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Non-locally modular regular types in classifiable theories

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Abstract

We introduce the notion of strong p -semiregularity and show that in a classifiable theory, if p is a regular type that is not locally modular then any p -semiregular type is strongly p -semiregular. Moreover, for any such p -semiregular type, “domination implies isolation” which allows us to prove the following: Suppose that T is countable, classifiable and M is any model. If $p \in S(M)$ is regular but not locally modular and b is any realization of p then every model N containing M that is dominated by b over M is both constructible and minimal over Mb .

1 Introduction

In obtaining the uncountable spectrum of any classifiable theory T in [2], localizations of ω -stability near certain regular types were considered. A regular type $p \in S(M)$ over a countable M is *locally totally transcendental* (locally t.t.) if it is not orthogonal to a $q \in S(M)$ that is strongly regular and for which there is a constructible (and hence prime) model over M and any realization of q . There are examples of depth zero non-trivial regular types in classifiable theories which are not locally t.t (see for instance Example 2.4). We intend to consider the manner in which models dominated by such types are constructed in future papers. In this paper, we concentrate on *non-locally modular* regular types p and prove that they are all locally t.t.

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in a very strong way. The two main results build on the dichotomy theorem of Hrushovski and Shelah in [4]. Here, we prove that if a stationary $q \in S(A)$ is p -semiregular then

- q is strongly p -semiregular (see Definition 3.1); and
- q is depth-zero like and “domination implies isolation” (DI), (see Definitions 5.13 and 6.2) hence if q is based on a model M then there is a constructible, minimal model $N \supseteq Mb$ over Mb for any realization b of $q|_M$, and moreover, any $N \supseteq Mb$ that is dominated by b over M is constructible and minimal over Mb .

It is this last result which produces in particular the following easy to state theorem:

Theorem 5.20 *Suppose that M is any model of a classifiable theory. If $p \in S(M)$ is regular but not locally modular and b is any realization of p , then every model N containing M that is dominated by b over M is both constructible and minimal over Mb .*

We assume that the reader is familiar with superstability and adopt the usual convention when working with stable theories that we are working in a large, saturated model \mathfrak{C} of the theory and all models mentioned are small elementary submodels of \mathfrak{C} , sets are small subsets of \mathfrak{C} and tuples are from \mathfrak{C} . We will also assume that T eliminates imaginaries i.e. $T = T^{eq}$. This paper has an extensive Appendix that records a number of definitions and facts from basic geometric stability theory. Section 2 contains some background information and gives an example indicating the necessity of non-local modularity in Theorem 5.20. Section 3 introduces the notion of strongly p -semiregular strong types and proves that when p is any non-locally modular regular type, every p -semiregular strong type is strongly p -semiregular. We provide some applications of this in Section 4 before proving our main theorem in Section 5. The paper proper concludes with an examination of other circumstances under which domination implies isolation in Section 6.

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For the whole of this article we assume T is (at least) stable.

2 A brief historical background

One of the major accomplishments of stability theory was Shelah’s proof of the Main Gap in [10] where the notion of a classifiable theory was introduced.

Definition 2.1 A complete theory T in a countable language is *classifiable* if T is superstable, has prime models over pairs (PMOP) and does not have the dimension order property (NDOP)

The definitions of these terms, along with equivalents we will use are given in Definition A.27 and Facts A.28 and A.29. The significance of classifiability is given by two results. On one hand, in [9], building extensively on [10], Shelah and Buechler prove that if T is classifiable, then every model of T is constructible¹ and minimal over an independent tree of countable elementary substructures. By contrast, if a countable theory is not classifiable, then $I(T, \kappa) = 2^\kappa$ for every uncountable cardinal κ . The reader is cautioned that these two properties are not exclusive – a classifiable, deep theory has both the structure theorem and has maximally many non-isomorphic models in every uncountable cardinality.

After Shelah defined PMOP, Harrington gave an alternate treatment of PMOP that was developed in [1]. Following the notation preceding Fact A.29, call a type $p \in S(A_1 A_2)$ over an independent triple (A_0, A_1, A_2) of sets V -dominated if, for every extension (B_0, B_1, B_2) of (A_0, A_1, A_2) , and for any realization c of p with $c \downarrow_{A_1 A_2} B_0$, we have $c \downarrow_{A_1 A_2} B_1 B_2$. Then, as in Fact A.29(4), Harrington notes that the statement “Every V -dominated type is isolated” is equivalent to PMOP for countable, superstable theories with NDOP. The form of this statement is indicative of what we do here. In Section 6 we investigate conditions under which ‘domination implies isolation’ in various settings.

Since the proof of the structure theorem, there has been a considerable amount of work analyzing the ‘fine structure’ of classifiable theories. The fine structure to a large extent revolves around understanding the leaves of classifying trees. The leaves are controlled by depth zero types and so we remind the reader of the definition (for more definitions and classical results consult the Appendix).

Definition 2.2 A regular type p in a superstable theory is said to have *depth zero* if for any a-model M on which p is based and any realization b of $p|_M$, any non-algebraic type q over $M[b]$ (the a-prime model over Mb), is non-orthogonal to M i.e. p does not support a regular type.

A *leaf* is a triple (M, b, N) where $M \preceq N$, $b \in N$, $tp(b/M)$ is regular and depth zero, and N is dominated by b over M . The computation of the uncountable spectrum of a countable theory depends on, among other things, understanding the isomorphism types of leaves

¹See Section A.1 of the Appendix.

that appear in the classifying trees of models. Through the coarseness of cardinal arithmetic, in [2], it was not necessary to determine all possible isomorphism types of leaves in order to determine the uncountable spectra of countable theories. Still, the techniques available from classification theory are suitable to do this and demonstrate a deeper understanding of the structure of models of classifiable theories. In this paper and subsequent papers, we intend to explore the isomorphism types of leaves (M, b, N) at least when $tp(b/M)$ is non-trivial; in a classifiable theory, non-trivial types necessarily have depth zero (Fact A.28). Our main result here is that if (M, b, N) is a leaf with $tp(b/M)$ regular and non-locally modular, then there is only one choice for the isomorphism type of N over Mb . This leads to an amusing corollary to Theorem 5.20.

Corollary 2.3 *If T is any countable, complete theory and its spectrum of uncountable models is any of 4, 5, 7, or 10 (in the notation of Theorem 6.1 of [2]) then an infinite group is interpretable in models of T . In particular, if $I(T, \aleph_\alpha) = \min\{2^{\aleph_\alpha}, \beth_2\}$ (i.e., the spectrum of $Th(\mathbb{Z}, +)$), then T must interpret an infinite group.*

Proof. Hrushovski proves that if T does not interpret an infinite group, then every non-trivial regular type is non-locally modular, see e.g., 7.3.3 of [7]. Thus, by Theorem 1b of [4] and Theorem 5.20 here, any non-trivial regular type $p \in S(M)$ is locally t.t. in M . So, in the notation of [2], for any $n \geq 1$, ‘non-trivial failure’ $NTF(n)$ is impossible, hence $TT(n)$ holds if and only if $TF(n)$ fails. The result follows by considering the case analysis for each spectrum given in Section 5 of [2]. \square

The results of this paper are concerned with non-locally modular regular types over models. It is natural to ask for other circumstances under which leaves are constructible. This is always possible if T is ω -stable, as such a theory has constructible models over any set. However, we close this section with an example of a classifiable theory that has leaves (M, b, N) for which N is not constructible over Mb .

Example 2.4 The language will consist of countably many sorts $\{U_n : n \geq 1\}$, a collection of unary relations $\{R_\eta^n : \eta \in 2^{<\omega}, n \geq 1\}$, and on each sort U_n , there is a binary operation $+_n : U_n^2 \rightarrow U_n$. The language also includes projection functions $f_n : U_{n+1} \rightarrow U_n$ for each $n \geq 1$. The canonical model of our theory in this language is as follows:

1. U_n is interpreted as the product of n copies of 2^ω ;
2. $+_n$ is interpreted as coordinate-wise addition modulo 2;

3. If $b \in U_n$ codes the n -tuple $\nu_1 \dots, \nu_n$, then $R_\eta^n(b)$ holds iff η is an initial segment of ν_n , and
4. f_n is the projection onto the first n coordinates.

The theory T of this structure is classifiable; in fact, it is superstable and unidimensional but not ω -stable. Now suppose that M is a model of T and N is an elementary weight one extension of M - the type of thing that would happen with leaves on a classifying tree. The claim is that if $b \in N \setminus M$ is any finite tuple then N is not constructible over Mb . To see this, suppose we have such a b . By the presence of addition in all the sorts and the model M , we can assume that b is a singleton in some sort U_n . But then, if one considers the preimage of b under f_n , one sees that this formula in the sort U_{n+1} does not contain an isolated type - the predicates R_η^{n+1} preclude this.

This example is suggestive of the result we will prove in subsequent papers: if we don't restrict ourselves to finite tuples then of course N is determined by making a coordinated choice of elements from each sort. In the example, all the regular types are locally modular (non-trivial). It is not clear in advance that this is important but in this paper we will show that if (M, b, N) is a leaf in a countable, classifiable theory and $\text{tp}(b/M)$ is *not* locally modular, then N is constructible over Mb .

3 Strongly p -semiregular types

Definitions of p -semiregular types, p -simplicity and other related notions can be found in the Appendix.

Definition 3.1 A stationary type $q \in S(B)$ is *strongly p -semiregular* of weight $k > 0$ if q is p -semiregular of p -weight k and there is a p -simple formula $\theta(x) \in q$ of weight k such that if d realizes θ and $C \supseteq B$ with $d \downarrow_B C$ and $w_p(d/C) = w_p(q)$, then $\text{tp}(d/C) = q|C$, the non-forking extension of q to $S(C)$.

The goal of this entire section is to prove the following Theorem. Its proof is patterned after the argument in [4], where Hrushovski and Shelah prove that in a classifiable theory, every non-locally modular stationary, regular type is strongly regular (i.e., strongly p -semiregular of p -weight one).

Theorem 3.2 *If T is classifiable and p is a non-locally modular regular type, then every stationary p -semiregular type is strongly p -semiregular.*

Remark 3.3 By the open mapping theorem, this notion is parallelism invariant. In particular, if $d \perp E$ with $\text{tp}(d/\emptyset)$ stationary, then if $\text{tp}(d/E)$ is strongly p -semiregular via $\theta(x, e)$, then $\text{tp}(d/\emptyset)$ will be strongly p -semiregular via $\bigvee d_r y \theta(x, y)$, the (finite) disjunction of θ -definitions of each $r \in S_\theta(\mathfrak{C})$ consistent with $\text{tp}(e/\emptyset)$.

We will use this Remark as justification for freely adding independent parameters in many places.

3.1 p -triples

In this subsection, T is superstable and p is a stationary regular type over \emptyset . The reader is encouraged to review the definitions and basic facts in Sections A.2 and A.3 of the Appendix. We adopt the data structure of a p -triple and then show that a given p -triple can be massaged to get matching p -triples with more and more desirable properties. The key will be to obtain a *minimal p -triple* as a normal cover of a given one.

Definition 3.4 • A p -triple is a sequence (a, b, C) such that $a, b \in D(p, C)$, i.e., $\text{stp}(ab/C)$ is p -simple, with both $w_p(a/Cb) > 0$ and $w_p(b/Ca) > 0$.

- A p -triple (a, b, C) is *normal* if the three strong types $\text{stp}(ab/C)$, $\text{stp}(a/Cb)$, and $\text{stp}(b/Ca)$ are all p -semiregular.
- A p -triple (a, b, C) is *p -disjoint* if $\text{cl}_p(Ca) \cap \text{cl}_p(Cb) = \text{cl}_p(C)$.
- Two p -triples (a, b, C) , (a', b', C') are *matching* if $w_p(ab/C) = w_p(a'b'/C')$, $w_p(a/Cb) = w_p(a'/C'b')$, and $w_p(b/Ca) = w_p(b'/C'a')$.

We seek p -triples (a, b, C) that embed a realization of a given p -semiregular strong type in its first component.

Definition 3.5 Suppose $\text{stp}(e/\emptyset)$ is p -semiregular. A p -triple (a, b, C) *envelops* e if $e \in \text{dcl}(a)$ and $e \perp Cb$.

There is a canonical way of extending a p -disjoint p -triple (a, b, C) to a matching, normal p -disjoint p -triple (a', b', C') .

Definition 3.6 A *soft extension* of a p -triple (a, b, C) is a p -triple (a', b', C') such that

1. $a'b'C' \subseteq \text{dcl}(abC)$;
2. $C \subseteq C' \subseteq \text{cl}_p(C)$ with $C' \setminus C$ finite;
3. $a \subseteq a' \subseteq \text{cl}_p(C'a)$ and $b \subseteq b' \subseteq \text{cl}_p(C'b)$.

Note that $\text{cl}_p(C') = \text{cl}_p(C)$, $\text{cl}_p(C'a') = \text{cl}_p(Ca)$, and $\text{cl}_p(C'b') = \text{cl}_p(Cb)$ in any soft extension.

Lemma 3.7 (Normalization) *Given any p -triple (a, b, C) , there is a soft extension to a matching normal p -triple (a', b', C') , called a normalization of (a, b, C) . If (a, b, C) is p -disjoint, then (a', b', C') will be p -disjoint as well. Moreover, if $\text{stp}(e/\emptyset)$ is p -semiregular and (a, b, C) envelops e , then (a', b', C') also envelops e .*

Proof. First, apply Lemma A.20 to $\text{stp}(ab/C)$ to get $C' \subseteq \text{dcl}(Cab) \cap \text{cl}_p(C)$ such that $\text{stp}(ab/C')$ is p -semiregular. Note that $\text{cl}_p(C') = \text{cl}_p(C)$, $\text{cl}_p(C'a) = \text{cl}_p(Ca)$, and $\text{cl}_p(C'b) = \text{cl}_p(Cb)$. Next, apply the Lemma to $\text{stp}(b/aC')$ to get a' and $\text{stp}(b/C'a')$ p -semiregular, and finally apply the Lemma to $\text{stp}(a'/C'b)$ to get b' . It follows from Lemma A.21 that (a', b', C') is normal and matches (a, b, C) . The preservation of p -disjointness is clear because of the equality of the p -closed sets mentioned above. Finally, suppose (a, b, C) envelops e . Since $a \subseteq a'$, $e \in \text{dcl}(a')$. Also, since $C'b' \subseteq \text{cl}_p(Cb)$, $\text{stp}(e/\emptyset)$ p -semiregular and $e \perp Cb$ implies $e \perp C'b'$, so (a', b', C') envelops e as well. \square

Next we describe three ways of extending a given p -triple (a, b, C) to a larger, matching (a', b', C') that preserves p -disjointness and enveloping.

Definition 3.8 A *simple extension* of a p -triple (a, b, C) is any of:

1. (a^*, b, C) , where $a \subseteq a^* \subseteq \text{cl}_p(Ca)$;
2. (a, b^*, C) , where $b \subseteq b^* \subseteq \text{cl}_p(Cb)$;
3. (a, b, C^*) , where $C^* \supseteq C$ and $C^* \perp_C ab$.

Lemma 3.9 *Given any p -disjoint, normal p -triple (a, b, C) , the normalization (a', b', C') of any simple extension is a matching p -disjoint extension of (a, b, C) . Moreover, if $\text{stp}(e/\emptyset)$ is p -semiregular and (a, b, C) envelops e , then (a', b', C') envelops e as well.*

Proof. That all three species of simple extensions are matching is clear. Next, we show that each of the simple extensions preserves p -disjointness. This is clear for the first two, as $\text{cl}_p(Ca^*) = \text{cl}_p(Ca)$ in the first case and $\text{cl}_p(Cb^*) = \text{cl}_p(Cb)$ in the second. As (a, b, C) normal implies $\text{stp}(ab/C)$ is p -semiregular, p -disjointness of the third species is preserved by Lemma A.24. If (a, b, C) envelops e , then obviously $e \in \text{dcl}(a^*)$ for any of the three species of simple extensions. For the first species, $e \perp Cb$ holds vacuously. For the second we have $Cb^* \subseteq \text{cl}_p(Cb)$, so $e \perp Cb^*$ follows from $\text{stp}(e/\emptyset)$ p -semiregular, and

for the third, since $eb \downarrow_C C^*$, we obtain $e \downarrow C^*b$ by the transitivity of non-forking.

The Lemma now follows from Lemma 3.7. \square

Definition 3.10 Suppose that (a, b, C) is a p -disjoint normal p -triple.

- A *normal cover* is any p -disjoint normal (a', b', C') obtained as a sequence of extensions as in Lemma 3.9.
- The *strength* of (a, b, C) which we denote by $\alpha(a, b, C)$, is equal to $R^\infty(a/bb'C)$, where b' is (any) element satisfying $\text{stp}(b'/Ca) = \text{stp}(b/Ca)$ and $b' \downarrow_{C_a} b$. (This is well-defined, as $\text{stp}(abb'/C)$ is independent of our choice of b' .)
- A normal p -triple (a, b, C) is *minimal* if $\alpha(a, b, C) \leq \alpha(a', b', C')$ for all of its normal covers (a', b', C') .

