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▶ **To cite this version:**

Adrien Kassel, Thierry Lévy. Covariant Symanzik Identities. 2019. hal-02333747v1

HAL Id: hal-02333747
https://hal.science/hal-02333747v1

Preprint submitted on 25 Oct 2019 (v1), last revised 12 May 2020 (v2)

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COVARIANT SYMANZIK IDENTITIES

ADRIEN KASSEL AND THIERRY LÉVY

Abstract. Classical isomorphism theorems due to Dynkin, Eisenbaum, Le Jan, Sznitman, establish equalities in law between local times of random paths or ensemble of paths and the square of discrete Gaussian free fields on graphs. We extend these results to the case of real or complex vector bundles of arbitrary rank over graphs endowed with a connection, by providing identities in law between functionals of the covariant Gaussian free fields and holonomies of random paths. As an application, we give a formula for computing moments of a large class of random, in general non-Gaussian, sections in terms of holonomies of random paths with respect to an annealed random gauge field, in the spirit of Symanzik's foundational work on the subject.

Contents

Introduction

Hay que sentarse a la orilla
del pozo de la sombra
y pescar luz caída
con paciencia[1]

Pablo Neruda

This paper is concerned with some relationships between covariant random vector fields and holonomies of random paths on finite graphs.

At the root of our study is the by now classical subject of isomorphism theorems relating local times of either Markovian paths or Poissonian ensembles of Markovian loops to squares of Gaussian free fields. This subject of study originated in an approach of Symanzik to constructive quantum field theory [Sym69], which was initially implemented by Brydges, Fröhlich, Spencer,

Date: October 25, 2019.

Key words and phrases. Discrete potential theory, Laplacian on vector bundles, Gaussian free field, random walks, Feynman-Kac formula, Poissonnian ensembles of Markovian loops, local times, isomorphism theorems, discrete gauge theory, holonomy.

[1]We must sit on the rim/of the well of darkness/and fish for fallen light/with patience.

and Sokal [BFS82, BFS83] in the context of classical spin systems, and later developped by Dynkin who expressed these physics ideas in the formal language of Markov processes [Dyn83, Dyn84a, Dyn84b]. Further developments followed, with the isomorphisms of Eisenbaum [Eis95], Le Jan [LJ08], and Sznitman [Szn12a], among others. See [MR06, LJ11, Szn12b] and the references therein for expository accounts of these results and their relation to the earlier Ray–Knight theorems. The introduction of the Brownian loop soup by Lawler and Werner [LW04] was followed by [LJ10], and the whole subject has seen further recent developments, attested among many other papers by [Lup16, Wer15, QW15, Zha14, DL16].

Our work however does not build on these previous results, but instead considers a more general geometric setup, which can be specialised to obtain back these classical results. The general framework that we consider is that of real or complex vector bundles of arbitrary rank over graphs. The case of vector bundles of low rank was investigated by Kenyon [Ken11], and the slightly different but related case of graph coverings in a recent work by Le Jan [LJ16].

In this geometric framework, the usual Gaussian free field is naturally replaced by its covariant, or vector valued, version, and a natural class of multiplicative functionals of paths is provided by holonomy. However, the multiplicative functionals of interest in the classical isomorphism theorems are of a different nature, namely exponential functionals of local times. We reconcile these two classes by defining an extended notion of holonomy, twisted by an external potential.

The relations that we prove between holonomies and fields extend the known relations between random paths and Gaussian fields, since the class of functionals obtained from twisted holonomies captures more faithfully, thanks to the non-commutativity of the gauge groups, the actual geometry of paths, whereas local times ignore much of the chronological unfolding of events in a trajectory.

Sections 1–3 are devoted to setting the basic setup of vector bundles over graphs which we use throughout. The main novelty consists in the introduction, in Section 3, of a notion of twisted holonomy (Definition 3.5) which, we believe, is a fruitful extension in our geometric framework of classical exponential functionals of local times.

One of the most useful result for our study is a covariant Feyman–Kac formula (Theorem 4.1) which we prove in Section 4. This formula can be thought of as a discrete analogue of a continuum version which can be traced back at least to the work of Norris [Nor92] (see also [AHKHK89]) in stochastic differential geometry, and which can also be found, in a different guise for discrete time walks, in earlier works of Brydges, Fröhlich, and Seiler [BFS79] on lattice gauge theories as we discovered after completing our work.

The last three sections contain the main results of the paper. Section 6 presents a very general formulation of both Dynkin's (Theorem 6.1) and Eisenbaum's (Theorem 6.2) isomorphism theorems. Our formulation is targetted at giving more insight into the relation between these isomorphism theorems and the probabilistic formulation of potential theory, see for instance (38).

Section 7 presents a unified formulation (Theorem 7.6) of the isomorphisms of Le Jan and Sznitman in the context of vector bundles.

Finally, Section 8 builds on the previous sections to prove a statement (Theorem 8.3) relating moments of a large class of random sections to holonomies of random paths under an annealed gauge field.

Acknowledgements. We thank, for their hospitality and support, the Forschungsinstitut für Mathematik (FIM) in Zurich, the Centre Interfacultaire Bernoulli (CIB) in Lausanne, and the ETH Zurich where parts of this work were completed.

1. Graphs

1.1. **Graphs with a well.** In this first section, we give a precise definition of the graphs that we will consider throughout this paper. One aspect in which this definition differs from that used for example in [Szn12b] or [LJ11] is that we allow more than one edge between two vertices of a graph. We also allow loop edges, that is, edges with identical end points.

Definition 1.1. *A* graph *is a quintuple* $\mathsf{G} = (\mathsf{U}, \mathsf{E}, s, t, i)$ *consisting of*

- *two finite sets* U *and* E *such that* $\mathsf{U} \neq \varnothing$,
- *two maps* $s, t : \mathsf{E} \to \mathsf{U}$, *and*
- *a fixed-point free involution* i *defined on a subset* $\mathsf{E}^{\updownarrow}$ *of* E, *such that the equality* $t \circ i = s$ *holds on this subset.*

If $\mathsf{E}^{\updownarrow} = \mathsf{E}$, *the graph is said to be* symmetric.

The elements of U are the *vertices* of the graph, the elements of E its *edges*. Each edge e is oriented and joins its source $s(e)$ to its target $t(e)$.

The involution i induces an equivalence relation on $\mathsf{E}^{\updownarrow}$ and the elements of the set

$$(\mathsf{E}^{\updownarrow}/i) \cup (\mathsf{E} \setminus \mathsf{E}^{\updownarrow})$$

are called, if need be, the *geometric edges* of the graph. Accordingly, two edges of a pair $\{e, i(e)\}$, where $e \in \mathsf{E}^{\updownarrow}$, are understood as being the two orientations of one and the same geometric edge. In contrast, we think of an element of $\mathsf{E} \setminus \mathsf{E}^{\updownarrow}$ as the only existing orientation of the corresponding geometric edge. Thus, in a sense which will be made precise later (see Section 3.2), the elements of $\mathsf{E}^{\updownarrow}$ should be counted as one half of an edge, and the elements of $\mathsf{E} \setminus \mathsf{E}^{\updownarrow}$ as one full edge (see also Figure 1 below).

We will often write $\underline{e} = s(e)$, $\overline{e} = t(e)$ and $e^{-1} = i(e)$ for the source, target, and inverse of an edge. We will also write a graph simply as a pair (U, E) and not mention the maps s, t, i.

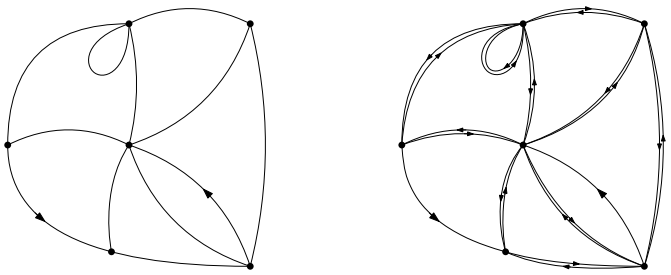

FIGURE 1. The graph depicted on the left has 6 vertices and 12 geometric edges. Out of these 12 edges, 2 have a fixed orientation and the other 10 admit the two orientations. The picture on the right shows the 22 edges of this graph.

Consider a graph $\mathsf{G} = (\mathsf{U}, \mathsf{E}, s, t, i)$. Let V be a subset of U. We define the subset

$$\mathsf{E}_{|\mathsf{V}} = \{e \in \mathsf{E} : s(e) \in \mathsf{V} \text{ and } t(e) \in \mathsf{V}\}$$

of E. The maps s, t and i restrict to V and $\mathsf{E}_{|\mathsf{V}}$ and we define the subgraph $\mathsf{G}_{|\mathsf{V}}$ to be the graph $(\mathsf{V}, \mathsf{E}_{|\mathsf{V}}, s, t, i)$.

In this paper, we will always work on graphs with a well, in the sense of the following definition.

Definition 1.2. *A* graph with a well *is a graph* $\mathsf{G} = (\mathsf{U}, \mathsf{E})$, *the set of vertices of which is partitioned into two non-empty subsets* V *and* W *such that*

- *the subgraph* $\mathsf{G}_{|\mathsf{V}}$ *is symmetric,*

- *every vertex of* W *is the target of at least one edge issued from a vertex of* V*,*
- *no edge is issued from any vertex of* W*.*

In this situation, the set W *is called the* well *of the graph, and the elements of* V *are called its* proper *vertices. The subset*

$$\partial\mathsf{V} = \{x \in \mathsf{V} : \exists e \in \mathsf{E}, s(e) = x \text{ and } t(e) \in \mathsf{W}\}$$

of V *is called the* rim *of the well. The elements of* $\mathsf{V} \setminus \partial\mathsf{V}$ *are called* inner *vertices. The subgraph* $\mathsf{G}_{|\mathsf{V}}$ *is called the* proper subgraph *of* G*.*

Note that there exists at most one partition of the set of vertices of a graph $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ which satisfies the three properties above. Indeed, W must be the set of vertices of the graph which are the target of at least one edge and the source of none. For the graph to be a graph with a well, there remains to check that neither W nor $\mathsf{U} \setminus \mathsf{W}$ are empty, and that the restriction of G to $\mathsf{U} \setminus \mathsf{W}$ is symmetric.

In this paper, we shall always denote by $\mathsf{U} = \mathsf{V} \cup \mathsf{W}$ the partition of the set of vertices of a graph with a well.

Graphs with a well arise naturally as follows (see also Figure 2 below). Consider a symmetric graph and a subset V of the set of vertices of this graph. Define W to be the set of all vertices of the graph that are not in V but that are the target of at least one edge issued from a vertex of V. Define then U as $\mathsf{V} \cup \mathsf{W}$ and E as the set of edges which either join two vertices of V or a vertex of V to a vertex of W. Thus, although we shall not pursue this line of thought in the present paper, graphs with a well provide a convenient framework to work with increasing exhaustions of an infinite graphs by finite subgraphs.

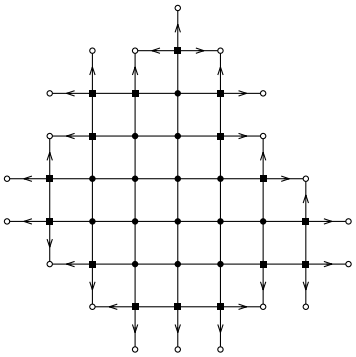

FIGURE 2. In this picture, proper vertices are black and vertices of the well are white. Vertices of the rim are square and inner vertices are round.

1.2. **Weights and measures.** We will gradually introduce a certain amount of structure on the graphs that we consider. We start with a weighting of the edges.

Definition 1.3. *Let* $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ *be a graph. A* conductance *on* G *is a positive real-valued function* $\chi : \mathsf{E} \to (0, +\infty)$ *such that* $\chi \circ i = \chi$ *on the set* $\mathsf{E}^{\updownarrow}$*.*

We think of χ as a measure on the set of edges of the graph. We shall usually write χ_e instead of $\chi(e)$ for the conductance of an edge e. The pair (G, χ) is called a *weighted graph.*

Let $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ be a graph with a well endowed with a conductance χ. We say that a function $\lambda : \mathsf{U} \to (0, +\infty)$ is a *reference measure* on G if

$$\forall x \in \mathsf{V},\ \lambda_x = \sum_{e \in \mathsf{E} : \underline{e} = x} \chi_e. \tag{1}$$

Let us emphasise that λ takes a positive value at every vertex, that this value is completely determined by the conductance at a proper vertex, and that this value is arbitrary at every vertex of the well. In the situation where the graph with a well arises by restriction from a larger graph as explained at the end of Section 1.1, the value of λ at a vertex of the well is understood as being the sum of the conductances of the edges of the original graph that were issued from that vertex, and that are now forgotten.

The part of λ that is due to the conductances of edges joining a proper vertex to a vertex of the well plays a special role. We define, for every vertex x,

$$\kappa_x = \sum_{e\in\mathsf{E}:\underline{e}=x,\overline{e}\in\mathsf{W}} \chi_e.$$

By definition, the support of the measure κ, that is, the set of vertices x such that $\kappa_x > 0$, is the rim of the well.

In this paper and unless explicitly stated otherwise, by a *weighted graph with a well*, we will always mean the data of the collection of objects $(\mathsf{U} = \mathsf{V} \cup \mathsf{W}, \mathsf{E}, s, t, i, \chi, \lambda, \kappa)$ as defined in this section, and with this notation. We will moreover always assume that the graph is connected, in the sense explained at the end of Section 2.1.

Let us conclude this section by a few terminological remarks. Firstly, the reader may find that there is a confusion between measures and functions in our presentation, as we took advantage of the discrete nature of our framework to identify the measures χ, λ, κ with their densities with respect to the counting measures on E and U. We adopted these identifications because they allow for a simpler and lighter notation, but we also believe that they hide to some extent the true nature of the objects which one manipulates. Therefore, we invite the reader to check periodically that our statements are consistent in this respect and that our formulas are, in the physical sense, homogeneous.

Secondly, the objects that we are considering are very classical and receive several different names in the literature. The vertex or set of vertices that we call the well is sometimes called the *sink*, and sometimes also the *cemetery*, and the authors who use the latter name call the quantity that we denote by κ the *killing rate*. We prefer to avoid a morbid terminology, and, following Neruda [Ner73], find wells and their rims more inspiring than sinks. However, we respect the tradition and keep the notation κ. An electric analogy is also often used in this context, accounting for the name of the conductance. In this terminology, the well should be though of as an electric *ground*. This may be the place to mention that, depending on the analogy which one choses, the conductance of an edge can be understood as the inverse of an electric resistance, the inverse of a length, or the section of a water pipe.

2. Paths and loops

Paths in graphs will play a crucial role in this study. The reader will soon notice that it is not only important for us to know which vertices a path visits, but also which edges it traverses, and because this is not the most widespread point of view, we give complete definitions of the objects that we consider. We will mainly study paths indexed by continuous time, but it is convenient to define paths indexed by discrete time first.

2.1. **Discrete paths.**

Let $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ be a graph. Let us introduce the fibre products

$$\mathsf{U} \times_{\mathsf{G}} \mathsf{E} = \{(x, e) \in \mathsf{U} \times \mathsf{E} : x = s(e)\} \text{ and } \mathsf{E} \times_{\mathsf{G}} \mathsf{U} = \{(e, x) \in \mathsf{E} \times \mathsf{U} : t(e) = x\}. \tag{2}$$

We extend this notation in the obvious way to alternated Cartesian products of U and E.

Definition 2.1. *Let $n \geq 0$ be an integer. The set of* discrete paths *of length n in G is the set*

$$\mathsf{DP}_n(\mathsf{G}) = \mathsf{U} \times_{\mathsf{G}} (\mathsf{E} \times_{\mathsf{G}} \mathsf{U})^{\times_{\mathsf{G}} n}. \tag{3}$$

The set of discrete paths on G is the disjoint union

$$\mathsf{DP}(\mathsf{G}) = \bigcup_{n \geq 0} \mathsf{DP}_n(\mathsf{G}). \tag{4}$$

For example, a discrete path of length 0 is simply a vertex, and a discrete path of length 2 is a well-chained sequence $(x_0, e_1, x_1, e_2, x_2)$ of vertices and edges. The *length* of a path, sometimes called its *combinatorial length*, is the number of edges traversed by this path.

A discrete path of positive length k is of course completely characterised by the sequence of k edges which it traverses. Nevertheless, despite the fact that edges will play for us a more important role than is usual, we will still pay a lot of attention to the sequence of vertices visited by paths, and it is useful to keep explicit track of this information in their definition.

The initial and final vertex of a discrete path $p = (x_0, e_1, \ldots, x_n)$ are respectively denoted by $\underline{p} = x_0$ and $\overline{p} = x_n$.

Definition 2.2. *A* discrete loop *is a discrete path with identical initial and final vertices. The set of discrete loops of length n is denoted by $\mathsf{DL}_n(\mathsf{G})$ and the set of all discrete loops by*

$$\mathsf{DL}(\mathsf{G}) = \bigcup_{n \geq 0} \mathsf{DL}_n(\mathsf{G}).$$

Note that $\mathsf{DL}_0(\mathsf{G}) = \mathsf{DP}_0(\mathsf{G}) = \mathsf{U}$.

A graph is usually said to be *connected* if any two vertices x and y are respectively the starting and finishing point of some discrete path. However, a graph with a well is never connected in this sense, because no edge starts from any vertex of the well. Thus, with our usual notation, we will say that a graph with a well G is connected if its proper subgraph $\mathsf{G}_{|\mathsf{V}}$ is connected. We will always assume that graphs with a well are connected in this sense.

2.2. **Continuous paths.** A path indexed by continuous time is a discrete path which spends, at each vertex that it visits, a certain positive amount of time. Here is the formal definition.

Definition 2.3. *Let $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ be a graph. A* continuous path *in G is an element of the set*

$$\mathsf{P}(\mathsf{G}) = \bigcup_{n \geq 0} \left(\mathsf{DP}_n(\mathsf{G}) \times (0, +\infty)^n \times (0, +\infty]\right).$$

A continuous loop *in G is an element of the set*

$$\mathsf{L}(\mathsf{G}) = \bigcup_{n \geq 0} \left(\mathsf{DL}_n(\mathsf{G}) \times (0, +\infty)^n \times (0, +\infty]\right).$$

If $p = (x_0, e_1, \ldots, x_n)$ is a discrete path, $\tau_0, \ldots, \tau_{n-1}$ are positive reals and τ_n is an element of $(0, +\infty]$, then we will denote the continuous path $(p, \tau_0, \ldots, \tau_n)$ by

$$\gamma = ((x_0, \tau_0), e_1, (x_1, \tau_1), \ldots, e_n, (x_n, \tau_n)).$$

This path is understood as the trajectory of a particle which starts from x_0, spends time τ_0 at x_0, jumps to x_1 through the edge e_1, spends time τ_1 at x_1, and so on. The path p is said to be the discrete path underlying the continuous path γ. Note that a continuous loop spends at least two intervals of time at its starting point, one at the beginning and the other at the end of its course.

The times $\tau_0, \ldots, \tau_n$ are called the *holding times* of γ. The *lifetime* of γ is

$$\tau(\gamma) = \tau_0 + \ldots + \tau_n \in (0, +\infty].$$

To the path γ, we associate a right-continuous function from $[0, \tau(\gamma))$ to U, which we also denote by γ. It is defined as follows: for every $t \in [0, \tau(\gamma))$, there is a unique $k \in \{0, \dots, n\}$ such that $\tau_0 + \ldots + \tau_{k-1} \leq t < \tau_0 + \ldots + \tau_k$, and we set $\gamma_t = x_k$. If $t \in [0, \tau_0)$, then $k = 0$ and $\gamma_t = x_0$. Informally,

$$\gamma_t = \sum_{k=0}^{n} x_k \mathbb{1}_{[\tau_0+\ldots+\tau_{k-1}, \tau_0+\ldots+\tau_k)}(t).$$

For every $t \in (0, \tau(\gamma))$, we define the restriction of the continuous path γ to $[0, t)$ by setting

$$\gamma_{|[0,t]} = ((x_0, \tau_0), e_1, \dots, e_{k-1}, (x_{k-1}, \tau_{k-1}), e_k, (x_k, t - (\tau_0 + \ldots + \tau_{k-1}))),$$

where k is the same integer as before. Note that the right-continuous function from $[0, t)$ to V associated to the continuous path $\gamma_{|[0,t]}$ is equal to the restriction to $[0, t)$ of the right-continuous function from $[0, \tau(\gamma))$ to V associated to γ.

Finally, if the lifetime of γ is finite and γ traverses only edges of $\mathsf{E}^{\updownarrow}$, we define the *time-reversal*, or *inverse* of γ as the path

$$\gamma^{-1} = ((x_n, \tau_n), e_n^{-1}, (x_{n-1}, \tau_{n-1}), \dots, e_1^{-1}, (x_0, \tau_0)).$$

The lifetime of γ^{-1} is the same as that of γ, and the right-continuous function from $[0, \tau(\gamma))$ to U associated to γ^{-1} is defined by

$$\gamma^{-1}(t) = \lim_{s \uparrow \tau - t} \gamma(s).$$

2.3. **Discrete time random walk.** We will now describe the natural random walk on a weighted graph with a well. It is a random continuous path on the graph, of which we will start by describing the underlying random discrete path.

