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CYRIL GRUNSPAN AND RICARDO PEREZ-MARCO

Abstract. We compute the revenue ratio of the Trail Stubborn mining strategy in the Bitcoin network and compare its profitability to other block-withholding strategies. We use for this martingale techniques and a classical analysis of the hiker problem. In this strategy the attacker could find himself mining in a shorter fork, but we prove that for some parameter values it is still profitable to not give up. This confirms previous numerical studies.

1. Introduction

In our previous article [4] we gave a rigorous foundation for the profitability of alternative mining strategies in the Bitcoin network [7]. As for games with repetition, it depends on the proper analysis of the revenue and the duration over attack cycles. More precisely, the expected revenue $\mathbb{E}[R]$ and expected duration $\mathbb{E}[\tau]$ over an attack cycle give the "Revenue Ratio"

$$\Gamma = \frac{\mathbb{E}[R]}{\mathbb{E}[\tau]}$$

This is the correct benchmark for the profitability of the strategy.

This analysis was also carried out previously by the authors in [4] to the "Selfish Mining" (SM) strategy from [1] and in [5] to "Lead-Stubborn Mining" (LSM) and "Equal Fork Stubborn Mining" (EFSM) strategies from [8]. In these articles we found for these strategies the exact mathematical formula for the revenue ratios and we compared its profitability in parameter space.

The main technique for these derivations is the application of martingale techniques introduced in [4] that yield, using Doob's Stopping Time Theorem, the expected duration of the attack cycles. We assume in the computation of the Revenue Ratio that there is no difficulty adjustment inside the attack cycles, or that $E[\tau]$ is much shorter than the period of difficulty adjustment so that its effect in the cycles can be neglected. But, as we proved in [4], the effect of these attacks is to slow down the

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network, and it is only after a difficulty adjustment that these "block withholding strategies" can become profitable. Then the profitability can also be read on the apparent hashrate. So these rogue strategies are an exploit on the difficulty adjustment formula and we gave in [4] an improvement proposal of the Bitcoin protocol to fix the difficulty adjustment formula.

In this article we apply again these new techniques to another block withholding strategy: The "Trail-Stubborn Mining" strategy from [8]. In this strategy, the block withholder miner does not give up when the honest chain takes over some block advantage, but instead keeps mining on top of his secret chain and only gives up if the advantage of the honest chain reaches $A \geq 1$ blocks. We talk about "A-Trail-Stubborn Mining" strategy or TSM_A in short.

We denote by $b > 0$ be the block reward, and τ_0 the average inter-block validation time for the total network (around 10 minutes for the Bitcoin network). We denote by q (resp. p) the relative hashing power of the attacker (resp. honest miners) and $\lambda = q/p < 1$. Let γ be the fraction of the honest network that the attacker attracts to mine on top of his fork. For a miner that after a difficulty adjustment has a Revenue Ratio $\tilde{\Gamma}$ we define his apparent hashrate $\tilde{q}$ by

$$\tilde{q} = \frac{\tilde{\Gamma} \cdot \tau_0}{b} \ .$$

The apparent hashrate of a miner can also be defined after a difficulty adjustment as the average proportion of blocks mined by the miner in the official blockchain. Our main Theorem is:

Theorem 1 (A-Trail-Stubborn mining)**.** *Let $A \geq 1$. The revenue ratio of the "A-Trail-Stubborn mining" strategy is*

$$\Gamma = \frac{q + \frac{(1-\gamma)pq(p-q)}{(p+pq-q^2)[A+1]} \left(\left([A-1] + \frac{1}{p} \frac{P_A(\lambda)}{[A+1]} \right) \lambda^2 - \frac{2}{\sqrt{1-4(1-\gamma)pq}+p-q} \right)}{1 + \frac{(1-\gamma)pq}{p+pq-q^2}(A+1) \left(\frac{[2]}{[A+1]} - \frac{2}{A+1} \right)} \frac{b}{\tau_0}$$

where $[n] = \frac{1-\lambda^n}{1-\lambda}$ for $n \in \mathbb{N}$, and $P_A(\lambda) = \frac{1-A\lambda^{A-1}+A\lambda^{A+1}-\lambda^{2A}}{(1-\lambda)^3}$.

After a difficulty adjustment, the apparent hashrate of the stubborn miner is

$$\tilde{q} = \frac{q + \frac{(1-\gamma)pq(p-q)}{(p+pq-q^2)[A+1]} \left(\left([A-1] + \frac{1}{p} \frac{P_A(\lambda)}{[A+1]} \right) \lambda^2 - \frac{2}{\sqrt{1-4(1-\gamma)pq}+p-q} \right)}{\frac{p+pq-q}{p+pq-q^2} + \frac{(1-\gamma)pq}{p+pq-q^2}(A+\lambda) \left(\frac{1}{[A+1]} - \frac{1}{A+\lambda} \right)}$$

The polynomial $1 - AX^{A-1} + AX^{A+1} - X^{2A}$ vanish at $X = 1$, as well as its two first derivatives, hence $P_A(X)$ is a polynomial in $\mathbb{Z}[X]$. Making $A = 1$ in Theorem 1 we get Theorem 1 of [5] as a particular case. Indeed, we have $\mathrm{TSM}_1 = \mathrm{LSM}$, i.e. 1-Trail-Stubborn Mining and Lead-Stubborn Mining strategies are the same. Before proving Theorem 1 we need to study a classical refinement of the Gambler's Ruin Problem: The hiker problem.

2. The hiker problem.

We consider a hiker on $[0, M]$ with $M \in \mathbb{N}$, $M \geq 2$. His position is denoted by the random process $(\mathbf{X}_n)_{n\in\mathbb{N}}$. The transition probability from i to j are

$$P(i, j) = \mathbb{P}[\mathbf{X}_{n+1} = j | \mathbf{X}_n = i] = p\,\mathbf{1}_{i=j-1} + q\,\mathbf{1}_{i=j+1}$$

for $(i, j) \in [1, M-1] \times [0, M]$. It is independent of $n \in \mathbb{N}$. We make the assumption that 0 and M are absorbing boundaries: $P(0, 0) = P(M, M) = 1$. The problem is studied in [2] where it is proved that with probability 1 the hiker exits $[1, M-1]$. We need more precise information.

Definition 2.1. *For $k \in \{0, M\}$ and $m \in [0, M]$, let $\nu_{m,k} \in \mathbb{N} \cup \{\infty\}$ be the stopping time defined by*

$$\nu_{m,k} = \inf\{n; \mathbf{X}_n = k | \mathbf{X}_0 = m\}$$

We denote $\nu_m = \nu_{m,0} \wedge \nu_{m,M}$, i.e. ν_m is the stopping time for exiting $[1, M-1]$ starting from m. From [2], we have $\nu_m < +\infty$ almost surely. The condition $\nu_m = \nu_{m,0}$ is equivalent to the realization of the event "the hiker exits $[1, M-1]$ at 1".

Theorem 2.2. *We have:*

$$\mathbb{E}[\nu_m] = \frac{M}{p-q} \cdot \left(\frac{1-\lambda^m}{1-\lambda^M} - \frac{m}{M} \right)$$

$$\mathbb{P}[\nu_m = \nu_{m,0}] = \frac{\lambda^m - \lambda^M}{1-\lambda^M}$$

$$\mathbb{E}[\nu_m | \nu_m = \nu_{m,0}] = \frac{m\lambda^m - (2M-m)\lambda^M + (2M-m)\lambda^{M+m} - m\lambda^{2M}}{p(1-\lambda)(\lambda^m - \lambda^M)(1-\lambda^M)}$$

The first two equations are well known classical results that can be found in [2] p. 314 and p.317. The last equation is from [10] and is not so classical and, to be self-contained, we give another proof in Appendix B.

