreasing, we have the following property. If $I, J \subset \Sigma$ with $I \cap J = \emptyset$ and $I \cup J \neq \Sigma$ (not corresponding to a diagonal coefficient), then

$$(71) \quad v(\Phi_{I,J}) \geq \kappa(I) - 1.$$

We set $\rho = \rho_\Sigma^* \det^{-m}$. The above properties can be used to prove:

Lemma 7.21. *Let $f = {}^t(f^I)$ be a ρ -quasi-periodic function with $\iota_\Sigma(f) \in \mathfrak{D}_\Sigma$. Then, if $I \subsetneq \Sigma$, $v(f^I) \geq \kappa(I)$.*

Proof. By the proof of Proposition 4.13, we have

$$f = \Phi_\rho g$$

where $g = {}^t(g^I) \in \mathbb{K}_\Sigma[[u]]^{N \times 1}$. Since the entries of Φ_ρ are in $\mathcal{T}^\circ(\mathbb{K}_\Sigma)$ (valuations in $] - 1, 0] \cup \{\infty\}$) we see, inductively, that $g^I \in u\mathbb{K}_\Sigma[[u]]$ if $I \subsetneq \Sigma$ (while $g^\Sigma \in \mathbb{K}_\Sigma[[u]]$) and (71) allows to conclude. \square

This generalises Corollary 7.3 and, quantitatively, it strengthens Lemma 7.19.

We come back to Σ be a finite subset of \mathbb{N}^* with $s > 0$ elements. Let $r \geq 0$ be the unique integer such that $r(q-1) + 1 \leq s \leq (r+1)(q-1)$. We have the elementary result:

Lemma 7.22. *If $I \subsetneq \Sigma$ then $q^r \kappa(I) \geq |I|$.*

Proof. A simple computation suffices to justify this result; it is left to the reader. \square

We deduce:

Theorem 7.23. *Let $f \in M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$. Then, $\tau^r(f) \in M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$.*

Proof. Thanks to the alternative condition for strong regularity (67) and Lemma 7.21, the property is verified taking into account the elementary inequality of Lemma 7.22. \square

Note that if $s = 1$, every Drinfeld modular form for ρ_t^* is strongly regular, which is a restatement of Theorem 3.9 of [37]. It is clear that a strongly regular function is also regular. We have

$$M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) \subset M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) \subset M_w^1(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma),$$

and the inclusions are in general strict. However, as an immediate consequence of Theorem 7.23, we have:

Corollary 7.24. *If $s = |\Sigma| < q$, then $M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$. For any s , $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma)$ is of finite dimension over \mathbb{K}_Σ .*

In particular, one can check that, in the above hypotheses,

$$(72) \quad \mathcal{E}(s; \rho_\Sigma^*) = \bigotimes_{i \in \Sigma} \mathcal{E}(1; \rho_{t_i}^*).$$

In fact, the formula 72 holds also for $s = q$.

7.4.2. Non-existence of negative weight modular forms. We deduce the next result which asserts, in particular, that there are no non-zero modular forms of negative weight:

Corollary 7.25. *We have $M_w(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \{0\}$ for $w < 0$, for $w = 0$ and $m \neq 0$, or for $w = 0$ and $\Sigma \neq \emptyset$.*

Proof. Note that $M_w^\dagger(\rho_\Sigma^* \det^{-m}; \mathbb{K}_\Sigma) = \{0\}$ if $w < 0$. Hence we obtain the first assertion, combining with Theorem 7.23. The other properties are easy. \square

8. HARMONIC RELATIONS AND EISENSTEIN SERIES

In this section we move to a different and perhaps quite independent aspect of our work. In [32] we highlighted an algebra structure on *multiple zeta series* in the Tate algebras \mathbb{T}_Σ (or more precisely, in $\mathbb{E}_\Sigma \subset \mathbb{T}_\Sigma$) generalizing Thakur's multiple zeta values (see for example [1, 44]). We will see, in this section, that this algebra structure is in some sense ubiquitous in the theory of the Eisenstein series that we develop in the present paper. In particular, it is an important source of explicit relations connecting such functions.

The results of this section will cover various aspects of a harmonic product formula (Theorem 8.2 and complements) generalizing [32, Theorems 2.3, 3.1]. We recall that, as usual, Σ denotes a non-empty finite subset of \mathbb{N}^* of cardinality s . Let L/\mathbb{F}_q be a field extension. We consider:

- (1) Injective \mathbb{F}_q -linear maps $\delta_i : A \rightarrow L$, for $i \in \Sigma$.
- (2) For $\alpha_{i,j} \in \mathbb{N}$ ($i \in \Sigma$ and $j = 1, \dots, r$), maps $\sigma_j : A \rightarrow L$ defined by $\sigma_j(a) := \prod_{i \in \Sigma} \delta_i(a)^{\alpha_{i,j}}$. We shall refer to such maps as to *semi-characters* $A \rightarrow L$.
- (3) Another injective \mathbb{F}_q -linear map $\gamma : A \rightarrow L$ (we adopt the notation γ_a for the evaluation of γ in $a \in A$).

We consider a semi-character $\sigma = \prod_{i \in \Sigma} \delta_i^{\alpha_i}$ with linear maps δ_i , $i \in \Sigma$. A *factorization* of σ is a decomposition in product of semi-characters $\sigma = \psi\phi$ with $\psi = \prod_{i \in \Sigma} \delta_i^{\beta_i}$ and $\phi = \prod_{i \in \Sigma} \delta_i^{\gamma_i}$ and with $\beta_i + \gamma_i = \alpha_i$ for all $i \in \Sigma$. It is obvious that we have only finitely many such factorizations.

Together with the objects that we have introduced so far, we consider, for integers $n_i \in \mathbb{N}^*$ with $i = 1, \dots, r$ (*admissible*) *composition arrays* ⁽⁶⁾

$$(73) \quad \mathcal{C} := \begin{pmatrix} \sigma_1 & \cdots & \sigma_r \\ n_1 & \cdots & n_r \end{pmatrix}.$$

Sometimes, when $r = 1$, we write $(n; \sigma)$ instead of $\binom{\sigma}{n}$. If $\mathcal{C} = \binom{\mathbf{1}}{n_1} \cdots \binom{\mathbf{1}}{n_r}$ we abridge it to $\mathcal{C} = (n_1, \dots, n_r)$. For a composition array as in (73), we introduce the *twisted power sum*

$$S_d(\mathcal{C}) := \sum_{\substack{d_1 > \cdots > d_r \geq 0 \\ a_1, \dots, a_r \in A^+ \\ \deg_\theta(a_i) = d_i, \forall i = 1, \dots, r}} \frac{\sigma_1(a_1) \cdots \sigma_r(a_r)}{\gamma_{a_1}^{n_1} \cdots \gamma_{a_r}^{n_r}} \in L.$$

These twisted power sums generalize the classical power sums as in Thakur's [43], as well as the twisted power sums of [36]. We shall show the following simple generalization of [32, Theorem 3.1]:

⁶The attribute *admissible* indicates the fact that $n_i > 0$ for all i . In this paper, we do not study the more general case in which $n_i \in \mathbb{Z}$.