Clearly, by superstability and the transitivity of being a normal cover, every normal p -triple (a, b, C) has a minimal, normal cover (a', b', C') . In fact, one can find one with the additional property that $a' \in \text{dcl}(C'a)$. At present, this improvement does not seem to be necessary.

Lemma 3.11 *Given any p -disjoint, normal p -triple (a, b, C) , there is a matching minimal, normal, p -disjoint p -triple (a', b', C') that is a normal cover of (a, b, C) . If $\text{stp}(e/\emptyset)$ is p -semiregular and (a, b, C) envelops e , then (a', b', C') envelops e as well.*

Proof. Among all normal covers (a', b', C') of (a, b, C) , choose the one of smallest strength. \square

We record how the minimality assumption will be used in the proof of Theorem 3.2.

Lemma 3.12 *Suppose (a, b, C) is a minimal normal p -triple with strength α . For every $\overline{C} \supseteq C$ with $\overline{C} \downarrow_C ab$ and every B with $b \subseteq B \subseteq \text{cl}_p(\overline{C}b)$, $R^\infty(a/\overline{C}BB') = \alpha$ for some/every B' with $\text{stp}(B'/\overline{C}a) = \text{stp}(B/\overline{C}a)$ and $B \downarrow_{C_a} B'$.*

Proof. This is immediate since (a, b, \overline{C}) is a simple extension of (a, b, C) , and (a, B, \overline{C}) is a simple extension of (a, b, \overline{C}) , hence (a, B, \overline{C}) is a normal cover of (a, b, C) . \square

We close with two lemmas concerning p -disjointness.

Lemma 3.13 *Suppose $\text{stp}(a/C)$ and $\text{stp}(b/C)$ are both p -semiregular and $a \downarrow_C b$. Then the p -triple (a, b, C) is p -disjoint.*

Proof. Choose any $e \in \text{cl}_p(Ca) \cap \text{cl}_p(Cb)$. Then $w_p(e/Ca) = w_p(e/Cb) = 0$. By Lemma A.21(1), $\text{stp}(b/Ca)$ is p -semiregular, so $b \downarrow_C e$. But this, coupled with $w_p(e/Cb) = 0$ implies $e \in \text{cl}_p(C)$. \square

Lemma 3.14 *Suppose (a_1, b_1, C) and (a_2, b_2, C) are both normal and p -disjoint. If, moreover, $a_1 b_1 \downarrow_C a_2 b_2$, then the p -triple $(a_1 a_2, b_1 b_2, C)$ is also p -disjoint.*

Proof. In light of Lemma A.24, then p -disjointness of (a_1, b_1, C) implies that $\text{cl}_p(a_1 a_2 b_2 C) \cap \text{cl}_p(b_1 a_2 b_2 C) \subseteq \text{cl}_p(a_2 b_2 C)$, so

$$\text{cl}_p(a_1 a_2 C) \cap \text{cl}_p(b_1 b_2 C) \subseteq \text{cl}_p(a_2 b_2 C)$$

Arguing in reverse, the p -disjointness of (a_2, b_2, C) yields

$$\text{cl}_p(a_1 a_2 C) \cap \text{cl}_p(b_1 b_2 C) \subseteq \text{cl}_p(a_1 b_1 C)$$

Furthermore, it follows immediately from Lemma 3.13 that

$$\text{cl}_p(a_1 b_1 C) \cap \text{cl}_p(a_2 b_2 C) \subseteq \text{cl}_p(C)$$

and the result follows. \square

3.2 Proof of Theorem 3.2

We proceed to prove Theorem 3.2, following the proof of Proposition 3.1 in [4]. For the whole of this subsection, assume T is classifiable and p is a (stationary) regular, non-locally modular type p over \emptyset . By Fact A.28, p has depth zero. We will often use Fact A.22, which says that every p -semiregular type contains a p -simple formula D inside which p -weight is continuous and definable. Fix some e such that $\text{tp}(e/\emptyset)$ is stationary and p -semiregular with $w_p(e) = k \geq 1$. Our goal is to show that $\text{tp}(e/\emptyset)$ is strongly p -semiregular. For the whole of this subsection, we work inside D^{eq} , which suffices. Thus, all types under consideration will be p -simple.

The following Proposition is the content of Lemma 3.2 of [4].

Proposition 3.15 *For every realization d of p , there is a minimal, normal, p -disjoint p -triple (a, b, C) enveloping d . Moreover, $w_p(a/C) = w_p(b/C) = 2$, $w_p(ab/C) = 3$, and $w_p(a/bC) = w_p(b/aC) = 1$.*

Proof. By the first paragraph of the proof of Lemma 3.2 in [4], there is a p -triple (a, b, M) , where M is an a -model, a and b each consist of two M -independent realizations of $p|M$, with $w_p(a/Mb) = w_p(b/Ma) = 1$ and the p -triple (a, b, M) being p -disjoint.

By p -weight computations, at least one of $\{a_1, a_2\}$ must be independent from b over M , so by passing to an automorphism of \mathfrak{C} , we may assume that $d \subseteq a$ and $d \perp Mb$. Thus, (a, b, M) envelops d . Now, apply the Normalization Lemma 3.7, and then choose a minimal normal cover via Lemma 3.11. \square

The following generalization is the point of all these definitions.

Proposition 3.16 *For e as above, there is a minimal, normal, p -disjoint (a, b, C) enveloping e such that $w_p(ab/C) = 3k$, $w_p(a/C) = w_p(b/C) = 2k$, and $w_p(a/bC) = w_p(b/aC) = k$.*

Proof. First, by p -semiregularity, choose E and an E -independent sequence $\bar{d} = (d_i : i < k)$ of realizations of $p|E$ such that $e \perp E$ and e and \bar{d} are domination equivalent over E .

By Proposition 3.15, choose a triple (a_0, b_0, C_0) enveloping d_0 as there. Since d_0 realizes $p|C_0$ and $p|E$, we may assume $E \subseteq C_0$ and \bar{d} is independent over C_0 . Next, choose $(a_i b_i : i < k)$ to be C_0 -independent with $\text{stp}(a_i b_i d_i / C_0) = \text{stp}(a_0 b_0 d_0 / C_0)$ for each i . In particular, (a_i, b_i, C_0) envelops d_i for each $i < k$. As notation, let \bar{a} denote $(a_i : i < k)$ and \bar{b} denote $(b_i : i < k)$. Note that by the independence, $b_i \perp_{C_0 a_i} \bar{a}$ for each i .

By iterating Lemma 3.14, we have that the p -triple (\bar{a}, \bar{b}, C_0) is p -disjoint. As well, it follows from the independence that $w_p(\bar{a}/C_0) = w_p(\bar{b}/C_0) = 2k$, $w_p(\bar{a}\bar{b}/C_0) = 3k$, and $w_p(\bar{a}/C_0\bar{b}) = w_p(\bar{b}/C_0\bar{a}) = k$. It also follows that (\bar{a}, \bar{b}, C_0) is normal.

As $w_p(e/d_0, \dots, d_{k-1}) = 0$, we have $w_p(e/\bar{a}C_0) = 0$. Thus, the p -triple $(e\bar{a}, \bar{b}, C_0)$ is a simple extension of (\bar{a}, \bar{b}, C_0) enveloping e . By Lemmas 3.7 and 3.11 there is a matching, minimal, normal cover (a, b, C) of $(e\bar{a}, \bar{b}, C_0)$ that is p -disjoint and envelops e . \square

From now on, fix a p -triple (a, b, C) as in Proposition 3.16.

We continue along the lines of Section 3 of [4], with our Proposition 3.16 taking the place of their Lemma 3.2. The following Lemma takes the place of their Lemma 3.3.

Lemma 3.17 *Choose any b' realizing $\text{stp}(b/Ca)$ with $b' \perp_{C_a} b$. Then:*

1. $w_p(a/bb'C) = 0$;

2. $b' \perp_C b$;
3. There is a formula $\rho(x, b, b')$ (possibly with hidden parameters from C) isolating $\text{tp}(a/Cbb')$.
4. For any a' and any b'' satisfying $\text{stp}(b''/C) = \text{stp}(b/C)$ and $b'' \perp_C b$, if $\rho(a', b, b'')$ holds, then $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$.

Proof. (1) Here is where p -disjointness plays a leading role. Recalling the notation from the proof of Proposition 3.16, (a, b, C) is a normal cover of $(e\bar{a}, \bar{b}, C_0)$, and for each $i < k$ we have $w_p(a_i/b_iC_0) = 1$ and $b_i \perp_{C_0a_i} \bar{a}$. We claim that $a_i \not\perp_{C_0b_i} b'$ for each $i < k$. Since $\text{stp}(a_i/b_iC_0)$ is p -semiregular, the claim implies that $w_p(a_i/b_i b' C) = 0$ for each i , hence $w_p(a/bb' C) = 0$, completing the verification of (1).

To establish the claim, fix $i < k$. The forking is attained in three steps. First, by p -disjointness we have

$$\text{cl}_p(Ca) \cap \text{cl}_p(Cb) \subseteq \text{cl}_p(C)$$

Thus, $\text{acl}(C_0a_i) \cap \text{acl}(C_0b_i) \subseteq \text{cl}_p(C)$. Coupling this with the fact that $\text{stp}(a_i b_i/C_0)$ is p -semiregular implies $a_i b_i \perp_{C_0} \text{cl}_p(C)$, hence

$$\text{acl}(a_i C_0) \cap \text{acl}(b_i C_0) \subseteq \text{acl}(C_0)$$

Second, from the proof of Proposition 3.16, $b_i \perp_{C_0a_i} \bar{a}$. By the normality of (a_i, b_i, C_0) , $\text{stp}(b_i/C_0a_i)$ is p -semiregular, so since $Ca \subseteq \text{cl}_p(C_0\bar{a})$ we have $b_i \perp_{C_0a_i} a$. Our assumption was that $b \perp_{Ca} b'$, so in particular, $b_i \perp_{Ca} b'$. Thus, by the transitivity of non-forking

$$b' \perp_{Ca_i} b_i$$

Finally, since $\text{stp}(b'/Ca) = \text{stp}(b/Ca)$, there is $b'_i \in b'$ (corresponding to b_i) such that $b'_i \not\perp_{C_0} a_i$, hence

$$b' \not\perp_{C_0} a_i b_i$$

Combining the last three displayed expressions with Lemma A.26 (where b' takes the role of X) we obtain $a_i \not\perp_{C_0b_i} b'$.

(2) Given (1), this is analogous to the p -weight computation given in Lemma 3.3(b) of [4], just multiplied by k . First, $w_p(abb'/C) = w_p(a/C) + 2w_p(b/Ca) = 4k$. Using (1), $4k = w_p(abb'/C) = w_p(bb'/C)$. Since $w_p(b/C) = w_p(b'/C) = 2k$, we conclude that $w_p(bb'/C) = w_p(b/C) + w_p(b'/C)$. As $\text{stp}(b/C)$ is p -semiregular, this implies $b \perp_C b'$.

(3) This is analogous to 3.3(c) of [4]. We employ Fact A.29(4) and the notation immediately preceding it. By (2), (C, Cb, Cb') is an independent triple, so it suffices to show that $\text{tp}(a/Cbb')$ is V -dominated. Choose any independent triple (\bar{C}, B, B') extending (C, b, b') , i.e., $\bar{C} \supseteq C$ with $\bar{C} \downarrow_C abb'$ and $B \supseteq \bar{C}b$, $B' \supseteq \bar{C}b'$ with $B \downarrow_C B'$. We show that $a \downarrow_{\bar{C}bb'} BB'$ by partitioning BB' into its p -weight zero part and its p -semiregular part. More precisely, let $B_0 := \text{dcl}(B) \cap \text{cl}_p(\bar{C}b)$ and $B'_0 := \text{dcl}(B') \cap \text{cl}_p(\bar{C}b')$. Since $b \downarrow_{\bar{C}a} b'$, there is an automorphism $\sigma \in \text{Aut}(\mathfrak{C})$ fixing $\text{acl}(\bar{C}a)$ pointwise and $\sigma(b) = b'$. Note that σ maps $\text{cl}_p(\bar{C}b)$ onto $\text{cl}_p(\bar{C}b')$, so we can find B_1 with $B_0 \subseteq B_1 \subseteq \text{cl}_p(\bar{C}b)$ and $B'_0 \subseteq \sigma(B_1)$. As our original normal triple (a, b, C) is minimal, it follows from Lemma 3.12 that $R^\infty(a/\bar{C}B_1\sigma(B_1)) = R^\infty(a/Cbb')$. In particular,

$$a \downarrow_{\bar{C}bb'} B_0B'_0$$

Continuing, recall that we are working entirely within a p -simple formula D . Thus, by Lemma A.20, both $\text{stp}(B/B_0)$ and $\text{stp}(B'/B'_0)$ are p -semiregular, with at least one of positive p -weight. As well, $\bar{C} \subseteq B_0 \cap B'_0$ and $B \downarrow_C B'$, hence $\text{stp}(BB'/B_0B'_0)$ is p -semiregular of positive p -weight as well. By (1) $w_p(a/B_0B'_0) = 0$, so $a \downarrow_{B_0B'_0} B_1B'_1$. Thus, $\text{stp}(a/Cbb')$ is V -dominated, and hence by Fact A.29(4) there is $\rho(x, b, b')$ (possibly with hidden parameters from C) isolating $\text{tp}(a/Cbb')$.

(4) By (2), any such b'' satisfies $\text{tp}(bb''/C) = \text{tp}(bb'/C)$, so $\rho(a', b, b'')$ implies $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$. \square

Continuing, we get an analogue of Lemma 3.4[4].

Lemma 3.18 *There is a formula $\alpha(x)$ over Cb such that, for any a' , $\alpha(a')$ implies:*

1. $w_p(a'/C) \leq 2k$, $w_p(a'/Cb) \leq k$, $w_p(b/Ca') \leq k$; and
2. For all b' , if $\text{stp}(b'/Ca') = \text{stp}(b/Ca')$ and $w_p(b'/Ca'b) \geq k$, then both $\rho(a', b, b')$ and $w_p(a'/Cbb')$ = 0 hold.

Proof. First, all of the inequalities involving p -weight are definable by Lemma A.22. In particular, (1) holds. As for (2), we first note that it holds for a . Choose any b' with $\text{stp}(b'/Ca) = \text{stp}(b/Ca)$ and $w_p(b'/Cab) \geq k$. Since $w_p(b/Ca) = k$ by Proposition 3.16, we conclude that $w_p(b'/Ca) = w_p(b/Ca) = k$. As both $\text{stp}(b/Ca)$ and $\text{stp}(b'/Ca)$ are p -semiregular, we conclude that $b \downarrow_{Ca} b'$. Thus $\rho(a, b, b') = 0$ and $w_p(a/Cbb') = 0$ hold by Lemma 3.17. As the implication holds for every b' and since ‘having the same strong type’ is describable by a set of formulas, it follows by compactness that a Cb -definable formula $\alpha(x)$ exists. \square

Next, Lemma 3.5 of [4] becomes:

Lemma 3.19 *For any a' , if $w_p(a'/Cb) = k$ and $\alpha(a')$ holds, then $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$.*

Proof. Fix any such a' and choose b' such that $\text{stp}(b'/Ca') = \text{stp}(b/Ca')$ and $b' \perp_{Ca'} b$. We first compute $w_p(a'b/C) = w_p(a'/bC) + w_p(b/C) = k + 2k = 3k$. Since $w_p(a'/C) \leq 2k$, we have $w_p(b/Ca') = k$. Since $\text{stp}(b'/Ca') = \text{stp}(b/Ca')$, we have $w_p(b'/Ca') = k$ as well. Thus, by Lemma 3.18 we conclude that $w_p(a/Cbb') = 0$ and $\rho(a', b, b')$. But then

$$w_p(bb'/C) = w_p(a/C) + w_p(b/Ca) + w_p(b'/Cab) - w_p(a'/Cbb') = 2k + k + k - 0 = 4k$$

As $\text{stp}(b/C)$ is p -semiregular with $w_p(b/C) = 2k$, the same holds for $\text{stp}(b'/C)$, hence $b \perp_C b'$. Thus, $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$ by Lemma 3.17(4). \square

We now finish the proof of Theorem 3.2. Recall that $\text{tp}(e/\emptyset)$ is stationary and p -semiregular with $w_p(e/\emptyset) = k$. Since (a, b, C) envelops e , $e \perp_C b$ and there is a 0-definable function h such that $h(a) = e$. Let $\alpha^*(x) := \exists x'(\alpha(x') \wedge h(x') = x)$. We claim that $\text{tp}(e/Cb)$ is k -strongly regular via $\alpha^*(x)$. To see this, suppose $\alpha^*(e')$ holds and $w_p(e'/Cb) = k$. Choose $a' \in h^{-1}(e')$ with $\alpha(a')$ holding. Then $w_p(a'/Cb) = k$ as well, so $\text{tp}(a'/Cb) = \text{tp}(a/Cb)$ by Lemma 3.19. In particular, $\text{tp}(e'/Cb) = \text{tp}(e/Cb)$, so $\text{tp}(e/Cb)$ is k -strongly regular. In light of Remark 3.3, so is $\text{tp}(e/\emptyset)$. \square

4 Applications of Theorem 3.2

In this brief section, we give three applications of Theorem 3.2, although they will not be used in the proof of our main results.

