GEOMETRIC ORBITAL INTEGRALS AND THE CENTER OF THE ENVELOPING ALGEBRA

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Abstract. The purpose of this paper is to extend the explicit geometric evaluation of semisimple orbital integrals for smooth kernels for the Casimir operator obtained by the first author to the case of kernels for arbitrary elements in the center of the enveloping algebra.

Contents

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1. Introduction

In [B11, Chapter 6], one of us established a geometric formula for the semisimple orbital integrals of smooth kernels associated with the Casimir. The purpose of this paper is to extend this formula to the smooth kernels where more general elements of the center of the enveloping algebra also appear.

Let us briefly describe our main result in more detail. Let G be a connected real reductive group, and let $\mathfrak{g}$ be its Lie algebra. Let $\theta \in \mathrm{Aut}(G)$ be a Cartan involution, and let $K \subset G$ be the corresponding maximal compact subgroup with Lie algebra $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the associated Cartan splitting. Let B be a symmetric nondegenerate bilinear form on $\mathfrak{g}$ which is G and θ invariant, positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Let $X = G/K$ be the associated symmetric space, a Riemannian manifold with parallel nonpositive curvature.

Let $\rho^E : K \to U(E)$ be a finite dimensional unitary representation of K, and let $F = G \times_K E$ be the corresponding vector bundle on X. Then G acts on the left on $C^\infty(X, F)$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Then $Z(\mathfrak{g})$ acts on $C^\infty(X, F)$ and its action commutes with the left action of G. Among the elements of $Z(\mathfrak{g})$, there is the Casimir $C^{\mathfrak{g}}$, whose action on $C^\infty(X, F)$ is denoted $C^{\mathfrak{g},X}$.

Let $\mathcal{S}^{\mathrm{even}}(\mathbf{R})$ denote the even real functions on $\mathbf{R}$ that lie in the Schwartz space $\mathcal{S}(\mathbf{R})$. Let $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ be such that if $\widehat{\mu} \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ is its Fourier transform, there is $C > 0$, and for any $k \in \mathbf{N}$, there is $c_k > 0$ such that

$$\left|\widehat{\mu}^{(k)}(y)\right| \leq c_k \exp\left(-Cy^2\right). \tag{1.1}$$

If $A \in \mathbf{R}$, $\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)$ is a well-defined operator with a smooth kernel.

If $\gamma \in G$ is semisimple, as explained in [B11, Section 6.2], the orbital integral $\mathrm{Tr}^{[\gamma]}\left[\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\right]$ is well-defined, and it only depends on the conjugacy class of γ in G. After conjugation, we can write γ in the form $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k^{-1})\, a = a$. If $Z(\gamma) \subset G$ is the centralizer of γ with Lie algebra $\mathfrak{z}(\gamma)$, then θ acts on $Z(\gamma)$, and $Z(\gamma)$ is a possibly nonconnected reductive group. Let $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$ be the associated Cartan splitting.

Let $I^{\cdot}(\mathfrak{g})$ be the algebra of invariant polynomials on $\mathfrak{g}^*$, and let $\tau_{\mathrm{D}} : I^{\cdot}(\mathfrak{g}) \simeq Z(\mathfrak{g})$ denote the Duflo isomorphism [Du70, Thorme V.2]. If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, let $I^{\cdot}(\mathfrak{h}, \mathfrak{g})$ denote the algebra of polynomials on $\mathfrak{h}^*$ that are invariant under the corresponding algebraic Weyl group, so that we have the canonical identification $I^{\cdot}(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h}, \mathfrak{g})$[1]. There is a Harish-Chandra isomorphism $\phi_{\mathrm{HC}} : Z(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h}, \mathfrak{g})$. By [Du70, Lemme V.1], the Duflo and Harish-Chandra isomorphisms are known to be compatible.

There is a canonical projection $\mathfrak{g} \to \mathfrak{z}(\gamma)$[2], that induces a corresponding projection $I^{\cdot}(\mathfrak{g}) \to I^{\cdot}(\mathfrak{z}(\gamma))$. If $L \in Z(\mathfrak{g})$, let $L^{\mathfrak{z}(\gamma)}$ denote the differential operator on $\mathfrak{z}(\gamma)$ canonically associated with the projection of $\tau_{\mathrm{D}}^{-1} L$ on $I^{\cdot}(\mathfrak{z}(\gamma))$. In particular, up to a constant, $-(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}$ extends to the standard Laplacian on the Euclidean vector space $\mathfrak{z}_i(\gamma) = \mathfrak{p}(\gamma) \oplus i\mathfrak{k}(\gamma)$.

Following [B11, Chapter 5], in Definition 2.6, we define a smooth function $\mathcal{J}_\gamma : i\mathfrak{k}(\gamma) \to \mathbf{C}$. Let δ_a be the Dirac mass at $a \in \mathfrak{p}(\gamma)$. Then

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1} e^{-Y_0^{\mathfrak{k}}}\right)\right] \otimes \delta_a$$

is a distribution on $\mathfrak{z}_i(\gamma)$. [3]

Our main result, which is repeated in the text as Theorem 9.1 is as follows.

Theorem 1.1. *The following identity holds:*

$$\begin{aligned} (1.2)\quad & \mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\right] \\ & = L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)} + A}\right)\left[\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1} e^{-Y_0^{\mathfrak{k}}}\right)\right] \delta_a\right](0)\,. \end{aligned}$$

When $L = 1$, our theorem was already established in [B11, Theorem 6.2.2].

The proofs in [B11] used a construction of a new object, the hypoelliptic Laplacian. Here, we will only need the results of [B11].

Our proof is done in two steps. In a first step, using the results of [B11], we prove Theorem 1.1 when $\gamma \in G$ is regular. In this case, using the properties of the Harish-Chandra isomorphism [HC66], the proof is relatively easy.

[1] This isomorphism is usually written in its complex version $I^{\cdot}(\mathfrak{g}_{\mathbf{C}}) \simeq I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$. In Subsections 3.3 and 6.3, the corresponding real version is derived. Such considerations will also apply to other complex isomorphisms.

[2] This projection is defined in Subsection 8.2.

[3] In the sequel, $\otimes$ will be omitted.

When γ is nonregular, we combine our result for γ regular with limit arguments due to Harish-Chandra on the behavior of orbital integrals when γ' regular converges to γ. In both steps, remarkable and non-trivial properties of the function $\mathcal{J}_\gamma$ are used. One of these properties is that one essential piece of $\mathcal{J}_\gamma$ can be calculated only in terms of imaginary roots.

This paper is organized as follows. In Section 2, we describe the geometric setting, and we explain the formula for the semisimple orbital integrals that was obtained in [B11].

In Section 3, we recall some of the properties of Cartan subalgebras, Cartan subgroups, and of the corresponding root systems.

In Section 4, we express the restriction of the function $\mathcal{J}_\gamma$ to Cartan subalgebras in terms of a positive root system.

In Section 5, we specialize the results of the previous section to the case where γ is regular. We prove a crucial and unexpected smooth dependence of $\mathcal{J}_\gamma$ on γ.

In Section 6, we explain in some detail the Harish-Chandra isomorphism.

In Section 7, we establish Theorem 1.1 when γ is regular.

In Section 8, when γ is non necessarily regular, we study the limit of $\mathcal{J}_{\gamma'}$, and the limit of our formula for regular orbital integrals as γ' regular converges to γ in a suitable sense.

In Section 9, using the results of the previous section, we establish Theorem 1.1 in full generality.

Finally, in Section 10, we prove that our formula is compatible to the index theory for Dirac operators, and also with known results on Dirac cohomology [HuP02].

2. Geometric formulas for orbital integrals and the Casimir

In this Section, we explain the geometric formula given in [B11, Chapter 6] for the semisimple orbital integrals associated with the proper smooth kernels for the Casimir.

This section is organized as follows. In Subsection 2.1, we introduce the real reductive group G, its maximal compact subgroup K, the Lie algebras $\mathfrak{g}, \mathfrak{k}$, and the symmetric space $X = G/K$.

In Subsection 2.2, we recall the definition of semisimple elements in G, and of the corresponding displacement function.

In Subsection 2.3, we introduce the enveloping algebra $U(\mathfrak{g})$, and the Casimir element $C^{\mathfrak{g}} \in U(\mathfrak{g})$.

In Subsection 2.4, given a unitary representation of K, we construct the corresponding vector bundle F on X, and the elliptic operator $C^{\mathfrak{g},X}$ which is just the action of $C^{\mathfrak{g}}$ on $C^{\infty}(X,F)$.

In Subsection 2.5, given $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ such that its Fourier transform has the proper Gaussian decay, if $A \in \mathbf{R}$, we recall the definition of the semisimple orbital integrals associated with the smooth kernel for $\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)$. Among these kernels, there is the heat kernel for $C^{\mathfrak{g},X}$.

In Subsection 2.6, if $\gamma \in G$ is semisimple, if $Z(\gamma) \subset G$ is its centralizer with Lie algebra $\mathfrak{z}(\gamma)$, if $\mathfrak{k}(\gamma)$ is the compact part of $\mathfrak{z}(\gamma)$, we recall the definition of the function $\mathcal{J}_{\gamma}$ on $i\mathfrak{k}(\gamma)$ given in [B11, Theorem 5.5.1].

In Subsection 2.7, we study the behavior of $\mathcal{J}_{\gamma}$ when replacing by γ by γ^{-1}, and also by complex conjugation.

Finally, in Subsection 2.8, we state the geometric formula obtained in [B11] for the above orbital integrals, in which the function $\mathcal{J}_{\gamma}$ plays a key role.

2.1. **Reductive groups and symmetric spaces.** Let G be a connected reductive real Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let $\theta \in \mathrm{Aut}(G)$ be a Cartan involution. Then θ acts as an automorphism of $\mathfrak{g}$. Let $K \subset G$ be the fixed point set of θ. Then K is a compact connected subgroup of G, which is a maximal compact subgroup. If $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of K, then $\mathfrak{k}$ is the fixed point set of θ in $\mathfrak{g}$. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of θ corresponding to the eigenvalue -1, so that we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{2.1}$$

Put

$$m = \dim \mathfrak{p}, \qquad n = \dim \mathfrak{k}, \tag{2.2}$$

so that

$$\dim \mathfrak{g} = m + n. \tag{2.3}$$

Let B be a G and θ invariant bilinear symmetric nondegenerate form on $\mathfrak{g}$. Then (2.1) is a B-orthogonal splitting. We assume that B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Let $\langle\,\rangle = -B(\cdot,\theta\cdot)$ be the corresponding scalar product on $\mathfrak{g}$. Let B^* be the bilinear symmetric form on $\mathfrak{g}^* = \mathfrak{p}^* \oplus \mathfrak{k}^*$ which is dual to B.

Let $\omega^{\mathfrak{g}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}$. By (2.1), $\omega^{\mathfrak{g}}$ splits as

$$\omega^{\mathfrak{g}} = \omega^{\mathfrak{p}} + \omega^{\mathfrak{k}}. \tag{2.4}$$

Let $X = G/K$ be the corresponding symmetric space. Then $p : G \to X = G/K$ is a K-principal bundle, and $\omega^{\mathfrak{k}}$ is a connection form. Also the tangent bundle TX is given by

$$TX = G \times_K \mathfrak{p}. \tag{2.5}$$

Then TX is equipped with the scalar product $\langle\,\rangle$ induced by B, so that X is a Riemannian manifold. The connection ∇^{TX} on TX which is induced by $\omega^{\mathfrak{k}}$ is the Levi-Civita connection of TX, and its curvature is parallel and nonpositive. Also G acts isometrically on the left on X, and θ acts as an isometry of X.

By [Kn86, Proposition 1.2], any element $\gamma \in G$ factorizes uniquely in the form

$$\gamma = e^a k^{-1}, \qquad a \in \mathfrak{p}, \qquad k \in K. \tag{2.6}$$

If $\gamma, g \in G$, set

$$C(g)\,\gamma = g\gamma g^{-1}. \tag{2.7}$$

Then $C(g)$ is an automorphism of G. Its derivative at the identity is the adjoint representation $g \in G \to \mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$. The derivative of this last map is given by $a \in \mathfrak{g} \to \mathrm{ad}(a) \in \mathrm{End}(\mathfrak{g})$, with $\mathrm{ad}(a)\,b = [a, b]$. If $\gamma \in G$, the fixed point set of $C(\gamma)$ is the centralizer $Z(\gamma) \subset G$, whose Lie algebra $\mathfrak{z}(\gamma)$ is given by

$$\mathfrak{z}(\gamma) = \ker\left(1 - \mathrm{Ad}(\gamma)\right). \tag{2.8}$$

If $f \in \mathfrak{g}$, let $Z(f) \subset G$ be the stabilizer of f. Its Lie algebra $\mathfrak{z}(f) \subset \mathfrak{g}$ is given by

$$\mathfrak{z}(f) = \ker \mathrm{ad}(f). \tag{2.9}$$

In the sequel, if M is a Lie group, we denote by M^0 the connected component of the identity.

2.2. **Semisimple elements and their displacement function.** Let d be the Riemannian distance on X. By [BaGS85, 6.1], d is a convex function on $X \times X$. If $\gamma \in G$, let d_γ be the corresponding displacement function on X, i.e.,

$$d_\gamma(x) = d(x, \gamma x). \tag{2.10}$$

If $g \in G$, then

$$d_{C(g)\gamma}(gx) = d_\gamma(x). \tag{2.11}$$

Moreover,

$$d_{\theta(\gamma)}(\theta x) = d_\gamma(x). \tag{2.12}$$

Set

$$m_\gamma = \inf d_\gamma. \tag{2.13}$$

Let $X(\gamma) \subset X$ be the closed subset where d_γ reaches its minimum. By [BaGS85, p. 78 and 1.2], $X(\gamma)$ is a closed convex subset, d_γ is smooth on $X \setminus X(\gamma)$ and has no critical points on $X \setminus X(\gamma)$. Also by (2.11), (2.12),

$$\begin{aligned} &X(C(g)\gamma) = gX(\gamma), && X(\theta\gamma) = \theta X(\gamma), \\ &m_{C(g)\gamma} = m_\gamma, && m_{\theta(\gamma)} = m_\gamma. \end{aligned} \tag{2.14}$$

By [E96, Definition 2.19.21], γ is said to be semisimple if $X(\gamma)$ is nonempty. If γ is semisimple, then $C(g)\gamma$ and $\theta(\gamma)$ are semisimple. Also γ is said to be elliptic if it is semisimple and $m_\gamma = 0$. Elliptic elements are exactly the group elements that are conjugate to elements of K. Finally, γ is said to be hyperbolic if it is conjugate to $e^a, a \in \mathfrak{p}$.

By [Ko73, Proposition 2.1], [BaGS85, Theorems 2.19.23 and 2.19.24], $\gamma \in G$ is semisimple if and only if it factorizes as $\gamma = he = eh$, with commuting hyperbolic h and elliptic e. Also e, h are uniquely determined by γ, and

$$Z(\gamma) = Z(h) \cap Z(e). \tag{2.15}$$

Set

$$x_0 = p1. \tag{2.16}$$

Theorem 2.1. *Let $\gamma \in G$ be semisimple. If $g \in G, x = pg \in X$, then $x \in X(\gamma)$ if and only if there exist $a \in \mathfrak{p}, k \in K$ such that $\mathrm{Ad}(k)\, a = a$, and*

$$\gamma = C(g)\left(e^a k^{-1}\right). \tag{2.17}$$

Also $C(g)\, e^a \in G, C(g)\, k \in G$ are uniquely determined by γ. If $g_t = ge^{ta}$, then $t \in [0,1] \to y_t = pg_t$ is the unique geodesic connecting x and γx. Moreover

$$m_\gamma = |a|. \tag{2.18}$$

If $\gamma \in G$ is semisimple, then $x_0 \in X(\gamma)$ if and only if there exist $a \in \mathfrak{p}, k \in K$ such that

$$\gamma = e^a k^{-1}, \qquad a \in \mathfrak{p}, \qquad \mathrm{Ad}(k)\, a = a. \tag{2.19}$$

Also a, k are uniquely determined by (2.19).

Proof. The first part of our theorem was established in [B11, Theorem 3.1.2]. By taking $g = 1$ in the first part, we obtain the second part. □

Let $\gamma \in G$ be a semisimple element written as in (2.19). By [B11, Proposition 3.2.8 and eqs. (3.3.4), (3.3.6)],

(2.20)
$$Z(e^a) = Z(a), \quad Z(\gamma) = Z(a) \cap Z(k), \qquad \mathfrak{z}(\gamma) = \mathfrak{z}(a) \cap \mathfrak{z}(k).$$

By (2.19), $a \in \mathfrak{z}(\gamma)$, and by (2.20), $\mathfrak{z}(\gamma) \subset \mathfrak{z}(a)$, so that a is an element of the center of $\mathfrak{z}(\gamma)$.

Clearly,

$$\theta(\gamma) = e^{-a}k^{-1}. \tag{2.21}$$

Therefore $\theta(\gamma) \in Z(\gamma)$, so that the above centralizers and Lie algebras are preserved by θ. Set

$$K(\gamma) = Z(\gamma) \cap K. \tag{2.22}$$

By [B11, Theorem 3.3.1], we have the identity

$$K^0(\gamma) = Z^0(\gamma) \cap K, \tag{2.23}$$

and $K^0(\gamma)$ is a maximal compact subgroup of $Z^0(\gamma)$.

Put

$$\mathfrak{p}(\gamma) = \mathfrak{p} \cap \mathfrak{z}(\gamma), \qquad \mathfrak{k}(\gamma) = \mathfrak{k} \cap \mathfrak{z}(\gamma). \tag{2.24}$$

Then $\mathfrak{k}(\gamma)$ is the Lie algebra of $K(\gamma)$. We use similar notation for the Lie algebras $\mathfrak{z}(k), \mathfrak{z}(a)$. We have the Cartan decompositions of Lie algebras,

(2.25)
$$\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma), \quad \mathfrak{z}(k) = \mathfrak{p}(k) \oplus \mathfrak{k}(k), \qquad \mathfrak{z}(a) = \mathfrak{p}(a) \oplus \mathfrak{k}(a).$$

Then B restricts to a nondegenerate form on $\mathfrak{z}(\gamma), \mathfrak{z}(k), \mathfrak{z}(a)$, so that $Z(\gamma), Z(k), Z(a)$ are possibly nonconnected reductive subgroups of G. By [B11, Theorem 3.3.1], we have the identification of finite groups,

$$Z^0(\gamma) \setminus Z(\gamma) = K^0(\gamma) \setminus K(\gamma). \tag{2.26}$$

Let $\mathfrak{z}^\perp(\gamma), \mathfrak{z}^\perp(a)$ be the orthogonal spaces to $\mathfrak{z}(\gamma), \mathfrak{z}(a)$ in $\mathfrak{g}$ with respect to B. We have splittings

$$\mathfrak{z}^\perp(\gamma) = \mathfrak{p}^\perp(\gamma) \oplus \mathfrak{k}^\perp(\gamma), \qquad \mathfrak{z}^\perp(a) = \mathfrak{p}^\perp(a) \oplus \mathfrak{k}^\perp(a). \tag{2.27}$$

Let $\mathfrak{z}_a^\perp(\gamma)$ denote the orthogonal to $\mathfrak{z}(\gamma)$ in $\mathfrak{z}(a)$. We still have a splitting

$$\mathfrak{z}_a^\perp(\gamma) = \mathfrak{p}_a^\perp(\gamma) \oplus \mathfrak{k}_a^\perp(\gamma). \tag{2.28}$$

Now we recall a result established in [B11, Theorem 3.3.1].

Theorem 2.2. *The set $X(\gamma)$ is preserved by θ. Moreover,*

$$X(\gamma) = X(e^a) \cap X(k). \tag{2.29}$$

Also $X(\gamma)$ is a totally geodesic submanifold of X. In the geodesic coordinate system centered at $x_0 = p1$, then

$$X(\gamma) = \mathfrak{p}(\gamma). \tag{2.30}$$

The actions of $Z^0(\gamma), Z(\gamma)$ on $X(\gamma)$ are transitive. More precisely the maps $g \in Z^0(\gamma) \to pg \in X, g \in Z(\gamma) \to pg \in X$ induce the identification of $Z^0(\gamma)$-manifolds,

$$X(\gamma) = Z^0(\gamma)/K^0(\gamma) = Z(\gamma)/K(\gamma). \tag{2.31}$$

Now we will establish a simple important consequence of Theorem 2.2.

Theorem 2.3. *Let γ be a semisimple element of G as in (2.19). Let γ' be another semisimple element of G such that*

$$\gamma' = e^{a'} k'^{-1}, \qquad a' \in \mathfrak{p}, \qquad \mathrm{Ad}(k')\, a' = a'. \tag{2.32}$$

Then there exists $g \in G$ such that $\gamma' = C(g)\gamma$ if and only if there exists $k'' \in K$ such that $C(k'')\gamma = \gamma'$, in which case

$$a' = \mathrm{Ad}(k'')\, a, \qquad k' = C(k'')\, k. \tag{2.33}$$

Proof. Assume that $\gamma' = C(g)\gamma$. By (2.14), we get

$$X(\gamma') = gX(\gamma). \tag{2.34}$$

By Theorem 2.1, $x_0 \in X(\gamma) \cap X(\gamma')$. By (2.34), $gx_0 \in X(\gamma')$. By Theorem 2.2, there exists $h \in Z(\gamma')$ such that

$$hgx_0 = x_0, \tag{2.35}$$

which is equivalent to

$$k'' = hg \in K. \tag{2.36}$$

Since $h \in Z(\gamma')$, we conclude that $C(k'')\gamma = \gamma'$. Using the uniqueness of decomposition in (2.32), equation (2.33) follows. The proof of our theorem is completed. □

2.3. **Enveloping algebra and the Casimir.** We identify $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on G. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Then $U(\mathfrak{g})$ can be identified with the algebra of left-invariant differential operators on G. Let $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$.

If E is a finite dimensional real or complex vector space, and if $\rho^E : \mathfrak{g} \to \mathrm{End}(E)$ is a morphism of Lie algebras, the map ρ^E extends to a morphism $U(\mathfrak{g}) \to \mathrm{End}(E)$.

Among the elements of $Z(\mathfrak{g})$, there is the Casimir element $C^{\mathfrak{g}}$. If $e_1, \ldots, e_{m+n}$ is a basis of $\mathfrak{g}$, and if $e_1^*, \ldots, e_{m+n}^*$ is the dual basis of $\mathfrak{g}$ with respect to B, then [4]

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m+n} e_i^* e_i. \tag{2.37}$$

If we consider instead the Lie algebra $(\mathfrak{k}, B|_{\mathfrak{k}})$, $C^{\mathfrak{k}} \in Z(\mathfrak{k})$ denotes the associated Casimir element.

If $e_1, \ldots, e_m$ is a basis of $\mathfrak{p}$, and if $e_1^*, \ldots, e_m^*$ is the dual basis of $\mathfrak{p}$ with respect to $B|_{\mathfrak{p}}$, set

$$C^{\mathfrak{p}} = -\sum_{i=1}^{m} e_i^* e_i. \tag{2.38}$$

Then $C^{\mathfrak{p}} \in U(\mathfrak{g})$. Using (2.37), (2.38), we get

$$C^{\mathfrak{g}} = C^{\mathfrak{p}} + C^{\mathfrak{k}}. \tag{2.39}$$

Also $C^{\mathfrak{p}}$ and $C^{\mathfrak{k}}$ commute.

If $\rho^E : \mathfrak{g} \to \mathrm{End}(E)$ is taken as above, put

$$C^{\mathfrak{g},E} = \rho^E\left(C^{\mathfrak{g}}\right). \tag{2.40}$$

Under the above conditions, we can define $C^{\mathfrak{p},E}, C^{\mathfrak{k},E}$.

Since $\mathfrak{g}$ is itself a representation of $\mathfrak{g}$, $C^{\mathfrak{g},\mathfrak{g}}$ is the action of $C^{\mathfrak{g}}$ on $\mathfrak{g}$. Since $\mathfrak{k}$ acts on $\mathfrak{p}, \mathfrak{k}$, $C^{\mathfrak{k},\mathfrak{p}}, C^{\mathfrak{k},\mathfrak{k}}$ are also well-defined.

Proposition 2.4. *The following identity holds:*

$$\mathrm{Tr}\left[C^{\mathfrak{g},\mathfrak{g}}\right] = 3\mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{p}}\right] + \mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{k}}\right]. \tag{2.41}$$

Proof. By (2.39), we get

$$\mathrm{Tr}\left[C^{\mathfrak{g},\mathfrak{g}}\right] = \mathrm{Tr}\left[C^{\mathfrak{p},\mathfrak{g}}\right] + \mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{g}}\right]. \tag{2.42}$$

Let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathfrak{p}$, and let $e_{m+1}, \ldots, e_n$ be an orthonormal basis of $\mathfrak{k}$. Then

$$\mathrm{Tr}\left[C^{\mathfrak{p},\mathfrak{g}}\right] = -\sum_{\substack{1\le i\le m\\ 1\le j\le m+n}} \left|[e_i, e_j]\right|^2, \tag{2.43}$$

$$\mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{p}}\right] = -\sum_{\substack{1\le i\le m\\ m+1\le j\le m+n}} \left|[e_i, e_j]\right|^2 = -\sum_{1\le i,j\le m} \left|[e_i, e_j]\right|^2.$$

[4] In [Kn86, Section 8.3], the Casimir is defined with the opposite sign. We have adopted the sign conventions of [B11], which are closer to analysis.

By (2.43), we deduce that

$$\mathrm{Tr}\left[C^{\mathfrak{p},\mathfrak{g}}\right] = 2\mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{p}}\right]. \tag{2.44}$$

Also

$$\mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{g}}\right] = \mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{p}}\right] + \mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{k}}\right]. \tag{2.45}$$

By (2.42), (2.44), and (2.45), we get (2.41). The proof of our proposition is completed. □

Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra. Then we have the splitting $\mathfrak{h} = \mathfrak{h}_{\mathfrak{p}} \oplus \mathfrak{h}_{\mathfrak{k}}$. [5] Let $R \subset \mathfrak{h}^*_{\mathbf{C}}$ be the corresponding root system. Let R_+ be a positive root system. If $R_- = -R_+$, R is the disjoint union of R_+ and R_-. Let $\rho^{\mathfrak{g}} \in \mathfrak{h}^*_{\mathbf{C}}$ be the half sum of the positive roots. Here $\rho^{\mathfrak{g}} \in \mathfrak{h}^*_{\mathfrak{p}} \oplus i\mathfrak{h}^*_{\mathfrak{k}}$. By Kostant's strange formula [Ko76], we have the identity

$$B^*\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right) = -\frac{1}{24}\mathrm{Tr}\left[C^{\mathfrak{g},\mathfrak{g}}\right]. \tag{2.46}$$

By (2.41), (2.46), we get

$$B^*\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right) = -\frac{1}{8}\mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{p}}\right] - \frac{1}{24}\mathrm{Tr}\left[C^{\mathfrak{k},\mathfrak{k}}\right]. \tag{2.47}$$

Using the notation in [B11, eq. (2.6.11)] [6], by (2.47), we obtain

$$B^*\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right) = -\frac{1}{4}B^*\left(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}\right). \tag{2.48}$$

2.4. **The elliptic operator $C^{\mathfrak{g},X}$.** Let E be a finite dimensional Hermitian vector space, and let $\rho^E : K \to U(E)$ denote a unitary representation of K. The Casimir $C^{\mathfrak{k},E}$ is a self-adjoint nonpositive endomorphism of E. If ρ^E is irreducible, then $C^{\mathfrak{k},E}$ is a scalar.

Let F be the vector bundle on X,

$$F = G \times_K E. \tag{2.49}$$

Then F is a Hermitian vector bundle on X, which is equipped with a canonical connection ∇^F. Also $C^{\mathfrak{k},E}$ descend to a parallel section $C^{\mathfrak{k},F}$ of $\mathrm{End}(F)$. Moreover, G acts on $C^\infty(X,F)$, so that if $g \in G, s \in C^\infty(X,F)$, if g_* denotes the lift of the action of g to F,

$$gs(x) = g_* s\left(g^{-1}x\right). \tag{2.50}$$

The Casimir operator $C^{\mathfrak{g}}$ descends to a second order elliptic operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X,F)$, which commutes with G. Let Δ^X denote

[5] More details will be given in Section 3 on Cartan subalgebras and root systems.

[6] The definition of $\kappa^{\mathfrak{g}}$ is not needed. The formula is given for later reference.

the classical Bochner Laplacian acting on $C^\infty(X,F)$. By [B11, eq. (2.13.2)], the splitting (2.39) of $C^{\mathfrak{g}}$ descends to the splitting of $C^{\mathfrak{g},X}$,

$$C^{\mathfrak{g},X} = -\Delta^X + C^{\mathfrak{k},F}. \tag{2.51}$$

2.5. **Orbital integrals and the Casimir.** Let $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$, and let $\widehat{\mu} \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ be its Fourier transform, i.e.,

$$\widehat{\mu}(y) = \int_{\mathbf{R}} e^{-2i\pi yx}\mu(x)\,dx. \tag{2.52}$$

We assume that there exists $C > 0$ such that for any $k \in \mathbf{N}$, there is $c_k > 0$ such that

$$\left|\widehat{\mu}^{(k)}(y)\right| \leq c_k \exp\left(-Cy^2\right). \tag{2.53}$$

The above condition is verified if $\widehat{\mu}$ has compact support. For $t > 0$, it is also verified by the Gaussian function e^{-tx^2}.