The *transition matrix* of a weighted graph with a well is the matrix $P \in M_{\mathsf{U},\mathsf{E}}(\mathbb{R})$ such that, for all $x \in \mathsf{U}$ and $e \in \mathsf{E}$,

$$P_{x,e} = \begin{cases} \frac{\chi_e}{\lambda_x} & \text{if } \underline{e} = x, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.4. *Let G be a weighted graph with a well. For every vertex $x \in \mathsf{U}$, there is a unique probability measure $\mathbb{Q}_x$ on the countable set $\mathsf{DP}(\mathsf{G})$ such that for every discrete path $(x_0, e_1, \dots, e_n, x_n)$, one has*

$$\mathbb{Q}_x(\{(x_0, e_1, \dots, e_n, x_n)\}) = \mathbb{1}_{\{x\}}(x_0)\mathbb{1}_{\mathsf{W}}(x_n) \prod_{k=0}^{n-1} P_{x_k, e_{k+1}}. \tag{5}$$

Note that the probability $\mathbb{Q}_x$ is supported by the set of paths joining x to the well.

Proof. The only thing to check is that the total mass of $\mathbb{Q}_x$ is 1. Let us introduce the vertex transition matrix Q defined, for all $x, y \in \mathsf{U}$, by

$$Q_{x,y} = \sum P_{x,e},$$

where the sum extends over all edges e of E such that $\underline{e} = x$ and $\overline{e} = y$.

Each row of Q corresponding to a proper vertex has sum 1. Each row corresponding to a vertex of the well is identically zero. Moreover, the fact that G is connected implies that there exists a power of Q in which all entries of all proper rows are positive. The rows corresponding to the well are always 0. These facts and an elementary argument of linear algebra imply that the spectral radius of Q is strictly smaller than 1.

Let us define the columns $V = (\mathbb{1}_{\mathsf{V}}(x))_{x\in\mathsf{U}}$ and $W = (\mathbb{1}_{\mathsf{W}}(x))_{x\in\mathsf{U}}$. For each vertex x, the total mass of $\mathbb{Q}_x$ is

$$\mathbb{Q}_x(1) = \sum_{n\geq 0}(Q^n W)_x. \tag{6}$$

On the other hand, the identity $Q(V+W) = V$ can be rewritten $(I-Q)V = QW$, so that for all $n \geq 0$, we have $(Q^n - Q^{n+1})V = Q^{n+1}W$. Summing over n and using the fact that Q^n tends to 0 as n tends to infinity, we find $\sum_{n\geq 0} Q^n W = V + W$, a column of 1. □

2.4. **Continuous time random walk.** The definition 2.3 of the set $\mathsf{P}(\mathsf{G})$ of continuous paths on G as a countable union of Cartesian products of a finite set and finitely many intervals suggests the definition of a σ-field on $\mathsf{P}(\mathsf{G})$ which we adopt but do not deem necessary to write down explicitly.

Definition 2.5. *Let G be a weighted graph with a well. For every $x \in \mathsf{U}$, we denote by $\mathbb{P}_x$ the unique probability measure on $\mathsf{P}(\mathsf{G})$ such that, for every bounded measurable function F on $\mathsf{P}(\mathsf{G})$,*

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma)\, d\mathbb{P}_x(\gamma) = \sum_{n\geq 0} \sum_{\substack{p\in \mathsf{DP}_n(\mathsf{G}) \\ p=(x_0,e_1,\dots,x_n)}} \mathbb{Q}_x(\{p\})$$

$$\int_{(0,+\infty)^n} F\big(((x_0,\tau_0),e_1,\dots,(x_{n-1},\tau_{n-1}),e_n,(x_n,+\infty))\big) e^{-\tau_0-\dots-\tau_{n-1}}\, d\tau_0\dots d\tau_{n-1}. \tag{7}$$

In words, a sample of $\mathbb{P}_x$ can be obtained by first sampling $\mathbb{Q}_x$, in order to obtain a discrete path $(x_0, e_1, \dots, e_n, x_n)$, and then sampling n independent exponential random variables $\tau_0, \dots, \tau_{n-1}$ of parameter 1. The continuous path

$$((x_0,\tau_0),e_1,\dots,e_{n-1},(x_{n-1},\tau_{n-1}),e_n,(x_n,+\infty))$$

has the distribution $\mathbb{P}_x$.

Note that $\mathbb{P}_x$ is supported by continuous paths with infinite lifetime. Moreover, for all $t > 0$, on the event $\{\gamma_t \notin \mathsf{W}\}$, the inverse of $\gamma_{|[0,t]}$ is well defined (see the end of Section 2.2).

The following lemma expresses the reversibility, or more precisely the λ-reversibility of the continuous time random walk on G.

Lemma 2.6. *Let G be a weighted graph with a well. Let $x, y \in \mathsf{V}$ be proper vertices of G. Let $t > 0$ be a positive real. Let F be a bounded measurable function on $\mathsf{P}(\mathsf{G})$. Then*

$$\lambda_x \int_{\mathsf{P}(\mathsf{G})} F(\gamma^{-1})\mathbb{1}_{\{\gamma_t=y\}}\, d\mathbb{P}_x(\gamma) = \lambda_y \int_{\mathsf{P}(\mathsf{G})} F(\gamma)\mathbb{1}_{\{\gamma_t=x\}}\, d\mathbb{P}_y(\gamma).$$

Proof. The left-hand side is equal to

$$e^{-t}\sum_{n=0}^{\infty} \sum_{\substack{p\in\mathsf{DP}_n \\ p=(x_0,e_1,\dots,x_n) \\ x_0=x, x_n=y}} \lambda_x P_{x_0,e_1}\dots P_{x_{n-1},e_n} \int_{0<t_1<\dots<t_n<t} F(((x_n,t-t_n),e_n^{-1},\dots,(x_1,t_2-t_1),e_1^{-1},(x_0,t_1)))\, dt_1\dots dt_n.$$

Using the λ-reversibility of the discrete random walk, that is, the relation $\lambda_{x_0}P_{x_0,e_1}\dots P_{x_{n-1},e_n} = \lambda_{x_n}P_{x_n,e_n^{-1}}\dots P_{x_1,e_1^{-1}}$, and performing the change of variables $s_i = t - t_i$ for $i \in \{1,\dots,n\}$, we find an expression of the right-hand side. □

2.5. **Measures on paths and loops.** In this section, we will use the probability measures $\mathbb{P}_x$ to construct two families of measures on $\mathbb{P}(\mathsf{G})$ which will play a central role in this work. We assume again that G is a weighted graph with a well.

The first family of measures is defined as follows.

Definition 2.7. *Let $x, y \in \mathsf{V}$ be proper vertices. The measure $\nu_{x,y}$ is the measure on $\mathsf{P}(\mathsf{G})$ such that for all non-negative measurable function F on $\mathsf{P}(\mathsf{G})$, we have*

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma) d\nu_{x,y}(\gamma) = \int_{\mathsf{P}(\mathsf{G})\times(0,+\infty)} F(\gamma_{|[0,t]}) \mathbb{1}_{\{\gamma_t = y\}} \frac{1}{\lambda_y} \, d\mathbb{P}_x(\gamma) dt.$$

The measure ν_x is defined by

$$\nu_x = \sum_{y \in \mathsf{V}} \nu_{x,y} \lambda_y,$$

so that for all bounded measurable function F on $\mathsf{P}(\mathsf{G})$, we have

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma) d\nu_x(\gamma) = \int_{\mathsf{P}(\mathsf{G})} \int_0^{\tau(\gamma)} F(\gamma_{|[0,t]}) \, dt d\mathbb{P}_x(\gamma).$$

Finally, the measure ν is defined by

$$\nu = \sum_{x \in \mathsf{V}} \nu_x.$$

In Section 4, we shall compute various integrals with respect to these measures, and we will see in particular that they are finite.

The second family of measures is the following, and was introduced in [LJ11].

Definition 2.8. *Let $x, y \in \mathsf{V}$ be proper vertices. The measure $\mu_{x,y}$ is the measure on $\mathsf{P}(\mathsf{G})$ such that for all non-negative measurable function F on $\mathsf{P}(\mathsf{G})$, we have*

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma) d\mu_{x,y}(\gamma) = \int_{\mathsf{P}(\mathsf{G})\times(0,+\infty)} F(\gamma_{|[0,t]}) \mathbb{1}_{\{\gamma_t = y\}} \frac{1}{\lambda_y} \, d\mathbb{P}_x(\gamma) \frac{dt}{t}.$$

The measure μ_x is defined by

$$\mu_x = \sum_{y \in \mathsf{V}} \mu_{x,y} \lambda_y,$$

so that for all bounded measurable function F on $\mathsf{P}(\mathsf{G})$, we have

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma) d\mu_x(\gamma) = \int_{\mathsf{P}(\mathsf{G})} \int_0^{\tau(\gamma)} F(\gamma_{|[0,t]}) \, \frac{dt}{t} d\mathbb{P}_x(\gamma).$$

Finally, the measure μ is defined by

$$\mu = \sum_{x \in \mathsf{V}} \mu_x.$$

The measures $\mu_{x,y}$ are finite when $x \neq y$, but the measures $\mu_{x,x}$ are infinite, because of the contribution of short loops, which in the present discrete setting are constant loops. Let us split the measure μ in a way that allows us to isolate this divergence.

Definition 2.9. *The measures $\mu^{\circledcirc}$ and $\tilde{\mu}$ are defined as follows:*

$$\mu^{\circledcirc} = \sum_{x \in \mathsf{V}} \mu_{x,x} \lambda_x \quad \text{and} \quad \tilde{\mu} = \sum_{x,y \in \mathsf{V}, x \neq y} \mu_{x,y} \lambda_y,$$

so that $\mu^{\circledast}$ is the restriction of μ to the set $\mathsf{L}(\mathsf{G})$ of continuous loops. We now split the measure $\mu^{\circledast}$ as

$$\mu^{\circledast} = \mu^{\cdot} + \mu^{\circ},$$

where $\mu^{\cdot}$ is the restriction of $\mu^{\circledast}$ (and hence of μ) to the set of constant loops, and μ° its restriction to the set of non-constant loops.

Note that the measure $\mu^{\cdot}$ has a very simple expression: for every measurable function F on $\mathsf{P}(\mathsf{G})$, we have

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma) d\mu^{\cdot}(\gamma) = \sum_{x \in \mathsf{V}} \int_0^{+\infty} F((x,t)) e^{-t} \frac{dt}{t}.$$

The measure $\mu^{\cdot}$ is in particular infinite, and we shall see that $\mu - \mu^{\cdot} = \mu^{\circ} + \tilde{\mu}$ is finite.

The following straightforward consequence of Lemma 2.6 will be useful.

Lemma 2.10. *With the notation of Definitions 2.7 and 2.8, the respective images by the map $\gamma \mapsto \gamma^{-1}$ of the measures $\nu_{x,y}$ and $\mu_{x,y}$ are the measures $\nu_{y,x}$ and $\mu_{y,x}$. The measure $\mu^{\circledast}$ is invariant under the map $\gamma \mapsto \gamma^{-1}$.*

3. Vector bundles

In this section, we introduce the notion which plays the main role in our study, namely that of vector bundle over a graph.

3.1. Bundles and bundle-valued forms.

Let $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ be a graph. Let us choose a base field $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$. Let $r \geq 1$ be an integer.

Definition 3.1. *A $\mathbb{K}$-vector bundle of rank r over G is a collection $\mathsf{F} = ((\mathsf{F}_x)_{x \in \mathsf{U}}, (\mathsf{F}_e)_{e \in \mathsf{E}})$ of vector spaces over $\mathbb{K}$ which all have the same dimension r, and such that for all edge $e \in \mathsf{E}^{\updownarrow}$, $\mathsf{F}_e = \mathsf{F}_{e^{-1}}$.*

We say that the bundle F is *Euclidean* (if $\mathbb{K} = \mathbb{R}$), or *Hermitian* (if $\mathbb{K} = \mathbb{C}$), if each of the vector spaces of which it consists are Euclidean, or Hermitian. The bilinear, or sesquilinear form on the vector space F_x (resp. F_e) is denoted by $\langle \cdot, \cdot \rangle_x$ (resp. $\langle \cdot, \cdot \rangle_e$). When $\mathbb{K} = \mathbb{C}$, we take this form to be antilinear in the first variable and linear in the second, according to the physicists' convention. We also denote by $\| \cdot \|_x$ and $\| \cdot \|_e$ the associated norms.

Most of our results will have the same form in the real and complex cases, up to the adjustment of a few constants. In order to uniformise the results, it turns out to be useful to introduce the parameter

$$\beta = \dim_{\mathbb{R}} \mathbb{K}. \tag{8}$$

In fact, this parameter will not be used until Section 5.1, and will be redefined there.

Definition 3.2. *Let F be a vector bundle over a graph $\mathsf{G} = (\mathsf{U}, \mathsf{E})$. Let S be a subset of U The space of* sections *of F over S, or* 0-forms *on S with values in F, is the vector space*

$$\Omega^0(\mathsf{S}, \mathsf{F}) = \bigoplus_{x \in \mathsf{S}} \mathsf{F}_x.$$

If S_1 and S_2 are two subsets of U such that $\mathsf{S}_1 \subset \mathsf{S}_2$, we identify $\Omega^0(\mathsf{S}_1, \mathsf{F})$ with the subset of $\Omega^0(\mathsf{S}_2, \mathsf{F})$ consisting of sections that vanish identically on $\mathsf{S}_2 \setminus \mathsf{S}_1$. We denote by π_{S_1} the linear operator on $\Omega^0(\mathsf{S}_2, f)$ such that for every section f of F over S_2, and for every $x \in \mathsf{S}_2$,

$$(\pi_{\mathsf{S}_1} f)(x) = \mathbb{1}_{\mathsf{S}_1}(x) f(x). \tag{9}$$

The space of 1-forms *on* G *with values in* F *is the space*

$$\Omega^1(\mathsf{E},\mathsf{F}) = \left\{\omega \in \bigoplus_{e\in\mathsf{E}} \mathsf{F}_e : \forall e \in \mathsf{E}^{\updownarrow}, \omega(e^{-1}) = -\omega(e)\right\}.$$

It would of course be possible to define 1-forms on subsets of E, but we will not use this construction in this work.

A fundamental example of a vector bundle over a graph G is the trivial vector bundle of rank r, denoted by $\mathbb{K}^r_{\mathsf{G}}$, which is such that for all $a \in \mathsf{V}\cup\mathsf{E}$, the fibre $(\mathbb{K}^r_{\mathsf{G}})_a$ over a is $\mathbb{K}^r$. When $r = 1$, the sections of this bundle are exactly the functions over V, and 1-forms with values in this bundle are the usual scalar-valued 1-forms on the graph G, for example in the sense of [LJ11].

3.2. **Hermitian structures.** Let us now assume that G is a weighted graph with a well. What matters here is not so much the well as the availability of the measures χ on E and λ on U. In what follows, we use the word *Hermitian* in place of *Euclidean or Hermitian.*

For all $f_1, f_2 \in \Omega^0(\mathsf{U},\mathsf{F})$, we set

$$(f_1, f_2)_{\Omega^0} = \sum_{x\in\mathsf{U}} \lambda_x \langle f_1(x), f_2(x)\rangle_x.$$

This endows $\Omega^0(\mathsf{U},\mathsf{F})$ with the structure of a Hermitian space.

In the case where F is the rank 1 trivial bundle $\mathbb{K}_{\mathsf{G}}$, of which sections are simply functions on the graph, we use the notation $(\cdot,\cdot)$ without any subscript for the Hermitian structure on $\Omega^0(\mathsf{U},\mathbb{K}_{\mathsf{G}})$.

There is also a Hermitian scalar product on the space of 1-forms with values in F. To define it, we need to associate to each edge $e \in \mathsf{E}$ the symmetry factor

$$s_e = \begin{cases} \frac{1}{2} & \text{if } e \in \mathsf{E}^{\updownarrow}, \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Thus, the symmetry factor is equal to $\frac{1}{2}$ for each edge which joins two proper vertices, and to 1 for each edge which joins a proper vertex to a vertex of the well.

For all $\omega_1, \omega_2 \in \Omega^1(\mathsf{E},\mathsf{F})$, we define

$$(\omega_1,\omega_2)_{\Omega^1} = \sum_{e\in\mathsf{E}} s_e \chi_e \langle \omega_1(e), \omega_2(e)\rangle_e. \tag{11}$$

3.3. **Connections.** Recall from (2) the definition of the fibre products $\mathsf{U}\times_{\mathsf{G}}\mathsf{E}$ and $\mathsf{E}\times_{\mathsf{G}}\mathsf{U}$.

Definition 3.3. *A* connection *on the Hermitian (resp. Euclidean) vector bundle* F *is a collection* h *of unitary (resp. orthogonal) isomorphisms between some of the vector spaces which constitute* F*, namely the data,*

- *for each* $(x,e) \in \mathsf{U}\times_{\mathsf{G}}\mathsf{E}$*, of a unitary isomorphism* $h_{e,x} : \mathsf{F}_x \to \mathsf{F}_e$*, and*
- *for each* $(e,x) \in \mathsf{E}\times_{\mathsf{G}}\mathsf{U}$*, of a unitary isomorphism* $h_{x,e} : \mathsf{F}_e \to \mathsf{F}_x$*,*

such that for all $(x,e) \in \mathsf{U}\times_{\mathsf{G}}\mathsf{E}^{\updownarrow}$*, the relation* $h_{x,e^{-1}} = h_{e,x}^{-1}$ *holds.*

This definition is illustrated in Figure 3 below. We shall denote the set of connections on F by $\mathcal{A}(\mathsf{F})$.

Our notation for the isomorphisms which constitute a connection may seem complicated. To remember it, one should notice that, whether a and b be vertices or edges, $h_{a,b}$ sends the fibre over b to the fibre over a. Moreover, although this is redundant, the edge is written with the orientation which is consistent with the parallel transport expressed by $h_{a,b}$.

As a useful example, let us mention that $\mathbb{K}^r_{\mathsf{G}}$, the trivial bundle of rank r carries a connection that we call the canonical connection, for which all the isomorphisms $h_{a,b}$ are the identity of $\mathbb{K}^r$.

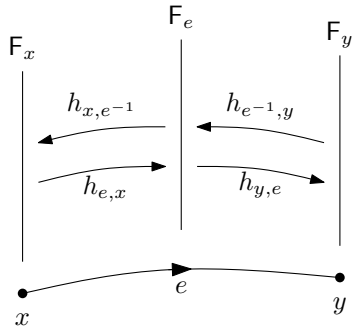

FIGURE 3. The isomorphisms $h_{e,x}$ and $h_{y,e}$ correspond to parallel transport along the oriented edge e.

3.4. **Holonomy.** Let h be a connection on the vector bundle F over the graph G. For each edge $e \in \mathsf{E}$, we define the holonomy along e as the isomorphism

$$h_e = h_{\overline{e},e} \circ h_{e,\underline{e}} : \mathsf{F}_{\underline{e}} \to \mathsf{F}_{\overline{e}}.$$

The holonomy of the connection h along a discrete path $p = (x_0, e_1, \dots, e_n, x_n)$ is defined as

$$\mathsf{hol}_h(p) = h_{e_n} \circ \cdots \circ h_{e_1} : \mathsf{F}_{x_0} \to \mathsf{F}_{x_n}.$$

This definition extends to the case of a continuous path γ with underlying discrete path p (see Section 2.2) by setting

$$\mathsf{hol}_h(\gamma) = \mathsf{hol}_h(p).$$

3.5. **Twisted holonomy.** The definition of the holonomy along a continuous path which we just gave is natural, but it depends only on the discrete path which underlies it. It is perhaps one of the contributions of the present work to propose a definition of the holonomy along a continuous-time trajectory in a graph which is genuinely dependent on its continuous time structure.

In order to give this definition, we need to introduce a new ingredient. From the bundle F, we can form a new bundle $\mathrm{End}(\mathsf{F})$ such that for all $a \in \mathsf{U} \cup \mathsf{E}$, the fibre of $\mathrm{End}(\mathsf{F})$ over a is

$$\mathrm{End}(\mathsf{F})_a = \mathrm{End}(\mathsf{F}_a),$$

the vector space of $\mathbb{K}$-linear endomorphisms of F_a. We give a name to some sections of this new bundle.

Definition 3.4. *Let G be a weighted graph with a well. Let F be a Euclidean (resp. Hermitian) vector bundle over G. A* potential *on F is an element $H \in \Omega^0(\mathsf{V}, \mathrm{End}(\mathsf{F}))$, that is, a section of the vector bundle $\mathrm{End}(\mathsf{F})$ over V such that for every vertex $x \in \mathsf{V}$, the operator $H_x \in \mathrm{End}(\mathsf{F}_x)$ is symmetric (resp. Hermitian).*

Let us emphasise that, according to our convention regarding sections of bundles over subsets of U, a potential on F vanishes at every vertex of the well.

We can now give an enhanced definition of the holonomy along a continuous path.

Definition 3.5. *Let h be a connection and H a potential on a Hermitian vector bundle F over a graph G. Let $\gamma = ((x_0, \tau_0), e_1, \dots, e_n, (x_n, \tau_n))$ be a continuous path in G. Assume that $\tau_n < \infty$ or $H_{x_n} = 0$. The* holonomy of h twisted by H along γ *is the linear map*

$$\mathsf{hol}_{h,H}(\gamma) = e^{-\tau_n H_{x_n}} \circ h_{e_n} \circ \cdots \circ e^{-\tau_1 H_{x_1}} \circ h_{e_1} \circ e^{-\tau_0 H_{x_0}} : \mathsf{F}_{x_0} \to \mathsf{F}_{x_n}.$$

Let us emphasise that the twisted holonomy does not behave well with respect to the time-reversal of paths: in general, even if the path γ^{-1} is defined (see the end of Section 2.2), it may be that

$$\mathsf{hol}_{h,H}(\gamma^{-1}) \neq \mathsf{hol}_{h,H}(\gamma)^{-1}.$$

For example, if $\gamma = ((x,\tau))$ is a constant path at a vertex x such that $H_x \neq 0$, then $\gamma^{-1} = \gamma$ but $\mathsf{hol}_{h,H}(\gamma^{-1}) = e^{-\tau H_x} \neq e^{\tau H_x} = \mathsf{hol}_{h,H}(\gamma)^{-1}$. Note however that the equality

$$\mathsf{hol}_{h,H}(\gamma^{-1}) = \mathsf{hol}_{h,H}(\gamma)^* \tag{12}$$

holds.