Theorem 8.1. *Let σ, ψ be two semi-characters and m, n two positive integers. For any α, β semi-characters and $i, j \in \mathbb{N}^*$ there is an element $f_{\alpha, \beta, i, j} \in \mathbb{F}_p$ such that, for all $d \geq 0$,*

$$S_d \begin{pmatrix} \sigma \\ m \end{pmatrix} S_d \begin{pmatrix} \psi \\ n \end{pmatrix} - S_d \begin{pmatrix} \sigma\psi \\ m+n \end{pmatrix} = \sum_{\substack{\alpha\beta=\sigma\psi \\ i+j=m+n}} f_{\alpha, \beta, i, j} S_d \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix}.$$

In the theorem, the sum is on the factorizations of the semi-character $\sigma\psi$ and the decompositions $n+m=i+j$, so there are only finitely many terms in it.

Further, let us assume that: (1) L is endowed with a valuation $\nu : L \rightarrow \mathbb{Q} \cup \{\infty\}$, it is complete for this valuation, (2) that $\nu(\delta_{i,j}(a)) \in \{0, \infty\}$ for all i, j and $a \in A$, and (3) that, moreover, $\gamma_a^{-1} \rightarrow 0$ as a runs in A (for the valuation). Then, the series

$$(74) \quad f_A(\mathcal{C}) := \sum_{d \geq 0} S_d(\mathcal{C})$$

converges in L for any composition array \mathcal{C} as in (73). Let n be a positive integer, and let $\sigma : A \rightarrow L$ be a semi-character such that ν is trivial over its image. We denote by \mathcal{F}_n^σ the \mathbb{F}_p -sub-vector space of L generated by the elements $f_A(\frac{\sigma_1}{n_1} \dots \frac{\sigma_r}{n_r})$ with $r > 0$, $\prod_i \sigma_i = \sigma$, $\sum_i n_i = n$ (with $n_i > 0$ for all i). We also set $\mathcal{F}_0^* := \mathbb{F}_p$. We consider the sum $\mathcal{F} := \sum_{n, \sigma} \mathcal{F}_n^\sigma$. The above result can be used, in a lengthy but straightforward way very similar to that of [32], to prove the next result.

Theorem 8.2. *For all $m, n > 0$ and σ, ψ semi-characters, We have that $\mathcal{F}_m^\sigma \mathcal{F}_n^\psi \subset \mathcal{F}_{m+n}^{\sigma\psi}$, and the \mathbb{F}_p -vector space \mathcal{F} is an \mathbb{F}_p -algebra.*

Before going further, we shall give the two main examples of the above settings that will be considered in this paper (actually, we will mainly consider the second one, described in §8.2).

8.1. Multiple zeta values. We consider variables $\underline{t}_\Sigma = \{t_i : i \in \Sigma\}$ and the field $L = \mathbb{K}_\Sigma := \widehat{K(\underline{t}_\Sigma)}_{v_\infty}$ obtained by completing $K(\underline{t}_\Sigma)$ with respect to the Gauss valuation ν extending the valuation v_∞ of K . We consider further the injective \mathbb{F}_q -algebra morphisms $\delta_i(a) := \chi_{t_i}(a)$ for all $i \in \Sigma$ to build our semi-characters. Finally, we choose γ the identity map, so that for all $a \in A$, $\gamma_a = a \in L$. Then, we are in the settings of [32]. In the notations of *ibid.*, we have $\zeta_A(\mathcal{C}) = f_A(\mathcal{C})$ for any \mathcal{C} as in (73). In fact, these elements belong, more precisely, to $\mathbb{E}_\Sigma \subset \mathbb{K}_\Sigma$. If we consider the particular case of composition arrays \mathcal{C} as in (73) such that the semi-characters σ_i are all equal to the trivial semi-character $\mathbf{1}$, then it is easy to see that the series $\zeta_A(\mathcal{C}) \in K_\infty$ are the *multiple zeta values* of Thakur (the reader can find more in the papers [1, 44], but the literature is more ample).

8.2. A-periodic multiple sums. We generalise the viewpoint of Chen in [10] in connection with analogues of multiple Eisenstein series in the Drinfeldian settings. We use the same variables \underline{t}_Σ of the example 1 above, and the same semi-characters. However, we choose this time

$$\gamma_a := e_C(az), \quad a \in A.$$

This choice leads us to work in the field $L = K(\underline{t}_\Sigma)((u))$ which is complete for the valuation $\nu = v$, giving the order at $u = 0$ of a formal power series of u . In this case, for \mathcal{C} as in (73), we set $\varphi_A(\mathcal{C}) = f_A(\mathcal{C})$. Explicitly:

$$\varphi_A(\mathcal{C}) = \sum_{\substack{d_1 > \dots > d_r \geq 0 \\ a_1, \dots, a_r \in A^+ \\ \deg_\theta(a_i) = d_i, \forall i=1, \dots, r}} \sigma_1(a_1) \cdots \sigma_r(a_r) u_{a_1}^{n_1} \cdots u_{a_r}^{n_r} \in L,$$

with $u_a = e_C(az)^{-1}$, which defines a formal series of $\mathbb{C}_\infty[[u]]$, converging for u in a non-empty disk of radius ≤ 1 containing 0. In fact, we can go a step further by considering semi-characters $\tilde{\sigma}$ of the type:

$$\tilde{\sigma}(a) = a^{n_0} \prod_{i \in \mathbb{N}^*} \chi_{t_i}(a)^{n_i}, \quad a \in A,$$

with $n_i \in \mathbb{N}$ for all i and $n_i = 0$ for all but finitely many i . Extending the admissible composition arrays \mathcal{C} in (73) in the obvious way it is easy to see that the series $\varphi_A(\mathcal{C})$ are convergent for the valuation v .

Remark 8.3. We will see in §5 how these sums can be, in certain cases, related to the first entries of modular forms with values in the modules $\mathbb{E}_\Sigma^{1 \times 2^s}$. For instance, observe that if $|\Sigma| \equiv n \pmod{q-1}$ and $n > 0$, then, by the results of §7.1 and more particularly, Proposition 7.2, $\varphi_A(n; \sigma_\Sigma)$ occurs in the first entry of an Eisenstein series $\mathcal{E}(n; \rho_\Sigma^*)$. This brings back to a generalisation of formulas first observed by Gekeler. For instance, in our notation, [15, Formula (6.3)] rewrites as

$$\mathcal{E}(k; \mathbf{1}) = -\zeta_A(k) - \tilde{\pi}^k \sum_i c_i^{(k)} \varphi_A(i; \mathbf{1}),$$

where $q-1 \mid k$, $\mathcal{E}(k; \mathbf{1}) = \sum'_{a,b} (az+b)^{-k}$ is the Eisenstein series of weight k , and the coefficients $c_i^{(k)}$ are those of the k -th Goss' polynomial $G_k(X) = \sum_i c_i^{(k)} X^i \in K[X]$ associated to the discrete A -module $\tilde{\pi}A$ of \mathbb{C}_∞ [15, §3]. A more explicit example is given by the series $\varphi_A(1; \chi_t)$, which has been also discussed in [28, 37] (with different notations). Note that after Proposition 7.2, the first entry of $\mathcal{E}(1; \chi_t)$ is $-\tilde{\pi} \varphi_A(1; \chi_t)$. This is related to the sequence of *extremal quasi-modular forms* $(x_k)_{k \geq 0}$ introduced in [8], where the initial explicit elements are $x_0 = -E$, $x_1 = -Eg - h$, in the notations of [15], and where E is the normalized false Eisenstein series of weight 2 already used in §6.7, which is a quasi-modular form in the sense of [7]. It is easy to see that, for all $u \in \mathbb{C}_\infty$ with $|u|$ small enough, $f := \varphi_A(1; \chi_t)$ defines an entire function of the variable t . Then, $\tau^k(f)_{t=\theta} = -x_k$ for all $k \in \mathbb{N}$. If $k = 0$, we deduce Gekeler's series expansion [15, p. 686]:

$$(75) \quad E = \sum_{a \in A^+} au_a,$$

while taking $f_{t=\theta^q k}$ for $k \geq 1$ we get Petrov's sequence of Hecke eigenforms $F_k = \sum_{a \in A^+} a^{q^k} u_a$ of weight $q^k + 1$ and type 1, notably the initial values $F_1 = h$ and $F_2 = hg^q$ (see [40, §3.2] and the proof of Theorem 3.6 *ibid.*).

