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SUBSTITUTIVE NORMAL SETS

CHRISTIAN MAUDUIT

Let $u = (u_n)_{n \in \mathbb{N}}$ be an infinite sequence of integers. Define $B(u)$ as the set of those real α 's such that the sequence $(u_n \alpha)_{n \in \mathbb{N}}$ is uniformly distributed modulo 1. A set B of real numbers is said to be normal if there exists a sequence u such that $B = B(u)$. G. Rauzy has given a complete characterization of normal sets (cf. [4]), namely, a set B is normal if and only if (i) $0 \notin B$, $B+1=B$; (ii) $\forall q \in \mathbb{Z}^*$, $qB \subset B$; (iii) there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous real functions such that $\lim_{n \rightarrow \infty} f_n(x) = 0$ exactly for those x 's which are elements of B .

We will say that a set B of real numbers is Σ -normal (or a substitutive normal set) if there exists an increasing

sequence u generated by an irreducible substitution such that $B = B(u)$. We give the following characterization of Σ -normal sets (cf. [2]):

THEOREM: B is a Σ -normal set if and only if $R \setminus B$ is an algebraic extension of finite degree of Q .

I. SUBSTITUTION OVER A FINITE ALPHABET

Let $g \geq 2$, $A = (a_1, \dots, a_g)$ an alphabet and Σ a substitution over A (cf. [6]).

Σ is given by the set $\Sigma(a_i) = a_{\sigma_1(1)}, \dots, a_{\sigma_1(l_1(1))}$ ($i \in \{1, \dots, g\}$), where $l_i(n)$ is the length of the word $\Sigma^n(a_i)$ ($n \in \mathbb{N}$) and σ_1 a map from $\{1, \dots, l_1(1)\}$ to $\{1, \dots, g\}$.

We will suppose that Σ is irreducible, that is to say that for any $(i, j) \in \{1, \dots, g\}^2$, there exists an integer n such that the word $\Sigma^n(a_i)$ contains the letter a_j .

We remark that $L(n) = M^n \mathbb{T}(1, \dots, 1)$, where

$L(n) = \mathbb{T}(l_1(n), \dots, l_g(n))$ and

$$M = (M_{ij})_{(i,j) \in \{1, \dots, g\}^2} \text{ with } M_{ij} = \text{card}(\sigma_1^{-1}(\{j\})).$$

Definition: Let $m \in \mathbb{A}^{\mathbb{N}}$ and $t \in \{1, \dots, g\}$. We call support of m with respect to a_t and we denote by $[m]_t$, the increasing sequence of those integers n such that a_t is

the $(n+1)$ -th letter of the word m .

We say that a sequence u of integers is generated by a substitution Σ if there exists $m \in A^{\mathbb{N}}$ such that $\Sigma(m) = m$ and $t \in \{1, \dots, g\}$ such that u is the support of m with respect to a_t .

II. λ FUNCTIONS ASSOCIATED TO Σ

Statistical properties of sequences generated by Σ are determined by the behaviour of the following functions.

If $\sigma_1^{-1}(\{j\})$ is non empty, let us denote by $k_1, \dots, k_{M_{ij}}$ the increasing sequence of its elements.

For any $v \in \{1, \dots, M_{ij}\}$ we put $\lambda_{ij}^{(v)} = \sum_{k' < k_v} \ell_{\sigma_1(k')}$

$$\Lambda_{ij} = \lambda_{ij}^{(v)}, v \in \{1, \dots, M_{ij}\} \setminus \{0\} \text{ and } \Lambda = \bigcup_{(i,j) \in \{1, \dots, g\}^2} \Lambda_{ij}.$$

Now we define the set E :

$$E = \{\alpha \in \mathbb{R}, \exists k \in \mathbb{Z}^*, \forall \lambda \in \Lambda \lim_{n \rightarrow \infty} e(k\lambda(n)\alpha) = 1\}$$

(where $e(x) = \exp(2i\pi x)$) and we show that

$$E = \{\alpha \in \mathbb{R}, \exists k \in \mathbb{Z}^*, \forall i \in \{1, \dots, g\} \lim_{n \rightarrow \infty} e(k\lambda_i(n)\alpha) = 1\}$$

is an algebraic extension of finite degree of \mathbb{Q} .

Example 1. If Σ is a substitution of constant length (i.e. $\ell_i(1) = r$ for any $i \in \{1, \dots, g\}$), that is to say if u is generated by a finite automaton (cf. [1]), we have $\Lambda = \{\lambda\}$ with $\lambda(n) = r^n$, and so $E = \mathbb{Q}$.