If $A \in \mathbf{R}$, the operator $\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)$ is self-adjoint with a smooth kernel $\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)(x,x')$ with respect to the Riemannian volume dx' on X. As explained in [B11, Section 6.2], using finite propagation speed for the wave equation, condition (2.53) implies that there exist $C > 0, c > 0$ such that if $x, x' \in X$, then

$$\left|\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)(x,x')\right| \leq Ce^{-cd^2(x,x')}. \tag{2.54}$$

If $\widehat{\mu}$ has compact support, then $\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)(x,x')$ vanishes when $d(x,x')$ is large enough.

As explained in [B11, Section 6.2], the above condition guarantees that if $\gamma \in G$ is semisimple, the orbital integral $\mathrm{Tr}^{[\gamma]}\left[\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$ is well-defined. Let us give more details on our conventions.

Let $\gamma \in G$ be taken as in (2.19). Let $N_{X(\gamma)/X}$ be the orthogonal bundle to $TX(\gamma)$ in TX. By [B11, eq. (3.4.1)], we have the identity

$$N_{X(\gamma)/X} = Z^0(\gamma) \times_{K^0(\gamma)} \mathfrak{p}^\perp(\gamma). \tag{2.55}$$

Let $\mathcal{N}_{X(\gamma)/X}$ be the total space of $N_{X(\gamma)/X}$. By [B11, Theorem 3.4.1], the normal geodesic coordinate system based at $X(\gamma)$ gives a smooth identification of $\mathcal{N}_{X(\gamma)/X}$ with X. Let dx, dy, df be the Riemannian volume forms on $X, X(\gamma), N_{X(\gamma)/X}$. Then $dydf$ is a volume form on $\mathcal{N}_{X(\gamma)/X}$. Let $r(f)$ denote the corresponding Jacobian, so that we have the identity of volume forms on X,

$$dx = r(f)\,dydf. \tag{2.56}$$

By [B11, eq. (3.4.36)], there are constants $C > 0, C' > 0$ such that

$$r(f) \leq Ce^{C'|f|}. \tag{2.57}$$

By [B11, Theorem 3.4.1], there exists $C_\gamma > 0$ such that for $f \in \mathfrak{p}^{\perp}(\gamma), |f| \geq 1$,

$$d_\gamma\left(e^f x_0\right) \geq |a| + C_\gamma |f|. \tag{2.58}$$

As explained in [B11, eq. (4.2.6)], by (2.54), (2.58), there exist $C_\gamma > 0, c_\gamma > 0$ such that if $f \in \mathfrak{p}^{\perp}(\gamma)$, then

$$\left|\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\left(\gamma^{-1}e^f x_0, e^f x_0\right)\right| \leq C_\gamma \exp\left(-c_\gamma |f|^2\right). \tag{2.59}$$

We denote by γ_* the action of γ on F. More precisely, if $x \in X$, γ_* maps F_x into $F_{\gamma x}$.

In [B11, Definition 4.2.2], the orbital integral

$$\mathrm{Tr}^{[\gamma]}\left[\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\right]$$

is defined by the formula

$$\begin{aligned}\mathrm{Tr}^{[\gamma]}&\left[\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\right] \\ &= \int_{\mathfrak{p}^{\perp}(\gamma)} \mathrm{Tr}\left[\gamma_* \mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\left(\gamma^{-1}e^f x_0, e^f x_0\right)\right] r(f)\, df.\end{aligned} \tag{2.60}$$

Equations (2.57), (2.59) guarantee that the integral in (2.60) converges.

Let dk be the Haar measure on K such that $\mathrm{Vol}(K) = 1$. Then $dg = dxdk$ is a Haar measure on G. Let dy be the Riemannian volume form on $X(\gamma)$. Let dk'^0 be the Haar measure on $K^0(\gamma)$ such that $\mathrm{Vol}(K^0(\gamma)) = 1$. Then $dz^0 = dydk'^0$ is a Haar measure on $Z^0(\gamma)$. Let dv^0 be the volume on $Z^0(\gamma) \backslash G$ such that $dg = dz^0 dv^0$.

As explained in [B11, Section 4.2], the smooth kernel

$$\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)(x, x')$$

lifts to a smooth function on G with values in $\mathrm{End}(E)$, denoted

$$\mu^E\left(\sqrt{C^{\mathfrak{g},X} + A}\right)(g),$$

and by [B11, eq. (4.2.11)], we have the identity

$$\begin{aligned}\mathrm{Tr}^{[\gamma]}&\left[\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\right] \\ &= \int_{Z^0(\gamma)\backslash G} \mathrm{Tr}^E\left[\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\left(\left(v^0\right)^{-1}\gamma v^0\right)\right] dv^0.\end{aligned} \tag{2.61}$$

This definition of orbital integrals coincides with the definition given by Selberg [Se56, p. 66].

2.6. **The function $\mathcal{J}_\gamma$.** We use the assumptions in Subsection 2.2 and the corresponding notation. In particular $\gamma \in G$ is a semisimple element as in (2.19).

Then $\mathrm{Ad}(\gamma)$ preserves $\mathfrak{z}(a), \mathfrak{z}^\perp(a)$. Also $\mathrm{Ad}(k^{-1})$ preserves $\mathfrak{z}_a^\perp(\gamma)$. If $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, $\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)$ preserves $\mathfrak{z}_a^\perp(\gamma)$. The splitting (2.28) is preserved by $\mathrm{Ad}(k^{-1})$ and $\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)$.

If $x \in \mathbf{R}$, put

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}. \tag{2.62}$$

If $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, $\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)$ acts as an antisymmetric endomorphism of $\mathfrak{p}(\gamma), \mathfrak{k}(\gamma)$, so that its eigenvalues are either 0, or they come by nonzero conjugate imaginary pairs. If $Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, put[7]

$$\begin{aligned}\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right) &= \left[\det\left(\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)\right)\right]^{1/2},\\ \widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{k}(\gamma)}\right) &= \left[\det\left(\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{k}(\gamma)}\right)\right)\right]^{1/2}.\end{aligned} \tag{2.63}$$

The square root in (2.63) is the positive square root of a positive real number.

We follow [B11, Theorem 5.5.1], while slightly changing the notation.

Definition 2.5. If $Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, put

$$\mathcal{L}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{\det\left(1 - \mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_a^\perp(\gamma)}}{\det\left(1 - \mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_a^\perp(\gamma)}}. \tag{2.64}$$

Set

$$\mathcal{M}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \left[\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_a^\perp(\gamma)}}\mathcal{L}_\gamma\left(Y_0^{\mathfrak{k}}\right)\right]^{1/2}. \tag{2.65}$$

The fact that the square root in (2.65) is unambiguously defined is established in [B11, Section 5.5]. Let us explain the details. First we make $Y_0^{\mathfrak{k}} = 0$. Then

$$\begin{aligned}&\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_a^\perp(\gamma)}}\frac{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}_a^\perp(\gamma)}}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_a^\perp(\gamma)}}\\ &\qquad = \left[\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_a^\perp(\gamma)}}\right]^2.\end{aligned} \tag{2.66}$$

[7]This fits with the classical notation in the theory of characteristic classes.

The conventions in [B11] say that the square root of (2.66) is the obvious positive square root, i.e.,

$$\mathcal{M}_\gamma(0) = \frac{1}{\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_a^\perp(\gamma)}}. \tag{2.67}$$

Using analyticity in the variable $Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, the choice of the square root in (2.65) determines a choice of the square root in (2.68). This point will be discussed at length in Section 4. No choice of a Cartan subalgebra or of a positive root system is needed at this stage.

Definition 2.6. Let $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)$ be the smooth function of $Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$,

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}^\perp(a)}\right|^{1/2}} \frac{\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{k}(\gamma)}\right)} \mathcal{M}_\gamma\left(Y_0^{\mathfrak{k}}\right). \tag{2.68}$$

With the conventions in [B11, Chapter 5], where instead a function $J_\gamma\left(Y_0^{\mathfrak{k}}\right)$ is defined on $\mathfrak{k}(\gamma)$, we have

$$J_\gamma\left(Y_0^{\mathfrak{k}}\right) = \mathcal{J}_\gamma\left(iY_0^{\mathfrak{k}}\right). \tag{2.69}$$

By [B11, eq. (5.5.11)] or by (2.68), if $Y_0^{\mathfrak{k}} \in i\mathfrak{k}$, then

$$\mathcal{J}_1\left(Y_0^{\mathfrak{k}}\right) = \frac{\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}}\right)}{\widehat{A}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{k}}\right)}. \tag{2.70}$$

2.7. **Some properties of the function $\mathcal{J}_\gamma$.** Let $\rho^{E^*} : K \to U(E^*)$ denote the representation of K which is dual to the representation ρ^E.

Proposition 2.7. *If $Y_0^{\mathfrak{k}} \in i\mathfrak{k}(\gamma)$, then*

(2.71)

$$\mathcal{J}_{\gamma^{-1}}\left(Y_0^{\mathfrak{k}}\right) = \mathcal{J}_\gamma\left(-Y_0^{\mathfrak{k}}\right), \quad \mathrm{Tr}^{E^*}\left[\rho^{E^*}\left(ke^{-Y_0^{\mathfrak{k}}}\right)\right] = \mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{Y_0^{\mathfrak{k}}}\right)\right],$$
$$\overline{\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)} = \mathcal{J}_\gamma\left(-Y_0^{\mathfrak{k}}\right), \quad \overline{\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]} = \mathrm{Tr}^{E^*}\left[\rho^{E^*}\left(k^{-1}e^{Y_0^{\mathfrak{k}}}\right)\right].$$

Proof. If $f \in \mathrm{End}(\mathfrak{g})$, let $\widetilde{f} \in \mathrm{End}(\mathfrak{g})$ denote the adjoint of f with respect to B. We have the identity

$$\mathrm{Ad}\left(\gamma^{-1}\right) = \widetilde{\mathrm{Ad}\left(\gamma\right)}. \tag{2.72}$$

By (2.72), we deduce that

$$\det\left(1-\mathrm{Ad}\left(\gamma^{-1}\right)\right)|_{\mathfrak{z}^\perp(a)} = \det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}^\perp(a)}. \tag{2.73}$$

A similar argument shows that

$$\det\left(1-\mathrm{Ad}\left(ke^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_a^{\perp}(\gamma)}=\det\left(1-\mathrm{Ad}\left(k^{-1}e^{Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_a^{\perp}(\gamma)},$$
$$\det\left(1-\mathrm{Ad}\left(ke^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_a^{\perp}(\gamma)}=\det\left(1-\mathrm{Ad}\left(k^{-1}e^{Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_a^{\perp}(\gamma)}. \tag{2.74}$$

By (2.64), (2.65), (2.68), (2.73), and (2.74), we get the first identity in (2.71). The second identity in (2.71) is trivial.

If $Y_0^{\mathfrak{k}}\in i\mathfrak{k}(\gamma)$,

$$\overline{\det\left(1-\mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_a^{\perp}(\gamma)}}=\det\left(1-\mathrm{Ad}\left(k^{-1}e^{Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{p}_a^{\perp}(\gamma)},$$
$$\overline{\det\left(1-\mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_a^{\perp}(\gamma)}}=\det\left(1-\mathrm{Ad}\left(k^{-1}e^{Y_0^{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}_a^{\perp}(\gamma)}. \tag{2.75}$$

By (2.75), we get the third identity in (2.71). Since $Y_0^{\mathfrak{k}}\in i\mathfrak{k}(\gamma)$, the fourth identity is trivial. The proof of our proposition is completed. □

2.8. **A geometric formula for the orbital integrals associated with the Casimir.** Note that $i\mathfrak{k}(\gamma)$ is naturally an Euclidean vector space. If $Y_0^{\mathfrak{k}}\in i\mathfrak{k}(\gamma)$, we denote by $\left|Y_0^{\mathfrak{k}}\right|$ its Euclidean norm. More precisely, if $Y_0^{\mathfrak{k}}\in i\mathfrak{k}(\gamma)$, then

$$\left|Y_0^{\mathfrak{k}}\right|^2=B\left(Y_0^{\mathfrak{k}},Y_0^{\mathfrak{k}}\right). \tag{2.76}$$

By [B11, eq. (6.1.1)], there exist $c>0, C>0$ such that if $Y_0^{\mathfrak{k}}\in i\mathfrak{k}(\gamma)$,

$$\left|\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\right|\leq c\exp\left(C\left|Y_0^{\mathfrak{k}}\right|\right). \tag{2.77}$$

In the sequel, $\int_{i\mathfrak{k}(\gamma)}$ denotes integration on the real vector space $i\mathfrak{k}(\gamma)$.

Let $dY_0^{\mathfrak{k}}$ be the Euclidean volume on $i\mathfrak{k}(\gamma)$. Set $p=\dim\mathfrak{p}(\gamma), q=\dim\mathfrak{k}(\gamma)$. Now we state the result obtained in [B11, Theorem 6.1.1]. Our reformulation takes equation (2.48) into account.

Theorem 2.8. *For $t>0$, the following identity holds:*

(2.78)

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X}/2\right)\right]=\exp\left(-tB^*\left(\rho^{\mathfrak{g}},\rho^{\mathfrak{g}}\right)/2\right)\frac{\exp\left(-|a|^2/2t\right)}{(2\pi t)^{p/2}}$$
$$\int_{i\mathfrak{k}(\gamma)}\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

Let $B|_{\mathfrak{z}(\gamma)}$ be the restriction of B to $\mathfrak{z}(\gamma)$, and let $B^*|_{\mathfrak{z}(\gamma)}$ be the corresponding quadratic form on $\mathfrak{z}^*(\gamma)$. Let $\Delta^{\mathfrak{z}(\gamma)}$ denote the associated generalized Laplacian on $\mathfrak{z}(\gamma)$. We can extend $\Delta^{\mathfrak{z}(\gamma)}$ to an operator acting via constant holomorphic vector fields on $\mathfrak{z}(\gamma)_{\mathbf{C}}$.

Put

$$\mathfrak{z}_i(\gamma) = \mathfrak{p}(\gamma) \oplus i\mathfrak{k}(\gamma). \tag{2.79}$$

Then $B|_{\mathfrak{z}_i(\gamma)}$ is a scalar product on $\mathfrak{z}_i(\gamma)$. The generalized Laplacian $\Delta^{\mathfrak{z}(\gamma)}$ restricts on $\mathfrak{z}_i(\gamma)$ to the standard Euclidean Laplacian of $\mathfrak{z}_i(\gamma)$.

We take $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ as in Subsection 2.5. If $f \in \mathfrak{z}_i(\gamma)$, let

$$\mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + A}\right)(f)$$

be the smooth convolution kernel for $\mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + A}\right)$ on $\mathfrak{z}_i(\gamma)$ with respect to the volume form associated with the scalar product of $\mathfrak{z}_i(\gamma)$. Using (2.53) and finite propagation speed for the wave equation, there exist $C > 0, c > 0$ such that if $f \in \mathfrak{z}_i(\gamma)$, then

$$\left|\mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + A}\right)(f)\right| \leq Ce^{-c|f|^2}. \tag{2.80}$$

Let δ_a be the Dirac mass at $a \in \mathfrak{p}(\gamma)$. Then

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a$$

is a distribution on $\mathfrak{z}_i(\gamma)$, to which the smooth convolution kernel

$$\mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + A}\right)$$

can be applied. By definition,

$$\begin{aligned}\mu&\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + A}\right) \\ &\qquad\left[\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0) \\ &= \int_{i\mathfrak{k}(\gamma)} \mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + A}\right)\left(-Y_0^{\mathfrak{k}}, -a\right) \\ &\qquad\qquad \mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right] dY_0^{\mathfrak{k}}.\end{aligned} \tag{2.81}$$

In the right hand-side of (2.81), $\left(-Y_0^{\mathfrak{k}}, -a\right)$ can as well be replaced by $\left(Y_0^{\mathfrak{k}}, a\right)$.

In [B11, Theorem 6.2.2], the following extension of Theorem 2.8 was established.

Theorem 2.9. *The following identity holds:*

$$\mathrm{Tr}^{[\gamma]}\left[\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]=\mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)}+B^{*}\left(\rho^{\mathfrak{g}},\rho^{\mathfrak{g}}\right)+A}\right)\left[\mathcal{J}_{\gamma}\left(Y_{0}^{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_{0}^{\mathfrak{k}}}\right)\right]\delta_{a}\right](0)\,. \tag{2.82}$$

3. Cartan subalgebras, Cartan subgroups, and root systems

The purpose of this Section is to recall basic facts on Cartan subalgebras, on Cartan subgroups, and on root systems.

This section is organized as follows. In Subsection 3.1, we state some elementary facts of linear algebra.

In Subsection 3.2, we recall the definition of Cartan subalgebras.

In Subsection 3.3, we introduce the corresponding root system, and the associated algebraic Weyl group.

In Subsection 3.4, we define the real and the imaginary roots.

In Subsection 3.5, we construct a positive root system.

In Subsection 3.6, when the Cartan subalgebra is fundamental, we compare the root system of $\mathfrak{k}$ with the root system of $\mathfrak{g}$.

In Subsection 3.7, we introduce the Cartan subgroups, and of the corresponding regular elements.

In Subsection 3.8, we relate semisimple elements in G to Cartan subgroups.

In Subsection 3.9, we describe the characters of Cartan subgroups associated with a root system.

In Subsection 3.10, we give some properties of the real and imaginary roots with respect to semisimple elements in G.

Finally, in Subsection 3.11, we give a well-known formula that relates the action of invariant differential operators on the Lie algebra $\mathfrak{g}$ and on a Cartan subalgebra $\mathfrak{h}$.

We make the same assumptions as in Section 2, and we use the corresponding notation.

3.1. Linear algebra.

Let V be a finite dimensional real vector space. The symmetric algebras $S^{\cdot}(V), S^{\cdot}(V^{*})$ are the algebras of polynomials on V^{*}, V. If $v \in V$, v acts as a derivation of $S^{\cdot}(V^{*})$. More generally, $S^{\cdot}(V)$ acts on $S^{\cdot}(V^{*})$, and this action identifies $S^{\cdot}(V)$ with the algebra $D^{\cdot}(V)$ of real partial differential operators on V with constant coefficients. In particular, if $B^{*} \in S^{2}(V)$ is a bilinear symmetric form on V^{*}, the associated element in $D^{\cdot}(V)$ is the corresponding Laplacian Δ^{V}. If

B^* is positive, Δ^V is just a classical Laplacian. If B^* is negative, then Δ^V is the negative of a classical Laplacian on V.

Let $V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of V, a complex vector space. Its complex dual is given by $V^*_{\mathbf{C}} = V^* \otimes_{\mathbf{R}} \mathbf{C}$. The algebras $S^{\cdot}(V_{\mathbf{C}}), S^{\cdot}(V^*_{\mathbf{C}})$ are the algebras of complex polynomials on V^*, V. Note that

$$S^{\cdot}(V_{\mathbf{C}}) = S^{\cdot}(V) \otimes_{\mathbf{R}} \mathbf{C}, \qquad S^{\cdot}(V^*_{\mathbf{C}}) = S^{\cdot}(V^*) \otimes_{\mathbf{R}} \mathbf{C}. \tag{3.1}$$

Put

$$D^{\cdot}(V_{\mathbf{C}}) = D(V) \otimes_{\mathbf{R}} \mathbf{C}, \qquad D^{\cdot}(V^*_{\mathbf{C}}) = D^{\cdot}(V^*) \otimes_{\mathbf{R}} \mathbf{C}. \tag{3.2}$$

Then $D^{\cdot}(V_{\mathbf{C}}), D^{\cdot}(V^*_{\mathbf{C}})$ are the complexifications of $D^{\cdot}(V), D^{\cdot}(V^*)$, and also the spaces of complex holomorphic differential operators with constant coefficients on $V_{\mathbf{C}}, V^*_{\mathbf{C}}$.

In particular, if $B^* \in S^2(V)$, Δ^V is now viewed as a holomorphic operator on $V_{\mathbf{C}}$, that coincides with the corresponding Laplacian Δ^V on V, and with $-\Delta^V$ on $iV \simeq V$. [8]

Also $S^{\cdot}(V^*) \subset C^{\infty}(V, \mathbf{R})$, and the action of $D^{\cdot}(V)$ extends to $C^{\infty}(V, \mathbf{R})$.

Let $S[[V^*]]$ be the algebra of formal power series $s = \sum_{i=0}^{+\infty} s^i$, $s^i \in S^i(V^*)$. Then $S[[V^*]]$ can be identified with the algebra $D[[V^*]]$ of differential operators of infinite order with constant coefficients on V^*. In particular, $S[[V^*]]$ acts on $S^{\cdot}(V)$.

3.2. **The Cartan subalgebras of $\mathfrak{g}$.** By [W88, Section 0.2], a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be a Cartan subalgebra if $\mathfrak{h}$ is maximal among the abelian subalgebras of $\mathfrak{g}$ whose elements act as semisimple endomorphisms of $\mathfrak{g}$. Cartan subalgebras are known to exist and have the same dimension r, which is called the complex rank of G. By [Kn02, Proposition 6.64], there is a finite family of nonconjugate Cartan subalgebras in $\mathfrak{g}$. By [W88, Lemma 2.3.3], in every conjugacy class of Cartan subalgebras, there is a unique θ-stable Cartan algebra, up to conjugation by K. Therefore there is a finite family of nonconjugate θ-stable Cartan subalgebras, up to conjugation by K. By [Kn02, Theorem 2.15], the Cartan subalgebras of $\mathfrak{g}_{\mathbf{C}}$ are unique up to automorphisms induced by the adjoint group $\mathrm{Ad}(\mathfrak{g}_{\mathbf{C}})$.

[8] On $\mathbf{C} \simeq \mathbf{R}^2$, when acting on holomorphic functions, the differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}$ coincide. In this sense, the differential operator $\frac{\partial}{\partial x}$ on $\mathbf{R}$ extends to the differential operator $\frac{\partial}{\partial z}$ on $\mathbf{C}$, and restricts to the operator $-i\frac{\partial}{\partial y}$ on the imaginary line $i\mathbf{R}$. The operator $\frac{\partial^2}{\partial x^2}$ on $\mathbf{R}$ restricts to the operator $-\frac{\partial^2}{\partial y^2}$ on $i\mathbf{R}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra. To the Cartan splitting of $\mathfrak{g}$ in (2.1) corresponds the splitting

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{p}} \oplus \mathfrak{h}_{\mathfrak{k}}. \tag{3.3}$$

In particular the restriction $B|_{\mathfrak{h}}$ of B to $\mathfrak{h}$ is nondegenerate. This is also the case if $\mathfrak{h}$ is any Cartan subalgebra.

Up to conjugation by K, there is a unique θ-stable Cartan subalgebra $\mathfrak{h}$, which is called fundamental, such that $\mathfrak{h}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}$. Let us give more details on its construction [Kn86, pp. 129, 131]. Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{z}(\mathfrak{t}) \subset \mathfrak{g}$ be the centralizer of $\mathfrak{t}$, i.e.,

$$\mathfrak{z}(\mathfrak{t}) = \{f \in \mathfrak{g}, [\mathfrak{t}, f] = 0\}. \tag{3.4}$$

Then $\mathfrak{h} = \mathfrak{z}(\mathfrak{t})$ is a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{t}$.

An element $f \in \mathfrak{g}$ is said to be semisimple if $\mathrm{ad}(f) \in \mathrm{End}(\mathfrak{g})$ is semisimple. If $\mathfrak{h}$ is a Cartan subalgebra, elements of $\mathfrak{h}$ are semisimple. Any semisimple element of $\mathfrak{g}$ lies in a Cartan subalgebra.

If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, let $\mathfrak{h}^{\perp}$ be the orthogonal to $\mathfrak{h}$ in $\mathfrak{g}$. We have the B-orthogonal splitting,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}, \tag{3.5}$$

and B is also nondegenerate on $\mathfrak{h}^{\perp}$. If $\mathfrak{h}$ is θ-stable, then $\mathfrak{h}^{\perp}$ is also θ-stable, and so it splits as

$$\mathfrak{h}^{\perp} = \mathfrak{h}_{\mathfrak{p}}^{\perp} \oplus \mathfrak{h}_{\mathfrak{k}}^{\perp}. \tag{3.6}$$

Let $\mathfrak{u} = i\mathfrak{p} \oplus \mathfrak{k}$ be the compact form of $\mathfrak{g}$. Then $\mathfrak{h}_{\mathfrak{u}} = i\mathfrak{h}_{\mathfrak{p}} \oplus \mathfrak{h}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{u}$. If $\mathfrak{h}$ is θ-stable, then $\mathfrak{h}_{\mathfrak{u}}$ is also θ-stable.

An element $f \in \mathfrak{g}$ is said to be regular if $\mathfrak{z}(f)$ is a Cartan subalgebra. Regular elements in $\mathfrak{g}$ are semisimple.

If $\mathfrak{h}$ is a Cartan subalgebra, if $f \in \mathfrak{h}$, $\mathrm{ad}(f)$ acts as an endomorphism of $\mathfrak{g}/\mathfrak{h}$. Then $f \in \mathfrak{h}$ is regular if and only if $\det \mathrm{ad}(f)|_{\mathfrak{g}/\mathfrak{h}} \neq 0$.

3.3. **A root system and the Weyl group.** Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra.

Let $R \subset \mathfrak{h}_{\mathbf{C}}^{*}$ be the root system associated with $\mathfrak{h}, \mathfrak{g}$ [Kn02, Section II.4]. If $\alpha \in R$, then $-\alpha \in R, \overline{\alpha} \in R$. If $\alpha \in R$, let $\mathfrak{g}_{\alpha} \subset \mathfrak{g}_{\mathbf{C}}$ be the weight space associated with α, which is of dimension 1. Then we have the splitting

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \bigoplus \oplus_{\alpha \in R} \mathfrak{g}_{\alpha}. \tag{3.7}$$

If $\alpha \in R$, then

$$\mathfrak{g}_{\overline{\alpha}} = \overline{\mathfrak{g}}_{\alpha}. \tag{3.8}$$

If $f \in \mathfrak{h}$, $\mathrm{ad}\,(f) \in \mathrm{End}\,(\mathfrak{g})$ is antisymmetric with respect to B, so that the $\mathfrak{g}_\alpha|_{\alpha\in R}$ are B-orthogonal to $\mathfrak{h}_{\mathbf{C}}$. If $\alpha, \beta \in R$, then $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are B-orthogonal except when $\beta = -\alpha$, and the pairing between $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ is nondegenerate, so that if $\alpha \in R$, the form B induces the identification

$$\mathfrak{g}_{-\alpha} \simeq \mathfrak{g}_\alpha^*. \tag{3.9}$$

Also

$$\mathfrak{h}_{\mathbf{C}}^{\perp} = \oplus_{\alpha\in R}\mathfrak{g}_\alpha. \tag{3.10}$$

If $\alpha \in R$, α takes real values on $\mathfrak{h}_{\mathfrak{p}}$, and imaginary values on $\mathfrak{h}_{\mathfrak{k}}$, i.e., $\alpha \in \mathfrak{h}_{\mathfrak{p}}^* \oplus i\mathfrak{h}_{\mathfrak{k}}^*$. Also θ preserves the splitting (3.5) of $\mathfrak{g}$, and it maps R into itself. More precisely, if $\alpha \in R$,

$$\theta\alpha = -\overline{\alpha}, \qquad \mathfrak{g}_{\theta\alpha} = \theta\mathfrak{g}_\alpha. \tag{3.11}$$

Let $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}}) \subset \mathrm{Aut}\,(\mathfrak{h}_{\mathbf{C}})$ be the algebraic Weyl group [Kn86, p. 131]. Then $R \subset i\mathfrak{h}_{\mathfrak{u}}^*$, and $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}}) \subset \mathrm{Aut}\,(\mathfrak{h}_{\mathfrak{u}})$, i.e., $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ preserves the real vector space $\mathfrak{h}_{\mathfrak{u}}$. Also θ acts as an automorphism of the Lie algebras $\mathfrak{g}, \mathfrak{u}$, and $W\,(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ is preserved by conjugation by θ.

In general, $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ does not preserve the real vector space $\mathfrak{h}$. Recall that if $h \in \mathfrak{h}_{\mathbf{C}}$, we can define its complex conjugate $\overline{h} \in \mathfrak{h}_{\mathbf{C}}$. If $u \in \mathrm{End}\,(\mathfrak{h}_{\mathbf{C}})$, its complex conjugate $\overline{u} \in \mathrm{End}\,(\mathfrak{h}_{\mathbf{C}})$ is such that if $h \in \mathfrak{h}_{\mathbf{C}}$, then

$$\overline{u}\,(h) = \overline{u\,(\overline{h})}. \tag{3.12}$$

Proposition 3.1. *If $w \in W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$, then*

$$\overline{w} = \theta w \theta^{-1}. \tag{3.13}$$

In particular, the group $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ is preserved by complex conjugation.

Proof. The group $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ is generated by the symmetries $s_\alpha, \alpha \in R$ with respect to the vanishing locus of the $\alpha \in R$. By (3.11), we deduce that if $\alpha \in R$,

$$\overline{s_\alpha} = \theta s_\alpha \theta^{-1}, \tag{3.14}$$

from which get (3.13). Since $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ is stable by conjugation by θ, the group $W\,(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ is preserved by complex conjugation.