8.3. Existence of the harmonic product. We prove Theorem 8.1. We will use the methods of [32, §3.1.2 and §3.1.3]. The following result can be found there.

Proposition 8.4. *Let Σ be a finite subset of \mathbb{N}^* . Consider U, V such that $U \sqcup V = \Sigma$. Let L/\mathbb{F}_q be a field extension and let us suppose that x_i ($i \in \Sigma$) are elements of L and let z be an element of $L \setminus \mathbb{F}_q$. Then, the following formula holds:*

$$\sum_{\mu, \nu \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} (x_i + \mu) \prod_{j \in V} (x_j + \nu)}{(z + \mu)(z + \nu)} = - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{k \in I} (x_k + \mu)}{(z + \mu)}.$$

With appropriate choices of the set Σ , of the subsets U, V , of the elements x_i and z and applying some power of an endomorphism of L which is $\mathbb{F}_q(x_i : i \in \Sigma)$ -linear and which sends z to z^q , and specialization of some x_i to z , we deduce:

Corollary 8.5. *Considering a finite set $\Sigma \subset \mathbb{N}^*$, a partition $\Sigma = U \sqcup V$, a positive integer N and a composition $N = \alpha + \beta$, for all $1 \leq k \leq N$ and $I \subset \Sigma$, there exists $c_{I,k} \in \mathbb{F}_p$ such that*

$$\sum_{\mu, \nu \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} (x_i + \mu) \prod_{j \in V} (x_j + \nu)}{(z + \mu)^\alpha (z + \nu)^\beta} = \sum_{\substack{k=1, \dots, N \\ I \subset \Sigma}} c_{I,k} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{i \in I} (x_i + \mu)}{(z + \mu)^k}.$$

We can now prove the Theorem 8.1. We recall that we have denoted by $A^+(d)$ the set of monic polynomials of degree d in A . We also denote by $A^+(< d)$ the set of monic polynomials of A which have degree $< d$. For $n \in A^+(d)$ and $m \in A^+(< d)$, we write

$$S_{n,m} = \{(n + \mu m, n + \nu); \mu, \nu \in \mathbb{F}_q, \mu \neq \nu\} \subset A^+(d) \times A^+(d) \setminus \Delta,$$

where Δ is the diagonal of $A^+(d) \times A^+(d)$. Similarly, we define for $n \in A^+(d)$ and $m \in A^+(< d)$:

$$S'_{n,m} = \{(n + \mu m, m); \mu \in \mathbb{F}_q\} \subset A^+(d) \times A^+(< d).$$

From [32, Lemmas 3.10 and 3.11] (see also Thakur's [44]), we deduce that the sets $S_{n,m}$ determine a partition of $A^+(d) \times A^+(d) \setminus \Delta$ and the sets $S'_{n,m}$ determine a partition of $A^+(d) \times A^+(< d)$. Moreover, $S'_{n,m} = S'_{n',m'}$ if and only if $S_{n,m} = S_{n',m'}$.

Now, let us choose $d > 0$. We write $\sigma\psi = \prod_{i \in \Sigma} \delta_i$ with δ_i an injective \mathbb{F}_q -linear map $A \rightarrow L$ for all $i \in \Sigma$ (there can be repetitions), and $\sigma = \prod_{i \in U} \delta_i$, $\psi = \prod_{i \in V} \delta_i$ with $U \sqcup V = \Sigma$. We have:

$$\begin{aligned} S_d \begin{pmatrix} \sigma \\ \alpha \end{pmatrix} S_d \begin{pmatrix} \psi \\ \beta \end{pmatrix} - S_d \begin{pmatrix} \sigma\psi \\ N \end{pmatrix} &= \\ &= \sum_{(a,b) \in A^+(d) \times A^+(d) \setminus \Delta} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta} \\ &= \sum_{(m,n) \in \mathcal{U}} \sum_{(a,b) \in S_{m,n}} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta}. \end{aligned}$$

We focus on the sub-sum corresponding to the choice of a set $S_{m,n}$. We want now to compute:

$$\begin{aligned}
& \sum_{(a,b) \in S_{m,n}} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta} = \\
&= \sum_{(\mu,\nu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\sigma(n + \mu m)\psi(n + \nu m)}{\gamma_{n+\mu m}^\alpha \gamma_{n+\nu m}^\beta} \\
&= \sum_{(\mu,\nu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} \delta_i(n + \mu m) \prod_{j \in V} \delta_j(n + \nu m)}{(\gamma_n + \mu \gamma_m)^\alpha (\gamma_n + \nu \gamma_m)^\beta} \\
&= \frac{\sigma(m)\psi(m)}{\gamma_m^N} \sum_{(\mu,\nu) \in \mathbb{F}_q^2 \setminus \Delta} \frac{\prod_{i \in U} \left(\frac{\delta_i(n)}{\delta_i(m)} + \mu \right) \prod_{j \in V} \left(\frac{\delta_j(n)}{\delta_j(m)} + \nu \right)}{\left(\frac{\gamma_n}{\gamma_m} + \mu \right)^\alpha \left(\frac{\gamma_n}{\gamma_m} + \nu \right)^\beta}.
\end{aligned}$$

Note that we have used the \mathbb{F}_q -linearity of δ_i for all $i \in \Sigma$ so that $\delta_i(n + \mu m) = \delta_i(n) + \mu \delta_i(m)$ and the injectivity, to divide by $\delta_i(m)$ which needs to be non-zero. Similarly, we have used the \mathbb{F}_q -linearity of the map $a \mapsto \gamma_a$ and the fact that $\gamma_n + \lambda \gamma_m$ does not vanish, because n, m , in the above computation, have distinct degrees. Applying the Corollary 8.5 with $x_i = \frac{\delta_i(n)}{\delta_i(m)}$ for $i \in \Sigma$ and $z = \frac{\gamma_n}{\gamma_m}$ which does not belong to \mathbb{F}_q , we obtain the identity:

$$\begin{aligned}
& \sum_{(a,b) \in S_{m,n}} \frac{\sigma(a)\psi(b)}{\gamma_a^\alpha \gamma_b^\beta} = \\
&= \sigma(m)\psi(m)\gamma_m^{-N} \sum_{\substack{I \subset \Sigma \\ k=1, \dots, N}} c_{I,k} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{i \in I} \left(\frac{\delta_i(n)}{\delta_i(m)} + \mu \right)}{\left(\frac{\gamma_n}{\gamma_m} + \mu \right)^k} \\
&= \sum_{\substack{I \sqcup J = \Sigma \\ k=1, \dots, N}} c_{I,k} \sum_{\mu \in \mathbb{F}_q} \frac{\prod_{i \in I} \delta_i(n + \mu m) \prod_{j \in J} \delta_j(m)}{\gamma_{n+\mu m}^k \gamma_m^{N-k}}.
\end{aligned}$$

The latter, is a sum over $S'_{m,n}$. In view of our previous observations, this concludes our proof of Theorem 8.2. The deduction of Theorem 8.2 from Theorem 8.1 is standard and we omit it. If we choose $\delta_i = \chi_{t_i}$ for $i \in \Sigma$ and $\gamma_a = e_C(az)$, and we follow closely the above proof of Theorem 8.2 in conjunction with [32, Theorem 3.1], we deduce the following explicit result that will be used later, with $\sigma_\Sigma = \prod_{i \in \Sigma} \chi_{t_i}$ and $\gamma_a = e_C(az)$ for $a \in A \setminus \{0\}$.