Corollary 2.3. *We have* $\lim_{M\to\infty} \mathbb{E}[\nu_m | \nu_m = \nu_{m,0}] = \frac{m}{p-q}$.

Definition 2.4. *We denote by $\mathcal{L}(n)$ (resp. $\mathcal{R}(n)$) the number of steps to the left (resp. right) realized by the hiker between $t = 0$ and $t = n$.*

In other terms, $\mathcal{L}(0) = \mathcal{R}(0) = 0$ and for $n \leq \nu_m$,

$$\mathcal{L}(n) = \mathcal{L}(n-1) + \mathbf{1}_{\mathbf{X}(n)=\mathbf{X}(n-1)-1}$$
$$\mathcal{R}(n) = \mathcal{R}(n-1) + \mathbf{1}_{\mathbf{X}(n)=\mathbf{X}(n-1)+1}$$

Note that $\mathcal{L}(n) + \mathcal{R}(n) = n$ for $n \leq \nu_m$.

Corollary 2.5. *We have*

$$\mathbb{E}[\mathcal{L}(\nu_m)|\nu_m = \nu_{m,0}] = \frac{m}{2} + \frac{m\lambda^m - (2M-m)\lambda^M + (2M-m)\lambda^{M+m} - m\lambda^{2M}}{2p(1-\lambda)(\lambda^m - \lambda^M)(1-\lambda^M)}$$
$$\mathbb{E}[\mathcal{R}(\nu_m)|\nu_m = \nu_{m,M}] = \frac{M-m}{2} + \frac{m\lambda^m - (2M-m)\lambda^M + (2M-m)\lambda^{M+m} - m\lambda^{2M}}{2p(1-\lambda)(\lambda^m - \lambda^M)(1-\lambda^M)}$$

Proof. If the hiker exits $[1, M-1]$ at 1 (resp. $M-1$), then $\mathcal{L}(\nu_m) = \mathcal{R}(\nu_m) + m$ (resp. $\mathcal{R}(\nu_m) = \mathcal{L}(\nu_m) + M - m$). So we have

$$\mathbb{E}[\mathcal{L}(\nu_m)|\nu_m = \nu_{m,0}] = \frac{m}{2} + \frac{\mathbb{E}[\nu_m|\nu_m = \nu_{m,0}]}{2}$$
$$\mathbb{E}[\mathcal{L}(\nu_m)|\nu_m = \nu_{m,M}] = \frac{M-m}{2} + \frac{\mathbb{E}[\nu_m|\nu_m = \nu_{m,0}]}{2}$$

and the result follows from Theorem 2.2. □

In particular, for $m = 2$ this gives:

$$\mathbb{E}[\mathcal{L}(\nu_2)|\nu_2 = \nu_{2,0}] = 1 + \frac{1}{p}.\frac{1 - (M-1)\lambda^{M-2} + (M-1)\lambda^M - \lambda^{2M-2}}{(1-\lambda)(1-\lambda^{M-2})(1-\lambda^M)} \tag{1}$$

3. Expected duration of the Trail-Stubborn Mining Strategy

3.1. **Notations and previous results.** We set, $\alpha = \frac{p}{\tau_0}, \alpha' = \frac{q}{\tau_0}$ and $\lambda = \frac{q}{p} < 1$. We note N and N' the two independent Poisson processes with parameters α and α' representing the number of blocks validated by the honest miners and the rogue miner. We denote by $T_1, T_2, \ldots$ (resp. $T'_1, T'_2, \ldots$) the inter-block validation time for the honest miners (resp. attackers).

We use some notations from [4]. In particular, for the stopping times:

$$\tau = \inf\{t \geq T_1;\, N(t) = N'(t) + \mathbf{1}_{T_1 < T'_1}\} \tag{2}$$

and

$$\tau_{LSM} = \tau + (T_{N(\tau)+1} \wedge T'_{N(\tau)+1}) \cdot \mathbf{1}_{T'_1 \leq T_1} \tag{3}$$

We proved in [4] that

$$\mathbb{E}[\tau] = \frac{p}{p-q}\tau_0,\ \mathbb{E}[\tau_{LSM}] = \left(\frac{p}{p-q} + q\right)\tau_0, \text{ and } \quad \mathbb{E}[N'(\tau)] = \alpha'\mathbb{E}[\tau] = \frac{pq}{p-q} \tag{4}$$

More precisely, for $n \geq 1$, we have

$$\mathbb{P}[N'(\tau) = n] = C_{n-1}(pq)^n \tag{5}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the n-th Catalan number, whose generating series is $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

At the end of an attack cycle, the revenue of a miner following the Lead Stubborn Mining strategy is denoted by $R(\tau_{LSM})$.

3.2. **Description of the A-Trail-Stubborn mining strategy.** At the beginning of an attack cycle both, the rogue miner and the honest miners start mining on top of the same common block. Then, either the first block is discovered by the honest miners, and the attack cycle ends, or the attacker is the first one validating a block. Then, he keeps mining secretly until he is being caught up by the honest miners. During this period, each time the honest miners publish a new block, the rogue miner broadcasts the part of his fork sharing the same height. Once he has been caught up, there is a "decisive competition" to decide which fork prevails. In this competition, the rogue miner does not withold his block. There are two cases depending on who the winner is. Either the new block is mined on top of a block validated by the rogue miner (by himself or by a fraction γ of the honest miners) and then the attack cycle ends immediatly. Otherwise, the attacker has a fork which is one block behind the official blockchain. We call this event $\mathbf{\Sigma}$. His delay is defined as the difference between the height of the official blockchain and his fork. Then, he keeps mining until his delay exceeds a fixed threeshold $A \geq 1$, or ends up leading the official blockchain by one block. Then, in both cases, the cycle attack ends. The trailing mining strategy is a repetition of these attack cycles.

3.3. **Stopping time.** We denote by ξ the stopping time of an attack cycle corresponding to the Trail-Stubborn mining strategy (TSM). At the end of an attack cycle the revenue of a rogue miner following TSM is denoted by $R(\xi)$. Note that when $T_1' < T_1$, there is a "decisive round" which starts at τ and ends at τ_{LSM}. So, $\tau_{LSM} \leq \xi$ and from 0 to τ_{LSM} the two strategies τ_{LSM} and ξ are the same.

3.4. **Blocks of the rogue miner in the official blockchain.** For $t \geqslant 0$, we denote by $Z(t)$ the number of blocks mined by a miner following the Trail-Stubborn Mining strategy at t-time and present in the official blockchain. We have that $t \mapsto Z(t)$ is

non-decreasing. Before $t \leq \tau$, the two strategies "Trail-Stubborn Mining Strategy" and"Lead Stubborn Mining Strategy" are the same. So, by [5] we know that

$$\mathbb{E}[Z(\tau)|N'(\tau) = n] = n - \frac{1-(1-\gamma)^n}{\gamma} \tag{6}$$

Lemma 3.1. *The following conditions are equivalent to* $\mathbf{\Sigma}$*:*

(i) $\tau_{LSM} < \xi$*;*
(ii) $R(\tau_{LSM}) < N'(\tau_{LSM})b$*;*
(iii) $(T'_1 < T_1) \wedge (T_{N(\tau)+1} < T'_{N(\tau)+1}) \wedge$ *(the* $(N(\tau)+1)$*-th honest block is found on top of a block mined by a honest miner).*