Another proof is as follows. Observe that there is a canonical identification of complex vector spaces $\varphi : \mathfrak{h}_{\mathbf{C}} \simeq \mathfrak{h}_{\mathfrak{u},\mathbf{C}}$, but the complex conjugations on $\mathfrak{h}_{\mathbf{C}}$ and on $\mathfrak{h}_{\mathfrak{u},\mathbf{C}}$ are not the same. More precisely, if $h \in \mathfrak{h}_{\mathbf{C}}$,

$$\overline{\varphi h} = \varphi\theta\overline{h} = \theta\varphi\overline{h}. \tag{3.15}$$

If $w \in W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$, then

$$w|_{\mathfrak{h}_{\mathbf{C}}} = \varphi^{-1} w|_{\mathfrak{h}_{\mathfrak{u},\mathbf{C}}} \varphi. \tag{3.16}$$

Recall that w is a real automorphism of the real vector space $\mathfrak{h}_{\mathfrak{u}}$. By (3.15), (3.16), we get

$$\overline{w}|_{\mathfrak{h}_{\mathbf{C}}} = \overline{\varphi}^{-1} w|_{\mathfrak{h}_{\mathfrak{u},\mathbf{C}}} \overline{\varphi}. \tag{3.17}$$

By (3.15)–(3.17), we get (3.13). The proof of our proposition is completed. □

3.4. **Real roots and imaginary roots.** Let $R^{\mathrm{re}} \subset R$ be the roots $\alpha \in R$ such that $\theta\alpha = -\alpha$, let R^{im} be the roots $\alpha \in R$ such that $\theta\alpha = \alpha$. These are respectively the real roots and the imaginary roots. Imaginary roots vanish on $\mathfrak{h}_{\mathfrak{p}}$, real roots vanish on $\mathfrak{h}_{\mathfrak{k}}$. By [Kn86, p. 349], the set of complex roots $R^{\mathrm{c}} \subset R$ is defined to be

$$R^{\mathrm{c}} = R \setminus \left(R^{\mathrm{re}} \cup R^{\mathrm{im}}\right). \tag{3.18}$$

Proposition 3.2. *If $\alpha \in R$, the map $f \in \mathfrak{g}_{\mathbf{C}} \to \overline{f} \in \mathfrak{g}_{\mathbf{C}}$ induces an antilinear isomorphism from $\mathfrak{g}_{\alpha}$ into $\mathfrak{g}_{-\theta\alpha}$, and the map $f \in \mathfrak{g}_{\mathbf{C}} \to \theta\overline{f} \in \mathfrak{g}_{\mathbf{C}}$ induces an antilinear isomorphism from $\mathfrak{g}_{\alpha}$ into $\mathfrak{g}_{-\alpha}$. If $\alpha \in R^{\mathrm{re}}$, $\mathfrak{g}_{\alpha}$ is the complexification of a real vector subspace of $\mathfrak{h}^{\perp}$.*

Proof. If $b \in \mathfrak{h}$, $f \in \mathfrak{g}_{\alpha}$, then

$$[b, f] = \langle \alpha, b \rangle f. \tag{3.19}$$

By taking the conjugate of (3.19), we obtain

$$\left[b, \overline{f}\right] = \langle -\theta\alpha, b \rangle \overline{f}. \tag{3.20}$$

By (3.20), we get the first part of our proposition. By composing this isomorphism with θ, we obtain the second part. If $\alpha \in R^{\mathrm{re}}$, then $-\theta\alpha = \alpha$, so that $\mathfrak{g}_{\alpha}$ is real. The proof of our proposition is completed. □

Definition 3.3. Put

$$\mathfrak{i} = \ker \mathrm{ad}\,(\mathfrak{h}_{\mathfrak{p}}) \cap \mathfrak{h}^{\perp}, \qquad \mathfrak{r} = \ker \mathrm{ad}\,(\mathfrak{h}_{\mathfrak{k}}) \cap \mathfrak{h}^{\perp}. \tag{3.21}$$

Then θ acts on $\mathfrak{i}, \mathfrak{r}$, so that we have the splittings,

$$\mathfrak{i} = \mathfrak{i}_{\mathfrak{p}} \oplus \mathfrak{i}_{\mathfrak{k}}, \qquad \mathfrak{r} = \mathfrak{r}_{\mathfrak{p}} \oplus \mathfrak{r}_{\mathfrak{k}}. \tag{3.22}$$

Proposition 3.4. *The vector spaces $\mathfrak{i}, \mathfrak{r}$ are orthogonal in $\mathfrak{h}^{\perp}$. Moreover,*

$$\mathfrak{i}_{\mathbf{C}} = \oplus_{\alpha \in R^{\mathrm{im}}} \mathfrak{g}_{\alpha}, \qquad \mathfrak{r}_{\mathbf{C}} = \oplus_{\alpha \in R^{\mathrm{re}}} \mathfrak{g}_{\alpha}. \tag{3.23}$$

If $\alpha \in R^{\mathrm{im}}$, then either $\mathfrak{g}_{\alpha} \subset \mathfrak{p}_{\mathbf{C}}$, or $\mathfrak{g}_{\alpha} \subset \mathfrak{k}_{\mathbf{C}}$.

Proof. If $f \in \mathfrak{h}^{\perp}$, $f \in \mathfrak{i} \cap \mathfrak{r}$, then f commutes with $\mathfrak{h}$. Since $\mathfrak{h}$ is a Cartan subalgebra, $f = 0$, so that $\mathfrak{i} \cap \mathfrak{r} = 0$. If $\alpha \in R \setminus R^{\mathrm{im}}$, α does not vanish identically on $\mathfrak{h}_{\mathfrak{p}}$, and its vanishing locus in $\mathfrak{h}_{\mathfrak{p}}$ is a hyperplane. So one can find $b_{\mathfrak{p}} \in \mathfrak{h}_{\mathfrak{p}} \setminus 0$ such that for any $\alpha \in R \setminus R^{\mathrm{im}}$, $\langle \alpha, b_{\mathfrak{p}} \rangle \neq 0$. Then

$$\mathfrak{i} = \ker \mathrm{ad}\,(b_{\mathfrak{p}}) \cap \mathfrak{h}^{\perp}. \tag{3.24}$$

Since $\mathfrak{i} \cap \mathfrak{r} = 0$, $\mathrm{ad}\,(b_{\mathfrak{p}})$ acts as an invertible morphism of $\mathfrak{r}$. Therefore any element of $\mathfrak{r}$ lies in the image of $\mathrm{ad}\,(b_{\mathfrak{p}})$. Since $\mathrm{ad}\,(b_{\mathfrak{p}})$ is symmetric in the classical sense, $\mathfrak{i}$ and $\mathfrak{r}$ are orthogonal. Equation (3.23) is elementary. If $\alpha \in R^{\mathrm{im}}$, the action of $\mathfrak{h}$ on $\mathfrak{g}_{\alpha}$ factors through $\mathfrak{h}_{\mathfrak{k}}$. Also $\mathrm{ad}\,(\mathfrak{h}_{\mathfrak{k}})$ preserves the splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Therefore if $\alpha \in R^{\mathrm{im}}$, either $\mathfrak{g}_{\alpha} \subset \mathfrak{p}_{\mathbf{C}}$, or $\mathfrak{g}_{\alpha} \subset \mathfrak{k}_{\mathbf{C}}$. The proof of our proposition is completed. □

Definition 3.5. Put

$$R^{\mathrm{im}}_{\mathfrak{p}} = \left\{\alpha \in R^{\mathrm{im}}, \mathfrak{g}_{\alpha} \subset \mathfrak{p}_{\mathbf{C}}\right\}, \quad R^{\mathrm{im}}_{\mathfrak{k}} = \left\{\alpha \in R^{\mathrm{im}}, \mathfrak{g}_{\alpha} \subset \mathfrak{k}_{\mathbf{C}}\right\}. \tag{3.25}$$

Then

$$R^{\mathrm{im}} = R^{\mathrm{im}}_{\mathfrak{p}} \cup R^{\mathrm{im}}_{\mathfrak{k}}. \tag{3.26}$$

Let $\mathfrak{c}$ denote the orthogonal to $\mathfrak{i} \oplus \mathfrak{r}$ in $\mathfrak{h}^{\perp}$. Again $\mathfrak{c}$ splits as

$$\mathfrak{c} = \mathfrak{c}_{\mathfrak{p}} \oplus \mathfrak{c}_{\mathfrak{k}}. \tag{3.27}$$

Moreover, we have the orthogonal splitting

$$\mathfrak{h}^{\perp} = \mathfrak{i} \oplus \mathfrak{r} \oplus \mathfrak{c}. \tag{3.28}$$

Proposition 3.6. *The following identity holds:*

$$\mathfrak{c}_{\mathbf{C}} = \oplus_{\alpha \in R^{\mathfrak{c}}} \mathfrak{g}_{\alpha}. \tag{3.29}$$

Proof. This follows from equations (3.10), (3.23), and (3.28). □

Now we give a result taken from [W88, Lemma 2.3.5].

Proposition 3.7. *A θ-stable Cartan subalgebra $\mathfrak{h}$ is fundamental if and only if there are no real roots.*

Proof. If $\alpha \in R$, then $\alpha \in R^{\mathrm{re}}$ if and only if when $f \in \mathfrak{g}_{\alpha}$,

$$[\mathfrak{h}_{\mathfrak{k}}, f] = 0. \tag{3.30}$$

If $\mathfrak{h}$ is fundamental, by (3.4), then $f \in \mathfrak{h}_{\mathbf{C}}$, so that $f = 0$, which proves there are no real roots. Conversely, if there is $\alpha \in R^{\mathrm{re}}$, then $\mathfrak{g}_{\alpha} \subset \mathfrak{z}\,(\mathfrak{h}_{\mathfrak{k}})_{\mathbf{C}}$, so that $\mathfrak{z}\,(\mathfrak{h}_{\mathfrak{k}})$ is not equal to $\mathfrak{h}$, and $\mathfrak{h}$ is not fundamental. The proof of our proposition is completed. □

Proposition 3.8. *The vector spaces* $\mathfrak{i}_\mathfrak{p}, \mathfrak{i}_\mathfrak{k}, \mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{k}$ *have even dimension. Also* $H \cap K$ *preserves these vector spaces, and the corresponding determinants are equal to* 1. *Also* $\mathfrak{r}_\mathfrak{p}, \mathfrak{r}_\mathfrak{k}$ *(resp.* $\mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{k}$*) have the same dimension, and the actions of* $H \cap K$ *on these two vector spaces are equivalent. In particular, we have the identity*

$$\dim \mathfrak{p} - \dim \mathfrak{h}_\mathfrak{p} = \dim \mathfrak{i}_\mathfrak{p} + \frac{1}{2} \dim \mathfrak{r} + \frac{1}{2} \dim \mathfrak{c}. \tag{3.31}$$

Proof. If $\alpha \in R \setminus R^{\mathrm{re}}$, the vanishing locus of α in $\mathfrak{h}_\mathfrak{k}$ is a hyperplane. Therefore we can find $f_\mathfrak{k} \in \mathfrak{h}_\mathfrak{k} \setminus 0$ such for any $\alpha \in R \setminus R^{\mathrm{re}}$, $\langle \alpha, f_\mathfrak{k} \rangle \neq 0$, which just says that $\mathrm{ad}\,(f_\mathfrak{k})$ acts as an invertible endomorphism of $\mathfrak{i}, \mathfrak{c}$. This endomorphism preserves their $\mathfrak{p}$ and $\mathfrak{k}$ components, and it is classically antisymmetric. This is only possible if these vector spaces are even dimensional. If $k \in H \cap K$, $\mathrm{Ad}\,(k^{-1})$ preserves these vector spaces and commutes with $\mathrm{ad}\,(f_\mathfrak{k})$. Therefore the eigenspaces associated with the eigenvalue -1 are preserved by $\mathrm{ad}\,(f_\mathfrak{k})$, so they are even dimensional. This forces the determinant of $\mathrm{Ad}\,(k^{-1})$ to be equal to 1 on each of these vector spaces.

We choose $b_\mathfrak{p} \in \mathfrak{h}_\mathfrak{p} \setminus 0$ such that for any $\alpha \in R \setminus R^{\mathrm{im}}$, $\langle \alpha, b_\mathfrak{p} \rangle \neq 0$. Therefore $\mathrm{ad}\,(b_\mathfrak{p})$ acts as an automorphism of $\mathfrak{r}, \mathfrak{c}$ that exchanges the corresponding $\mathfrak{p}$ and $\mathfrak{k}$ parts, and commutes with $\mathrm{Ad}\,(k^{-1})$.

By (3.28), we get

$$\mathfrak{h}_\mathfrak{p}^\perp = \mathfrak{i}_\mathfrak{p} \oplus \mathfrak{r}_\mathfrak{p} \oplus \mathfrak{c}_\mathfrak{p}. \tag{3.32}$$

Using the results we already established and (3.32), we get (3.31). The proof of our proposition is completed. □

3.5. **A positive root system.** Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra, and let R denote the corresponding root system. Let $R_+ \subset R$ be a positive root system. Set

$$R_+^{\mathrm{re}} = R_+ \cap R^{\mathrm{re}}, \qquad R_+^{\mathrm{im}} = R_+ \cap R^{\mathrm{im}}, \qquad R_+^{\mathrm{c}} = R_+ \cap R^{\mathrm{c}}, \tag{3.33}$$

so that

$$R_+ = R_+^{\mathrm{re}} \cup R_+^{\mathrm{im}} \cup R_+^{\mathrm{c}}. \tag{3.34}$$

In the whole paper, we choose R_+ such that $-\theta$ preserves $R_+ \setminus R_+^{\mathrm{im}}$. Equivalently, we assume that if $\alpha \in R_+ \setminus R_+^{\mathrm{im}}$, then $\overline{\alpha} \in R_+$.

Let us explain how to do this. If $\alpha \in R \setminus R^{\mathrm{im}}$, the vanishing locus of α in $\mathfrak{h}_\mathfrak{p}$ is a hyperplane, and so there is $b_\mathfrak{p} \in \mathfrak{h}_\mathfrak{p}, |b_\mathfrak{p}| = 1$ such that for any $\alpha \in R \setminus R^{\mathrm{im}}$, $\langle \alpha, b_\mathfrak{p} \rangle \neq 0$. The same argument shows that there is $b_\mathfrak{k} \in h_\mathfrak{k}, |b_\mathfrak{k}| = 1$ such that for $\alpha \in R^{\mathrm{im}}$, $\langle \alpha, b_\mathfrak{k} \rangle \neq 0$. For $\epsilon > 0$, $b_\pm = \pm b_\mathfrak{p} + i\epsilon b_\mathfrak{k} \in \mathfrak{h}_\mathfrak{p} \oplus i\mathfrak{h}_\mathfrak{k}$, and θ interchanges b_+ and b_-. Also for $\epsilon > 0$

small enough, for $\alpha \in R$, the real numbers $\langle \alpha, b_{\pm} \rangle$ do not vanish, and if $\alpha \in R \setminus R^{\mathrm{im}}$, they have opposite signs. Put

$$R_+ = \{\alpha \in R, \langle \alpha, b_+ \rangle > 0\}. \tag{3.35}$$

Then R_+ is a positive root system such that $-\theta$ preserves $R_+ \setminus R_+^{\mathrm{im}}$.

Note that $-\theta$ acts without fixed points on R_+^{c}, so that $\left|R_+^{\mathrm{c}}\right|$ is even.

Definition 3.9. Put

$$\mathfrak{c}_{+,\mathbf{C}} = \oplus_{\alpha \in R_+^{\mathrm{c}}} \mathfrak{g}_\alpha, \qquad \mathfrak{c}_{-,\mathbf{C}} = \oplus_{\alpha \in -R_+^{\mathrm{c}}} \mathfrak{g}_\alpha. \tag{3.36}$$

Proposition 3.10. *The vector spaces $\mathfrak{c}_{+,\mathbf{C}}, \mathfrak{c}_{-,\mathbf{C}}$ are the complexifications of real Lie subalgebras $\mathfrak{c}_+, \mathfrak{c}_-$ of $\mathfrak{g}$, which have the same even dimension, and are such that*

$$\mathfrak{c} = \mathfrak{c}_+ \oplus \mathfrak{c}_-, \qquad \mathfrak{c}_- = \theta \mathfrak{c}_+. \tag{3.37}$$

Also B vanishes on $\mathfrak{c}_+, \mathfrak{c}_-$ and induces the identification,

$$\mathfrak{c}_- \simeq \mathfrak{c}_+^*. \tag{3.38}$$

The projections on $\mathfrak{p}, \mathfrak{k}$ map $\mathfrak{c}_\pm$ into $\mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{k}$ isomorphically. Finally, the actions of $H \cap K$ on $\mathfrak{c}_+, \mathfrak{c}_-, \mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{k}$ are equivalent.

Proof. By Proposition 3.2, $\mathfrak{c}_{+,\mathbf{C}}, \mathfrak{c}_{-,\mathbf{C}}$ are stable by conjugation, and so they are complexifications of real vector spaces $\mathfrak{c}_+, \mathfrak{c}_-$. The fact that these are Lie subalgebras is obvious. Since $\left|R_+^{\mathrm{c}}\right|$ is even, these vector spaces are even dimensional, and also they have the same dimension. By Proposition 3.2, θ induces an isomorphism of $\mathfrak{c}_+$ into $\mathfrak{c}_-$. Using the considerations that follow (3.9), we find that B vanishes on $\mathfrak{c}_+, \mathfrak{c}_-$, and we obtain (3.38). By Proposition 3.8, $\mathfrak{c}_+, \mathfrak{c}_-, \mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{k}$ have the same even dimension. The projections on $\mathfrak{p}, \mathfrak{k}$ are given respectively by $\frac{1}{2}(1 \mp \theta)$. Since θ exchanges $\mathfrak{c}_+$ and $\mathfrak{c}_-$, they restrict to isomorphisms on $\mathfrak{c}_+, \mathfrak{c}_-$. By Proposition 3.8, we know that the actions $H \cap K$ on $\mathfrak{c}_\mathfrak{p}, \mathfrak{c}_\mathfrak{k}$ are equivalent. Since the adjoint action of $H \cap K$ commutes with θ, the corresponding representations of $H \cap K$ on these vector spaces are equivalent. The proof of our proposition is completed. □

If $f \in \mathfrak{h}$, then

$$\det \mathrm{ad}\,(f)_{\mathfrak{h}^\perp} = \prod_{\alpha \in R} \langle \alpha, f \rangle. \tag{3.39}$$

Definition 3.11. Let $\pi^{\mathfrak{h},\mathfrak{g}} \in S^{\cdot}(\mathfrak{h}_{\mathbf{C}}^*)$ be such that if $h \in \mathfrak{h}_{\mathbf{C}}$, then

$$\pi^{\mathfrak{h},\mathfrak{g}}(h) = \prod_{\alpha \in R_+} \langle \alpha, h \rangle. \tag{3.40}$$

By (3.39), if $f \in \mathfrak{h}$,

$$\det \mathrm{ad}\,(f)_{|_{\mathfrak{h}^\perp}} = \pi^{\mathfrak{h},\mathfrak{g}}(f)\,\pi^{\mathfrak{h},\mathfrak{g}}(-f)\,. \tag{3.41}$$

Also $f \in \mathfrak{h}$ is regular if and only if $\pi^{\mathfrak{h},\mathfrak{g}}(f) \neq 0$.

Proposition 3.12. *The function $\pi^{\mathfrak{h},\mathfrak{g}}$ vanishes identically on $\mathfrak{h}_{\mathfrak{k}}$ if and only if $\mathfrak{h}$ is not fundamental.*

Proof. Assume that $\mathfrak{h}$ is not fundamental. By Proposition 3.7, there are real roots, and so there are real positive roots. If $\alpha \in R^{\mathrm{re}}_+$, then α vanishes on $\mathfrak{h}_{\mathfrak{k}}$, and so $\pi^{\mathfrak{h},\mathfrak{g}}$ vanishes on $\mathfrak{h}_{\mathfrak{k}}$. Conversely, if $\pi^{\mathfrak{h},\mathfrak{g}}$ vanishes on $\mathfrak{h}_{\mathfrak{k}}$, one of the $\alpha \in R_+$ has to vanish identically on $\mathfrak{h}_{\mathfrak{k}}$, so that $\alpha \in R^{\mathrm{re}}_+$, and $\mathfrak{h}$ is not fundamental. The proof of our proposition is completed. □

3.6. **The case when $\mathfrak{h}$ is fundamental and the root system of $(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k})$.** In this Subsection, we assume that $\mathfrak{h}$ is a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}$. By (3.18), we get

$$R^{\mathrm{c}} = R \setminus R^{\mathrm{im}}. \tag{3.42}$$

By Proposition 3.4, $\mathfrak{r} = 0$. By (3.28), we have the orthogonal splitting,

$$\mathfrak{h}_{\mathfrak{p}}^{\perp} = \mathfrak{i}_{\mathfrak{p}} \oplus \mathfrak{c}_{\mathfrak{p}}, \qquad \mathfrak{h}_{\mathfrak{k}}^{\perp} = \mathfrak{i}_{\mathfrak{k}} \oplus \mathfrak{c}_{\mathfrak{k}}. \tag{3.43}$$

Also $\mathfrak{i}_{\mathfrak{p}}, \mathfrak{i}_{\mathfrak{k}}$ have even dimension, $\mathfrak{c}_{\mathfrak{p}}, \mathfrak{c}_{\mathfrak{k}}$ have the same even dimension, and the action of $H \cap K$ on these last two vector spaces are conjugate.

The roots in R do not vanish identically on $\mathfrak{h}_{\mathfrak{k}}$. We will now reinforce the choice of positive roots made in Subsection 3.5. We may and we will assume that $b_{\mathfrak{k}} \in \mathfrak{h}_{\mathfrak{k}}, |b_{\mathfrak{k}}| = 1$ has been chosen so that if $\alpha \in R$, $\langle \alpha, b_{\mathfrak{k}} \rangle \neq 0$.

As we saw in Subsection 3.5, $-\theta$ acts without fixed points on R^{c}_+. Also if $\alpha \in R^{\mathrm{c}}_+$, $-\theta\alpha|_{\mathfrak{h}_{\mathfrak{k}}} = -\alpha|_{\mathfrak{h}_{\mathfrak{k}}}$, so that the nonzero real numbers $\langle \alpha, ib_{\mathfrak{k}} \rangle$ and $\langle -\theta\alpha, ib_{\mathfrak{k}} \rangle$ have opposite signs.

Set

$$R^{\mathrm{im}}_{\mathfrak{k},+} = R^{\mathrm{im}}_{\mathfrak{k}} \cap R_+, \qquad R^{\mathrm{c}}_{++} = \left\{ \alpha \in R^{\mathrm{c}}_+, \langle \alpha, ib_{\mathfrak{k}} \rangle > 0 \right\}. \tag{3.44}$$

Definition 3.13. Let $R(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k})$ be the root system associated with the pair $(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k})$. If $R_+(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k})$ is a positive root system for $(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k})$, if $h_{\mathfrak{k}} \in \mathfrak{h}_{\mathfrak{k},\mathbf{C}}$, put

$$\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}}(h_{\mathfrak{k}}) = \prod_{\beta \in R_+(\mathfrak{h}_{\mathfrak{k}},\mathfrak{k})} \langle \beta, h_{\mathfrak{k}} \rangle\,. \tag{3.45}$$

Then $\left[\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}}\right]^2(h_{\mathfrak{k}})$ does not depend on the choice of $R_+(\mathfrak{h}_{\mathfrak{k}},\mathfrak{k})$. The arguments above (3.44) show that if $h\in\mathfrak{h}_{\mathfrak{k}}$, then

$$\prod_{\alpha\in R^{\mathrm{c}}_+}\langle\alpha,h_{\mathfrak{k}}\rangle\geq 0. \tag{3.46}$$

Proposition 3.14. *The map $\alpha\in R^{\mathrm{im}}_{\mathfrak{k}}\cup R^{\mathrm{c}}_+\to\alpha|_{\mathfrak{h}_{\mathfrak{k}}}$ is injective, and gives the identification*

$$R(\mathfrak{h}_{\mathfrak{k}},\mathfrak{k})=R^{\mathrm{im}}_{\mathfrak{k}}\cup R^{\mathrm{c}}_+. \tag{3.47}$$

A positive root system $R_+(\mathfrak{h}_{\mathfrak{k}},\mathfrak{k})$ for $(\mathfrak{h}_{\mathfrak{k}},\mathfrak{k})$ is given by

$$R_+(\mathfrak{h}_{\mathfrak{k}},\mathfrak{k})=R^{\mathrm{im}}_{\mathfrak{k},+}\cup R^{\mathrm{c}}_{++}. \tag{3.48}$$

If $h_{\mathfrak{k}}\in\mathfrak{h}_{\mathfrak{k},\mathbf{C}}$, then

$$\left[\pi^{h_{\mathfrak{k}},\mathfrak{k}}(h_{\mathfrak{k}})\right]^2=(-1)^{\frac{1}{2}|R^{\mathrm{c}}_+|}\left[\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}}\langle\alpha,h_{\mathfrak{k}}\rangle\right]^2\prod_{\alpha\in R^{\mathrm{c}}_+}\langle\alpha,h_{\mathfrak{k}}\rangle\,. \tag{3.49}$$

Proof. By (3.28), we get

$$\mathfrak{h}_{\mathfrak{k}}^{\perp}=\mathfrak{i}_{\mathfrak{k}}\oplus\mathfrak{c}_{\mathfrak{k}}, \tag{3.50}$$

and the above splitting is preserved by $\mathfrak{h}_{\mathfrak{k}}$. The weights for this action on $\mathfrak{i}_{\mathfrak{k}}$ are given by $R^{\mathrm{im}}_{\mathfrak{k}}$. By Proposition 3.10, $\mathfrak{c}_{\mathfrak{k}}$ and $\mathfrak{c}_+$ are equivalent under the action of $\mathfrak{h}_{\mathfrak{k}}$. By the first equation in (3.36), the weights for the action of $\mathfrak{h}_{\mathfrak{k}}$ on $\mathfrak{c}_+$ are given by the restriction of R^{c}_+ to $\mathfrak{h}_{\mathfrak{k}}$. Since the weights for the action of $\mathfrak{h}_{\mathfrak{k}}$ on $\mathfrak{c}_{\mathfrak{k}}$ are nonzero and of multiplicity 1, the map $\alpha\in R^{\mathrm{im}}_{\mathfrak{k}}\cup R^{\mathrm{c}}_+\to\alpha|_{\mathfrak{h}_{\mathfrak{k}}}$ gives the identification in (3.47). By (3.47), we get (3.48). Using (3.45) and the above results, we get (3.49). The proof of our proposition is completed. □

Remark 3.15. The results contained in Proposition 3.14 will play an important role in the proof of the limit results of Subsection 8.1.

3.7. **Cartan subgroups and regular elements.** If $\mathfrak{h}$ is a Cartan subalgebra, the associated Cartan subgroup $H\subset G$ is the stabilizer of $\mathfrak{h}$. Then H is a Lie subgroup of G, with Lie algebra $\mathfrak{h}$.

Assume that $\mathfrak{h}$ is θ-stable. Then θ restricts to an involution of H, and (3.3) is the corresponding Cartan splitting of $\mathfrak{h}$. Also B restricts to a H and θ invariant symmetric nondegenerate bilinear form $B|_{\mathfrak{h}}$ on $\mathfrak{h}$, so that H is a reductive subgroup of G.

We still assume $\mathfrak{h}$ to be θ-stable. Let $Z_G(H)\subset G$ be the centralizer of H, and let $N_G(H)\subset G$ be its normalizer. Then $Z_G(H)$ is included

in H, it is just the center $Z(H)$ of H. As in [Kn86, p. 131], the analytic Weyl group $W(H:G)$ is defined as the quotient

$$W(H:G) = N_G(H)/Z_G(H). \tag{3.51}$$

Put

$$Z_K(H) = Z_G(H) \cap K, \qquad N_K(H) = N_G(H) \cap K. \tag{3.52}$$

Then $N_K(H)/Z_K(H)$ embeds in $W(H:K)$. By [Kn86, p. 131], this embedding is an isomorphism, i.e.,

$$W(H:G) = N_K(H)/Z_K(H). \tag{3.53}$$

By [Kn86, eq. (5.6)], $W(H:G) \subset W(\mathfrak{h}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$.

By [Kn86, p. 130], an element $\gamma \in G$ is said to be regular if $\mathfrak{z}(\gamma)$ is a Cartan subalgebra. If H is the corresponding Cartan subgroup, then $\gamma \in H$. By [Kn86, Theorem 5.22], the set $G^{\mathrm{reg}} \subset G$ of regular elements is open and conjugation invariant. More precisely, if $H_1, \ldots, H_\ell$ denotes the finite family of nonconjugate Cartan subgroups, by [Kn86, Theorem 5.22], G^{reg} splits as the disjoint union of open sets

$$G^{\mathrm{reg}} = \cup_{i=1}^{\ell} G^{\mathrm{reg}}_{H_i}, \tag{3.54}$$

where $G^{\mathrm{reg}}_{H_i}$ denote the open set of elements of G^{reg} that are conjugate to an element of H_i.