Theorem 8.6. *The following formula holds, for all $\Sigma \subset \mathbb{N}^*$ and $U \sqcup V = \Sigma$:*

$$f_A \begin{pmatrix} \sigma_U \\ 1 \end{pmatrix} f_A \begin{pmatrix} \sigma_V \\ 1 \end{pmatrix} - f_A \begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix} = f_A \begin{pmatrix} \sigma_U & \sigma_V \\ 1 & 1 \end{pmatrix} + f_A \begin{pmatrix} \sigma_V & \sigma_U \\ 1 & 1 \end{pmatrix} - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} f_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix}.$$

9. SOME CONJECTURES

In our general setting we do not have explicit formulas for the dimensions of the \mathbb{L}_Σ -vector spaces $M_w(\rho_\Sigma^*, \mathbb{L}_\Sigma)$. One of the reasons is that an essential tool is still missing: a *valence formula*. This is due to the particular kind of functions we are dealing with; analytic functions from a rigid analytic space with values in Banach algebras which are not necessarily affinoid. Waiting the time in which such tools will be available, we give here some conjectures which allow to produce examples of relations which can be in certain cases individually verified by explicit computations.

This section provides perspectives suggested by numerical investigations we did for modular forms associated to the representations ρ_Σ^* . Conjectures 9.1, 9.2 and 9.9 together provide a collection of (sometimes provable) identities between our Eisenstein series, introduced in §7.1.

9.1. A conjecture for zeta values in Tate algebras. We first discuss a conjecture dealing with zeta values in Tate algebras (8); for a little while we switch our attention to this setting. Recall from the introduction that $q = p^e$ with $e > 0$. Hence $\tau = \mu^e$ where μ is the \mathbb{F}_p -linear automorphism of \mathbb{C}_∞ given by $c \mapsto c^p$ for $c \in \mathbb{C}_\infty$, which can be extended $\mathbb{F}_p(\underline{t}_\Sigma)$ -linearly to \mathbb{K}_Σ for any finite set Σ . We introduce the following \mathbb{F}_p -algebra

$$\mathbb{I} := \mathbb{F}_p \left[\mu^m(\zeta_A(1, \chi_{t_i})) : \begin{matrix} i \in \mathbb{N}^* \\ m \in \mathbb{Z} \end{matrix} \right] \subset \bigcup_{k \geq 0} \mathbb{F}_p[t_i : i \in \mathbb{N}^*][[\theta^{-\frac{1}{p^k}}]].$$

We set $\zeta_A(0) := 1$. The \mathbb{F}_p -algebra \mathbb{I} is thus generated by all the μ -twists (negative or positive) of the functions $\zeta_A(1, \chi_{t_i})$ for $i \in \Sigma$. It is very important to allow negative values for m , and for this reason this \mathbb{F}_p -algebra carries a structure of inversive μ -difference algebra.

Conjecture 9.1. *For all $n \in \mathbb{N}^*$ and $\Sigma \subset \mathbb{N}^*$ such that $|\Sigma| \equiv n \pmod{q-1}$ we have a unique expansion*

$$(76) \quad \zeta_A(n; \sigma_\Sigma) = \sum_{\substack{0 \leq k \leq n \\ k \equiv 0 \pmod{q-1}}} \zeta_A(k) \eta_k, \quad \eta_k \in \mathbb{I}.$$

Recall that in our conventions, $\zeta_A(k) = \zeta_A(k; \mathbf{1})$. We are going to give some examples of relations along the predictions of this conjecture. Note that the factors η_k need not to lie in $\mathbb{F}_p[t_i : i \in \mathbb{N}^*][(\frac{1}{\theta})]$. However, there exists $l \in \mathbb{N}$ such that $\mu^l(\eta_k) \in \mathbb{F}_p[t_i : i \in \mathbb{N}^*][(\frac{1}{\theta})]$ for

all $k \equiv 0 \pmod{q-1}$ in the range $0 \leq k \leq n$ and all the terms involved are products of zeta values. Since $\mu^l(\zeta_A(k; \sigma_\Sigma)) = \zeta_A(kp^l; \sigma_\Sigma)$, the identity (76) is equivalent to an algebraic identity of zeta values as in (8) defined over \mathbb{F}_p . We recall that Thakur conjectures in [44, §5.3] that the only \mathbb{F}_p -relations among his multiple zeta values in K_∞ are those which come from the harmonic product.

Conjecture 9.2. *The only \mathbb{F}_p -algebraic relations in \mathbb{I} are those coming from the harmonic product.*

A more general conjecture, encompassing Thakur's and the above, can be stated but lies far outside the purposes of the present text and we will not mention it. After Conjecture 9.2, all the algebraic relations defined over \mathbb{F}_p between the elements $\zeta_A(n; \sigma_\Sigma)$ with $n \equiv |\Sigma| \pmod{q-1}$ can be derived from the harmonic product and for each zeta value $\zeta_A(n; \sigma_\Sigma)$ it should be possible to derive explicit formulas like in (76) by using the harmonic product of Theorem 8.6 (or in [32]). However, carrying this program might be very difficult in practice due to the combinatorial computations involved. The challenge is to introduce other techniques to tackle it. We produce some isolated evidences to give more credibility to it.

9.1.1. *Some evidences of truth.* The first example we give is a formula which holds with $n = q + 1$. It is very easy to prove by a direct application of Theorem 8.6.

Lemma 9.3. *The following formula holds:*

(77)

$$\zeta_A(q+1, \sigma_\Sigma) = \zeta_A(1, \chi_{t_1})\zeta_A(q, \chi_{t_2}) + \zeta_A(q, \chi_{t_1})\zeta_A(1, \chi_{t_2}) - \zeta_A(q-1)\zeta_A(1, \chi_{t_1})\zeta_A(1, \chi_{t_2}).$$