We would expect the definition of the twisted holonomy to seem rather strange to some readers, especially after the comment that we just made, and we hope that the results that will be proved in this paper will convince them of its interest. In the meantime, let us devote the next paragraph to a discussion of two points of view from which, we hope, this definition can be found natural.

3.6. **Twisted holonomy and hidden loops.** The first point of view is quite informal and inspired by quantum mechanics. We already alluded to the fact that a path in a graph can be seen as a discrete model for the time evolution of a particle in space. In this picture, the connection represents the way in which an ambient gauge field acts on the state of the particle as it moves around. If we now understand the operator H_x, up to a factor i or $i/\hbar$, as the Hamiltonian of the particle at the point x, then the term $e^{-\tau_x H_x}$ represents the evolution of the state of the particle as it spends a stretch of time τ_x at the vertex x, and the twisted holonomy along the path followed by the particle simply represents the subsequent modification of its state.

From a second and more mathematical point of view, the twisted holonomy as we defined it can be seen, at least in the case where the operators H_x are non-negative, as an averaged version of the ordinary holonomy in a larger graph, which has a small loop attached to each vertex.

To explain this, let us consider a weighted graph with a well G. Let us associate to G a new graph $\mathsf{G}^\circ = (\mathsf{U}, \mathsf{E}^\circ)$ which has the same vertices as G, and an extended set of edges

$$\mathsf{E}^\circ = \mathsf{E} \cup \bigcup_{x \in \mathsf{V}} \{l_x, l_x^{-1}\},$$

where the edge l_x and its inverse have source and target x. We call these new edges *looping edges*.

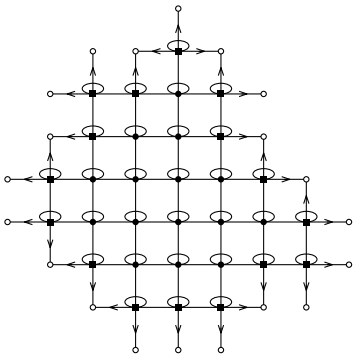

FIGURE 4. The graphs G° obtained from the graph depicted in Figure 2.

Let us assume that for every $x \in \mathsf{V}$, the operator H_x is Hermitian non-negative on F_x. For each $x \in \mathsf{V}$, let us associate to the edge l_x a positive jump rate r_x and a holonomy h_{l_x}, in such

a way that

$$H_x = r_x \left(2\mathrm{Id}_{\mathsf{F}_x} - (h_{l_x} + h_{l_x}^{-1})\right).$$

It is indeed the case that every Hermitian non-negative operator can be written, although not uniquely, as $r(2\mathrm{Id} - (U + U^{-1}))$ with $r > 0$ and U unitary. One can for example take any r larger than one quarter of the largest eigenvalue of H, and $U = \exp i \arccos(1 - \frac{H}{2r})$.

Let us now consider the continuous time random walk in the graph G° which, when it stands at a vertex x, jumps from x through the edge $e \in \mathsf{E}$ at rate $P_{x,e}$, and through each of the looping edges l_x and l_x^{-1} at rate r_x. It will be convenient to use probabilistic language, and for this, we will denote this random continuous path on G° by Γ°. Let us emphasise that Γ° is not exactly a random walk on the graph G° in the sense of Section 2.3, because it jumps at rate $1 + 2r_x$ from the vertex x, whereas the random walk on a weighted graph normally jumps at rate 1.

The reason for our change of jumping rate in the definition of Γ° is the following: from Γ°, we can recover the usual random walk, which we will denote by Γ, in the original graph G, by simply ignoring the jumps across the loop-edges of G°. More formally, there is a shearing map $S : \mathsf{P}(\mathsf{G}^\circ) \to \mathsf{P}(\mathsf{G})$ which forgets the jumps across the looping edges of $\mathsf{E}^\circ \setminus \mathsf{E}$, and we are claiming that $S(\Gamma^\circ)$ and Γ have the same distribution. We reckon that a formal definition of Γ° and the map S, followed by a formal proof of this claim, would take a lot of space and time, and bring little additional light on the present discussion.

Let us now compare the ordinary holonomy hol_h° of the connection h in the graph G° with the holonomy $\mathsf{hol}_{h,H}$ of h twisted by H in G.

Proposition 3.6. *For all $t \geq 0$ and all $x \in \mathsf{V}$, the following equality almost surely holds in* $\mathrm{Hom}(\mathsf{F}_x, \mathsf{F}_{X_t})$*:*

$$\mathbb{E}_x[\mathsf{hol}_h^\circ(\Gamma^\circ_{|[0,t]})|S(\Gamma^\circ_{|[0,t]})] = \mathsf{hol}_{h,H}(S(\Gamma^\circ_{|[0,t]})).$$

The way in which we want to read this equality is the following: for every path γ in G with lifetime t, the holonomy of h twisted by H along γ is given by

$$\mathsf{hol}_{h,H}(\gamma) = \mathbb{E}[\mathsf{hol}_h^\circ(\Gamma_t^\circ)|S(\Gamma^\circ_{|[0,t]}) = \gamma],$$

the average over a set of non-observed paths in the larger graph G° of an ordinary unitary holonomy, and of which the observed path γ is an approximation.

Proof. Conditional on $S(\Gamma^\circ_{|[0,t]}) = ((x_0, \tau_0), e_1, \ldots, (x_n, \tau_n))$, the path $\Gamma^\circ_{|[0,t]}$ is equal to $S(\Gamma^\circ_{|[0,t]})$ to which have been grafted, at each of its stays at one of the vertices $x_1, \ldots, x_n$, an independent Poissonian set of excursions through the loop-edges at this vertex.

For each $k \in \{0, \ldots, n\}$, let N_k be an independent Poisson variable with parameter $2r_{x_k}\tau_k$ and let β_{x_k,τ_k} be a random discrete loop of length N_k based at x_k, which traverses N_k times one of the edges l_x and l_x^{-1} (and no other), independently with equal probability at each jump. Then

$$\mathbb{E}_x[\mathsf{hol}_h^\circ(X^\circ_{|[0,t]})|S(X^\circ_{|[0,t]}) = ((x_0, \tau_0), e_1, \ldots, (x_n, \tau_n))] = \\ \mathbb{E}[\mathsf{hol}_h^\circ(\beta_{x_n,\tau_n})] \circ h_{e_n} \circ \mathbb{E}[\mathsf{hol}_h^\circ(\beta_{x_{n-1},\tau_{n-1}})] \circ \ldots \circ h_{e_1} \circ \mathbb{E}[\mathsf{hol}_h^\circ(\beta_{x_0,\tau_0})].$$

On the other hand, for each $k \in \{0, \ldots, n\}$,

$$\begin{aligned}
\mathbb{E}[\mathsf{hol}_h^\circ(\beta_{x_k,\tau_k})] &= e^{-2r_{x_k}\tau_k} \sum_{n=0}^{\infty} \frac{(2r_{x_k}\tau_k)^n}{n!} \frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} h_{l_{x_k}}^m h_{l_{x_k}^{-1}}^{n-m} \\
&= e^{-\tau_k r_{x_k}(2\mathrm{Id}_{\mathsf{F}_{x_k}} - (h_{l_{x_k}} + h_{l_{x_k}^{-1}}))} \\
&= e^{-\tau_k H_{x_k}},
\end{aligned}$$

and the expected result follows. □

3.7. **Occupation measures and local times.** A particular case of interest of the twisted holonomy is that where for every vertex x, the operator H_x is scalar. In this case, the twisted holonomy of a path can be related to its occupation measure, or to its local time, which we now define.

Definition 3.7. *Let $\gamma = ((x_0, \tau_0), e_1, \dots, e_n, (x_n, \tau_n))$ be a continuous path. The* occupation measure *of γ is the measure $\vartheta(\gamma)$ on U such that for all vertex x,*

$$\vartheta_x(\gamma) = \sum_{k=0}^{n} \delta_{x,x_k} \tau_k.$$

The local time *of γ is the density of its occupation measure with respect to the reference measure λ: for every vertex x,*

$$\ell_x(\gamma) = \frac{\vartheta_x(\gamma)}{\lambda_x} = \frac{1}{\lambda_x} \sum_{k=0}^{n} \delta_{x,x_k} \tau_k.$$

From Section 7 on, we will consider Poissonian ensembles of paths. We will then use the following extended definition of the occupation measures and local times: if $\mathcal{P}$ is a set of paths, we define, for all vertex x,

$$\ell_x(\mathcal{P}) = \sum_{\gamma \in \mathcal{P}} \ell_x(\gamma) \text{ and } \vartheta_x(\mathcal{P}) = \sum_{\gamma \in \mathcal{P}} \vartheta_x(\gamma).$$

The following lemma is a straightforward consequence of the definition of the twisted holonomy. In its statement, we use the same notation for a scalar operator and its unique eigenvalue.

Lemma 3.8. *Assume that for all vertex x, the operator H_x is scalar on F_x, that is, equal to $H_x \mathrm{Id}_{\mathsf{F}_x}$ for some $H_x \in \mathbb{K}$. Let γ be a continuous path. Assume that $\tau(\gamma) < +\infty$ or $H_{\overline{\gamma}} = 0$. Then one has the equality*

$$\mathsf{hol}_{h,H}(\gamma) = e^{-\sum_{x \in \mathsf{V}} H_x \vartheta_x(\gamma)} \mathsf{hol}_h(\gamma). \tag{13}$$

This lemma shows how the twisted holonomy, which, in the classical language of the theory of Markov processes, could be called a multiplicative functional, extends the classical definition of exponential functionals of local times.

3.8. **Differentials.** Let us consider a graph $\mathsf{G} = (\mathsf{U}, \mathsf{E})$, a vector bundle F over G and a connection h on F.

Definition 3.9. *The* differential *is the linear operator*

$$d : \Omega^0(\mathsf{U}, \mathsf{F}) \to \Omega^1(\mathsf{E}, \mathsf{F}) \subset \bigoplus_{e \in \mathsf{E}} \mathsf{F}_e$$

defined by setting, for every $f \in \Omega^0(\mathsf{U}, \mathsf{F})$ and every $e \in \mathsf{E}$,

$$(df)(e) = h_{e^{-1},\overline{e}} f(\overline{e}) - h_{e,\underline{e}} f(\underline{e}).$$

The range of d is contained in $\Omega^1(\mathsf{E}, \mathsf{F})$: indeed, if the inverse of the edge e is defined, then replacing e by e^{-1} in this definition exchanges $\underline{e}$ and $\overline{e}$, so that $(df)(e^{-1}) = -(df)(e)$ as expected.

The definition of d depends on the connection h, but we prefer not to use the notation d^h, which we find too heavy.

Let us now assume that the graph G is a weighted graph with a well. Recall from (10) the definition of the symmetry factor s_e of an edge e.

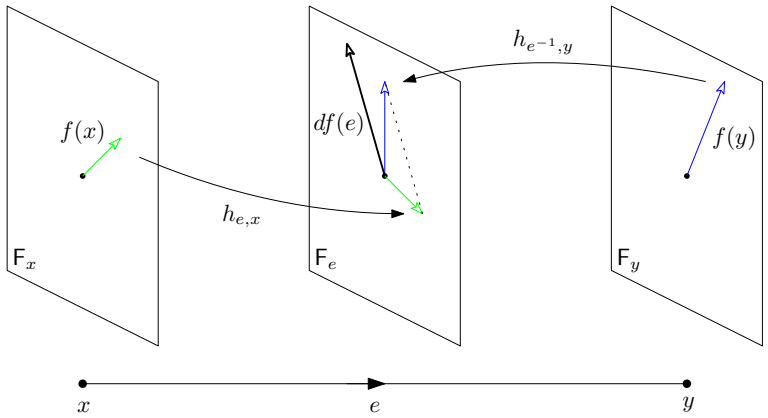

FIGURE 5. The computation of $df(e)$.

Definition 3.10. *The* codifferential *is the operator*

$$d^* : \Omega^1(\mathsf{E}, \mathsf{F}) \to \Omega^0(\mathsf{U}, \mathsf{F})$$

defined by setting, for all $\omega \in \Omega^1(\mathsf{E}, \mathsf{F})$ *and all* $x \in \mathsf{U}$,

$$d^*\omega(x) = \frac{1}{\lambda_x} \sum_{e \in \mathsf{E} : \overline{e} = x} s_e \chi_e h_{x,e} \omega(e) - \frac{1}{\lambda_x} \sum_{e \in \mathsf{E} : \underline{e} = x} s_e \chi_e h_{x,e^{-1}} \omega(e).$$

The following lemma justifies the notation that we used for the codifferential.

Lemma 3.11. *The operators* d *and* d^* *are adjoint of each other with respect to the Hermitian forms* $(\cdot,\cdot)_{\Omega^0}$ *and* $(\cdot,\cdot)_{\Omega^1}$.

Let us emphasise that the proof of this lemma depends on the fact that the connection h consists in unitary operators.

Proof. Let $f \in \Omega^0(\mathsf{U}, \mathsf{F})$ and $\omega \in \Omega^1(\mathsf{E}, \mathsf{F})$. We have

$$\begin{aligned}
(f, d^*\omega)_{\Omega^0} &= \sum_{x \in \mathsf{V}} \lambda_x \langle f(x), d^*\omega(x) \rangle_x \\
&= \sum_{x \in \mathsf{V}} \sum_{e \in \mathsf{E} : \overline{e} = x} s_e \chi_e \langle f(x), h_{x,e} \omega(e) \rangle_x - \sum_{x \in \mathsf{V}} \sum_{e \in \mathsf{E} : \underline{e} = x} s_e \chi_e \langle f(x), h_{x,e^{-1}} \omega(e) \rangle_x \\
&= \sum_{e \in \mathsf{E}} s_e \chi_e \langle f(\overline{e}), h_{\overline{e},e} \omega(e) \rangle_{\overline{e}} - \sum_{e \in \mathsf{E}} s_e \chi_e \langle f(\underline{e}), h_{\underline{e},e^{-1}} \omega(e) \rangle_{\underline{e}} \\
&= \sum_{e \in \mathsf{E}} s_e \chi_e \langle h_{e^{-1},\overline{e}} f(\overline{e}), \omega(e) \rangle_e - \sum_{e \in \mathsf{E}} s_e \chi_e \langle h_{e,\underline{e}} f(\underline{e}), \omega(e) \rangle_e \\
&= \sum_{e \in \mathsf{E}} s_e \chi_e \langle df(e), \omega(e) \rangle_e = (df, \omega)_{\Omega^1}.
\end{aligned}$$

We used the unitarity of h between the third and the fourth line. □

Note that if x is a proper vertex of G, that is, not an element of the well, then the value at x of the codifferential of a 1-form ω can be written more simply as

$$d^*\omega(x) = - \sum_{e \in \mathsf{E} : \underline{e} = x} P_{x,e} h_{x,e^{-1}} \omega(e). \tag{14}$$

3.9. **Laplacians.** To the situation that we are describing since the beginning of this paper, is associated a Laplace operator, which is our main object of interest. A simple definition of this operator, acting on $\Omega^0(\mathsf{U},\mathsf{F})$, would be to set

$$\overline{\Delta}_h = d^* \circ d \in \mathrm{End}(\Omega^0(\mathsf{U},\mathsf{F})).$$

However, the Laplacian that we will use most of the time is a compression of this operator to the subspace $\Omega^0(\mathsf{V},\mathsf{F})$, which consists of sections that vanish on the well. In the following definition, we will use the notation π_{V} defined by (9).

Definition 3.12. *Let G be a weighted graph with a well. Let F be a vector bundle over G. Let h be a connection on F. The h-Laplacian, or simply Laplacian, on $\Omega^0(\mathsf{V},\mathsf{F})$ is the linear mapping*

$$\Delta_h = \pi_{\mathsf{V}} \circ d^* \circ d_{|\Omega^0(\mathsf{V},\mathsf{F})} \in \mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F})).$$

Consider $f \in \Omega^0(\mathsf{V},\mathsf{F})$ and choose $x \in \mathsf{V}$. Thanks to (14), we have

$$\begin{aligned}(\Delta_h f)(x) &= -\sum_{e\in\mathsf{E}:\underline{e}=x} P_{x,e} h_{x,e^{-1}}(df)(e)\\ &= \sum_{e\in\mathsf{E}:\underline{e}=x} P_{x,e}(f(x) - h_{e^{-1}} f(\overline{e}))\\ &= f(x) - \sum_{e\in\mathsf{E}:\underline{e}=x} P_{x,e} h_{e^{-1}} f(\overline{e}). \qquad (15)\end{aligned}$$

This last expression will be useful in the sequel.

Using the Laplacian, we can define the covariant Dirichlet energy functional on $\Omega^0(\mathsf{U},\mathsf{F})$.

Definition 3.13. *Let f be an element of $\Omega^0(\mathsf{V},\mathsf{F})$. The* covariant Dirichlet energy *of f is the real number*

$$\mathcal{E}_h(f) = (f, \Delta_h f)_{\Omega^0}.$$

A short computation yields

$$\mathcal{E}_h(f) = \frac{1}{2}\sum_{e\in\mathsf{E}} \chi_e \|f(\underline{e}) - h_{e^{-1}} f(\overline{e})\|_e^2, \qquad (16)$$

which makes it obvious that $\mathcal{E}_h(f)$ is non-negative.

Proposition 3.14. *The operator Δ_h is Hermitian, non-negative, and invertible on $\Omega^0(\mathsf{V},\mathsf{F})$.*

Proof. It follows from Lemma 3.11 that the operator $\pi_{\mathsf{V}} \circ d^* \circ d \circ \pi_{\mathsf{V}}$ is Hermitian and non-negative on $\Omega^0(\mathsf{U},\mathsf{F})$. Moreover, it leaves the subspace $\Omega^0(\mathsf{V},\mathsf{F})$ invariant. Thus, Δ_h, which is the restriction of this operator to $\Omega^0(\mathsf{V},\mathsf{F})$, is Hermitian and non-negative. There remains to prove that it is invertible.

Consider a section f such that $\Delta_h f = 0$. Consider a vertex x on the rim of the well, that is, a vertex which is joined to a vertex of the well. Since f vanishes identically on the well, (16) and the fact that $\mathcal{E}_h(f) = 0$ imply that $f(x) = 0$. Then, since $f(\overline{e}) = h_e f(\underline{e})$ for every edge e, and since the proper subgraph $\mathsf{G}_{|\mathsf{V}}$ is connected, f vanishes identically. □

In the special case of the trivial bundle $\mathbb{K}_{\mathsf{G}}$ over G endowed with the canonical connection (see the end of Section 3.3), we will denote the Laplacian simply by Δ.

3.10. **The smallest eigenvalue of the Laplacian.** It follows from Lemma 3.14 that the lowest eigenvalue of Δ_h is positive. In this section, we give a lower bound on this first eigenvalue. Let us define

$$\sigma_h = \min \operatorname{Spec} \Delta_h = \min_{f \in \Omega^0(\mathsf{V},\mathsf{F})\setminus\{0\}} \frac{(f, \Delta_h f)_{\Omega^0}}{(f,f)_{\Omega^0}} \,. \tag{17}$$

On the trivial bundle $\mathbb{K}$ with the usual Laplacian Δ, we denote this quantity by σ.

Proposition 3.15 (Discrete Kato's lemma)**.** *The following inequality holds:*

$$\sigma_h \geq \sigma > 0 \,.$$

Proof. The inequality $\sigma > 0$ is a special case of Lemma 3.14. Let us prove that $\sigma_h \geq \sigma$. For this, consider a section $f \in \Omega^0(\mathsf{V},\mathsf{F})$. For all $e \in \mathsf{E}$, we have

$$\|f(\underline{e}) - h_{e^{-1}} f(\overline{e})\|_{\underline{e}} \geq \big| \|f(\underline{e})\|_{\underline{e}} - \|h_{e^{-1}} f(\overline{e})\|_{\underline{e}} \big| = \big| \|f(\underline{e})\|_{\underline{e}} - \|f(\overline{e})\|_{\overline{e}} \big| \,,$$

because h_e is unitary. Using (16), this implies the inequality

$$(f, \Delta_h f)_{\Omega^0} \geq (\|f\|, \Delta \|f\|) \,,$$

where $\|f\|$ is the function on V defined by $\|f\|(x) = \|f(x)\|_x$. Since $(f,f)_{\Omega^0} = (\|f\|, \|f\|)$, the inequality $\sigma_h \geq \sigma$ follows. □

3.11. **Generalized Laplacians.** We will consider linear operators on the space of sections of F that are similar to, but more general than the Laplacian Δ_h. In analogy with the case of a Hermitian vector bundle over a Riemannian manifold, for which there exists many second order differential operators with the same symbol, which all deserve to be called Laplacians, and which differ by a term of order 0, we set the following definition.

