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Distinguishing numbers and distinguishing indices of oriented graphs

Kahina MESLEM¹ Éric SOPENA^{2,3}

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Abstract

A distinguishing r -vertex-labelling (resp. r -edge-labelling) of an undirected graph G is a mapping λ from the set of vertices (resp. the set of edges) of G to the set of labels $\{1, \dots, r\}$ such that no non-trivial automorphism of G preserves all the vertex (resp. edge) labels. The distinguishing number $D(G)$ and the distinguishing index $D'(G)$ of G are then the smallest r for which G admits a distinguishing r -vertex-labelling or r -edge-labelling, respectively. The distinguishing chromatic number $D_\chi(G)$ and the distinguishing chromatic index $D_{\chi'}(G)$ are defined similarly, with the additional requirement that the corresponding labelling must be a proper colouring.

These notions readily extend to oriented graphs, by considering arcs instead of edges. In this paper, we study the four corresponding parameters for oriented graphs whose underlying graph is a path, a cycle, a complete graph or a bipartite complete graph. In each case, we determine their minimum and maximum value, taken over all possible orientations of the corresponding underlying graph, except for the minimum values for unbalanced complete bipartite graphs $K_{m,n}$ with $m = 2, 3$ or 4 and $n > 3, 6$ or 13 , respectively, or $m \geq 5$ and $n > 2^m - \lfloor \frac{m}{2} \rfloor$, for which we only provide upper bounds.

Keywords: Distinguishing number; Distinguishing index; Distinguishing chromatic number; Distinguishing chromatic index; Automorphism group; Oriented graph; Complete bipartite graph.

MSC 2010: 05C20, 20B25.

1 Introduction

All graphs considered in this paper are simple. For a graph G , we denote by $V(G)$ its set of vertices, and by $E(G)$ its set of edges.

An r -vertex-labelling of a graph G is a mapping λ from $V(G)$ to the set of labels $\{1, \dots, r\}$. An r -vertex-colouring of G is a proper r -vertex-labelling of G , that is, an r -vertex-labelling λ such that $\lambda(u) \neq \lambda(v)$ for every edge uv of G . The *chromatic number* $\chi(G)$ of G is then the smallest number of labels (called colours in that case) needed for a vertex-colouring of G . Similarly, an r -edge-labelling of G is a mapping λ' from $E(G)$ to the set of labels $\{1, \dots, r\}$, and

¹LaROMaD, Faculty of mathematics, University of Sciences and Technology Houari Boumediene (USTHB), B.P. 32 El-Alia, Bab-Ezzouar, 16111 Algiers, Algeria.

²Univ. Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR5800, F-33400 Talence, France.

³Corresponding author. Eric.Sopena@labri.fr

	Graph G	$\chi(G)$	$D(G)$	$D_\chi(G)$	$OD^-(G)$	$OD^+(G)$	$OD_\chi^-(G)$	$OD_\chi^+(G)$
1.	$P_{2n},$ $n \geq 1$	2	2	2	1 (Th. 5)	1 (Th. 6)	2 (Th. 5)	2 (Th. 6)
2.	$P_{2n+1},$ $n \geq 1$	2	2	3	1 (Th. 5)	2 (Th. 6)	2 (Th. 5)	3 (Th. 6)
3.	C_4	2	3	4	1 (Th. 5)	2 (Th. 7)	2 (Th. 5)	4 (Th. 7)
4.	C_5	3	3	3	1 (Th. 5)	2 (Th. 7)	3 (Th. 5)	3 (Th. 7)
5.	C_6	2	2	4	1 (Th. 5)	2 (Th. 7)	2 (Th. 5)	3 (Th. 7)
6.	$C_{2n},$ $n \geq 4$	2	2	3	1 (Th. 5)	2 (Th. 7)	2 (Th. 5)	3 (Th. 7)
7.	$C_{2n+1},$ $n \geq 3$	3	2	3	1 (Th. 5)	2 (Th. 7)	3 (Th. 5)	3 (Th. 7)
8.	$K_n,$ $n \geq 3$	n	n	n	1 (Th. 5)	2 (Th. 8)	n (Th. 5)	n (Th. 8)
9.	$K_{1,n},$ $n \geq 2$	2	n	$n + 1$	$\lceil \frac{n}{2} \rceil$ (Th. 9)	n (Th. 9)	$1 + \lceil \frac{n}{2} \rceil$ (Th. 9)	$n + 1$ (Th. 9)
10.	$K_{n,n},$ $n \geq 2$	2	$n + 1$	$2n$	1 (Th. 12)	n (Th. 12)	2 (Th. 12)	$2n$ (Th. 12)
11.	$K_{m,n},$ $n > m \geq 2$	2	n	$m + n$	Th. 15 and Cor. 22	n (Th. 15)	Th. 15 and Cor. 22	$m + n$ (Th. 15)

Table 1: Table of results for $OD^-(G)$, $OD^+(G)$, $OD_\chi^-(G)$ and $OD_\chi^+(G)$.

an r -edge-colouring of G is a proper r -edge-labelling of G , that is, an r -edge-labelling λ' such that $\lambda'(e) \neq \lambda'(e')$ for every two adjacent edges e and e' (that is, such that e and e' have one vertex in common). The *chromatic index* $\chi'(G)$ of G is then the smallest number of labels (or colours) needed for an edge-colouring of G .

An *automorphism* ϕ of a graph G is an edge-preserving mapping from $V(G)$ to $V(G)$, that is, such that $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(G)$. For a given vertex or edge-labelling of G , an automorphism ϕ of G is λ -preserving if $\lambda(\phi(u)) = \lambda(u)$ for every vertex u of G , or $\lambda(\phi(uv)) = \lambda(uv)$ for every edge uv of G , respectively. A vertex or edge-labelling λ of G is *distinguishing* if the only λ -preserving automorphism of G is the identity, that is, the labelling λ breaks all the symmetries of G . Such a distinguishing vertex or edge-labelling is *optimal* if G does not admit any vertex or edge-labelling using less colours.

The *distinguishing number*, *distinguishing chromatic number*, *distinguishing index* and *distinguishing chromatic index* of a graph G , denoted by $D(G)$, $D_\chi(G)$, $D'(G)$ and $D_{\chi'}(G)$, respectively, are then defined as the smallest r for which G admits a distinguishing r -vertex-labelling, a distinguishing r -vertex-colouring, a distinguishing r -edge-labelling or a distinguishing r -edge-colouring, respectively. Distinguishing numbers and distinguishing chromatic numbers have been introduced by Albertson and Collins in [3] and Collins and Trenk in [9], respectively, while distinguishing indices and distinguishing chromatic indices have been introduced by Kalinowski and Piłśniak in [15] (these two parameters are often denoted χ_D and χ'_D instead of D_χ and $D_{\chi'}$, respectively).

A graph G is *rigid* (or *asymmetric*) if the only automorphism of G is the identity. Therefore, $D(G) = 1$ if and only if G is rigid and, similarly, $D'(G) = 1$ if and only if G is rigid. Moreover, for every such graph G , $D_\chi(G) = \chi(G)$ and $D_{\chi'}(G) = \chi'(G)$. Note here that being rigid is not a necessary condition for any of these two equalities to hold (consider the path of order 2 or the path of order 3, respectively).