Lemma 4.1 *Suppose T is superstable. If H is an infinitely definable connected group (over A) whose generic type is strongly p -semiregular, then H is definable over A .*

Proof. Let r be the generic type of H and $\varphi(x) \in r$ be a formula such that r is the unique type of p -weight k in φ . By superstability, there is a definable group H' with connected component H . The formula $\varphi(x)$ contains the principal generic, hence H' is a finite union of translates of φ . Each translate of φ contains a unique type of p -weight equal to k . Every other generic type of H' must also be of p -weight k , so there are only finitely many generic types in H' . This

means that H , its connected component, has finite index in H' , hence that H itself is in fact definable (H is a closed subgroup in H' , if it has finite index it must be also open). \square

Together with Theorem 3.2, this immediately yields the following corollary.

Corollary 4.2 *Let T be classifiable and p a non-locally modular regular type. If H is an infinitely definable connected group (over A), with generic p -semiregular, then H is definable over A .*

The second application connects binding group constructions with isolation of p -semiregular types, in the spirit of [5] or more recently [6]. First, we record an easy remark.

Remark 4.3 In general, if there is a B -definable group G with a B -definable transitive action on a complete type q over B , then q is isolated over B : Let e realize q , consider the formula $\varphi(x) \in L(Be)$ asserting $\exists g \in G(x = g.e)$. Then $\varphi(x)$ holds if and only if $x \models q$. So q is isolated by a formula over Be . On the other hand, q is B -invariant, so in fact, q is isolated by a formula over B .

Let us now recall some definitions and notation.

Definition 4.4 Let $B = \text{acl}(B)$ and let $p, q \in S(B)$ with p regular. We say that q is p -internal if there is $D \supseteq B$ such that for some $a \models q|D$, $a \in \text{dcl}(D \cup p(\mathfrak{C}))$.

The following existence result is Corollary 7.4.6 of [7].

Proposition 4.5 *Let T be stable. Suppose B is algebraically closed and $p, q \in S(B)$ are non-orthogonal with p regular. Then for any a realizing q , there is $a' \in \text{dcl}(Ba) \setminus B$ such that $\text{tp}(a'/B)$ is p -internal.*

Lemma 4.6 *Let T be stable. Suppose $B = \text{acl}(B) \supseteq M$, $p \in S(M)$ regular and $q \in S(B)$ is p -semiregular. If $q \perp^a M$ then $a \not\downarrow_B p(\mathfrak{C})$ for any a realizing q .*

Proof. Let a realize q and let $E \subseteq p(\mathfrak{C})$ be finite. Let $E_1 \subseteq E$ be a maximal M -independent subset of E , and let E_2 be a maximal subset of E_1 satisfying $E_2 \not\downarrow_M B$. Note that E_2 realizes a Morley sequence $(p|B)^{(k)}$ of the non-forking extension of p to B .

Claim. $w_p(E/E_2B) = 0$.

Proof. First, choose any $e \in E_1 \setminus E_2$. Because maximality implies that $eE_2 \not\downarrow_M B$, $\text{tp}(e/BE_2)$ is a forking extension of a regular

type, hence $w_p(e/E_2B) = 0$. It follows that $w_p(E_1/E_2B) = 0$ as well. Now choose any $e \in E \setminus E_1$. By the maximality of E_1 , $e \not\perp_M E_1$ so $w_p(e/E_1) = 0$. Thus, $w_p(e/E_1B) = 0$. As this holds for every $e \in E \setminus E_1$, $w_p(E/E_1B) = 0$. Combining the two arguments yields $w_p(E/E_2B) = 0$. \square

Because $\text{tp}(E_2/B)$ does not fork over M and $q \perp^a M$, we have that $a \perp_B E_2$. As q is p -semiregular, that $a \perp_B E$ follows immediately from the Claim. \square

Theorem 4.7 (Binding group) ([7] 7.4.8 or [8] 2.2.20) *Suppose T is stable. Let $B = \text{acl}(B)$ and let $p, q \in S(B)$ with p regular and q p -internal. Let $G := \text{Aut}(q(\mathfrak{C})/B \cup p(\mathfrak{C}))$. Then G and its action are infinitely definable in the following sense: there is G_1 an infinitely definable group over B and a B -definable action of G_1 on $q(\mathfrak{C})$ such that, as permutation groups of $q(\mathfrak{C})$ over $B \cup p(\mathfrak{C})$, G and G_1 are isomorphic. Additionally, if $a \perp_B p(\mathfrak{C})$ for all a realizing q , then G and hence also G_1 act transitively on $q(\mathfrak{C})$.*

Note that by stability, the group G is also the group of restrictions to $q(\mathfrak{C})$ of the automorphisms of \mathfrak{C} fixing $B \cup p(\mathfrak{C})$ pointwise. G is infinite if and only if q is not algebraic over $B \cup p(\mathfrak{C})$. The next proposition tells us that we get a transitive action from the connected component of G_1 .

Proposition 4.8 *If an infinitely definable group G is defined over $B = \text{acl}(B)$ and there is a B -definable, transitive action of G on a complete type $q \in S(B)$, then the restriction of the action to the connected component G^0 of G is also transitive.*

Proof. For notational simplicity suppose $B = \emptyset$.

Claim 1. There is some pair (e, e^*) realizing $q \otimes q$ and some $h \in G^0$ such that $h.e = e^*$.

Proof. Choose a set of representatives $R \subseteq G(\mathfrak{C})$ such that every $g \in G$ can be written as ch for some $c \in R$ and $h \in G^0$, i.e., R contains an element of every G^0 -coset of G . As the index $[G : G^0]$ is bounded, we may choose R of bounded size (2^{\aleph_0} if the language is countable). Choose any e realizing $q|_R$ and any e' realizing $q|_{Re}$.

By the transitivity of the action, as both e, e' realize q , choose $g \in G$ such that $g.e = e'$. By choice of R , choose $c \in R$ and $h \in G^0$ such that $g = ch$ and put $e' := h.e$. We need to show that $e^* \perp e$. For this, note that

$$e^* = (c^{-1}g).e = c^{-1}.e' \in \text{dcl}(Re')$$

But $e' \downarrow Re$ and $e \downarrow R$ imply $e'R \downarrow e$, hence $e^* \downarrow e$ as required. \square

Claim 2. For every (e, e') realizing $q \otimes q$ there is $h \in G^0$ such that $h.e = e'$.

Proof. Homogeneity of \mathfrak{C}/B . Fix (e, e^*) and h as in Claim 1, and let (e_1, e_2) be any other realization of $q \otimes q$. Choose an automorphism σ of \mathfrak{C} , fixing B pointwise with $\sigma(e) = e_1$ and $\sigma(e^*) = e_2$. As G^0 and the action are B -definable, $\sigma(h) \in G^0$ and $\sigma(h).e_1 = e_2$. \square

To complete the proof of the Proposition, choose any $e, f \in q(\mathfrak{C})$. Choose e^* realizing $q|\{e, f\}$. By Claim 2, choose $h_1 \in G^0$ such that $h_1.e = e^*$ and choose $h_2 \in G^0$ such that $h_2.e^* = f$. Then $h_2h_1 \in G^0$ and $(h_2h_1).e = f$, so G^0 acts transitively on $q(\mathfrak{C})$. \square

The following results give a sufficient condition for a non-locally modular type to be isolated. These results will be used as part of the forthcoming work of the authors on the analysis of weight one models in classifiable theories. The following definition appears already in [5].

Definition 4.9 Suppose T is superstable. We say that $q \in S(B)$ is *c-isolated* (following Hrushovski-Shelah in [5]) if there is a formula $\theta(x) \in q$ such that $R^\infty(\theta(x)) = R^\infty(q) = \alpha$ and furthermore, for any $r \in S(B)$, $\theta(x) \in r$ implies $R^\infty(r) = \alpha$.

Proposition 4.10 Suppose T is superstable. Let $q \in S(B)$ be *p-strongly semiregular* and *c-isolated* via the formula $\varphi(x)$ and let G_1 be an infinitely definable group over B , with a B -definable, transitive action of G_1 on $q(\mathfrak{C})$. Then q is isolated.

Proof. By the usual construction (e.g., Lemma 1.6.19 in [7]) find a B -definable supergroup $H \supseteq G_1$, a B -definable set X containing q , and a B -definable, transitive action of H on X which extends the action of G_1 on $q(\mathfrak{C})$, and such that $X \subset \varphi(\mathfrak{C})$.

So X has the property that every type over B in X has the same R^∞ rank as q , say α , that every type in X is p -simple, of p -weight at most equal to k , the p -weight of q , and that q is the unique type in X of p -weight exactly k . We show that X isolates q .

Claim 1. Let e realize q , and let $h \in H$ be independent from e over B , then $h.e$ also realizes q .

Proof. Letting $d = h.e$, d and e are inter-definable over Bh . Since e realizes $q|Bh$, $w_p(e/Bh) = k$, hence $w_p(d/B) \geq k$. By our assumptions on X , $w_p(d/B) = k$, so d realizes q . \square

Claim 2. Let h be any element of H , not necessarily independent from e , and let $d = h.e$. Then d realizes q .

Proof. Let g be a generic in H , independent from h, e (and hence from h, e, d) over B . It follows that $g^{-1}h$ and e are independent over $B : g^{-1} \underset{Bh}{\perp} e$ by choice of g , hence $(g^{-1}h) \underset{Bh}{\perp} e$. As g^{-1} is generic and independent from h , $g^{-1}h$ is also independent from h (see e.g., 1.6.9(iv) of [7]), so it follows that $g^{-1}h$ and e are independent over B . Hence by Claim 1, $f := (g^{-1}h).e$ realizes q .

We next show g and f are independent over B : Since $f = g^{-1}.d$, d and f are inter-definable over Bg , hence they have same ∞ -rank over Bg . By c -isolation, $R^\infty(d/B) = \alpha$. By our choice of g , $g \underset{B}{\perp} d$, hence $\alpha = R^\infty(d/B) = R^\infty(d/Bg) = R^\infty(f/Bg)$. As $f \in X$, $R^\infty(f/B) \leq \alpha$, so $R^\infty(f/Bg) = R^\infty(f/B)$, implying f and g are independent over B .

Thus, by Claim 1, $g.f$ must realize q , but $g.f = d$. \square

As the action of H on X is transitive, Claim 2 implies that any $d \in X$ must realize q . That is, X isolates q . \square

Corollary 4.11 *Let T be classifiable with $M \subseteq_{na} \mathfrak{C}$ and $B \supseteq M$ algebraically closed and suppose p is a non-locally modular regular type with $p \not\perp M$. If $q = \text{tp}(a/B)$ is p -semiregular, c -isolated, but $q \perp^a M$, then there is $a' \in \text{dcl}(Ba) \setminus B$ such that $q' := \text{tp}(a'/B)$ is isolated.*

Proof. Since $M \subseteq_{na} \mathfrak{C}$ and $p \not\perp M$ is regular, there is a regular $p_0 \in S(M)$ non-orthogonal to p by Fact A.5. To ease notation, we may assume that our p is the non-forking extension of p_0 to $S(B)$. By Proposition 4.5 choose $a' \in \text{dcl}(Ba) \setminus B$ such that $q' := \text{tp}(a'/B)$ is p -internal. It is easily checked that q' remains c -isolated and p -semiregular and that $q' \perp^a M$. Since T is classifiable and p is non-locally modular, q' is strongly p -semiregular by Theorem 3.2.

As $q \perp^a M$, by Lemma 4.6 we have $a' \underset{B}{\perp} p_0(\mathfrak{C})$, hence $a' \underset{B}{\perp} p(\mathfrak{C})$ as well. Thus, by Lemma 4.7, there is an infinitely B -definable group G_1 with a B -definable, transitive action on $q'(\mathfrak{C})$. Hence q' is isolated by Proposition 4.10. \square

Note that in Corollary 4.11 we used two conditions on the type q' to prove isolation: the strong p -semiregularity condition goes up to non forking extensions, but the c -isolation does not necessarily.

We finish this section with a third way of proving isolation, without group actions, in the case of a strongly regular type. It is not clear if this method could be generalized to the case of strong p -semiregularity.

Proposition 4.12 *Suppose that T superstable, $M \subseteq_{na} \mathfrak{C}$, $B \supseteq M$ is algebraically closed, and $q \in S(B)$ is strongly regular, depth zero, and that $q \perp^a M$. Then either $q \perp M$ or q is isolated.*

Proof. If q is trivial, $q \perp^a M$ implies that $q \perp M$ (see, e.g., X 7.3(6) of [10]). So we can suppose that q is non-trivial. Henceforth, assume that q is non-orthogonal to M . By Fact A.5 choose a regular $p \in S(M)$ with $q \not\perp p$. As $q \perp^a M$, in particular, over B , q is almost orthogonal to $p^{(\omega)}$. As q , and hence p , is non-trivial of depth 0, use Fact A.22 to choose $\varphi \in q$ that is p -simple, such that p -weight is defined and continuous in φ . As q is strongly regular, by strengthening φ we may additionally assume q is the only type over B containing φ of positive p -weight.

Now choose n least such that there are $\bar{a} = (a_1, \dots, a_n)$ with each a_i realizing φ and $\text{tp}(\bar{a}/B)$ is not almost orthogonal to $p^{(\omega)}$. We know that such a finite n exists, since q non-orthogonal to p implies that some $q^{(\ell)}$ is not almost orthogonal to $p^{(\omega)}$ (see e.g., 4.3.1(iii) of [7]). Remark: Here, however, we are minimizing n without assuming \bar{a} is B -independent.

Note that $n \geq 2$. Indeed, if a_1 realizes q , then a_1/B is almost orthogonal to $p^{(\omega)}$ by assumption. On the other hand, if a_1 realizes φ but not q , then $w_p(a_1/B) = 0$, so a_1 cannot fork over B with any independent set of realizations of p .

Once n is fixed, choose k such that \bar{a}/B is not almost orthogonal to $p^{(k)}$. To save writing, let $n = m + 1$ and $r(\bar{y}) := (p|B)^{(k)}$. Choose a specific realization \bar{c} of r such that $\bar{a} \not\perp_B \bar{c}$, and choose an $L(B)$ -formula $\theta(\bar{x}, \bar{y}) \in \text{tp}(\bar{a}\bar{c}/B)$ witnessing the forking. Let

$$\gamma(x_0) := \varphi(x_0) \wedge d_{r\bar{y}} \left[\exists x_1 \dots \exists x_m \left(\bigwedge \varphi(x_i) \wedge \theta(x_0, x_1, \dots, x_m, \bar{y}) \right) \right]$$

As B is algebraically closed, γ is over B . We argue that γ isolates q .

To see this, choose any b_0 realizing γ . Choose \bar{d} realizing $r|Bb_0$, and choose witnesses b_1, \dots, b_m . Thus, the n elements b_0, \dots, b_m each realize φ and $\theta(\bar{b}, \bar{d})$ holds. We argue that in fact, every b_i realizes q . Let $I = \{i \leq m : b_i \text{ realizes } q\}$. Let \bar{b}_I be the subsequence of \bar{b} induced by I . By way of contradiction, assume $|I| < m + 1 = n$. By the minimality of n , we must have $\bar{b}_I \not\perp_B \bar{d}$. But also, by our choice of φ , if $i \notin I$, then $w_p(b_i/B) = 0$. Thus, $w_p(\bar{b}/B\bar{b}_I) = 0$. But \bar{d} is a Morley sequence in p , hence $\text{stp}(\bar{d}/B\bar{b}_I)$ is p -semiregular. Thus, $\bar{b} \perp_{B\bar{b}_I} \bar{d}$. It follows by transitivity that $\bar{b} \perp_B \bar{d}$, which is contradicted by $\theta(\bar{b}, \bar{d})$. \square

Corollary 4.13 *Let T be classifiable, $M \subseteq_{\text{na}} \mathfrak{C}$, $B \supseteq M$ algebraically closed, and let $q \in S(B)$ be regular, but non locally modular. If $q \not\perp M$ but $q \perp^a M$, then q is isolated.*

Proof. By the classifiability of T , q must also be strongly regular of depth zero, so Proposition 4.12 applies. \square

5 Constructible, minimal models over realizations of non-locally modular types

Recall our global assumption that all theories T are (at least) stable. The basic definitions and facts about locally modular regular types can be found in the Appendix (Section A.2).

5.1 Witnesses to non-modularity

We describe a minimal counterexample to non-modularity in our context.

Definition 5.1 Let M be any model (not necessarily an a-model) and let $p \in S(M)$ be regular. A quadruple (a, b, c, d) is *4-dependent* if

1. Any three elements realize $p^{(3)}$, but
2. $w_p(abcd/M) = 3$.

A *witness to non-modularity for p over M* is a set of parallel lines, i.e., some 4-dependent quadruple (a, b, c, d) such that $\text{cl}_p(Mab) \cap \text{cl}_p(Mcd) = \text{cl}_p(M)$.