Proof. We have the following formulas, where $\Sigma = \{1, 2\}$, and where we also observe the formula $\zeta(1; \chi_{t_1})\zeta_A(1; \chi_{t_2}) = \zeta_A(2; \sigma_\Sigma)$:

$$\begin{aligned} \zeta_A(1, \chi_{t_1})\zeta_A(q, \chi_{t_2}) &= \zeta_A\left(\begin{smallmatrix} \sigma_\Sigma \\ q+1 \end{smallmatrix}\right) + \zeta_A\left(\begin{smallmatrix} \sigma_\Sigma & \mathbf{1} \\ 2 & q-1 \end{smallmatrix}\right) + \zeta_A\left(\begin{smallmatrix} \chi_{t_2} & \chi_{t_1} \\ 2 & q-1 \end{smallmatrix}\right) \\ \zeta_A(1, \chi_{t_2})\zeta_A(q, \chi_{t_1}) &= \zeta_A\left(\begin{smallmatrix} \sigma_\Sigma \\ q+1 \end{smallmatrix}\right) + \zeta_A\left(\begin{smallmatrix} \sigma_\Sigma & \mathbf{1} \\ 2 & q-1 \end{smallmatrix}\right) + \zeta_A\left(\begin{smallmatrix} \chi_{t_1} & \chi_{t_2} \\ 2 & q-1 \end{smallmatrix}\right) \\ \zeta_A(q-1)\zeta_A(2, \sigma_\Sigma) &= \zeta_A\left(\begin{smallmatrix} \sigma_\Sigma \\ q+1 \end{smallmatrix}\right) + 2\zeta_A\left(\begin{smallmatrix} \sigma_\Sigma & \mathbf{1} \\ 2 & q-1 \end{smallmatrix}\right) - \zeta_A\left(\begin{smallmatrix} \chi_{t_2} & \chi_{t_1} \\ 2 & q-1 \end{smallmatrix}\right) - \\ &\quad - \zeta_A\left(\begin{smallmatrix} \chi_{t_1} & \chi_{t_2} \\ 2 & q-1 \end{smallmatrix}\right). \end{aligned}$$

The formula (77) follows easily. □

Next, we focus on the case $n = 1$ so that we can now suppose that $|\Sigma| = m(q-1) + 1$ with $m \geq 0$. We know from [2, 4] that

$$(78) \quad \zeta_A(1; \sigma_\Sigma) = \frac{(-1)^m \tilde{\pi} \mathbb{B}_\Sigma}{\omega_\Sigma}, \quad |\Sigma| \equiv 1 \pmod{q-1}, \quad |\Sigma| > 1,$$

where $\mathbb{B}_\Sigma \in A[t_\Sigma]$ (\mathbb{B} stands for 'Bernoulli') is a monic polynomial in θ of degree $m - 1$ when $m \geq 1$ and $\omega_\Sigma = \prod_{i \in \Sigma} \omega(t_i) \in \mathbb{T}_\Sigma^\times$. If $m = 0$, the conjecture is clearly verified thanks to the formula (63). If $m = 1$ then $\mathbb{B}_\Sigma = 1$ by [4, Corollary 7.3] so that

$$\zeta_A(1, \sigma_\Sigma) = \tau^{-1} \left(\prod_{i \in \Sigma} \zeta_A(1, \chi_{t_i}) \right) \in \mathbb{I}$$

confirming Conjecture 9.1 also in this case.

To describe the case $m = 2$ (so that $|\Sigma| = 2q - 1$) we shall introduce the notation

$$\mathcal{L}_U^{(m)} := \tau^m \left(\prod_{i \in U} \zeta_A(1, \chi_{t_i}) \right),$$

for $U \subset \Sigma$. Then it is possible to show the following explicit formula:

$$\zeta_A(1, \sigma_\Sigma) = \sum_{\substack{\Sigma = U_1 \sqcup U_2 \\ |U_1| = q-1 \\ |U_2| = q}} \mathcal{L}_{U_1}^{(-1)} \mathcal{L}_{U_2}^{(-2)},$$

where \sqcup denotes disjoint union. Now, recall that the right-hand side is equal to $\frac{\widehat{\pi} \mathbb{B}_\Sigma^*}{\omega_\Sigma}$, with

$$\mathbb{B}_\Sigma^* = - \sum_{\substack{U_2 \subset \Sigma \\ |U_2| = q}} \prod_{i \in U_2} (t_i - \theta^{\frac{1}{q}}),$$

while the left-hand side is easily seen to be equal to $\frac{\widehat{\pi} \mathbb{B}_\Sigma}{\omega_\Sigma}$, with

$$\mathbb{B}_\Sigma = \theta - \sum_{\substack{V \subset \Sigma \\ |V| = q}} \prod_{i \in V} t_i = -e_q(t_i - \theta^{\frac{1}{q}} : i \in \Sigma)$$

(with e_n denoting here the n -th elementary symmetric polynomial), and it is easy to see that $\mathbb{B}_\Sigma = \mathbb{B}_\Sigma^*$ (all the terms defined over $\mathbb{F}_p[\theta^{\frac{1}{q}}]$ but not over $\mathbb{F}_p[\theta]$ cancel. The latter formula in the case $m = 2$ suggests the following:

Conjecture 9.4. *For all $m \geq 0$ and for all q large enough depending on m , the following formula holds:*

$$(79) \quad \zeta_A(1; \sigma_\Sigma) = \sum_{\substack{U_1 \sqcup \dots \sqcup U_m = \Sigma \\ q^{-1}|U_1| + \dots + q^{-m}|U_m| = 1}} \mathcal{L}_{U_1}^{(-1)} \dots \mathcal{L}_{U_m}^{(-m)}.$$

Although Conjecture 9.2 predicts that such formulas can all be derived from the harmonic product, the above conjecture has been verified by Ngo Dac for $m = 1, 2, 3, 4$ in [12] with a method which *does not use* the harmonic product. In this range of values for m , the formula holds regardless the value of q .

Finally, we give a last explicit formula which illustrate the case $n = 2$.

Lemma 9.5. *The following formula holds:*

$$(80) \quad \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1} \\ |V| \equiv 1 \pmod{q-1}}} \zeta_A \begin{pmatrix} \sigma_U \\ 1 \end{pmatrix} \zeta_A \begin{pmatrix} \sigma_V \\ 1 \end{pmatrix} = 2\zeta_A \begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix}.$$

If the characteristic p of \mathbb{F}_q is not equal to 2, (80) yields yet another evidence for Conjecture 9.1. If $p = 2$, the formula does not give any expression of $\zeta_A(2; \sigma_\Sigma)$. However, in this case, $\zeta_A(2; \sigma_\Sigma) = \mu(\zeta_A(1; \sigma_\Sigma))$ and Conjecture 9.4 provides a different formula to express, after nested applications of the endomorphism μ , $\zeta_A(2; \sigma_\Sigma)$ in terms of elements $\zeta_A(1; \chi_{t_i})$.

Proof of Lemma 9.5. We set $m = \alpha(q-1) + 2$ and $n = \alpha(q-1) + 1$, for $\alpha \geq 0$. We claim that

$$(81) \quad \sum_{\substack{k \equiv 0 \pmod{q-1} \\ 0 < k \leq \alpha}} \binom{n}{k} \equiv 0 \pmod{p},$$

$$(82) \quad \sum_{\substack{k \equiv 1 \pmod{q-1} \\ 0 \leq k \leq \alpha}} \binom{m}{k} \equiv 2 \pmod{p}.$$

To see this we consider more generally $N \in \mathbb{N}$ and we write $N = \alpha(q-1) + l$ with $\alpha \geq 0$ and $0 \leq l \leq q-2$. Let λ, μ be in \mathbb{F}_q . Then,

$$(\lambda + \mu)^l = (\lambda + \mu)^N = \sum_{r=0}^N \binom{N}{r} \lambda^r \mu^{N-r} = \sum_{r_0=0}^{q-2} \lambda^{r_0} \mu^{\nu(r_0)} \underbrace{\sum_{\substack{r \equiv r_0 \pmod{q-1} \\ 0 \leq r \leq N}} \binom{N}{r}}_{=: \beta_{r_0}},$$

where $\nu(r_0)$ is the unique integer in $\{0, \dots, q-2\}$ such that $l - r_0 \equiv \nu(r_0) \pmod{q-1}$. Setting further $\lambda = 1$, we have the polynomial

$$P(X) = (X+1)^l - \sum_{r_0=0}^{q-2} \beta_{r_0} X^{\nu(r_0)} \in \mathbb{F}_p[X],$$

which vanishes identically over \mathbb{F}_q , and has degree $\leq q-2$. This implies that it is identically zero. Taking $N = m = \alpha(q-1) + 2$ we have $l = 2$ and computing the coefficient of X in P , we deduce (81). Taking $N = n = \alpha(q-1) + 1$ and computing the constant term of P , we deduce (82). This shows the claim. We can complete the proof of formula 80. We use