	Graph G	$\chi'(G)$	$D'(G)$	$D_{\chi'}(G)$	$OD'^-(G)$	$OD'^+(G)$	$OD_{\chi'}^-(G)$	$OD_{\chi'}^+(G)$
1.	$P_{2n},$ $n \geq 2$	2	2	3	1 (Th. 5)	1 (Th. 6)	2 (Th. 5)	2 (Th. 6)
2.	$P_{2n+1},$ $n \geq 1$	2	2	2	1 (Th. 5)	2 (Th. 6)	2 (Th. 5)	2 (Th. 6)
3.	C_4	2	3	4	1 (Th. 5)	2 (Th. 7)	2 (Th. 5)	3 (Th. 7)
4.	C_5	3	3	3	1 (Th. 5)	2 (Th. 7)	3 (Th. 5)	3 (Th. 7)
5.	C_6	2	2	4	1 (Th. 5)	2 (Th. 7)	2 (Th. 5)	3 (Th. 7)
6.	$C_{2n},$ $n \geq 4$	2	2	3	1 (Th. 5)	2 (Th. 7)	2 (Th. 5)	3 (Th. 7)
7.	$C_{2n+1},$ $n \geq 3$	3	2	3	1 (Th. 5)	2 (Th. 7)	3 (Th. 5)	3 (Th. 7)
8.	K_3	3	3	3	1 (Th. 5)	2 (Th. 8)	3 (Th. 5)	3 (Th. 8)
9.	K_4	3	3	5	1 (Th. 5)	2 (Th. 8)	3 (Th. 5)	3 (Th. 8)
10.	K_5	5	3	5	1 (Th. 5)	2 (Th. 8)	5 (Th. 5)	5 (Th. 8)
11.	$K_{2n},$ $n \geq 3$	$2n - 1$	2	$2n - 1$	1 (Th. 5)	2 (Th. 8)	$2n - 1$ (Th. 5)	$2n - 1$ (Th. 8)
12.	$K_{2n+1},$ $n \geq 3$	$2n + 1$	2	$2n + 1$	1 (Th. 5)	2 (Th. 8)	$2n + 1$ (Th. 5)	$2n + 1$ (Th. 8)
13.	$K_{1,n},$ $n \geq 3$	n	n	n	$\lceil \frac{n}{2} \rceil$ (Th. 9)	n (Th. 9)	n (Th. 9)	n (Th. 9)
14.	$K_{3,3}$	3	3	5	1 (Th. 12)	2 (Th. 12)	3 (Th. 12)	4 (Th. 12)
15.	$K_{n,n},$ $4 \leq n \leq 6$	n	2	$n + 1$	1 (Th. 12)	2 (Th. 12)	n (Th. 12)	$n + 1$ (Th. 12)
16.	$K_{n,n},$ $n \geq 7$	n	2	$n + 1$	1 (Th. 12)	2 (Th. 12)	n (Th. 12)	n (Th. 12)
17.	$K_{m,n},$ $n > m \geq 2$	n	[11], [13]	n	Th. 15 and Cor. 22	$D'(K_{m,n})$ (Th. 15)	Th. 15 and Cor. 22	n (Th. 15)

Table 2: Table of results for $OD'^-(G)$, $OD'^+(G)$, $OD_{\chi'}^-(G)$ and $OD_{\chi'}^+(G)$.

Our aim in this paper is to study the distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index of several classes of oriented graphs. By oriented graphs, we mean here antisymmetric digraphs, that is, digraphs with no directed cycle of length at most 2, or, equivalently, digraphs obtained from undirected graphs by giving to each of their edges one of its two possible orientations. All the notions of vertex-labelling, vertex-colouring, edge-labelling, edge-colouring, automorphism, distinguishing labelling, distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index, readily extends to oriented graphs by simply considering arcs instead of edges. For each undirected graph G with m edges, and each distinguishing parameter, we will study both the minimum and maximum possible value of the parameter, taken over all of the 2^m possible orientations of G .

Distinguishing numbers of digraphs have been studied in a few papers (see [2, 17, 18, 19, 20, 21]), while distinguishing numbers or indices of various classes of undirected graphs have attracted a lot of attention (see for instance [1, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16]). Up to our knowledge, distinguishing chromatic number, distinguishing index and distinguishing chromatic

index of digraphs have not been considered yet.

Our paper is organised as follows. In Section 2, we formally introduce the main definitions and give some basic results. We then consider simple classes of graphs, namely paths, cycles and complete graphs in Section 3, and complete bipartite graphs in Sections 4 and 5. We finally propose some directions for future work in Section 6.

Our results concerning distinguishing numbers and distinguishing indices are summarized in Tables 1 and 2, respectively, where P_n , C_n and K_n denote the path, the cycle and the complete graph of order n , respectively, and $K_{m,n}$ denotes the complete bipartite graph whose parts have size m and n .

2 Preliminaries

An *oriented graph* is a digraph with no loops and no pairs of opposite arcs. For an oriented graph \vec{G} , we denote by $V(\vec{G})$ and $A(\vec{G})$ its set of vertices and its set of arcs, respectively. Let \vec{G} be an oriented graph and u a vertex of \vec{G} . The *out-degree* of u in \vec{G} , denoted $d_{\vec{G}}^+(u)$, is the number of arcs in $A(\vec{G})$ of the form uv , and the *in-degree* of u in \vec{G} , denoted $d_{\vec{G}}^-(u)$, is the number of arcs in $A(\vec{G})$ of the form vu . The *degree* of u , denoted $d_{\vec{G}}(u)$, is then defined by $d_{\vec{G}}(u) = d_{\vec{G}}^+(u) + d_{\vec{G}}^-(u)$. If uv is an arc in \vec{G} , u is an *in-neighbour* of v and v is an *out-neighbour* of u . We denote by $N_{\vec{G}}^+(u)$ and $N_{\vec{G}}^-(u)$ the set of out-neighbours and the set of in-neighbours of u in \vec{G} , respectively. Hence, $d_{\vec{G}}^+(u) = |N_{\vec{G}}^+(u)|$ and $d_{\vec{G}}^-(u) = |N_{\vec{G}}^-(u)|$. A *source vertex* is a vertex with no in-neighbours, while a *sink vertex* is a vertex with no out-neighbours. Let v and w be two neighbours of u . We say that v and w *agree on u* if either both v and w are in-neighbours of u , or both v and w are out-neighbours of u , and that v and w *disagree on u* otherwise. For a subset S of $V(\vec{G})$, we denote by $G[S]$ the sub-digraph of \vec{G} induced by S , which is obviously an oriented graph.

For any finite set Ω , Id_{Ω} denotes the identity permutation acting on Ω . Since the set Ω will always be clear from the context, we will simply write Id instead of Id_{Ω} in the following.

An *automorphism* of an oriented graph \vec{G} is an arc-preserving permutation of its vertices, that is, a one-to-one mapping $\phi : V(\vec{G}) \rightarrow V(\vec{G})$ such that $\phi(u)\phi(v)$ is an arc in \vec{G} whenever uv is an arc in \vec{G} . The set of automorphisms of \vec{G} is denoted $\text{Aut}(\vec{G})$. The *order* of an automorphism ϕ is the smallest integer $k > 0$ for which $\phi^k = Id$. An automorphism ϕ of an oriented graph \vec{G} is *non-trivial* if $\phi \neq Id$. A vertex u of \vec{G} is *fixed* by ϕ if $\phi(u) = u$.

We now introduce the distinguishing parameters we will consider. Let λ be a vertex-labelling of an oriented graph \vec{G} . Recall first that an automorphism ϕ of \vec{G} is λ -*preserving* if $\lambda(\phi(u)) = \lambda(u)$ for every vertex u of \vec{G} , and that a vertex-labelling λ of \vec{G} is *distinguishing* if the only λ -preserving automorphism of \vec{G} is the identity. A distinguishing vertex-colouring is then a distinguishing proper vertex-labelling. Similarly, an automorphism ϕ of an oriented graph \vec{G} is λ -*preserving*, for a given arc-labelling λ of G , if $\lambda(\phi(\vec{uv})) = \lambda(\vec{uv})$ for every arc \vec{uv} of \vec{G} , and an arc-labelling λ of \vec{G} is *distinguishing* if the only λ -preserving automorphism of \vec{G} is the identity. A distinguishing arc-colouring is then a distinguishing proper arc-labelling. We then define the four following distinguishing parameters of an oriented graph \vec{G} .

1. The *oriented distinguishing number* of \vec{G} , denoted $OD(\vec{G})$, is the smallest number of labels needed for a distinguishing vertex-labelling of \vec{G} .

2. The *oriented distinguishing chromatic number* of \vec{G} , denoted $OD_\chi(\vec{G})$, is the smallest number of labels needed for a distinguishing vertex-colouring of \vec{G} .
3. The *oriented distinguishing index* of \vec{G} , denoted $OD'(\vec{G})$, is the smallest number of labels needed for a distinguishing arc-labelling of \vec{G} .
4. The *oriented distinguishing chromatic index* of \vec{G} , denoted $OD_{\chi'}(\vec{G})$, is the smallest number of labels needed for a distinguishing arc-colouring of \vec{G} .

Using these parameters, we can define eight new distinguishing parameters of an undirected graph G .

1. The *minimum oriented distinguishing number* of G , denoted $OD^-(G)$, is the smallest oriented distinguishing number of its orientations.
2. The *maximum oriented distinguishing number* of G , denoted $OD^+(G)$, is the largest oriented distinguishing number of its orientations.
3. The *minimum oriented distinguishing chromatic number* of G , denoted $OD_\chi^-(G)$, is the smallest oriented distinguishing chromatic number of its orientations.
4. The *maximum oriented distinguishing chromatic number* of G , denoted $OD_\chi^+(G)$, is the largest oriented distinguishing chromatic number of its orientations.
5. The *minimum oriented distinguishing index* of G , denoted $OD'^-(G)$, is the smallest oriented distinguishing index of its orientations.
6. The *maximum oriented distinguishing index* of G , denoted $OD'^+(G)$, is the largest oriented distinguishing index of its orientations.
7. The *minimum oriented distinguishing chromatic index* of G , denoted $OD_{\chi'}^-(G)$, is the smallest oriented distinguishing chromatic index of its orientations.
8. The *maximum oriented distinguishing chromatic index* of G , denoted $OD_{\chi'}^+(G)$, is the largest oriented distinguishing chromatic index of its orientations.