If $\gamma \in H$, $\mathrm{Ad}(\gamma)$ acts on $\mathfrak{g}$ and fixes $\mathfrak{h}$. Since $\mathrm{Ad}(\gamma)$ preserves B, it also acts on $\mathfrak{h}^{\perp}$, so that $1 - \mathrm{Ad}(\gamma)$ acts on $\mathfrak{h}^{\perp}$. Then γ is regular if and only this endomorphism is invertible, i.e., $\det(1 - \mathrm{Ad}(\gamma))|_{\mathfrak{h}^{\perp}} \neq 0$.

3.8. **Cartan subgroups and semisimple elements.** The following result is established in [Va77, Part I, Section 2.3, Theorem 4].

Proposition 3.16. *A group element $\gamma \in G$ is semisimple if and only if it lies in a Cartan subgroup.*

Let us give a direct proof of part of our proposition. Le $\mathfrak{h}$ be a θ-stable Cartan subalgebra, and let H be the corresponding Cartan subgroup. If $\gamma \in H$, then $\mathfrak{h} \subset \mathfrak{z}(\gamma)$. Moreover, γ can be written uniquely in the form

$$\gamma = e^a k^{-1}, \qquad a \in \mathfrak{h}_{\mathfrak{p}}, \qquad k \in H \cap K. \tag{3.55}$$

Since $a \in \mathfrak{h}_{\mathfrak{p}}, k \in H$, then $\mathrm{Ad}(k)\, a = a$, which guarantees that γ is semisimple in G.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let $H \subset G$ be the associated Cartan subgroup. If $\gamma \in H$, then $\mathfrak{h} \subset \mathfrak{z}(\gamma)$, so that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{z}(\gamma)$. In particular G and $Z^0(\gamma)$ have the same complex rank.

Proposition 3.17. *Any θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{z}(\gamma)$ is also a Cartan subalgebra of $\mathfrak{g}$.*

Proof. As we saw in Subsection 2.2, $Z^0(\gamma)$ is a connected reductive group and θ induces on $Z^0(\gamma)$ a corresponding Cartan involution. Since $\mathfrak{h}_0$ is commutative and θ-stable, and since its action on $\mathfrak{g}$ preserves B, it acts on $\mathfrak{g}$ by semisimple endomorphisms of $\mathfrak{g}$. Since G and $Z^0(\gamma)$ have the same complex rank, $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$. The proof of our proposition is completed. □

3.9. **Root systems and their characters.** Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra. We use the notation of the previous subsections.

Take $\gamma \in H$. As we saw after Proposition 3.16, if $\gamma \in H$, we can write γ uniquely in the form

$$\gamma = e^a k^{-1}, \qquad a \in \mathfrak{h}_{\mathfrak{p}}, \qquad k \in H \cap K, \tag{3.56}$$

so that

$$\mathrm{Ad}(k)\, a = a. \tag{3.57}$$

Let $R(\gamma), R(a)$ be the root systems associated with $(\mathfrak{h}, \mathfrak{z}(\gamma)), (\mathfrak{h}, \mathfrak{z}(a))$. We will denote with extra subscripts the corresponding real, imaginary, and complex roots.

Theorem 3.18. *If $\gamma \in H$, for any $\alpha \in R$, $\mathrm{Ad}(\gamma)$ preserves the 1-dimensional complex line $\mathfrak{g}_\alpha$. For every $\alpha \in R$, there is a character $\xi_\alpha : H \to \mathbf{C}^*$ such that $\mathrm{Ad}(\gamma)$ acts on $\mathfrak{g}_\alpha$ by multiplication by $\xi_\alpha(\gamma)$. If $\alpha \in R$,*

$$\xi_\alpha \xi_{-\alpha} = 1. \tag{3.58}$$

If $\alpha \in R$, if $f \in \mathfrak{h}, k \in H \cap K$, then

$$\xi_\alpha\left(e^f\right) = e^{\langle \alpha, f\rangle}, \qquad |\xi_\alpha(k)| = 1. \tag{3.59}$$

In particular, if $\gamma \in H$ is taken as in (3.56), then

$$\xi_\alpha(\gamma) = e^{\langle \alpha, a\rangle} \xi_\alpha\left(k^{-1}\right), \qquad \xi_{-\theta\alpha}(\gamma) = \overline{\xi_\alpha(\gamma)}. \tag{3.60}$$

If $\alpha \in R^{\mathrm{re}}$, then $\xi_\alpha(\gamma) \in \mathbf{R}^*$, if $\alpha \in R^{\mathrm{im}}$, then $|\xi_\alpha(\gamma)| = 1$. If $\alpha \in R^{\mathrm{re}}$, the restriction of ξ_α to $H \cap K$ takes its values in $\{-1, +1\}$.

Also

$$\det(1 - \mathrm{Ad}(\gamma))|_{\mathfrak{h}^\perp} = \prod_{\alpha \in R} (1 - \xi_\alpha(\gamma)), \tag{3.61}$$

and γ is regular if and only if for any $\alpha \in R$, $\xi_\alpha(\gamma) \neq 1$.

The following identities hold:

$$R(\gamma)=\{\alpha\in R,\xi_\alpha(\gamma)=1\},\qquad R(a)=\{\alpha\in R,\langle\alpha,a\rangle=0\},$$

(3.62)

$$R^{\mathrm{re}}(\gamma)=R(\gamma)\cap R^{\mathrm{re}},\ R^{\mathrm{im}}(\gamma)=R(\gamma)\cap R^{\mathrm{im}},\ R^{\mathrm{c}}(\gamma)=R(\gamma)\cap R^{\mathrm{c}},$$

$$R^{\mathrm{re}}(a)=R(a)\cap R^{\mathrm{re}},\ R^{\mathrm{im}}(a)=R(a)\cap R^{\mathrm{im}},\ R^{\mathrm{c}}(a)=R(a)\cap R^{\mathrm{c}}.$$

Also $R_+(\gamma)=R(\gamma)\cap R_+, R_+(a)=R(a)\cap R_+$ *are positive root systems for* $(\mathfrak{h},\mathfrak{z}(\gamma)),(\mathfrak{h},\mathfrak{z}(a))$.

Proof. If $\gamma\in H$, then $\mathrm{Ad}(\gamma)$ fixes $\mathfrak{h}$, and so if $h\in\mathfrak{h}$, we have the commutation relation in $\mathrm{End}(\mathfrak{g})$,

$$[\mathrm{Ad}(\gamma),\mathrm{ad}(h)]=0. \tag{3.63}$$

By (3.7), (3.63), we deduce that for any $\alpha\in R$, $\mathrm{Ad}(\gamma)$ preserves $\mathfrak{g}_\alpha$. Since $\mathfrak{g}_\alpha$ is a complex line, H acts on $\mathfrak{g}_\alpha$ via a character ξ_α.

Since $\mathrm{Ad}(\gamma)$ preserves B, if $f,f'\in\mathfrak{g}_{\mathbf{C}}$, we get

$$B(\mathrm{Ad}(\gamma)f,f')=B\left(f,\mathrm{Ad}(\gamma)^{-1}f'\right). \tag{3.64}$$

Take $\alpha\in R$. By (3.64), if $f\in\mathfrak{g}_\alpha, f'\in\mathfrak{g}_{-\alpha}$, then

$$\xi_\alpha(\gamma)B(f,f')=\xi_{-\alpha}^{-1}(\gamma)B(f,f'). \tag{3.65}$$

As we saw in Subsection 3.3, if $\alpha\in R$, the pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ via B is nondegenerate. By (3.65), we get (3.58).

The first equation in (3.59) is trivial. Since ξ_α restricts to a character of the compact group $H\cap K$, we get the second equation in (3.59). The first equation in (3.60) follows from the previous considerations. Since $\theta(\gamma)=e^{-a}k^{-1}$, and since θ maps $\mathfrak{g}_\alpha$ into $\mathfrak{g}_{\theta\alpha}$, we obtain the second equation in (3.60). From this second equation, we deduce that if $\alpha\in R^{\mathrm{re}}$, then $\xi_\alpha(\gamma)$ is real, and if $\alpha\in R^{\mathrm{im}}$, then $|\xi_\alpha(\gamma)|=1$. If $\gamma\in H\cap K,\alpha\in R^{\mathrm{re}}$, we know that $\xi_\alpha(\gamma)\in\mathbf{R}^*,|\xi_\alpha(\gamma)|=1$, so that $\xi_\alpha(\gamma)=\pm1$.

Equations (3.61), (3.62) are trivial. By (3.61), γ is regular if and only if for $\alpha\in R$, $\xi_\alpha(\gamma)\neq1$.

Now we proceed as in [BL99, Theorem 1.38]. If $\kappa\subset\mathfrak{h}$ is a positive Weyl chamber for $(\mathfrak{h},\mathfrak{g})$, the forms in R do not vanish on κ, so that κ is included in a $\mathfrak{z}(\gamma)$ Weyl chamber. It follows that $R(\gamma)\cap R_+$ is a positive root system on $(\mathfrak{h},\mathfrak{z}(\gamma))$. The same argument is valid for $R(a)$. The proof of our theorem is completed. □

3.10. **Real roots, imaginary roots, and semisimple elements.** We still take γ as in Subsection 3.9. When taking the intersection of $\mathfrak{i},\mathfrak{r},\mathfrak{c}$ with $\mathfrak{z}(\gamma),\mathfrak{z}(a),\mathfrak{z}(k)$, this will be indicated with a parenthesis containing the corresponding argument. The intersection with

$\mathfrak{z}^{\perp}(\gamma), \mathfrak{z}^{\perp}(a), \mathfrak{z}^{\perp}(k)$ will be denoted with an extra $\perp$. These vector spaces also have a $\mathfrak{p}$ and a $\mathfrak{k}$ component.

By construction,

$$R^{\mathrm{im}}(\gamma) = R^{\mathrm{im}}(k). \tag{3.66}$$

As in (3.25), (3.26), we get

$$R^{\mathrm{im}}(\gamma) = R^{\mathrm{im}}_{\mathfrak{p}}(\gamma) \cup R^{\mathrm{im}}_{\mathfrak{k}}(\gamma), \quad R^{\mathrm{im}}(k) = R^{\mathrm{im}}_{\mathfrak{p}}(k) \cup R^{\mathrm{im}}_{\mathfrak{k}}(k). \tag{3.67}$$

To make the notation simpler, in (3.67), we did not use instead the notation $\mathfrak{p}(\gamma), \mathfrak{k}(\gamma), \mathfrak{p}(k), \mathfrak{k}(k)$.

Proposition 3.19. *The following identities holds:*

$$\mathfrak{i} \subset \mathfrak{z}(a), \qquad \mathfrak{i}(\gamma) = \mathfrak{i}(k), \qquad \mathfrak{i}^{\perp}(k) \subset \mathfrak{z}_a^{\perp}(\gamma). \tag{3.68}$$

Also

$$\mathfrak{i}(k)_{\mathbf{C}} = \oplus_{\alpha \in R^{\mathrm{im}}(k)} \mathfrak{g}_{\alpha}, \qquad \mathfrak{i}^{\perp}(k)_{\mathbf{C}} = \oplus_{\alpha \in R^{\mathrm{im}} \setminus R^{\mathrm{im}}(k)} \mathfrak{g}_{\alpha}. \tag{3.69}$$

Moreover,

$$\begin{aligned} &\mathfrak{z}(a)_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \bigoplus \oplus_{\alpha \in R(a)} \mathfrak{g}_{\alpha}, \\ &\mathfrak{z}^{\perp}(a) \subset \mathfrak{r} \oplus \mathfrak{c}, \\ &\mathfrak{z}^{\perp}(a)_{\mathbf{C}} = \oplus_{\alpha \in R \setminus R(a)} \mathfrak{g}_{\alpha}. \end{aligned} \tag{3.70}$$

If γ is regular, then

$$\mathfrak{i}(k) = 0. \tag{3.71}$$

Proof. By the first identity in (3.21), since $a \in \mathfrak{h}_{\mathfrak{p}}$, we get the first identity in (3.68). Combining the third identity in (2.20) with this first identity, we get the second identity in (3.68). The third identity in (3.68) is a consequence of the first two. By (3.7), we get (3.69) and the first and the third equations in (3.70), the second equation being a consequence of the first equation in (3.68). Also γ is regular if and only if $\mathfrak{z}(\gamma) = \mathfrak{h}$. Since $\mathfrak{h} \cap \mathfrak{i} = 0$, by the second identity in (3.68), we get (3.71). The proof of our proposition is completed. □

3.11. **Cartan subalgebras and differential operators.** Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra. There is a natural projection $\mathfrak{g}^* \to \mathfrak{h}^*$.

By (3.5), there is a well-defined projection $\mathfrak{g} \to \mathfrak{h}$. To the splitting (3.5) corresponds the dual splitting

$$\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{h}^{*\perp}. \tag{3.72}$$

The projections $S^{\cdot}(\mathfrak{g}^*) \to S^{\cdot}(\mathfrak{h}^*), S^{\cdot}(\mathfrak{g}) \to S^{\cdot}(\mathfrak{h})$ associated with (3.5), (3.72) are just the restriction r of polynomials on $\mathfrak{g}$ to $\mathfrak{h}$, or of polynomials on $\mathfrak{g}^*$ to $\mathfrak{h}^*$.

The Lie algebra $\mathfrak{g}$ acts as an algebra of derivations on $S^{\cdot}(\mathfrak{g}^*)$.

Definition 3.20. Let $I^{\cdot}(\mathfrak{g}^*) \subset S^{\cdot}(\mathfrak{g}^*)$ be the algebra of invariant elements in $S^{\cdot}(\mathfrak{g}^*)$, i.e., the algebra of the elements of $S^{\cdot}(\mathfrak{g}^*)$ on which the derivations associated with $\mathfrak{g}$ vanish. Let $I^{\cdot}(\mathfrak{h}^*_{\mathbf{C}}, \mathfrak{g}^*_{\mathbf{C}})$ be the algebra of $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$-invariant elements in $S^{\cdot}(\mathfrak{h}^*_{\mathbf{C}})$.

Recall that

$$S^{\cdot}(\mathfrak{h}^*_{\mathbf{C}}) = S^{\cdot}(\mathfrak{h}^*)_{\mathbf{C}}\,. \tag{3.73}$$

In particular $S^{\cdot}(\mathfrak{h}^*_{\mathbf{C}})$ is equipped with a natural conjugation.

Proposition 3.21. *The algebra $I^{\cdot}(\mathfrak{h}^*_{\mathbf{C}}, \mathfrak{g}^*_{\mathbf{C}})$ is preserved under complex conjugation. There is a real algebra $I^{\cdot}(\mathfrak{h}^*, \mathfrak{g}^*) \subset S^{\cdot}(\mathfrak{h}^*)$ such that*

$$I^{\cdot}(\mathfrak{h}^*_{\mathbf{C}}, \mathfrak{g}^*_{\mathbf{C}}) = I^{\cdot}(\mathfrak{h}^*, \mathfrak{g}^*)_{\mathbf{C}}\,. \tag{3.74}$$

The map $r : S^{\cdot}(\mathfrak{g}^*) \to S^{\cdot}(\mathfrak{h}^*)$ induces the canonical isomorphism

$$r : I^{\cdot}(\mathfrak{g}^*) \simeq I^{\cdot}(\mathfrak{h}^*, \mathfrak{g}^*)\,. \tag{3.75}$$

Proof. By Proposition 3.1, $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$ is preserved by conjugation. Therefore $I^{\cdot}(\mathfrak{h}^*_{\mathbf{C}}, \mathfrak{g}^*_{\mathbf{C}})$ is preserved by conjugation, which gives (3.74). From the obvious isomorphism

$$r : I^{\cdot}(\mathfrak{g}^*_{\mathbf{C}}) \to I^{\cdot}(\mathfrak{h}^*_{\mathbf{C}}, \mathfrak{g}^*_{\mathbf{C}})\,, \tag{3.76}$$

we get (3.75). The proof of our proposition is completed. □

What we did for $\mathfrak{g}^*$ can also be done for $\mathfrak{g}$. The same argument as in (3.75) leads to the identification

$$r : I^{\cdot}(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h}, \mathfrak{g})\,. \tag{3.77}$$

As we saw in Subsection 3.1, $S^{\cdot}(\mathfrak{g})$ can be identified with the algebra $D^{\cdot}(\mathfrak{g})$ of real differential operators on $\mathfrak{g}$ with constant coefficients, so that $I^{\cdot}(\mathfrak{g})$ is identified with the algebra $D^{\cdot}_I(\mathfrak{g})$ of real differential operators with constant coefficients on $\mathfrak{g}$ which commute with the above $\mathfrak{g}$-derivations. Similarly $I^{\cdot}(\mathfrak{h}, \mathfrak{g})$ can be identified with the algebra $D^{\cdot}_I(\mathfrak{h}, \mathfrak{g})$ of real differential operators on $\mathfrak{h}$ with constant coefficients that are $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$-invariant.

Let $R_+ \subset R$ be a positive root system as in Subsection 3.5. Recall that the associated polynomial $\pi^{\mathfrak{h},\mathfrak{g}} \in S^{\cdot}(\mathfrak{h}^*_{\mathbf{C}})$ was defined in (3.40). If $A \in I^{\cdot}(\mathfrak{g}) = D_I(\mathfrak{g})$, if $f \in I^{\cdot}(\mathfrak{g}^*)$, then $Af \in I^{\cdot}(\mathfrak{g}^*)$, so that $r(Af) \in I^{\cdot}(\mathfrak{h}^*, \mathfrak{g}^*)$. Also $r(A) \in I^{\cdot}(\mathfrak{h}, \mathfrak{g}) = D_I(\mathfrak{h}, \mathfrak{g})$. By [HC57a, Lemmas 6 and 8], if $f \in I^{\cdot}(\mathfrak{g}^*)$,

$$r(Af) = \frac{1}{\pi^{\mathfrak{h},\mathfrak{g}}} r(A)\, \pi^{\mathfrak{h},\mathfrak{g}} r f. \tag{3.78}$$

Let $C^{\infty,\mathfrak{g}}(\mathfrak{g}, \mathbf{R})$ be the vector space of smooth real functions on $\mathfrak{g}$ that vanish under the above $\mathfrak{g}$-derivations. Then (3.78) extends to $f \in C^{\infty,\mathfrak{g}}(\mathfrak{g}, \mathbf{R})$.

4. Root systems and the function $\mathcal{J}_\gamma$

The purpose of this Section is to give a drastically simplified version of the function $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)$ introduced in Definition 2.6. This will be done by expressing this function in terms of a positive root system. Imaginary roots will play an essential role in this expression. In particular, the function $\mathcal{L}_\gamma$ introduced in Definition 2.5 will turn out not to depend on a.

This section is organized as follows. In Subsection 4.1, if $\mathfrak{h}$ is a θ-stable Cartan subalgebra and H is the corresponding Cartan subgroup, if $\gamma \in H$, we give explicit formulas for the determinant of $1-\mathrm{Ad}\,(\gamma)$ on various subspaces in terms of a positive root system.

In Subsection 4.2, we establish our formula for $\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)$ using the root system.

We use the assumptions and the notation of Section 3.

4.1. **The determinant of $1-\mathrm{Ad}\,(\gamma)$.** Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra, and let $H \subset G$ be the corresponding Cartan subgroup. Put

$$\mathfrak{h}_+^\perp = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha, \qquad \mathfrak{h}_-^\perp = \bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha}. \tag{4.1}$$

By (3.10), we get

$$\mathfrak{h}_{\mathbf{C}}^\perp = \mathfrak{h}_+^\perp \oplus \mathfrak{h}_-^\perp. \tag{4.2}$$

Let $\gamma \in H$ be written as in (3.56). By Theorem 3.18, we obtain

$$\det \mathrm{Ad}\,(\gamma)\,|_{\mathfrak{h}_+^\perp} = \prod_{\alpha \in R_+} \xi_\alpha\,(\gamma)\,. \tag{4.3}$$

We write (4.3) in the form

$$\det \mathrm{Ad}\,(\gamma)\,|_{\mathfrak{h}_+^\perp} = \prod_{\alpha \in R_+^{\mathrm{c}}} \xi_\alpha\,(\gamma) \prod_{\alpha \in R_+^{\mathrm{re}}} \xi_\alpha\,(\gamma) \prod_{\alpha \in R_+^{\mathrm{im}}} \xi_\alpha\,(\gamma)\,. \tag{4.4}$$

By the considerations we made in Subsection 3.5, $-\theta$ acts without fixed points on R_+^{c}. By Theorem 3.18, in the right-hand side of (4.4), the first term is positive, the second is a product of nonzero real numbers, and the third term is a product of complex numbers of module 1.

If $\alpha \in R_+$, we choose a square root $\xi_\alpha^{1/2}\,(k^{-1})$ of $\xi_\alpha\,(k^{-1})$. In view of the second identity in (3.60), if $\alpha \in R_+^{\mathrm{c}}$, we may and we will assume that

$$\xi_{-\theta\alpha}^{1/2}\left(k^{-1}\right) = \overline{\xi_\alpha^{1/2}\,(k^{-1})}. \tag{4.5}$$

For $\alpha \in R_+$, we choose the square root $\xi_\alpha^{1/2}(\gamma)$ so that

$$\xi_\alpha^{1/2}(\gamma) = e^{\langle \alpha, a\rangle/2}\xi_\alpha^{1/2}\left(k^{-1}\right). \tag{4.6}$$

By (4.5), (4.6), if $\alpha \in R_+^{\mathrm{c}}$, then

$$\xi_{-\theta\alpha}^{1/2}(\gamma) = \overline{\xi_\alpha^{1/2}(\gamma)}. \tag{4.7}$$

A square root of $\det \mathrm{Ad}(\gamma)|_{\mathfrak{h}_+^\perp}$ in (4.3) is given by

$$\det \mathrm{Ad}(\gamma)|_{\mathfrak{h}_+^\perp}^{1/2} = \prod_{\alpha \in R_+} \xi_\alpha^{1/2}(\gamma). \tag{4.8}$$

By proceeding as in (4.4), we can rewrite (4.8) in the form

$$\det \mathrm{Ad}(\gamma)|_{\mathfrak{h}_+^\perp}^{1/2} = \prod_{\alpha \in R_+^{\mathrm{c}}} \xi_\alpha^{1/2}(\gamma) \prod_{\alpha \in R_+^{\mathrm{re}}} \xi_\alpha^{1/2}(\gamma) \prod_{\alpha \in R_+^{\mathrm{im}}} \xi_\alpha^{1/2}(\gamma). \tag{4.9}$$

Using (4.6), (4.7), we find that the first product in the right hand-side of (4.9) is positive, the second product is either a nonzero real number, or the product of $\sqrt{-1}$ by a nonzero real number, and the third product is of module 1.

Definition 4.1. Put

$$\epsilon_D(\gamma) = \mathrm{sgn} \prod_{\alpha \in R_+^{\mathrm{re}} \setminus R_+^{\mathrm{re}}(\gamma)} \left(1 - \xi_\alpha^{-1}(\gamma)\right). \tag{4.10}$$

Recall that if $\alpha \in R^{\mathrm{re}}$, then $\xi_\alpha(k^{-1}) = \pm 1$.

Proposition 4.2. *The following identity holds:*

$$\epsilon_D(\gamma) = \mathrm{sgn} \prod_{\alpha \in R_+^{\mathrm{re}} \setminus R_+^{\mathrm{re}}(a)} \left(1 - \xi_\alpha^{-1}(\gamma)\right). \tag{4.11}$$

Proof. If $\alpha \in R(a)$, by (3.60), $\xi_\alpha(\gamma) = \xi_\alpha(k^{-1})$. If $\alpha \notin R(\gamma)$, by (3.62), $\xi_\alpha(\gamma) \neq 1$. By Theorem 3.18, if $\alpha \in R^{\mathrm{re}}(a) \setminus R^{\mathrm{re}}(\gamma)$, we have $\xi_\alpha(\gamma) = -1$, so that $1 - \xi_\alpha^{-1}(\gamma) = 2$. This completes the proof of our proposition. □

Theorem 4.3. *The following identities hold:*

$$\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)\big|_{\mathfrak{z}^{\perp}(a)} = (-1)^{|R_+\setminus R_+(a)|}\prod_{\alpha\in R_+\setminus R_+(a)}\left(\xi_{\alpha}^{1/2}\left(\gamma\right)-\xi_{\alpha}^{-1/2}\left(\gamma\right)\right)^2,$$

(4.12)

$$\left|\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)\big|_{\mathfrak{z}^{\perp}(a)}\right|^{1/2} = \epsilon_D\left(\gamma\right)\prod_{\alpha\in R_+\setminus R_+(a)}\left(\xi_{\alpha}^{1/2}\left(\gamma\right)-\xi_{\alpha}^{-1/2}\left(\gamma\right)\right)$$

$$\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\xi_{\alpha}^{-1/2}\left(k^{-1}\right),$$

$$\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)\big|_{\mathfrak{z}_a^{\perp}(\gamma)} = (-1)^{|R_+(a)\setminus R_+(\gamma)|}$$

$$\prod_{\alpha\in R_+(a)\setminus R_+(\gamma)}\left(\xi_{\alpha}^{1/2}\left(\gamma\right)-\xi_{\alpha}^{-1/2}\left(\gamma\right)\right)^2.$$

Proof. Using Theorem 3.18 and the third identity in (3.70), we get

$$\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)\big|_{\mathfrak{z}^{\perp}(a)} = \prod_{\alpha\in R_+\setminus R_+(a)}\left(1-\xi_{\alpha}\left(\gamma\right)\right)\left(1-\xi_{\alpha}^{-1}\left(\gamma\right)\right), \tag{4.13}$$

from which the first equation in (4.12) follows.

By proceeding as in Subsection 3.5, we find that $-\theta$ acts on $R_+\setminus\left(R_+\left(a\right)\cup R_+^{\mathrm{re}}\right)$ without fixed points, so that $\left|R_+\setminus\left(R_+\left(a\right)\cup R_+^{\mathrm{re}}\right)\right|$ is even, and so

$$(-1)^{|R_+\setminus R_+(a)|} = (-1)^{\left|R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)\right|}. \tag{4.14}$$

The same arguments also show that

$$\prod_{\alpha\in R_+\setminus\left(R_+(a)\cup R_+^{\mathrm{re}}\right)}\left(\xi_{\alpha}^{1/2}\left(\gamma\right)-\xi_{\alpha}^{-1/2}\left(\gamma\right)\right) \tag{4.15}$$

is a positive number, and also that

$$\prod_{\alpha\in R_+\setminus\left(R_+(a)\cup R_+^{\mathrm{re}}\right)}\left(1-\xi_{\alpha}\left(\gamma\right)\right)\left(1-\xi_{\alpha}^{-1}\left(\gamma\right)\right) \tag{4.16}$$

$$= \prod_{\alpha\in R_+\setminus\left(R_+(a)\cup R_+^{\mathrm{re}}\right)}\left(\xi_{\alpha}^{1/2}\left(\gamma\right)-\xi_{\alpha}^{-1/2}\left(\gamma\right)\right)^2.$$

Moreover, we have the identity of nonzero real numbers,

$$\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\left(1-\xi_\alpha(\gamma)\right)\left(1-\xi_\alpha^{-1}(\gamma)\right) = (-1)^{\left|R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)\right|}\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\left(1-\xi_\alpha^{-1}(\gamma)\right)^2\xi_\alpha(\gamma). \tag{4.17}$$

By Theorem 3.18, if $\alpha\in R_+^{\mathrm{re}}$, then

$$\xi_\alpha(\gamma)=e^{\langle\alpha,a\rangle}\xi_\alpha\left(k^{-1}\right),\qquad \xi_\alpha\left(k^{-1}\right)=\pm 1. \tag{4.18}$$

Using Proposition 4.2 and (4.17), we get

$$\left|\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\left(1-\xi_\alpha(\gamma)\right)\left(1-\xi_\alpha^{-1}(\gamma)\right)\right|^{1/2} = \epsilon_D(\gamma)\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\left(1-\xi_\alpha^{-1}(\gamma)\right)e^{\langle\alpha,a\rangle/2}. \tag{4.19}$$

Equation (4.19) can be rewritten in the form

$$\left|\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\left(1-\xi_\alpha(\gamma)\right)\left(1-\xi_\alpha^{-1}(\gamma)\right)\right|^{1/2} = \epsilon_D(\gamma)\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\left(\xi_\alpha^{1/2}(\gamma)-\xi_\alpha^{-1/2}(\gamma)\right)\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\xi_\alpha^{-1/2}\left(k^{-1}\right). \tag{4.20}$$

By (4.13)–(4.20), we get the second identity in (4.12).