The following proposition follows from Section A.2 of the appendix.

Proposition 5.2 *Over any a-model M , if $p \in S(M)$ is a regular type, then p is non-locally modular if and only if there is a witness to non-modularity for p over M .*

Lemma 5.3 *Suppose M is any countable model, and $p \in S(M)$ is regular, but not locally modular. There is an ϵ -finite set E (E is contained in the algebraic closure of some finite set) such that for any countable model M' containing $M \cup E$, there is a witness (a, b, c, d) to non-modularity over M' .*

Proof. Let $M^* \supseteq M$ be any a-model, and let q denote the non-forking extension of p to M^* . As q is not locally modular, it follows from Proposition 5.2 that there is a witness (a, b, c, d) to non-modularity over M^* . Choose a countable, ϵ -finite E over which $\text{tp}(abcd/M^*)$ is based and stationary. To see that E suffices, choose any countable M' containing $M \cup E$. As M^* is sufficiently saturated, we may assume that $M' \preceq M^*$. We argue that (a, b, c, d) is a witness to non-modularity over M' . To see this, note that $abcd \downarrow_{M'} M^*$. Thus, (a, b, c, d) is 4-dependent with respect to the non-forking extension

of p to M' . For the final clause, let $C := \text{dcl}(M'abcd) \cap \text{cl}_p(M')$. Note that $abcd \downarrow_C M^*$ and $\text{stp}(abcd/C)$ is p -semiregular by Criterion A.18. Thus, $\text{cl}_p(M^*ab) \cap \text{cl}_p(M^*cd) = \text{cl}_p(M^*)$ implies that $\text{cl}_p(Cab) \cap \text{cl}_p(Ccd) = \text{cl}_p(C)$ by Proposition A.24. However, $\text{cl}_p(C) = \text{cl}_p(M')$, $\text{cl}_p(Cab) = \text{cl}_p(M'ab)$, and $\text{cl}_p(Ccd) = \text{cl}_p(M'cd)$, so we finish. \square

5.2 A definable witness to non-modularity

In this section we assume throughout that T is classifiable. Under this assumption, we obtain a definable witness to non-modularity over an arbitrary model M on which a non-locally modular type is based. The starting point is the following, which melds Theorem 1b of [4] with Theorems A.22 and A.28 from the appendix.

Theorem 5.4 *If T is classifiable, B is algebraically closed, and $p \in S(B)$ is a regular, non-locally modular type, then there is a formula $\theta \in p$ such that (recall Definition A.12):*

1. θ is p -simple of p -weight one;
2. p -weight is defined and continuous inside θ ; and
3. p is strongly regular via the formula θ , i.e., for every $C \supseteq B$ and every $e \in \theta(\mathfrak{C})$, if $w_p(e/C) > 0$, then $\text{tp}(e/C)$ is the non-forking extension of p to C .

For the next few Lemmas, fix a model M and non-locally modular $p \in S(M)$. Also fix $\theta(x, \bar{m}) \in p$ as in Theorem 5.4 with $\theta(x, \bar{y})$ 0-definable.

Lemma 5.5 *Suppose $p \in S(M)$ is regular and (a, b, c, d) is 4-dependent over M . Then there is some $h \subseteq M$ and an \bar{m} -definable formula $R(x, y, z, w, u) \in \text{tp}(abcdh)$ such that for any (a', b', c', d') with some triple realizing $p^{(3)}$ and for any $h' \subseteq M$, if $R(a', b', c', d', h')$ holds, then (a', b', c', d') is 4-dependent over M .*

Proof. First, as $w_p(a/bcdM) = 0$ and $\theta(a)$ holds, choose $h_1 \subseteq M$ and $\alpha_1(x, y, z, w, u) \in \text{tp}(abcdh_1)$ such that for any a', b', c', d' and any $h'_1 \subseteq M$, if $\alpha_1(a', b', c', d', h'_1)$ holds, then $\theta(a')$ holds and $w_p(a'/b'c'd'M) = 0$. Similarly, choose $h_2, h_3, h_4 \subseteq M$ and $\alpha_2, \alpha_3, \alpha_4$ such that e.g., $\alpha_2(a', b', c', d', h'_2)$ implies $\theta(b')$ and $w_p(b'/a'c'd'M) = 0$. Put $h := h_1 h_2 h_3 h_4$ and put $R(x, y, z, w, u) := \bigwedge_{i=1}^4 \alpha_i$.

We argue that R is as claimed. By symmetry, choose $b'c'd'$ realizing $p^{(3)}$ and $h' \subseteq M$. We verify that for every $a' \in \mathfrak{C}$, if $R(a', b', c', d', h')$

holds, then (a', b', c', d') is 4-dependent over M . To see this, since $\alpha_1(a', b', c', d', h')$ holds, we have $\theta(a')$ and $w_p(a'/b'c'd'M) = 0$. Because of α_2 holding, $w_p(b'/a'c'd'M) = 0$. As $b'c'd'$ realizes $p^{(3)}$, we must have $w_p(a'/M) > 0$. Thus, by strong regularity, $\text{tp}(a'/M) = p$. To verify 4-dependence over M , it remains to show that each of $a'b'c'$, $a'b'd'$ and $a'c'd'$ realize $p^{(3)}$. We check this for $a'b'c'$, with the other two following by symmetry. As $\alpha_4(a', b', c', d', h')$ holds, $w_p(d'/a'b'c'd'h') = 0$. But, as $b'c'd'$ realizes $p^{(3)}$ this implies $d' \not\downarrow_{Mb'c'} a'$. As $\text{tp}(d'/Mb'c')$ is regular, this implies $a' \not\downarrow_M b'c'$, so $a'b'c'$ realizes $p^{(3)}$. \square

Next, among all 4-dependent quadruples (a, b, c, d) , we want to distinguish those that are witnesses to non-modularity over M . This is the content of the next two Lemmas.

Lemma 5.6 *Suppose $p \in S(M)$ is a regular, non-locally modular type and (a, b, c, d) is a witness to non-modularity over M . Then for every e realizing $p|Mabcd$ and for every $f \in \theta(\mathfrak{C})$ that satisfies $b \in \text{cl}_p(Maef)$, we have (c, d, e, f) realizes $p^{(4)}$ and $w_p(ab/Mcdef) = 0$.*

Proof. First, since $\theta(f)$ holds, $\text{tp}(f/M)$ is p -simple. As b realizes p , $b \not\downarrow_M ae$, but $b \in \text{cl}_p(Maef)$, we must have $w_p(f/M) > 0$, hence $\text{tp}(f/M) = p$. Thus, all six elements a, b, c, d, e, f realize p . Next, note that $w_p(abcdef/M) = 4$, since $w_p(abcd/M) = 3$ (by 4-dependence), $e \not\downarrow_M abcd$, and $w_p(f/Mabcde) = 0$ by exchange. Next, we show that $w_p(cdef/M) = 4$, i.e., that (c, d, e, f) realizes $p^{(4)}$. By way of contradiction, assume that this were not the case, i.e., that $w_p(cdef/M) \leq 3$. We compute the following p -weights:

- $w_p(ef/Mabcd) = 1$ [it is ≥ 1 because of e , but < 2 since $f \in \text{cl}_p(Mabe)$].
- $w_p(ef/Mcd) = 1$ [it is ≥ 1 from the former line, but if it was $= 2$, then we would have $w_p(cdef/M) = 4$].
- $w_p(ef/Mab) = 1$ [it is > 0 because of e , but < 2 because $f \in \text{cl}_p(Mabe)$].

As both $\text{stp}(ef/\text{cl}_p(Mab))$ and $\text{stp}(ef/\text{cl}_p(Mcd))$ are p -semiregular,

$$ef \not\downarrow_{\text{cl}_p(Mab)} abcd \quad \text{and} \quad ef \not\downarrow_{\text{cl}_p(Mcd)} abcd$$

Hence,

$$Cb(ef/Mabcd) \subseteq \text{cl}_p(Mab) \cap \text{cl}_p(Mcd) = \text{cl}_p(M)$$

with the last equality holding since (a, b, c, d) is a witness to non-modularity. This would imply $ef \downarrow_{\text{cl}_p(M)} abcd$, which contradicts $f \in \text{cl}_p(Mabe)$.

Finally, since $w_p(abcdef/M) = w_p(cdef/M) = 4$, it follows immediately that $w_p(ab/Mcdef) = 0$. \square

By contrast:

Lemma 5.7 *Suppose (a, b, c, d) is 4-dependent over M , but $\text{cl}_p(Mab) \cap \text{cl}_p(Mcd) \neq \text{cl}_p(M)$. Then there are ef such that $\text{stp}(ef/Mab) = \text{stp}(cd/Mab)$ with e realizing $p|Mabcd$, but $w_p(cdef/M) = 3$.*

Proof. Choose $g \in (\text{cl}_p(Mab) \cap \text{cl}_p(Mcd)) \setminus \text{cl}_p(M)$. Since $g \in \text{cl}_p(Mab)$, $\text{tp}(g/M)$ is p -simple. As $g \notin \text{cl}_p(M)$, $w_p(g/M) > 0$, but since $c \notin \text{cl}_p(Mab)$, we cannot have $w_p(g/M) \geq 2$. Then $w_p(g/M) = 1$ and $w_p(cd/Mg) = 1$. Now choose any ef such that $\text{stp}(ef/Mabg) = \text{stp}(cd/Mabg)$ with $ef \downarrow_{Mabg} cd$. Then e realizes $p|Mabcd$ and $w_p(ef/Mg) = 1$. So

$$w_p(cdef/M) \leq w_p(cdefg/M) \leq w_p(cd/Mg) + w_p(ef/Mg) + w_p(g/M) = 3$$

\square

Lemma 5.8 *Suppose $p \in S(M)$ is regular and that (a, b, c, d) witnesses non-modularity over M . Fix an $L(M)$ -formula $\theta(x, \bar{m}) \in p$ from Theorem 5.4. Then there is some $h \subseteq M$ and an \bar{m} -definable $R^*(x, y, z, w, u) \in \text{tp}(a, b, c, d, h)$ such that, for any $h' \subseteq M$, any $b'c'd'$ realizing $p^{(3)}$, and any $a' \in \mathfrak{C}$, if $R^*(a', b', c', d', h')$ holds, then (a', b', c', d') witnesses the non-modularity of p over M .*

Proof. As (a, b, c, d) is 4-dependent over M choose $h \subseteq M$ and $R(x, y, z, w, u)$ as in Lemma 5.5. Choose any $e \models p|Mabcd$. Then, for every f , if $R(a, b, e, f, h)$ holds, then since $w_p(b/ae fM) = 0$ we have $a \in \text{cl}_p(Mcdef)$ by Lemma 5.6. Thus, by compactness (and since p -weight 0 formulas are closed under finite disjunctions) there is an L -formula $\gamma(x, y, z, w, v_1, v_2, u)$ such that

- $\gamma(a, b, c, d, e, f, h)$ holds for any $e \models p|Mabcd$ and any f for which $R(a, b, e, f, h)$ holds; and
- For all $(a', b, c', d', e', f', h')$, $\gamma(a', b', c', d', e', f', h')$ holding implies $w_p(a'/c'd'e'f'h') = 0$.

Put $R^*(x, y, z, w, u) :=$

$$R(x, y, z, w, u) \wedge d_p v_1 \forall v_2 [R(x, y, v_1, v_2, u) \rightarrow \gamma(x, y, z, w, v_1, v_2, u)]$$

From above, $R^* \in \text{tp}(a, b, c, d, h)$. Now choose any $h' \subseteq M$ and (a', b', c', d') such that $R^*(a', b', c', d', h')$ holds. By Lemma 5.5 we know that (a', b', c', d') is 4-dependent over M . By way of contradiction, suppose that (a', b', c', d') is not a witness to the non-modularity of M . Choose any $e' \models p|M a' b' c' d'$. By Lemma 5.7, there is f' such that $\text{stp}(e' f' / M a' b') = \text{stp}(c' d' / M a' b')$ but $w_p(c' d' e' f' / M) = 3$. Since $R(a', b', c', d', h')$ holds, we have $R(a', b', e', f', h')$ holding as well. Thus, $\gamma(a', b', c', d', e', f', h')$ holds, hence $w_p(a' / c' d' e' f' M) = 0$. By $R(a', b', e', f', h')$ again, we have $w_p(b' / a' e' f' M) = 0$, so $w_p(a' b' c' d' e' f' / M) = 3$. But this is impossible since $b' c' d'$ realizes $p^{(3)}$ and $e' \models p|M b' c' d'$. \square

Proposition 5.9 *If T is classifiable, then for any model M and any $p \in S(M)$ that is regular but not locally modular, there is a witness (a, b, c, d) to the non-modularity of p over M . Thus, there is an $L(M)$ -formula $S(x, y, z, w) \in \text{tp}(abcd/M)$ such that for any $b' c' d'$ realizing $p^{(3)}$, $S(x, b', c', d')$ is consistent and (a', b', c', d') is a witness to non-modularity of p over M for every realization a' of $S(x, b', c', d')$.*

Proof. It suffices to show that a witness to the non-modularity of p exists, since the second sentence follows from this via Lemma 5.8. Choose $\theta(x, \bar{m}) \in p$ as in Theorem 5.4 and choose any bcd realizing $p^{(3)}$. Choose any a -model $M^* \succeq M$ independent from bcd over M . By Proposition 5.2 there is a witness (a_0, b_0, c_0, d_0) to the non-modularity of $p|M^*$ over M^* . As $\text{tp}(bcd/M^*) = \text{tp}(b_0 c_0 d_0/M^*)$, there is some a such that (a, b, c, d) is a witness to the non-modularity of $p|M^*$ over M^* . Choose an \bar{m} -definable $R^*(x, y, z, w, h)$ with $\bar{h} \subseteq M^*$ from Lemma 5.8.

So $\exists x R^*(x, b, c, d, h)$ holds. As $bcd \downarrow_M h$, finite satisfiability gives some $h' \subseteq M$ for which $\exists x R^*(x, b, c, d, h')$. Choose any $a' \in R^*(\mathfrak{C}, b, c, d, h')$ with $a' \downarrow_{M^{bcd}} M^*$. By Lemma 5.8 (a', b, c, d) witnesses the non-modularity of $p|M^*$ over M^* , but also $abcd \downarrow_M M^*$. It follows by non-forking calculus that (a', b, c, d) is 4-dependent over M . As well, it follows from the ‘easy half’ of Lemma A.24 (not requiring p -semiregularity) that $\text{cl}_p(a' b M) \cap \text{cl}_p(c d M) = \text{cl}_p(M)$. [In more detail, if there were a ‘bad’ g , then taking g' such that $\text{stp}(g' / a' b c d M) = \text{stp}(g / a' b c d M)$ but $g' \downarrow_{M^{a' b c d}} M^*$, we would have $g' \in (\text{cl}_p(a' b M^*) \cap \text{cl}_p(c d M^*)) \setminus \text{cl}_p(M^*)$, contradicting (a', b, c, d) a witness to the non-modularity of $p|M^*$ over M^* .] \square

The following Corollary codifies what we will use in the subsequent sections.

Corollary 5.10 *Suppose T is classifiable, $M \preceq N \models T$, and $p \in S(M)$ is regular but not locally modular. If there are $(b, c, d) \in N \setminus M$ realizing $p^{(3)}$, then there is $a \in N \setminus M$ such that (a, b, c, d) is a witness to the non-modularity of p over M .*

Proof. Choose an $L(M)$ -formula S as in Proposition 5.9. Then $\mathfrak{C} \models \exists x S(x, b, c, d)$ so $N \models \exists x S(x, b, c, d)$ by elementarity. By Proposition 5.9, any $a \in N$ satisfying $S(a, b, c, d)$ suffices. \square

5.3 Depth-zero like types and minimality

Recall the definition of a regular type p being of depth zero given in Definition 2.2. By working with a -models, it is easily checked that a regular type being of depth zero is preserved under non-orthogonality.

Lemma 5.11 *Suppose T is superstable.*

1. *For M any model, if $p = \text{tp}(c/M)$ is regular of depth zero, then for any model M' dominated by c over M , every regular q non-orthogonal to M' is non-orthogonal to M .*
2. *Suppose $M \subseteq_{na} N$, $c \in N \setminus M$ is such that $p = \text{tp}(c/M)$ is regular of depth zero, and let $M' \preceq N$ be any model dominated by c over M . Then M' is minimal over Mc and $M' \subseteq_{na} N$.*

Proof. (1) Fix M, c, M' and q , and suppose $q \in S(A)$. By superstability choose a finite $B \subseteq M'$ such that $q \not\perp B$. Choose any a -model $M^* \succeq M$ with $M^* \downarrow_{M'} ABc$. By Fact A.8(2), Bc is dominated by c over M^* , so by Fact A.8(3) there is an a -prime model $M^*[c]$ containing B . As $q \not\perp M^*[c]$ and since $M^*[c]$ is an a -model, there is a regular $q' \in S(M^*[c])$ with $q' \not\perp q$. As $\text{tp}(c/M^*)$ also has depth zero, q' (and hence q) is non-orthogonal to M^* . As A is independent from M^* over M , we conclude that $q \not\perp M$.