Let G be an undirected graph, \vec{G} be any orientation of G , and λ be any vertex or edge-labelling of G (which can also be considered as a vertex or arc-labelling of \vec{G} , respectively). Observe that every automorphism of \vec{G} is an automorphism of G . From this observation and the definition of our distinguishing parameters, we directly get the following result.

Proposition 1 *For every undirected graph G ,*

1. $OD^-(G) \leq OD^+(G) \leq D(G)$,
2. $OD'^-(G) \leq OD'^+(G) \leq D'(G)$,
3. $\chi(G) \leq OD_\chi^-(G) \leq OD_\chi^+(G) \leq D_\chi(G)$,
4. $\chi'(G) \leq OD_{\chi'}^-(G) \leq OD_{\chi'}^+(G) \leq D_{\chi'}(G)$,
5. $D(G) \leq D_\chi(G)$, $OD^-(G) \leq OD_\chi^-(G)$, and $OD^+(G) \leq OD_\chi^+(G)$,
6. $D'(G) \leq D_{\chi'}(G)$, $OD'^-(G) \leq OD_{\chi'}^-(G)$, and $OD'^+(G) \leq OD_{\chi'}^+(G)$.

As observed in the previous section, for every undirected graph G , $D(G) = 1$ if and only if G is rigid, and $D_\chi(G) = \chi(G)$ if G is rigid. This property obviously also holds for oriented graphs: for every oriented graph \vec{G} , $OD(\vec{G}) = 1$ if and only if \vec{G} is rigid, and $OD_\chi(\vec{G}) = \chi(\vec{G})$ if \vec{G} is rigid. We thus have the following result.

Proposition 2 *If G is an undirected graph that admits a rigid orientation, then $OD^-(G) = 1$, $OD'^-(G) = 1$, $OD_\chi^-(G) = \chi(G)$ and $OD_{\chi'}^-(G) = \chi'(G)$.*

3 Paths, cycles and complete graphs

The distinguishing number of undirected paths, cycles and complete graphs has been determined by Albertson and Collins in [3], while their distinguishing chromatic number was given by Collins and Trenk in [9]. The following theorem summarizes these results.

Theorem 3 ([3], [9])

1. For every integer $n \geq 1$, $D(P_{2n}) = D_\chi(P_{2n}) = 2$.
2. For every integer $n \geq 1$, $D(P_{2n+1}) = 2$ and $D_\chi(P_{2n+1}) = 3$.
3. $D(C_4) = 3$ and $D_\chi(C_4) = 4$.
4. $D(C_5) = 3$ and $D_\chi(C_5) = 3$.
5. $D(C_6) = 2$ and $D_\chi(C_6) = 4$.
6. For every integer $n \geq 7$, $D(C_n) = 2$ and $D_\chi(C_n) = 3$.
7. For every integer $n \geq 1$, $D(K_n) = D_\chi(K_n) = n$.

On the other hand, the distinguishing index and distinguishing chromatic index of undirected paths, cycles and complete graphs, have been determined by Kalinowski and Piłśniak in [15], and Alekhani and Soltani in [5]. The following theorem summarizes these results.

Theorem 4 ([15], [5])

1. $D'(P_2) = D_{\chi'}(P_2) = 1$.
2. For every integer $n \geq 2$, $D'(P_{2n}) = 2$ and $D_{\chi'}(P_{2n}) = 3$.
3. For every integer $n \geq 1$, $D'(P_{2n+1}) = D_{\chi'}(P_{2n}) = 2$.
4. $D'(C_4) = 3$ and $D_{\chi'}(C_4) = 4$.
5. $D'(C_5) = 3$ and $D_{\chi'}(C_5) = 3$.
6. $D'(C_6) = 2$ and $D_{\chi'}(C_6) = 4$.
7. For every integer $n \geq 7$, $D'(C_n) = 2$ and $D_{\chi'}(C_n) = 3$.
8. $D'(K_3) = 3$ and $D_{\chi'}(K_3) = 3$.
9. $D'(K_4) = D'(K_5) = 3$ and $D_{\chi'}(K_4) = D_{\chi'}(K_5) = 5$.
10. For every integer $n \geq 3$, $D'(K_{2n}) = 2$ and $D_{\chi'}(K_n) = 2n - 1$.
11. For every integer $n \geq 3$, $D'(K_{2n+1}) = 2$ and $D_{\chi'}(K_n) = 2n + 1$.

It is not difficult to observe that every path, cycle or complete graph, admits a rigid orientation. We thus get the following result, which proves columns $OD^-(G)$ and $OD_\chi^-(G)$ of Table 1 for lines 1 to 8, and columns $OD'^-(G)$ and $OD_{\chi'}^-(G)$ of Table 2 for lines 1 to 12. We say that an oriented path, or an oriented cycle, is *directed* if all its arcs have the same direction.

Theorem 5 *If G is an undirected path, cycle, or complete graph, then G admits a rigid orientation. Therefore, $OD^-(G) = OD'^-(G) = 1$, $OD_\chi^-(G) = \chi(G)$ and $OD_{\chi'}^-(G) = \chi'(G)$.*

Proof. Since all directed paths and transitive tournaments are rigid, and every orientation of the cycle C_n , $n \geq 3$, obtained from the directed cycle by reversing exactly one arc is rigid, the first statement holds. The second statement then directly follows from Proposition 2. \square

We now consider the parameters $OD^+(G)$, $OD_\chi^+(G)$, $OD'^+(G)$ and $OD_{\chi'}^+(G)$ for G being a path, a cycle, or a complete graph. For paths, we have the following result, which proves columns $OD^+(G)$ and $OD_\chi^+(G)$ of Table 1 and columns $OD'^+(G)$ and $OD_{\chi'}^+(G)$ of Table 2, for lines 1 and 2.

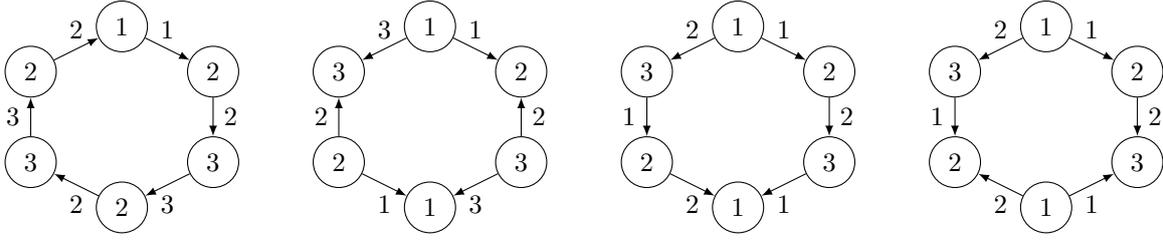


Figure 1: Distinguishing vertex and edge-colourings of non-rigid orientations of C_6 .

Theorem 6 Let P_n denote the path of order n . We then have $OD^+(P_n) = OD'^+(P_n) = 1$ and $OD_\chi^+(P_n) = OD_{\chi'}^+(P_n) = 2$ if n is even, and $OD^+(P_n) = OD'^+(P_n) = OD_{\chi'}^+(P_n) = 2$ and $OD_\chi^+(P_n) = 3$ otherwise.

Proof. Let us denote by $P_n = v_1 \dots v_n$, $n \geq 2$, the undirected path of order n . Note that the only non-trivial automorphism of P_n is the permutation π that exchanges vertices x_i and x_{n-i+1} for every i , $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

If n is even, every orientation of P_n is rigid, which gives $OD^+(P_n) = OD'^+(P_n) = 1$ and $OD_\chi^+(P_n) = OD_{\chi'}^+(P_n) = 2$ by Proposition 2.