Example 2. If Σ is defined by $\Sigma(a_1) = a_1 a_2 a_2$ and $\Sigma(a_2) = a_2 a_1$, we have $\Lambda_{11} = \Lambda_{22} = \emptyset$, $\Lambda_{12} = \{\ell_1, \ell_1 + \ell_2\}$ and $\Lambda_{21} = \{\ell_2\}$ with $\ell_1(0) = \ell_2(0) = 1$, $\ell_1(1) = 3$, $\ell_2(1) = 2$ and for any $i \in \{1, 2\}$ $\ell_i(n+1) = 2\ell_i(n) + \ell_i(n-1)$. We deduce from this that $E = \mathbb{Q}(\sqrt{2})$.

III. NORMAL SET WITH RESPECT TO A SUBSTITUTIVE SEQUENCE

First, we have to show that if u is generated by Σ , then $B(u) = \mathbb{R} \setminus E$. We recall that, according to Weyl's criterium (cf. [5]), the sequence $(u_n \cdot \alpha)_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if, for any $k \in \mathbb{Z}^*$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} e(k u_n \alpha) = 0.$$

Let us consider the Weyl sums $S_i(\alpha, n) = \sum_{p \in \{\Sigma^n(a_i)\}_t} e(p\alpha)$

and the vector $w(\alpha, n) = (w_i(\alpha, n))_{i \in \{1, \dots, g\}}$

with $w_i(\alpha, n) = \frac{S_i(\alpha, n)}{S_i(0, n)}$.

Then $w(\alpha, n+1) = A(\alpha, n) w(\alpha, n)$ where $A(\alpha, n)$ is a $g \times g$ -matrix. We remark that every coefficient of $A(\alpha, n)$ depending on α is a real linear combination of $e(\lambda(n)\alpha)$ with $\lambda \in \Lambda$.

So, if $\alpha \in \mathbb{R} \setminus E$ the arguments of these complex numbers are "well distributed" modulo 2π , and we show that $\lim_{n \rightarrow \infty} B(\alpha, n) = 0$ (we put $B(\alpha, n) = A(\alpha, n) A(\alpha, n-1) \dots A(\alpha, 0)$).

We deduce from this that $w(\alpha, n) \in o(1)$ and then

$$\sum_{p \in [m]_{\mathbb{Z}}} e(p\alpha) \in o(N), \text{ i.e. } (u_n \cdot \alpha)_{n \in \mathbb{N}} \text{ is uniformly}$$

distributed modulo 1.

Conversely, let $\alpha \in E$.

As $(A(\alpha, n) - A(0, n))$ converges to 0 with an exponential speed, $(B(\alpha, n) - B(0, n))$ is smaller than some constant related to the spectrum of M . By iterating Σ , this constant can be made arbitrarily small.

$$\text{As } w(\alpha, n+1) = B(\alpha, n) w(\alpha, n)$$

$$= \mathbb{T}(1, \dots, 1) + (B(\alpha, n) - B(0, n)) \mathbb{T}(0, \dots, 0, 1, 0, \dots, 0),$$

it follows that $\underline{\lim} |w_i(\alpha, n)| > 0$ and so the sequence

$(u_n \cdot \alpha)_{n \in \mathbb{N}}$ is not uniformly distributed modulo 1.

IV. CONVERSE

Let F be an algebraic extension of finite degree of \mathbb{Q} . We know that there exists a Pisot number θ such that $F = \mathbb{Q}(\theta)$ (cf. [3]).

On the other hand, we can show that for any Pisot number θ of degree g , there exists an integer v and an irreducible non negative integral $g \times g$ matrix $M = (M_{ij})_{(i,j) \in \{1, \dots, g\}^2}$ such that the spectrum contains only θ^v and its conjugates.

Now consider the substitution Σ given by

$$\forall i \in \{1, \dots, g\}, \Sigma(a_i) = \underbrace{a_1 \dots a_1}_{M_{i1} \text{ times}} \dots \underbrace{a_j \dots a_j}_{M_{ij} \text{ times}} \dots \underbrace{a_g \dots a_g}_{M_{ig} \text{ times}}$$

By iterating Σ , we can suppose $M_{ij} > 0$. Then, let m be the fixed point of Σ beginning with a_1 and $u = [m]_1$ be the sequence generated by m and a_1 .

Using the same method as in § III, we show that

$$E = \mathbb{Q}(\theta^v).$$

As θ is a Pisot number, $\mathbb{Q}(\theta^v) = \mathbb{Q}(\theta)$ and

$$B(u) = \mathbb{R} \setminus E = \mathbb{R} \setminus \mathbb{Q}(\theta^v) = \mathbb{R} \setminus \mathbb{Q}(\theta) = \mathbb{R} \setminus F.$$

So, $\mathbb{R} \setminus F$ is a Σ -normal set.

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