Theorem 8.6, which tells us that if $U \sqcup V = \Sigma$ with $|U| \equiv |V| \equiv 1 \pmod{q-1}$,

$$\zeta_A \begin{pmatrix} \sigma_U \\ 1 \end{pmatrix} \zeta_A \begin{pmatrix} \sigma_V \\ 1 \end{pmatrix} - \zeta_A \begin{pmatrix} \sigma_\Sigma \\ 2 \end{pmatrix} = \zeta_A \begin{pmatrix} \sigma_U & \sigma_V \\ 1 & 1 \end{pmatrix} + \zeta_A \begin{pmatrix} \sigma_V & \sigma_U \\ 1 & 1 \end{pmatrix} - \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} \zeta_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix}.$$

We sum these identities over all such partitions $\Sigma = U \sqcup V$. First of all, the number of such partitions is equal to

$$\sum_{k=0}^{\alpha} \binom{s}{k(q-1)+1}$$

which is congruent to 2 modulo p by (81). Let

$$f : \mathcal{P}(\Sigma)^2 \rightarrow L$$

be any map with values in a field L of characteristic p , where $\mathcal{P}(\Sigma)$ is the set of subsets of Σ . Then,

$$\begin{aligned} & \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1}}} \left(\sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subset U \text{ or } J \subset V}} f(I, J) - f(U, V) - f(V, U) \right) = \\ &= \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1}}} \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1} \\ J \subsetneq U \text{ or } J \subsetneq V}} f(I, J) \\ &= \sum_{\substack{I \sqcup J = \Sigma \\ |J| \equiv 1 \pmod{q-1}}} f(I, J) \sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1} \\ U \supseteq J \text{ or } V \supseteq J}} 1, \end{aligned}$$

which vanishes by (82). Observing that we can choose $f(I, J) = \zeta_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix}$ terminates the proof. \square

9.1.2. *More about Conjecture 9.4.* We conclude this sub-section with some complements about Conjecture 9.4. It is not hard to show that it is equivalent to the following:

Conjecture 9.6. *Assuming that $m \geq 2$ and that q is large enough, depending on m , we have the formula*

$$\zeta_A(1, \sigma_\Sigma) = \sum_{r=0}^{m-2} \sum_{\substack{U \sqcup V \sqcup \Sigma' = \Sigma \\ |V| = q-r-1 \\ |U| = rq}} \tau^{-1}(\zeta_A(1, \sigma_{\Sigma'})) \mathcal{L}_U^{(-2)} \mathcal{L}_V^{(-1)}.$$

The interest of Conjecture 9.6 is that it can be considered in parallel with analogue classical formulas by Euler. We recall that the well-known Riccati's differential equation $f' = -1 - f^2$ satisfied by the cotangent function $f(x) = \cot(x)$ implies, via the formula $-\frac{\pi x}{2} \cot(\pi x) = \sum_{i \geq 0} \zeta(2i)x^{2i}$:

$$\left(n + \frac{1}{2}\right) \zeta(2n) = \sum_{i=1}^{n-1} \zeta(2i)\zeta(2n-2i), \quad n > 1.$$

Note that the coefficients in the quadratic expression on the right-hand side are all equal to 1.

Conjecture 9.4 implies nice formulas for the polynomials $\mathbb{B}_\Sigma \in A[\underline{t}_\Sigma]$ (when $|\Sigma| > q$). Indeed, observe that for all $m \geq 1$,

$$(83) \quad \tau^{-m}((\theta - t)\omega)^{-1} = (t - \theta^{\frac{1}{q^{m-1}}}) \cdots (t - \theta^{\frac{1}{q}})\omega^{-1}.$$

Hence,

$$\tau^{-m}(\zeta_A(1, \chi_t)) = -\frac{\tilde{\pi}^{\frac{1}{q^m}} (t - \theta^{\frac{1}{q^{m-1}}}) \cdots (t - \theta^{\frac{1}{q}})}{\omega}, \quad m \geq 1.$$

Setting $b_m^* := (t - \theta^{\frac{1}{q^{m-1}}}) \cdots (t - \theta^{\frac{1}{q}})$ (again for $m \geq 1$) and $B_m^*(\underline{t}_\Sigma) = \prod_{i \in \Sigma} b_m^*(t_i)$, we thus have:

Conjecture 9.7. *The following formula holds.*

$$\mathbb{B}_\Sigma = (-1)^{m-1} \sum_{\substack{U_1 \sqcup \cdots \sqcup U_m = \Sigma \\ q^{-1}|U_1| + \cdots + q^{-m}|U_m| = 1}} B_1^*(\underline{t}_{U_1}) \cdots B_m^*(\underline{t}_{U_m}).$$

Similarly, Conjecture 9.6 is equivalent to the following.

Conjecture 9.8. *The following formula holds, for $|\Sigma| = m(q-1) + 1$ with $m \geq 2$.*

$$\mathbb{B}_\Sigma = \sum_{r=0}^{m-2} (-1)^{r+1} \sum_{\substack{U \sqcup V \sqcup \Sigma' = \Sigma \\ |U| = qr \\ |V| = q-r-1 \\ |\Sigma'| = (m-r-1)(q-1)+1}} \tau^{-1}(\mathbb{B}_{\Sigma'}) \prod_{i \in U \sqcup \Sigma'} (t_i - \theta^{\frac{1}{q}}).$$

9.2. Realisations of an algebra of multiple zeta values. We now state another conjecture which allows, in combination with Conjectures 9.1 and 9.2, to describe what we can expect for the relations between Eisenstein series $\mathcal{E}(n; \rho_\Sigma^*)$ of §7.1. As we said already, this is useful because we do not know yet how to compute the dimensions of the spaces $M_w(\rho_\Sigma^*)$. Returning to the setting of §8 we now write \mathcal{Z}_ζ for the \mathbb{F}_p -algebra $\mathcal{F} = \sum_{n, \sigma} \mathcal{F}_{n, \sigma}$ where $\mathcal{F}_{n, \sigma}$ is the \mathbb{F}_p -subvector space of $\mathbb{F}_p[t_i : i \in \mathbb{N}][[\frac{1}{\theta}]]$ (with the Gauss norm $\|\cdot\|$ extending $|\cdot|$) generated by the sums $f_A(\mathcal{C})$ of (74) in the settings of §8.1 modified so that the semi-characters σ involved in the compositions arrays (73) are maps from A to $\mathbb{F}_q[t_i : i \in \mathbb{N}]$

defined by

$$(84) \quad \sigma(a) = \prod_{i \in \mathbb{N}} \chi_{t_i}(a)^{n_i}, \quad a \in A,$$

with $n_i \in \mathbb{N}$ and $n_i = 0$ for all but finitely many $i \in \mathbb{N}$ (so the variable t_0 is allowed). In this case we prefer to write $\zeta_A(\mathcal{C})$ instead of $f_A(\mathcal{C})$. The algebra \mathcal{Z}_ζ is the \mathbb{F}_p -algebra of the *multiple zeta values* (in Tate algebras). Additionally, it is not difficult to show that \mathcal{Z}_ζ is graded by n and σ .