Suppose now that n is odd. In that case, every orientation such that the edges $v_i v_{i+1}$ and $v_{n-i} v_{n-i+1}$ have opposite directions is not rigid, which implies that the value of the four considered distinguishing parameters is strictly greater than 1. Since $D(P_n) = D'(P_n) = D_{\chi'}(P_n) = 2$, we get $OD^+(P_n) = OD'^+(P_n) = OD_{\chi'}^+(P_n) = 2$ by Proposition 1. Finally, since every 2-vertex-colouring of P_n is preserved by π , we have $OD_\chi^+(P_n) > 2$ and thus, since $D_\chi(P_n) = 3$, $OD_\chi^+(P_n) = 3$ by Proposition 1. \square

For cycles, we have the following result, which proves columns $OD^+(G)$ and $OD_\chi^+(G)$ of Table 1 and columns $OD'^+(G)$ and $OD_{\chi'}^+(G)$ of Table 2, for lines 3 to 7 (the case of the 3-cycle is covered by Theorem 8).

Theorem 7

1. For every integer $n \geq 4$, $OD^+(C_n) = OD'^+(C_n) = 2$.
2. $OD_\chi^+(C_4) = 4$ and, for every integer $n \geq 5$, $OD_\chi^+(C_n) = 3$.
3. For every integer $n \geq 4$, $OD_{\chi'}^+(C_n) = 3$.

Proof. Let us denote by $C_n = v_1 \dots v_n v_1$, $n \geq 4$, the undirected cycle of order n . Since the directed cycle \vec{C}_n is not rigid, the value of the four considered distinguishing parameters is strictly greater than 1.

Let us first consider distinguishing labellings and let \vec{C} be any orientation of C_n . The 2-vertex-labelling λ of \vec{C} defined by $\lambda(v_1) = \lambda(v_2) = 1$ and $\lambda(v_i) = 2$ for every i , $3 \leq i \leq n$, is clearly distinguishing since v_1 and v_2 must be fixed by every λ -preserving automorphism of \vec{C} . We thus get $OD^+(C_n) = 2$ for every $n \geq 4$. Similarly, the 2-edge-labelling λ' of \vec{C} that assigns colour 1 to exactly one arc of \vec{C} is distinguishing since the end-vertices of the arc coloured with 1 must be fixed by every λ -preserving automorphism of \vec{C} . Therefore, $OD'^+(C_n) = 2$ for every $n \geq 4$.

Let us now consider distinguishing colourings. Suppose first that n is odd. In that case, since $\chi(C_n) = \chi'(C_n) = 3$, we get $OD_\chi^+(C_n) = D_\chi(C_n) = 3$ and $OD_{\chi'}^+(C_n) = D_{\chi'}(C_n) = 3$ by Proposition 1 and Theorem 3 or 4.

Suppose now that n is even. Note that every 2-vertex or 2-arc-colouring of the directed cycle \vec{C}_n is ρ^2 -preserving, where ρ denotes the automorphism of \vec{C}_n defined by $\rho(v_i) = v_{i+1}$ for every i , $1 \leq i \leq n-1$, and $\rho(v_n) = v_1$. This implies $OD_\chi^+(C_n) > 2$ and $OD_{\chi'}^+(C_n) > 2$. We thus get $OD_\chi^+(C_n) = D_\chi(C_n) = 3$ and $OD_{\chi'}^+(C_n) = D_{\chi'}(C_n) = 3$ for every even $n \geq 8$, thanks to Proposition 1 and Theorem 3 or 4.

Consider now the cycle $C_4 = v_1v_2v_3v_4v_1$. Note that, up to symmetries, C_4 has only two non-rigid orientations, namely the directed cycle \vec{C}_4 and the orientation \vec{C}_4' with arcs $\overrightarrow{v_1v_2}$, $\overrightarrow{v_3v_2}$, $\overrightarrow{v_3v_4}$ and $\overrightarrow{v_1v_4}$. Moreover, observe that $\text{Aut}(\vec{C}_4')$ has three non-trivial automorphisms, one exchanging v_1 and v_3 , one exchanging v_2 and v_4 , and the third one exchanging both these pairs of vertices. This implies that every distinguishing vertex-colouring of \vec{C}_4' must use four colours, and thus $OD_\chi^+(C_4) = 4$. To see that $OD_{\chi'}^+(C_4) = 3$, it suffices to observe that, by colouring the arcs of \vec{C}_4 or \vec{C}_4' cyclically with colours 1213, we obtain a distinguishing 3-arc-colouring.

Consider finally the cycle C_6 . Up to symmetries, C_6 has four distinct non-rigid orientations, depicted in Figure 1. It is not difficult to check that the vertex and arc-colourings described in the same figure are all optimal, which gives $OD_\chi^+(C_6) = OD_{\chi'}^+(C_6) = 3$.

This concludes the proof. \square

The distinguishing number of tournaments has been studied by Albertson and Collins in [2], where it is proved in particular that $D(T_n) \leq 1 + \left\lceil \frac{\lfloor \log n \rfloor}{2} \right\rceil$ for every tournament T_n of order n . Moreover, they conjectured that $D(T) \leq 2$ for every tournament T . As observed by Godsil in 2002 (this fact is mentioned by Lozano in [20]), this conjecture follows from Gluck's Theorem [12].

For every integer $n \geq 3$, the values of $OD^-(K_n)$, $OD_\chi^-(K_n)$, $OD'^-(K_n)$ and $OD_{\chi'}^-(K_n)$ are already given by Theorem 5. The values of the other parameters are given by the following result, which proves columns $OD^+(G)$ and $OD_\chi^+(G)$ of Table 1, line 8, and columns $OD'^+(G)$ and $OD_{\chi'}^+(G)$ of Table 2, lines 8 to 12.

Theorem 8 *For every integer $n \geq 3$, $OD^+(K_n) = OD'^+(K_n) = 2$, $OD_\chi^+(K_n) = n$, and $OD_{\chi'}^+(K_n) = \chi'(K_n)$.*

Proof. Since every complete graph K_n , $n \geq 3$, admits a non rigid orientation (consider for instance the orientation \vec{K}_n of K_n obtained from the transitive orientation of K_n with directed path $x_1 \dots x_n$, by reversing the arc $x_{n-2}x_n$, so that x_{n-2} , x_{n-1} and x_n form a directed cycle; the permutation (x_{n-2}, x_{n-1}, x_n) is then clearly a non trivial automorphism of \vec{K}_n), we get $OD^+(K_n) \geq 2$ and $OD'^+(K_n) \geq 2$.

Since, as mentioned above, $D(T) \leq 2$ for every tournament T , we get, by Proposition 1, $OD^+(K_n) \leq 2$ and thus $OD^+(K_n) = 2$. By Proposition 1, we also have $OD'^+(K_n) \leq D'(K_n)$, and thus $OD'^+(K_n) = 2$ if $n \geq 6$. If $n = 3$, then the only non rigid orientation of K_3 is the directed cycle. By assigning colour 1 to one arc and colour 2 to the two other arcs, we get a distinguishing arc-labelling, so that $OD^+(K_3) = 2$. If $n \in \{4, 5\}$, observe that each non rigid orientation of K_n contains a transitive triangle. By assigning colour 1 to the three arcs of this triangle and colour 2 to all the other arcs, we get a distinguishing arc-labelling, so that $OD'^+(K_4) = OD'^+(K_5) = 2$.

Finally, by Proposition 1, $\chi(K_n) \leq OD_\chi^+(K_n) \leq D_\chi(K_n)$ and $\chi'(K_n) \leq OD_{\chi'}^+(K_n) \leq D_{\chi'}(K_n)$, which gives, thanks to Theorems 3 and 4, $OD_\chi^+(K_n) = n$, and $OD_{\chi'}^+(K_n) = \chi'(K_n)$ if $n \neq 4$, respectively. Now, there are only two orientations of K_4 that admit a non trivial automorphism, one with a source vertex and the other one with a sink vertex, and the three other vertices in both of them inducing a directed 3-cycle. In each case, every 3-edge-colouring is clearly distinguishing, which gives $OD_{\chi'}^+(K_4) = 3 = \chi'(K_4)$. \square

4 Complete bipartite graphs: easy cases

In this section, we consider “easy cases” of complete bipartite graphs, namely $K_{1,n}$ and $K_{n,n}$ for every $n \geq 3$ (the cases $n \in \{1, 2\}$ correspond to P_2 , P_3 or C_4 , already considered in the previous section).

Concerning stars $K_{1,n}$, it is not difficult to get the following result, which proves line 9 of Table 1 and line 13 of Table 2.