(2) We first show that M' is minimal over Mc . By way of contradiction, assume there were a proper $M'' \prec M'$ containing Mc . By Fact A.4 there is a regular type $q \in S(M'')$ realized in M' . By (1), $q \not\perp M$, so as $M \subseteq_{na} M'$, it follows from the 3-model Lemma (Fact A.7) that there is some $a \in M' \setminus M$ with $a \downarrow_{M'} M''$. But this contradicts M'' being dominated by c over M . Thus, M' is minimal over Mc . To see that $M' \subseteq_{na} N$, as $M \subseteq_{na} N$, apply Fact A.6, taking A to be M' , to obtain $M^* \subseteq_{na} N$ containing M' with M^* dominated by M' over M . As M' was dominated by c over M , the same holds for M^* by Fact A.8(1). From the previous sentences, M^* is minimal over Mc , hence $M^* = M'$. \square

The following Lemma extends this to independent tuples of depth zero types.

Lemma 5.12 *Suppose T is superstable and $M \subseteq_{\text{na}} N$. For every n , if $\{c_i : i < n\} \subseteq N$ are M -independent with $\text{tp}(c_i/M)$ regular of depth zero for each i , then a model $M^* \preceq N$ containing $M \cup \{c_i : i < n\}$ and dominated by $\{c_i : i < n\}$ over M exists. Moreover, any such M^* satisfies the following properties:*

1. M^* is minimal over $M \cup \{c_i : i < n\}$;
2. $M^* \subseteq_{\text{na}} N$;
3. If q is regular and $q \not\perp M^*$, then $q \not\perp M$.

Proof. For any n , the existence of a model $M^* \preceq N$ dominated by c_0, \dots, c_{n-1} over M is immediate by Fact A.6. We argue by induction on n that (1)–(3) hold for any model $M^* \preceq N$ dominated by c_0, \dots, c_{n-1} over M . For $n = 1$, this is given by Lemma 5.11. Assume these conclusions hold for sets of size n , and choose c_0, \dots, c_n from N satisfying the hypotheses. Let M^* be dominated by $\{c_i : i \leq n\}$ over M . We first show M^* is minimal over $M \cup \{c_i : i \leq n\}$. Choose any $M' \preceq M^*$ containing $M \cup \{c_i : i \leq n\}$. As $M \subseteq_{\text{na}} M'$ we work inside M' . Let $M_0 \preceq M'$ be dominated by $\{c_i : i < n\}$ over M . By our inductive hypothesis, both $M_0 \subseteq_{\text{na}} M'$ and any regular type q non-orthogonal to M_0 is non-orthogonal to M . By Fact A.6, choose $M_1 \preceq M'$ to be dominated by c_n over M_0 . By Lemma 5.11, any regular type non-orthogonal to M_1 is non-orthogonal to M_0 . We argue that $M_1 = M^*$, which implies our desired $M' = M^*$. For this, suppose $M_1 \neq M^*$. By Fact A.4, there is some $d \in M^* \setminus M_1$ with $q := \text{tp}(d/M_1)$ regular. From above, q is also non-orthogonal to M_0 and hence to M . So, by the 3-model Lemma (Fact A.7) applied to the triple (M, M_1, M^*) , there would be $e \in M^*$ such that $\text{tp}(e/M)$ is regular and non-orthogonal to q , with $e \downarrow M_1$. As $\{c_i : i \leq n\}$, this implies $e \downarrow_M c_0, \dots, c_n$, contradicting M^* being dominated by $\{c_i : i \leq n\}$ over M . Thus, M^* is minimal over $M \cup \{c_i : i \leq n\}$, proving (1).

Next, by Fact A.6 there is a model $N' \subseteq_{\text{na}} N$ that contains and is dominated by M^* over M . By Fact A.8(1), N' is dominated by $\{c_i : i \leq n\}$ over M , so by (1) applied to N' , it is minimal over $M \cup \{c_i : i \leq n\}$. Thus, $N' = M^*$, so $M^* \subseteq_{\text{na}} N$, giving (2).

Finally, choose any regular q non-orthogonal to $M^* = M_1$, using the notation in (1). Since $\text{tp}(c_n/M_0)$ has depth zero, q is non-orthogonal to M_0 by Lemma 5.11, hence is non-orthogonal to M by our inductive hypothesis. \square

The following definition extends the concept of depth zero to both finite, independent tuples of depth zero types as well as to types dominated by such tuples.

Definition 5.13 A strong type p is *depth-zero like* if every regular type q non-orthogonal to p is of depth zero.

As examples, if a regular type p has depth zero, then any p -semiregular type q is depth zero-like. The following Proposition uses classifiability to obviate the need for $M \subseteq_{\text{na}} \mathfrak{C}$ in Lemma 5.12(1).

Proposition 5.14 *Suppose p is depth zero-like.*

1. *If T is superstable and M is an a -model on which p is based, then for any realization b of $p|_M$ and every model $N \supseteq Mb$ that is dominated by b over M we have:*
 - (a) *every regular type q non-orthogonal to N is non-orthogonal to M : and*
 - (b) *N is minimal over Mb .*
2. *If T is classifiable, then (1) holds for every model M on which p is based.*

Proof. (1) Assume T is superstable. Fix an a -model M on which p is based and a realization b of $p|_M$. We first show (a) holds when $N = M[b]$, any a -prime model over Mb . To see this, choose a maximal M -independent set $\{c_1, \dots, c_n\} \subseteq M[b]$ with $\text{tp}(c_i/M)$ regular. (n is finite because $\text{tp}(b/M)$ has finite weight.) As each c_i forks with b over M , $\text{tp}(c_i/M)$ is of depth zero. Let $M[c_1, \dots, c_n] \preceq M[b]$ be any model dominated by $\{c_1, \dots, c_n\}$ over M . We claim that $M[c_1, \dots, c_n] = M[b]$. For, if not, then by Fact A.4 there would be some $d \in M[b] \setminus M[c_1, \dots, c_n]$ with $r := \text{tp}(d/M[c_1, \dots, c_n])$ regular. By Lemma 5.12(3), we would have $r \not\perp M$, hence by Fact A.7 there would be $e \in M[b] \setminus M$ with $\text{tp}(e/M)$ regular, but $e \underset{M}{\perp} c_1, \dots, c_n$, contradicting the maximality of $\{c_1, \dots, c_n\}$. Thus, (a) holds for $M[b] = M[c_1, \dots, c_n]$ by Lemma 5.12(3). For the general case, take $N \supseteq Mb$ to be any model dominated by b over M . Choose any regular q non-orthogonal to N . By superstability, choose a finite $d \subseteq N$ such that $q \not\perp Mbd$. As bd is dominated by b over M , apply Fact A.8(3) to find an a -prime $M[b]$ over Mb containing d . Then $q \not\perp M[b]$, hence $q \not\perp M$ by the special case above.

For (b), choose any model $N \supseteq Mb$ that is dominated by b over M . Choose any $N' \preceq N$ containing Mb and assume by way of contradiction that $N' \neq N$. By Fact A.4, choose $c \in N \setminus N'$ such that $q = \text{tp}(c/N')$ is regular. By (a) applied to N' and M we have $q \not\perp M$.

So, by the 3-model Lemma (Fact A.7) , there is $c^* \in N - M$ that does not fork with N' over M . As $b \in N'$ we conclude that $c^* \not\downarrow_M b$, which contradicts N being dominated by b over M .

(2) Now assume that T is classifiable. Fix any model M on which p is based and fix a realization b of $p|_M$. Suppose N is dominated by b over M . To show that N is minimal over Mb , choose any $N_0 \preceq N$ containing Mb . To see that $N_0 = N$, choose any $c \in N$ and we will conclude that $c \in N_0$. To start, choose any a-model $M^* \succeq M$ with $b \downarrow_M M^*$. By Fact A.8(2), N is dominated by b over M^* . By PMOP, choose a constructible model N_2 over $N \cup M^*$. By Fact A.9(2) N_2 is dominated by N over M^* , hence N_2 is also dominated by b over M^* by Fact A.8(1). It follows from (1) that N_2 is minimal over M^*b . Also, $c \in N_2$.

As $N_0 \preceq N$, by PMOP again we can find $N_1 \preceq N_2$ that is constructible over $N_0 \cup M^*$. The minimality of N_2 over M^*b implies that $N_2 = N_1$, hence $c \in N_1$. As N_1 is atomic over $N_0 M^*$, we have that $\text{tp}(c/N_0 M^*)$ is isolated. However, as $c N_0$ is dominated by b over M , the fact that $b \downarrow_M M^*$ implies that $c N_0 \not\downarrow_M M^*$. As $M \preceq N_0$, we have $c \not\downarrow_{N_0} M^*$. The Open Mapping Theorem implies that $\text{tp}(c/N_0)$ is isolated, hence $c \in N_0$. \square

5.4 The main theorem

Lemma 5.12 suggests the following notation. If $M \subseteq_{\text{na}} \mathfrak{C}$ and $\{c_i : i \leq n\}$ are M -independent and realize regular, depth zero types over M , then the notation $M(c_0, \dots, c_n)$ refers to any model dominated by c_0, \dots, c_n over M . The reader is cautioned that even when $\text{tp}(b/M) = \text{tp}(c/M)$ are the same regular type, the notations $M(b)$ and $M(c)$ represent possibly different isomorphism types of models dominated over M by b or c , respectively.

Now suppose that T is classifiable, so in particular, a constructible model N exists over any independent triple of models.

Lemma 5.15 *Suppose T is classifiable, $\{c_i : i < n\}$ independent over a model M and for each i , $\text{tp}(c_i/M)$ has depth zero and $M(c_i)$ is dominated by c_i over M . Then the constructible model N over $\bigcup \{M(c_i) : i < n\}$ is dominated by $\{c_i : i < n\}$ over M . Thus, N is minimal over $M \cup \{c_i : i < n\}$ and its isomorphism type over M is uniquely determined by the isomorphism types $\{M(c_i) : i < n\}$ over M .*

Proof. As $\{c_i : i < n\}$ are M -independent and as each $M(c_i)$ is dominated by c_i over M , it follows by forking calculus that $\{M(c_i) :$

$i < n$ are M -independent. Thus, by PMOP, a constructible model N over $A := \bigcup\{M_i : i < n\}$ exists. By Fact A.9(2) N is dominated by A over M . Thus, N is dominated by $\{c_i : i < n\}$ over M by Fact A.8(1). By Lemma 5.12(1), N is minimal over $M \cup \{c_i : i < n\}$. As N is constructible over A , its isomorphism type over M depends only on the isomorphism type of A over M . By independence and M being a model, the isomorphism type over A over M is determined by the set of isomorphism types of its component pieces. \square

In light of Lemma 5.15, we write such an N as $M(c_0, \dots, c_{n-1})$. In particular, if $\text{tp}(c/M)$ has depth zero and $M(c)$ is chosen, we let $M(c)^{(4)}$ denote the prime model over four M -independent copies of $M(c)$. It follows from Lemma 5.15 that the isomorphism type of $M(c)^{(4)}$ over M is uniquely determined by the type of $M(c)$ over M .

For the next few results, recall that by Fact A.28, any non-locally modular type has depth zero. Proposition 5.16 is the most technical result of this paper.

Proposition 5.16 *Suppose T is classifiable, $M \subseteq_{\text{na}} \mathfrak{C}$, $p \in S(M)$ is regular but not locally modular. Then for any realizations b, c of p and any choice of models $M(b)$ and $M(c)$ as above, $M(b)$ elementarily embeds into $M(c)^{(4)}$ over M .*

Proof. Without loss of generality, we may assume b and c are independent. Fix choices for $M(b)$ and $M(c)$. Choose d realizing $p|Mbc$ and choose $M(d)$ to be isomorphic to $M(c)$ over M . Let $M(bcd)$ denote the prime model over $M(b)M(c)M(d)$. Since p is not locally modular and as (b, c, d) realizes $p^{(3)}$, Corollary 5.10 gives an $a \in M(bcd)$ such that (a, b, c, d) is a witness to non-modularity over M . Since $M \subseteq_{\text{na}} M(bcd)$, we can use Fact A.6 to find some $M(a) \subseteq_{\text{na}} M(bcd)$. As $M(a), M(b), M(c) \preceq M(bcd)$, by PMOP there is a prime model $M^* \preceq M(bcd)$ over $M(a)M(b)M(c)$. Clearly, M^* is dominated by abc over M and (a, b, c) realizes $p^{(3)}$, so we can write it as $M(abc)$.

Claim 1. $M^* = M(bcd)$.

Proof. If not, then use Fact A.4 to choose $g \in M(bcd) \setminus M^*$ to realize a regular type q . By Lemma 5.12 we have $q \not\perp M$, so by the 3-model Lemma (Fact A.7) there would be $h \in M(bcd)$ such that $\text{tp}(h/M)$ is regular and non-orthogonal to q , with $h \underset{M}{\perp} M^*$. This h is dominated by bcd over M , so $\text{tp}(h/M)$ must be non-orthogonal to p . But then h, a, b, c are independent realizations of regular types non-orthogonal to p , contradicting $w_p(M(bcd)/M) = 3$. \square

So M^* is equal to both $M(abc)$ and $M(bcd)$. Next, let $M(ab) \preceq M(abc)$ be the unique model that is prime over $M(a) \cup M(b)$. For

the moment we work over $M(ab)$. Choose e to realize $p|Mabcd$ and fix an isomorphism

$$\Phi : M(abc) \rightarrow M(abe)$$

fixing $M(ab)$ pointwise with $\Phi(c) = e$. Put $M(e) := \Phi(M(c))$. As both d and $M(d)$ are contained in $M(abc)$, we let $f := \Phi(d)$ and $M(f) := \Phi(M(d))$. Let N^* be prime over $M(abc) \cup M(abe)$ over $M(ab)$. Note that N^* is dominated by ce over $M(ab)$.

We now use the fact that (a, b, c, d) is a witness to non-modularity. By construction, ef and cd have the same type over $M(ab)$, so it follows from Lemma 5.6 that $\{c, d, e, f\}$ are independent over M . It follows that the four models $M(c), M(d), M(e), M(f)$ are M -independent. Since $M(d)$ was chosen to be isomorphic to $M(c)$ over M , $M(e) = \Phi(M(c))$, and $M(f) = \Phi(M(d))$, the four models are pairwise isomorphic over M .

As each of $M(c), M(d), M(e), M(f)$ are contained in N^* , let $N \preceq N^*$ be constructible over $M(c) \cup M(d) \cup M(e) \cup M(f)$. Note that N is isomorphic to $M(c)^{(4)}$ over M and $M(b) \preceq M(ab) \preceq N^*$. Thus, the Proposition is proved once we establish the following claim.

Claim 2. $N = N^*$.

Proof. If not, then by Fact A.4 choose $g \in N^* \setminus N$ with $q := \text{tp}(g/N)$ regular. By Lemma 5.12, q is non-orthogonal to M . Thus, by the 3-model Lemma (Fact A.7), there is $h \in N^* \setminus M$ such that $\text{tp}(h/M)$ is regular and non-orthogonal to q , with $h \underset{M}{\perp} N$. We split into cases depending on the non-orthogonality class of q .

First, assume that q is non-orthogonal to p . On one hand, $\{h, c, d, e, f\} \subseteq N^*$ consists of 5 independent realizations of regular types non-orthogonal to p . On the other hand, $w_p(M(ab)/M) = 2$ and $w_p(N^*/M(ab)) = 2$, so $w_p(N^*/M) = 4$, which is a contradiction.

Finally, assume that q is orthogonal to p . Then clearly $\text{tp}(h/M(ab))$ does not fork over M . But N^* is dominated by ce over $M(ab)$, and by the orthogonality, $h \underset{M(ab)}{\perp} ce$. Thus, by transitivity, $h \underset{M}{\perp} N^*$, which is absurd since $h \in N^* \setminus M$. \square

Corollary 5.17 *T* classifiable. Suppose $M \subseteq_{\text{na}} \mathfrak{C}$ is countable and $p := \text{tp}(c/M)$ is regular but not locally modular. Then for each n , $Q_n := \{q \in S_n(Mc) : \bar{d}c \text{ is dominated by } c \text{ over } M \text{ when } \bar{d} \models q\}$ is countable.