Since on $\mathfrak{z}_a^{\perp}(\gamma)$, $\mathrm{Ad}(\gamma)$ acts like $\mathrm{Ad}(k^{-1})$, the proof of the third identity in (4.12) is the same as the proof of the first identity, which completes the proof of our theorem. □

Remark 4.4. By the first two equations in (4.12), we deduce that

$$\operatorname{sgn}\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}^{\perp}(a)}=(-1)^{|R_+\setminus R_+(a)|}\prod_{\alpha\in R_+^{\mathrm{re}}\setminus R_+^{\mathrm{re}}(a)}\xi_\alpha^{-1}\left(k^{-1}\right). \tag{4.21}$$

Also

$$|R_+\setminus R_+(a)|=\dim\mathfrak{p}^{\perp}(a). \tag{4.22}$$

Using (4.22), we can rewrite (4.21) in the form

$$\operatorname{sgn}\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}^{\perp}(a)}=(-1)^{\dim\mathfrak{p}^{\perp}(a)}\det\mathrm{Ad}(k)|_{\mathfrak{r}_{\mathfrak{p}}^{\perp}(a)}. \tag{4.23}$$

Using Proposition 3.8, we get

$$\det \mathrm{Ad}\,(k)\,|_{\mathfrak{r}_{\mathfrak{p}}^{\perp}(a)} = \det \mathrm{Ad}\,(k)\,|_{\mathfrak{p}^{\perp}(a)}. \tag{4.24}$$

By (4.23), (4.24), we get

$$\operatorname{sgn} \det\left(1-\mathrm{Ad}\,(\gamma)\right)|_{\mathfrak{z}^{\perp}(a)} = (-1)^{\dim \mathfrak{p}^{\perp}(a)} \det \mathrm{Ad}\,(k)\,|_{\mathfrak{p}^{\perp}(a)}, \tag{4.25}$$

a result already established in [B11, Proposition 5.4.1].

Let $\mathfrak{i}^{\perp}$ be the orthogonal space to $\mathfrak{i}$ in $\mathfrak{h}^{\perp}$. By (2.20), $\mathfrak{z}(\gamma) \subset \mathfrak{z}(a)$, and by (3.68), $\mathfrak{i} \subset \mathfrak{z}(a)$. Therefore,

$$\mathfrak{z}^{\perp}(a) \subset \mathfrak{z}^{\perp}(\gamma) \cap \mathfrak{i}^{\perp}. \tag{4.26}$$

Similarly, we have the inclusion

$$R_{+}(\gamma) \cup R_{+}^{\mathrm{im}} \subset R_{+}(a). \tag{4.27}$$

Theorem 4.5. *The following identities hold:*

$$\begin{aligned}\det\left(1-\mathrm{Ad}\,(\gamma)\right)|_{\mathfrak{z}^{\perp}(\gamma)\cap\mathfrak{i}^{\perp}} &= (-1)^{\left|R_{+}\backslash\left(R_{+}(\gamma)\cup R_{+}^{\mathrm{im}}\right)\right|} \\ &\prod_{\alpha\in R_{+}\backslash\left(R_{+}(\gamma)\cup R_{+}^{\mathrm{im}}\right)} \left(\xi_{\alpha}^{1/2}(\gamma)-\xi_{\alpha}^{-1/2}(\gamma)\right)^{2},\end{aligned} \tag{4.28}$$

$$\begin{aligned}\left|\det\left(1-\mathrm{Ad}\,(\gamma)\right)|_{\mathfrak{z}^{\perp}(\gamma)\cap\mathfrak{i}^{\perp}}\right|^{1/2} &= \epsilon_{D}(\gamma) \\ &\prod_{\alpha\in R_{+}\backslash\left(R_{+}(\gamma)\cup R_{+}^{\mathrm{im}}\right)} \left(\xi_{\alpha}^{1/2}(\gamma)-\xi_{\alpha}^{-1/2}(\gamma)\right) \prod_{\alpha\in R_{+}^{\mathrm{re}}\backslash R_{+}^{\mathrm{re}}(\gamma)} \xi_{\alpha}^{-1/2}\left(k^{-1}\right).\end{aligned}$$

Proof. The proof of the first identity in (4.28) is the same as the proof of the first identity in (4.12) that was given in Theorem 4.3. Instead of (4.14), we get

$$(-1)^{\left|R_{+}\backslash\left(R_{+}(\gamma)\cup R_{+}^{\mathrm{im}}\right)\right|} = (-1)^{\left|R_{+}^{\mathrm{re}}\backslash R_{+}^{\mathrm{re}}(\gamma)\right|}. \tag{4.29}$$

If in (4.15), we replace $R_{+}\backslash\left(R_{+}(a)\cup R_{+}^{\mathrm{re}}\right)$ by $R_{+}\backslash\left(R_{+}(\gamma)\cup R_{+}^{\mathrm{re}}\cup R_{+}^{\mathrm{im}}\right)$, the conclusions remain valid. Similarly, (4.17), (4.19) remain valid when replacing $R_{+}^{\mathrm{re}}\setminus R_{+}^{\mathrm{re}}(a)$ by $R_{+}^{\mathrm{re}}\setminus R_{+}^{\mathrm{re}}(\gamma)$. This completes the proof of our theorem. □

4.2. **Evaluation of the function $\mathcal{J}_{\gamma}$ on $i\mathfrak{h}_{\mathfrak{k}}$.** Let $\mathfrak{h}\subset\mathfrak{g}$ be a θ-stable Cartan subalgebra. Take $\gamma\in H$. Then γ is semisimple, and $\mathfrak{h}\subset\mathfrak{z}(\gamma)$. In particular, $\mathfrak{h}_{\mathfrak{k}}\subset\mathfrak{k}(\gamma)$, so that functions defined on $i\mathfrak{k}(\gamma)$ restrict to functions on $i\mathfrak{h}_{\mathfrak{k}}$.

Recall that the function $\mathcal{L}_{\gamma}\left(Y_{0}^{\mathfrak{k}}\right), \mathcal{M}_{\gamma}\left(Y_{0}^{\mathfrak{k}}\right)$ on $i\mathfrak{k}(\gamma)$ were defined in Definition 2.5.

Definition 4.6. If $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, put

$$\mathscr{L}_{k^{-1}}(h_{\mathfrak{k}}) = \frac{\det\left(1 - \mathrm{Ad}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)|_{\mathfrak{i}_{\mathfrak{k}}^{\perp}(k)}}{\det\left(1 - \mathrm{Ad}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)|_{\mathfrak{i}_{\mathfrak{p}}^{\perp}(k)}}. \tag{4.30}$$

Like the function $\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)$ in (2.68), the function $\mathscr{L}_{k^{-1}}(h_{\mathfrak{k}})$ is a smooth function of $h_{\mathfrak{k}}$, which verifies estimates similar to (2.77). Exactly the same arguments as in [B11, Section 5.5] and after (2.65) show that there is an unambiguously defined square root

$$\mathscr{M}_{k^{-1}}(h_{\mathfrak{k}}) = \left[\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{i}^{\perp}(k)}}\mathscr{L}_{k^{-1}}(h_{\mathfrak{k}})\right]^{1/2}. \tag{4.31}$$

This square root is positive for $h_{\mathfrak{k}} = 0$.

Theorem 4.7. *If $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, then*

$$\frac{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{k}(\gamma)}\right)} = \frac{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{i}_{\mathfrak{p}}(k)}\right)}{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{i}_{\mathfrak{k}}(k)}\right)}, \tag{4.32}$$

$$\mathcal{L}_{\gamma}(h_{\mathfrak{k}}) = \mathscr{L}_{k^{-1}}(h_{\mathfrak{k}}).$$

In particular $\mathcal{L}_{\gamma}(h_{\mathfrak{k}})$ does not depend on a.

If $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, we have the identity,

$$\mathcal{J}_{\gamma}(h_{\mathfrak{k}}) = \frac{1}{\left|\det\left(1 - \mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}^{\perp}(\gamma)\cap\mathfrak{i}^{\perp}}\right|^{1/2}}\frac{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{i}_{\mathfrak{p}}(k)}\right)}{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{i}_{\mathfrak{k}}(k)}\right)}\mathscr{M}_{k^{-1}}(h_{\mathfrak{k}}). \tag{4.33}$$

This identity can be written in the form,

$$\begin{aligned}\mathcal{J}_{\gamma}(h_{\mathfrak{k}}) &= \frac{(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)\right|}\epsilon_D(\gamma)\prod_{\alpha\in R^{\mathrm{re}}_{+}\setminus R^{\mathrm{re}}_{+}(\gamma)}\xi_{\alpha}^{1/2}\left(k^{-1}\right)}{\prod_{\alpha\in R_{+}\setminus R_{+}(\gamma)}\left(\xi_{\alpha}^{1/2}(\gamma) - \xi_{\alpha}^{-1/2}(\gamma)\right)}\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\widehat{A}\left(\langle\alpha, h_{\mathfrak{k}}\rangle\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\widehat{A}\left(\langle\alpha, h_{\mathfrak{k}}\rangle\right)} \\ &\quad\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}\setminus R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right) - \xi_{\alpha}^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right) - \xi_{\alpha}^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}.\end{aligned} \tag{4.34}$$

Proof. By (3.28), we get

$$\mathfrak{i}^{\perp} = \mathfrak{r} \oplus \mathfrak{c}. \tag{4.35}$$

Also $\mathfrak{i}^{\perp}$ splits as

$$\mathfrak{i}^{\perp} = \mathfrak{i}_{\mathfrak{p}}^{\perp} \oplus \mathfrak{i}_{\mathfrak{k}}^{\perp}. \tag{4.36}$$

By Proposition 3.8, as representations of $H \cap K$, $\mathfrak{i}_{\mathfrak{p}}^{\perp}$ and $\mathfrak{i}_{\mathfrak{k}}^{\perp}$ are equivalent, so that (4.32) holds.

Observe that $\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_a^{\perp}(\gamma)\cap\mathfrak{i}^{\perp}}>0$, and so this number has a positive square root. Moreover,

$$\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}^{\perp}(\gamma)\cap\mathfrak{i}^{\perp}}=\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}^{\perp}(a)}\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_a^{\perp}(\gamma)\cap\mathfrak{i}^{\perp}}. \tag{4.37}$$

By (2.64), (2.65), (2.68), (4.32), and (4.37), we get (4.33).

Clearly,

$$\frac{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{i}_{\mathfrak{p}}(k)}\right)}{\widehat{A}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{i}_{\mathfrak{k}}(k)}\right)}=\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\widehat{A}\left(\langle\alpha,h_{\mathfrak{k}}\rangle\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\widehat{A}\left(\langle\alpha,h_{\mathfrak{k}}\rangle\right)}. \tag{4.38}$$

By proceeding as in the proof of the third identity in (4.12), we get

$$\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{i}^{\perp}(k)}=(-1)^{\left|R^{\mathrm{im}}_{+}\setminus R^{\mathrm{im}}_{+}(k)\right|}\prod_{\alpha\in R^{\mathrm{im}}_{+}\setminus R^{\mathrm{im}}_{+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}\right)-\xi_{\alpha}^{-1/2}\left(k^{-1}\right)\right)^{2}. \tag{4.39}$$

The same argument shows that

$$\mathscr{L}_{k^{-1}}\left(h_{\mathfrak{k}}\right)=(-1)^{\left|R^{\mathrm{im}}_{+}\setminus R^{\mathrm{im}}_{+}(k)\right|}\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}\setminus R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi_{\alpha}^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)^{2}}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi_{\alpha}^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)^{2}}. \tag{4.40}$$

By (4.39), (4.40), and keeping in mind the fact that we take the properly positive square root in (4.31), we get

$$\mathscr{M}_{k^{-1}}\left(h_{\mathfrak{k}}\right)=\frac{(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}/R^{\mathrm{im}}_{\mathfrak{p},+}(k)\right|}}{\prod_{\alpha\in R^{\mathrm{im}}_{+}\setminus R^{\mathrm{im}}_{+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}\right)-\xi_{\alpha}^{-1/2}\left(k^{-1}\right)\right)}\frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}\setminus R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi_{\alpha}^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\left(\xi_{\alpha}^{1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)-\xi_{\alpha}^{-1/2}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)}. \tag{4.41}$$

In the first product in the right-hand side of (4.41), we may as well replace k^{-1} by γ.

By combining the second identity in (4.28), (4.33), (4.38), and (4.41), we get (4.34). The proof of our theorem is completed. □

5. The function $\mathcal{J}_\gamma$ when γ is regular

The purpose of this Section is to study extra properties of the function $\mathcal{J}_\gamma$ when γ is regular.

This section is organized as follows. In Subsection 5.1, if $\mathfrak{h}$ is a θ-stable Cartan subalgebra and if H is the corresponding Cartan subgroup, if $\gamma \in H$, we describe a neighborhood of γ in H.

In Subsection 5.2, if $\gamma \in H$, we define the γ-regular elements in $\mathfrak{h}$, which are such that a small perturbation of γ by a γ-regular element is regular.

In Subsection 5.3, following Harish-Chandra [HC65], we introduce the function D_H on H. This function is an analogue of the denominator in the Lefschetz formulas.

Finally, in Subsection 5.4, we specialize the formula obtained in Theorem 4.7 for $\mathcal{J}_\gamma(h_\mathfrak{k})$ to the case where $\gamma \in H^{\mathrm{reg}}$. As a consequence, we prove the unexpected result that the function $(\gamma, h_\mathfrak{k}) \in H^{\mathrm{reg}} \times i\mathfrak{h}_\mathfrak{k} \to \mathcal{J}_\gamma(h_\mathfrak{k}) \in \mathbf{C}$ is smooth.

We make the same assumptions and we use the same notation as in Section 4.

5.1. A neighborhood of γ in H.

If $b \in \mathfrak{h}$, b splits as

$$b = b_\mathfrak{p} + b_\mathfrak{k}, \qquad b_\mathfrak{p} \in \mathfrak{h}_\mathfrak{p}, \qquad b_\mathfrak{k} \in \mathfrak{h}_\mathfrak{k}. \tag{5.1}$$

Put

$$\gamma' = \gamma e^b. \tag{5.2}$$

Then $\gamma' \in H \cap Z(\gamma)$.

Set

$$a' = a + b_\mathfrak{p}, \qquad k' = k e^{-b_\mathfrak{k}}. \tag{5.3}$$

Then

$$\gamma' = e^{a'} k'^{-1}, \quad a' \in \mathfrak{h}_\mathfrak{p}, \qquad k' \in H \cap K(\gamma), \qquad \mathrm{Ad}(k')\, a' = a'. \tag{5.4}$$

Also $\mathrm{Ad}(\gamma')$ preserves the splitting $\mathfrak{g} = \mathfrak{z}(\gamma) \oplus \mathfrak{z}^\perp(\gamma)$. Since $1 - \mathrm{Ad}(\gamma)$ is invertible on $\mathfrak{z}^\perp(\gamma)$, we conclude that for $\epsilon > 0$ small enough, if $|b| \leq \epsilon$,

$$\mathfrak{h} \subset \mathfrak{z}(\gamma') \subset \mathfrak{z}(\gamma). \tag{5.5}$$

Let H^{reg} be the set of regular elements in H. Assume temporarily that $\gamma \in H^{\mathrm{reg}}$, i.e., $\mathfrak{z}(\gamma) = \mathfrak{h}$. By (5.5), for $\epsilon > 0$ small enough if $|b| \leq \epsilon$, then

$$\mathfrak{z}(\gamma') = \mathfrak{h}, \tag{5.6}$$

i.e., $\gamma' \in H^{\mathrm{reg}}$, which is a trivial conclusion. Since $b \in \mathfrak{h}$, we conclude that $\gamma' \in Z(\gamma), \gamma \in Z(\gamma')$. A priori, $Z(\gamma)$ and $Z(\gamma')$ may be distinct. Still we have the obvious identity

$$Z^0(\gamma) = Z^0(\gamma') = H^0. \tag{5.7}$$

5.2. **The γ-regular elements in $\mathfrak{h}$.** We no longer assume γ to be regular. By (3.70), we get

$$\mathfrak{z}_a^{\perp}(\gamma)_{\mathbf{C}} = \oplus_{\alpha \in R(a) \setminus R(\gamma)} \mathfrak{g}_\alpha. \tag{5.8}$$

Let $\mathfrak{h}_a^{\perp}$ denote the orthogonal space to $\mathfrak{h}$ in $\mathfrak{z}(a)$. Then we have the splitting

$$\mathfrak{h}_a^{\perp} = \mathfrak{h}_{a,\mathfrak{p}}^{\perp} \oplus \mathfrak{h}_{a,\mathfrak{k}}^{\perp}. \tag{5.9}$$

By (3.70), we get

$$\mathfrak{h}_{a,\mathbf{C}}^{\perp} = \oplus_{\alpha \in R(a)} \mathfrak{g}_\alpha. \tag{5.10}$$

Definition 5.1. An element $h \in \mathfrak{h}$ is said to be γ-regular if for $\alpha \in R(\gamma)$, $\langle \alpha, h \rangle \neq 0$.

The γ-regular elements in $\mathfrak{h}$ are exactly the regular elements in $\mathfrak{h}$ viewed as a Cartan subalgebra of $\mathfrak{z}(\gamma)$. The γ-regular elements lie in the complement of a finite family of hyperplanes in $\mathfrak{h}$.

Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{z}(\gamma)$, we define the function $\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}$ on $\mathfrak{h}_{\mathbf{C}}$ as in (3.40), i.e.,

$$\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(h) = \prod_{\alpha \in R_+(\gamma)} \langle \alpha, h \rangle. \tag{5.11}$$

Then $h \in \mathfrak{h}$ is γ-regular if and only if $\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(h) \neq 0$.

Now we use the notation of Subsection 5.1.

Proposition 5.2. *There exists $\epsilon > 0$ such that if $b \in \mathfrak{h}$ is γ-regular, and $|b| \leq \epsilon$, if $\gamma' = \gamma e^b$, then $\gamma' \in H^{\mathrm{reg}}$.*

Proof. For $\epsilon > 0$ small enough, (5.5) holds, so that

$$\mathfrak{z}(\gamma') = \mathfrak{z}(\gamma) \cap \mathfrak{z}\left(e^b\right). \tag{5.12}$$

By (3.7), we get

$$\mathfrak{z}(\gamma)_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \bigoplus \oplus_{\alpha \in R(\gamma)} \mathfrak{g}_\alpha. \tag{5.13}$$

For $\alpha \in R(\gamma)$, e^b acts on $\mathfrak{g}_\alpha$ by multiplication by $e^{\langle \alpha, b \rangle}$. For $\epsilon > 0$ small enough, if b is γ-regular and $|b| \leq \epsilon$, for $\alpha \in R(\gamma)$, $e^{\langle \alpha, b \rangle} \neq 1$. By (5.12), (5.13), we conclude that under the given conditions on b, $\mathfrak{z}(\gamma') = \mathfrak{h}$, i.e., γ' is regular. The proof of our proposition is completed. □

5.3. **The function $D_H(\gamma)$.** Here, we follow Harish-Chandra [HC65, Section 19].

Definition 5.3. If $\gamma \in H^{\mathrm{reg}}$, put[9]

$$D_H(\gamma) = \prod_{\alpha \in R_+} \left(\xi_\alpha^{1/2}(\gamma) - \xi_\alpha^{-1/2}(\gamma)\right). \tag{5.14}$$

Using (3.61) and proceeding as in the proof of Theorem 4.3, we get

$$\det\left(1 - \mathrm{Ad}(\gamma)\right)|_{\mathfrak{h}^\perp} = (-1)^{|R_+|} D_H^2(\gamma). \tag{5.15}$$

By (5.15), we deduce that if $\gamma \in H^{\mathrm{reg}}$, then

$$\left|\det\left(1 - \mathrm{Ad}(\gamma)\right)|_{\mathfrak{h}^\perp}\right| = \left|D_H(\gamma)\right|^2, \tag{5.16}$$

so that $D_H(\gamma) \neq 0$.

5.4. **The function $\mathcal{J}_\gamma$ when γ is regular.** In this subsection, we assume that $\gamma \in H^{\mathrm{reg}}$, i.e., $D_H(\gamma) \neq 0$.

By (4.30), (4.31), we get

$$\mathscr{L}_{k^{-1}}(h_{\mathfrak{k}}) = \frac{\det\left(1 - \mathrm{Ad}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)|_{\mathfrak{i}_{\mathfrak{k}}}}{\det\left(1 - \mathrm{Ad}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)|_{\mathfrak{i}_{\mathfrak{p}}}}, \tag{5.17}$$

$$\mathscr{M}_{k^{-1}}(h_{\mathfrak{k}}) = \left[\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{i}}} \mathscr{L}_{k^{-1}}(h_{\mathfrak{k}})\right]^{1/2}.$$

By (4.10), $\epsilon_D(\gamma)$ is given by

$$\epsilon_D(\gamma) = \mathrm{sgn} \prod_{\alpha \in R_+^{\mathrm{re}}} \left(1 - \xi_\alpha^{-1}(\gamma)\right). \tag{5.18}$$

The function $\epsilon_D(\gamma)$ is locally constant on H^{reg}.

Theorem 5.4. *If $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, we have the identity,*

$$\mathcal{J}_\gamma(h_{\mathfrak{k}}) = \frac{1}{\left|\det\left(1 - \mathrm{Ad}(\gamma)\right)|_{\mathfrak{i}^\perp}\right|^{1/2}} \mathscr{M}_{k^{-1}}(h_{\mathfrak{k}}). \tag{5.19}$$

[9] In [HC65, Section 18], Harish-Chandra assumes G to be acceptable, i.e., $\rho^{\mathfrak{g}}$ is assumed to be a weight, so that $D_H(\gamma)$ can be globally defined. Here, we only need a local definition of $D_H(\gamma)$, and we do not need this assumption.

This identity can be written in the form,

$$\mathcal{J}_\gamma(h_\mathfrak{k}) = \frac{(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|}\epsilon_D(\gamma)\prod_{\alpha\in R^{\mathrm{re}}_+}\xi^{1/2}_\alpha(k^{-1})}{D_H(\gamma)} \frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}}\left(\xi^{1/2}_\alpha\left(k^{-1}e^{-h_\mathfrak{k}}\right)-\xi^{-1/2}_\alpha\left(k^{-1}e^{-h_\mathfrak{k}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}}\left(\xi^{1/2}_\alpha\left(k^{-1}e^{-h_\mathfrak{k}}\right)-\xi^{-1/2}_\alpha\left(k^{-1}e^{-h_\mathfrak{k}}\right)\right)}. \tag{5.20}$$

The function $(\gamma, h_\mathfrak{k}) \in H^{\mathrm{reg}} \times i\mathfrak{h}_\mathfrak{k} \to \mathcal{J}_\gamma(h_\mathfrak{k}) \in \mathbf{C}$ *is smooth.*

Proof. The first part of our theorem is a trivial consequence of Theorem 4.7. For $b_\mathfrak{k} \in \mathfrak{h}_\mathfrak{k}$, for $|b_\mathfrak{k}|$ small enough, we take

$$\xi^{1/2}_\alpha\left(k'^{-1}\right) = e^{\langle\alpha, b_\mathfrak{k}/2\rangle}\xi^{1/2}_\alpha\left(k^{-1}\right). \tag{5.21}$$

By (4.5), (5.21), we deduce that if $\alpha \in R^{\mathrm{c}}_+$, then

$$\xi^{1/2}_{-\theta\alpha}\left(k'^{-1}\right) = \overline{\xi^{1/2}_\alpha\left(k'^{-1}\right)}. \tag{5.22}$$

The stated smoothness is an obvious consequence of the above formulas. The proof of our theorem is completed. □

6. The Harish-Chandra isomorphism

In this section, if $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, we describe the Harish-Chandra isomorphism of algebras $\phi_{\mathrm{HC}} : Z(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h}, \mathfrak{g})$. Also we explain the action of $Z(\mathfrak{g})$ on $C^\infty(X, F)$, and we introduce certain semisimple orbital integrals in which $Z(\mathfrak{g})$ appears.

This section is organized as follows. In Subsection 6.1, we introduce the center of the enveloping algebra $Z(\mathfrak{g})$.

In Subsection 6.2, we recall some properties of the complex Harish-Chandra isomorphism $\phi_{\mathrm{HC}} : Z(\mathfrak{g}_\mathbf{C}) \simeq I^{\cdot}(\mathfrak{h}_\mathbf{C}, \mathfrak{g}_\mathbf{C})$, including some aspects of its construction.

In Subsection 6.3, we show that there is a real form of the Harish-Chandra isomorphism $\phi_{\mathrm{HC}} : Z(\mathfrak{h}) \simeq I^{\cdot}(\mathfrak{h}, \mathfrak{g})$.

In Subsection 6.4, we recall the relation of the Harish-Chandra isomorphism to the Duflo isomorphism that was established in [Du70].

In Subsection 6.5, we consider the case of the Casimir.

In Subsection 6.6, we describe the action of $Z(\mathfrak{g})$ on $C^\infty(X, F)$.

Finally, in Subsection 6.7, we consider the orbital integrals in which $Z(\mathfrak{g})$ appears.

6.1. **The center of the enveloping algebra.** Recall that the enveloping algebra $U(\mathfrak{g})$ was introduced in Subsection 2.3. Then $U(\mathfrak{g})$ is a filtered algebra, and the corresponding Gr is just the algebra of polynomials $S^{\cdot}(\mathfrak{g})$ on $\mathfrak{g}^*$.

Note that $\mathfrak{g}$ acts by derivations on $U(\mathfrak{g})$. Recall that $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$, i.e., it is the kernel of the above derivations.

Observe that G acts both on the left and on the right on $C^\infty(G, \mathbf{R})$ by the formula

$$\gamma_L s(g) = s\left(\gamma^{-1} g\right), \qquad \gamma_R s(g) = s(g\gamma), \tag{6.1}$$

and these two actions commute. They are intertwined by the involution induced by the involution $g \to \sigma g = g^{-1}$. Let $D_L(G)$ be the Lie algebras of left-invariant real differential operators on G. As we saw in Subsection 2.3, $U(\mathfrak{g})$ can be identified with $D_L(G)$. The algebra $D_L(G)$ commutes with the left action of G.

If $\mathfrak{g}_-$ is the Lie algebra $\mathfrak{g}$ with the negative of the original Lie bracket, the isomorphism of $\mathfrak{g}$ $f \to -f$ identifies $\mathfrak{g}$ and $\mathfrak{g}_-$. This isomorphism is induced by the involution σ.

Let $U(\mathfrak{g}_-)$ be the enveloping algebra associated with $\mathfrak{g}_-$. Then $U(\mathfrak{g}_-)$ can be identified with the algebra of right-invariant real differential operators $D_R(G)$. This algebra commutes with the right action of G. Also the isomorphism $f \to -f$ induces an identification of $U(\mathfrak{g})$ and $U(\mathfrak{g}_-)$. This identification is still induced by σ.

We equip $U(\mathfrak{g}), U(\mathfrak{g}_-)$ with the antiautomorphism $*$ which is just the adjoint in the classical L_2 sense when identifying $U(\mathfrak{g}), U(\mathfrak{g}_-)$ with $D_L(G), D_R(G)$. This involution extends to a $\mathbf{C}$-linear involution of $U(\mathfrak{g}_{\mathbf{C}}), U(\mathfrak{g}_{-,\mathbf{C}})$.

By definition, $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ is the subalgebra of $D_L(G)$ which commutes with right multiplication. Equivalently

$$Z(\mathfrak{g}) = D_L(G) \cap D_R(G). \tag{6.2}$$

Note that $*$ induces an automorphism of $Z(\mathfrak{g})$, which is an involution, and which we still denote $*$.

The isomorphism of $U(\mathfrak{g})$ with $U(\mathfrak{g}_-)$ which was described before is the one induced by σ. It induces the obvious isomorphism of $D_L(G)$ with $D_R(G)$. This way, we obtain an automorphism σ of $Z(\mathfrak{g})$, which is also an involution.

Clearly,

$$Z(\mathfrak{g}_{\mathbf{C}}) = Z(\mathfrak{g})_{\mathbf{C}}. \tag{6.3}$$

Equivalently, $Z(\mathfrak{g}_{\mathbf{C}})$ is equipped with a complex conjugation, and $Z(\mathfrak{g})$ is the algebra of complex conjugation invariants in $Z(\mathfrak{g}_{\mathbf{C}})$. Also $*$ and σ extend to complex automorphisms of $Z(\mathfrak{g}_{\mathbf{C}})$.

6.2. **The complex form of the Harish-Chandra isomorphism.** Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable Cartan subalgebra. By [Kn86, Theorem 8.18], there is a canonical Harish-Chandra isomorphism of filtered algebras,

$$\phi_{\mathrm{HC}} : Z(\mathfrak{g}_{\mathbf{C}}) \simeq I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}). \tag{6.4}$$

We need to describe the Harish-Chandra isomorphism in more detail. We fix a positive root R_+ as in Subsection 3.5. Put

$$\mathscr{P} = \sum_{\alpha \in R_+} U(\mathfrak{g}_{\mathbf{C}})\, \mathfrak{g}_{\alpha}. \tag{6.5}$$

Observe that $S^{\cdot}(\mathfrak{h}_{\mathbf{C}}) = U(\mathfrak{h}_{\mathbf{C}})$, and also that $U(\mathfrak{h}_{\mathbf{C}}) \subset U(\mathfrak{g}_{\mathbf{C}})$, so that $S^{\cdot}(\mathfrak{h}_{\mathbf{C}}) \subset U(\mathfrak{g}_{\mathbf{C}})$. By [Kn86, Lemma 8.17], we get

$$S^{\cdot}(\mathfrak{h}_{\mathbf{C}}) \cap \mathscr{P} = 0, \qquad Z(\mathfrak{g}_{\mathbf{C}}) \subset S^{\cdot}(\mathfrak{h}_{\mathbf{C}}) \oplus \mathscr{P}. \tag{6.6}$$

Let ϕ_{1,R_+} be the projection from $Z(\mathfrak{g}_{\mathbf{C}})$ on $S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$.