If one of these conditions is satisfied then $N'(\tau_{LSM}) = N'(\tau)$*,* $N(\tau_{LSM}) = N(\tau)+1$ *and* $Z(\tau_{LSM}) = Z(\tau)$*. We have that* $\mathbf{\Sigma}$ *is* τ_{LSM}*-measurable and* $\mathbb{P}[\mathbf{\Sigma}] = (1-\gamma)pq$*.*

Proof. If (i) holds then $T'_1 < T_1$ otherwise $\tau_{LSM} = \tau = \xi = T_1$. Moreover, at τ_{LSM}, at least one block mined by the rogue miner has not been recognized by the official blockchain (otherwise, the cycle ends at τ_{LSM} and $\xi = \tau_{LSM}$). So, $R(\tau_{LSM}) < N'(\tau_{LSM})b$. If (ii) is true then $0 < N'(\tau_{LSM})$. So, the miner has at least mined a block during the attack cycle. So, $T'_1 < T_1$ (otherwise as before $\tau_{LSM} = \tau = \xi = T_1$ and $N'(\tau_{LSM}) = 0$)). Moreover, the rogue miner has lost the "decisive competition". Otherwise, we have $R(\tau_{LSM}) = N'(\tau_{LSM})b$. Also, by the same argument, the block found by the honest miners during this round cannot have been validated on top of a block mined by the rogue miner. So, we get (iii). Finally, if (iii) holds, then the rogue miner has lost the "decisive competition". Hence, $\tau_{LSM} < \xi$ by definition of an attack cycle and so (i), (ii) and (iii) are equivalent. Also, if one of these conditions is satisfied, then the rogue miner did not mine a block during the period $[\tau, \tau_{LSM}]$ whereas the honest miner has found exactly one. So, $N'(\tau_{LSM}) = N'(\tau)$ and $N(\tau_{LSM}) = N(\tau)+1$. Moreover the block found by the honest miners has been found on top of an honest block by (iii). So we have, $Z(\tau_{LSM}) = Z(\tau)$. By (ii), $\mathbf{\Sigma}$ is τ_{LSM}-measurable. Moreover, the condition $\{T'_1 < T_1\}$ occurs with probability q and the two last conditions of (iii) occur with probability $(1-\gamma)p$. Therefore, we have $\mathbb{P}[\mathbf{\Sigma}] = (1-\gamma)pq$. □

3.5. **Trail-mining.** We consider that after a possible second phase of the attack cycle (after τ_{LSM}), the rogue miner following TSM_A will give up if his delay exceeds A with $A \geqslant 1$. Note that at the beginning of this second phase, the delay of the miner is 1. So, $\mathrm{TSM}_1 = \mathrm{LSM}$. Note also that in order to win, it is not enough for the miner to catch-up the official blockchain. He needs to lead it by 1 block. So, his delay is in between -1 and A. Therefore, he behaves as the hiker studied in Section 2 with delay $\mathbf{X}_n - 1$ and $M = A+1$.

Proposition 3.2. *We have*

$$\xi = \tau_{LSM} + \sigma.\mathbf{1}_{\mathbf{\Sigma}}$$

where σ is the stopping time defined by

$$\sigma = \inf\{t \in \mathbb{R}_+^*; (\tilde{N}'(t) = \tilde{N}(t) + 2) \vee (\tilde{N}(t) = \tilde{N}'(t) + A - 1)\}$$

with $\tilde{N}(t) = N(t + \tau_{LSM}) - N(\tau_{LSM})$ and $\tilde{N}'(t) = N'(t + \tau_{LSM}) - N'(\tau_{LSM})$.

In particular, we have that $\tilde{N}$ and $\tilde{N}'$ are two independent Poisson processes with respective parameters α and α', and σ is independent with $\mathbf{\Sigma}$.

Proof. The stopping time of the Trail-Stubborn Mining Strategy is the same as the stopping time of the Lead Stubborn Mining strategy studied in [5] except when the stubborn miner has been first mining a block, then has been caught-up by the honest miners, and at has lost the final "competition" (when a fraction $(1-\gamma)p$ of the honest miners finds a new block on top of a honest block). We have called $\mathbf{\Sigma}$ this event. If it occurs, then the stubborn miner keeps on mining until he catches up the honest miners or his delay becomes too big. In this case, he needs to catch-up the honest miners and also lead the official blockchain by 1. The start time of this possible second round is τ_{LSM} with $N'(\tau_{LSM}) = N(\tau_{LSM}) - 1$ and the miner will stop at $\tau_{LSM} + t$ with $N'(t + \tau_{LSM}) = N(t + \tau_{LSM}) + 1$ or $N(t + \tau_{LSM}) = N'(t + \tau_{LSM}) + A$. The first equality is equivalent to $N'(t + \tau_{LSM}) - N'(\tau_{LSM}) = N(t + \tau_{LSM}) - N(\tau_{LSM}) + 2$ and the second is equivalent to $N(t+\tau_{LSM}) - N(\tau_{LSM}) = N'(t+\tau_{LSM}) - N'(\tau_{LSM}) + A - 1$. Moreover, by the strong Markov property σ is independent with τ_{LSM}. So, by Lemma 3.1, it is also independent with $\mathbf{\Sigma}$. □

Note that the condition $(\tilde{N}'(\sigma) = \tilde{N}(\sigma)+2) \vee (\tilde{N}(\sigma) = \tilde{N}'(\sigma)+A-1)$ is equivalent to $X(\sigma) \in \{0, A+1\}$ with $X(t) = N(t) - N'(t) + 2$. So we have that the miner is a hiker on $[0, M]$ as studied in section 2 starting from $\mathbf{X}_0 = 2$ with $M = A + 1$. In Appendix A we prove the following Proposition:

Proposition 3.3. *We have*

$$\mathbb{E}[\sigma] = \frac{A+1}{p-q}\left(\frac{1-\lambda^2}{1-\lambda^{A+1}} - \frac{2}{A+1}\right)\tau_0$$

Proposition 3.4. *We have*

$$\frac{\mathbb{E}[\xi]}{\tau_0} = \frac{p}{p-q} + q + (A+1)\cdot\frac{(1-\gamma)pq}{p-q}\cdot\left(\frac{1-\lambda^2}{1-\lambda^{A+1}} - \frac{2}{A+1}\right)$$

Proof. By Proposition 3.2, we have

$$\mathbb{E}[\xi] \;=\; \mathbb{E}[\tau_{LSM}] + \mathbb{P}[\mathbf{\Sigma}]\cdot\mathbb{E}[\sigma]$$

So, we get the result using (4), Lemma 3.1 and Proposition 3.3. □

4. Revenue ratio of the Trail-Stubborn Mining Strategy

Proposition 4.1. *We have:*

$$R(\xi) = R(\tau_{LSM})\cdot\mathbf{1}_{\xi=\tau_{LSM}}+(N'(\tau)+\mathcal{L}(\nu_2))b\cdot\mathbf{1}_{(\xi>\tau_{LSM})\wedge(\nu_2=\nu_{2,0})}+Z(\tau)b\cdot\mathbf{1}_{(\xi>\tau_{LSM})\wedge(\nu_2=\nu_{2,A+1})}$$

In this Proposition $\mathcal{L}(\nu_2)$ is the number of blocks validated by the rogue miner during the second phase of the strategy. The event $(\xi > \tau_{LSM}) \wedge (\nu_2 = \nu_{2,0})$ (resp. $(\xi > \tau_{LSM}) \wedge (\nu_2 = \nu_{2,A+1})$) means that the cycle is made of two distinct phases: in the first one the rogue miner looses the first phase of the attack, and in the second one he wins (resp. looses) the second phase.