Proof. Fix a model $M(c)$ dominated by c over M and fix an $n \geq 1$. For any $q \in Q_n$, choose a realization \bar{d}_q of q and let N_q be any countable, ℓ -constructible model over $M\bar{c}\bar{d}_q$. By Fact A.9(2)

N_q is dominated by $\bar{d}_q c$ (and hence by c) over M . By Proposition 5.16, choose an elementary embedding $f_q : N_q \rightarrow M(c)^{(4)}$ fixing M pointwise. If Q_n were uncountable, there would be distinct $q \neq q'$ with $f_q(\bar{d}_q c) = f_{q'}(\bar{d}_{q'} c)$. Thus, $\text{tp}(\bar{d}_q c/M) = \text{tp}(\bar{d}_{q'} c/M)$, i.e., $\text{tp}(\bar{d}_q/Mc) = \text{tp}(\bar{d}_{q'}/Mc)$, contradicting $q \neq q'$. \square

The following technical Lemma, which is true in any superstable theory, is implicit in the proof of Lemma 5.3 of [9], but is included here at the request of the referee.

Lemma 5.18 *T superstable. Suppose $M \subseteq_{na} \mathfrak{C}$ and $A \supseteq M$. For every non-algebraic $\psi(x, a_0)$ (with $a_0 \subseteq A$), suppose $\varphi(x, a)$ is of least R^∞ -rank satisfying the following conditions:*

- $a \subseteq A$, $\varphi(x, a) \vdash \psi(x, a_0)$, and $\varphi(x, a)$ consistent, but non-algebraic.

Then cA is dominated by A over M for every c realizing $\varphi(x, a)$.

Proof. The driving force is the equivalent of $M \subseteq_{na} \mathfrak{C}$ preferred by Lemma 2.9(c) of [5]. Choose any such $\varphi(x, a)$ and by way of contradiction, assume that cA is not dominated by A over M for some c with $\varphi(c, a)$. Choose an element $b \in \mathfrak{C}$ such that $b \downarrow_M A$, but $b \not\downarrow_M Ac$. Possibly enlarging a within A , choose an $L(M)$ -formula $\theta(x, y, z) \in \text{tp}(cab/M)$ such that $\theta(x, a, b)$ forks over Ma . Fix a finite $F \subseteq M$ containing the hidden parameters from θ with $cab \downarrow_F M$. Then $a \downarrow_F Mb$ and $\varphi(x, a) \wedge \theta(x, a, b)$ is consistent and forks over Fa . Thus, by Lemma 2.9(c) of [5], there is some $b' \in M$ and some $L(M)$ -formula $\theta'(x, a, b')$ such that $\varphi(x, a) \wedge \theta'(x, a, b')$ is consistent and forks over Fa . Thus, $R^\infty(\varphi(x, a) \wedge \theta(x, a, b')) < R^\infty(\varphi(x, a))$. As $b' \subseteq A$, this contradicts the minimality of $R^\infty(\varphi(x, a))$. \square

Theorem 5.19 *T classifiable. Suppose $M \subseteq_{na} \mathfrak{C}$ is countable and $p := \text{tp}(c/M)$ is regular but not locally modular. Then there is a constructible, minimal model over Mc .*

Proof. Assume by way of contradiction that there is no constructible model over Mc . As Mc is countable, there is a finite \bar{b} such that $\text{tp}(\bar{b}/Mc)$ is isolated, and a consistent, non-algebraic formula $\theta(x)$ over $M\bar{b}c$ that has no complete extension $\psi(x) \vdash \theta(x)$ over $M\bar{c}$. Among all consistent, non-algebraic formulas $\varphi(x, \bar{b}c) \vdash \theta(x)$, choose one of minimal R^∞ -rank. By Lemma 5.18 $\bar{d}\bar{b}c$ is dominated by $\bar{b}c$ over M for every realization of $\varphi(x, \bar{b}c)$. However, $\text{tp}(\bar{b}/Mc)$ is isolated, hence $\bar{b}c$ is dominated by c over M by Fact A.9(2). Thus, by Fact A.8(1), $\bar{d}\bar{b}c$ is dominated by c over M for every d realizing

$\varphi(x, \bar{b}c)$. However, $\varphi(x, \bar{b}c)$ has no complete extension, so there are a perfect set of complete types in $S_k(M\bar{b}c)$ extending $\varphi(x, \bar{b}c)$, where $k = \text{lg}(x)$. Thus, there are uncountably many distinct types in Q_n (where $n = k + \text{lg}(\bar{b})$) contradicting Corollary 5.17.

Thus, a constructible model N over Mc exists. As N is dominated by c over M by Fact A.9(2), its minimality over M follows from Lemma 5.11(2). \square

The following strengthening, obviating the requirements that M be countable and $M \subseteq_{na} \mathfrak{C}$, is the main result of this section.

Theorem 5.20 *Let M be any model of a classifiable theory T . If $p \in S(M)$ is regular but not locally modular and b is any realization of p then every model N containing Mb that is dominated by b over M is both constructible and minimal over Mb .*

Proof. As T is classifiable and p is non-locally modular, then p has depth zero by Fact A.28. We first prove the Theorem when M is countable. Choose any $N \succeq M$ that is dominated by b over M , e.g., an ℓ -constructible model over Mb . By Proposition 5.14(2), N is minimal over Mb , hence N must be countable as well.

To show that N is constructible over Mb , choose any countable $M^* \succeq M$ with $N \downarrow_M M^*$ such that $M^* \subseteq_{na} \mathfrak{C}$. By PMOP, choose a countable N^* that is prime over $N \cup M^*$. As $N \downarrow_M M^*$ and N is dominated by b over M , N is also dominated by b over M^* by Fact A.8(2). As N^* is atomic over $M^* \cup N$, it is dominated by N over M^* by Fact A.9(2), hence N^* is dominated by b over M^* by Fact A.8(2). By Lemma 5.11 N^* is minimal over M^*b . By Theorem 5.19, as $M^* \subseteq_{na} \mathfrak{C}$ there is a countable S , constructible and minimal over M^*b . As S is prime over M^*b , we may assume $S \preceq N^*$. As N^* is minimal over M^*b , $N^* = S$, so N^* is constructible over M^*b . To finish, since $N \subset N^*$, N (as a set) is atomic over M^*b . By the Open mapping theorem, it follows that N is already atomic over Mb . As N is a countable, it is constructible over Mb .

Now suppose M is arbitrary. Choose any $N \supseteq Mb$ that is dominated by b over M . By Proposition 5.14(2), N is minimal over Mb . To show that N is constructible over Mb , consider the pair (N, M) in a language with a predicate U for M . Take a countable elementary substructure in the pair language, $(N', M') \preceq (N, M)$ with $b \in N'$. It follows that, in the original language, N' and M are independent over M' .

As $N' \subset N$, b dominates N' over M . By the independence of N' and M over M' , b also dominates N' over M' by Fact A.8(2). As N' is countable, it follows from the argument above that N' is

constructible over $M'b$ by a construction sequence $\bar{c} = \langle c_\alpha : \alpha < \beta \rangle$. By Fact A.10, \bar{c} is also a construction sequence over Mb . Now by PMOP, take $N^* \preceq N$, a constructible model over N' and M , inside N . As $Mb\bar{c} = N'$ as a set, it follows that N^* is constructible over Mb . As N is minimal over Mb , $N^* = N$ so N is constructible over Mb . \square

Corollary 5.21 *T classifiable. Suppose that $M \preceq N$ and N/M has weight one, non-orthogonal to a non-locally modular type p . Then N is both constructible and minimal over Mb for any element $b \in N \setminus M$.*

Proof. Choose any $b \in N \setminus M$. Since $M \prec N$, by Fact A.4, choose $a \in N \setminus M$ such that $p' := \text{tp}(a/M)$ is regular. As N/M is weight one, non-orthogonal to p , $p' \not\perp p$, so p' is also non-locally modular (see e.g., 7.2.4 of [7]). By Theorem 5.20, N is constructible and minimal over Ma . In particular, $\text{tp}(b/Ma)$ is isolated by some $L(M)$ -formula $\alpha(a, y)$. Dually, we have the following.

Claim. $\text{tp}(a/Mb)$ is isolated.

Proof. As $p' \in S(M)$ is non-locally modular, choose $\theta(x) \in p'$ as in Theorem 5.4. Also, since N/M has weight one, $a \not\perp_M b$. Choose an $L(M)$ -formula $\beta(x, b)$ witnessing this forking. Put

$$\psi(x, b) := \theta(x) \wedge \alpha(x, b) \wedge \beta(x, b)$$

Clearly, $\psi(a, b)$ holds and we show it isolates $\text{tp}(a/Mb)$. For this, choose any $L(M)$ -formula $\delta(x, y)$ and assume $\delta(a, b)$ holds. As $N \preceq \mathfrak{C}$, it suffices to show that any $a' \in N$ realizing $\psi(x, b)$ satisfies $\delta(x, b)$. So choose any $a' \in \psi(N, b)$. Since $\theta(a')$ holds, $\text{tp}(a'/M)$ is p -simple. Since $\beta(a', b)$ holds, $\text{tp}(a'/Mb)$ forks over M , so $a' \notin M$. Thus, as N/M has weight one, $a \not\perp_M a'$. Since $\text{tp}(a/M)$ is regular, this implies $w_p(a'/M) > 0$, so by Theorem 5.4(3), $\text{tp}(a'/M) = p' = \text{tp}(a/M)$. Finally, as $\alpha(a, y)$ isolates $\text{tp}(b/Ma)$, we have $\forall y(\alpha(a, y) \rightarrow \delta(a, y)) \in \text{tp}(a/M)$. Thus, $\forall y(\alpha(a', y) \rightarrow \delta(a', y))$ holds as well. Since $\alpha(a', b)$ was assumed to hold, we conclude that $\delta(a', b)$ holds. \square

Given the Claim, N is constructible over Mb , since from above it is constructible over Mab and $\text{tp}(a/Mb)$ is isolated. To see that N is minimal over Mb , note that in fact, N is minimal over Ma' for any realization of p' in N . So, choose any model $N' \preceq N$ containing Mb . By the Claim there is some $a' \in N'$ realizing p' , hence $N' = N$. \square

6 When domination implies isolation

We begin this section with a recasting of Theorem 5.20.

Corollary 6.1 *Suppose T is classifiable, A is any set, $p \in S(A)$ is a regular, stationary, non-locally modular type, and b is any realization of p . Then for any e , if be is dominated by b over A , $\text{tp}(e/Ab)$ is isolated.*

Proof. Let $M \subseteq_{\text{na}} \mathfrak{C}$ be free from b over A and choose an ℓ -constructible model $N \supseteq Mbe$ over Mbe . As be is dominated by b over M , hence N is dominated by b over M . By Theorem 5.20, N is constructible, hence atomic over Mb , so $\text{tp}(e/Mb)$ is isolated. Also, $e \downarrow_{Ab} Mb$, hence $\text{tp}(e/Ab)$ is isolated by the Open Mapping Theorem. \square

This result suggests the following definition.

Definition 6.2 A strong type p satisfies *DI* (read ‘domination implies isolation’) if, for every set A on which p is based and stationary and for every realization b of $p|A$, for every $c \in \mathfrak{C}$, if bc is dominated by b over A , then $\text{tp}(c/Ab)$ is isolated.

In the remainder of this section, we explore this notion in classifiable theories. Among depth zero-like types, the notion of DI has many equivalents.

Proposition 6.3 *Suppose T is classifiable. The following are equivalent for a depth zero-like strong type p :*

1. p is DI;
2. For every countable M on which p is based and for every b realizing $p|M$, and for every n , the isolated types in $S_n(Mb)$ are dense;
3. For every countable M on which p is based, for every b realizing $p|M$, there is a constructible model N over Mb . Moreover, every model N that is dominated by b over M is constructible over Mb ;
4. Same as (3), but for every model M on which p is based;
5. There is some a -model M on which p is based and some b realizing $p|M$ for which there is a constructible model N over Mb .

Proof. We prove (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1). Then (4) \Rightarrow (5) is trivial and we will show (5) \Rightarrow (2).

(1) \Rightarrow (2). Assume (1) and choose a countable M , b , and n as in (2). Let $\varphi(x, b, m)$ be any consistent formula with $\text{lg}(x) = n$. Choose any model N that is dominated by b over M (e.g., an ℓ -constructible one). Choose any $c \in N$ realizing $\varphi(x, b, m)$. As cb is dominated by b over M , it follows from (1) that $\text{tp}(c/Mb)$ is isolated.

(2) \Rightarrow (3). Fix M and b as in (3). As M is countable and the isolated types are dense, it follows from Vaught that a constructible model N over Mb exists. For the final sentence, let N^* be any model dominated by b over M . As N is prime over Mb , we may assume $N \preceq N^*$. But, as N^* is minimal over Mb by Proposition 5.14(2), $N = N^*$, so N^* is constructible over Mb .

(3) \Rightarrow (4). Fix M and b as in (4). Choose a countable model $M_0 \preceq M$ such that $b \downarrow_{M_0} M$. By (3), let N_0 be constructible over M_0b . Fix a construction sequence $\bar{c} = \langle c_\alpha : \alpha < \beta \rangle$ for N_0 over M_0b . By Fact A.10 \bar{c} is a construction sequence over Mb . Now, by PMOP, let N be constructible over N_0M . By concatenation, it follows that N is constructible over Mb . For the final sentence, let N^* be any model dominated by b over M . As N is prime over Mb , we may assume $N \preceq N^*$. But then $N = N^*$, again by Proposition 5.14(2).

(4) \Rightarrow (1). Choose any set A on which p is based and stationary and let b be any realization of $p|A$. Choose any element $c \in \mathfrak{C}$ such that bc is dominated by b over A . Choose any model $M \supseteq A$ satisfying $M \downarrow_A bc$. It follows that bc is dominated by b over M . Choose any model N dominated by bc over M . By transitivity we have that N is dominated by b over M . Thus, by (4), N is constructible, and hence atomic over Mb . In particular, $\text{tp}(c/Mb)$ is isolated. Thus, $\text{tp}(c/Ab)$ is also isolated by the Open Mapping Theorem.

(4) \Rightarrow (5) is immediate.

(5) \Rightarrow (2). Let M^* and b^* be any a-model and witness exemplifying (5). Given any countable M on which p is based and b realizing $p|M$, the saturation of M^* implies there is an $M' \preceq M^*$ such that $\text{tp}(Mb) = \text{tp}(M'b^*)$. Thus, without loss, assume $M \preceq M^*$ and $b = b^*$. As b realizes $p|M^*$, we have that $b \downarrow_M M^*$. Let $\varphi(x, b, m)$ be any consistent formula with $m \in M$. By (5), choose a constructible (and hence atomic) N over M^*b . Choose any $c \in N$ realizing $\varphi(x, b, m)$. Then $\text{tp}(c/M^*b)$ is isolated, hence $\text{tp}(c/Mb)$ is as well by the Open Mapping Theorem. \square

Proposition 6.4 *Let T be classifiable.*

1. *Suppose $\text{stp}(b/A)$ is depth zero-like and DI and bc is dominated by b over A . Then $\text{stp}(bc/A)$ is both depth zero-like and DI.*
2. *Suppose $\text{tp}(bc/M)$ is depth zero-like and DI, and $\text{tp}(c/Mb)$ is isolated. Then $\text{tp}(b/M)$ is depth zero-like and DI as well.*

3. If p and q are both depth zero-like and DI , then so is $p \otimes q$.

Proof. (1) That $\text{stp}(bc/A)$ is depth zero-like is clear. As for DI , choose an a -model $M \supseteq A$ with $b \downarrow_A M$. By Fact A.8 bc is dominated by b over M , hence there is an a -prime model $M[b]$ over Mb with $c \in M[b]$. As $\text{tp}(b/M)$ is DI , $M[b]$ is constructible over Mb by Proposition 6.3(4). Thus, $\text{tp}(c/Mb)$ is isolated, so $M[b]$ is also constructible over Mbc . Hence $\text{stp}(bc/A)$ is DI by Proposition 6.3(5).

(2) That $\text{tp}(b/M)$ is depth zero-like is immediate. For DI , choose an a -model M^* as in Proposition 6.3(5) witnessing that $\text{tp}(bc/M)$ is DI . Without loss, we may assume $M \preceq M^*$, so $bc \downarrow_M M^*$. It follows from Lemma A.9(3) that $\text{tp}(c/M^*b)$ is isolated as well. As $\text{tp}(bc/M^*)$ is DI , let N^* be constructible over M^*bc . As $\text{tp}(c/M^*b)$ is isolated, N^* is also constructible over M^*b . Thus, $\text{tp}(b/M^*)$ is DI by Proposition 6.3(5).

(3) As both p and q are depth zero-like, it is immediate that $p \otimes q$ is as well. As for DI , let M be any model on which $p \otimes q$ is based and let (c_1, c_2) realize $p \otimes q$. As both p and q are DI , there is a constructible model N_1 over Mc_1 and a constructible model N_2 over Mc_2 . As $c_1 \downarrow_M c_2$, domination implies $N_1 \downarrow_M N_2$. It follows from Fact A.10 that the set N_2 is constructible over Mc_1c_2 . By Fact A.10 again, the set N_1 is constructible over N_2c_1 . Concatenating these construction sequences, we get that the set $N_1 \cup N_2$ is constructible over Mc_1c_2 . Finally, by PMOP there is a model N^* that is constructible over N_1N_2 . It follows that N^* is constructible over Mc_1c_2 , so $p \otimes q$ is DI by Proposition 6.3(4). \square

We can combine several of our results in the following Corollary which both generalizes Corollary 5.21 and extends Theorem 3.2.