Theorem 9 *For every integer $n \geq 3$, $OD^-(K_{1,n}) = OD'^-(K_{1,n}) = \lceil \frac{n}{2} \rceil$, $OD^+(K_{1,n}) = OD'^+(K_{1,n}) = n$, $OD_{\chi}^-(K_{1,n}) = 1 + \lceil \frac{n}{2} \rceil$, $OD_{\chi}^+(K_{1,n}) = n+1$, and $OD_{\chi'}^-(K_{1,n}) = OD_{\chi'}^+(K_{1,n}) = n$.*

Proof. Let x be the central vertex of $K_{1,n}$ and $\{y_1, \dots, y_n\}$ the set of neighbours of x . Let \vec{K} denote any orientation of $K_{1,n}$. Observe that every automorphism of \vec{K} can only permute in-neighbours of x , and out-neighbours of x , x being fixed. Hence, by considering the orientation of $K_{1,n}$ for which $d^+(x) = n$, we get $OD^+(K_{1,n}) = OD'^+(K_{1,n}) = n$.

On one other hand, the minimum value of $OD^-(K_{1,n})$ and $OD'^-(K_{1,n})$ is attained when the number of in-neighbours and the number of out-neighbours of x differ by at most one, which gives $OD^-(K_{1,n}) = OD'^-(K_{1,n}) = \lceil \frac{n}{2} \rceil$. Similarly, since all in-neighbours (resp. all out-neighbours) must have distinct colours in every distinguishing vertex-colouring of $K_{1,n}$, and these colours have to be distinct from the colour of x , we get $OD_{\chi}^-(K_{1,n}) = 1 + \lceil \frac{n}{2} \rceil$ (when the number of in-neighbours and out-neighbours of x differ by at most one), and $OD_{\chi}^+(K_{1,n}) = n+1$ (when all arcs are out-going arcs from x).

Finally, since all arc-colourings of every orientation of $K_{1,n}$ are distinguishing, we get $OD_{\chi'}^-(K_{1,n}) = OD_{\chi'}^+(K_{1,n}) = \chi'(K_{1,n}) = n$. \square

The distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index of balanced complete bipartite graphs $K_{n,n}$ have been studied by Collins and Trenk [9], Kalinowski and Piłśniak [15], and Alikhani and Soltani [5]. The following theorem summarizes these results.

Theorem 10 ([9], [15], [5])

1. $D(K_{3,3}) = 4$, $D_{\chi}(K_{3,3}) = 6$, $D'(K_{3,3}) = 3$ and $D_{\chi'}(K_{3,3}) = 5$.
2. For every $n \geq 4$, $D(K_{n,n}) = n+1$, $D_{\chi}(K_{n,n}) = 2n$, $D'(K_{n,n}) = 2$ and $D_{\chi'}(K_{n,n}) = n+1$.

Let us denote by $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ the bipartition of $K_{n,n}$. It is a well-known fact the every proper edge-colouring of $K_{n,n}$ corresponds to a Latin square of order n , the colour of the edge $x_i y_j$, $1 \leq i, j \leq n$, corresponding to the symbol in row i and column j . A Latin square L is *asymmetric* [23] if, for every three permutations α , β and γ on the set $\{1, \dots, n\}$, the Latin square L' defined by $L'[\alpha(i), \beta(j)] = \gamma(L[i, j])$ for every i, j , $1 \leq i, j \leq n$ is equal to L if and only if α , β and γ are all the identity. In other words, if an edge-colouring λ of $K_{n,n}$ corresponds to an asymmetric Latin square, then the only λ -preserving automorphism of $K_{n,n}$ that preserves the bipartition is the identity. Therefore, if \vec{K} is the orientation of $K_{n,n}$ defined by $N_{\vec{K}}^+(x_i) = Y$ for every i , $1 \leq i \leq n$, and λ is any n -arc-colouring of \vec{K} corresponding to an asymmetric Latin square, then none of the non-trivial automorphisms of \vec{K} is λ -preserving.

The following result, concerning asymmetric Latin squares and due to K.T. Phelps [22], will be useful for our next result.

Theorem 11 ([22]) *For every integer $n \geq 7$, there exists an asymmetric Latin square of order n . Moreover, the smallest asymmetric Latin squares have order 7.*

For balanced complete bipartite graphs, we have the following result, which proves line 10 of Table 1 and lines 14, 15 and 16 of Table 2.

Theorem 12 *For every integer $n \geq 3$, $OD^-(K_{n,n}) = OD'^-(K_{n,n}) = 1$, $OD^+(K_{n,n}) = n$, $OD'^+(K_{n,n}) = 2$, $OD_\chi^-(K_{n,n}) = 2$, $OD_{\chi'}^-(K_{n,n}) = n$ and $OD_\chi^+(K_{n,n}) = 2n$. Moreover, $OD_{\chi'}^+(K_{n,n}) = n + 1$ if $3 \leq n \leq 6$, and $OD_{\chi'}^+(K_{n,n}) = n$ if $n \geq 7$.*

Proof. Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ denote the bipartition of $K_{n,n}$, $n \geq 3$.

Observe first that the orientation of $K_{n,n}$ given by $N^+(x_i) = \{y_j : j \leq i\}$ for every i , $1 \leq i \leq n$, is rigid, since all vertices in X have distinct in-degrees. Therefore, thanks to Proposition 2, we get $OD^-(K_{n,n}) = OD'^-(K_{n,n}) = 1$, $OD_\chi^-(K_{n,n}) = \chi(K_{n,n}) = 2$ and $OD_{\chi'}^-(K_{n,n}) = \chi'(K_{n,n}) = n$ for every $n \geq 3$.

Consider now the orientation \vec{K} of $K_{n,n}$ given by $N^+(x_i) = Y$ for every i , $1 \leq i \leq n$. Clearly, for every two permutations π_X of X and π_Y of Y , the product $\pi_X \pi_Y$ is an automorphism of \vec{K} . Therefore, all vertices of \vec{K} must be assigned distinct colours by every distinguishing vertex colouring, which gives $OD_\chi^+(K_{n,n}) = 2n$.

The n -vertex-labelling φ of $K_{n,n}$ defined by $\varphi(x_i) = \varphi(y_i) = i$ for every i , $1 \leq i \leq n$, is distinguishing for every orientation of $K_{n,n}$ since every label is used on two vertices that are connected by an arc. Hence, $OD^+(K_{n,n}) \leq n$. For the orientation \vec{K} of $K_{n,n}$ defined in the previous paragraph, every distinguishing vertex-labelling must assign distinct labels to every vertex in each part, and thus $OD^+(K_{n,n}) = n$.

Consider now the 2-edge-labelling φ' of $K_{n,n}$ defined by $\varphi'(x_i y_j) = 1$ if and only if $j < i$, for every i, j , $1 \leq i, j \leq n$. Note that every two vertices in each part have a distinct number of incident edges with label 1. Therefore, reasoning similarly as in the previous paragraph, we get $OD'^+(K_{n,n}) = 2$.

If $n \geq 7$, it follows from Theorem 11 that the orientation \vec{K} of $K_{n,n}$ admits a distinguishing n -arc-colouring, and thus $OD_{\chi'}^+(K_{n,n}) = \chi'(K_{n,n}) = n$. Finally, if $3 \leq n \leq 6$, consider the $(n+1)$ -edge-colouring λ of $K_{n,n}$ defined by $\lambda(x_i y_j) = i + j \pmod{n+1}$ for every i, j , $1 \leq i, j \leq n$. Since, in each part, the set of colours used on the incident edges of every two vertices are distinct (the colour i does not appear on the edges incident with x_i , and the colour j does not appear on the edges incident with y_j), we get that λ is a distinguishing arc-colouring of every orientation of $K_{n,n}$, and thus $OD_{\chi'}^+(K_{n,n}) \leq n + 1$. On the other hand, it follows from Theorem 11 that for every n -edge-colouring λ of $K_{n,n}$, there exists a non trivial λ -preserving automorphism of $K_{n,n}$ such that every vertex is mapped to a vertex in the same part. Considering the orientation \vec{K} of $K_{n,n}$ defined above, this gives $OD_{\chi'}^+(K_{n,n}) > n$, and thus $OD_{\chi'}^+(K_{n,n}) = n + 1$.

This completes the proof. \square

5 Complete bipartite graphs: other cases

We consider in this section the remaining cases of complete bipartite graphs, that is, unbalanced complete bipartite graphs $K_{m,n}$ with $2 \leq m < n$.

The distinguishing number and the distinguishing chromatic number of unbalanced complete bipartite graphs have been determined by Collins and Trenk [9], while their distinguishing chromatic index has been given by Alekhani and Soltani [5]. The following theorem summarizes these results.