Proof. If $R(\tau_{LSM}) < N'(\tau_{LSM})b$, then the miner tries to catch-up the official blockchain. He is in the position of a hiker starting from $\mathbf{X}_0 = 2$ and winning when $\nu_2 = \nu_{2,0}$. Each move to the left (towards 0) corresponds to a new block mined by the stubborn miner. So, if he succeeds (case $\nu_2 = \nu_{2,0}$), then he earns a reward $(N'(\tau_{LSM})+\mathcal{L}(\nu_2))b$. If he fails (case $\nu_2 = \nu_{2,A+1}$) then he earns only $Z(\tau_{LSM})b$ and the attack cycle ends. Otherwise, the strategy ends at τ_{LSM} and $R(\xi) = R(\tau_{LSM})$. The result then follows from Lemma 3.1. □

Now we compute the expected revenue of the A-Trail-Stubborn Mining Strategy in an attack cycle.

Proposition 4.2. *We have*

$$\frac{\mathbb{E}[R(\xi)]}{b} = \left(\frac{p+pq-q^2}{p-q}\right)q + (1-\gamma)pq\cdot$$
$$\cdot\left[\left(1+\frac{1}{p}\cdot\frac{1-A\lambda^{A-1}+A\lambda^{A+1}-\lambda^{2A}}{(1-\lambda)(1-\lambda^{A-1})(1-\lambda^{A+1})}\right)\frac{\lambda^2-\lambda^{A+1}}{1-\lambda^{A+1}} - \frac{2p}{\sqrt{1-4(1-\gamma)pq}+p-q}\frac{1-\lambda^2}{1-\lambda^{A+1}}\right]$$

Proof. Consider the events, for $n \in \mathbb{N}$, $E_n = \{N'(\tau) = n\}$, $F = \{R(\tau_{LSM}) = N'(\tau_{LSM})b\}$ and $G = \{\nu_2 = \nu_{2,0}\}$. From [4] and [5] we have

$$\mathbb{P}[E_n] = p\,\mathbf{1}_{n=0} + (pq)^n C_{n-1}\mathbf{1}_{n>0}$$
$$\mathbb{P}[G] = \frac{\lambda^2-\lambda^{A+1}}{1-\lambda^{A+1}}$$

and for $n > 0$,

$$\mathbb{P}[E_n \cap F] = \mathbb{P}[E_n](q+\gamma p)$$

Note also that

- If E_0 occurs then $R(\xi) = 0$.

- If $E_n \cap F$ occurs (with $n > 0$) then $R(\xi) = (n+1)b$ with probability $\frac{q}{q+\gamma p}$ and $R(\xi) = nb$ with probability $\frac{\gamma p}{q+\gamma p}$.
- If $E_n \cap \bar{F} \cap \bar{G}$ occurs then $R(\xi) = Z(\tau)b$.
- If $E_n \cap \bar{F} \cap G$ occurs then $R(\xi) = (n + \mathcal{L}(\nu_2))b$.

So, by conditioning on $E_0, E_n \cap F, E_n \cap \bar{F} \cap G, E_n \cap \bar{F} \cap \bar{G}$ and using (6) and Corollary 2.5 together with $\sum_{n \geqslant 0} \mathbb{P}[E_n] = 1$ we have:

$$\begin{aligned}
\frac{\mathbb{E}[R(\xi)]}{b} &= 0 \cdot \mathbb{P}[E_0] + \sum_{n>0} ((n+1)q + n\gamma p)\, \mathbb{P}[E_n] \\
&\quad + \sum_{n>0} \left(n - \frac{1-(1-\gamma)^n}{\gamma} \right) (1-\gamma)p \frac{1-\lambda^2}{1-\lambda^{A+1}}\, \mathbb{P}[E_n] \\
&\quad + \sum_{n>0} \left(n + \mathbb{E}[\mathcal{L}(\nu_2) | \nu_2 = \nu_{2,0}] \right) (1-\gamma)p \frac{\lambda^2 - \lambda^{A+1}}{1-\lambda^{A+1}}\, \mathbb{P}[E_n] \\
&= \mathbb{E}[N'(\tau)] + \frac{(1-\gamma)p}{\gamma} \frac{1-\lambda^2}{1-\lambda^{A+1}} (1-\gamma)pq\, C((1-\gamma)pq) \\
&\quad + \left(q - \frac{(1-\gamma)p}{\gamma} \frac{1-\lambda^2}{1-\lambda^{A+1}} + (1-\gamma)p \mathbb{E}[\mathcal{L}(\nu_2) | \nu_2 = \nu_{2,0}] \frac{\lambda^2 - \lambda^{A+1}}{1-\lambda^{A+1}} \right) (1 - \mathbb{P}[E_0]) \\
&= \left(\frac{p}{p-q} + q \right) q - (1-\gamma)pq \frac{1-\lambda^2}{1-\lambda^{A+1}} \frac{[1 - (1-\gamma)pC((1-\gamma)pq)]}{\gamma} \\
&\quad + (1-\gamma)pq \mathbb{E}[\mathcal{L}(\nu_2) | \nu_2 = \nu_{2,0}] \frac{\lambda^2 - \lambda^{A+1}}{1-\lambda^{A+1}}
\end{aligned}$$

Moreover for $q > 0$, we have

$$\begin{aligned}
1 - (1-\gamma)p\, C((1-\gamma)pq) &= 1 - \frac{1 - \sqrt{1 - 4(1-\gamma)pq}}{2q} \\
&= \frac{\sqrt{1 - 4(1-\gamma)pq} - (p-q)}{2q} \\
&= \frac{1 - 4pq + 4\gamma pq - (p^2 - 2pq + q^2)}{2q \left[\sqrt{1 - 4(1-\gamma)pq} + p - q \right]} \\
&= \frac{2p\gamma}{\sqrt{1 - 4(1-\gamma)pq} + p - q}
\end{aligned}$$

and we get the result using (1). □

Proposition 3.4 and Proposition 4.2 give the revenue ratio of the strategy and the first part of Theorem 1.

5. Difficulty adjustment

Proposition 5.1. *We have*

$$\mathbb{E}[N(\xi)\vee N'(\xi)]=\frac{pq+p-q}{p-q}+\frac{(1-\gamma)pq}{p-q}\left((Ap+q)\frac{1-\lambda^2}{1-\lambda^{A+1}}-1\right)$$

Proof. We keep the same notations as in the proof of Proposition 4.2. Note that

- If E_0 occurs, then $N(\xi)\vee N'(\xi)=1$.
- If $E_n\cap F$ occurs $(n>0)$, then $N(\xi)\vee N'(\xi)=n+1$.
- If $E_n\cap\bar F\cap G$ occurs $(n>0)$, then $N(\xi)\vee N'(\xi)=n+\mathcal{L}(\nu_2)$.
- If $E_n\cap\bar F\cap\bar G$ occurs $(n>0)$, then $N(\xi)\vee N'(\xi)=N(\xi)=n+1+\mathcal{R}(\nu_2)$.