Recall from Definition 3.4 that we call potential a section of the vector bundle $\operatorname{End}(\mathsf{F})$ over V such that H_x is Hermitian (or, in the real case, symmetric) for each $x \in \mathsf{V}$. There is a natural inclusion $\Omega^0(\operatorname{End}(\mathsf{F})) \subset \operatorname{End}(\Omega^0(\mathsf{F}))$ thanks to which a potential can act on a section of F. Concretely, if H is a potential and f a section of F, then the section Hf is simply defined by the fact that for all $x \in \mathsf{V}$,

$$(Hf)(x) = H_x(f(x)).$$

Definition 3.16. *Let F be a Hermitian vector bundle over a graph with a well G, endowed with a connection h. Let H be a potential on F. The* generalised Laplacian *on F associated to h and H, or (h,H)-Laplacian, is the following linear endomorphism of $\Omega^0(\mathsf{V},\mathsf{F})$:*

$$\Delta_{h,H} = \Delta_h + H.$$

Imitating Definition 3.13, we define the generalised covariant Dirichlet energy of a section f as the number

$$\begin{aligned} \mathcal{E}_{h,H}(f) &= (f, (\Delta_h + H) f)_{\Omega^0} \\ &= \frac{1}{2} \sum_{e \in \mathsf{E}} \chi_e \|f(\underline{e}) - h_{e^{-1}} f(\overline{e})\|_e^2 + \sum_{x \in \mathsf{V}} \lambda_x \langle f(x), H_x f(x) \rangle_x . \end{aligned} \tag{18}$$

Recall from the previous section that we denote by σ_h the smallest eigenvalue of Δ_h, and that $\sigma_h > 0$. If $H_x > -\sigma_h$ for every vertex x, then the operator $\Delta_{h,H}$ is invertible. This is in particular the case as soon as $H_x \geq 0$ for every vertex x.

3.12. **Gauge transformations.** Let us conclude this section by a study of the action of the group of automorphisms of our vector bundles on the various objects that we defined.

For every Hermitian (resp. Euclidean) vector space F, let us denote by $\mathrm{U}(F)$ the group of unitary (resp. orthogonal) linear transformations of F.

Definition 3.17. *The* gauge group *of the vector bundle* F *over* $\mathsf{G} = (\mathsf{U}, \mathsf{E})$ *is the group*

$$\mathsf{Aut}(\mathsf{F}) = \left\{ j \in \prod_{x \in \mathsf{U}} \mathrm{U}(\mathsf{F}_x) \times \prod_{e \in \mathsf{E}} \mathrm{U}(\mathsf{F}_e) : j_e = j_{e^{-1}} \text{ for all } e \in \mathsf{E}^{\updownarrow} \right\}.$$

The elements of the gauge group are called gauge transformations.

The gauge group $\mathsf{Aut}(\mathsf{F})$ acts on the set $\mathcal{A}(\mathsf{F})$ of connections on F, as follows. Let h be a connection on F. Let j be a gauge transformation. The connection $j \cdot h$ is defined by

$$(j \cdot h)_{e,x} = j_e \circ h_{e,x} \circ j_x^{-1} \quad \text{for } (e, x) \in \mathsf{E} \times_{\mathsf{G}} \mathsf{U} \text{ and}$$
$$(j \cdot h)_{x,e} = j_x \circ h_{x,e} \circ j_e^{-1} \quad \text{for } (x, e) \in \mathsf{U} \times_{\mathsf{G}} \mathsf{E}.$$

The next formula expresses the modification of the holonomy along a continuous path induced by the action of a gauge transformation on the connection. Suppose h is a connection, j a gauge transformation, and γ a path, discrete or continuous, in our graph. Then

$$\mathsf{hol}_{j \cdot h}(\gamma) = j_{\overline{\gamma}} \circ \mathsf{hol}_h(\gamma) \circ j_{\underline{\gamma}}^{-1}.$$

Let us write the corresponding formula for twisted holonomies. For this, we need to explain how the gauge group acts on the space of potentials. Let H be a potential and j a gauge transformation. We define $j \cdot H$ as the potential such that

$$(j \cdot H)_x = j_x \circ H_x \circ j_x^{-1}.$$

Then, for all continuous path γ, we have the relation

$$\mathsf{hol}_{j \cdot h, j \cdot H}(\gamma) = j_{\overline{\gamma}} \circ \mathsf{hol}_{h,H}(\gamma) \circ j_{\underline{\gamma}}^{-1}.$$

The gauge group also acts naturally on $\Omega^0(\mathsf{U}, \mathsf{F})$: for all gauge transformation j and all section f, the section $j \cdot f$ is defined by

$$(j \cdot f)(x) = j_x(f(x)).$$

This action extends to an action by conjugation on $\mathrm{End}(\Omega^0(\mathsf{U}, \mathsf{F}))$: if B is an endomorphism of $\Omega^0(\mathsf{U}, \mathsf{F})$ and f is a section, then

$$(j \cdot B)(f) = j \cdot (B(j^{-1} \cdot f)).$$

These actions allow us to express the way in which the Laplacians are acted on by the gauge group: with our current notation,

$$j \cdot \Delta^{h,H} = \Delta^{j \cdot h, j \cdot H}.$$

Since the gauge group acts on each fibre by unitary transformations, it follows that the Dirichlet energy satisfies

$$\mathcal{E}_{j \cdot h, j \cdot H}(j \cdot f) = \mathcal{E}_{h,H}(f).$$

4. A covariant Feynman-Kac formula

4.1. **A Feynman–Kac formula.** In this section, we will prove our first theorem, which is a covariant version of the Feynman–Kac formula and which will be one of the most useful results for our study.

Theorem 4.1 (Feynman–Kac formula)**.** *Let G be a weighted graph with a well. Let F be a vector bundle over G, endowed with a connection h and a potential H. Let X be the random walk on G defined in Section 2.3.*

Let f be an element of $\Omega^0(\mathsf{V},\mathsf{F})$. For all $x \in \mathsf{V}$ and all $t \geq 0$, the following equality holds:

$$\left(e^{-t\Delta_{h,H}} f\right)(x) = \int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma_{|[0,t]}^{-1}) f(\gamma_t) \mathbb{1}_{\{\tau(\gamma)>t, \gamma_t \in \mathsf{V}\}} \, d\mathbb{P}_x(\gamma).$$

The indicator of the event $\{\gamma_t \in \mathsf{V}\}$ is there to make sure that the inverse of the path $\gamma_{|[0,t]}$ is well defined. In any case, f vanishes identically on the well, and there should be no contribution to the right-hand side coming from paths that reach the well.

Proof. We will use the fact that for any two endomorphisms A and B of a finite dimensional vector space and all real $t > 0$,

$$e^{t(A+B)} = \sum_{k=0}^{\infty} \int_{0<t_0<\ldots<t_{k-1}<t} e^{t_0 A} B e^{(t_1-t_0)A} \ldots B e^{(t-t_{k-1})A} \, dt_0 \ldots dt_{k-1},$$

a formula that can be checked by expanding the exponentials on the right-hand side in power series, computing the Eulerian integrals that appear, and observing that one recovers the left-hand side. In our context, we use this formula in the following way: we first write $e^{-t\Delta_{h,H}} = e^{-t} e^{-tH+t(\mathrm{Id}-\Delta_h)}$ and deduce that

$$e^{-t\Delta_{h,H}} = e^{-t} \sum_{k=0}^{\infty} \int_{0<t_0<\ldots<t_{k-1}<t} e^{-t_0 H} (\mathrm{Id}-\Delta_h) e^{-(t_1-t_0)H} \ldots (\mathrm{Id}-\Delta_h) e^{-(t-t_{k-1})H} \, dt_0 \ldots dt_{k-1}.$$

Let f be a section of F and x a vertex of G. Using the expression (15) of Δ_h, we find

$$(e^{-t\Delta_{h,H}} f)(x) = e^{-t} \sum_{k=0}^{\infty} \sum_{e_1,\ldots,e_k \in \mathsf{E}} P_{x,e_1} P_{\overline{e_1},e_2} \ldots P_{\overline{e_{k-1}},e_k}$$

$$\int_{0<t_0<\ldots<t_{k-1}<t} e^{-t_0 H_x} h_{e_1^{-1}} e^{-(t_1-t_0)H_{\overline{e_1}}} \ldots h_{e_k^{-1}} e^{-(t-t_{k-1})H_{\overline{e_k}}} f(\overline{e_k}) \, dt_0 \ldots dt_{k-1}. \tag{19}$$

On the other hand, according to (7),

$$\int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma_{|[0,t]}^{-1}) f(\gamma_t) \mathbb{1}_{\{\tau(\gamma)>t, \gamma_t \in \mathsf{V}\}} \, d\mathbb{P}_x(\gamma) = \sum_{\substack{0 \leq k < n \\ p \in \mathsf{DP}_n(\mathsf{G}) \\ p=(x_0,e_1,\ldots,e_n,x_n)}} \mathbb{Q}_x(\{p\})$$

$$\int_{0<t_0<\ldots<t_{k-1}<t<t_k<\ldots<t_{n-1}} e^{-t_0 H_x} h_{e_1^{-1}} e^{-(t_1-t_0)H_{x_1}} \ldots h_{e_k^{-1}} e^{-(t-t_{k-1})H_{x_k}} f(x_k) e^{-t_{n-1}} \, dt_0 \ldots dt_{n-1}$$

$$= e^{-t} \sum_{k=0}^{\infty} \sum_{e_1,\ldots,e_k \in \mathsf{E}} P_{x,e_1} P_{\overline{e_1},e_2} \ldots P_{\overline{e_{k-1}},e_k} \left(\sum_{l=0}^{\infty} \sum_{e'_1,\ldots,e'_l \in \mathsf{E}} P_{\overline{e_k},e'_1} P_{\overline{e'_1},e'_2} \ldots P_{\overline{e'_{l-1}},e'_l} \mathbb{1}_{\mathsf{W}}(\overline{e'_l}) \right)$$

$$\int_{0<t_0<\ldots<t_{k-1}<t} e^{-t_0 H_x} h_{e_1^{-1}} e^{-(t_1-t_0)H_{\overline{e_1}}} \ldots h_{e_k^{-1}} e^{-(t-t_{k-1})H_{\overline{e_k}}} f(\overline{e_k}) \, dt_0 \ldots dt_{k-1}.$$

The sum over l between the brackets is the total mass of the probability measure $\mathbb{Q}_{\overline{e_k}}$, that is, 1, and we recover the right-hand side of (19). □

We will often use a slightly different, but equivalent, version of the Feynman–Kac formula. In order to state it, let us introduce the following notation. The vector space of linear endomorphisms of $\Omega^0(\mathsf{V},\mathsf{F})$ is isomorphic to

$$\mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F})) = \mathrm{End}\Big(\bigoplus_{x\in\mathsf{V}} \mathsf{F}_x\Big) = \bigoplus_{x,y\in\mathsf{V}} \mathrm{Hom}(\mathsf{F}_y,\mathsf{F}_x).$$

This decomposition is nothing more than the block decomposition of the matrix representing an endomorphism of $\Omega^0(\mathsf{V},\mathsf{F})$, each block corresponding to one of the fibres of the bundle. For all linear operator $B \in \mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F}))$ and all $x, y \in \mathsf{V}$, we will denote by $B_{x,y}$ the linear map from F_y to F_x which appears in the right-hand side of the decomposition above.

Proposition 4.1 immediately implies the following.

Corollary 4.2. *Under the assumptions of Theorem 4.1, and for all $x, y \in \mathsf{V}$, the following equality holds in* $\mathrm{Hom}(\mathsf{F}_y,\mathsf{F}_x)$*:*

$$\big(e^{-t\Delta_{h,H}}\big)_{x,y} = \int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1}_{|[0,t]})\mathbb{1}_{\{\tau(\gamma)>t,\gamma_t=y\}}\, d\mathbb{P}_x(\gamma).$$

Provided we are careful enough about the source and target spaces of the linear operators that we are writing, we can express the corollary in the following more compact form:

$$e^{-t\Delta_{h,H}} = \bigoplus_{x\in\mathsf{V}} \big(e^{-t\Delta_{h,H}}\big)_{x,y} = \int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1}_{|[0,t]})\mathbb{1}_{\{\tau(\gamma)>t\}}\, d\mathbb{P}_x(\gamma).$$

4.2. **The Green section.** We devote this very short section to the definition of the covariant analogue of the Green function. For this, let us define the operator Λ on $\Omega^0(\mathsf{U},\mathsf{F})$ such that for all section f and all vertex $x \in \mathsf{V}$,

$$(\Lambda f)(x) = \lambda_x f(x).$$

Definition 4.3. *Let G be a weighted graph with a well. Let F be a fibre bundle over G. Let h be a connection on F and H be a potential on F. The* Green section *associated with h and H is the operator*

$$G_{h,H} = (\Lambda \circ \Delta_{h,H})^{-1} \in \mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F})).$$

Let us make a brief comment about the distinction between sections and measures. The operator Λ takes a section and multiplies it by a measure. The result of this operation is an object that can be paired pointwise with a section, using the Hermitian scalar product of each fibre, thus producing a scalar function that can be integrated over the graph. It would thus be fair to say that Λ sends $\Omega^0(\mathsf{U},\mathsf{F})$ into its dual space, and, since $\Delta_{h,H}$ is a genuine operator on $\Omega^0(\mathsf{V},\mathsf{F})$, that $G_{h,H}$ sends the dual of $\Omega^0(\mathsf{V},\mathsf{F})$ into $\Omega^0(\mathsf{V},\mathsf{F})$ itself. In a sense that we will not make very precise, the kernel of $G_{h,H}$ is thus the covariant analogue of a function of two variables.

4.3. **Two elementary algebraic identities.** The Feynman-Kac formula relates the semigroup of the generalised Laplacian to the average of the twisted holonomy along the paths of the random walk in the graph. In this paragraph, we will derive two consequences of this formula which will be the bases of our proofs of the isomorphism theorems. The two formulas which we will now prove ultimately rely on the following elementary lemma.

Lemma 4.4. *Let A and B be two positive Hermitian matrices of the same size. Then*

$$\int_0^\infty e^{-tA}\,dt = A^{-1} \tag{20}$$

and

$$\int_0^\infty \frac{e^{-tA} - e^{-tB}}{t} dt = \log B - \log A\,. \tag{21}$$

Proof. It suffices to prove that for all positive real a,

$$\int_0^\infty \frac{e^{-ta} - e^{-t}}{t} dt = -\log a\,.$$

Now, for all $\varepsilon > 0$, $\int_\varepsilon^\infty \frac{e^{-ta}}{t}\,dt = \int_{a\varepsilon}^\infty \frac{e^{-t}}{t}\,dt$ and the integral that we want to compute is equal to $\int_{a\varepsilon}^{\varepsilon} \frac{e^{-t}}{t} = \int_{a\varepsilon}^{\varepsilon} \frac{1}{t} + O(\varepsilon)$, from which the result follows immediately. □

From the Feynman-Kac formula (Theorem 4.1), the definition of the measure ν (Definition 2.7), and (20), we deduce the following proposition.

Proposition 4.5. *For all $x, y \in \mathsf{V}$ the equality*

$$\int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1})\,d\nu_{x,y}(\gamma) = \big(G_{h,H}\big)_{x,y} \tag{22}$$

holds in $\mathrm{Hom}(\mathsf{F}_y, \mathsf{F}_x)$. *Moreover, we have the following equality in* $\mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F}))$*:*

$$\int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1})\,d\nu(\gamma) = \Delta_{h,H}^{-1}. \tag{23}$$

Similarly, from Theorem 4.1, Definition 2.8 and (21), we deduce the following.

Proposition 4.6. *Let h, h' be two connections on F and H, H' be two non-negative potentials. We have the following identity in* $\mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F}))$*:*

$$\int_{\mathsf{P}(\mathsf{G})} \big(\mathsf{hol}_{h,H}(\gamma^{-1}) - \mathsf{hol}_{h',H'}(\gamma^{-1})\big)\,d\mu(\gamma) = \log \Delta_{h',H'} - \log \Delta_{h,H}\,. \tag{24}$$

It turns out that (24) is difficult to use in its present form, and that it is often more convenient to use the less detailed but still informative equation obtained from it by taking the trace of both sides. Here, by the trace, we mean the usual trace in the space of endomorphisms of $\Omega^0(\mathsf{V},\mathsf{F})$. Thus, if B is such an endomorphism, we call trace of B the number

$$\mathrm{Tr}(B) = \sum_{x\in\mathsf{V}} \mathrm{Tr}_{\mathsf{F}_x}(B_{x,x}).$$

Corollary 4.7. *Let h be a connection and H a non-negative potential. We have the identity*

$$\int_{\mathsf{L}(\mathsf{G})} \big(\mathrm{Tr}\,\mathsf{hol}_{h,H}(\gamma^{-1}) - \mathrm{Tr}\,\mathsf{hol}_{h}(\gamma^{-1})\big)\,d\mu^{\circledast}(\gamma) = \log \frac{\det \Delta_h}{\det \Delta_{h,H}}\,. \tag{25}$$

The reader may wonder why we wrote Proposition 4.6 as we did with a difference of holonomies, and what the integral of the twisted holonomy along a random path with respect to the measure μ is. The answer is that this integral is ill defined, because of the presence of the infinite measure $\mu^\bullet$ within μ. However, it makes sense to compute this integral against $\mu - \mu^\bullet$. The result is given by the next proposition.

Proposition 4.8. *Let h be a connection and H a positive potential. In* $\mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F}))$*, we have the identity*

$$\int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1}) d(\mu-\mu^\bullet)(\gamma) = -\log\Delta_{h,H} + \log(\mathrm{Id}+H)\,. \tag{26}$$

Proof. Consider x and y in V. We have

$$\begin{aligned}\text{l.h.s. of (26)} &= \frac{1}{\lambda_y}\int_{\mathsf{P}(\mathsf{G})\times(0,+\infty)} \mathsf{hol}(\gamma^{-1}_{|[0,t]})\mathbb{1}_{\{\tau(\gamma)>t,\gamma_t=y\}}(1-\mathbb{1}_{\{\gamma_{|[0,t]}=(x,t)\}})\,d\mathbb{P}_x(\gamma)\frac{dt}{t}\\ &= \frac{1}{\lambda_y}\int_0^\infty \left((e^{-t\Delta_{h,H}})_{x,y} - \delta_{x,y}e^{-t}e^{-tH_x}\right)\frac{dt}{t}\\ &= \frac{1}{\lambda_y}\left(\int_0^\infty \frac{e^{-t\Delta_{h,H}}-e^{-t(\mathrm{Id}+H)}}{t}\,dt\right)_{x,y}.\end{aligned}$$

Summing over y with respect to the reference measure λ and over x with respect to the counting measure, and using (21), we find the expected result. □

5. The covariant Gaussian free field

In this section, we consider a weighted graph with a well G, over which we are given a vector bundle F with a connection h and a potential H. In this setting, we will construct a Gaussian probability measure on $\Omega^0(\mathsf{V},\mathsf{F})$.

5.1. **Probability measures on $\Omega^0(\mathsf{V},\mathsf{F})$.** Recall from Section 3.11 that if H_x is a non-negative Hermitian operator on F_x for every $x\in\mathsf{V}$, then the operator $\Delta_{h,H}$ is positive Hermitian on $\Omega^0(\mathsf{V},\mathsf{F})$. We are going to use it to define a probability measure on $\Omega^0(\mathsf{V},\mathsf{F})$.

Let us first discuss Lebesgue measures on $\Omega^0(\mathsf{V},\mathsf{F})$. Let us agree that the natural Lebesgue measure on a Euclidean or Hermitian space is that which gives measure 1 to any real cube the edges of which have length 1. With this convention, there is a natural Lebesgue measure Leb_{Ω^0} on $\Omega^0(\mathsf{V},\mathsf{F})$ and, for each $x\in\mathsf{V}$, a natural Lebesgue measure Leb_x on F_x. Considering the way in which the Euclidean or Hermitian structures on $\Omega^0(\mathsf{V},\mathsf{F})$ on one hand and on the fibres of F on the other hand are related, through the measure λ, we find the equality

$$\mathrm{Leb}_{\Omega^0} = \bigotimes_{x\in\mathsf{V}}\left(\lambda_x^{\frac{\beta}{2}r}\,\mathrm{Leb}_x\right),$$

where, in order to treat the Euclidean and Hermitian cases simultaneously, we used the constant

$$\beta = \dim_{\mathbb{R}}\mathbb{K}, \tag{27}$$

equal to 1 in the Euclidean case and to 2 in the Hermitian case.

Let us denote by $|\mathsf{V}|$ the cardinality of V. We define the probability measure $\mathbb{P}^{h,H}$ on $\Omega^0(\mathsf{V},\mathsf{F})$ by setting

$$d\mathbb{P}^{h,H}(f) = \left(\frac{\det\Delta_{h,H}}{\pi^{r|\mathsf{V}|}}\right)^{\frac{\beta}{2}} e^{-\frac{1}{2}(f,\Delta_{h,H}f)_{\Omega^0}}\,d\mathrm{Leb}_{\Omega^0}(f). \tag{28}$$

We shall denote by Φ the canonical process (that is, the identity map) on $\Omega^0(\mathsf{V},\mathsf{F})$, so that for all bounded measurable function $F:\Omega^0(\mathsf{V},\mathsf{F})\to\mathbb{R}$, we have

$$\mathbb{E}^{h,H}[F(\Phi)] = \int_{\Omega^0(\mathsf{V},\mathsf{F})} F(f)\,d\mathbb{P}^{h,H}(f).$$

The random section Φ is called the *covariant Gaussian free field* on G associated with h and H.

We will often consider the Gaussian free field associated to a connection h and the zero potential. In that case, we will use the notation $\mathbb{E}^h$ instead of $\mathbb{E}^{h,0}$.

5.2. **Covariance.** The measure $\mathbb{P}^{h,H}$ is Gaussian on $\Omega^0(\mathsf{V},\mathsf{F})$ and we will now compute its covariance function. Let us introduce some notation which will be useful to express this covariance, and several of the results that we shall prove.