Recall that $S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$ is the algebra of polynomials on $\mathfrak{h}^*_{\mathbf{C}}$, and that $\rho^{\mathfrak{g}} \in \mathfrak{h}^*_{\mathbf{C}}$ is the half sum of the roots in R_+. Let ϕ_{2,R_+} be the filtered automorphism of $S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$ that is such that if $f \in S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$, if $h^* \in \mathfrak{h}^*_{\mathbf{C}}$, then

$$\phi_{2,R_+} f(h^*) = f(h^* - \rho^{\mathfrak{g}}). \tag{6.7}$$

The fundamental result of Harish-Chandra [HC56, Lemmas 18-20], [Kn86, Theorem 8.18] is that $\phi_{2,R_+}\phi_{1,R_+}$ maps $Z(\mathfrak{g}_{\mathbf{C}})$ onto $I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$, that it induces an isomorphism of filtered algebras that does not depend on the choice of R_+. This is exactly the Harish-Chandra isomorphism $\phi_{\mathrm{HC}} : Z(\mathfrak{g}_{\mathbf{C}}) \simeq I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$.

Now we proceed as in Subsection 3.11, i.e., we identify $I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ with the algebra $D^{\cdot}_I(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ of holomorphic differential operators on $\mathfrak{h}_{\mathbf{C}}$ with constant complex coefficients which are $W(\mathfrak{h}_{\mathbf{C}} : \mathfrak{g}_{\mathbf{C}})$-invariant. The same arguments as in Subsection 3.11 show that there is an algebra $D^{\cdot}_I(\mathfrak{h}, \mathfrak{g})$ of real differential operators with constant coefficients on $\mathfrak{h}$ such that

$$D^{\cdot}_I(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}) = D^{\cdot}_I(\mathfrak{h}, \mathfrak{g})_{\mathbf{C}}, \tag{6.8}$$

and that $I^{\cdot}(\mathfrak{h}, \mathfrak{g})$ can be identified with $D^{\cdot}_I(\mathfrak{h}, \mathfrak{g})$.

We will now use the assumptions and notation of Subsection 3.7. Let $C^{\infty, G}(G^{\mathrm{reg}}, \mathbf{C})$ denote the Ad-invariant smooth complex functions on the open set G^{reg}.

Let $C^{\infty,W(H:G)}\left(H^{\mathrm{reg}},\mathbf{C}\right)$ be the smooth $W\left(H:G\right)$-invariant functions on H^{reg}. There is a restriction map

$$r: C^{\infty,G}\left(G^{\mathrm{reg}},\mathbf{C}\right)\to C^{\infty,W(H:G)}\left(H^{\mathrm{reg}},\mathbf{C}\right).$$

Observe that $Z\left(\mathfrak{g}_{\mathbf{C}}\right)$ acts on $C^{\infty,G}\left(G^{\mathrm{reg}},\mathbf{C}\right)$, and $I^{\cdot}\left(\mathfrak{h}_{\mathbf{C}},\mathfrak{g}_{\mathbf{C}}\right)$ acts on $C^{\infty,W(H:G)}\left(H^{\mathrm{reg}},\mathbf{C}\right)$.

Let $L\in Z\left(\mathfrak{g}_{\mathbf{C}}\right)$. By [HC65, Lemma 13], [Kn86, Theorem 10.33], if $f\in C^{\infty,G}\left(G^{\mathrm{reg}},\mathbf{C}\right)$, on H^{reg}, we have the identity

$$rLf=\frac{1}{D_H}\left(\phi_{\mathrm{HC}}L\right)D_H rf. \tag{6.9}$$

6.3. **The real form of the Harish-Chandra isomorphism.** The involution $h\to -h$ induces an involution of $I^{\cdot}\left(\mathfrak{h}_{\mathbf{C}},\mathfrak{g}_{\mathbf{C}}\right)\simeq D_I^{\cdot}\left(\mathfrak{h}_{\mathbf{C}},\mathfrak{g}_{\mathbf{C}}\right)$. If N counts the degree in $I^{\cdot}\left(\mathfrak{h}_{\mathbf{C}},\mathfrak{g}_{\mathbf{C}}\right)$, this involution is just $(-1)^N$. We still denote this involution by $*$.

In Proposition 3.21, we proved that $I^{\cdot}\left(\mathfrak{h}_{\mathbf{C}},\mathfrak{g}_{\mathbf{C}}\right)$ is preserved by complex conjugation. At the end of Subsection 6.1, we proved that $Z\left(\mathfrak{g}_{\mathbf{C}}\right)$ is also preserved by complex conjugation. Observe that θ acts on $Z\left(\mathfrak{g}_{\mathbf{C}}\right), I^{\cdot}\left(\mathfrak{h}_{\mathbf{C}},\mathfrak{g}_{\mathbf{C}}\right)$ and preserves $Z\left(\mathfrak{g}\right), I^{\cdot}\left(\mathfrak{h},\mathfrak{g}\right)$.

Theorem 6.1. *If $L\in Z\left(\mathfrak{g}_{\mathbf{C}}\right)$, then*

(6.10)

$$\phi_{\mathrm{HC}}\left(L^*\right)=\left(\phi_{\mathrm{HC}}L\right)^*,\quad \phi_{\mathrm{HC}}\left(\overline{L}\right)=\overline{\phi_{\mathrm{HC}}\left(L\right)},\qquad \phi_{\mathrm{HC}}\theta L=\theta\phi_{\mathrm{HC}}L.$$

On $Z\left(\mathfrak{g}_{\mathbf{C}}\right)$, the involutions σ and $*$ coincide. Finally, ϕ_{HC} induces an isomorphism of real filtered algebras:

$$Z\left(\mathfrak{g}\right)\simeq I^{\cdot}\left(\mathfrak{h},\mathfrak{g}\right). \tag{6.11}$$

Proof. The first equation in (6.10) was established by Harish-Chandra [HC56, Lemma 20]. For the proof of the next two equations, we will follow Harish-Chandra, and use the notation in Subsection 6.2.

Observe that $\overline{R}_+$ is also a positive root system. More precisely, by (3.34), we get

$$\overline{R}_+=\overline{R}_+^{\mathrm{im}}\cup\overline{R}_+^{\mathrm{re}}\cup\overline{R}_+^{\mathrm{c}}. \tag{6.12}$$

From the properties of R_+, (6.12) can be rewritten in the form

$$\overline{R}_+=\left(-R_+^{\mathrm{im}}\right)\cup R_+^{\mathrm{re}}\cup R_+^{\mathrm{c}}. \tag{6.13}$$

Let $\overline{\mathscr{P}}$ denote the conjugate of $\mathscr{P}$ in $U\left(\mathfrak{g}_{\mathbf{C}}\right)$. By (3.8), $\overline{\mathscr{P}}$ is just the object defined in (6.5) associated with $\overline{R}_+$. We deduce that if $L\in Z\left(\mathfrak{g}_{\mathbf{C}}\right)$, we have the identity in $S^{\cdot}\left(\mathfrak{h}_{\mathbf{C}}\right)$,

$$\phi_{1,\overline{R}_+}\overline{L}=\overline{\phi_{1,R_+}L}. \tag{6.14}$$

Also $\overline{\rho^{\mathfrak{g}}} \in \mathfrak{h}^*_{\mathbf{C}}$ is the half sum of the roots in $\overline{R}_+$. By (6.13), if $f \in S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$, then

$$\phi_{2,\overline{R}_+}\overline{f} = \overline{\phi_{2,R_+} f}. \tag{6.15}$$

By (6.14), (6.15), we get the identity in $S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$,

$$\phi_{2,\overline{R}_+}\phi_{1,\overline{R}_+}\overline{L} = \overline{\phi_{2,R_+}\phi_{1,R_+} L}. \tag{6.16}$$

Let us now use Harish-Chandra's result described after (6.7). For $L \in Z(\mathfrak{g}_{\mathbf{C}})$, we can rewrite (6.16) as an identity in $S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$,

$$\phi_{\mathrm{HC}}\overline{L} = \overline{\phi_{\mathrm{HC}} L}. \tag{6.17}$$

But Harish-Chandra gives more, namely that the image of ϕ_{HC} is exactly $I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$. By (6.17), $I^{\cdot}(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ is preserved by complex conjugation, which we already knew by Proposition 3.21, and we also obtain the second equation in (6.10), and (6.11).

Also $\theta R_+ = -\overline{R}_+$ is a positive root system, and the corresponding half-sum of roots is given by $\theta\rho^{\mathfrak{g}}$. As in (6.14), if $L \in Z(\mathfrak{g}_{\mathbf{C}})$, then

$$\phi_{1,\theta R_+}\theta L = \theta\phi_{1,R_+} L. \tag{6.18}$$

If $f \in S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$, then

$$\phi_{2,\theta R_+}\theta f = \theta\phi_{2,R_+} f. \tag{6.19}$$

By (6.18), (6.19), we conclude that

$$\phi_{2,\theta R_+}\phi_{1,\theta R_+}\theta L = \theta\phi_{2,R_+}\phi_{1,R_+} L. \tag{6.20}$$

Using again the result of Harish-Chandra, from (6.20), we obtain the third equation in (6.10).

If $f, h \in C^{\infty,c}(G, \mathbf{R})$, the convolution $f * h \in C^{\infty,c}(G, \mathbf{R})$ is defined by the formula,

$$f * h(g) = \int_G f\left(g^{-1}g'\right) h(g')\, dg'. \tag{6.21}$$

If $A \in D_R(G), B \in D_L(G)$, we get easily

$$f * Ah = A(f * h), \quad f * Bh = (B^* f) * h, \quad B(f * h) = ((\sigma B) f) * h. \tag{6.22}$$

By (6.2), (6.22), we conclude that if $L \in Z(\mathfrak{g}_{\mathbf{C}})$, then

$$(L^* f) * h = ((\sigma L) f) * h. \tag{6.23}$$

from which we get

$$L^* f = \sigma L f. \tag{6.24}$$

This completes the proof of our theorem. □

6.4. **The Duflo and the Harish-Chandra isomorphisms.** Here, $S^{\cdot}[[\mathfrak{g}^*]]$ denotes the algebra of formal power series $\alpha = \sum_{i=0}^{+\infty} \alpha_i, \alpha_i \in S^i[\mathfrak{g}^*]$. Then $\mathfrak{g}$ still acts on $S^{\cdot}[[\mathfrak{g}^*]]$ as an algebra of derivations. Let $I^{\cdot}[[\mathfrak{g}^*]]$ be the subalgebra of invariant elements in $S^{\cdot}[[\mathfrak{g}^*]]$.

As in Subsection 3.1, $S^{\cdot}[[\mathfrak{g}^*]]$ can be identified with the algebra $D^{\cdot}[[\mathfrak{g}^*]]$ of formal real partial differential operators with constant coefficients on $\mathfrak{g}^*$, and $I^{\cdot}[[\mathfrak{g}^*]]$ with the algebra of formal real invariant differential operators with constant coefficients $D_I^{\cdot}[[\mathfrak{g}^*]]$, which acts on $S^{\cdot}(\mathfrak{g})$.

Then $\widehat{A}^{-1}(\mathrm{ad}(\cdot)) \in I^{\cdot}[[\mathfrak{g}^*]]$. In the sequel, we view $\widehat{A}^{-1}(\mathrm{ad}(\cdot))$ as an element of $D_I^{\cdot}[[\mathfrak{g}^*]]$.

Let τ_{PBW} be the Poincar-Birkhoff-Witt isomorphism of filtered vector spaces $S^{\cdot}(\mathfrak{g}) \simeq U(\mathfrak{g})$. Then τ_{PBW} induces an identification of filtered vector spaces $I^{\cdot}(\mathfrak{g}) \simeq Z(\mathfrak{g})$.

Definition 6.2. Put

$$\tau_{\mathrm{D}} = \tau_{\mathrm{PBW}} \widehat{A}^{-1}(\mathrm{ad}(\cdot)) : S^{\cdot}(\mathfrak{g}) \to U(\mathfrak{g}). \tag{6.25}$$

Then τ_{D} is an isomorphism of filtered vector spaces, which commutes with θ.

A result by Duflo [Du70, Thorme V.2] asserts that when restricted to $I^{\cdot}(\mathfrak{g})$, τ_{D} induces an isomorphism of filtered algebras,

$$I^{\cdot}(\mathfrak{g}) \simeq Z(\mathfrak{g}). \tag{6.26}$$

By [Du70, Lemme V.1], we have the commutative diagram

$$\begin{array}{ccc} I^{\cdot}(\mathfrak{g}_{\mathbf{C}}) & \xrightarrow{\tau_{\mathrm{D}}} & Z(\mathfrak{g}_{\mathbf{C}}) \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle \phi_{\mathrm{HC}}} & \\ & I(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}) & \end{array} . \tag{6.27}$$

By Theorem 6.1 and by (6.27), we get the commutative diagram

$$\begin{array}{ccc} I(\mathfrak{g}) & \xrightarrow{\tau_{\mathrm{D}}} & Z(\mathfrak{g}) \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle \phi_{\mathrm{HC}}} & \\ & I(\mathfrak{h}, \mathfrak{g}) & \end{array} , \tag{6.28}$$

and the morphisms in (6.28) commute with θ.

6.5. **The case of the Casimir.** Note that $B^*|_{\mathfrak{h}} \in I^2(\mathfrak{h}, \mathfrak{g})$ corresponds to the Laplacian $\Delta^{\mathfrak{h}}$ on $\mathfrak{h}$ associated with $B|_{\mathfrak{h}}$. The following result of Harish-Chandra is established in [Kn02, Example 5.64] as a consequence of the constructions in Subsection 6.2.

Proposition 6.3. *We have the identity:*

$$\phi_{\mathrm{HC}} C^{\mathfrak{g}} = -\Delta^{\mathfrak{h}} + B^{*}\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right). \tag{6.29}$$

Proposition 6.4. *The following identity holds:*

$$\tau_{\mathrm{D}}^{-1} C^{\mathfrak{g}} = -B^{*} + B^{*}\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right). \tag{6.30}$$

Proof. Clearly,

$$\widehat{A}^{-1}(x) = 1 + \frac{1}{24} x^{2} + \ldots \tag{6.31}$$

Also $B^{*} \in I^{2}(\mathfrak{g})$. Let $e_{1} \ldots, e_{m+n}$ be a basis of $\mathfrak{g}$, and let $e_{1}^{*}, \ldots, e_{m+n}^{*}$ be the basis of $\mathfrak{g}$ which is dual with respect to B. By (6.31), we get

$$\widehat{A}^{-1}(\mathrm{ad}(\cdot)) B^{*} = B^{*} + \frac{1}{24} \mathrm{Tr}^{\mathfrak{g}}\left[\mathrm{ad}(e_{i}) \mathrm{ad}(e_{j})\right] B^{*}\left(e_{i}^{*}, e_{j}^{*}\right). \tag{6.32}$$

Equation (6.32) can be written in the form

$$\widehat{A}^{-1}(\mathrm{ad}(\cdot)) B^{*} = B^{*} - \frac{1}{24} \mathrm{Tr}^{\mathfrak{g}}\left[C^{\mathfrak{g},\mathfrak{g}}\right]. \tag{6.33}$$

By (2.46), we can rewrite (6.33) in the form

$$\widehat{A}^{-1}(\mathrm{ad}(\cdot)) B^{*} = B^{*} + B^{*}\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right). \tag{6.34}$$

By (6.25), (6.34), we get

$$\tau_{\mathrm{D}} B^{*} = -C^{\mathfrak{g}} + B^{*}\left(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}\right), \tag{6.35}$$

which is equivalent to (6.30). The proof of our proposition is completed. □

Remark 6.5. Using (6.28), Propositions 6.3 and 6.4 can be derived from each other.

6.6. **The action of $Z(\mathfrak{g})$ on $C^{\infty}(X, F)$.** Note that G acts on the left on $C^{\infty}(G, E)$ as in (6.1), and there is a corresponding action of $D_{R}(G)$. Also K acts on $C^{\infty}(G, E)$ by the formula

$$k_{R} s(g) = \rho^{E}(k)\, s(gk), \tag{6.36}$$

and this action of K commutes with the left action of G. Moreover, we have the identity

$$C^{\infty}(X, F) = \left[C^{\infty}(G, E)\right]^{K}. \tag{6.37}$$

Then $D_{R}(G)$ commutes with the action of K on $C^{\infty}(G, E)$. As a subalgebra of $D_{R}(G)$, $Z(\mathfrak{g})$ also acts on $C^{\infty}(G, E)$, and its action commutes with the left action of G and with the right action of K. The action of $D_{R}(G)$ descends to $C^{\infty}(X, F)$, so that the action of $Z(\mathfrak{g})$ descends to $C^{\infty}(X, F)$ and commutes with the left action of G.

6.7. **The semisimple orbital integrals involving $Z(\mathfrak{g})$.** Let $\mathscr{S}$ be the algebra of differential operators acting on $C^\infty(X,F)$ with uniformly bounded coefficients together with their derivatives of any order.[10]

Definition 6.6. Let $C^{\infty,b}(X,F)$ be the vector space of smooth sections of F on X which are bounded together with their covariant derivatives of any order.

Let $\mathcal{Q}$ be the space of smooth kernels $Q(x,x')|_{x,x'\in X}$ acting on $C^{\infty,b}(X,F)$ and commuting with the left action of G such that there exists $C>0$, and for any $S,S'\in\mathscr{S}$, there exists $C_{S,S'}>0$ for which

$$|SQS'(x,x')|\leq C_{S,S'}\exp\left(-Cd^2(x,x')\right). \tag{6.38}$$

The same arguments as in [B11, Proposition 4.1.2] shows that the vector space $\mathcal{Q}$ is an algebra with respect to the composition of operators.

In particular $D_R(G)$ commutes with Q, and so $Z(\mathfrak{g})$ commutes with Q.

Proposition 6.7. *If $L\in Z(\mathfrak{g}), Q\in\mathcal{Q}$, then $LQ\in\mathcal{Q}$.*

Proof. Since $L\in Z(\mathfrak{g})$, L commutes with the left action of G, and so LQ commutes with this action of G. We fix $x_0=p1\in G$. Since LQ commutes with G, and G acts isometrically on X, to establish (6.38) for LQ, we may as well take $x=x_0$. If $U\in\mathfrak{g}$, and if U^X is the corresponding vector field on X, since U^X is a Jacobi field along the geodesics in X, there exist $C>0, c>0$ such that

$$\left|U^X(x)\right|\leq C\exp\left(cd(x_0,x)\right). \tag{6.39}$$

The above estimate is also valid for the corresponding covariant derivatives. From (6.38), we get the estimate (6.38) for LQ when $x=x_0$. The proof of our proposition is completed. □

Let $\gamma\in G$ be semisimple. By (2.58), (6.38), we have the analogue of (2.59), i.e., if $f\in\mathfrak{p}^\perp(\gamma)$, then

$$\left|Q\left(\gamma^{-1}e^f x_0,e^f x_0\right)\right|\leq C_\gamma\exp\left(-c_\gamma|f|^2\right). \tag{6.40}$$

In [B11, Definition 4.2.2], if $\gamma\in G$ is semisimple, if $Q\in\mathcal{Q}$, the orbital integral $\mathrm{Tr}^{[\gamma]}[Q]$ is defined by a formula similar to (2.60), i.e.,

$$\mathrm{Tr}^{[\gamma]}[Q]=\int_{\mathfrak{p}^\perp(\gamma)}\mathrm{Tr}\left[\gamma_* Q\left(\gamma^{-1}e^f x_0,e^f x_0\right)\right]r(f)\,df. \tag{6.41}$$

[10] These are linear combinations of operators $\nabla^F_{U_1}\ldots\nabla^F_{U_k}$, where $U_1,\ldots,U_k$ are smooth bounded vector fields with uniformly bounded covariant derivatives of any order.

The estimates (2.57), (6.40) guarantees that the integral in (6.41) is well-defined.

By Proposition 6.7, if $L \in Z(\mathfrak{g})$, $LQ \in \mathcal{Q}$, and so $\mathrm{Tr}^{[\gamma]}[LQ]$ is also well-defined.

7. The center of $U(\mathfrak{g})$ and the regular orbital integrals

The purpose of this Section is to evaluate the orbital integrals for kernels of the form $L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)$ associated with regular elements in G, when $L \in Z(\mathfrak{g})$. To establish our formula, we will use the main result of [B11] described in Theorem 2.9, the smoothness properties of the function $\mathcal{J}_\gamma(h_\mathfrak{t})$ that were obtained in Section 5, and the Harish-Chandra isomorphism in the form given in equation (6.9).

This section is organized as follows. In Subsection 7.1, we recall the classical result of Harish-Chandra [HC57b, Theorem 3], [HC66, Section 18] that expresses certain orbital integrals on H^{reg} via the action of $Z(\mathfrak{g})$ on the orbital integral as a function of γ.

In Subsection 7.2, using Theorem 2.9, we obtain our formula.

7.1. The algebra $Z(\mathfrak{g})$ and the regular orbital integrals.

Let $L \in Z(\mathfrak{g}), Q \in \mathcal{Q}$, by Proposition 6.7, $LQ \in \mathcal{Q}$. If $f \in C^\infty(G^{\mathrm{reg}}, \mathbf{C})$, Lf is a smooth function on G^{reg}. For greater clarity, this function will be denoted instead $L_\gamma f$.

Now we give another proof of a result of Harish-Chandra [HC57b, Theorem 3], [HC66, Section 18], [Kn86, Proposition 11.9].

Proposition 7.1. *If $Q \in \mathcal{Q}$, the map $\gamma \in G^{\mathrm{reg}} \to \mathrm{Tr}^{[\gamma]}[Q]$ is smooth. If $L \in Z(\mathfrak{g})$, we have the identity of smooth functions on G^{reg}:*

$$\mathrm{Tr}^{[\gamma]}[LQ] = (\sigma L)_\gamma \mathrm{Tr}^{[\gamma]}[Q]. \tag{7.1}$$

Proof. The map $(\gamma, g) \in H^{\mathrm{reg}} \times G/H \to g^{-1}\gamma g \in G^{\mathrm{reg}}$ is locally a diffeomorphism. Since $\mathrm{Tr}^{[\gamma]}[Q]$ is invariant by conjugation, to obtain the required smoothness, it is enough to prove that $\gamma \in H^{\mathrm{reg}} \to \mathrm{Tr}^{[\gamma]}[Q] \in \mathbf{C}$ is smooth.

Put

$$K^0(H) = H^0 \cap K. \tag{7.2}$$

As we saw in Subsection 2.2, $K^0(H)$ is a maximal compact subgroup of H^0.

Using the notation in Subsection 5.1, if $\gamma \in H^{\mathrm{reg}}$, and if $\gamma' \in H$ is close to γ, then $\gamma' \in H^{\mathrm{reg}}$ and (5.7) holds. Using equation (2.31) in Theorem 2.2, we deduce that

$$X(\gamma') = H^0/K^0(H). \tag{7.3}$$

In particular $X(\gamma')$ does not depend on γ', and $\mathfrak{p}^{\perp}(\gamma') = \mathfrak{h}_{\mathfrak{p}}^{\perp}$. By (6.41), we get

$$\mathrm{Tr}^{[\gamma']}[Q] = \int_{\mathfrak{h}_{\mathfrak{p}}^{\perp}} \mathrm{Tr}\left[\gamma'_* Q\left(\gamma'^{-1} e^f x_0, e^f x_0\right)\right] r(f)\, df. \tag{7.4}$$

For $\gamma' \in H$ close enough to $\gamma \in H$, it is elementary to make the estimate in [B11, Theorem 3.4.1], which was explained in (2.58), uniform, so that there exists $C > 0$ such that if $\gamma' \in H^{\mathrm{reg}}$ is close enough to γ, if $f \in \mathfrak{h}_{\mathfrak{p}}^{\perp}, |f| \geq 1$,

$$d_{\gamma'}\left(e^f x_0\right) \geq C|f|. \tag{7.5}$$

By combining (6.38) with $S = 1, S' = 1$ and (7.5), for $\gamma' \in H$ close enough to γ, there exist $C > 0, c > 0$ such that if $f \in \mathfrak{h}_{\mathfrak{p}}^{\perp}, |f| \geq 1$, we get

$$\left|\gamma_* Q\left(\gamma^{-1} e^f x_0, e^f x_0\right)\right| \leq C \exp\left(-c|f|^2\right). \tag{7.6}$$

Using dominated convergence, by (2.57), (7.6), we deduce that $\mathrm{Tr}^{[\gamma]}[Q]$ is a continuous function of $\gamma \in H^{\mathrm{reg}}$.

Let us now prove the above function is smooth on H^{reg}. The argument is essentially the same as before, by combining the estimates in (6.38) with S arbitrary, together with uniform estimates given in (6.39).

Equation (7.1) just reflects the fact that σL is the image of L by the map $g \to g^{-1}$. □

7.2. **A geometric formula for the regular orbital integrals.** In the sequel we take the function $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ as in Subsection 2.5. Let $A \in \mathbf{R}, L \in Z(\mathfrak{g})$.

Put

$$\mathfrak{h}_i = \mathfrak{h}_{\mathfrak{p}} \oplus i\mathfrak{h}_{\mathfrak{k}}. \tag{7.7}$$

Recall that $\phi_{\mathrm{HC}} L \in D_I'(\mathfrak{h})$. This differential operator acts on smooth functions on $\mathfrak{h}$, but as explained in Subsection 3.1, it also acts on smooth functions on $\mathfrak{h}_i$.

In the next statement, the smooth kernel $(\phi_{\mathrm{HC}} L)\, \mu\left(\sqrt{\phi_{\mathrm{HC}} C^{\mathfrak{g}} + A}\right)$ on $\mathfrak{h}_i$ acts on the distribution

$$\mathcal{J}_\gamma(h_{\mathfrak{k}})\, \mathrm{Tr}^E\left[\rho^E\left(k^{-1} e^{-h_{\mathfrak{k}}}\right)\right] \delta_a.$$

Theorem 7.2. *The following identity holds on H^{reg}:*

$$\begin{aligned} &\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X} + A}\right)\right] \\ &\quad = (\phi_{\mathrm{HC}} L)\, \mu\left(\sqrt{\phi_{\mathrm{HC}} C^{\mathfrak{g}} + A}\right)\left[\mathcal{J}_\gamma(h_{\mathfrak{k}})\, \mathrm{Tr}^E\left[\rho^E\left(k^{-1} e^{-h_{\mathfrak{k}}}\right)\right] \delta_a\right](0). \end{aligned} \tag{7.8}$$

Proof. If $\gamma \in H^{\mathrm{reg}}$, then $\mathfrak{z}(\gamma) = \mathfrak{h}$, and $\mathfrak{k}(\gamma) = \mathfrak{h}_{\mathfrak{k}}$. When $L = 1$, our theorem is just Theorem 2.9 combined with Proposition 6.3. When $L \in Z(\mathfrak{g})$ is arbitrary, we use equation (6.9) and Proposition 7.1. Here $\phi_{\mathrm{HC}}\sigma L$ is a differential operator on $\mathfrak{h}$. We find that on H^{reg},

$$\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right] = \frac{1}{D_H(\gamma)}\left(\phi_{\mathrm{HC}}\sigma L\right)\left[D_H(\gamma)\, r\mathrm{Tr}^{[\gamma]}\left[\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right]. \tag{7.9}$$

By Theorem 6.1, we get

$$\phi_{\mathrm{HC}}\sigma L = \left(\phi_{\mathrm{HC}}L\right)^{*}. \tag{7.10}$$

We combine Theorem 2.9 and equations (6.29), (7.9), and (7.10). We get

$$\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right] = \frac{1}{D_H(\gamma)}\left(\phi_{\mathrm{HC}}L\right)^{*} \left[D_H(\gamma)\, r\mu\left(\sqrt{\phi_{\mathrm{HC}}C^{\mathfrak{g}}+A}\right)\left[\mathcal{J}_\gamma(h_{\mathfrak{k}})\,\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]\delta_a\right](0)\right]. \tag{7.11}$$

For greater clarity, we fix $\gamma \in H^{\mathrm{reg}}$, and for $b \in \mathfrak{h}$ with $|b|$ small enough, we take $\gamma' = \gamma e^{b}$ as in (5.2), so that the differential operator $\left(\phi_{\mathrm{HC}}L\right)^{*}$ acts on the variable $b \in \mathfrak{h}$. This action will be now be denoted $\left(\phi_{\mathrm{HC}}L\right)^{*}_{b}$. Also we use the notation of Subsection 5.1.

By equation (5.3) and equation (5.20) in Theorem 5.4, and we get

$$D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}}) = (-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|}\epsilon_D(\gamma')\prod_{\alpha\in R^{\mathrm{re}}_{+}}\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}}\right) \frac{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}}\left(\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right)}{\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}}\left(\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right)}. \tag{7.12}$$

By the same argument as in (5.21), (5.22), if $\alpha \in R^{\mathrm{re}}_{+}$, we can choose $\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}}\right)$ so that for $|b_{\mathfrak{k}}|$ small enough,

$$\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}}\right) = \xi_\alpha^{1/2}\left(k^{-1}\right), \tag{7.13}$$

i.e., (7.13) does not depend on b. Similarly, $\epsilon_D(\gamma')$ is locally constant on H^{reg}. Therefore, the product of these terms in the right hand-side of (7.12) is unaffected by the action of $\left(\phi_{\mathrm{HC}}L\right)^{*}_{b}$ in the right-hand side of (7.11).