Similarly, we write \mathcal{Z}_φ for the \mathbb{F}_p algebra $\mathcal{F} = \sum_{n,\sigma} \mathcal{F}_{n,\sigma}$ where $\mathcal{F}_{n,\sigma}$ is this time the \mathbb{F}_p -subvector space of $\mathbb{F}_p[\theta][t_i : i \in \mathbb{N}^*][[u]]$ (with the v -valuation) generated by the sums $f_A(\mathcal{C})$ of (8.2) modified along the settings of §8.2 where additionally, the composition arrays include semi-characters $\tilde{\sigma} : A \rightarrow A[t_i : i \in \mathbb{N}^*]$ such as:

$$(85) \quad \tilde{\sigma}(a) = a^{n_0} \prod_{i \in \mathbb{N}^*} \chi_{t_i}(a)^{n_i}, \quad a \in A,$$

with $n_i \in \mathbb{N}^*$ satisfying conditions similar to those of (84) (so this time the variable t_0 is excluded, but the variable θ is included). We have said that \mathcal{Z}_ζ and \mathcal{Z}_φ are \mathbb{F}_p -algebras; this of course follows from Theorem 8.2. However, we do not know how to prove that \mathcal{Z}_φ is graded by the vector spaces $\mathcal{F}_{n,\sigma}$ like \mathcal{Z}_ζ . The algebra \mathcal{Z}_ζ is the algebra of *A-periodic multiple sums*. Note the $\mathbb{F}_q[t_i : i \in \mathbb{N}^*]$ -linear map $\mathbb{F}_q[t_i : i \in \mathbb{N}] \rightarrow A[t_i : i \in \mathbb{N}^*]$ defined by $t_0 \mapsto \theta$ induces a map on semi-characters associating σ as in (84) to $\tilde{\sigma}$ as in (85) and therefore induces a correspondence $\zeta_A(\mathcal{C}) \leftrightarrow \varphi_A(\mathcal{C})$. We can state our conjecture:

Conjecture 9.9. *The correspondence $\zeta \leftrightarrow \varphi$ induces an isomorphism of \mathbb{F}_p -algebras $\mathcal{Z}_\zeta \cong \mathcal{Z}_\varphi$.*

Conjecture 9.9 describes the algebras \mathcal{Z}_ζ and \mathcal{Z}_φ , hypothetically, as distinct realisations of a single \mathbb{F}_p -algebra. In fact, infinitely many such conjectural realisations of this \mathbb{F}_p -algebra should exist (the reader can imagine them by just replacing the Carlitz exponential in the definition of the uniformiser u by analytic families of exponential functions for more general Drinfeld A -modules), but we only focus on these two in this text. Conjecture 9.9 implies that \mathcal{Z}_φ is graded, just because it is known that \mathcal{Z}_ζ is. Moreover, all the identities for multiple zeta values in \mathcal{Z}_ζ of §9.1 (conjectural or proved) transfer to analogue identities for multiple A -periodic sums, many of which can be proved directly (e.g. Lemmas 9.3 and 9.5).

9.3. Identities for modular forms. We need the following:

Lemma 9.10. *For all Σ the representations $\rho_\Sigma, \rho_\Sigma^*$ are irreducible over the vector space $\mathbb{F}_q(\underline{t}_\Sigma)^{1 \times N}$.*

Proof. This can be easily deduced from the arguments in [31]. □

In particular, for any $I \subset \Sigma$, the map $M_w(\rho; \mathbb{L}_\Sigma) \rightarrow \text{Hol}_{\mathbb{K}_\Sigma}(\Omega \rightarrow \mathbb{L}_\Sigma)$ defined by $f \mapsto f^I$ (projection on the I -th coordinate) for $\rho = \rho_\Sigma$ or $\rho = \rho_\Sigma^*$ is injective. This allows us to deduce identities for modular forms by using the harmonic product structure introduced

in §8. To illustrate it with an example we choose $k \in \Sigma$ and we write $\Sigma' := \Sigma \setminus \{k\}$. For all $s = 1, \dots, q$ and $\Sigma = \{1, \dots, s\}$, the harmonic product formula of Theorem 8.6 yields inductively

$$\varphi_A(s-1, \sigma_{\Sigma'}) \varphi_A(1, \chi_{t_k}) = \varphi_A(s, \sigma_{\Sigma}),$$

which implies (72). This formula can also be written more explicitly in the following way:

$$\prod_{i \in \Sigma} \left(\sum_{a \in A^+} \chi_{t_i}(a) u_a \right) = \sum_{a \in A^+} \sigma_{\Sigma}(a) u_a^s.$$

9.3.1. *Serre's derivatives of Eisenstein series.* We return to the operators $\partial_n^{(w)}(f)$ introduced in §6.7. We suppose that $\Sigma \subset \mathbb{N}^*$ is such that $s = |\Sigma| \equiv 1 \pmod{q-1}$ and we study the u -expansion of the first entry (indexed by \emptyset) of

$$\partial_1^{(1)}(\mathcal{E}(1; \rho_{\Sigma}^*)) \in S_3(\rho_{\Sigma}^* \det^{-1}; \mathbb{K}_{\Sigma}).$$

By Proposition 7.2, the first entry of $\mathcal{E}(1; \rho_{\Sigma}^*)$ is equal to $-\tilde{\pi} \varphi_A(1; \sigma_{\Sigma})$. We compute, by setting $\Sigma' = \Sigma \sqcup \{0\}$:

$$\begin{aligned} \partial_1^{(1)}(f_A(1; \sigma_{\Sigma})) &= \\ &= \sum_{a \in A^+} \sigma_{\Sigma}(a) a u_a^2 - \sum_{a \in A^+} a u_a \sum_{b \in A^+} \sigma_{\Sigma}(b) u_b \\ &= -[\varphi_A(1; \chi_{t_0}) \varphi_A(1; \sigma_{\Sigma}) - \varphi_A(2; \sigma_{\Sigma'})]_{t_0=\theta}. \end{aligned}$$

Hence:

Lemma 9.11. *We have the formula:*

$$\partial_1^{(1)}(f_A(1; \sigma_{\Sigma})) = \left[\sum_{\substack{I \sqcup J = \Sigma' \\ |J| \equiv 1 \pmod{q-1} \\ J = \{0\} \text{ or } J \subset \Sigma'}} \varphi_A \begin{pmatrix} \sigma_I & \sigma_J \\ 1 & 1 \end{pmatrix} - \varphi_A \begin{pmatrix} \chi_{t_0} & \sigma_{\Sigma} \\ 1 & 1 \end{pmatrix} - \varphi_A \begin{pmatrix} \sigma_{\Sigma} & \chi_{t_0} \\ 1 & 1 \end{pmatrix} \right]_{t_0=\theta}.$$

Proof. This follows directly from Theorem 8.6 (we interpret Serre's derivatives in terms of specialisations of the harmonic relations of §8). \square

In particular we have:

Lemma 9.12. *If $s = |\Sigma| \leq q-1$, then $\partial_1^{(s)}(\mathcal{E}(s; \rho_{\Sigma}^*)) = 0$.*

We propose, if $s = |\Sigma| \equiv 1 \pmod{q-1}$:

Conjecture 9.13. *The form $\partial_1^{(1)}(\mathcal{E}(1; \sigma_{\Sigma}))$ and the last entry of $\mathcal{P}_{3,1}(G)$ are proportional with a proportionality factor in $\mathbb{L}_{\Sigma}^{\times}$.*

In the conjecture, G is as in Proposition 5.19. Similarly, the forms $\partial_1^{(q-1)}(g)$ and h are proportional. Additionally, it is plausible that in the case $s = q$, $\partial_n^{(q-1)}(\mathcal{E}(1; \sigma_{\Sigma})) \neq 0$ for $n = 1, \dots, q-2$.