Consider a Euclidean or Hermitian vector space $(V,(\cdot,\cdot))$. Recall that if V is Hermitian, we take the Hermitian scalar product to be linear in the second variable. We define the *conjugation map* to be the following:

$$\begin{aligned} V &\longrightarrow V^* \\ v &\longmapsto \overline{v} = (v,\cdot). \end{aligned}$$

This map is an antilinear isomorphism between V and its dual, and it satisfies, for every scalar z and every vector v, the relation $\overline{zv} = \overline{z}\,\overline{v}$.

If V and W are Euclidean or Hermitian spaces, and if v and w are elements of V and W respectively, then $w\otimes\overline{v}$ is an element of $W\otimes V^* \simeq \mathrm{Hom}(V,W)$. More specifically, $w\otimes\overline{v}$ is the linear map of rank 1 from V to W such that for all $u\in V$,

$$(w\otimes\overline{v})(u) = (v,u)w.$$

With this notation, a standard Gaussian computation yields the following identity.

Proposition 5.1. *Let F be a vector bundle over G, endowed with a connection h and a potential H. Let Φ be the associated covariant Gaussian free field on G. Then*

$$\mathbb{E}^{h,H}[\Phi\otimes\overline{\Phi}] = \Delta_{h,H}^{-1}.$$

In other words, for all $f,g\in\Omega^0(\mathsf{V},\mathsf{F})$,

$$\mathbb{E}^{h,H}\left[(f,\Phi)_{\Omega^0}(\Phi,g)_{\Omega^0}\right] = (f,\Delta_{h,H}^{-1}g)_{\Omega^0}. \tag{29}$$

Note that this formulation is true in the Euclidean case as well as in the Hermitian case.

It is often useful to have an expression of the covariance of the values of the Gaussian free field in two specific fibres of F. Passing from the space of sections to the individual fibres involves the reference measure λ, so that the inverse of the Laplacian will be replaced by the Green section.

Proposition 5.2. *Let F be a vector bundle over G, endowed with a connection h and a potential H. Let Φ be the covariant Gaussian free field on G. Let $G_{h,H}$ be the Green section of F. For all $x,y\in\mathsf{V}$, we have the following identity in $\mathrm{Hom}(\mathsf{F}_y,\mathsf{F}_x)$:*

$$\mathbb{E}^{h,H}\left[\Phi_x\otimes\overline{\Phi_y}\right] = (G_{h,H})_{x,y}. \tag{30}$$

Proof. Let us choose two vertices x and y in V and two vectors $\xi\in\mathsf{F}_x$ and $\eta\in\mathsf{F}_y$. Applying Proposition 5.1 to the sections $f=\xi\mathbb{1}_{\{x\}}$ and $g=\eta\mathbb{1}_{\{y\}}$, which vanish respectively everywhere except at x and y, and satisfy $f(x)=\xi$ and $g(y)=\eta$, we find

$$\lambda_x\lambda_y\mathbb{E}^{h,H}\left[\langle\xi,\Phi_x\rangle_x\langle\Phi_y,\eta\rangle_y\right] = \lambda_x\langle\xi,\left(\Delta_{h,H}^{-1}\right)_{x,y}\eta\rangle_x.$$

This equality can be written

$$\mathbb{E}^{h,H}\left[\langle\xi,\Phi_x\rangle_x\langle\Phi_y,\eta\rangle_y\right] = \langle\xi,\left(\Delta_{h,H}^{-1}\Lambda^{-1}\right)_{x,y}\eta\rangle_x,$$

or even

$$\left\langle\xi,\mathbb{E}^{h,H}\left[\Phi_x\otimes\overline{\Phi_y}\right]\eta\right\rangle_x = \langle\xi,\left(G_{h,H}\right)_{x,y}\eta\rangle_x,$$

which, because it is true for all ξ and η, implies the result. □

5.3. **Laplace transform.** For future reference, let us record the Laplace transform and a formula for the higher moments of the covariant Gaussian free field. We keep the notation of the previous section.

The next proposition follows from (29) and the standard computation of a Gaussian integral. Note that it is written in such a way as to be true in the real case as well as in the complex case.

Proposition 5.3. *For all $f \in \Omega^0(\mathsf{V},\mathsf{F})$, we have*

$$\mathbb{E}^{h,H}\left[\left|e^{\frac{1}{2}(f,\Phi)_{\Omega^0}}\right|^2\right] = e^{\frac{1}{2}(f,\Delta_{h,H}^{-1}f)_{\Omega^0}}.$$

The following formula, which is a reformulation of Wick's theorem (see [Jan97]) in our context, combined with (23), will be useful in the proof of the covariant Symanzik identity (Theorem 8.3). It is one of the results where the real and complex case are quite different.

Proposition 5.4. *Assume that $\mathbb{K} = \mathbb{R}$. Then, for all integer $k \geq 0$ and all $f_1, \dots, f_k \in \Omega^0(\mathsf{V},\mathsf{F})$, we have*

$$\mathbb{E}^{h,H}\left[\prod_{i=1}^{k}(f_i,\Phi)_{\Omega^0}\right] = \sum_{\pi}\prod_{\{i,j\}\in\pi}\int_{\mathsf{P}(\mathsf{G})}\lambda_{\underline{\gamma}}\langle f_i(\underline{\gamma}),\mathsf{hol}_{h,H}(\gamma^{-1})f_j(\overline{\gamma})\rangle_{\underline{\gamma}}\,d\nu(\gamma), \tag{31}$$

where the sum is over all partitions of the set $\{1,\dots,k\}$ by pairs. In particular, this expectation is zero if k is odd.

Assume that $\mathbb{K} = \mathbb{C}$. Then, for all integers $k, l \geq 0$, all $f_1, \dots, f_k$ and $f'_1, \dots, f'_l$ in $\Omega^0(\mathsf{V},\mathsf{F})$, we have

$$\mathbb{E}^{h,H}\left[\prod_{i=1}^{k}(f_i,\Phi)_{\Omega^0}\prod_{j=1}^{l}(\Phi,f'_j)_{\Omega^0}\right] = \sum_{\sigma}\prod_{i=1}^{k}\int_{\mathsf{P}(\mathsf{G})}\lambda_{\underline{\gamma}}\langle f_i(\underline{\gamma}),\mathsf{hol}_{h,H}(\gamma^{-1})f'_{\sigma(i)}(\overline{\gamma})\rangle_{\underline{\gamma}}\,d\nu(\gamma), \tag{32}$$

where the sum is over all bijections from $\{1,\dots,k\}$ to $\{1,\dots,l\}$. In particular, this expectation is zero if $k \neq l$.

5.4. **Square of the shifted Gaussian free field.** In this section, and in preparation for the proof of Theorem 7.6, we record a useful lemma which expresses the Laplace transform of the square of the shifted Gaussian free field. This is a generalisation of [Szn12b, Proposition 2.14] to the present covariant setting. Recall the definition of σ_h from Proposition 3.15.

Proposition 5.5. *Let h be a unitary connection and H a potential on F. Assume that $H > -\sigma_h$. For all $f \in \Omega^0(\mathsf{V},\mathsf{F})$, we have*

$$\mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}}\right] = \mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi,H\Phi)_{\Omega^0}}\right]e^{\frac{1}{2}\left(\Delta_h f,(\Delta_{h,H}^{-1}-\Delta_h^{-1})\Delta_h f\right)_{\Omega^0}}. \tag{33}$$

Proof. Let us compute the left-hand side of the equality to prove. We will successively apply the definition (28) of the measure $\mathbb{P}^h$, writing $d\varphi$ instead of $d\mathrm{Leb}_{\Omega^0}(\varphi)$, then perform the usual completion of the square, then use the invariance by translation of the Lebesgue measure. We find

$$\begin{aligned}\left(\frac{\pi^{r|\mathsf{V}|}}{\det\Delta_h}\right)^{\frac{\beta}{2}}\mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}}\right] &= \int_{\Omega^0(\mathsf{V},\mathsf{F})}e^{-\frac{1}{2}(\varphi+f,H(\varphi+f))_{\Omega^0}}e^{-\frac{1}{2}(\varphi,\Delta_h\varphi)_{\Omega^0}}d\varphi\\ &= \int_{\Omega^0(\mathsf{V},\mathsf{F})}e^{-\frac{1}{2}\left(\varphi+\Delta_{h,H}^{-1}Hf,\Delta_{h,H}(\varphi+\Delta_{h,H}^{-1}Hf)\right)_{\Omega^0}}d\varphi\;e^{\frac{1}{2}\left(f,(H\Delta_{h,H}^{-1}H-H)f\right)_{\Omega^0}}\\ &= \int_{\Omega^0(\mathsf{V},\mathsf{F})}e^{-\frac{1}{2}\left(\varphi,\Delta_{h,H}\varphi\right)_{\Omega^0}}d\varphi\;e^{\frac{1}{2}\left(\Delta_h f,(\Delta_{h,H}^{-1}-\Delta_h^{-1})\Delta_h f\right)_{\Omega^0}}.\end{aligned}$$

In the second exponential of the last line, we used the equality

$$a(a+b)^{-1}a - a = (a+b-b)(a+b)^{-1}(a+b-b) - a = b(a+b)^{-1}b - b$$

which is true in any associative algebra and that we applied in this instance to $a = H$ and $b = \Delta_h$ in the algebra $\mathrm{End}(\Omega^0(\mathsf{V},\mathsf{F}))$.

Now, in the integral on the right-hand side we recognise, up to the normalisation factor by which we multiplied both sides, the expectation on the right-hand side of (33). □

6. Isomorphism theorems of Dynkin and Eisenbaum

In this section, we explain how the two classical isomorphism theorems of Dynkin [Dyn84a] and Eisenbaum [Eis95] extend to the present setting, in a way which incorporates the geometry of the vector bundle over the graph. Our main tool will be the following combination of (22) and (30), which reads, for any two vertices x and y of a weighted graph with a well,

$$\mathbb{E}^{h,H}\left[\Phi_x \otimes \overline{\Phi_y}\right] = (G_{h,H})_{x,y} = \int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1})\, d\nu_{x,y}(\gamma). \tag{34}$$

6.1. **Dynkin's isomorphism.** Our generalisation of Dynkin's isomorphism is the following. We use the notation $\int_{\nu_{x,y}}$ to denote the integral with respect to the finite measure $\nu_{x,y}$. Recall from (27) the definition of the constant β.

Theorem 6.1. *Let G be a weighted graph with a well. Let F be a Hermitian or Euclidean vector bundle over G. Let h be a connection on F and H a potential on F. For all vertices x, y of V, the following identity holds in $\mathrm{Hom}(\mathsf{F}_y, \mathsf{F}_x)$:*

$$\mathbb{E}^h \otimes \int_{\nu_{x,y}} \left[e^{-\frac{1}{2}(\Phi, H\Phi)_{\Omega^0}}\, \mathsf{hol}_{h,H}(\gamma^{-1}) \right] = \mathbb{E}^h \left[e^{-\frac{1}{2}(\Phi, H\Phi)_{\Omega^0}}\, \Phi_x \otimes \overline{\Phi_y} \right]. \tag{35}$$

Proof. By definition of the measure $\mathbb{P}^{h,H}$, we have

$$\frac{\mathbb{E}^h \left[e^{-\frac{1}{2}(\Phi, H\Phi)_{\Omega^0}} \Phi_x \otimes \overline{\Phi_y} \right]}{\mathbb{E}^h \left[e^{-\frac{1}{2}(\Phi, H\Phi)_{\Omega^0}} \right]} = \mathbb{E}^{h,H}\left[\Phi_x \otimes \overline{\Phi_y}\right],$$

which equals $\int_{\nu_{x,y}} \left[\mathsf{hol}_{h,H}(\gamma^{-1})\right]$ by (34). The result follows immediately. □

Let us explain how successive specialisations of this theorem will allow us to recover Dynkin's isomorphism in its classical version. Let us consider first the case where H is a scalar potential, that is, the case where for each vertex $x \in \mathsf{V}$, the operator H_x has a unique eigenvalue, which we also denote by H_x. In that case, according to Lemma 3.8, the twisted holonomy along a continuous path can be expressed in terms of its classical holonomy and an exponential functional of its local time, and (35) becomes

$$\mathbb{E}^h \otimes \int_{\nu_{x,y}} \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x (\|\Phi_x\|^2 + \ell_x(\gamma))} \mathsf{hol}_h(\gamma^{-1}) \right] = \mathbb{E}^h \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x \|\Phi_x\|^2} \Phi_x \otimes \overline{\Phi_y} \right].$$

Going one step further in the specialisation, let us apply this formula to the case of the trivial real bundle of rank 1 endowed with the trivial connection. In that case, Φ is a random real valued function on V and we find one of the classical formulations of Dynkin's isomorphism, namely

$$\mathbb{E} \otimes \int_{\nu_{x,y}} \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x (|\Phi_x|^2 + \ell_x(\gamma))} \right] = \mathbb{E}\left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x |\Phi_x|^2} \Phi_x \Phi_y \right].$$

It appears that Dynkin's theorem is one of these deep results that, with the benefit of a few decades of hindsight and the appropriate set of preliminary definitions, become almost tautological. Nevertheless, from our point of view, it remains definitely not trivial in that it expresses the

covariance of the covariant Gaussian free field as a weighted sum of holonomies along paths in our graph. For example, if we pick a vertex x, then the covariance matrix of the random vector Φ_x in the vector space F_x is given by

$$\mathbb{E}^{h,H}\left[\Phi_x \otimes \overline{\Phi_x}\right] = \int_{\mathsf{L}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1})\, d\nu_{x,x}(\gamma) \in \mathrm{End}(\mathsf{F}_x).$$

Let us make a few comments on this expression. Firstly, the left-hand side makes it clear that the endomorphism of F_x which both sides of this equality express is Hermitian and non-negative. If $H = 0$, the Hermitian character of the right-hand side can be easily recognised from the invariance of the measure $\nu_{x,x}$ by inversion, by writing

$$\int_{\mathsf{L}(\mathsf{G})} \mathsf{hol}_h(\gamma^{-1})\, d\nu_{x,x}(\gamma) = \int_{\mathsf{L}(\mathsf{G})} \frac{1}{2}\left(\mathsf{hol}_h(\gamma) + \mathsf{hol}_h(\gamma)^{-1}\right)\, d\nu_{x,x}(\gamma).$$

Secondly, let us compute an example of this formula. Let G be the graph with two vertices: x and w, and three edges: two looping edges e and e^{-1} based at x, and one edge joining x to w. Of course, x is a proper vertex and w constitutes the well of this graph.

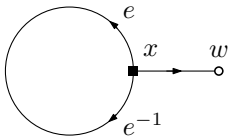

FIGURE 6. A very simple graph.

Let χ_e denote the conductance of e, and κ_x the conductance of the edge that joins x to w. A computation similar to the one that we did in the proof of Proposition 3.6 yields

$$\int_{\mathsf{L}(\mathsf{G})} \mathsf{hol}_h(\gamma^{-1})\, d\nu_{x,x}(\gamma) = (\chi_e(2\mathrm{Id}_{\mathsf{F}_x} - (h_e + h_e^{-1})) + \kappa_x \mathrm{Id}_{\mathsf{F}_x})^{-1}.$$

As we explained there, this operator can be made, by appropriate choices of χ_e, κ_x and h_e, equal to any positive Hermitian operator on F_x. This shows that the presence of the connection can make the Gaussian free field as anisotropic as one wishes at a given vertex, and that this anisotropy is an effect of the non-triviality of the holonomy of the connection.

6.2. **Eisenbaum's isomorphism.** We will now state a generalisation of Eisenbaum's isomorphism. Our understanding is that this theorem gives a probabilistic expression in terms of the Gaussian free field of the difference between the inverses of the Laplacians associated to a same connection but to different potentials (see (38) below).

Theorem 6.2. *Let G be a weighted graph with a well. Let F be a Hermitian or Euclidean vector bundle over G. Let h be a connection on F and H a potential on F. Let f be a section of F. Then the following identity holds in $\Omega^0(\mathsf{V},\mathsf{F})$:*

$$\mathbb{E}^h \otimes \int_\nu \left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}} \mathsf{hol}_{h,H}(\gamma^{-1})\right] \Delta_h f = \mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}}(\Phi+f)\right]. \tag{36}$$

More concretely, for all vertex x, we have

$$\sum_{y\in\mathsf{V}} \lambda_y \int_{\nu_{x,y}} \mathsf{hol}_{h,H}(\gamma^{-1})\big((\Delta_h f)(y)\big)\, d\nu_{x,y}(\gamma) = \frac{\mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}}(\Phi_x + f_x)\right]}{\mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}}\right]}. \tag{37}$$

Before we prove the theorem, let us make a few comments. The first remark is that, although the left-hand side of (37) depends in a manifestly linear way on f, it is not clear that the right-hand side does. This is however the case, as will be clear from the proof. This feature is shared by Eisenbaum's original statement, which we shall deduce from our theorem later.

A second remark is that one of the prominent features of Eisenbaum's theorem, namely the use of unconditioned measures on the trajectories of the Markov process, seems to have disappeared in our statement. We shall explain below how to fill this apparent gap between our formulation and Eisenbaum's original statement.

A third remark is that, in the right-hand side of (37), f can be taken out of the expectation, thus producing, thanks to (23), a statement of the form

$$\Delta_{h,H}^{-1}\Delta_h f = f + \dots$$

where the dots hide a non-obviously linear functional of f. Applying this formula not to f but to $\Delta_h^{-1}f$, we find

$$\Delta_{h,H}^{-1}f = \Delta_h^{-1}f + \frac{\mathbb{E}^h\big[e^{-\frac{1}{2}(\Phi+\Delta_h^{-1}f, H(\Phi+\Delta_h^{-1}f))_{\Omega^0}}\Phi\big]}{\mathbb{E}^h\big[e^{-\frac{1}{2}(\Phi+\Delta_h^{-1}f, H(\Phi+\Delta_h^{-1}f))_{\Omega^0}}\big]}, \tag{38}$$

the announced probabilistic expression of the difference between the inverses of two Laplacians.

Let us now turn to the proof of the theorem.

Proof. Using successively (23) and Proposition 5.1, we find

$$\left(\int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1})\, d\nu(\gamma)\Delta_h f\right)(x) = \Delta_{h,H}^{-1}(\Delta_h f)(x) = \mathbb{E}^{h,H}\left[(\Phi, \Delta_h f)_{\Omega^0}\Phi_x\right].$$

We now use the following Gaussian lemma. Let (X, Y) be an $(r+1)$-dimensional Gaussian vector such that X is r-dimensional and Y is 1-dimensional. Assume that the vector (X, Y) is symmetric, that is, centred in the real case, and, in the complex case, such that $(e^{i\theta}X, e^{i\theta}Y)$ has the same distribution as (X, Y) for every real θ. Then

$$\mathbb{E}[\overline{Y}X] = \frac{\mathbb{E}[e^{\Re Y}X]}{\mathbb{E}[e^{\Re Y}]}.$$

We apply this lemma $X = \Phi_x$ and $Y = (\Delta_h f, \Phi)_{\Omega^0}$. The Gaussian vector (X, Y) is symmetric because Φ and $-\Phi$ have the same distribution and, in the complex case, the same distribution as $e^{i\theta}\Phi$ for all real θ. We find

$$\left(\int_{\mathsf{P}(\mathsf{G})} \mathsf{hol}_{h,H}(\gamma^{-1})\, d\nu(\gamma)\Delta_h f\right)(x) = \frac{\mathbb{E}^{h,H}\left[e^{\Re(\Phi,\Delta_h f)_{\Omega^0}}\Phi_x\right]}{\mathbb{E}^{h,H}\left[e^{\Re(\Phi,\Delta_h f)_{\Omega^0}}\right]}.$$

Replacing, by an affine change of variables, Φ by $\Phi + f$ in the numerator and the denominator, we find that this quotient is equal to

$$\frac{\mathbb{E}^{h,H}\left[e^{-\Re(\Phi,Hf)_{\Omega^0}}(\Phi_x + f_x)\right]}{\mathbb{E}^{h,H}\left[e^{-\Re(\Phi,Hf)_{\Omega^0}}\right]} = \frac{\mathbb{E}^h\big[e^{-\frac{1}{2}(\Phi+f, H(\Phi+f))_{\Omega^0}}(\Phi_x + f_x)\big]}{\mathbb{E}^h\big[e^{-\frac{1}{2}(\Phi+f, H(\Phi+f))_{\Omega^0}}\big]},$$

as expected. □

The next lemma will help us to bridge the gap between Theorem 6.2 and Eisenbaum's original isomorphism. In order to state it, let us introduce a piece of notation.

For every continuous path γ, define

$$T_\gamma = \sup\{t \geq 0 : \gamma_t \in \mathsf{V}\} = \inf\{t \geq 0 : \gamma_t \in \mathsf{W}\},$$

the hitting time of the well by γ. For every vertex x, the path $\gamma_{|[0,T_\gamma]}$ is $\mathbb{P}_x$-almost surely a path in the proper subgraph $\mathsf{G}_{|\mathsf{V}}$. This path joins x to a point $\gamma_{T_\gamma^-}$ of $\partial\mathsf{V}$, the rim of the well.