Also we have the identity,

$$\mathrm{Tr}^{E}\left[\rho^{E}\left(k'^{-1}e^{-h_{\mathfrak{k}}}\right)\right] = \mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right]. \tag{7.14}$$

The right-hand sides of (7.12) and (7.14) depend on $b_{\mathfrak{k}} - h_{\mathfrak{k}}$. When only considering the action of $(\phi_{\mathrm{HC}} L)_b^*$ in the variable $b_{\mathfrak{k}}$, this action can instead be transferred to the variable $h_{\mathfrak{k}}$ with a correcting sign. This argument still does not take into account the fact that $(\phi_{\mathrm{HC}} L)_b^*$ also acts in the variable $b_{\mathfrak{p}}$. However, differentiating a smooth kernel at the terminal point is equivalent to compose the smooth kernel with the same change of signs as before. By combining (7.11)–(7.14), this ultimately explains the disappearance of $*$, and to equation (7.8). The proof of our theorem is completed. □

8. The function $\mathcal{J}_\gamma$ and the limit of regular orbital integrals

In this section, we verify the compatibility of our formula for regular orbital integrals of Theorem 7.2 to the limit theorems obtained by Harish-Chandra for such orbital integrals. Key properties of the function $\mathcal{J}_\gamma$ play a key role in the proofs.

This section is organized as follows. In Subsection 8.1, given $\gamma \in H$ not necessarily regular, we study the function $\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})$ for $\gamma' \in H^{\mathrm{reg}}$ close to γ.

In Subsection 8.2, if $L \in Z(\mathfrak{g})$, we define the proper image $L^{\mathfrak{z}(\gamma)} \in D_I^{\cdot}(\mathfrak{z}(\gamma))$.

In Subsection 8.3, using a formula by Rossmann, we express a smooth kernel involving $\Delta^{\mathfrak{h}}$ as the restriction of another kernel for $\mathfrak{z}(\gamma)$.

Finally, in Subsection 8.4, we compute the limit of orbit integrals as $\gamma' \in H^{\mathrm{reg}}$ converges to $\gamma \in H$.

8.1. The function $\mathcal{J}_\gamma$ when γ is not regular.

Let $\gamma \in G$ be a semisimple element as in (2.19). Then $Z^0(\gamma) \subset G$ is a reductive Lie group. Let $\mathfrak{h} \subset \mathfrak{z}(\gamma)$ be a θ-stable Cartan subalgebra of $\mathfrak{z}(\gamma)$. As we saw in Proposition 3.17, $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let $H \subset G$ be the corresponding Cartan subgroup. Recall that the function $\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}$ on $\mathfrak{h}_{\mathbf{C}}$ was introduced in (5.11).

Definition 8.1. An element $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$ is said to be γ im-regular if for $\alpha \in R^{\mathrm{im}}(k)$, $\langle \alpha, h_{\mathfrak{k}} \rangle \neq 0$.

The vanishing locus of an imaginary root being a hyperplane in $i\mathfrak{h}_{\mathfrak{k}}$, the set of γ im-regular elements has full Lebesgue measure.

We use the conventions of Section 4, where we explained in particular how to choose the $\xi_\alpha^{1/2}(\gamma)|_{\alpha \in R_+}$. We extend the definition of $D_H(\gamma)$ for $\gamma \in H^{\mathrm{reg}}$ in Definition 5.3 to general elements $\gamma \in H$ by the formula

$$D_H(\gamma) = \prod_{\alpha \in R_+ \setminus R_+(\gamma)} \left(\xi_\alpha^{1/2}(\gamma) - \xi_\alpha^{-1/2}(\gamma)\right). \tag{8.1}$$

By Theorem 3.18, if $\alpha \in R_+^{\mathrm{re}}(\gamma) \cup R_+^{\mathrm{im}}(k)$, then $\xi_\alpha(k^{-1}) = 1$, and $\xi_\alpha^{1/2}(k^{-1}) = \xi_\alpha^{-1/2}(k^{-1}) = \pm 1$, so that $\prod_{\alpha \in R_+^{\mathrm{re}}(\gamma) \cup R_+^{\mathrm{im}}(k)} \xi_\alpha^{1/2}(k^{-1})$ is equal to ± 1.

If $\mathfrak{h}$ is a θ-stable fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, we will use the notation introduced in Subsection 3.6, except that $\mathfrak{g}$ is now replaced by $\mathfrak{z}(\gamma)$, and the pair $(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k})$ is replaced by the pair $(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k}(\gamma))$. Let $R(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k}(\gamma))$ denote the associated root system. Let $R_+(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k}(\gamma))$ denote a positive root system. As in (3.45), if $h_{\mathfrak{k}} \in \mathfrak{h}_{\mathfrak{k}}$, set

$$\pi^{\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k}(\gamma)}(h_{\mathfrak{k}}) = \prod_{\beta \in R_+(\mathfrak{h}_{\mathfrak{k}}, \mathfrak{k}(\gamma))} \langle \beta, h_{\mathfrak{k}} \rangle. \tag{8.2}$$

By (3.49), we get

$$\left[\pi^{h_{\mathfrak{k}}, \mathfrak{k}(\gamma)}(h_{\mathfrak{k}})\right]^2 = (-1)^{\frac{1}{2}\left|R_+^{\mathrm{c}}(\gamma)\right|} \left[\prod_{\alpha \in R_{\mathfrak{k},+}^{\mathrm{im}}(k)} \langle \alpha, h_{\mathfrak{k}} \rangle\right]^2 \prod_{\alpha \in R_+^{\mathrm{c}}(\gamma)} \langle \alpha, h_{\mathfrak{k}} \rangle. \tag{8.3}$$

We no longer assume $\mathfrak{h}$ to be fundamental in $\mathfrak{z}(\gamma)$.

In the sequel, we use the notation of Subsection 5.1. In particular $\gamma \in H$ is fixed, and for $b \in \mathfrak{h}$, $\gamma' = \gamma e^b$. In particular, equations (5.3), (5.4) hold.

We establish the following important result.

Theorem 8.2. *For $\epsilon > 0$ small enough, there exist $c > 0, C > 0$ such that for $b \in \mathfrak{h}$ γ-regular with $|b| \leq \epsilon$, $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, then*

$$\left|\pi^{\mathfrak{h}, \mathfrak{z}(\gamma)}(b_{\mathfrak{p}} + h_{\mathfrak{k}}) D_H(\gamma') \mathcal{J}_{\gamma'}(h_{\mathfrak{k}})\right| \leq C \exp\left(c\left|h_{\mathfrak{k}}\right|\right). \tag{8.4}$$

If $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$ is not γ im-regular, the left-hand side of (8.4) vanishes.

If $\mathfrak{h}$ is not the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, for $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, as $b \in \mathfrak{h}$ γ-regular tends to 0,

$$\pi^{\mathfrak{h}, \mathfrak{z}(\gamma)}(b_{\mathfrak{p}} + h_{\mathfrak{k}}) D_H(\gamma') \mathcal{J}_{\gamma'}(h_{\mathfrak{k}}) \to 0. \tag{8.5}$$

If $\mathfrak{h}$ is the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, if $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$ is γ im-regular, as $b \in \mathfrak{h}$ γ-regular tends to 0,

$$\begin{aligned}(8.6)\quad \pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(b_{\mathfrak{p}}+h_{\mathfrak{k}})\,D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})&\\ \to(-1)^{\frac{1}{2}\left|R^{\mathrm{c}}_{+}(\gamma)\right|+\left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right|}&\left[\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}(\gamma)}(h_{\mathfrak{k}})\right]^2\\ &\prod_{\alpha\in R^{\mathrm{im}}_{+}(k)}\xi_\alpha^{1/2}\left(k^{-1}\right)D_H(\gamma)\,\mathcal{J}_\gamma(h_{\mathfrak{k}}).\end{aligned}$$

Proof. By (5.11), we get

$$(8.7)\qquad \pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(b_{\mathfrak{p}}+h_{\mathfrak{k}})=\prod_{\alpha\in R_+(\gamma)}\langle\alpha,b_{\mathfrak{p}}+h_{\mathfrak{k}}\rangle.$$

We can rewrite (8.7), in the form

(8.8)

$$\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(b_{\mathfrak{p}}+h_{\mathfrak{k}})=\prod_{\alpha\in R^{\mathrm{re}}_{+}(\gamma)}\langle\alpha,b_{\mathfrak{p}}\rangle\prod_{\alpha\in R^{\mathrm{im}}_{+}(k)}\langle\alpha,h_{\mathfrak{k}}\rangle\prod_{\alpha\in R^{\mathrm{c}}_{+}(\gamma)}\langle\alpha,b_{\mathfrak{p}}+h_{\mathfrak{k}}\rangle.$$

By (8.8), we deduce that if $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$ is not γ im-regular, (8.8) vanishes.

If $\alpha \in R^{\mathrm{im}}(k)$, then $\xi_\alpha(k^{-1})=1$, so that $\xi_\alpha^{1/2}(k^{-1})=\pm 1$. Since $\alpha \in R(\gamma)$, if $b \in \mathfrak{h}$ is γ-regular, then $\langle\alpha,b_{\mathfrak{k}}\rangle \neq 0$. If $\epsilon>0$ is small enough, if $b \in \mathfrak{h}$ is γ-regular, and $|b_{\mathfrak{k}}| \leq \epsilon$, if $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$, then $1-e^{-\langle\alpha,b_{\mathfrak{k}}-h_{\mathfrak{k}}\rangle}\neq 0$, so that the following expression is well-defined,

$$\begin{aligned}(8.9)\quad \prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}&\frac{\langle\alpha,h_{\mathfrak{k}}\rangle}{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}\\ &=\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{\langle\alpha,h_{\mathfrak{k}}\rangle}{1-e^{-\langle\alpha,b_{\mathfrak{k}}-h_{\mathfrak{k}}\rangle}}\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right).\end{aligned}$$

Equation (8.9) can be rewritten in the form

$$\begin{aligned}(8.10)\quad \prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}&\frac{\langle\alpha,h_{\mathfrak{k}}\rangle}{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}\\ &=\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{\langle\alpha,h_{\mathfrak{k}}\rangle/2}{\sinh\left(\langle\alpha,b_{\mathfrak{k}}-h_{\mathfrak{k}}\rangle/2\right)}\xi_\alpha^{-1/2}\left(k^{-1}\right).\end{aligned}$$

When $h_{\mathfrak{k}}$ is γ im-regular, we have an identity similar to (8.10),

$$\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\frac{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}{\langle\alpha,h_{\mathfrak{k}}\rangle} = \prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\frac{\sinh\left(\langle\alpha,b_{\mathfrak{k}}-h_{\mathfrak{k}}\rangle/2\right)}{\langle\alpha,h_{\mathfrak{k}}\rangle/2}\xi_\alpha^{1/2}\left(k^{-1}\right). \tag{8.11}$$

If $x\in\mathbf{R}, y\in\mathbf{R}$,

$$\left|1-e^{x+iy}\right|\geq\left|1-e^{x}\right|. \tag{8.12}$$

By (8.12), we deduce that

$$\left|\sinh\left((x+iy)/2\right)\right|\geq\left|\sinh\left(x/2\right)\right|. \tag{8.13}$$

By (8.13), there exists $C>0$ such that if $e^{x+iy}\neq 1$,

$$\left|\frac{x}{\sinh\left((x+iy)/2\right)}\right|\leq C. \tag{8.14}$$

By (8.10), (8.14), if $h_{\mathfrak{k}}\in i\mathfrak{h}_{\mathfrak{k}}$ is γ im-regular, we get

$$\left|\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{\langle\alpha,h_{\mathfrak{k}}\rangle}{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}\right|\leq C. \tag{8.15}$$

The γ-regularity of b does not play any role in the above estimate.

If $\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)$, then $\xi_\alpha\left(k^{-1}\right)\neq 1$, so that for $\epsilon>0$ small enough, if $|b_{\mathfrak{k}}|\leq\epsilon$, the complex numbers $\xi_\alpha\left(k^{-1}e^{b_{\mathfrak{k}}}\right)$ which have module 1, stay away from 1. Given $\eta\in\left]0,\pi\right[$, there is $C_\eta>0$ such that if $x\in\mathbf{R}_+, y\in[\eta,2\pi-\eta]$, then

$$\left|xe^{iy}-x^{-1}e^{-iy}\right|\geq C_\eta. \tag{8.16}$$

By (8.16), we deduce that for $\epsilon>0$ small enough, if $|b_{\mathfrak{k}}|\leq\epsilon$, and $h_{\mathfrak{k}}\in i\mathfrak{h}_{\mathfrak{k}}$, then

$$\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{1}{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)} = \prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}{1-e^{-\langle\alpha,b_{\mathfrak{k}}-h_{\mathfrak{k}}\rangle}\xi_\alpha^{-1}\left(k^{-1}\right)} \tag{8.17}$$

is well-defined, and moreover,

$$\left|\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}\setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{1}{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}\right|\leq C. \tag{8.18}$$

By combining (5.20), (8.8), (8.15), and (8.18), we get (8.4), and also the fact that if $h_{\mathfrak{k}}\in i\mathfrak{h}_{\mathfrak{k}}$ is not γ im-regular, the left-hand side of (8.4) vanishes.

By Proposition 3.7, $\mathfrak{h}$ is the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$ if and only if $R^{\mathrm{re}}(\gamma)$ is empty.

If $\mathfrak{h}$ is not the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, $R^{\mathrm{re}}_+(\gamma)$ is nonempty. Using (5.20), (8.8) and the previous bounds, we get (8.5).

From now on, we assume that $\mathfrak{h}$ is the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$. By (4.10), since $R^{\mathrm{re}}_+(\gamma)$ is empty, we get

$$\epsilon_D(\gamma)=\mathrm{sgn}\prod_{\alpha\in R^{\mathrm{re}}_+}\left(1-\xi_\alpha^{-1}(\gamma)\right). \tag{8.19}$$

For $\alpha\in R^{\mathrm{re}}_+$, then $\xi_\alpha(\gamma)\neq 1$ so that for $\epsilon>0$ small enough, if $|b_{\mathfrak{p}}|\leq\epsilon$,

$$\epsilon_D(\gamma')=\epsilon_D(\gamma). \tag{8.20}$$

Also if $\alpha\in R^{\mathrm{re}}_+$, $\xi_\alpha(k')=\xi_\alpha(k)$. By the above, it follows that for $b\in\mathfrak{h}$ γ-regular close enough to 0,

$$\epsilon_D(\gamma')\prod_{\alpha\in R^{\mathrm{re}}_+}\xi_\alpha^{1/2}\left(k'^{-1}\right)=\epsilon_D(\gamma)\prod_{\alpha\in R^{\mathrm{re}}_+}\xi_\alpha^{1/2}\left(k^{-1}\right). \tag{8.21}$$

By (8.10), (8.11), and (8.14), if $h_{\mathfrak{k}}\in i\mathfrak{h}_{\mathfrak{k}}$ is γ im-regular, if $b\in\mathfrak{h}$ γ-regular tends to 0,

$$\begin{aligned}&\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\frac{\langle\alpha,h_{\mathfrak{k}}\rangle}{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}\\&\quad\to(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}(k)\right|}\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\widehat{A}\left(\langle\alpha,h_{\mathfrak{k}}\rangle\right)\xi_\alpha^{-1/2}\left(k^{-1}\right),\\&\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\frac{\xi_\alpha^{1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)}{\langle\alpha,h_{\mathfrak{k}}\rangle}\\&\quad\to(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right|}\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\frac{\xi_\alpha^{1/2}\left(k^{-1}\right)}{\widehat{A}\left(\langle\alpha,h_{\mathfrak{k}}\rangle\right)}.\end{aligned} \tag{8.22}$$

By (5.20), by the considerations after (8.1), by (8.3), (8.8), (8.21), and (8.22), we deduce that if $h_{\mathfrak{k}} \in i\mathfrak{h}_{\mathfrak{k}}$ is γ im-regular, then

$$\begin{aligned}(8.23)\quad & \pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(b_{\mathfrak{p}}+h_{\mathfrak{k}}\right) D_H\left(\gamma'\right) \mathcal{J}_{\gamma'}\left(h_{\mathfrak{k}}\right) \\ & \quad \to \epsilon_D(\gamma)(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|+\left|R^{\mathrm{im}}_{+}(k)\right|+\frac{1}{2}\left|R^{\mathrm{c}}_{+}(\gamma)\right|} \\ & \prod_{\alpha \in R^{\mathrm{re}}_{+} \cup R^{\mathrm{im}}_{+}(k)} \xi_\alpha^{1/2}\left(k^{-1}\right)\left[\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}(\gamma)}\left(h_{\mathfrak{k}}\right)\right]^2 \frac{\prod_{\alpha \in R^{\mathrm{im}}_{\mathfrak{p},+}(k)} \widehat{A}\left(\langle \alpha, h_{\mathfrak{k}}\rangle\right)}{\prod_{\alpha \in R^{\mathrm{im}}_{\mathfrak{k},+}(k)} \widehat{A}\left(\langle \alpha, h_{\mathfrak{k}}\rangle\right)} \\ & \quad \frac{\prod_{\alpha \in R^{\mathrm{im}}_{\mathfrak{k},+} \setminus R^{\mathrm{im}}_{\mathfrak{k},+}(k)}\left(\xi_\alpha^{1/2}\left(k^{-1} e^{-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1} e^{-h_{\mathfrak{k}}}\right)\right)}{\prod_{\alpha \in R^{\mathrm{im}}_{\mathfrak{p},+} \setminus R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\left(\xi_\alpha^{1/2}\left(k^{-1} e^{-h_{\mathfrak{k}}}\right)-\xi_\alpha^{-1/2}\left(k^{-1} e^{-h_{\mathfrak{k}}}\right)\right)}.\end{aligned}$$

By comparing (4.34) and the right-hand side of (8.23), we get (8.6). The proof of our theorem is completed. □

8.2. **The Lie algebra $\mathfrak{z}(\gamma)$ and the isomorphisms of Harish-Chandra and Duflo.** Here, we will use results contained in Subsections 3.11, 6.1, and 6.4.

Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{z}(\gamma)$. We have the Harish-Chandra and Duflo isomorphisms of filtered algebras:

$$(8.24)\qquad \phi_{\mathrm{HC}} : Z(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h},\mathfrak{g}), \qquad \tau_{\mathrm{D}} : I^{\cdot}(\mathfrak{g}) \simeq Z(\mathfrak{g}).$$

As explained in Subsection 6.4, the above isomorphisms are compatible.

By (3.77), we have the isomorphisms,

$$(8.25)\qquad r : I^{\cdot}(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h},\mathfrak{g}), \qquad r : I^{\cdot}(\mathfrak{z}(\gamma)) \simeq I^{\cdot}(\mathfrak{h},\mathfrak{z}(\gamma)).$$

Recall that we have the splitting

$$(8.26)\qquad \mathfrak{g} = \mathfrak{z}(\gamma) \oplus \mathfrak{z}^{\perp}(\gamma).$$

Let $r_{\mathfrak{z}(\gamma)}$ denote the projection $\mathfrak{g} \to \mathfrak{z}(\gamma)$. This map induces a corresponding morphism of $\mathbf{Z}$-graded algebras $r_{\mathfrak{z}(\gamma)} : I^{\cdot}(\mathfrak{g}) \to I^{\cdot}(\mathfrak{z}(\gamma))$. Let i be the obvious morphism $I^{\cdot}(\mathfrak{h},\mathfrak{g}) \to I^{\cdot}(\mathfrak{h},\mathfrak{z}(\gamma))$.

We have the commutative diagram

$$(8.27)\qquad \begin{array}{ccc} I^{\cdot}(\mathfrak{g}) & \xrightarrow{r} & I^{\cdot}(\mathfrak{h},\mathfrak{g}) \\ \downarrow{\scriptstyle r_{\mathfrak{z}(\gamma)}} & & \downarrow{\scriptstyle i} \\ I^{\cdot}(\mathfrak{z}(\gamma)) & \xrightarrow{r} & I^{\cdot}(\mathfrak{h},\mathfrak{z}(\gamma)) \end{array}.$$

Definition 8.3. If $L \in Z(\mathfrak{g})$, let $L^{\mathfrak{z}(\gamma)} \in I^{\cdot}(\mathfrak{z}(\gamma))$ be given by

$$(8.28)\qquad L^{\mathfrak{z}(\gamma)} = r_{\mathfrak{z}(\gamma)} \tau_{\mathrm{D}}^{-1}(L).$$

The map $L \in Z(\mathfrak{g}) \to L^{\mathfrak{z}(\gamma)} \in I^{\cdot}(\mathfrak{z}(\gamma))$ is a morphism of filtered algebras.

Proposition 8.4. *If $L \in Z(\mathfrak{g})$, the following identity holds:*

$$L^{\mathfrak{z}(\gamma)} = r^{-1} i \phi_{\mathrm{HC}} L. \tag{8.29}$$

Proof. This follows from (6.28), (8.27), and (8.28). □

As we saw in Subsections 3.1 and 3.11, we have the identification

$$I^{\cdot}(\mathfrak{z}(\gamma)) \simeq D_I^{\cdot}(\mathfrak{z}(\gamma)). \tag{8.30}$$

In the sequel, when there is no ambiguity, if $L \in Z^{\cdot}(\mathfrak{g})$, $L^{\mathfrak{z}(\gamma)}$ will be considered as an element of $D_I^{\cdot}(\mathfrak{z}(\gamma))$.

Proposition 8.5. *The following identity holds:*

$$(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)} = -\Delta^{\mathfrak{z}(\gamma)} + B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}). \tag{8.31}$$

Proof. By Proposition 6.3 and by (8.29), we get (8.31). □

8.3. **An application of Rossmann's formula.** We know that $a \in \mathfrak{h}_{\mathfrak{p}}$. Since a is in the center of $\mathfrak{z}(\gamma)$, if $\alpha \in R(\gamma)$, then $\langle \alpha, a \rangle = 0$. By (5.11), we deduce that if $h \in \mathfrak{h}$,

$$\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(h + a) = \pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(h). \tag{8.32}$$

We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the form B, so that

$$S^{\cdot}(\mathfrak{h}^*) \simeq S^{\cdot}(\mathfrak{h}). \tag{8.33}$$

As we saw in Subsection 3.1, $S^{\cdot}(\mathfrak{h})$ can be identified with the algebra $D^{\cdot}(\mathfrak{h})$ of differential operators with constant coefficients on $\mathfrak{h}$. Let $\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)} \in D^{\cdot}(\mathfrak{h}_{\mathbf{C}})$ denote the differential operator on $\mathfrak{h}_{\mathbf{C}}$ associated with $\pi^{\mathfrak{h},\mathfrak{z}(\gamma)} \in S^{\cdot}(\mathfrak{h}_{\mathbf{C}})$.

Recall that $\mathfrak{h}_i$ was defined in (7.7). Let $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ be taken as in Subsection 2.5. If $A \in \mathbf{R}$, let $\mu\left(\sqrt{-\Delta^{\mathfrak{h}} + A}\right)(h)$ be the smooth convolution kernel on $\mathfrak{h}_i$ associated with the operator $\mu\left(\sqrt{-\Delta^{\mathfrak{h}} + A}\right)$. If $f \in C^{\infty,c}(\mathfrak{h}_{\mathfrak{p}} \oplus i\mathfrak{h}_{\mathfrak{k}}, \mathbf{R})$, then

$$\mu\left(\sqrt{-\Delta^{\mathfrak{h}} + A}\right) f(h) = \int_{\mathfrak{h}_{\mathfrak{p}} \oplus i\mathfrak{h}_{\mathfrak{k}}} \mu\left(\sqrt{-\Delta^{\mathfrak{h}} + A}\right)(h - h') f(h')\, dh'. \tag{8.34}$$

We use the corresponding notation on $\mathfrak{z}_i(\gamma) = \mathfrak{p}(\gamma) \oplus i\mathfrak{k}(\gamma)$.

Here, we use the notation of Subsection 8.2. In the sequel, $\phi_{\mathrm{HC}} L$, $L^{\mathfrak{z}(\gamma)}$ are viewed as differential operators in $D_I^{\cdot}(\mathfrak{h}), D_I^{\cdot}(\mathfrak{z}(\gamma))$. These differential operators can be composed with the above convolution kernels.

Proposition 8.6. *We have the identity of smooth functions on* $\mathfrak{h}_{\mathfrak{p}} \oplus i\mathfrak{h}_{\mathfrak{k}}$,

$$\begin{aligned}(8.35)\quad &\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(\phi_{\mathrm{HC}}L\right)\mu\left(\sqrt{-\Delta^{\mathfrak{h}}+A}\right)(h)\\ &\qquad=\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(-2\pi h\right)L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{-\Delta^{\mathfrak{z}(\gamma)}+A}\right)(h).\end{aligned}$$

Proof. In the proof, we will identify $\mathfrak{z}_i(\gamma) = \mathfrak{p}(\gamma) \oplus i\mathfrak{k}(\gamma)$ as a real vector space to $\mathfrak{u}(\gamma) = i\mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$, which is the compact form of $\mathfrak{z}(\gamma)$.

We identify the real Euclidean vector space $\mathfrak{h}_i$ to its dual by its scalar product. Let $\mathcal{F}^{\mathfrak{h}_i}$ denote the classical Fourier transform on $\mathfrak{h}_i$. If $f \in \mathcal{S}(\mathfrak{h}_i)$, if $h^* \in \mathfrak{h}_i$, then

$$(8.36)\qquad \mathcal{F}^{\mathfrak{h}_i} f(h^*) = \int_{\mathfrak{h}_i} \exp\left(-2i\pi B(h,h^*)\right) f(h)\, dh.$$

Put

$$(8.37)\qquad \breve{\mathcal{F}}^{\mathfrak{h}_i} f(h^*) = \mathcal{F}^{\mathfrak{h}_i}(-h^*).$$

Then

$$\begin{aligned}(8.38)\quad &\mathcal{F}^{\mathfrak{h}_i}\left[\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(\phi_{\mathrm{HC}}L\right)\mu\left(\sqrt{-\Delta^{\mathfrak{h}}+A}\right)\right](h^*)\\ &\qquad=\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(2i\pi h^*\right)\left(\phi_{\mathrm{HC}}L\right)\left(2i\pi h^*\right)\mu\left(\sqrt{4\pi^2 B(h^*,h^*)+A}\right).\end{aligned}$$

By (8.38), we get

$$\begin{aligned}(8.39)\quad &\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(\phi_{\mathrm{HC}}L\right)\mu\left(\sqrt{-\Delta^{\mathfrak{h}}+A}\right)(h)\\ &=\breve{\mathcal{F}}^{\mathfrak{h}_i}\left[\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(2i\pi h^*\right)\left(\phi_{\mathrm{HC}}L\right)\left(2i\pi h^*\right)\mu\left(\sqrt{4\pi^2 B(h^*,h^*)+A}\right)\right](h).\end{aligned}$$

We can define the Fourier transform $\mathcal{F}^{\mathfrak{z}_i(\gamma)}$ on the Euclidean vector space $\mathfrak{z}_i(\gamma)$, which is canonically identified to the Lie algebra $\mathfrak{u}(\gamma)$. The function $B(h^*,h^*)$ extends to a $\mathrm{Ad}(\mathfrak{u}(\gamma))$-invariant function on $\mathfrak{u}(\gamma)$. By Proposition 8.4, the $\mathrm{Ad}(\mathfrak{u}(\gamma))$-invariant function $L^{\mathfrak{z}(\gamma)}$ on $\mathfrak{z}_i(\gamma)$ restricts to the function $\phi_{\mathrm{HC}}L$ on $\mathfrak{h}_i$. By Rossmann's formula [R78, Theorem p. 209], [V79, Theorem p. 13], we get

$$\begin{aligned}(8.40)\quad &\breve{\mathcal{F}}^{\mathfrak{h}_i}\left[\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(2i\pi h^*\right)\left(\phi_{\mathrm{HC}}L\right)\left(2i\pi h^*\right)\mu\left(\sqrt{4\pi^2 B(h^*,h^*)+A}\right)\right](h)\\ &=\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(-2\pi h\right)\breve{\mathcal{F}}^{\mathfrak{z}_i(\gamma)}\left[L^{\mathfrak{z}(\gamma)}\left(2i\pi h^*\right)\mu\left(\sqrt{4\pi^2 B(h^*,h^*)+A}\right)\right](h).\end{aligned}$$

By (8.39), (8.40), we get (8.35). The proof of our proposition is completed. □

8.4. **The limit of certain orbital integrals.** We use the notation of Subsection 8.1. As we saw in Subsection 7.1, $\mathrm{Tr}^{[\gamma']}\left[\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$ is a smooth function of $\gamma' \in H^{\mathrm{reg}}$. Recall that $\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}$ is a differential operator on $\mathfrak{h}_{\mathbf{C}}$. To make our formulas clearer, we will denote its action in the variables b or h as $\overline{\pi}_b^{\mathfrak{h},\mathfrak{z}(\gamma)}$ or $\overline{\pi}_h^{\mathfrak{h},\mathfrak{z}(\gamma)}$.