So, by conditioning as before and using Corollary 2.5, we get

$$\begin{aligned}
\mathbb{E}[N(\xi)\vee N'(\xi)] &= 1\cdot\mathbb{P}[E_0]+\sum_{n>0}(n+1)(q+\gamma p)\,\mathbb{P}[E_n]\\
&+\sum_{n>0}\left(n+1+\frac{1}{2}\mathbb{E}[\nu_2|\nu_2=\nu_{2,0}]\right)(1-\gamma)p\,\mathbb{P}[E_n]\,\mathbb{P}[\nu_2=\nu_{2,0}]\\
&+\sum_{n>0}\left(n+1+\frac{A+1}{2}-1+\frac{1}{2}\mathbb{E}[\nu_2|\nu_2=\nu_{2,A+1}]\right)(1-\gamma)p\,\mathbb{P}[E_n]\,\mathbb{P}[\nu_2=\nu_{2,A+1}]\\
&=\mathbb{P}[E_0]+\sum_{n>0}(n+1)\mathbb{P}[E_n]+\sum_{n>0}\left(\frac{A+1}{2}-1\right)(1-\gamma)p\,\mathbb{P}[E_n]\,\mathbb{P}[\nu_2=\nu_{2,A+1}]\\
&+\sum_{n>0}(1-\gamma)p\,\mathbb{P}[E_n]\,\frac{\mathbb{E}[\nu_2]}{2}\\
&=\mathbb{E}[N'(\tau)]+1+(1-\gamma)pq\left(\left(\frac{A+1}{2}-1\right)\mathbb{P}[\nu_2=\nu_{2,A+1}]+\frac{\mathbb{E}[\nu_2]}{2}\right)\\
&=\frac{pq+p-q}{p-q}+\frac{(1-\gamma)pq}{p-q}((Ap+q)\,\mathbb{P}[\nu_2=\nu_{2,A+1}]-1)
\end{aligned}$$

□

Theorem 5.2. *The parameter δ updating the difficulty of the A-trail stubborn mining strategy is given by*

$$\delta=\frac{\frac{p+pq-q^2}{p-q}+(A+1)\cdot\frac{(1-\gamma)pq}{p-q}\cdot\left(\frac{1-\lambda^2}{1-\lambda^{A+1}}-\frac{2}{A+1}\right)}{\frac{pq+p-q}{p-q}+\frac{(1-\gamma)pq}{p-q}((Ap+q)\frac{1-\lambda^2}{1-\lambda^{A+1}}-1)}$$

Proof. From [4] we have that $\delta=\frac{\mathbb{E}[\xi]}{\mathbb{E}[N(\xi)\vee N'(\xi)]}\cdot\frac{1}{\tau_0}$. □

5.1. **Observations.** We denote by $\tilde{q}_A$ the long-term apparent hashrate of the A-Trail-Stubborn-mining strategy. As we have already observed, the lead-stubborn mining strategy LSM is a particular case of the A-Trail-Stubborn-mining strategy with $A = 1$ (in tis case, there is no possible second phase of the attack after τ_{LSM}). We note that Theorem 1 yields one of the results of [5]: if we choose $A = 1$ in Theorem 1, we get Theorem 1 of [5]. In Figure 1 below, we compare $\tilde{q}_{\text{LSM}}$ (the long-term apparent hashrate of the strategy LSM) with $\text{Max}\{\tilde{q}_A\,;\, A \geq 2\}$. Depending on (q, γ), this shows when a second phase of the attack increases the efficiency of the strategy LSM. In general, when γ is small, TSM is an amelioration of LSM.

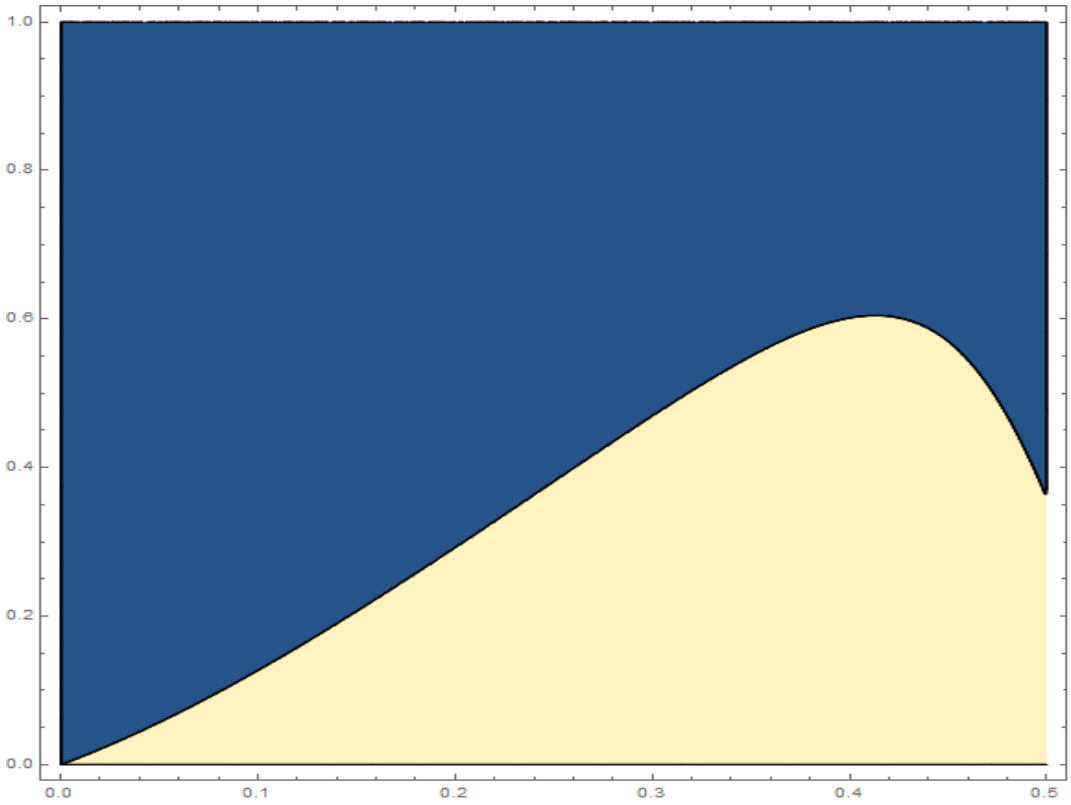

FIGURE 1. LSM vs A-Trail-Stubborn Mining strategy for $A \geq 2$

For γ greater than 20%, TSM with $A = 2$ dominates all other trailing strategies whatever q is. See Figure 2.

Also, if $\gamma = 0$, then

$$\widetilde{q}_{TSM} = \frac{\left(\frac{p+pq-q^2}{p-q}\right) q + pq\left[\left(1 + \frac{1}{p}\cdot\frac{1-A\lambda^{A-1}+A\lambda^{A+1}-\lambda^{2A}}{(1-\lambda)(1-\lambda^{A-1})(1-\lambda^{A+1})}\right)\frac{\lambda^2-\lambda^{A+1}}{1-\lambda^{A+1}} - \frac{p}{p-q}\frac{1-\lambda^2}{1-\lambda^{A+1}}\right]}{1 + \frac{pq}{p-q}\left((Ap+q)\frac{1-\lambda^2}{1-\lambda^{A+1}}\right)}$$

On the other hand, when $\gamma = 0$, the apparent hashrate after a difficulty adjustment for the selfish mining strategy is $\widetilde{q}_{SM} = \frac{pq^2+(p-q)(q+pq^2-p^2q)}{p^2q+p-q}$. It turns out that for $\gamma = 0$,

$$\lim_{q\to\frac{1}{2}} \widetilde{q}_{TSM} = 1 - \frac{1}{A+1} < 1 = \lim_{q\to\frac{1}{2}} \widetilde{q}_{SM}$$

Hence, when $q \to \frac{1}{2}$ and $\gamma \ll 1$, SM dominates all TSM strategies.