Lemma 6.3. *Let x be a proper vertex of G. Let F be a non-negative measurable function on $\mathsf{P}(\mathsf{G})$. Then*

$$\sum_{y\in\mathsf{V}} \kappa_y \int_{\mathsf{P}(\mathsf{G})} F(\gamma)\, d\nu_{x,y}(\gamma) = \int_{\mathsf{P}(\mathsf{G})} F(\gamma_{|[0,T_\gamma]})\, d\mathbb{P}_x(\gamma).$$

Proof. Let us compute the left-hand side. It is equal to

$$\sum_{y\in\mathsf{V}} \frac{\kappa_y}{\lambda_y} \int_0^\infty e^{-t} \sum_{n=0}^\infty \sum_{(x,e_1,x_1,\dots,x_{n-1},e_n,y)\in\mathsf{DP}_n(\mathsf{G})} P_{x,e_1}P_{x_1,e_2}\dots P_{x_{n-1},e_n} \int_{0<t_1<\dots<t_n<t} F((x,t_1),e_1,(x_1,t_2-t_1),\dots,(y,t-t_n))\, dt_1\dots dt_n,$$

that is, to

$$\sum_{\substack{(x,e_1,\dots,y)\in\mathsf{DP}_n(\mathsf{G})\\ e\in\mathsf{E}:\underline{e}=y,\overline{e}\in\mathsf{W}}} \mathbb{Q}_x((x,e_1,\dots,y,e,\overline{e})) \int_{(0,+\infty)^{n+1}} F((x,\tau_0),e_1,\dots,(y,\tau_n)) e^{-\tau_0-\dots-\tau_n}\, d\tau_0\dots d\tau_n,$$

in which we recognise the right-hand side. □

Using this lemma, we will now state a corollary of Theorem 6.2 from which it will be easy to deduce the classical statement.

Let us introduce the operator K on $\Omega^0(\mathsf{V},\mathsf{F})$ defined by

$$(Kf)(x) = \kappa_x f(x).$$

Corollary 6.4. *Let F be a Hermitian vector bundle over a weighted graph with a well G. Let h be a connection on F and H a potential on F. Let x be a vertex of G. Let b be a section of F over $\partial\mathsf{V}$. Define a global section f of F over V by setting*

$$f = G_h K b.$$

Then the following identity holds in F_x:

$$\mathbb{E}^h \otimes \mathbb{E}_x \left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}} \mathsf{hol}_{h,H}(\gamma^{-1}_{|[0,T_\gamma]})(b) \right] = \mathbb{E}^h \left[e^{-\frac{1}{2}(\Phi+f,H(\Phi+f))_{\Omega^0}} (\Phi_x + f_x) \right]. \tag{39}$$

Proof. According to Lemma 6.3 and (34),

$$\mathbb{E}_x[\mathsf{hol}_{h,H}(\gamma^{-1}_{|[0,T_\gamma]})(b)] = (G_{h,H}Kb)(x) = \Delta^{-1}_{h,H}\Delta_h f,$$

which, given (23), is computed by Theorem 6.2. □

Just as in the case of Dynkin's isomorphism, this result can be specialised to the original version of Eisenbaum's isomorphism. Assume that H is scalar in each fibre F_x, equal to $H_x\mathrm{Id}_{\mathsf{F}_x}$. In that case, observing that the full trajectory of the random walk and the trajectory stopped at the hitting time of the well have the same local time at every vertex of V, (39) becomes

$$\mathbb{E}^h \otimes \mathbb{E}_x \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x(\|\Phi_x+f_x\|^2+\ell_x(X))} \mathsf{hol}_h(\gamma^{-1}_{|[0,T\gamma]})(b) \right] = \mathbb{E}^h \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x\|\Phi_x+f_x\|^2} (\Phi_x + f_x) \right].$$

Assume now that the fibre bundle is the trivial real bundle of rank 1 and the connection is the trivial connection. Let b be the section, that is, the function, which is identically equal to a real

number s on $\partial\mathsf{V}$. In that case, $f = GKb = sG\kappa = s\mathbb{1}_{\mathsf{V}}$, the function identically equal to s on V. Then the formula above becomes

$$\mathbb{E}^h \otimes \mathbb{E}_x \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x((\Phi_x+s)^2+\ell_x(X))} \right] s = \mathbb{E}^h \left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}} \lambda_x H_x(\Phi_x+s)^2} (\Phi_x + s) \right].$$

Provided s is not zero, dividing by s yields one of the classical formulations of Eisenbaum's isomorphism.

Let us finally explain how the results of this section can be understood in terms of the resolution of a Dirichlet problem in G.

Recall from the beginning of Section 3.9 that there exists an uncompressed Laplacian $\overline{\Delta}_h = d^* \circ d$ acting on the space $\Omega^0(\mathsf{U}, \mathsf{F})$ of all sections of F, even those that do not vanish on the well. Adding a potential H to this Laplacian, we obtain the operator $\overline{\Delta}_{h,H} = \overline{\Delta}_h + H$.

Suppose that we are given a section $w \in \Omega^0(\mathsf{W}, \mathsf{F})$ of F over the well and we want to solve the Dirichlet problem with this boundary condition, that is, to find a section $f \in \Omega^0(\mathsf{U}, \mathsf{F})$ of F such that

$$\begin{cases} \overline{\Delta}_{h,H} f = 0 & \text{on } \mathsf{V}, \\ f = w & \text{on } \mathsf{W}. \end{cases} \tag{40}$$

Assuming that H vanishes on the well as usual, the Laplacians $\Delta_{h,H}$ and $\overline{\Delta}_{h,H}$ agree everywhere but on the rim and for every $x \in \partial\mathsf{V}$, we have

$$\overline{\Delta}_{h,H} f(x) = \Delta_{h,H} f(x) - \sum_{\underline{e}=x, \overline{e}\in\mathsf{W}} h_e^{-1} f(\overline{e}).$$

Thus, f solves the Dirichlet problem (40) if and only if it is satisfies

$$\Delta_{h,H} f(x) = \begin{cases} 0 & \text{if } x \in \mathsf{V} \setminus \partial\mathsf{V}, \\ b & \text{if } x \in \partial\mathsf{V}, \end{cases} \tag{41}$$

where b is the section of F over $\partial\mathsf{V}$ defined by

$$b(x) = \sum_{\underline{e}=x, \overline{e}\in\mathsf{W}} h_e^{-1} w(\overline{e}).$$

It appears that solving the Dirichlet problem for the uncompressed Laplacian in the graph with a boundary condition on the well amounts to applying the inverse of the compressed Laplacian to a certain section of F over the rim.

It is a classical fact that the solution of the Dirichlet problem in a domain with a certain boundary condition is the average of the Gaussian free field on this domain conditioned to satisfy this boundary condition. The relation (38) extends this result in the following sense: provided one knows how to solve the Dirichlet problem associated with the Laplacian Δ_h for a certain boundary condition, this corollary gives an expression in terms of the Gaussian free field of the solution of the Dirichlet problem with the same boundary condition but for any of the generalised Laplacians $\Delta_{h,H}$.

7. Isomorphism theorems of Le Jan and Sznitman

In this section, we continue our investigation of the way in which classical isomorphism theorems can be extended to the covariant setting and we turn to Le Jan's and Sznitman's isomorphism theorems, which relate the magnitude of the Gaussian free field to local times of Poissonian ensembles of loops and paths.

Just as our approach to the generalisation of the isomorphism theorems of Dynkin and Eisenbaum was based on one main equality, namely (23), the results of this section will ultimately be

based on (24). In particular, the measure μ in its various forms (see Definition 2.8) will play a prominent role in this study.

However, (24) turns out to be more difficult to use than (23) and we use instead the traced version of (24), namely (25), which we recall here for the convenience of the reader:

$$\int_{\mathsf{L}(\mathsf{G})} \left(\operatorname{Tr} \mathsf{hol}_{h,H}(\gamma^{-1}) - \operatorname{Tr} \mathsf{hol}_h(\gamma^{-1})\right) d\mu^{\circledcirc}(\gamma) = \log \frac{\det \Delta_h}{\det \Delta_{h,H}}\,. \tag{42}$$

$$\sum_{x\in\mathsf{V}} \int_{\mathsf{L}(\mathsf{G})\times(0,+\infty)} \left(\operatorname{Tr}_{\mathsf{F}_x}\left(\mathsf{hol}_{h,H}(\gamma^{-1}_{|[0,t]})\right) - \operatorname{Tr}_{\mathsf{F}_x}\left(\mathsf{hol}_h(\gamma^{-1}_{|[0,t]})\right)\right) \mathbb{1}_{\{\gamma_t=x\}}\, d\mathbb{P}_x(\gamma)\frac{dt}{t} = \log \frac{\det \Delta_h}{\det \Delta_{h,H}}. \tag{43}$$

7.1. **Overview of the approach.** Let us explain, in the Euclidean case, and with Le Jan's theorem in mind, how we are going to use (43). Firstly, we are going to exponentiate it and recognise, in the right-hand side, the expectation $\mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi,H\Phi)_{\Omega^0}}\right]$. The problem is to interpret the left-hand side

$$\exp \int_{\mathsf{L}(\mathsf{G})} \left(\operatorname{Tr} \mathsf{hol}_{h,H}(\gamma^{-1}) - \operatorname{Tr} \mathsf{hol}_h(\gamma^{-1})\right) d\mu^{\circledcirc}(\gamma). \tag{44}$$

A probabilistic interpretation of this quantity relies on Campbell's formula for Poisson point processes, which goes as follows. Given a diffuse σ-finite Borel measure m on a Polish space $\mathcal{X}$, let us denote by X the Poisson point process on $\mathcal{X}$ with intensity m. Then the Campbell formula asserts that for every Borel function f on $\mathcal{X}$ such that $f \geq 1$,

$$\mathbb{E}_{\Pi_m}\left[\prod_{x\in X} f(x)\right] = \exp \int_{\mathcal{X}} (f-1)\, dm.$$

In order to put (43) in the form of Cambpell's formula, we will proceed as follows. Firstly, we will use the fact that the measure $\mu^{\circledcirc}$ is invariant under the the path reversal map $\gamma \mapsto \gamma^{-1}$ (see Lemma 2.10) and the fact that the twisted holonomy is turned into its adjoint by composition by the same map (see (12)), to say that the imaginary part of the integral in (43) vanishes and that we can replace traces by their real parts.

The second step one would like to take is a simple algebraic manipulation leading to the form

$$\exp \int_{\mathsf{L}(\mathsf{G})} \left(\frac{\Re\operatorname{Tr} \mathsf{hol}_{h,H}(\gamma^{-1})}{\Re\operatorname{Tr} \mathsf{hol}_h(\gamma^{-1})} - 1\right) d(\Re\operatorname{Tr} \mathsf{hol}_h(\gamma^{-1})\mu^{\circledcirc}). \tag{45}$$

This form has two drawbacks. The first is that the quotient of real part of traces is difficult to interpret. With Le Jan's work [LJ11] in mind, we would like to understand it as the exponential of some linear functional of H and the local time of γ. However, the non-commutativity of the present setting makes it difficult to extract from this quotient the contribution of H. The second drawback is the fact that we are now integrating with respect to a signed measure.

We will not offer any serious alternative to the problem of working with a signed measure: we will simply write it as the difference of two positive measures, and use Campbell's formula for each of them. We will thus obtain a quotient of two instances of Campbell's formula.

On the other hand, we offer a solution to the problem of the interpretation of the quotient of real parts of traces of holonomies. It consists in lifting the integral from the space of loops to a larger space, on which holonomies, or rather quotients of holonomies become scalar quantities. This requires a detailed explanation, which will occupy us for the next few sections.

7.2. **Splitting of vector bundles.** The main new piece of structure that we need in our study of Le Jan's and Sznitman's isomorphisms is a decomposition of each fibre of the vector bundle F as an orthogonal direct sum of linear subspaces.

Definition 7.1. *Let G be a graph. Let F be a vector bundle over G. A* splitting *of F, or* colouring *of F, is a collection $\mathbb{I} = \{(I_x, (\mathsf{F}_x^i)_{i\in I_x}) : x \in \mathsf{V}\}$ in which, for each $x \in \mathsf{V}$, I_x is a set and $\{\mathsf{F}_x^i : i \in I_x\}$ is a family of pairwise orthogonal linear subspaces of F_x such that*

$$\mathsf{F}_x = \bigoplus_{i\in I_x}^{\perp} \mathsf{F}_x^i \,. \tag{46}$$

For each $x \in \mathsf{V}$ and $i \in I_x$, we denote by $\pi_x^i : \mathsf{F}_x \to \mathsf{F}_x^i$ the orthogonal projection.

A special case of splitting is the trivial splitting, in which each fibre of F is simply written as being equal to itself. As trivial as it is, this splitting will be useful for us, and we will denote it by $\mathbb{T} = \{(\{x\}, \mathsf{F}_x) : x \in \mathsf{V}\}$.

Another special case is the case of complete splittings, in which each fibre is written as an orthogonal direct sum of lines.

There is a natural partial ordering of the set of all splittings of a fibre bundle : we say that the splitting $\mathbb{I}$ is finer than the splitting $\mathbb{I}'$ if for each vertex x and each $i \in I_x$, there exists $i' \in I'_x$ such that $\mathsf{F}_x^i \subset \mathsf{F}_x^{i'}$. The trivial splitting is the maximum of this order, and the complete splittings are its minimal elements.

We say that a splitting $\mathbb{I}$ and a potential H on F are *adapted* to each other if for all $x \in \mathsf{V}$ and all $i \in I_x$, the space F_x^i is an eigenspace of H_x. In this case, we will denote by H_x^i the unique eigenvalue of the restriction of H_x to F_x^i, so that for all $x \in \mathsf{V}$, we have

$$H_x = \sum_{i\in I_x} H_x^i \pi_x^i.$$

For example, to say that a potential is adapted to the trivial splitting means that it is scalar on each fibre. This is a kind of potential that we already considered twice, in order to specialise Theorems 6.1 and 6.2 respectively to the classical Dynkin and Eisenbaum isomorphisms.

To every potential H we can associate its *eigensplitting*, the splitting of F obtained by writing each fibre F_x as the direct sum of the eigenspaces of H_x. This splitting is of course adapted to H and it is, among all splittings adapted to H, the one that is maximal for the partial order that we just described.

7.3. **Splitting of the Gaussian free field.** A splitting of F allows us, among other things, to split the Gaussian free field on G.

Let h be a connection and H be a potential on our vector bundle F over G. Let Φ be the Gaussian free field on F associated to h and H. Let $\mathbb{I}$ be a splitting of F. For each vertex x and each $i \in I_x$, we define

$$\Phi_x^i = \pi_x^i(\Phi_x). \tag{47}$$

We meet here for the first time a new kind of field, namely $(\Phi_x^i)_{x\in\mathsf{V}, i\in I_x}$, indexed not only by the vertices of G, but also, at each vertex x, by elements of I_x, which we will call *colours*. Without giving too precise a meaning to this term, we will speak of a coloured field.

This coloured field $(\Phi_x^i)_{x\in\mathsf{V}, i\in I_x}$ is a Gaussian random element of the direct sum $\bigoplus_{x\in\mathsf{V}, i\in I_x} \mathsf{F}_x^i$. It is centred, and an application of (30) allows us to compute its covariance: for all $x, y \in \mathsf{V}$ and all $i \in I_x$, $j \in I_y$,

$$\mathbb{E}^{h,H}\big[\Phi_x^i \otimes \overline{\Phi_y^j}\big] = \pi_x^i \circ (G_{h,H})_{x,y} \circ \pi_y^j, \tag{48}$$

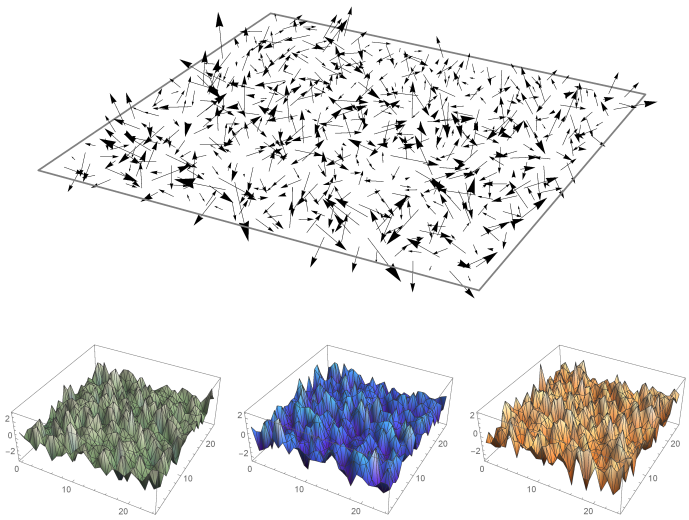

FIGURE 7. A sample of a covariant Gaussian free field Φ in a trivial real bundle of rank 3 along with its components Φ^1, Φ^2, Φ^3 (interpolated linearly). The connection was chosen independently on each edge and uniformly from the orthogonal group $O(3, \mathbb{R})$. The underlying graph is a 20×20 square grid where the rim of the well is the external boundary of the grid.

an equality to be read in $\mathrm{Hom}(\mathsf{F}_y^j, \mathsf{F}_x^i)$.

The case where $\mathbb{I}$ is a complete splitting is of particular interest. Let us consider this case, and let us also assume that the set of colours I_x is the same for each vertex $x \in \mathsf{V}$, namely $I_x = \{1, \dots, r\}$. Let us finally choose a unit vector u_x^i in each line F_x^i. Then, for each $i \in \{1, \dots, r\}$, we can define a scalar random field Φ^i on V by setting, for all $x \in \mathsf{V}$, and with a conflict of notation that we do not deem too serious,

$$\Phi_x^i = \langle u_x^i, \Phi_x \rangle_x .$$

Each of the r scalar fields $\Phi^i, i \in \{1, \dots, r\}$ is a centred Gaussian field and for all $i \in \{1, \dots, r\}$, the covariance of the scalar field Φ^i is given, for all $x, y \in \mathsf{V}$, by

$$\mathbb{E}^{h,H}\left[\Phi_x^i \overline{\Phi_y^i}\right] = \langle u_x^i, \left(G_{h,H}\right)_{x,y} u_y^i \rangle_x \, .$$

Although this is clear from (48), we would like to stress that the fields $\Phi^1, \dots, \Phi^r$ are correlated, due to the presence of the connection h and the potential H (see Figure 7).

7.4. **Coloured paths, coloured loops and coloured local time.** We are now going to explain how the choice of a splitting of the vector bundle allows us to define an enriched version of the space of loops on the graph. From the point of view of paths that we will adopt now, we prefer to think of each subspace of a fibre of F as a colour, and of the splitting itself as a colouring.

Let $\mathbb{I}$ be a colouring of F, that is, a splitting of F. We define the space $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$ of $\mathbb{I}$-*coloured paths* on G as the set of all sequences

$$\eta = ((x_0, \tau_0, i_0), e_1, (x_1, \tau_1, i_1), \dots, e_n, (x_n, \tau_n, i_n))$$

such that

$$\gamma = ((x_0, \tau_0), e_1, (x_1, \tau_1), \dots, e_n, (x_n, \tau_n))$$

is an element of $\mathsf{P}(\mathsf{G})$ and, for each $k \in \{0, \dots, n\}$, we have $i_k \in I_{x_k}$.

The coloured path η will be said to have the colour i_k when it visits the vertex x_k. Note that a coloured path can have different colours at successive visits of the same vertex.

A $\mathbb{T}$-coloured path, where $\mathbb{T}$ is the trivial colouring, is nothing but a path which, at each vertex that it visits, has the only existing colour at that vertex. Accordingly, we will identify freely the spaces $\mathsf{P}^{\mathbb{T}}(\mathsf{G})$ and $\mathsf{P}(\mathsf{G})$.

We say that the coloured path η written above is a coloured loop if $x_n = x_0$ and $i_n = i_0$. We denote by $\mathsf{L}^{\mathbb{I}}(\mathsf{G})$ the set of coloured loops on G.

By analogy with Definition 3.7, we define, for all $x \in \mathsf{V}$ and all $i \in I_x$, the occupation measure and the local time of η with colour i at the vertex $x \in \mathsf{V}$ by

$$\vartheta_x^i(\eta) = \sum_{k=0}^{n} \delta_{x,x_k} \delta_{i,i_k} \tau_k \quad \text{and} \quad \ell_x^i(\eta) = \frac{1}{\lambda_x} \vartheta_x^i(\gamma).$$

In our naive quantum mechanical picture of paths on the graph, a coloured path describes not only the motion of a particle, but the successive states in which this particle can be found at the successive vertices that it visits. If the splitting that we are considering is the eigensplitting of a potential H, then these states can be measured at each vertex x by the observable H_x.

Let us conclude this section by saying something about the way in which the set of $\mathbb{I}$-coloured paths depends on the colouring $\mathbb{I}$. Let us consider two colourings $\mathbb{I}$ and $\mathbb{I}'$ of F such that $\mathbb{I}$ is finer than $\mathbb{I}'$. Extending the metaphor of colours, there is a bleaching map $\mathsf{b}_{\mathbb{I}',\mathbb{I}} : \mathsf{P}^{\mathbb{I}}(\mathsf{G}) \to \mathsf{P}^{\mathbb{I}'}(\mathsf{G})$ which, at each vertex x visited by a path, changes the colour $i \in I_x$ into the unique colour $i' \in I'_x$ such that $\mathsf{F}_x^i \subset \mathsf{F}_x^{i'}$. If $\mathbb{I}''$ is a third colouring such that $\mathbb{I}'$ is finer than $\mathbb{I}''$, then the relation $\mathsf{b}_{\mathbb{I}'',\mathbb{I}'} \circ \mathsf{b}_{\mathbb{I}',\mathbb{I}} = \mathsf{b}_{\mathbb{I}'',\mathbb{I}}$ holds.

The special case where $\mathbb{I}'$ is the trivial colouring will be of particular interest, and we will use the simpler notation $[\eta] = \mathsf{b}_{\mathbb{T},\mathbb{I}}(\eta)$ for the element of $\mathsf{P}(\mathsf{G})$ obtained by forgetting altogether the colours of an $\mathbb{I}$-coloured path η.

7.5. **Amplitudes.** We will now use the colourings of a path to decompose the trace of the holonomy along that path, and ultimately to simplify the expression (45).