Corollary 6.5 *T classifiable. Suppose p is a regular, non-trivial, depth zero, DI type (for example a non-locally modular regular type) and let $\text{tp}(a/M)$ be p -semiregular of weight k . Then there is a constructible, minimal model N over Ma .*

Proof. Choose an a -model M^* independent from a over M and choose an M^* -independent tuple $\bar{c} = \langle c_i : i < k \rangle$ of realizations of $p|M^*$ such that a and \bar{c} are domination equivalent over M^* .

As $\text{tp}(a/M^*)$ is p -semiregular, it is depth-zero like. By Proposition 6.4(3) and (1), both $\text{tp}(\bar{c}/M^*)$ and $\text{tp}(a\bar{c}/M^*)$ are depth zero-like and DI . To show that $\text{tp}(a/M^*)$ is DI , by Proposition 6.4(2) it suffices to show $\text{tp}(\bar{c}/M^*a)$ is isolated.

To see the isolation, first note that since $\text{tp}(\bar{c}/M^*)$ is DI , there is an $L(M^*)$ -formula $\alpha(x, \bar{c})$ isolating $\text{tp}(a/M^*\bar{c})$. Next, choose $L(M^*)$ -formulas $\varphi(x) \in \text{tp}(a/M^*)$ and $\theta(y) \in p|M^*$ as in Fact A.22 and

Theorem 5.4, respectively. As $w_p(a/M^*\bar{c}) = 0$, the definability of p -weight inside $\varphi(x)$ implies there is a formula $\beta(a, \bar{y}) \in \text{tp}(\bar{c}/M^*a)$ such that $w_p(a/M^*\bar{d}) = 0$ for all \bar{d} realizing $\beta(a, \bar{y})$. Put

$$\psi(a, \bar{y}) := \bigwedge_{i < k} \theta(y_i) \wedge \alpha(a, \bar{y}) \wedge \beta(a, \bar{y})$$

Visibly, $\psi(a, \bar{y}) \in \text{tp}(\bar{c}/M^*a)$. To see that it isolates $\text{tp}(\bar{c}/M^*a)$, choose any $L(M^*)$ -formula $\delta(x, \bar{y})$ and suppose $\delta(a, \bar{c})$ holds. Choose any \bar{d} such that $\psi(a, \bar{d})$ holds and it suffices to show that $\delta(a, \bar{d})$ holds. For this, we first show that \bar{d} realizes $(p|M^*)^{(k)}$. Since $\theta(d_i)$ holds, $\text{tp}(d_i/M^*)$ is p -simple of p -weight ≤ 1 for each $i < k$. However, $w_p(a/M^*) = k$ and $\beta(a, \bar{d})$ implies that $w_p(a/M^*\bar{d}) = 0$. It follows that we must have $w_p(d_i/M^*) = 1$ for each $i < k$ and moreover, $\{d_i : i < k\}$ is independent over M^* . As $p|M^*$ is strongly regular, i.e., by Theorem 5.4(3), each d_i realizes $p|M^*$, hence $\text{tp}(\bar{d}/M^*) = (p|M^*)^{(k)} = \text{tp}(\bar{c}/M^*)$. To finish, since $\alpha(x, \bar{c})$ isolates $\text{tp}(a/M^*\bar{c})$ and since $\delta(a, \bar{c})$ holds, $\forall x(\alpha(x, \bar{y}) \rightarrow \delta(x, \bar{y})) \in \text{tp}(\bar{c}/M^*)$, so we have $\forall x(\alpha(x, \bar{d}) \wedge \delta(x, \bar{d}))$ holding. Thus $\delta(a, \bar{d})$ holds, so $\text{tp}(\bar{c}/M^*a)$ is isolated.

As $\text{tp}(a/M^*)$ is depth-zero like and DI, the same is true of $\text{tp}(a/M)$. Let N be any ℓ -constructible model over Ma . By Fact A.9(2), N is dominated by a over M . So Proposition 6.3(4) implies that N is constructible over Ma and Proposition 5.14(2) gives the minimality of N over Ma . \square

A Appendix

In this appendix, we bring together for the reader's convenience, many of the basic definitions and facts from classification theory and geometric stability theory that are used throughout the paper. Many of these can be found in e.g., Chapters 1,7, 8 of [7]. We assume that the reader is familiar with stability theory, independence and the basics of superstability. Throughout this appendix, T will be (at least) a stable theory.

Definition A.1 • We say that M is an a -model if every strong type over every finite $B \subseteq M$ is realized in M . In the original notation of Shelah ([10]) this corresponds to $\mathbf{F}_{\aleph_0}^a$ -saturation.

- Let $M \preceq N$ be models of T , N is an na -extension of M , denoted $M \subseteq_{na} N$, if for every formula $\varphi(x, y)$, for every tuple a from M and every finite subset F of M , if N contains a solution to $\varphi(x, a)$ not in M , then M contains a solution to $\varphi(x, a)$ that is not algebraic over F .

Obviously, if M is an a -model, then $M \subseteq_{na} \mathfrak{C}$, hence $M \subseteq_{na} N$ for any $N \succeq M$. We see below (e.g., Facts A.5 and A.6) many of the desirable attributes of working over a -models are reflected in na -substructures. The utility of this notion is that by an easy Löwenhiem-Skolem argument, models $M \subseteq_{na} \mathfrak{C}$ of size $|T|$ exist, whereas typically a -models have larger cardinality.

Definition A.2 • If $B \subseteq A$, types $p \in S(A)$ and $q \in S(B)$ are *almost orthogonal*, denoted $p \perp^a q$, if for all a realizing p and b realizing q , if $b \perp_B A$, then $a \perp_B b$. p is almost orthogonal to the set B , $p \perp^a B$ if $p \perp^a q$ for every $q \in S(B)$.

- Two stationary types $p \in S(A)$ and $q \in S(B)$ are *orthogonal*, denoted $p \perp q$, if $p|C \perp^a q$ for all $C \supseteq AB$. $p \in S(A)$ is *orthogonal to a set B* , $p \perp B$, if $p \perp q$ for every $q \in S(B)$.
- A set C is *dominated by E over D* if for all b such that b is independent from E over D , b is independent from C over D .
- A non-algebraic stationary type p is *regular* if it is orthogonal to all its forking extensions.
- A non-algebraic stationary type $p \in S(A)$ is *strongly regular* if there is a formula $\varphi(x) \in p$ such that, for any $B \supseteq A$, every stationary type $q \in S(B)$ containing $\varphi(x)$ is either equal to $p|B$ or orthogonal to p .

Note that if $B \subseteq A$ and $p \in S(A)$, then $p \perp^a B$ if and only if Aa is dominated by A over B for some/every a realizing p .

We frequently use the following fact without mention (see e.g., 1.4.4.2(ii) of [7]). Because of it, many of our adjectives for regular types (e.g., local modularity, depth zero, p -weight) are really features of the non-orthogonality class of the regular type.

Fact A.3 *Non-orthogonality is an equivalence relation on set of regular types.*

We list two existence theorems for regular types between models, which are respectively Propositions 8.3.2 and 8.3.5 of [7].

Fact A.4 *If T is superstable and $M \prec N$ with $M \neq N$, then there is some $a \in N \setminus M$ with $\text{tp}(a/M)$ regular.*

Fact A.5 *If T is superstable, $M \subseteq_{na} N$, and q is a regular type with $\text{tp}(N/M) \not\perp q$, then there is an $a \in N \setminus M$ with $\text{tp}(a/M)$ regular and non-orthogonal to q .*

The following fact is a consequence of Lemma 5.3 in [9]:

Fact A.6 *Suppose T is superstable, $M \subseteq_{\text{na}} N$ are models of T and $M \subseteq A \subseteq N$. Then there exists a model N' , $M \subseteq A \subseteq N' \subseteq_{\text{na}} N$ such that N' is dominated by A over M .*

Fact A.7 [*The 3-model lemma*] (Proposition 8.3.6 in [7]) *Suppose T superstable. Let $M_0 \preceq M_1 \preceq M_2$ be models of T such that $M_0 \subseteq_{\text{na}} M_2$. Suppose $a \in M_2$ and $\text{tp}(a/M_1)$ is regular non-orthogonal to M_0 . Then there is $b \in M_2$ such that $\text{tp}(b/M_1)$ is regular and does not fork over M_0 and b is not independent from a over M_1 .*

A.1 Domination and isolation in stable theories

In this subsection we list a number of facts about domination and isolation that hold in arbitrary stable theories.

The first two of the following facts are obtained by forking calculus, and the third appears in 1.4.3.4 of [7].

Fact A.8 *Let T be stable.*

1. *For any a, b, c, D , if abc is dominated by bc over D and bc is dominated by c over D , then abc is dominated by c over D .*
2. *If $E \supseteq D$ and $ab \underset{D}{\downarrow} E$, then ab is dominated by b over D if and only if ab is dominated by b over E .*
3. *If M is an a -model, then bc is dominated by b over M if and only if $\text{stp}(b/Mc)$ is a -isolated if and only if c is contained in some a -prime model $M[b]$ over Mb .*

We now recall definitions and facts about isolation and constructibility.

A type $p \in S(A)$ is *isolated* if there is some formula $\varphi(x, a) \in p$ such that $\varphi(x, a) \vdash p$. A *construction sequence over A* is a sequence $\langle a_\alpha : \alpha < \beta \rangle$ such that $\text{tp}(a_\alpha/A \cup \{a_\gamma : \gamma < \alpha\})$ is isolated for every $\alpha < \beta$. A model N is *constructible over A* if there is a construction sequence over A whose union is N . If N is constructible over A then it is both prime and atomic over A . Any two constructible models over A are isomorphic over A .

If T is \aleph_0 -stable, then constructible models exist over every set A . In a superstable theory it is not always true that there are constructible models over all sets. Indeed, one of the main goals of this paper is to determine when constructible models over particular sets exist.

A weaker notion is *ℓ -isolation*. A type $p \in S(A)$ is *ℓ -isolated* if, for every formula $\varphi(x, y)$ there is a formula $\psi(x, a) \in p$ such

that $\psi(x, a) \vdash p \upharpoonright_{\varphi}$, the restriction of p to instances of $\pm\varphi(x, b)$ for $b \in A$. ℓ -construction sequences and N being ℓ -constructible over A are defined analogously. An advantage is that for any countable stable theory T , for any set A and any consistent $\varphi(x, a)$ with $a \subseteq A$, an ℓ -isolated $p \in S(A)$ extending $\varphi(x, a)$ exists (see e.g., IV 2.18(4) of [10]). By iterating this fact, ℓ -constructible models over A exist over any set A . The disadvantage is that there can be many non-isomorphic ℓ -constructible models over A . The following facts are well known.

Fact A.9 *Suppose T is stable, A, B are independent over a model M and $\text{tp}(c/MA)$ is ℓ -isolated. Then:*

1. $\text{tp}(c/MA) \vdash \text{tp}(c/MAB)$, so $\text{tp}(c/MAB)$ is ℓ -isolated via the same formulas witnessing the ℓ -isolation of $\text{tp}(c/MA)$;
2. Ac is dominated by A over M ; and
3. $\text{tp}(c/MA)$ is isolated if and only if $\text{tp}(c/MAB)$ is isolated.

Proof. (1) Choose any L -formula $\varphi(x, y)$ and assume $\theta(x, ma) \vdash \text{tp}_{\varphi}(c/MA)$. If $\theta(x, ma) \not\vdash \text{tp}_{\varphi}(c/MAB)$, then $\exists x_1 \exists x_2 (\theta(x_1, ma) \wedge \theta(x_2, ma) \wedge \neg[\varphi(x_1, mab) \leftrightarrow \varphi(x_2, mab)])$. As $A \downarrow_M B$, finite satisfiability would imply the existence of $b' \in M$ satisfying this, contradicting $\theta(x, ma) \vdash \text{tp}_{\varphi}(c/MA)$.

(2) Choose any e with $A \downarrow_M e$. A non-forking extension of $\text{tp}(c/MA)$ to $S(MAe)$ exists, and by (1) it is implied by $\text{tp}(c/MA)$. Thus, $c \downarrow_{MA} e$.

(3) Left to right is (1) and the converse is the Open Mapping Theorem. \square

Iterating this yields

Fact A.10 *If T is stable and A, B are independent over a model M , then for any sequence $\bar{c} = \langle c_{\alpha} : \alpha < \beta \rangle$, \bar{c} is a construction sequence over MA if and only if \bar{c} is a construction sequence over MAB . In particular, if N is constructible over MA , then N (as a set) is a construction sequence over MAB and hence is dominated by AB over M .*

The following will be useful under the assumption of PMOP (see Section A.5).

Fact A.11 *Suppose T is stable, (M_0, M_1, M_2) is an independent triple of models with $b \in M_1$, and suppose that M_1 is constructible over Mb and M^* is constructible over $M_1 \cup M_2$. If M_1 is dominated by b over M , then M^* is constructible over M_2b and dominated by b over M_2 .*

Proof. By Fact A.10 the set M_1 is a construction sequence over M_2b , so since M^* is constructible over M_1M_2 , M^* is constructible over M_2b simply by concatenating the two construction sequences. The domination is by Fact A.9(2). \square

A.2 p -simplicity and locally modular regular types

Let p be any (stationary) regular type, which for convenience we take to be over \emptyset . Consider $p(\mathfrak{C})$ the set of realizations of the type p , then $(p(\mathfrak{C}), \text{cl}_{\text{fork}})$ forms a homogeneous pre-geometry (see e.g., Chapter 7 of [7]), where $a \in \text{cl}_{\text{fork}}(B)$ means that a forks with B over \emptyset . Recall that a pre-geometry (G, cl) is *modular* if, for all closed sets $X, Y \subseteq G$ we have

$$\dim(\text{cl}(X \cup Y)) + \dim(X \cap Y) = \dim(X) + \dim(Y)$$

and is *locally modular* if the above holds whenever $\dim(X \cap Y) \neq 0$. By considering a minimal counterexample, it is well known that if (G, cl) is not modular, then one can find closed sets $X, Y, Z \subseteq G$, each of finite dimension, such that $X \supseteq Z, Y \supseteq Z, \dim(X/Z) = \dim(Y/Z) = 2, X \cap Y = Z$, but $\dim(\text{cl}(X \cup Y)/Z) = 3$.

Now suppose that M is an a-model, let $p|M$ denote the non-forking extension of p to $S(M)$. Since $p(M)$ contains closed sets of infinite dimension, the induced pre-geometry $(p|M(\mathfrak{C}), \text{cl}_M)$, where $a \in \text{cl}_M(B)$ iff a forks with B over M , is locally modular if and only if it is modular. Moreover, if $(p|M, \text{cl}_M)$ is not modular, then there exist four realizations (a_1, a_2, b_1, b_2) of $p|M$ such that (a_1, a_2) and (b_1, b_2) are each independent pairs over M , $\dim(a_1a_2, b_1, b_2/M) = 3$, but $\text{cl}_M(a_1a_2) \cap \text{cl}_M(b_1b_2) \cap p|M(\mathfrak{C}) = \emptyset$.

For many applications it is useful to work in a wider space than $(p(\mathfrak{C}), \text{cl}_{\text{fork}})$. We recall the definition of p -simplicity.

Definition A.12 Suppose that a regular type p is non-orthogonal to a set A .

- A strong type $\text{stp}(a/A)$ is *hereditarily orthogonal to p* if $\text{stp}(a/B)$ is orthogonal to p for every $B \supseteq A$.
- A strong type $\text{stp}(a/A)$ is *p -simple* if for some a-model M independent from a over A , there is an M -independent set $\{b_1, \dots, b_k\}$ of realizations of $p|M$ such that $\text{stp}(a/Mb_1, \dots, b_k)$ is hereditarily orthogonal to p . We say that $\text{stp}(a/A)$ is *p -simple of weight k* if k is least such.
- If $\text{stp}(a/A)$ is p -simple of p -weight k we write $w_p(a/A) = k$.

- A formula $\theta(x)$ over A is *p-simple of p-weight k* if every strong type extending θ is *p-simple* and k is the maximum of $\{w_p(a/A) : \theta(x) \in \text{stp}(a/A)\}$.
- For a formula $\theta(x)$ over A , put $(\theta_A)^{eq} := \text{dcl}^{eq}(A \cup \theta(\mathfrak{C}))$. [If $\theta(x)$ is *p-simple*, then $\text{stp}(b/A)$ is *p-simple* for every $b \in (\theta_A)^{eq}$.]
- For a *p-simple* $\theta(x)$ over A , we say *p-weight is definable and continuous inside $\theta(x)$* if, for any $b \in (\theta_A)^{eq}$ and any $c \in \mathfrak{C}^{eq}$, if $w_p(b/Ac) = n$, then there is some $\varphi(x, y) \in \text{tp}(bc/A)$ such that for any $b'c'$ realizing φ , $w_p(\varphi(x, c')) = n$ [hence $w_p(b'/Ac') \leq n$].