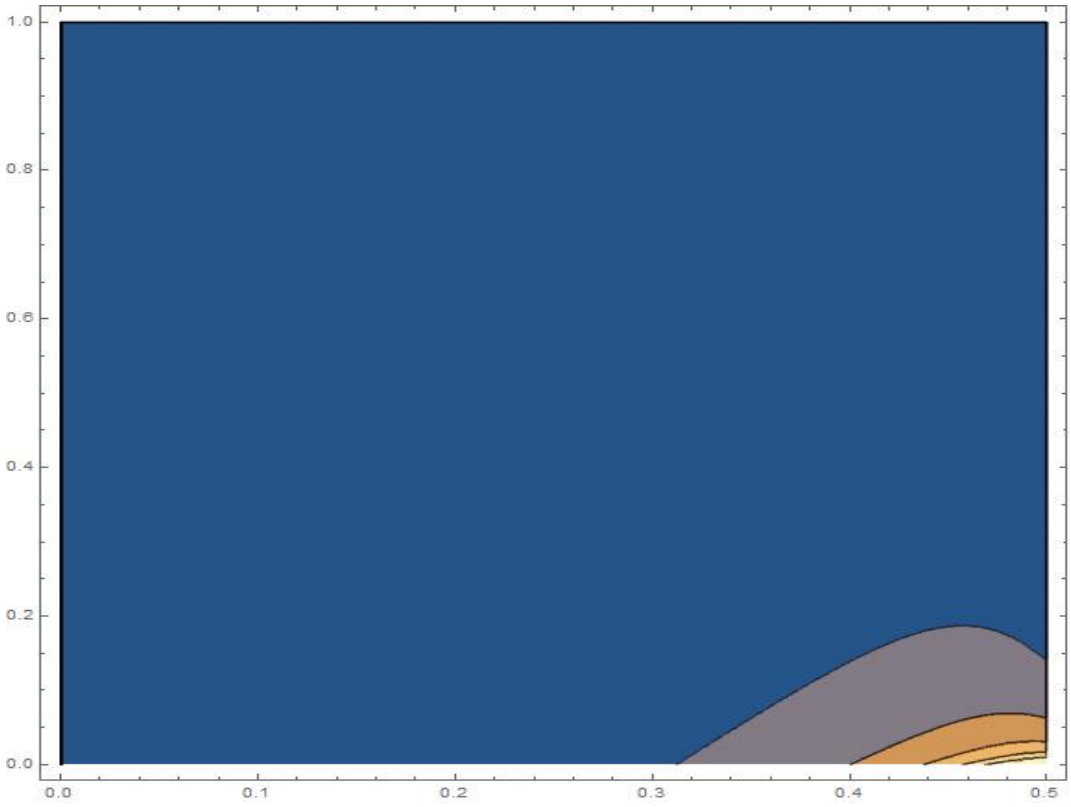

FIGURE 2. A-Trail-Stubborn Mining strategy for $A = 2, 3, 4, 5, 6, 7$

6. Mixed strategies.

6.1. **Weight of a mining strategy.** We consider a miner mining according to a strategy τ.

Definition 6.1. *The weight of a mining strategy is the average number of official blocks mined during an attack cycle. It is denoted by the greek letter μ.*

Note that if the strategy leads to a difficulty adjustment D, then we have: $\mu = \frac{\mathbb{E}[\tau]}{\tau_0 D}$.

6.2. **Apparent hashrate of a mixed strategy.** We consider now a miner implementing a mixed strategy. He starts mining according to strategy 1, then at the end of an attack cycle, he decides to follow another strategy 2, and so on until he comes back to strategy 1 after implementing n of different strategies. Thus the attack cycle is a given pattern of attack cycles of different strategies.

We denote by $\Gamma_1, \Gamma_2, \ldots, R_1, R_2, \ldots \tau_1, \tau_2, \ldots \tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots D_1, D_2 \ldots \mu_1, \mu_2, \ldots$ the revenue ratio, revenue, duration time, long-term apparent hashrate, difficulty adjustment and weight over an attack cycle of strategy $1, 2, \ldots$. We denote by $\Gamma, R, \tau, \tilde{\Gamma}, D$ and μ, the revenue ratio, revenue, duration time, long-term apparent hashrate, difficulty adjustment and weight after an attack cycle of the mixed strategy.

Theorem 6.2. *We have that $(D, \tilde{\Gamma})$ is barycenter of $(D_1, \tilde{\Gamma}_1), (D_2, \tilde{\Gamma}_2), \ldots$ weighted by $\mu_1, \mu_2, \ldots$.*

Proof. The number μ of official blocks mined after a whole attack cycle of the mixed strategy is $\mu = \mu_1 + \mu_2 + \ldots + \mu_n$. Moreover, $\mathbb{E}[\tau] = \mathbb{E}[\tau_1] + \mathbb{E}[\tau_2] + \ldots + \mathbb{E}[\tau_n]$. Therefore,

$$D = \sum_{i=0}^{n} \frac{\frac{\mathbb{E}[\tau_i]}{\tau_0 D_i}}{\mu} D_i = \sum_{i=0}^{n} \frac{\mu_i}{\mu} D_i$$

Similarly, we have

$$\tilde{\Gamma} = \Gamma D = \frac{\mathbb{E}[R]}{\mathbb{E}[\tau]} D = \frac{\sum_{i=1}^{n} \mathbb{E}[R_i]}{\mathbb{E}[\tau]} D = \frac{1}{\mu} \sum_{i=1}^{n} \mathbb{E}[R_i]$$

$$= \frac{1}{\mu} \sum_{i=1}^{n} \Gamma_i \mathbb{E}[\tau_i] = \frac{1}{\mu} \sum_{i=1}^{n} \tilde{\Gamma}_i \frac{\mathbb{E}[\tau_i]}{D_i} = \frac{1}{\mu} \sum_{i=1}^{n} \mu_i \tilde{\Gamma}_i$$

□

Corollary 6.3. *We have $\tilde{\Gamma} \leq \max(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \Gamma_n)$ with equality if and only if the strategy is not mixed.*

Therefore, there is no advantage in implementing mixed strategies.

7. Comparison with other strategies

We compare Trailing-Stubborn Mining strategies with $A = 2, 3, 4$ and other strategies HM, SM, LSM and EFSM studied in [5]. We observe that LSM is the dominant strategy in a very thin region between SM, EFSM and TSM2. Below to the right (but for γ not too small), the dominant strategy is TSM3. The strategy TSM4 is dominant in a very little domain with $q \approx 0.5$ and $\gamma \approx 5\%$. For γ less than 5% and large q (but less than 0.5), LSM is the dominant strategy confirming the observation at the end of section 5.