Theorem 13 ([9], [5]) *For every two integers m and n , $2 \leq m < n$,*

$$D(K_{m,n}) = n, \quad D_\chi(K_{m,n}) = m + n \quad \text{and} \quad D_{\chi'}(K_{m,n}) = n.$$

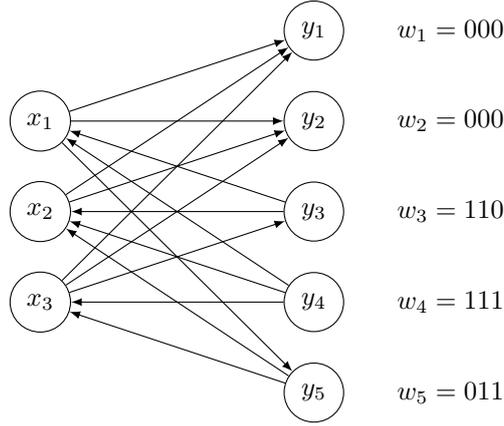


Figure 2: An orientation of the complete bipartite graph $K_{3,5}$.

As recalled in the previous section, the distinguishing index of balanced complete bipartite graphs $K_{n,n}$ has been given by Kalinowski and Piłśniak [15] (see Theorem 10). The distinguishing index of complete bipartite graphs $K_{m,n}$ with $n > m$ is less easy to determine. This has been done by Fisher and Isaak [11], and independently by Imrich, Jerebic and Klavžar [13].

Theorem 14 ([11], [13]) *Let m, n and r be integers such that $r \geq 2$ and $(r-1)^m < n \leq r^m$. We then have*

$$D'(K_{m,n}) = \begin{cases} r, & \text{if } n \leq r^m - \lceil \log_r m \rceil - 1, \\ r + 1, & \text{if } n \geq r^m - \lceil \log_r m \rceil + 1. \end{cases}$$

Moreover, if $n = r^m - \lceil \log_r m \rceil$, then $D'(K_{m,n})$ is either r or $r + 1$ and can be computed recursively in time $O(\log^*(n))$.

Before considering the eight distinguishing parameters for complete bipartite graphs we are interested in, we introduce some notation and definitions. For the complete bipartite graph $K_{m,n}$, $m < n$, we denote by (X, Y) the partition of its vertex set and let $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$. Moreover, for every orientation \vec{K} of $K_{m,n}$ and every integer d , $0 \leq d \leq m$, we will denote by Y_d the set of vertices in Y with out-degree d , that is, $Y_d = \{y \in Y \mid d_{\vec{K}}^+(y) = d\}$. For example, in the orientation of $K_{3,5}$ depicted in Figure 2, we have $Y_0 = \{y_1, y_2\}$, $Y_1 = \emptyset$, $Y_2 = \{y_3, y_5\}$ and $Y_3 = \{y_4\}$.

For general complete bipartite graphs, we have the following result.

Theorem 15 *For every two integers m and n , $2 \leq m < n$, the following holds.*

1. $OD^+(K_{m,n}) = OD_{\chi'}^+(K_{m,n}) = n$, $OD'^+(K_{m,n}) = D'(K_{m,n})$ and $OD_{\chi}^+(K_{m,n}) = m + n$.
2. If $K_{m,n}$ admits a rigid orientation, then $OD^-(K_{m,n}) = OD'^-(K_{m,n}) = 1$, $OD_{\chi}^-(K_{m,n}) = 2$ and $OD_{\chi'}^-(K_{m,n}) = n$.
3. If $K_{m,n}$ does not admit any rigid orientation, then $OD^-(K_{m,n}) \leq \lceil \frac{n}{m-1} \rceil$, $OD_{\chi}^-(K_{m,n}) \leq 1 + \lceil \frac{n}{m-1} \rceil$, $OD'^-(K_{m,n}) \leq D'(K_{m, \lceil \frac{n}{m-1} \rceil})$ and $OD_{\chi'}^-(K_{m,n}) = n$.

Proof. Consider the orientation \vec{K} of $K_{m,n}$ given by $N^+(x_i) = Y$ for every i , $1 \leq i \leq m$. In that case, since $m \neq n$, we have $\text{Aut}(\vec{K}) = \text{Aut}(K_{m,n})$, which implies $OD^+(K_{m,n}) = D(K_{m,n})$, $OD_{\chi'}^+(K_{m,n}) = D_{\chi'}(K_{m,n})$, $OD'^+(K_{m,n}) = D'(K_{m,n})$ and $OD_{\chi}^+(K_{m,n}) = D_{\chi}(K_{m,n})$. Equalities

in the first item then follows from Theorem 13. On one other hand, equalities in the second item directly follow from Proposition 2.

Suppose now that $K_{m,n}$ does not admit any rigid orientation. We first claim that we necessarily have $m < \lceil \frac{n}{m-1} \rceil$. Indeed, this follows from Lemmas 17, 19 and 20, proved later: if $m = 2, 3$ or 4 , then n is at least $4, 7$ or 14 , respectively (see Lemmas 17 and 19), and the claim holds, while $n > 2^m - \lceil \frac{m}{2} \rceil$ if $m \geq 5$ (see Lemma 20). It is then easy to check that, for every $m \geq 5$,

$$\left\lceil \frac{n}{m-1} \right\rceil > \left\lceil \frac{2^m - \lceil \frac{m}{2} \rceil}{m-1} \right\rceil > m.$$

For every non-trivial automorphism ϕ of $K_{m,n}$, and every two vertices u and v of $K_{m,n}$ with $\phi(u) = v$, we have $d^+(u) = d^+(v)$, and either $u, v \in X$ or $u, v \in Y$. Consider now any orientation \vec{K}' of $K_{m,n}$ satisfying the two following properties:

- P1. $d_{\vec{K}'}^+(y_j) = 1 + (j \bmod m - 1)$ for every $j, 1 \leq j \leq n$, and
- P2. $N_{\vec{K}'}^+(y_{i(m-1)}) = \{x_i\}$ for every $i, 1 \leq i \leq m$.

Thanks to Property P1, we have $1 \leq d_{\vec{K}'}^+(y_j) \leq m - 1$ for every $j, 1 \leq j \leq n$, and the cardinality of each set $Y_d, 1 \leq d \leq m - 1$, is at most $\lceil \frac{n}{m-1} \rceil$. Moreover, Property P2 ensures that the sets of in-neighbours of every two vertices in X are distinct, which implies that every automorphism of \vec{K}' that permutes vertices in X must also permute vertices in Y .

Consider now the $\lceil \frac{n}{m-1} \rceil$ -vertex-labelling λ of \vec{K}' defined by with $\lambda(x_i) = 1$ for every $i, 1 \leq i \leq m$, and $\lambda(y_j) = 1 + \lfloor \frac{j-1}{m-1} \rfloor$ for every $j, 1 \leq j \leq n$. This labelling assigns distinct labels to the vertices of each set $Y_d, 1 \leq d \leq m - 1$, and is thus distinguishing, which gives $OD^-(K_{m,n}) \leq \lceil \frac{n}{m-1} \rceil$. Moreover, by using one additional label for the vertices of X instead of label 1, the labelling λ becomes a distinguishing vertex-colouring, and thus $OD_{\chi}^-(K_{m,n}) \leq 1 + \lceil \frac{n}{m-1} \rceil$.

Let us now consider distinguishing arc-labellings of \vec{K}' . Observe that if the restriction of any arc-labelling λ of \vec{K}' to the subgraphs $\vec{K}'[X \cup Y_d]$ induced by $X \cup Y_d, 1 \leq d \leq m - 1$, is distinguishing for this subgraph, then λ is a distinguishing arc-labelling of \vec{K}' . Since for every $d, 1 \leq d \leq m - 1$, we have $OD'^-(\vec{K}'[X \cup Y_d]) \leq D'(K_{m, \lceil \frac{n}{m-1} \rceil})$, we get $OD'^-(K_{m,n}) \leq D'(K_{m, \lceil \frac{n}{m-1} \rceil})$.

Finally, from Proposition 1 and Table 2, we get

$$n = \chi'(K_{m,n}) \leq OD_{\chi'}^-(K_{m,n}) \leq D_{\chi'}(K_{m,n}) = n,$$

and thus $OD_{\chi'}^-(K_{m,n}) = n$, which completes the proof of the third item. \square

Our goal now is thus to determine for which values of m and n , $K_{m,n}$ admits a rigid orientation. Let us first introduce a few more notation. For a given orientation \vec{K} of $K_{m,n}$, we associate with each vertex y_i from Y the word $w_i = b_i^1 \cdots b_i^m$ on the alphabet $\{0, 1\}$, defined by $b_i^j = 0$ if $x_j y_i$ is an arc, and $b_i^j = 1$ otherwise. Figure 2 gives the word associated with each vertex from Y for the depicted orientation of $K_{3,5}$.