Let $\eta = ((x_0, \tau_0, i_0), e_1, (x_1, \tau_1, i_1), \dots, e_n, (x_n, \tau_n, i_n))$ be an $\mathbb{I}$-coloured path. Recall that we chose a connection h and a potential H on the vector bundle F. We define the *amplitude* of h along the $\mathbb{I}$-coloured path η twisted by H as the operator

$$\mathsf{amp}_{h,H}^{\mathbb{I}}(\eta) = \pi_{x_n}^{i_n} \circ e^{-\tau_n H_{x_n}} \circ h_{e_n} \circ \dots \circ \pi_{x_1}^{i_1} \circ e^{-\tau_1 H_{x_1}} \circ h_{e_1} \circ \pi_{x_0}^{i_0} \circ e^{-\tau_0 H_{x_0}} : \mathsf{F}_{x_0}^{i_0} \to \mathsf{F}_{x_n}^{i_n}.$$

This definition is designed for the following equality to hold: for every (non-coloured) path γ,

$$\sum_{\eta \in \mathsf{P}^{\mathbb{I}}(\mathsf{G}) : [\eta] = \gamma} \mathsf{amp}_{h,H}^{\mathbb{I}}(\eta) = \mathsf{hol}_{h,H}(\gamma)\,. \tag{49}$$

More generally, if $\mathbb{I}$ is finer than $\mathbb{I}'$, then for all $\eta' \in \mathsf{P}^{\mathbb{I}'}(\mathsf{G})$

$$\sum_{\eta \in \mathsf{P}^{\mathbb{I}}(\mathsf{G}) : \mathsf{b}_{\mathbb{I}',\mathbb{I}}(\eta) = \eta'} \mathsf{amp}_{h,H}^{\mathbb{I}}(\eta) = \mathsf{amp}_{h,H}^{\mathbb{I}'}(\eta')\,. \tag{50}$$

Note that for every $\mathbb{I}$-coloured path η, we have

$$\mathsf{amp}_{h,H}^{\mathbb{I}}(\eta^{-1}) = \mathsf{amp}_{h,H}^{\mathbb{I}}(\eta)^*, \tag{51}$$

the adjoint of $\mathsf{amp}_{h,H}^{\mathbb{I}}(\eta)$. In particular, if η is a coloured loop, then the traces $\operatorname{Tr} \mathsf{amp}_{h,H}^{\mathbb{I}}(\eta)$ and $\operatorname{Tr} \mathsf{amp}_{h,H}^{\mathbb{I}}(\eta^{-1})$ are conjugate complex numbers.

One of the reasons to introduce amplitudes is the following: if H and $\mathbb{I}$ are adapted, then

$$\mathsf{amp}^{\mathbb{I}}_{h,H}(\eta) = e^{-\sum_{x\in\mathsf{V},i\in I_x} H^i_x \vartheta^i_x(\eta)} \mathsf{amp}^{\mathbb{I}}_h(\eta), \tag{52}$$

a formula analogous to (13), but which holds without the assumption that H be scalar in each fibre. This indicates that, after lifting (45) from $\mathsf{L}(\mathsf{G})$ to $\mathsf{L}^{\mathbb{I}}(\mathsf{G})$, it will be much easier to deal with the quotient of holonomies, which will become a quotient of amplitudes. Before we implement this lifting procedure, we must say a word about the measures, indeed the signed measures that we will use on $\mathsf{L}^{\mathbb{I}}(\mathsf{G})$ and $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$.

7.6. **Signed measures on the sets of coloured loops and paths.** In this section, we will define two families of measures on sets of coloured paths. Firstly, we will define a signed measure $\mu^{\circledcirc,\mathbb{I}}_h$ on the set of $\mathbb{I}$-coloured loops, which in a sense lifts a measure on the set of loops absolutely continuous with respect $\mu^{\circledcirc}$ (see Definition 2.8), with a density depending on the connection h. Then, for every section f of F, we will define a signed measure $\nu^{\mathbb{I}}_{h,f}$ on the set of $\mathbb{I}$-coloured paths, which in a similar sense lifts and generalises the measures $\nu_{x,y}$ (see Definition 2.7).

Definition 7.2. *Let h be a connection on F. Let $\mathbb{I}$ be a colouring of F. The measure $\mu^{\circledcirc,\mathbb{I}}_h$ is defined on $\mathsf{L}^{\mathbb{I}}(\mathsf{G})$ by the fact that for all bounded non-negative measurable function F on $\mathsf{L}^{\mathbb{I}}(\mathsf{G})$,*

$$\int_{\mathsf{L}^{\mathbb{I}}(\mathsf{G})} F(\eta)\, d\mu^{\circledcirc,\mathbb{I}}_h(\eta) = \int_{\mathsf{L}(\mathsf{G})} \sum_{\eta\in\mathsf{L}^{\mathbb{I}}(\mathsf{G}):[\eta]=\gamma} F(\eta) \Re\mathrm{Tr}(\mathsf{amp}^{\mathbb{I}}_h(\eta^{-1}))\, d\mu^{\circledcirc}(\gamma).$$

To make it clear that the integral above exists, we can rewrite it as

$$\int_{\mathsf{L}(\mathsf{G})} \sum_{\eta\in\mathsf{L}^{\mathbb{I}}(\mathsf{G}):[\eta]=\gamma} F(\eta) \Re\mathrm{Tr}(\mathsf{amp}^{\mathbb{I}}_h(\eta^{-1}))\, d\mu^{\circ}(\gamma) + \sum_{x\in\mathsf{V}} \int_0^{+\infty} \sum_{i\in I_x} F((x,t,i)) e^{-t} \frac{dt}{t}, \tag{53}$$

so that the measure $\mu^{\circledcirc,\mathbb{I}}_h$ appears as the sum of an honest signed measure and a σ-finite positive measure.

In the particular case of the trivial splitting, the formula above defines a measure $\mu^{\circledcirc,\mathbb{T}}_h$ on $\mathsf{L}(\mathsf{G})$, which we will denote simply by $\mu^{\circledcirc}_h$ and which can be written as

$$d\mu^{\circledcirc}_h(\gamma) = \Re\mathrm{Tr}(\mathsf{hol}_h(\gamma^{-1}))\, d\mu^{\circledcirc}(\gamma).$$

The next proposition tells us how to lift to the space of coloured loops the integrals we are interested in.

Proposition 7.3. *Let F be a bounded non-negative function on $\mathsf{L}(\mathsf{G})$. Then*

$$\int_{\mathsf{L}(\mathsf{G})} F(\gamma)\, d\mu^{\circledcirc}_h(\gamma) = \int_{\mathsf{L}^{\mathbb{I}}(\mathsf{G})} F([\eta])\, d\mu^{\circledcirc,\mathbb{I}}_h(\eta). \tag{54}$$

Assume that $F(\gamma) = F(\gamma^{-1})$ for every loop γ. Then

$$\int_{\mathsf{L}(\mathsf{G})} F(\gamma) \mathrm{Tr}(\mathsf{hol}_{h,H}(\gamma^{-1}))\, d\mu^{\circledcirc}(\gamma) = \int_{\mathsf{L}^{\mathbb{I}}(\mathsf{G})} F([\eta]) e^{-\sum_{x\in\mathsf{V},i\in I_x} H^i_x \vartheta^i_x(\eta)}\, d\mu^{\circledcirc,\mathbb{I}}_h(\eta). \tag{55}$$

Proof. The first assertion is a direct consequence of the definition of $\mu^{\circledcirc,\mathbb{I}}_h$ and (49).

For the second assertion, we start from the right-hand side, apply the definition of $\mu^{\circledcirc,\mathbb{I}}_h$, use (52) and then (49). This yields the real part of the left-hand side of the equality that we wish to prove. To see that this left-hand side is indeed real, we use Lemma 2.10, which asserts that the measure $\mu^{\circledcirc}$ is invariant under the involution $\gamma \mapsto \gamma^{-1}$, and (51). □

Note that we can use (50) to generalise (54): if $\mathbb{I}$ is finer than $\mathbb{I}'$, then for every bounded non-negative measurable function on $\mathsf{L}^{\mathbb{I}'}(\mathsf{G})$, we have

$$\int_{\mathsf{L}^{\mathbb{I}}(\mathsf{G})} F \circ \mathsf{b}_{\mathbb{I}',\mathbb{I}} \, d\mu^{\circledast,\mathbb{I}} = \int_{\mathsf{L}^{\mathbb{I}'}(\mathsf{G})} F \, d\mu^{\circledast,\mathbb{I}'}. \tag{56}$$

In the statement of our main result, we will make use of the positive and negative parts of the measure $\mu_h^{\circledast,\mathbb{I}}$, which we will denote respectively by $\mu_h^{\circledast,\mathbb{I},+}$ and $\mu_h^{\circledast,\mathbb{I},-}$. Note that $\mu_h^{\circledast,\mathbb{I},+}$ contains the second term of the right-hand side of (53), corresponding to constant coloured loops, and is thus an infinite positive measure, whereas $\mu_h^{\circledast,\mathbb{I},-}$ is a finite positive measure supported by the set of non-constant coloured loops.

Let us mention a delicate point which arises from the fact that we are using signed measures. Although (56) expresses the compatibility of the family of measures $\mu_h^{\circledast,\mathbb{I}}$ with the bleaching maps $\mathsf{b}_{\mathbb{I}',\mathbb{I}}$, it is not the case that the measures $\mu_h^{\circledast,\mathbb{I},+}$, nor the measures $\mu_h^{\circledast,\mathbb{I},-}$ satisfy the same compatibility relations: in symbols, we have in general

$$\mu_h^{\circledast,\mathbb{I},+} \circ \mathsf{b}_{\mathbb{I}',\mathbb{I}}^{-1} \neq \mu^{\circledast,\mathbb{I}',+} \quad \text{and} \quad \mu_h^{\circledast,\mathbb{I},-} \circ \mathsf{b}_{\mathbb{I}',\mathbb{I}}^{-1} \neq \mu^{\circledast,\mathbb{I}',-},$$

but

$$(\mu_h^{\circledast,\mathbb{I},+} - \mu_h^{\circledast,\mathbb{I},-}) \circ \mathsf{b}_{\mathbb{I}',\mathbb{I}}^{-1} = (\mu^{\circledast,\mathbb{I}',+} - \mu^{\circledast,\mathbb{I}',-}).$$

This is because the amplitudes of all the $\mathbb{I}$-colourings of a path compatible with a given $\mathbb{I}'$-colouring of the same path need not have traces of the same sign. See Figure 8 below for a graphical explanation of this phenomenon in the case where $\mathbb{I}'$ is the trivial colouring. We shall come back to this point after the proof of Theorem 7.6 below.

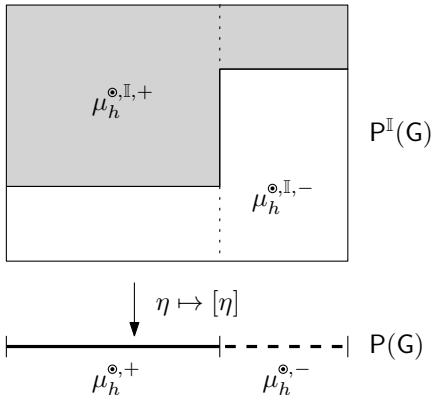

FIGURE 8. This schematical picture shows the supports of the positive and negative parts of the measures that we introduced on $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$ and $\mathsf{P}(\mathsf{G})$.

Let us turn to the definition of the second family of measures announced at the beginning of this section. It is a family of measures on the set of coloured paths that are not necessarily loops.

Definition 7.4. *Let h be a connection on F. Let $\mathbb{I}$ be a colouring of F. Let $f \in \Omega^0(\mathsf{V},\mathsf{F})$ be a section of F. The measure $\nu_{h,f}^{\mathbb{I}}$ is the measure on $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$ such that for all non-negative measurable function F on $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$,*

$$\int_{\mathsf{P}^{\mathbb{I}}(\mathsf{G})} F(\eta) \, d\nu_{h,f}^{\mathbb{I}}(\eta) = \int_{\mathsf{P}(\mathsf{G})} \sum_{\eta \in \mathsf{P}^{\mathbb{I}}(\mathsf{G}) : [\eta] = \gamma} \lambda_{\underline{\gamma}} F(\eta) \Re\big\langle (\Delta_h f)_{\underline{\gamma}}, \mathsf{amp}_h^{\mathbb{I}}(\eta^{-1})(\Delta_h f)_{\overline{\gamma}} \big\rangle_{\underline{\gamma}} \, d\nu(\gamma). \tag{57}$$

Since ν, in contrast to $\mu^{\circledcirc}$, is a finite measure, $\nu^{\mathbb{I}}_{h,f}$ is a genuine signed measure, with finite positive and negative parts.

Applied to the trivial splitting, this definition yields a measure $\nu^{\mathbb{T}}_{h,f}$ on $\mathsf{P}(\mathsf{G})$ which we denote simply by

$$d\nu_{h,f}(\gamma) = \lambda_{\underline{\gamma}} \Re\big\langle (\Delta_h f)(\underline{\gamma}), \mathsf{hol}_h(\gamma^{-1})(\Delta_h f)(\overline{\gamma})\big\rangle_{\underline{\gamma}}\, d\nu(\gamma).$$

Recall from Definition 2.7 that the definition of ν involves the value of λ at the final point of the path, so that the equation above is more symmetric than it looks, and could be written as

$$d\nu_{h,f}(\gamma) = \sum_{x,y\in\mathsf{V}} \lambda_x \lambda_y \Re\big\langle (\Delta_h f)(x), \mathsf{hol}_h(\gamma^{-1})(\Delta_h f)(y)\big\rangle_x\, d\nu_{x,y}(\gamma).$$

We have for the measures $\nu^{\mathbb{I}}_{h,f}$ the following formulas, analogous to those of Proposition 7.3.

Proposition 7.5. *Let F be a non-negative measurable function on $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$. Then*

$$\int_{\mathsf{P}(\mathsf{G})} F(\gamma)\, d\nu_{h,f}(\gamma) = \int_{\mathsf{P}^{\mathbb{I}}(\mathsf{G})} F([\eta])\, d\nu^{\mathbb{I}}_{h,f}(\eta) \tag{58}$$

and

$$\int_{\mathsf{P}(\mathsf{G})} \lambda_{\underline{\gamma}} F(\gamma) \big\langle (\Delta_h f)(\underline{\gamma}), \mathsf{hol}_{h,H}(\gamma^{-1})(\Delta_h f)(\overline{\gamma})\big\rangle_{\underline{\gamma}}\, d\nu(\gamma) = \int_{\mathsf{P}^{\mathbb{I}}(\mathsf{G})} F([\eta]) e^{-\sum_{x\in\mathsf{V}, i\in I_x} H^i_x \vartheta^i_x(\eta)}\, d\nu^{\mathbb{I}}_{h,f}(\eta). \tag{59}$$

Proof. The proof uses almost exactly the same arguments as the proof of Proposition 7.3. The only difference is the fact that the measure ν is not invariant under the map $\gamma \mapsto \gamma^{-1}$, but the measure $\lambda_{\underline{\gamma}} d\nu(\gamma)$, that is, the measure $\sum_{x,y\in\mathsf{V}} \lambda_x \lambda_y \nu_{x,y}$, is invariant. □

In the next section, we will make use of the positive and negative parts of this measure, which we will denote respectively by $\nu^{\mathbb{I},+}_{h,f}$ and $\nu^{\mathbb{I},-}_{h,f}$. A simple yet useful observation is that if f is the zero section, then these measures are the null measures.

The measures $\nu^{\mathbb{I}}_{h,f}$, where h and f are fixed and $\mathbb{I}$ varies in the set of splittings of the vector bundle F, satisfy the exact same compatibility relations with respect to the bleaching maps as the measures $\mu^{\circledcirc,\mathbb{I}}_h$, and the same precautions are in order when one considers the behaviour of the positive and negative parts of these measures with respect to the bleaching maps.

7.7. **Le Jan and Sznitman's isomorphisms.** Let us introduce a last piece of notation. Let $\mathbb{I}$ be a colouring of the fibre bundle F. Let $\mathcal{P}$ be a set of $\mathbb{I}$-coloured paths. We denote by $\ell^{\mathbb{I}}(\mathcal{P})$ the collection

$$\ell^{\mathbb{I}}(\mathcal{P}) = (\ell^i_x(\mathcal{P}))_{x\in\mathsf{V}, i\in I_x},$$

where for all $x \in \mathsf{V}$ and all $i \in I_x$,

$$\ell^i_x(\mathcal{P}) = \sum_{\eta\in\mathcal{P}} \ell^i_x(\eta).$$

Let now f be a section of F. We denote by $\|\cdot\|^{2,\mathbb{I}}(f)$ the collection

$$\|\cdot\|^{2,\mathbb{I}}(f) = (\|\pi^i_x(f)\|^2_x)_{x\in\mathsf{V}, i\in I_x}.$$

In the informal terminology that we introduced in Section 7.3, both $\ell^{\mathbb{I}}(\mathcal{P})$ and $\|\cdot\|^{2,\mathbb{I}}(f)$ are scalar coloured fields. The following theorem, which generalises the isomorphism theorems of Le Jan and Sznitman, states the equality in distribution of random scalar coloured fields of this nature.

Theorem 7.6. *Let G be a weighted graph with a well. Let F be a vector bundle over G. Let h be a connection on F. Let Φ be the Gaussian free field on G associated with h. Let $\mathbb{I}$ be a colouring of F. Let $f \in \Omega^0(\mathsf{V},\mathsf{F})$ be a section of F. Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be Poissonian ensembles in $\mathsf{L}^{\mathbb{I}}(\mathsf{G})$ with respective intensities $\frac{\beta}{2}\mu_h^{\circledast,\mathbb{I},+}$ and $\frac{\beta}{2}\mu_h^{\circledast,\mathbb{I},-}$. Let $\mathcal{E}_+$ and $\mathcal{E}_-$ be Poissonian ensembles in $\mathsf{P}^{\mathbb{I}}(\mathsf{G})$ with respective intensities $\frac{\beta}{2}\nu_{h,f}^{\mathbb{I},+}$ and $\frac{\beta}{2}\nu_{h,f}^{\mathbb{I},-}$. Assume that $\Phi, \mathcal{L}_+, \mathcal{L}_-, \mathcal{E}_+, \mathcal{E}_-$ are independent. Then the following equality holds in distribution:*

$$\ell^{\mathbb{I}}(\mathcal{L}_+ \cup \mathcal{E}_+) \overset{(d)}{=} \frac{1}{2}\|\cdot\|^{2,\mathbb{I}}(\Phi + f) + \ell^{\mathbb{I}}(\mathcal{L}_- \cup \mathcal{E}_-). \tag{60}$$

To be clear, the conclusion of the theorem is that the two random vectors

$$\{\ell_x^i(\mathcal{L}_+ \cup \mathcal{E}_+) : x \in \mathsf{V}, i \in I_x\} \text{ and } \left\{\frac{1}{2}\|\pi_x^i(\Phi_x + f_x)\|_x^2 + \ell_x^i(\mathcal{L}_- \cup \mathcal{E}_-) : x \in \mathsf{V}, i \in I_x\right\}$$

have the same distribution.

Proof. We start by proving the theorem in the case where $f = 0$. In this case, the ensembles $\mathcal{E}_+$ and $\mathcal{E}_-$ are almost surely empty.

Let us choose a potential H adapted to $\mathbb{I}$ and such that H_x is non-negative for all $x \in \mathsf{V}$. We start from (25), which we multiply by $\frac{\beta}{2}$ and of which we exponentiate both sides. On the right-hand side we find

$$\left(\frac{\det \Delta_h}{\det \Delta_{h,H}}\right)^{\frac{\beta}{2}} = \mathbb{E}^h\left[e^{-\frac{1}{2}(\Phi, H\Phi)_{\Omega^0}}\right].$$

On the left-hand side, we find, thanks to (55) applied once to the zero potential and once to the potential H,

$$\exp\frac{\beta}{2}\int_{\mathsf{L}^{\mathbb{I}}(\mathsf{G})}\left(e^{-\sum_{x\in\mathsf{V}, i\in I_x} H_x^i \vartheta_x^i(\eta)} - 1\right)\, d\mu_h^{\circledast,\mathbb{I}}(\eta).$$

Splitting the measure $\mu_h^{\circledast,\mathbb{I}}$ into its positive and negative parts and writing Campbell's formula for each part, we find that

$$\mathbb{E}\left[e^{-\sum_{x\in\mathsf{V}, i\in I_x} \lambda_x H_x^i \ell_x^i(\mathcal{L}_+)}\right] = \mathbb{E}^h\left[e^{-\frac{1}{2}\sum_{x\in\mathsf{V}, i\in I_x} \lambda_x H_x^i \|\pi_x^i(\Phi_x)\|_x^2}\right]\mathbb{E}\left[e^{-\sum_{x\in\mathsf{V}, i\in I_x} \lambda_x H_x^i \ell_x^i(\mathcal{L}_-)}\right], \tag{61}$$

where the first expectation (resp. the last) is taken with respect to the distribution of the Poisson point process with intensity $\frac{\beta}{2}\mu_h^{\circledast,\mathbb{I},+}$ (resp. $\frac{\beta}{2}\mu_h^{\circledast,\mathbb{I},-}$). Since this equality holds for arbitrary non-negative values of the numbers H_x^i, it says exactly that the Laplace transforms of both sides of (60) are equal.