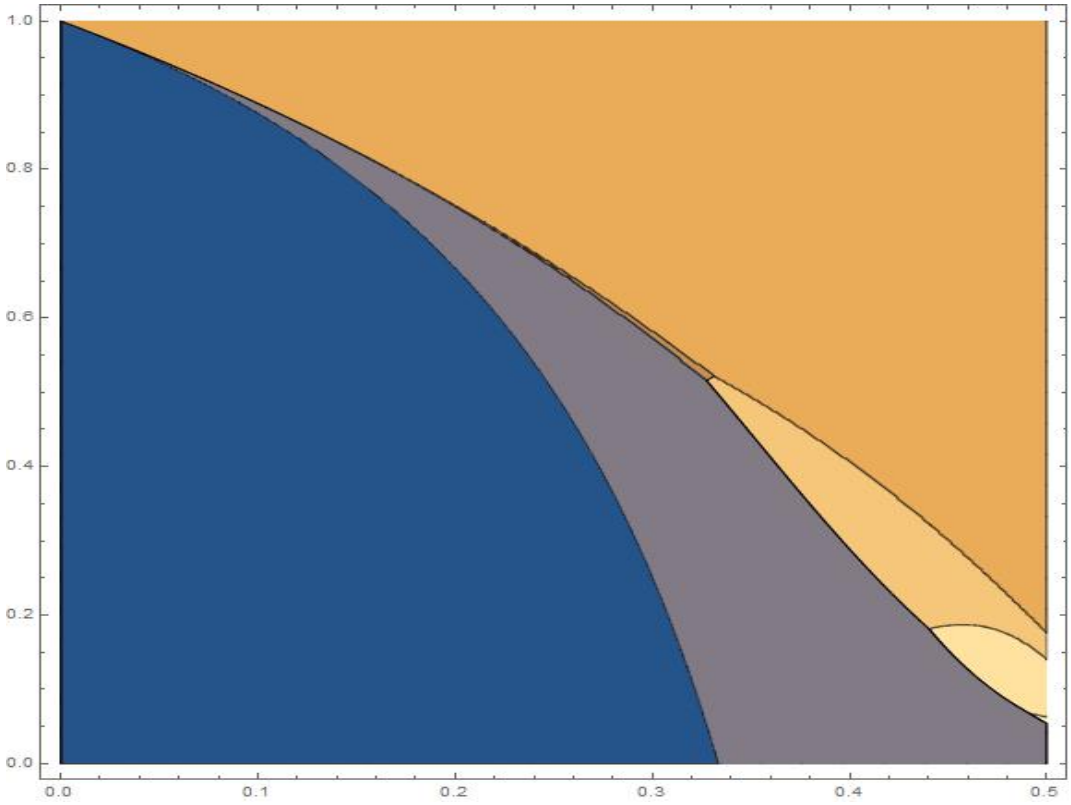

Figure 3. TSM with $A = 2, 3, 4$ vs other strategies (HM, SM, LSM, EFSM)

Appendix A. Poisson processes and random walk.

Let N and N' be two independent Poisson processes with parameters α and α' starting at 0: $N(0) = N'(0) = 0$. For $n, m, j, M \in \mathbb{N}$, with $m \leq j \leq M$, let

$$\begin{aligned}
\bar{S}_n &= \inf\{t \in \mathbb{R}_+; N(t) + N'(t) \geq n\} \\
\mathbf{X}_n &= (N - N')(\bar{S}_n) \\
\tau' &= \inf\{t \in \mathbb{R}_+; N'(t) - N(t) = m\} \\
\tau'' &= \inf\{t \in \mathbb{R}_+; N(t) - N'(t) = M - m\} \\
\tau &= \tau' \wedge \tau'' \\
\nu_{m,j} &= \inf\{i \in \mathbb{N}; \mathbf{X}_i = j - m\} \\
\nu_m &= \nu_{m,0} \wedge \nu_{m,M}
\end{aligned}$$

Theorem A.1. *We have that $(\mathbf{X}_n)_{n\in\mathbb{N}}$ is a random walk with a probability $p = \frac{\alpha}{\alpha+\alpha'}$ (resp. $q = \frac{\alpha'}{\alpha+\alpha'}$) to move to the right (resp. left). Moreover, if $\tau_0 = \frac{1}{\alpha+\alpha'}$, we have*

$$\begin{aligned}
\mathbb{P}[\tau = \tau'] = \mathbb{P}[\nu_m = \nu_{m,0}] &= \frac{\lambda^m - \lambda^M}{1 - \lambda^M} \\
\frac{\mathbb{E}[\tau]}{\tau_0} = \mathbb{E}[\nu_m] &= \frac{M}{p - q}\left(\frac{1 - \lambda^m}{1 - \lambda^M} - \frac{m}{M}\right)
\end{aligned}$$

Proof. For $n \in \mathbb{N}$, we have $\mathbf{X}_{n+1} = \mathbf{X}_n \pm 1$ and $\mathbb{P}[\mathbf{X}_{n+1} = \mathbf{X}_n + 1] = p$. So, $(\mathbf{X}_n)_{n\in\mathbb{N}}$ is a random walk. The two events $\{\tau = \tau'\}$ and $\{\nu_m = \nu_{m,0}\}$ are equal. So, they have the same probability. The computation of $\mathbb{P}[\nu_m = \nu_{m,0}]$ can be found in several places. See for example [2] p. 314. Using Doob's theorem, we have proved in [4] that $N(\tau), N'(\tau), \tau \in L^1$, $\mathbb{E}[N(\tau)] = \alpha\mathbb{E}[\tau]$ and $\mathbb{E}[N'(\tau)] = \alpha'\mathbb{E}[\tau]$. We also have:

$$\begin{aligned}
\mathbb{E}[N'(\tau)] &= \mathbb{E}[N'(\tau)|\tau = \tau']\,\mathbb{P}[\tau = \tau'] + \mathbb{E}[N'(\tau)|\tau = \tau'']\,\mathbb{P}[\tau = \tau''] \\
&= (m + \mathbb{E}[N(\tau)|\tau = \tau'])\,\mathbb{P}[\tau = \tau'] + (m - M + \mathbb{E}[N(\tau)|\tau = \tau''])\,\mathbb{P}[\tau = \tau''] \\
&= m\mathbb{P}[\tau = \tau'] + (m - M)\,\mathbb{P}[\tau = \tau''] + \mathbb{E}[N(\tau)]
\end{aligned}$$

So, we get

$$(\alpha - \alpha')\mathbb{E}[\tau] = (M - m)\mathbb{P}[\tau = \tau''] - m\mathbb{P}[\tau = \tau']$$

Therefore,

$$\frac{\mathbb{E}[\tau]}{\tau_0} = \frac{1}{p - q}\left((M - m)\mathbb{P}[\tau = \tau''] - m\mathbb{P}[\tau = \tau']\right) = \frac{M}{p - q}\left(\mathbb{P}[\tau = \tau''] - \frac{m}{M}\right)$$

and we get the result. See also [2] p. 317. □

Appendix B. Proof of a result of F. Stern.

We prove the last equation of Theorem 2.2 that is a result of F. Stern (see [10]).

An auxiliary sequence. We study first the sequence $(u_n)_{n\geq 0}$.

Definition B.1. *Let $(u_n)_{n\geq 0}$ be the sequence defined by induction by $u_0 = 1$ and for $n \geq 1$,*

$$u_n = \lambda \frac{1-\lambda^n}{1-\lambda^{n+2}} u_{n-1} + \frac{1}{p} \frac{1-\lambda^{n+1}}{1-\lambda^{n+2}}$$

The computation of u_1 and u_2 gives

$$\frac{u_0+u_1}{2} = \frac{1}{1-pq}$$
$$\frac{u_1+u_2}{2} = \frac{1}{1-2pq}$$

Let l be the solution to $l = \lambda l + \frac{1}{p}$, that is $l = \frac{1}{p-q} > 1$, and we have

$$\begin{aligned} l - u_n &= \lambda \frac{1-\lambda^n}{1-\lambda^{n+2}}(l-u_{n-1}) + \frac{2}{p}\frac{\lambda^{n+1}}{1-\lambda^{n+2}} \\ &\leq \lambda(l-u_{n-1}) + \frac{2}{p}\frac{\lambda^{n+1}}{1-\lambda} \end{aligned}$$

Then, by induction, we have

$$0 \leq l - u_n \leq \lambda^n (l-u_0) + \frac{2n}{p-q}\lambda^{n+1}$$

and therefore

$$\lim_{n\to\infty} u_n = \frac{1}{p-q}$$

We have a closed-form formula for u_n and its partial sums.