For every integer $m \geq 2$, we will denote by $\vec{K}K_m^*$ the (unique, up to isomorphism) orientation of the complete bipartite graph $K_{m,2^m}$ for which all the words associated with the vertices in Y are distinct, and by (X^*, Y^*) the corresponding bipartition of $V(\vec{K}K_m^*)$. This orientation will be called the *canonical orientation* of $K_{m,2^m}$. Observe that every vertex x in X^* has exactly 2^{m-1} in-neighbours and 2^{m-1} out-neighbours in $\vec{K}K_m^*$.

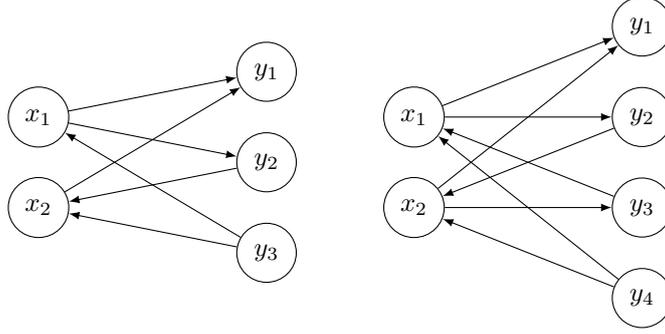


Figure 3: Orientations of $K_{2,3}$ and $K_{2,4}$ for the proof of Lemma 17.

Let \vec{K} be any orientation of $K_{m,n}$, $n \geq m \geq 2$. We say that two vertices u and v are *full twins* in \vec{K} if $N^+(u) = N^+(v)$ (which implies $N^-(u) = N^-(v)$). For example, y_1 and y_2 are full twins in the orientation of $K_{3,5}$ depicted in Figure 2. Observe that the existence of full twins in an orientation of a complete bipartite graph ensures that this orientation is not rigid.

Proposition 16 *Let \vec{K} be any orientation of $K_{m,n}$, $n > m \geq 2$. If there exist two full twins u and v in \vec{K} , then \vec{K} is not rigid. In particular, if $n > 2^m$, then $K_{m,n}$ does not admit any rigid orientation.*

Proof. It suffices to observe that the transposition (u, v) of $V(\vec{K})$ is an automorphism of \vec{K} . The second statement follows from the fact that the maximum number of distinct orientations of the edges incident with a vertex in Y is 2^m , so that every orientation of $K_{m,n}$, $n > 2^m$, necessarily contains a pair of full twin vertices. \square

Similarly, we say that u and v are *full antitwins* in \vec{K} if $N^+(u) = N^-(v)$ (which implies $N^-(u) = N^+(v)$).

Let now $\{x, x'\}$ be a pair of vertices from X . We say that y and y' are $\{x, x'\}$ -*antitwins* in \vec{K} if (i) the set of vertices $\{x, x', y, y'\}$ induces a directed 4-cycle, and (ii) y and y' agree on every vertex x'' from $X \setminus \{x, x'\}$. In particular, it means that there is no other directed 4-cycle containing both y and y' . Note that any two such $\{x, x'\}$ -antitwins have the same out-degree, and thus belong to the same subset Y_d of Y , for some integer d , $1 \leq d \leq m - 1$ (in particular, a source vertex or a sink vertex cannot have an $\{x, x'\}$ -antitwin). Moreover, observe that if $\{y, y'\}$ and $\{y, y''\}$ are both pairs of $\{x, x'\}$ -antitwins in \vec{K} , then y' and y'' are necessarily full twins in \vec{K} .

For example, in the orientation of $K_{3,5}$ depicted in Figure 2, y_1 and y_2 are full twins, y_1 and y_4 (or y_2 and y_4) are full antitwins, while y_3 and y_5 are $\{x_1, x_3\}$ -antitwins. Observe also that the canonical orientation $\vec{K} \vec{K}_m^*$ of $K_{m,2^m}$ contains no pair of full twins, that every vertex y from Y^* has one full antitwin, and that every vertex y from Y^* which is neither a source nor a sink has an $\{x, x'\}$ -antitwin for exactly $d^-(y) \times d^+(y)$ pairs of vertices $\{x, x'\}$.

For $m = 2$, we have the following result.

Lemma 17 *The complete bipartite graphs $K_{2,4}$ does not admit any rigid orientation, while $K_{2,3}$ admits a rigid orientation.*

Proof. Let \vec{K} be any orientation of $K_{2,4}$. By Proposition 16, we can assume that \vec{K} has no full twins, that is, \vec{K} is the orientation of $K_{2,4}$ depicted in Figure 3. It is then easy to check

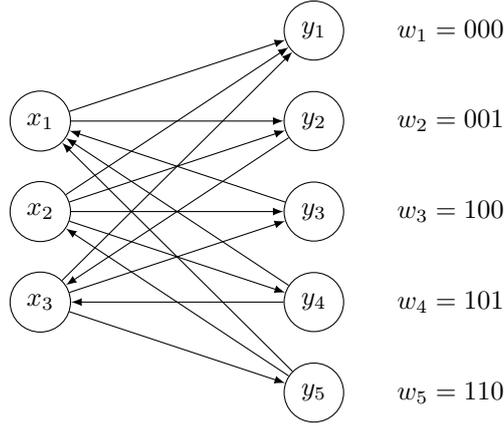


Figure 4: A twin-free orientation \vec{K} of the complete bipartite graph $K_{3,5}$.

that the permutation $(x_1, x_2)(y_2, y_3)$ is an automorphism of \vec{K} . A rigid orientation of $K_{2,3}$ is depicted in Figure 3. \square

We now want to characterize the values of m and n , with $n > m \geq 2$, for which $K_{m,n}$ admits a rigid orientation. For every such graph, by Proposition 2, we will then have $OD^-(K_{m,n}) = OD'^-(K_{m,n}) = 1$, $OD_{\chi}^-(K_{m,n}) = 2$ and $OD_{\chi'}^-(K_{m,n}) = n$.

Each orientation \vec{K} of a complete bipartite graph $K_{m,n}$, $n > m \geq 2$, having no full twins is a subdigraph of the canonical orientation $\vec{K}K_m^*$, obtained by deleting some vertices of Y^* . In the following, such an orientation \vec{K} will be described by the set $W(\vec{K})$ of words associated with these deleted vertices. Note that the cardinality of the set $W(\vec{K})$ is precisely $2^m - n$. Let $W(\vec{K}) = \{w_1, \dots, w_{2^m - n}\}$. Note that for every vertex x_i in X , $1 \leq i \leq m$, we have

$$d_{\vec{K}}^-(x_i) = 2^{m-1} - \sum_{j=1}^{2^m - n} w_j^i. \quad (1)$$

For example, $W(\vec{K}K_m^*) = \emptyset$, while the set of words describing the orientation \vec{K} of $K_{3,5}$ depicted in Figure 4 is $W(\vec{K}) = \{010, 011, 111\}$, and the in-degree of x_3 is $2^2 - (0 + 1 + 1) = 2$.

Since every orientation \vec{K} of $K_{m,n}$, $3 \leq m < n \leq 2^m$, having no full twins is a subdigraph of $\vec{K}K_m^*$, we get that for every two distinct vertices y and y' in Y , there exists a vertex x in X such that y and y' disagree on x . Therefore, every non-trivial automorphism of \vec{K} must act on X .

Observation 18 *For every orientation \vec{K} of $K_{m,n}$, $3 \leq m < n \leq 2^m$, having no full twins, the only automorphism of \vec{K} that fixes every vertex x of X is the identity. Therefore, if all vertices in X have distinct in-degrees (or, equivalently, distinct out-degrees), then \vec{K} is rigid.*

Thanks to this observation, we can solve the cases $m = 3$ and $m = 4$.

Lemma 19 *For every integer n , $4 \leq n \leq 6$, $K_{3,n}$ admits a rigid orientation. For every integer n , $5 \leq n \leq 13$, $K_{4,n}$ admits a rigid orientation.*

Proof. Consider the orientations \vec{K} , \vec{K}' and \vec{K}'' of $K_{3,4}$, $K_{3,5}$ and $K_{3,6}$, respectively, given by $W(\vec{K}) = \{110, 010, 000, 111\}$, $W(\vec{K}') = \{110, 010, 000\}$ and $W(\vec{K}'') = \{110, 010\}$. In each

1	2	3	4	5	6	7	j	
0	0	1	1	1	1	1	The set W_7	
0	1	0	0	1	1	1		
0	0	0	1	0	0	1		
0	0	0	0	0	1	0		
0							1	The sums $S^j(W_7)$

1	2	3	4	5	6	7	8	j
0	0	1	1	1	1	1	1	The set W_8
0	1	0	0	1	1	1	1	
0	0	0	1	0	0	1	1	
0	0	0	0	0	1	0	1	
0							1	The sums $S^j(W_8)$

Figure 5: The sets W_7 and W_8 for the proof of Lemma 20.

case, according to Equation (1), all vertices in X have distinct in-degrees, which implies, by Observation 18, that all these orientations are rigid.