9.4. Identities for Eisenstein series. We suppose that $|\Sigma| \equiv 2 \pmod{q-1}$ and we write $s = |\Sigma| = \alpha(q-1) + 2$, $\alpha \in \mathbb{N}$. We have:

Lemma 9.14. *The following formula holds:*

$$\sum_{\substack{U \sqcup V = \Sigma \\ |U| \equiv 1 \pmod{q-1} \\ |V| \equiv 1 \pmod{q-1}}} \mathcal{E}(1; \varphi_U) \otimes \mathcal{E}(1; \varphi_V) = 2\mathcal{E}(2; \varphi_\Sigma).$$

Proof. This is a simple combination of Lemma 9.5 and Lemma 9.10. \square

As a complement of Lemma 9.14 we propose the following conjecture.

Conjecture 9.15. *The forms $\mathcal{E}(1; \rho_U^*) \otimes \mathcal{E}(1; \rho_V^*)$, for $U \sqcup V = \Sigma$ and $|U| \equiv |V| \equiv 1 \pmod{q-1}$ generate the module $M_2(\rho_\Sigma^*; \mathbb{K}_\Sigma)$.*

The identity for Eisenstein series of Lemma 9.14 is defined over \mathbb{F}_p . However, in general, there are identities between Eisenstein series which are not defined over \mathbb{F}_p . Here is an explicit and instructive example. We use $\Sigma = \{1, 2\}$ and we suppose that $q > 2$.

Lemma 9.16. *The following identity holds:*

$$\mathcal{E}(q+1; \rho_\Sigma^*) = \mathcal{E}(1; \rho_{t_1}^*) \otimes \mathcal{E}(q; \rho_{t_2}^*) + \mathcal{E}(q; \rho_{t_1}^*) \otimes \mathcal{E}(1; \rho_{t_2}^*) + (\theta^q - \theta)^{-1} g\mathcal{E}(1, \rho_{t_1}^*) \otimes \mathcal{E}(1, \rho_{t_2}^*).$$

Proof. We note that since $q > 2$, $\mathcal{E}(2, \rho_\Sigma^*) = \mathcal{E}(1, \rho_{t_1}^*) \otimes \mathcal{E}(1, \rho_{t_2}^*)$ by (72). The first coordinates of the modular forms $\mathcal{E}(q+1; \varphi_\Sigma)$, $\mathcal{E}(1; \varphi_{t_1}) \otimes \mathcal{E}(q; \varphi_{t_2})$, $\mathcal{E}(q; \varphi_{t_1}) \otimes \mathcal{E}(1; \varphi_{t_2})$, $g\mathcal{E}(2, \varphi_\Sigma)$ are given by the following A -expansions (where $G_n(X)$ denotes the n -th Goss polynomial [15, §(3.4)]):

$$\begin{aligned} \mathcal{X} &:= \sum_{a \in A^+} \sigma_\Sigma(a) G_{q+1}(u_a), \\ \mathcal{Y}_1 &:= \left(\sum_{a \in A^+} \chi_{t_1}(a) u_a \right) \left(\sum_{b \in A^+} \chi_{t_2}(b) u_b^q \right) = \varphi_A(1, \chi_{t_1}) \varphi_A(q, \chi_{t_2}), \\ \mathcal{Y}_2 &:= \left(\sum_{a \in A^+} \chi_{t_2}(a) u_a \right) \left(\sum_{b \in A^+} \chi_{t_1}(b) u_b^q \right) = \varphi_A(q, \chi_{t_1}) \varphi_A(q, \chi_{t_2}), \\ \mathcal{Z} &:= \left(1 - (\theta^q - \theta) \sum_{a \in A^+} u_a^{q-1} \right) \left(\sum_{a \in A^+} \sigma_\Sigma(a) u_a^2 \right) = (1 - (\theta^q - \theta) \varphi_A(q-1)) \varphi_A(2, \sigma_\Sigma). \end{aligned}$$

Note that $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{F}_{q+1}^{\sigma_\Sigma}$ so they are homogeneous for the hypothetical graduation of $\mathcal{Z}_\varphi = \oplus_{n, \sigma} \mathcal{F}_n^\sigma$. On the other side, we have \mathcal{X} and \mathcal{Z} which are not homogeneous. Note indeed that if $G_n(X)$ denotes the n -th Goss polynomial (see [15, §(3.4)]), a simple computation yields $G_{q+1}(X) = X^{q+1} + (\theta^q - \theta)^{-1} X^2$. Hence

$$\mathcal{X} = (\theta^q - \theta)^{-1} \varphi_A(2; \sigma_\Sigma) + \varphi_A(q+1; \sigma_\Sigma)$$

which is not homogeneous for the weights of multiple A -periodic sums. By Conjecture 9.9, if the elements $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ and \mathcal{Z} are linearly dependent, a linear dependence relation must come from two homogeneous ones, more precisely one in $\mathcal{F}_{q+1}^{\sigma_\Sigma}$ and another one in $\mathcal{F}_2^{\sigma_\Sigma}$, both defined over \mathbb{F}_p . Through Conjecture 9.9 we see that these relations are indeed derived from (77) and the identity $\varphi_A(2; \sigma_\Sigma) = \varphi_A(1; \chi_{t_1})\varphi_A(1; \chi_{t_2})$. \square

The coefficient $(\theta^q - \theta)^{-1}$ in the formula of the lemma is therefore determined by the coefficient of X^2 in the Goss' polynomial $G_{q+1}(X)$.

We end this work with a conjectural formula which can be derived from Conjecture 9.4. We set, with $|\Sigma| = s = m(q - 1) + 1$:

$$\mathcal{E}_U^{(m)} := \tau^m \left(\bigotimes_{i \in U} \mathcal{E}(1; \rho_{t_i}^*) \right),$$

for $U \subset \Sigma$. Note that this needs not to represent a holomorphic function $\Omega \rightarrow \mathbb{L}_\Sigma^{N \times 1}$ for $N \geq 1$ if $m < 0$.

Conjecture 9.17. *For all $m \geq 0$ and for all q large enough depending on m , the following formula holds:*

$$(86) \quad \mathcal{E}(1; \rho_\Sigma^*) = \sum_{\substack{U_1 \sqcup \dots \sqcup U_m = \Sigma \\ q^{-1}|U_1| + \dots + q^{-m}|U_m| = 1}} \mathcal{E}_{U_1}^{(-1)} \otimes \dots \otimes \mathcal{E}_{U_m}^{(-m)}.$$

We note that (86) expresses the analytic function $\mathcal{E}(1; \rho_\Sigma^*)$ as a combination of non-analytic functions if $s \geq 2q - 1$. Clearly, Conjectures 9.4 and 9.9 imply Conjecture 9.17 (and the latter implies Conjecture 9.4). The cases $s = 1, q$ are obviously verified, see (72). The case $s = 2q - 1$ is at the moment still unsolved. The author was only able to see that the u -expansions of the \emptyset -coordinates of both sides in (86) agree up to a certain order but this is not enough to conclude.

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