We now turn to the proof of the general case. We proceed again to show the equality of the Laplace transforms of both sides of (60). Combining the special case of the theorem that we just proved, where $f = 0$, and Proposition 5.5, which was precisely designed to be used here, we conclude that it suffices to prove the equality

$$\mathbb{E}\left[e^{-\sum_{x\in\mathsf{V}, i\in I_x} \lambda_x H_x^i \ell_x^i(\mathcal{E}_+)}\right] = e^{\frac{1}{2}\left(\Delta_h f, (\Delta_{h,H}^{-1} - \Delta_h^{-1})\Delta_h f\right)_{\Omega^0}}\mathbb{E}\left[e^{-\sum_{x\in\mathsf{V}, i\in I_x} \lambda_x H_x^i \ell_x^i(\mathcal{E}_-)}\right].$$

Following backwards the same arguments that we used in the first part of this proof, we see that we must prove that

$$\int_{\mathsf{P}^{\mathbb{I}}(\mathsf{G})}\left(e^{-\sum_{x\in\mathsf{V}, i\in I_x} H_x^i \vartheta_x^i(\eta)} - 1\right)\, d\nu_{h,f}^{\mathbb{I}}(\eta) = \left(\Delta_h f, (\Delta_{h,H}^{-1} - \Delta_h^{-1})\Delta_h f\right)_{\Omega^0}. \tag{62}$$

By (23) and Definition 2.7, the right-hand side is equal to

$$\sum_{x,y\in\mathsf{V}} \lambda_x\lambda_y \int_{\mathsf{P}(\mathsf{G})} \left\langle (\Delta_h f)(x), (\mathsf{hol}_{h,H}(\gamma^{-1}) - \mathsf{hol}_h(\gamma^{-1}))(\Delta_h f)(y)\right\rangle_x d\nu_{x,y}(\gamma). \tag{63}$$

By (59), this is equal to the left-hand side of (62), as expected. □

Just as for the theorems of Dynkin and Eisenbaum, the classical theorems of Le Jan and Sznitman correspond to the special case of our theorem where $\mathsf{F} = \mathbb{R}_\mathsf{G}$ is the trivial real vector bundle of rank 1 endowed with the trivial connection. In that case, all measures are positive, and the ensembles $\mathcal{L}_-$ and $\mathcal{E}_-$ are almost surely empty. Let us write $\mathcal{L} = \mathcal{L}_+$ and $\mathcal{E} = \mathcal{E}_+$.

When $f = 0$, we recover Le Jan's theorem, which asserts, in our notation, that $\ell(\mathcal{L})$ has the same distribution as $\frac{1}{2}\Phi^2$. When f is constant equal to a real $\sqrt{2s}$, we have $\Delta f = \sqrt{2s}\frac{\kappa}{\lambda}$, and the measure $\frac{\beta}{2}\nu^{\mathbb{I}}_{h,f}$ becomes simply

$$s\sum_{x,y\in\mathsf{V}} \kappa_x\kappa_y\nu_{x,y}.$$

Then we recover Sznitman's theorem which asserts that $\ell(\mathcal{L}\cup\mathcal{E})$ has the same distribution as $\frac{1}{2}(\Phi + \sqrt{2s})^2$.

It is tempting to compare the conclusions of Theorem 7.6 for two different colourings $\mathbb{I}$ and $\mathbb{I}'$ such that $\mathbb{I}$ is finer than $\mathbb{I}'$. It seems that of the two statements associated with two such colourings, no one is logically stronger than the other.

Let us discuss without proof the case where $\mathbb{I}$ is a complete splitting such that $I_x = \{1,\dots,r\}$ for every vertex x, and $\mathbb{I}'$ the trivial splitting. Let us also assume, for the sake of simplicity, that $f = 0$. Then, using the equality $\|\Phi^1_x\|^2_x + \ldots + \|\Phi^r_x\|^2_x = \|\Phi_x\|^2_x$ for every x, one can prove the equality in distribution

$$\left\{\sum_{i=1}^r \ell^i_x(\mathcal{L}^{\mathbb{I}}_+) + \ell_x(\mathcal{L}_-) : x\in\mathsf{V}\right\} \overset{(d)}{=} \left\{\sum_{i=1}^r \ell^i_x(\mathcal{L}^{\mathbb{I}}_-) + \ell_x(\mathcal{L}_+) : x\in\mathsf{V}\right\},$$

where the ensembles corresponding to $\mathbb{I}$ carry a superscript $\mathbb{I}$, and the ensembles corresponding to $\mathbb{I}'$ carry no superscript. The common distribution of these random vectors can moreover be described as the distribution of the local time of a Poissonian ensemble of non-coloured paths with intensity equal to the image by the bleaching map $\eta\mapsto[\eta]$ of the measure $\mathbb{1}_{C^{\mathbb{I}}}\left|\mu^{\otimes,\mathbb{I}}_h\right|$, where

$$C^{\mathbb{I}} = \left\{\eta\in\mathsf{L}^{\mathbb{I}}(\mathsf{G}) : \mathrm{sgn}(\Re\mathrm{Tr}(\mathsf{amp}_h(\eta^{-1}))) = \mathrm{sgn}(\Re\mathrm{Tr}(\mathsf{hol}_h([\eta]^{-1})))\right\}.$$

In Figure 8, the set $C^{\mathbb{I}}$ is the union of the top left and bottom right rectangles.

8. Covariant Symanzik identity

In this last section, we investigate Symanzik's identity, which gives an expression of the moments of some non-Gaussian random sections of the vector bundle over our graph.

8.1. Non-Gaussian random sections.

The random fields that one considers in Symanzik's identity are annealed versions of the Gaussian free field: they are Gaussian with respect to a random potential. In our covariant situation, we let not only the potential, but also the connection be random.

Let as usual G be a weighted graph with a well and F a vector bundle over G. Recall that the set of connections on F is denoted by $\mathcal{A}(\mathsf{F})$. Let $\mathcal{H}(\mathsf{F})$ denote the set of potentials on F, in the sense of Definition 3.4. The space $\mathcal{A}(\mathsf{F})$ is a compact topological space and $\mathcal{H}(\mathsf{F})$ is a finite-dimensional vector space. We endow both of them with their Borel σ-field.

Definition 8.1. *Let $\mathbf{P}$ be a Borel probability measure on $\mathcal{A}(\mathsf{F}) \times \mathcal{H}(\mathsf{F})$. The measure $\mathbb{P}^{\mathbf{P}}$ is the probability measure on $\Omega^0(\mathsf{V},\mathsf{F})$ such that for all bounded measurable function F on $\Omega^0(\mathsf{V},\mathsf{F})$,*

$$\int_{\Omega^0(\mathsf{V},\mathsf{F})} F(f)\, d\mathbb{P}^{\mathbf{P}}(f) = \frac{1}{Z^{\mathbf{P}}} \int_{\Omega^0(\mathsf{V},\mathsf{F})\times\mathcal{A}(\mathsf{F})\times\mathcal{H}(\mathsf{F})} F(f) e^{-\frac{1}{2}(f,\Delta_{g,K} f)_{\Omega^0}}\, d\mathrm{Leb}_{\Omega^0}(f) d\mathbf{P}(g,K),$$

where

$$Z^{\mathbf{P}} = \pi^{\frac{\beta r|\mathsf{V}|}{2}} \int_{\mathcal{A}(\mathsf{F})\times\mathcal{H}(\mathsf{F})} \det \Delta_{g,K}^{-\frac{\beta}{2}}\, d\mathbf{P}(g,K). \tag{64}$$

If $\mathbf{P}$ is the Dirac mass at (h,H), then $\mathbb{P}^{\mathbf{P}} = \mathbb{P}^{h,H}$. In that case, we denote $Z^{\mathbf{P}}$ simply by $Z^{h,H}$, instead of $Z^{\delta_{(h,H)}}$.

Let us emphasise that in general, $\mathbb{P}^{\mathbf{P}}$ is not equal to $\int_{\mathcal{A}(\mathsf{F})\times\mathcal{H}(\mathsf{F})} \mathbb{P}^{g,K}\, d\mathbf{P}(g,K)$, but rather to

$$\mathbb{P}^{\mathbf{P}} = \frac{1}{Z^{\mathbf{P}}} \int_{\mathcal{A}(\mathsf{F})\times\mathcal{H}(\mathsf{F})} Z^{g,K} \mathbb{P}^{g,K}\, d\mathbf{P}(g,K). \tag{65}$$

If one is given a reference connection h and a reference potential H, then the definition of the measure $\mathbb{P}^{\mathbf{P}}$ can be rewritten as

$$\int_{\Omega^0(\mathsf{V},\mathbb{P})} F(f)\, d\mathbb{P}^{\mathbf{P}}(f) = \frac{1}{Z^{\mathbf{P}}} \int_{\Omega^0(\mathsf{V},\mathsf{F})} F(f) \left[\int_{\mathcal{A}(\mathsf{F})\times\mathcal{H}(\mathsf{F})} e^{-\frac{1}{2}(f,(\Delta_{g,K}-\Delta_{h,H})f)_{\Omega^0}}\, d\mathbf{P}(g,K) \right] d\mathbb{P}^{h,H}(f),$$

so that this measure appears as a perturbation of the distribution $\mathbb{P}^{h,H}$ of the covariant Gaussian free field. If one is interested in a particular perturbation, then one should look for a probability measure $\mathbf{P}$ which makes the expression between the brackets equal to this perturbation. This perturbative point of view was one of Symanzik's original motivations to prove the identity that bears his name.

Let us give another expression of the measure $\mathbb{P}^{\mathbf{P}}$ based on a computation of the partition function $Z^{\mathbf{P}}$.

Lemma 8.2. *Let $\mathcal{L}$ be a Poissonnian ensemble of loops with intensity $\frac{r\beta}{2}\mu^{\circledast}$. For every connection g and every potential K, the following equality holds:*

$$\mathbb{E}\Big[\prod_{\gamma\in\mathcal{L}} \mathrm{tr}(\mathsf{hol}_{g,K}(\gamma^{-1})) \Big] = \left(\frac{\det \Delta^r}{\det \Delta_{g,K}} \right)^{\frac{\beta}{2}} = (\pi^{-|\mathsf{V}|} \det \Delta)^{\frac{r\beta}{2}} Z^{g,K}.$$

Proof. By Campbell formula, the leftmost quantity is equal to

$$\exp \frac{\beta}{2} \int_{\mathsf{L}(\mathsf{G})} (\mathrm{Tr}(\mathsf{hol}_{g,K}(\gamma^{-1})) - r)\, d\mu^{\circledast}(\gamma).$$

In order to apply (25), we need to understand the scalar r as the trace of the holonomy of a connection along γ. Such a connection can be constructed by choosing a basis of each fibre of F and letting, for every vertex x and every edge e issued from x, the holonomy $h_{e,x}$ be the identity in the chosen bases of F_x and F_e. For such a connection, the holonomy along every loop based at a vertex x is the identity of F_x. Moreover, the Laplacian associated to such a connection is conjugated to the Laplacian of the trivial bundle of rank r endowed with the trivial connection. This accounts for the first equality. The second follows from (64). □

It follows immediately from this result that for every connection g and every potential K,

$$\frac{Z^{g,K}}{Z^{\mathbf{P}}} = \frac{\mathbb{E}\Big[\prod_{\gamma\in\mathcal{L}} \mathrm{tr}(\mathsf{hol}_{g,K}(\gamma^{-1})) \Big]}{\mathbf{E}\otimes\mathbb{E}\Big[\prod_{\gamma\in\mathcal{L}} \mathrm{tr}(\mathsf{hol}_{g',K'}(\gamma^{-1})) \Big]}. \tag{66}$$

8.2. **Symanzik's identity.** We can now state the covariant version of Symanzik's identity. Recall the definition of the measure ν (Definition 2.7).

Theorem 8.3. *Let* G *be a weighted graph with a well. Let* F *be a vector bundle over* G*. Let* $\mathbf{P}$ *be a Borel probability measure on* $\mathcal{A}(\mathsf{F}) \times \mathcal{H}(\mathsf{F})$*.*

Assume that $\mathbb{K} = \mathbb{R}$*. Let* $f_1, \dots, f_{2k}$ *be* $2k$ *sections of* F*. Then*

$$\mathbb{E}^{\mathbf{P}}\left[\prod_{i=1}^{2k}(f_i,\Phi)_{\Omega^0}\right] = \frac{\mathbf{E}\otimes\mathbb{E}\otimes\int_{\nu^{\otimes k}}\left[\prod_{\gamma\in\mathcal{L}}\mathrm{tr}(\mathsf{hol}_{g,K}(\gamma^{-1}))\sum_{\pi}\prod_{l=1}^{k}\lambda_{\underline{\gamma_l}}\langle f_{i_l}(\underline{\gamma_l}),\mathsf{hol}_{h,H}(\gamma^{-1})f_{j_l}(\overline{\gamma_l})\rangle_{\underline{\gamma_l}}\right]}{\mathbf{E}\otimes\mathbb{E}\left[\prod_{\gamma\in\mathcal{L}}\mathrm{tr}(\mathsf{hol}_{g,K}(\gamma^{-1}))\right]}, \tag{67}$$

where the sum is taken over all partitions $\pi = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\}$ *of* $\{1, \dots, 2k\}$ *by pairs.*

Assume that $\mathbb{K} = \mathbb{C}$*. Let* $f_1, \dots, f_k, f'_1, \dots, f'_k$ *be* $2k$ *sections of* F*. Then*

$$\mathbb{E}^{\mathbf{P}}\left[\prod_{i=1}^{k}(f_i,\Phi)_{\Omega^0}\prod_{i=1}^{k}(\Phi,f'_i)_{\Omega^0}\right] = \frac{\mathbf{E}\otimes\mathbb{E}\otimes\int_{\nu^{\otimes k}}\left[\prod_{\gamma\in\mathcal{L}}\mathrm{tr}(\mathsf{hol}_{g,K}(\gamma^{-1}))\sum_{\sigma}\prod_{l=1}^{k}\lambda_{\underline{\gamma_l}}\langle f_{l}(\underline{\gamma_l}),\mathsf{hol}_{h,H}(\gamma^{-1})f'_{\sigma(l)}(\overline{\gamma_l})\rangle_{\underline{\gamma_l}}\right]}{\mathbf{E}\otimes\mathbb{E}\left[\prod_{\gamma\in\mathcal{L}}\mathrm{tr}(\mathsf{hol}_{g,K}(\gamma^{-1}))\right]}, \tag{68}$$

where the sum is over all permutations of $\{1, \dots, k\}$*.*

An illustration of Theorem 8.3 is given in Figure 9.

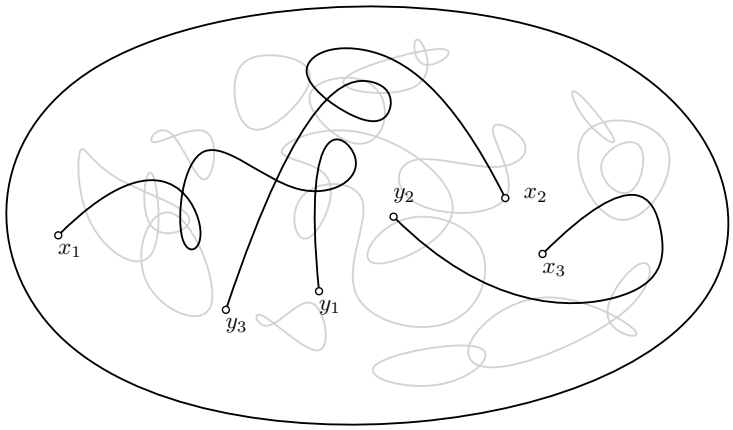

FIGURE 9. According to Symanzik's identity, illustrated here in the complex case, the correlation of the random section of F at the points x_1, x_2, x_3 and its conjugate at the points y_1, y_2, y_3 can be computed in terms of the traces of the holonomies along loops of a Poissonnian ensemble (here in gray) and the holonomies along paths connecting the x points to the y points.

Proof. The only difference between the real and complex cases is the form of Wick's theorem, which we stated in the present context as Proposition 5.4. Let us treat the complex case. Writing the definition of $\mathbb{P}^{\mathbf{P}}$ and using precisely Proposition 5.4, we find that the left-hand side of (68) is equal to

$$\frac{1}{Z^{\mathbf{P}}} \sum_{\sigma} \int_{\mathcal{A}(\mathsf{F})\times\mathcal{H}(\mathsf{F})} Z^{g,K} \int_{\mathsf{P}(\mathsf{G})^k} \prod_{l=1}^{k} \lambda_{\underline{\gamma_l}} \langle f_l(\underline{\gamma_l}), \mathsf{hol}_{h,H}(\gamma^{-1}) f'_{\sigma(l)}(\overline{\gamma_l}) \, d\nu(\gamma_1) \dots d\nu(\gamma_k) \, d\mathbf{P}(g,K).$$

It suffices to replace $\frac{Z^{g,K}}{Z^{\mathbf{P}}}$ by its value given by (66) to find the right-hand side of (68). □

Concluding remarks

We view our paper as setting a framework for further study in random spatial processes on vector bundles over graphs. As a conclusion, we mention without much detail a number of possible directions of future research.

We expect that, at least for smooth enough connections, our results should extend to the setup of Brownian paths on manifolds, where the squares of the covariant continuum Gaussian free field are replaced by appropriate Wick squares.

Our setup should also allow to study the random interlacement model [Szn12a], which can be investigated on an infinite lattice by using an exhausting sequence of finite graphs. We expect the analog of Theorem 7.6 to be true in that case too.

Loop percolation is the study of connectivity properties of the loop soup. On the hypercubic lattices, its phase transition with respect to an intensity parameter is now well understood. In our setup, there are many variants of loop percolation which could be defined and studied. In particular, since our parameter space is the space of connections, one can expect more complicated phase diagrams.

Our definition of twisted holonomies provides a covariant analogue of local times. Cover times also have strong relations to Gaussian free fields. Recently, Ding and Li [DL16] and Zhai [Zha14] used the refinement of Le Jan's isomorphism theorem by Lupu [Lup16] to prove results on cover times. Can one extend these results to our setup? What is the covariant analog of the cover time and its relation to the covariant Gaussian free field?

References

[AHKHK89] S. Albeverio, R. Hoegh-Krohn, H. Holden, and T. Kolsrud. *A covariant Feynman-Kac formula for unitary bundles over Euclidean space*, pages 1–12. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.

[BFS79] D. Brydges, J. Fröhlich, and E. Seiler. On the construction of quantized gauge fields. I. General results. *Annals of Physics*, 121(1):227–284, 1979.

[BFS82] D. Brydges, J. Fröhlich, and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.*, 83(1):123–150, 1982. MR648362

[BFS83] D. C. Brydges, J. Fröhlich, and A. D. Sokal. The random-walk representation of classical spin systems and correlation inequalities. II. The skeleton inequalities. *Comm. Math. Phys.*, 91(1):117–139, 1983. MR719815

[DL16] J. Ding and L. Li. Chemical distances for level-set percolation of two-dimensional discrete Gaussian free fields. 2016. arXiv:1605.04449.

[Dyn83] E. B. Dynkin. Markov processes as a tool in field theory. *J. Funct. Anal.*, 50(2):167–187, 1983. MR693227

[Dyn84a] E. B. Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes. *J. Funct. Anal.*, 55(3):344–376, 1984. MR734803

[Dyn84b] E. B. Dynkin. Local times and quantum fields. In *Seminar on stochastic processes, 1983 (Gainesville, Fla., 1983)*, volume 7 of *Progr. Probab. Statist.*, pages 69–83. Birkhäuser Boston, Boston, MA, 1984. MR902412

[Eis95] N. Eisenbaum. Une version sans conditionnement du théorème d'isomorphisme de Dynkin. In *Séminaire de Probabilités, XXIX*, volume 1613 of *Lecture Notes in Math.*, pages 266–289. Springer, Berlin, 1995. MR1459468

[Jan97] S. Janson. *Gaussian Hilbert spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997. MR1474726

[Ken11] R. Kenyon. Spanning forests and the vector bundle Laplacian. *Ann. Probab.*, 39(5):1983–2017, 2011. MR2884879

[LJ08] Y. Le Jan. Dynkin's isomorphism without symmetry. In *Stochastic analysis in mathematical physics*, pages 43–53. World Sci. Publ., Hackensack, NJ, 2008. MR2406020

[LJ10] Y. Le Jan. Markov loops and renormalization. *Ann. Probab.*, 38(3):1280–1319, 2010. MR2675000

[LJ11] Y. Le Jan. *Markov paths, loops and fields*, volume 2026 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 38th Probability Summer School held in Saint-Flour, 2008, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School]. MR2815763

[LJ16] Y. Le Jan. Markov loops, coverings and fields. 2016. arXiv:1602.02708.

[Lup16] T. Lupu. From loop clusters and random interlacements to the free field. *Ann. Probab.*, 44(3):2117–2146, 2016. MR3502602

[LW04] G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004. MR2045953

[MR06] M. B. Marcus and J. Rosen. *Markov processes, Gaussian processes, and local times*, volume 100 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. MR2250510

[Ner73] P. Neruda. *El mar y las campanas*. Editorial Losanda, Buenos Aires, 1973.

[Nor92] J. R. Norris. A complete differential formalism for stochastic calculus in manifolds. In *Séminaire de Probabilités, XXVI*, volume 1526 of *Lecture Notes in Math.*, pages 189–209. Springer, Berlin, 1992. MR1231995

[QW15] W. Qian and W. Werner. Decomposition of Brownian loop-soup clusters. 2015. arXiv:1509.01180.

[Sym69] K. Symanzik. Euclidean quantum field theory. In *Scuola internazionale di Fisica "Enrico Fermi", XLV Corso*, pages 152–223. Academic Press, 1969.

[Szn12a] A.-S. Sznitman. An isomorphism theorem for random interlacements. *Electron. Commun. Probab.*, 17:no. 9, 9, 2012. MR2892408

[Szn12b] A.-S. Sznitman. *Topics in occupation times and Gaussian free fields*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2012. MR2932978

[Wer15] W. Werner. On the spatial Markov property of soups of unoriented and oriented loops. 2015. arXiv:1508.03696.

[Zha14] A. Zhai. Exponential concentration of cover times. 2014. arXiv:1407.7617.

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