The following facts will be used repeatedly.

Fact A.13 *Suppose p is a regular type non-orthogonal to a set A .*

1. *If $A \subseteq B$ and $a \perp_A B$, then $\text{stp}(a/A)$ is hereditarily orthogonal to p if and only if $\text{stp}(a/B)$ is hereditarily orthogonal to p . [7.1.3' of [7]]*
2. *In particular, (taking B to be an a -model) $w_p(a/A) = 0$ if and only if $\text{stp}(a/A)$ is hereditarily orthogonal to p .*
3. *If $\text{stp}(a/A)$ and $\text{stp}(b/A)$ are *p-simple*, then so is $\text{stp}(ab/A)$. Moreover, $w_p(ab/A) = w_p(a/Ab) + w_p(b/A)$. [Lemmas 7.1.4(i) and 7.1.11 of [7].]*
4. *If $\text{stp}(a/A)$ is *p-simple* and $b \in \text{acl}(Aa)$, then $\text{stp}(b/A)$ is *p-simple*. [Lemma 7.1.4(ii) of [7]]*
5. *If $\text{stp}(a/A)$ is *p-simple* then $\text{stp}(a/B)$ is *p-simple* for every $B \supseteq A$. [Lemma 7.1.10 of [7]]*
6. *p is *p-simple* and $w_p(p) = 1$.*

Non-orthogonality to p gives the existence of *p-simple* types within the definable closure:

Fact A.14 *(Lemma 1.17, Chapter 7 in [7]). Let X be algebraically closed and $\text{tp}(a/X)$ be non-orthogonal to p . Then there is $e \in \text{dcl}(aX)$ such that $\text{tp}(e/X)$ is *p-simple of positive p-weight*.*

Following [3] and [7], for any set A with $p \not\perp A$, let

$$D(p, A) := \{a \in \mathfrak{C} : \text{stp}(a/A) \text{ is } p\text{-simple of finite } p\text{-weight}\}$$

We equip $D(p, A)$ with a closure operator cl_p , namely for a, B from $D(p, A)$, $a \in \text{cl}_p(B)$ if and only if $w_p(a/BA) = 0$. Technically, the closure relation cl_p depends on A , but much of the time we will take $A = \emptyset$, so we do not muddy our notation by referring to it explicitly. In light of Fact A.13, the closure space $(D(p, A), \text{cl}_p)$ admits a good dimension theory, but it is not a pre-geometry as the Exchange Axiom fails.

Definition A.15 Fix any set A with $p \not\perp A$, $D(p, A)$ is *modular* if $w_p(a/A) + w_p(b/A) = w_p(ab/A) + w_p((\text{cl}_p(a) \cap \text{cl}_p(b))/A)$ for all $a, b \in D(p, A)$.

As shown for example in 7.2.4 of [7]:

Fact A.16 *The regular type p is locally modular (as defined above) if and only if $D(p, A)$ is modular for all sets $A \not\perp p$.*

A.3 p -semiregular types

Within the space $D(p, A)$, it is useful to identify the p -semiregular types.

Definition A.17 $\text{stp}(a/A)$ is *p -semiregular of weight k* if it is p -simple and is (eventually) domination equivalent to $p^{(k)}$ for some finite $k \geq 1$, i.e., for some (equivalently, for all) a -models M independent from a over A , there is an M -independent sequence $\bar{b} = \langle b_1, \dots, b_k \rangle$ of realizations of the non-forking extension $p|M$ witnessing the p -simplicity of $\text{stp}(a/A)$, with a and \bar{b} domination equivalent over M (for any set X , $X \downarrow_M a$ if and only if $X \downarrow_M \bar{b}$.)

There is a natural Criterion for determining whether a p -simple type is p -semiregular. This Criterion appears as either Fact 1.4 of [4] or 7.1.18 of [7]:

Criterion A.18 *Suppose $\text{stp}(a/X)$ is p -simple of positive p -weight, and choose $Y \subseteq \text{dcl}(aX)$. Then $\text{stp}(a/Y)$ is p -semiregular of positive p -weight if and only if $a \notin \text{acl}(Y)$, but $w_p(e/Y) > 0$ for every $e \in \text{dcl}(aY) \setminus \text{acl}(Y)$.*

To get the existence of a p -semiregular type nearby a given p -simple type, we couple this with the following easy Lemma, whose proof only requires superstability.

Lemma A.19 *Suppose a and X are given with a finite, and Y is chosen arbitrarily such that $X \subseteq Y \subseteq \text{acl}(Xa)$. Then there is a finite sequence b from Y such that $Y \subseteq \text{acl}(Xb)$.*

Proof. Recursively construct a sequence $\langle b_i : i \rangle$ from Y of maximal length such that $a \not\perp_{B_i} b_i$, where $B_i := X \cup \{b_j : j < i\}$. Clearly, $R^\infty(a/B_i)$ is strictly decreasing with i , so any such sequence has finite length. But, for the sequence to terminate, it must be that $Y \subseteq \text{acl}(B_{i^*})$ for the terminal i^* . \square

Finally, we get our existence lemma.

Lemma A.20 *If $\text{stp}(a/X)$ is p -simple of positive p -weight, then there is a finite b from $\text{dcl}(aX) \cap \text{cl}_p(X)$ such that $\text{stp}(a/Xb)$ is p -semiregular and $w_p(a/Xb) = w_p(a/X)$.*

Proof. Let $Y = \text{dcl}(aX) \cap \text{cl}_p(X)$ and choose a finite b from Y such that $Y \subseteq \text{acl}(Xb)$. Now $\text{dcl}(Ya) = \text{dcl}(Xa)$, so if $e \in \text{dcl}(Ya) \setminus \text{acl}(Y)$, we must have $w_p(e/Y) > 0$, lest we would have $e \in Y$. Thus, Criterion A.18 for p -semiregularity applies. \square

Next we record ways in which an existing p -semiregular type is persistent.

Lemma A.21 *Suppose $\text{stp}(e/X)$ is p -semiregular.*

1. *If $e' \in \text{acl}(eX) \setminus \text{acl}(X)$, then $\text{stp}(e'/X)$ is p -semiregular;*
2. *If $\text{stp}(e/Y)$ is parallel to $\text{stp}(e/X)$, then $\text{stp}(e/Y)$ is p -semiregular of the same p -weight;*
3. *If $X \subseteq Y \subseteq \text{cl}_p(X)$, then $\text{stp}(e/Y)$ is p -semiregular of the same p -weight.*

Proof. (1) As $\text{dcl}(e'X) \subseteq \text{dcl}(eX)$, the result follows by Criterion A.18.

(2) This is immediate, as ‘domination equivalent to $p^{(k)}$ ’ is preserved.

(3) Since $\text{stp}(e/X)$ is p -semiregular, we automatically have $e \downarrow_X Y$, so (3) follows from (2). \square

Recalling Definition A.12, the following fact is Theorem 2(b) in [4].

Fact A.22 *Let T be superstable, let p be a non-trivial regular type of depth zero and let $\text{stp}(a/B)$ be p -semiregular. Then a lies in some $\text{acl}(B)$ -definable set D such that p -weight is continuous and definable inside D .*

A.4 p -disjointness

Definition A.23 Suppose $a, b \in D(p, C)$. We say that a and b are p -disjoint over C if $\text{cl}_p(Ca) \cap \text{cl}_p(Cb) = \text{cl}_p(C)$.

The next two Lemmas discuss the relationship between p -disjointness and forking, at least when $\text{stp}(ab/C)$ is p -semiregular.

Lemma A.24 *Suppose that $\text{stp}(ab/C)$ is p -semiregular, $C \subseteq D$ and $\text{stp}(ab/D)$ does not fork over C . Then $\text{cl}_p(Ca) \cap \text{cl}_p(Cb) = \text{cl}_p(C)$ if and only if $\text{cl}_p(Da) \cap \text{cl}_p(Db) = \text{cl}_p(D)$.*

Proof. First, assume there is a ‘bad element’ e for the triple (a, b, C) , that is $e \in \text{cl}_p(Ca) \cap \text{cl}_p(Cb) \setminus \text{cl}_p(C)$. As the existence of such an e is clearly determined by $\text{tp}(ab/C)$, by replacing D by some independent D^* realizing the same strong type as D over Cab , we may assume that $abe \downarrow_C D$. It follows immediately that $e \in [\text{cl}_p(Da) \cap \text{cl}_p(Db)] \setminus \text{cl}_p(D)$ so e is bad for (a, b, D) as well.

Conversely, if e is a ‘bad element’ for (a, b, D) , let $h := \text{Cb}(De/Cab)$. We first claim that $h \notin \text{cl}_p(C)$. If it were, then as $\text{stp}(ab/C)$ is p -semiregular, we would have $ab \downarrow_C h$. But, as $ab \downarrow_{C_h} De$, this would imply $ab \downarrow_D e$, contradicting $e \notin \text{cl}_p(D)$. Thus, $h \notin \text{cl}_p(C)$.

So, arguing by symmetry between a and b , it suffices to prove that $h \in \text{cl}_p(Ca)$. Choose a Morley sequence $\langle D_1 e_1, \dots, D_n e_n \rangle$ in $\text{stp}(De/Cab)$ with $D_1 e_1 = De$ such that $h \in \text{dcl}(e_1 \dots e_n D_1 \dots D_n)$. The standard argument yields

$$D_1, \dots, D_n \downarrow_C ab$$

As well, $h \in \text{acl}(Cab)$, hence $D_1, \dots, D_n \downarrow_{C_a} h$. Because of this, it suffices to prove that $\text{stp}(h/CaD_1 \dots D_n)$ is hereditarily orthogonal to p , i.e., has p -weight zero. However, for each i , $w_p(e_i/aD_i) = 0$, so $w_p(e_i/CaD_1 \dots D_n) = 0$ for each i . But $h \in \text{dcl}(e_1 \dots e_n D_1 \dots D_n)$, so $w_p(h/CaD_1 \dots D_n) = 0$. \square

Lemma A.25 *Suppose that $\text{stp}(ab/C)$ is p -semiregular and $\text{cl}_p(Ca) \cap \text{cl}_p(Cb) = \text{cl}_p(C)$. Then $\text{acl}(Ca) \cap \text{acl}(Cb) = \text{acl}(C)$.*

Proof. Choose any $e \in \text{acl}(Ca) \cap \text{acl}(Cb)$. As $\text{acl}(Ca) \subseteq \text{cl}_p(Ca)$ and $\text{acl}(Cb) \subseteq \text{cl}_p(Cb)$, our hypothesis implies that $e \in \text{cl}_p(C)$. However, as $\text{stp}(ab/C)$ is p -semiregular, this implies $ab \downarrow_C e$. As $e \in \text{acl}(abC)$, this implies $e \in \text{acl}(C)$ as desired. \square

Thanks to Lemma A.25 we will be able to apply the following general result below about forking to p -disjoint p -semiregular types.

Lemma A.26 *For all a, b, C , $\text{acl}(Ca) \cap \text{acl}(Cb) = \text{acl}(C)$ if and only if for every set X , if $X \downarrow_{C_a} b$ and $X \downarrow_{C_b} a$ both hold, then $X \downarrow_C ab$ holds as well.*

Proof. To ease notation, assume $C = \emptyset$. It suffices to prove this for finite sets X . For left to right, fix an X , and let D denote the canonical base of $\text{tp}(X/ab)$. On one hand, $D \subseteq \text{acl}(a)$, and on the other hand, $D \subseteq \text{acl}(b)$. Thus, by our assumption, $D \subseteq \text{acl}(\emptyset)$, implying that $X \downarrow ab$.

For the converse, choose any $h \in \text{acl}(a) \cap \text{acl}(b)$. Then, for trivial reasons we have $h \downarrow_a b$ and $h \downarrow_b a$, so by our hypothesis we have $h \downarrow ab$. But, as $h \in \text{acl}(ab)$, this implies $h \in \text{acl}(\emptyset)$. \square

A.5 Classifiable theories

Definition A.27 Let T be any stable theory.

- An *independent triple* is a sequence (A_0, A_1, A_2) satisfying $A_1 \cap A_2 = A_0$ and $A_1 \downarrow_{A_0} A_2$.
- A superstable theory *does not have the dimensional order property* (NDOP) if, for every independent triple $\mathcal{M} = (M_0, M_1, M_2)$ of \mathfrak{a} -models, the \mathfrak{a} -prime model M^* over $M_1 M_2$ (which exists in any superstable theory) is minimal among all \mathfrak{a} -models containing $M_1 M_2$.
- A superstable theory has *prime models over pairs* (PMOP) if, for any independent triple (M_0, M_1, M_2) of models, there is a constructible model over $M_1 M_2$.
- A complete theory T in a countable language is *classifiable* if T is superstable, has prime models over pairs (PMOP) and does not have the dimensional order property (NDOP).

The following are essential facts about NDOP. These appear as Lemma X 2.2 and Lemma X 7.2 of [10]. The second is also Fact 8.4.5 of [7].

Fact A.28 • T has NDOP if and only if, for $\mathcal{M} = (M_0, M_1, M_2)$ any independent triple of \mathfrak{a} -saturated models and M^* \mathfrak{a} -prime model over $M_1 M_2$, any regular type q non-orthogonal to M^* is either non-orthogonal to M_1 or M_2 .

- If T has NDOP, then any non-trivial regular type has depth zero.

For countable, superstable theories with NDOP, the condition PMOP has many equivalents. As notation due to Harrington, we say that an independent triple (B_0, B_1, B_2) *extends* the independent triple (A_0, A_1, A_2) if $B_0 \supseteq A_0$, $B_0 \downarrow_{A_0} A_1 A_2$, $B_1 \supseteq A_1 B_0$, $B_2 \supseteq A_2 B_0$, and $B_1 \downarrow_{B_0} B_2$. If (A_0, A_1, A_2) is any independent triple then a type $p \in S_n(A_1 A_2)$ is *V-dominated* if, for every independent triple (B_0, B_1, B_2) extending (A_0, A_1, A_2) , and every realization c of p , if $c \downarrow_{A_1 A_2} B_0$, then $c \downarrow_{A_1 A_2} B_1 B_2$. The best known equivalent of PMOP is NOTOP, the negation of the Omitting Types Order Property, see

e.g., Definition XII 4.2 of [10]. Although it is not mentioned in the statement of Fact A.29 below, Shelah says that T has the $(\aleph_0, 2)$ -*extension property* if there is a constructible model over any independent triple (M_0, M_1, M_2) of *countable* models. As the notion of NOTOP is not used in this paper, we merely note its equivalence to the notions (2)–(4) that are used.

Fact A.29 *The following are equivalent for a countable, superstable theory T with NDOP:*

1. T has NOTOP;
2. T has PMOP, i.e., for every independent triple (M_0, M_1, M_2) of models, a constructible model M^* over M_1M_2 exists;
3. For every independent triple (M_0, M_1, M_2) of models and every $n \geq 1$, every ℓ -constructible $p \in S_n(M_1M_2)$ is isolated;
4. For every independent triple (A_0, A_1, A_2) , every V -dominated $p \in S(A_1A_2)$ is isolated.

Proof. The equivalence of (1) and (2) are Theorem XII, 4.3 and Lemma XII 6.2 of [10].

(2) \Rightarrow (4): This is Lemma 8.5.10 of [7].

(4) \Rightarrow (3): Fix any independent triple (M_0, M_1, M_2) of models and any ℓ -isolated $p \in S(M_1M_2)$. To check that p is V -dominated, it suffices to take an independent triple (N_0, N_1, N_2) of models extending (M_0, M_1, M_2) . So choose such an (N_0, N_1, N_2) and choose any c realizing p with $c \downarrow_{M_1M_2} N_0$. By Lemma 1.3 of [1], $M_1M_2 \subseteq_{TV} N_1N_2$, i.e., every $L(M_1M_2)$ -formula $\varphi(x)$ with a solution in N_1N_2 has a solution in M_1M_2 . Thus, by [10], $p \vdash \text{tp}(c/N_1N_2)$, so in particular, $c \downarrow_{M_1M_2} N_1N_2$. Hence p is V -dominated, so p is isolated by (4).

(3) \Rightarrow (2): First, choose any independent triple (M_0, M_1, M_2) of *countable* models. As T is countable and stable, there is a countable ℓ -constructible model M^* over M_1M_2 . As each finite tuple from M^* is ℓ -isolated, it follows from (3) that M^* is atomic over M_1M_2 . As T and M^* are countable, this implies M^* is constructible over M_1M_2 . Thus, in Shelah's notation, we have proved that (3) implies that T has the $(\aleph_0, 2)$ -extension property. The whole of Section XII 5 of [10], culminating in Conclusion 5.14, is an inductive argument showing that this implies the $(\lambda, 2)$ -extension property for every infinite cardinal λ , i.e., that (2) holds. An alternate approach to this inductive argument is given in Theorem 3.5 of [1]. \square

Remark A.30 In fact, under the hypotheses of Fact A.29, it follows from Theorem XII 4.17 of [10] that any constructible model M^* occurring in the definition of PMOP is also minimal over M_1M_2 .

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