The same argument applies for the orientation \vec{K} of $K_{4,13}$ given by $W(\vec{K}) = \{1110, 1100, 1000\}$, and thus \vec{K} is a rigid orientation of $K_{4,13}$. Now, for each n , $5 \leq n \leq 12$, consider the set W_n obtained from the set $\{1110, 1100, 1000\}$ by adding $\lfloor \frac{13-n}{2} \rfloor$ pairs of full antitwins distinct from the pair $\{0000, 1111\}$, together with the word 0000 if n is even. This can always be done since the set $\{0, 1\}^4 \setminus \{1110, 1100, 1000\}$ contains five pairs of full antitwins. Again, in each case, we get that all vertices in X have distinct in-degrees so that, by Observation 18, all the corresponding orientations are rigid. \square

We are now able to prove the following result.

Lemma 20 *For every two integers m and k , $m \geq 5$, $\lceil \frac{m}{2} \rceil \leq k < 2^m - m$, $K_{m, 2^m - k}$ admits a rigid orientation.*

Proof. For every set of words $W = \{w_1, \dots, w_k\} \subseteq \{0, 1\}^m$, we denote by $S^i(W)$, $1 \leq i \leq m$, the sum of the i -th symbols of the words in W , that is, $S^i(W) = \sum_{\ell=1}^k w_\ell^i$.

We first consider the case $k = \lceil \frac{m}{2} \rceil$. We will construct a particular set $W_m = \{w_1, \dots, w_k\}$ of k words such that the orientation \vec{K} of $K_{m, 2^m - \lceil \frac{m}{2} \rceil}$ with $W(\vec{K}) = W_m$ is rigid. These k words are defined as follows (see Figure 5 for the sample sets W_7 and W_8):

- $w_1 = 00.1^{m-2}$,
- for every i , $2 \leq i \leq \lceil \frac{m}{2} \rceil - 1$, $w_i = 0^{2i-3}.100.1^{m-2i}$,
- $w_{\lceil \frac{m}{2} \rceil} = 0^{m-2}.10$ if m is odd, or $w_{\lceil \frac{m}{2} \rceil} = 0^{m-3}.101$ if m is even.

Observe that $S^1(W_m) = 0$, $S^{2i}(W_m) = S^{2i+1}(W_m) = i$ for every i , $1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$, and $S^m(W_m) = \frac{m}{2}$ if m is even. By Equation (1), this implies that the set of pairs of vertices in X having the same in-degree is $\{(x_{2i}, x_{2i+1}) : 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor\}$. Therefore, the restriction to X of every non-trivial automorphism of \vec{K} must be the product of at least one transposition corresponding to these pairs of vertices.

We claim that such a situation cannot occur, so that \vec{K} does not admit any non-trivial automorphism. Indeed, we cannot exchange x_{2i} and x_{2i+1} , $1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$, since the pair $\{y, y'\}$ of $\{x_{2i}, x_{2i+1}\}$ -antitwins, where y is the vertex associated with the word w_i , is such that $y \in W_m$, and thus y has been deleted, while $y' \notin W_m$, and thus y' still belongs to \vec{K} (hence, x_{2i} and x_{2i+1} disagree on y'). The orientation \vec{K} of $K_{m, 2^m-k}$ given by $W(\vec{K}) = W_m$ is thus rigid.

Let us now consider the case $\lfloor \frac{m}{2} \rfloor < k < 2^m - m$, and let $p = k - \lfloor \frac{m}{2} \rfloor$. We then construct a set of words W_k , by adding to the set W_m previously defined $\lfloor \frac{p}{2} \rfloor$ pairs of full antitwins distinct from the pair $\{0^m, 1^m\}$, together with the word 0^m if p is odd. Note that this is always possible since W_m contains $\lfloor \frac{m}{2} \rfloor$ words, and thus $\frac{1}{2}(2^m - 2 \lfloor \frac{m}{2} \rfloor)$ pairs of antitwins are available while we need at most $\frac{1}{2}(2^m - m - \lfloor \frac{m}{2} \rfloor)$ such pairs. Doing so, the sums $S^j(W_k)$, $1 \leq j \leq m$, satisfy the same properties as in the previous case so that, again, the set of pairs of vertices in X having the same in-degree is $\{(x_{2i}, x_{2i+1}) : 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor\}$. The same argument as before then allows us to conclude that the orientation \vec{K} of $K_{m, 2^m-k}$ given by $W(\vec{K}) = W_k$ is also rigid. This completes the proof. \square

In order to complete the study of complete bipartite graphs, we still have to consider the graphs $K_{m, 2^m-k}$ for every k , $0 \leq k < m$. The following result is useful for small values of k .

Lemma 21 *Let \vec{K} be any orientation of $K_{m,n}$, $3 \leq m < n \leq 2^m$, not containing full twins. If there exists two indices i and i' , $1 \leq i < i' \leq m$, such that $w_j^i = w_j^{i'}$ for every word $w_j \in W(\vec{K})$, then \vec{K} is not rigid. In particular, if $m \geq 3$ and $0 \leq k < \log_2(m)$, then $K_{m, 2^m-k}$ does not admit any rigid orientation.*

Proof. Observe that there are exactly $p = 2^{m-2}$ pairs of $\{x_i, x_{i'}\}$ -antitwins in Y in the canonical orientation $\vec{K}K_m^*$ of $K_{m,n}$. Let us denote the set of these pairs by $A_{x_i, x_{i'}} = \{(y_{j_1}, y'_{j_1}), \dots, (y_{j_p}, y'_{j_p})\}$. Since all vertices corresponding to words in $W(\vec{K})$ agree on x_i and $x_{i'}$, all vertices belonging to a pair of $A_{x_i, x_{i'}}$ belong to \vec{K} . Therefore, the permutation $(x_i, x_{i'})(y_{j_1}, y'_{j_1}) \cdots (y_{j_p}, y'_{j_p})$ is a non-trivial automorphism of \vec{K} , and thus \vec{K} is not rigid.

Finally, if $m \geq 3$ and $0 \leq k < \log_2(m)$, then, for every orientation \vec{K}' of $K_{m, 2^m-k}$, there necessarily exist two indices i and i' , $1 \leq i < i' \leq m$, such that $w_j^i = w_j^{i'}$ for every word $w_j \in W(\vec{K}')$, which implies that \vec{K}' is not rigid. \square

From Theorem 15(2) and Lemmas 17, 19 and 20, we finally get the following corollary.

Corollary 22 *For every two integers m and n , $m < n$, with either $m \geq 5$ and $n \leq 2^m - \lfloor \frac{m}{2} \rfloor$, or $(m, n) \in \{(2, 3), (3, 4), (3, 5), (3, 6)\} \cup \{(4, p), 5 \leq p \leq 13\}$, we have*

$$OD^-(K_{m,n}) = OD'^-(K_{m,n}) = 1, \quad OD_{\chi}^-(K_{m,n}) = 2 \quad \text{and} \quad OD_{\chi'}^-(K_{m,n}) = n.$$

6 Discussion

In this paper, we have studied the distinguishing number, the distinguishing index, the distinguishing chromatic number and the distinguishing chromatic index of oriented graphs.

We have determined the minimum and maximum values, taken over all possible orientations of the corresponding underlying graph, of these parameters for paths, cycles, complete graphs and bipartite complete graphs, except for the minimum values for unbalanced bipartite complete graphs $K_{m,n}$ not covered by Corollary 22, in which case we were only able to provide upper bounds (see Theorem 15(3)).

Following our work, and apart the question of considering other graph classes, the main question is thus to determine the minimum values of the distinguishing parameters of unbalanced bipartite complete graphs not covered by Corollary 22, that is, of $K_{m,n}$ with $m \geq 5$ and $n > 2^m - \lceil \frac{m}{2} \rceil$. In particular, it would be interesting to know which of those complete bipartite graphs admit a rigid orientation. By Lemma 21, we know that $K_{m,n}$, $m < n$, does not admit any rigid orientation when $n > 2^m - \log_2(m)$. We can also prove that $K_{m,n}$ does not admit any rigid orientation when $n = 2^m - 2^p$, for some $p \geq 2$. However, we do not have any complete characterization of these graphs yet.

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