Proposition B.2. *For $(m,n,M) \in \mathbb{N}^3$ with $2 \leq M$ and $1 \leq m \leq M-1$, we have :*

$$u_n = \frac{1-(2n+3)\lambda^{n+1}+(2n+3)\lambda^{n+2}-\lambda^{2n+3}}{p(1-\lambda)(1-\lambda^{n+1})(1-\lambda^{n+2})} \tag{7}$$

$$\sum_{i=M-1-m}^{M-2} u_i = \frac{m\lambda^m - (2M-m)\lambda^M + (2M-m)\lambda^{M+m} - m\lambda^{2M}}{p(1-\lambda)(\lambda^m-\lambda^M)(1-\lambda^M)} \tag{8}$$

Proof. We have that the right hand side of (7) equals to 1 when $n = 0$ and satisfies the induction from Definition B.1 when $n \in \mathbb{N}^*$. Hence, by induction, we get (7) for

all n. Therefore, equation (8) is true for $m = 1$ and any $M \geq 2$. Then equation (8) follows by induction on m using (7). □

Corollary B.3. *Let $M \in \mathbb{N}$ with $M \geqslant 2$. We have*

$$\frac{u_{M-3} + u_{M-2}}{2} = \frac{1}{p} \cdot \frac{1 - (M-1)\lambda^{M-2} + (M-1)\lambda^M - \lambda^{2M-2}}{(1-\lambda)(1-\lambda^{M-2})(1-\lambda^M)}$$

Proof of the last equation in Theorem 2.2.

Note that for any $t \in \mathbb{N}$ and $i \in [0, M-1]$

$$\mathbb{P}[\nu_m = \nu_{m,0} | \mathbf{X}_t = i] = \mathbb{P}[\nu_i = \nu_{i,0}] = \frac{\lambda^i - \lambda^M}{1 - \lambda^M}$$

is independent of t and m. Therefore, for any $t \in \mathbb{N}$ and $i \in [1, M-1]$,

$$\mathbb{P}[(\mathbf{X}_{t+1} = i \pm 1) | (\mathbf{X}_t = i) \wedge (\nu_m = \nu_{m,0})] = \frac{\mathbb{P}[(\mathbf{X}_{t+1} = i \pm 1) \wedge (\nu_m = \nu_{m,0}) | \mathbf{X}_t = i]}{\mathbb{P}[\nu_m = \nu_{m,0} | \mathbf{X}_t = i]}$$

$$= \mathbb{P}[(\mathbf{X}_{t+1} = i \pm 1) | \mathbf{X}_t = i] \cdot \frac{\mathbb{P}[\nu_m = \nu_{m,0} | \mathbf{X}_{t+1} = i \pm 1]}{\mathbb{P}[\nu_m = \nu_{m,0} | \mathbf{X}_t = i]}$$

is also independent of t and m. Let

$$\tilde{\mathbb{P}}(i, i \pm 1) = \mathbb{P}[(\mathbf{X}_{t+1} = i \pm 1) | (\mathbf{X}_t = i) \wedge (\nu_m = \nu_{m,0})]$$

More precisely, we have

$$\tilde{\mathbb{P}}(i, i+1) = p \cdot \frac{\lambda^{i+1} - \lambda^M}{\lambda^i - \lambda^M} \tag{9}$$

and

$$\tilde{\mathbb{P}}(i, i-1) = q \cdot \frac{\lambda^{i-1} - \lambda^M}{\lambda^i - \lambda^M} \tag{10}$$

We recognize in $\tilde{\mathbb{P}}$ the h-Doob transform of $\mathbb{P}$ with

$$h(i, j) = \frac{\lambda^j - \lambda^M}{\lambda^i - \lambda^M}$$

i.e., $\tilde{P}(i, j) = P(i, j)h(i, j)$. We check that

$$\tilde{\mathbb{P}}(i, i+1) + \tilde{\mathbb{P}}(i, i-1) = 1 \tag{11}$$

and

$$\tilde{\mathbb{P}}(M-1, M-2) = \tilde{\mathbb{P}}(0, 0) = 1 \tag{12}$$

The probability $\tilde{\mathbb{P}}$ is the probability $\mathbb{P}$ twisted by the condition that the hiker exits $[1, M-1]$ at 1. In particular, we have for all $m \in [0, M]$

$$\mathbb{E}[\nu_m | \nu_m = \nu_{m,0}] = \tilde{\mathbb{E}}[\nu_{m,0}] = \tilde{\mathbb{E}}[\nu_m] \tag{13}$$

where $\tilde{\mathbb{E}}$ is the expectation taken for the probability $\tilde{\mathbb{P}}$.

We define for $i \in [0, M-1]$,

$$v_{i,M} = \tilde{\mathbb{E}}[\nu_i]$$

and for $i \neq 0$

$$\alpha_{i,M} = v_{i,M} - v_{i-1,M} \tag{14}$$

Since $v_{0,M} = 0$, we have

$$v_{i,M} = \alpha_{1,M} + \ldots + \alpha_{i,M} \tag{15}$$

All these quantities depend on M since $\nu_m = \nu_{m,0} \wedge \nu_{m,M}$. Note that by (12), we have,

$$\alpha_{M-1,M} = 1 \tag{16}$$

Moreover, by (11) and the Markov property, we have

$$\begin{aligned} v_{i,M} &= \tilde{\mathbb{P}}(i, i-1) \cdot (1 + v_{i-1,M}) + \tilde{\mathbb{P}}(i, i+1) \cdot (1 + v_{i+1,M}) \\ &= \tilde{\mathbb{P}}(i, i-1) \cdot v_{i-1,M} + \tilde{\mathbb{P}}(i, i+1) \cdot (v_{i,M} + \alpha_{i+1,M}) + 1 \end{aligned}$$

So,

$$\tilde{\mathbb{P}}(i, i-1) \cdot v_{i,M} = \tilde{\mathbb{P}}(i, i-1) \cdot v_{i-1,M} + \tilde{\mathbb{P}}(i, i+1) \cdot \alpha_{i+1,M} + 1$$

and

$$\alpha_{i,M} = \frac{\tilde{\mathbb{P}}(i, i+1)}{\tilde{\mathbb{P}}(i, i-1)} \alpha_{i+1,M} + \frac{1}{\tilde{\mathbb{P}}(i, i-1)} \tag{17}$$

Hence we can compute $\alpha_{i,M}$ by induction on i from $i = M-1$ to $i = 1$ with the help of (9), (10) and (16). We get also $v_{i,M}$ by (15). Explicitly, set $u_{i,M} = \alpha_{M-1-i,M}$. Then, $u_{0,M} = 1$ and

$$\begin{aligned} u_{i,M} &= \frac{\tilde{\mathbb{P}}(M-i-1, M-i)}{\tilde{\mathbb{P}}(M-i-1, M-i-2)} u_{i-1,M} + \frac{1}{\tilde{\mathbb{P}}(M-i-1, M-i-2)} \\ &= \frac{p \cdot \frac{\lambda^{M-i} - \lambda^M}{\lambda^{M-i-1} - \lambda^M}}{q \cdot \frac{\lambda^{M-i-2} - \lambda^M}{\lambda^{M-i-1} - \lambda^M}} u_{i-1,M} + \frac{1}{q \cdot \frac{\lambda^{M-i-2} - \lambda^M}{\lambda^{M-i-1} - \lambda^M}} \\ &= \lambda \frac{1 - \lambda^i}{1 - \lambda^{i+2}} u_{i-1,M} + \frac{1}{p} \frac{1 - \lambda^{i+1}}{1 - \lambda^{i+2}} \end{aligned}$$

Therefore $u_{i,M} = u_i$ does not depend on M. Then we have by (15)

$$\tilde{\mathbb{E}}[\nu_m] = \sum_{i=M-1-m}^{M-2} u_i$$

and the result then follows from (8).

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[1]Send some anonymous and moderate satoshis to support our research at the pub.