Proposition 8.7. *The following identity holds on H^{reg}:*

$$\begin{aligned}(8.41)\quad &\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left[D_H(\gamma')\,\mathrm{Tr}^{[\gamma']}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right]\\ &\quad=(-1)^{|R_+(\gamma)|}\,\overline{\pi}_h^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(\phi_{\mathrm{HC}}L\right)\mu\left(\sqrt{\phi_{\mathrm{HC}}C^{\mathfrak{g}}+A}\right)\\ &\quad\left[D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right]\delta_{a'}\right](0)\end{aligned}$$

$$\begin{aligned}&=(-2\pi)^{|R_+(\gamma)|}\int_{i\mathfrak{h}_{\mathfrak{k}}}L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)(-a-b_{\mathfrak{p}}-h_{\mathfrak{k}})\\ &\quad\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(b_{\mathfrak{p}}+h_{\mathfrak{k}})\,D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right]dh_{\mathfrak{k}}.\end{aligned}$$

Proof. As we saw in Theorem 5.4, the function $(\gamma', h_{\mathfrak{k}}) \in H^{\mathrm{reg}} \times i\mathfrak{h}_{\mathfrak{k}} \to \mathcal{J}_{\gamma'}(h_{\mathfrak{k}}) \in \mathbf{C}$ is smooth. If $\gamma' \in H^{\mathrm{reg}}$, then $\mathfrak{z}(\gamma') = \mathfrak{h}$, so that by equation (7.8) in Theorem 7.2, we get

$$\begin{aligned}(8.42)\quad &\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left[D_H(\gamma')\,\mathrm{Tr}^{[\gamma']}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right]\\ &\quad=\left(\phi_{\mathrm{HC}}L\right)\mu\left(\sqrt{\phi_{\mathrm{HC}}C^{\mathfrak{g}}+A}\right)\\ &\quad\overline{\pi}_b^{\mathfrak{h},\mathfrak{z}(\gamma)}\left[D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right]\delta_{a'}\right](0).\end{aligned}$$

In the right hand-side of (8.42), the differential operator $\overline{\pi}_b^{\mathfrak{h},\mathfrak{z}(\gamma)}$ acts on the distribution on the right in the variables $b = (b_{\mathfrak{p}}, b_{\mathfrak{k}}) \in \mathfrak{h}$, and not on the variable $h_{\mathfrak{k}}$. The situation is actually strictly similar to what we already met in the proofs of Proposition 7.1 and Theorem 7.2. We obtain this way the first identity in (8.41).

By Proposition 8.6, using the conventions in (8.34), we get

$$\begin{aligned}(8.43)\quad &(-1)^{|R_+(\gamma)|}\,\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(\phi_{\mathrm{HC}}L\right)\mu\left(\sqrt{\phi_{\mathrm{HC}}C^{\mathfrak{g}}+A}\right)\\ &\quad\left[D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right]\delta_{a'}\right](0)\\ &=(-2\pi)^{|R_+(\gamma)|}\int_{i\mathfrak{h}_{\mathfrak{k}}}L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)(-a-b_{\mathfrak{p}}-h_{\mathfrak{k}})\\ &\quad\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}(a+b_{\mathfrak{p}}+h_{\mathfrak{k}})\,D_H(\gamma')\,\mathcal{J}_{\gamma'}(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{b_{\mathfrak{k}}-h_{\mathfrak{k}}}\right)\right]dh_{\mathfrak{k}}.\end{aligned}$$

By (8.32), we get

$$\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(a+b_{\mathfrak{p}}+h_{\mathfrak{k}}\right)=\pi^{\mathfrak{h},\mathfrak{z}(\gamma)}\left(b_{\mathfrak{p}}+h_{\mathfrak{k}}\right). \tag{8.44}$$

By the first identity in (8.41) and by (8.43), (8.44), we get the second identity in (8.41). The proof of our proposition is completed. □

Let $T(\gamma)$ be a maximal torus in $K^0(\gamma)$, let $\mathfrak{t}(\gamma)\subset\mathfrak{k}(\gamma)$ be the corresponding Lie algebra, and let $W(\mathfrak{t}(\gamma):\mathfrak{k}(\gamma))$[11] be the corresponding Weyl group.

Theorem 8.8. *If $\mathfrak{h}$ is not the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, as $b\in\mathfrak{h}$ γ-regular tends to 0, then*

$$\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left[D_H(\gamma')\,\mathrm{Tr}^{[\gamma']}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right]\to 0. \tag{8.45}$$

If $\mathfrak{h}$ is the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, as $b\in\mathfrak{h}$ γ-regular tends to 0, then

$$\begin{aligned}\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}&\left[D_H(\gamma')\,\mathrm{Tr}^{[\gamma']}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right]\\ &\to(-1)^{\frac{1}{2}(\dim\mathfrak{p}(\gamma)-\dim\mathfrak{h}_{\mathfrak{p}})}\frac{|W(\mathfrak{t}(\gamma):\mathfrak{k}(\gamma))|}{\mathrm{Vol}\left(K^0(\gamma)/T(\gamma)\right)}\prod_{\alpha\in R_+^{\mathrm{im}}(k)}\xi_\alpha^{1/2}\left(k^{-1}\right)\\ &D_H(\gamma)(2\pi)^{|R_+(\gamma)|}\int_{i\mathfrak{k}(\gamma)}L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\left(-a-Y_0^{\mathfrak{k}}\right)\\ &\qquad\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]dY_0^{\mathfrak{k}}.\end{aligned} \tag{8.46}$$

Proof. First, we consider the case where $\mathfrak{h}$ is not the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$. Using equations (2.80), (8.4), (8.5), (8.41), and dominated convergence, we get (8.45).

Assume now that $\mathfrak{h}$ is the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$. Using equations (2.80), (8.4), (8.6), (8.41), since the convergence in (8.6) takes place except on a Lebesgue negligible set of $i\mathfrak{h}_{\mathfrak{k}}$, we can use

[11] We use this notation instead of $W(\mathfrak{t}(\gamma)_{\mathbf{C}}:\mathfrak{k}(\gamma)_{\mathbf{C}})$ because this Weyl group is real.

dominated convergence, so that as $b \in \mathfrak{h}$ γ-regular tends to 0,

$$\begin{aligned}(8.47)\quad & \overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left[D_H(\gamma')\,\mathrm{Tr}^{[\gamma']}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right] \\ & \to (-1)^{|R_+(\gamma)|+\frac{1}{2}\left|R_+^{\mathrm{c}}(\gamma)\right|+\left|R_{\mathfrak{k},+}^{\mathrm{im}}(k)\right|}\prod_{\alpha\in R_+^{\mathrm{im}}(k)}\xi_\alpha^{1/2}\left(k^{-1}\right)D_H(\gamma) \\ & (2\pi)^{|R_+(\gamma)|}\int_{i\mathfrak{h}_{\mathfrak{k}}} L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)(-a-h_{\mathfrak{k}}) \\ & \qquad \mathcal{J}_\gamma(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}(\gamma)}(h_{\mathfrak{k}})\right]^2 dh_{\mathfrak{k}}.\end{aligned}$$

The operators $L^{\mathfrak{z}(\gamma)}$ and $(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}$ on $\mathfrak{z}(\gamma)$ are both $K(\gamma)$-invariant, and so the smooth function

$$L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\left(-a-Y_0^{\mathfrak{k}}\right)$$

on $i\mathfrak{k}(\gamma)$ is also $K(\gamma)$-invariant. This is also the case for the function

$$\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right].$$

Since $\mathfrak{h}$ is the fundamental Cartan subalgebra of $\mathfrak{z}(\gamma)$, $\mathfrak{h}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}(\gamma)$. By Weyl's integration formula, and taking into account the fact that on $i\mathfrak{h}_{\mathfrak{k}}$, $\left[\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}(\gamma)}(h_{\mathfrak{k}})\right]^2$ is nonnegative, we obtain

$$\begin{aligned}(8.48)\quad & \int_{i\mathfrak{k}(\gamma)} L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\left(-a-Y_0^{\mathfrak{k}}\right) \\ & \qquad \mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]dY_0^{\mathfrak{k}} \\ & = \frac{\mathrm{Vol}\left(K^0(\gamma)/T(\gamma)\right)}{\left|W\left(\mathfrak{t}(\gamma):\mathfrak{k}(\gamma)\right)\right|}\int_{i\mathfrak{h}_{\mathfrak{k}}} L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)(-a-h_{\mathfrak{k}}) \\ & \qquad \mathcal{J}_\gamma(h_{\mathfrak{k}})\,\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{h}_{\mathfrak{k}},\mathfrak{k}(\gamma)}(h_{\mathfrak{k}})\right]^2 dh_{\mathfrak{k}}.\end{aligned}$$

Using in particular Proposition 3.10 applied to $\mathfrak{z}(\gamma)$, we have the identities,

$$\begin{aligned}(8.49)\quad & |R_+(\gamma)| = \dim\mathfrak{c}_+(\gamma)+\frac{1}{2}\dim\mathfrak{i}(k), \\ & \left|R_{\mathfrak{k},+}^{\mathrm{im}}(k)\right| = \frac{1}{2}\dim\mathfrak{i}_{\mathfrak{k}}(k), \\ & \left|R_+^{\mathrm{c}}(\gamma)\right| = \dim\mathfrak{c}_+(\gamma).\end{aligned}$$

By (8.49), we deduce that

$$\begin{aligned}(8.50)\quad &|R_+(\gamma)| + \left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right| + \frac{1}{2}\left|R^{\mathrm{c}}_+(\gamma)\right| \\ &\qquad = \dim \mathfrak{c}_+(\gamma) + \frac{1}{2}\dim \mathfrak{i}_{\mathfrak{k}}(k) + \frac{1}{2}\dim \mathfrak{i}(k) + \frac{1}{2}\dim \mathfrak{c}_+(\gamma).\end{aligned}$$

By Proposition 3.10, $\mathfrak{c}_+(\gamma)$ has even dimension. This is also the case for $\mathfrak{i}_{\mathfrak{k}}(k)$. By (8.50), we get the equality $\bmod 2$,

$$(8.51)\quad |R_+(\gamma)| + \left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right| + \frac{1}{2}\left|R^{\mathrm{c}}_+(\gamma)\right| = \frac{1}{2}\dim \mathfrak{i}_{\mathfrak{p}}(k) + \frac{1}{4}\dim \mathfrak{c}(\gamma).$$

By equation (3.31) in Proposition 3.8 applied to $\mathfrak{z}(\gamma)$, since $\mathfrak{h}$ is fundamental in $\mathfrak{z}(\gamma)$, we get

$$(8.52)\qquad \dim \mathfrak{p}(\gamma) - \dim \mathfrak{h}_{\mathfrak{p}} = \dim \mathfrak{i}_{\mathfrak{p}}(k) + \frac{1}{2}\dim \mathfrak{c}(\gamma).$$

By (8.51), (8.52), we get

$$(8.53)\qquad (-1)^{|R_+(\gamma)| + \left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right| + \frac{1}{2}\left|R^{\mathrm{c}}_+(\gamma)\right|} = (-1)^{\frac{1}{2}(\dim \mathfrak{p}(\gamma) - \dim \mathfrak{h}_{\mathfrak{p}})}.$$

When $\gamma = 1$, the above identity had been established by [HC64, Lemma 18].

By (8.47), (8.48), and (8.53), we get (8.46). The proof of our theorem is completed. □

9. The final formula

In this section, we establish our final formula in the case of a non necessarily regular semisimple element $\gamma \in G$. Our formula extends both the formula in Theorem 2.9 valid for γ semisimple and $L = 1$, and the formula in Theorem 7.2 valid for γ regular. To establish our main result, we combine a fundamental result of Harish-Chandra with the results we obtained in Section 8. Also, along the lines of [B11, Chapter 6], we give a wave kernel formulation of our main result.

This section is organized as follows. In Subsection 9.1, we establish our main result.

In Subsection 9.2, as in [B11], we reformulate our main result in terms of wave kernels.

Finally, in Subsection 9.3, we verify our main formula is compatible to natural operations on orbital integrals.

9.1. **The general case.** Let $L \in Z(\mathfrak{g})$. Here, we take $\gamma \in G$ semisimple as in (2.19). We will extend Theorem 7.2 to non regular γ.

Theorem 9.1. *The following identity holds:*

$$\begin{aligned}(9.1)\quad &\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\\ &= L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\left[\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0).\end{aligned}$$

Proof. By a result of Harish-Chandra [HC66, Lemma 28], [Va77, Part II, Section 12.5, Theorem 13], we know that if $\mathfrak{h}$ is the fundamental Cartan subalgebra in $\mathfrak{z}(\gamma)$, there is a universal constant c_γ depending only on γ such that with the notation in Theorem 8.8, as $b \in \mathfrak{h}$ γ-regular tends to 0,

$$\begin{aligned}(9.2)\quad &\overline{\pi}^{\mathfrak{h},\mathfrak{z}(\gamma)}\left[D_H(\gamma')\,\mathrm{Tr}^{[\gamma']}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\right]\\ &\qquad\qquad\to c_\gamma \mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right].\end{aligned}$$

In Theorem 8.8, we gave another proof of the existence of the limit in (9.2). Also by the fundamental result of [B11] stated as Theorem 2.9, when $L = 1$, the integral in the right-hand side of (8.46) coincides with the orbital integral $\mathrm{Tr}^{[\gamma]}\left[\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$. To identify the constant c_γ, we only need to prove that one of these last orbital integrals does not vanish. It is enough to take E to be the trivial representation, and $\mu(x) = \exp(-x^2)$. Since the scalar heat kernel on X is positive, the corresponding orbital integrals do not vanish. So we find that

$$\begin{aligned}(9.3)\quad c_\gamma = (-1)^{\frac{1}{2}(\dim\mathfrak{p}(\gamma)-\dim\mathfrak{h}_{\mathfrak{p}})}&\frac{|W(\mathfrak{t}(\gamma):\mathfrak{k}(\gamma))|}{\mathrm{Vol}(K^0(\gamma)/T(\gamma))}\\ &(2\pi)^{|R_+(\gamma)|}\prod_{\alpha\in R_+^{\mathrm{im}}(k)}\xi_\alpha^{1/2}\left(k^{-1}\right)D_H(\gamma).\end{aligned}$$

When $\gamma = 1$, this computation has already been done by Harish-Chandra in [HC75, Section 37, Theorem 1]. [12] For the case of a general γ, this formula can also be derived from [HC66, p. 34] and from the reference given before.

[12] More precisely, if $G = KAN$ is the Iwasawa decomposition, when $\gamma = 1$, the constant obtained in [HC75] is $2^{\frac{\dim N}{2}}c_\gamma$. In [HC75, Section 7], Harish-Chandra uses another normalization for the Haar measure on G, which is adapted to the Iwasawa decomposition. By [HC75, p.202], the ratio of these two normalizations is given by $2^{\frac{\dim N}{2}}$, which explains the discrepancy.

By combining (8.46), (9.2), and (9.3), we get (9.1). The proof of our theorem is completed. □

9.2. **A microlocal version.** We still take $\gamma \in G$ semisimple as in (2.19).

We will proceed as in [B11, Section 6.3], to which we refer for more details. In the sequel we identify TX and T^*X by the metric.

Let $\mathrm{Tr}^{[\gamma]}\left[L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$ be the even distribution on $\mathbf{R}$ such that for any $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbf{R})$ with $\widehat{\mu}$ having compact support,

(9.4)
$$\mathrm{Tr}^{[\gamma]}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right] = \int_{\mathbf{R}} \widehat{\mu}(s)\,\mathrm{Tr}^{[\gamma]}\left[L\cos\left(2\pi s\sqrt{C^{\mathfrak{g},X}+A}\right)\right] ds.$$

The operator $L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)$ defines a distribution on $\mathbf{R}\times X\times X$. By finite propagation speed [CP81, section 7.8], [T81, section 4.4], its support is included in $(s,x,x'), |s| \geq d(x,x')$. Let $\mathcal{X}$ be the total space of TX. Let $s\in\mathbf{R}\to\varphi_s$ be the geodesic flow on $\mathcal{X}$. Let τ be the variable dual to s. By [Hö85, Theorem 23.1.4 and remark], the wave front set $\mathrm{WF}\left(L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right)$ of the distribution $L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)$ is the conic set in $\mathbf{R}^2\times T^*X\times T^*X$ generated by $(x',-Y') = \varphi_{\pm s}(x,Y), |Y|=1, \tau=\pm 1$. Conic means that the dilations by $\lambda>0$ are applied simultaneously to the variables Y, Y', τ.

As explained in [B11, Section 3.4], in the geodesic coordinate system centered at $x_0 = p1$, $\mathfrak{p}^{\perp}(\gamma)$ can be identified with a smooth submanifold $P^{\perp}(\gamma)$ of X. Let $N_{P^{\perp}(\gamma)/X}$ be the orthogonal bundle to $TP^{\perp}(\gamma)$ in TX.

Set

$$\Delta_X^{\gamma} = \left\{(x,\gamma x), x\in P^{\perp}(\gamma)\right\}. \tag{9.5}$$

Then Δ_X^{γ} is a smooth submanifold of $X\times X$. The conormal bundle to $\mathbf{R}\times\Delta_X^{\gamma}\subset\mathbf{R}\times X\times X$ is the set $((s,\tau),(x,Y),(x',Y'))\in\mathbf{R}^2\times\mathcal{X}\times\mathcal{X}$ such that $\tau=0, x\in P^{\perp}(\gamma), x'=\gamma x, \gamma^*Y'+Y\in N_{P^{\perp}(\gamma)/X}$.

By [Hö83, Theorem 8.2.10],

$$L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\Delta_X^{\gamma}$$

is a well-defined distribution on $\mathbf{R}\times X\times X$, and its wave front set is the formal sum of the wave front sets of the two above distributions. In particular the pushforward of the distribution $\mathrm{Tr}^F\left[\gamma L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$ by the projection $\mathbf{R}\times X\times X\to\mathbf{R}$ is well-defined. It will be denoted

$$\int_{\Delta_X^{\gamma}}\mathrm{Tr}^F\left[\gamma L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]. \tag{9.6}$$

This is an even distribution on $\mathbf{R}$.

Tautologically, we have the identity of even distributions on $\mathbf{R}$,

(9.7)
$$\mathrm{Tr}^{[\gamma]}\left[L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]=\int_{\Delta_X^\gamma}\mathrm{Tr}^F\left[\gamma L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right].$$

We have the result stablished in [B11, Proposition 6.3.1].

Proposition 9.2. *The singular support of* $\mathrm{Tr}^{[\gamma]}\left[L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$ *is included in* $s=\pm|a|$*, and the ordinary support is included in*

$$\{s\in\mathbf{R},|s|\geq|a|\}.$$

If $a=0$*, if* $\mathfrak{p}(\gamma)=0$*, the singular support of*

$$\mathrm{Tr}^{[\gamma]}\left[L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]$$

is empty.

We define the even distribution on $\mathbf{R}$,

(9.8)
$$L^{\mathfrak{z}(\gamma)}\cos\left(s\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\left[\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}\ e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0)$$

by the formula

$$\begin{aligned}(9.9)\quad &L^{\mathfrak{z}(\gamma)}\mu\left(\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\left[\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0)\\&=\int_{\mathbf{R}}\widehat{\mu}(s)\,L^{\mathfrak{z}(\gamma)}\cos\left(2\pi s\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)\\&\qquad\left[\mathcal{J}_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0).\end{aligned}$$

Let $z=\left(y,Y_0^{\mathfrak{k}}\right)$ be the generic element of $\mathfrak{z}_i(\gamma)=\mathfrak{p}(\gamma)\oplus i\mathfrak{k}(\gamma)$. Using finite propagation speed for the wave equation,

$$L^{\mathfrak{z}(\gamma)}\cos\left(s\sqrt{(C^{\mathfrak{g}})^{\mathfrak{z}(\gamma)}+A}\right)$$

is a distribution on $\mathbf{R}\times\mathfrak{z}_i(\gamma)\times\mathfrak{z}_i(\gamma)$ whose support is included in $(s,z,z')\,,|s|\geq|z'-z|$. Moreover, by [Hö85, Theorem 23.1.4 and remark], its wave front set is the conic set associated with $(y',-Y')=(y\pm sY,Y)\,,|Y|=1,\tau=\pm1$. Conic set means again that the dilations by $\lambda>0$ are applied to the variables Y,Y',τ.

Set

$$(9.10)\qquad H^\gamma=\{0\}\times(a,i\mathfrak{k}(\gamma))\subset\mathfrak{z}_i(\gamma)\times\mathfrak{z}_i(\gamma).$$

The wave front set associated with $\mathbf{R}\times H^{\gamma}\subset \mathbf{R}\times \mathfrak{z}_i\left(\gamma\right)\times \mathfrak{z}_i\left(\gamma\right)$ is such that $Y'^{\mathfrak{k}(\gamma)}=0,\tau=0$, so that the product

$$L^{\mathfrak{z}(\gamma)}\cos\left(s\sqrt{\left(C^{\mathfrak{g}}\right)^{\mathfrak{z}(\gamma)}+A}\right)H^{\gamma}$$

is well-defined.

The function $\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]$ can be viewed as a smooth function on the second copy of $\mathfrak{z}_i\left(\gamma\right)$ in $\mathfrak{z}_i\left(\gamma\right)\times\mathfrak{z}_i\left(\gamma\right)$. It lifts to a smooth function on $\mathfrak{z}_i\left(\gamma\right)\times\mathfrak{z}_i\left(\gamma\right)$.

Therefore,

$$L^{\mathfrak{z}(\gamma)}\cos\left(s\sqrt{\left(C^{\mathfrak{g}}\right)^{\mathfrak{z}(\gamma)}+A}\right)H^{\gamma}\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right] \tag{9.11}$$

is a well-defined distribution on $\mathbf{R}\times\mathfrak{z}_i\left(\gamma\right)\times\mathfrak{z}_i\left(\gamma\right)$. The pushforward of this distribution by the projection $\mathbf{R}\times\mathfrak{z}_i\left(\gamma\right)\times\mathfrak{z}_i\left(\gamma\right)\to\mathbf{R}$ is denoted

$$\int_{H^{\gamma}}L^{\mathfrak{z}(\gamma)}\cos\left(s\sqrt{\left(C^{\mathfrak{g}}\right)^{\mathfrak{z}(\gamma)}+A}\right)\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]. \tag{9.12}$$

This is an even distribution supported in $|s|\geq|a|$, with singular support included in $s=\pm|a|$. Note that if $a=0$ and if $\mathfrak{p}\left(\gamma\right)=0$, the singular support of this distribution is empty.

Theorem 9.3. *We have the identity of even distributions on* $\mathbf{R}$ *supported on* $|s|\geq|a|$ *with singular support included in* $\pm|a|$,

$$\begin{aligned}\int_{\Delta_X^{\gamma}}\mathrm{Tr}^{F}\left[\gamma L\cos\left(s\sqrt{C^{\mathfrak{g},X}+A}\right)\right]\\ =\int_{H^{\gamma}}L^{\mathfrak{z}(\gamma)}\cos\left(s\sqrt{\left(C^{\mathfrak{g}}\right)^{\mathfrak{z}(\gamma)}+A}\right)\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right].\end{aligned} \tag{9.13}$$

Proof. We use Theorem 9.1, and we proceed as in the proof of [B11, Theorem 6.3.2]. □

9.3. **Compatibility properties of the formula.** Let us give a direct proof that the right-hand side of (9.1) is invariant by conjugation of γ in G. Indeed let γ,γ' be two conjugate elements in G as in Theorem 2.3. By this theorem, they are also conjugate by an element k'' of K, and equation (2.33) holds. Since the character of the representation ρ^{E} is invariant by conjugation by elements of K, the right hand-sides of (9.1) associated with γ,γ' coincide.

We will denote the dependence of our orbital integrals on E with an extra superscript E. If $L\in Z\left(\mathfrak{g}\right)$, by Theorem 6.1, the L_2 transpose of

L is just $\sigma(L)$. Observe that $C^{\mathfrak{g},X}$ is symmetric, i.e., it is equal to its transpose. Then one has the easy formula

(9.14)
$$\mathrm{Tr}^{[\gamma^{-1}],E^*}\left[\sigma(L)\,\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]=\mathrm{Tr}^{[\gamma],E}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right],$$
$$\overline{\mathrm{Tr}^{[\gamma],E}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right]}=\mathrm{Tr}^{[\gamma],E^*}\left[L\mu\left(\sqrt{C^{\mathfrak{g},X}+A}\right)\right].$$

Using the identities in (2.71), we can recover (9.14) from (9.1).

Finally, it is easy to verify that, as it should be, our formula is unchanged when replacing γ, L by $\theta\gamma, \theta L$.

10. Orbital integrals and the index theorem

The purpose of this section is to verify the compatibility of our formula for orbital integrals to the index theorem of Atiyah-Singer [AS68a, AS68b], to the Lefschetz formulas of [ABo67, ABo68] for Dirac operators, to the index formula of Kawasaki [K79]. More precisely we extend to the case of an arbitrary L what was done in [B11, Chapter 7] in the case $L=1$. Also we verify the compatibility of our results to results of Huang-Pandžić [HuP02] who established the Vogan conjecture on Dirac cohomology.

This section is organized as follows. In Subsection 10.1, we construct the Dirac operator D^X on the symmetric space X.

In Subsection 10.2, we introduce the relevant notation when G and K have the same complex rank.

In Subsection 10.3, we evaluate the orbital integrals associated with the index theorem for Dirac operators when γ semisimple is nonelliptic, and also when $\gamma=1$.

In Subsection 10.4, when γ is elliptic, we consider again the case where the difference of complex ranks is still equal to 0.

In Subsection 10.5, we evaluate the orbital integrals associated with the index theorem for the Dirac operator.

Finally, in Subsection 10.6, we verify the compatibility of our results with the results of Huang-Pandžić [HuP02].

10.1. **The Dirac operator on X.** Here, we use the notation of Section 2. We assume that K is simply connected, and also that $\mathfrak{p}$ is even dimensional and oriented. Let $c(\mathfrak{p})$ be the Clifford algebra associated with $(\mathfrak{p}, B|_{\mathfrak{p}})$.

As explained in [B11, Section 7.2], the representation $\rho^{\mathfrak{p}}: K\to \mathrm{SO}(\mathfrak{p})$ lifts to a representation $K\to \mathrm{Aut}^{\mathrm{even}}(S^{\mathfrak{p}})$, where $S^{\mathfrak{p}}=S^{\mathfrak{p}}_+\oplus S^{\mathfrak{p}}_-$

is the $\mathbf{Z}_2$-graded Hermitian vector space of $\mathfrak{p}$-spinors. We have the identification of $\mathbf{Z}_2$-graded algebras,

$$c(\mathfrak{p}) \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{End}(S^{\mathfrak{p}}). \tag{10.1}$$

Set

$$S^{TX} = G \times_K S^{\mathfrak{p}}. \tag{10.2}$$

The $\mathbf{Z}_2$-graded vector bundle S^{TX} inherits a unitary connection $\nabla^{S^{TX}}$.

Let $\nabla^{S^{TX}\otimes F}$ be the connection on $S^{TX}\otimes F$ associated with $\nabla^{S^{TX}}, \nabla^F$.

Recall that $C^{\mathfrak{k},E}$ descends to a parallel section $C^{\mathfrak{k},F}$ of $\mathrm{End}(F)$. Here, $C^{\mathfrak{g},X}$ denotes the action of $C^{\mathfrak{g}}$ on $C^\infty\left(X, S^{TX}\otimes F\right)$.

Here, D^X denotes the Dirac operator acting on $C^\infty(X, S^{TX}\otimes F)$. If $e_1, \ldots, e_m$ is an orthonormal basis of TX then

$$D^X = \sum_1^n c(e_i)\, \nabla^{S^{TX}\otimes F}_{e_i}. \tag{10.3}$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}$. We will use the notation

$$\mathfrak{b} = \mathfrak{h}_{\mathfrak{p}}, \qquad\qquad \mathfrak{t} = \mathfrak{h}_{\mathfrak{k}}. \tag{10.4}$$

Then $\mathfrak{t} \subset \mathfrak{k}$ is the Lie algebra of a maximal torus $T \subset K$. Also $\dim \mathfrak{t}, \dim \mathfrak{h}$ are the complex ranks of K, G, and $\dim \mathfrak{b}$ is the difference of these complex ranks. Since m is even, $\dim \mathfrak{b}$ is also even. Let $\phi_{\mathrm{HC}} : Z(\mathfrak{g}) \simeq I^{\cdot}(\mathfrak{h}, \mathfrak{g})$ be the corresponding isomorphism of Harish-Chandra.

We fix a system of positive roots in $i\mathfrak{t}^*$ associated with the pair $(\mathfrak{t}, \mathfrak{k})$. In particular $\rho^{\mathfrak{k}} \in i\mathfrak{t}^*$ is calculated with respect to this system.

By [B11, eqs. (7.2.8) and (7.2.9)] and by (2.48), we get

$$D^{X,2} = C^{\mathfrak{g},X} - B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + B^*\left(\rho^{\mathfrak{k}}, \rho^{\mathfrak{k}}\right) - C^{\mathfrak{k},F}. \tag{10.5}$$

We may and we will assume that ρ^E is an irreducible representation of K with dominant weight $\lambda \in i\mathfrak{t}^*$. Then

$$C^{\mathfrak{k},E} = -B^*\left(\rho^{\mathfrak{k}} + \lambda, \rho^{\mathfrak{k}} + \lambda\right) + B^*\left(\rho^{\mathfrak{k}}, \rho^{\mathfrak{k}}\right). \tag{10.6}$$

By (10.5), (10.6), we get

$$D^{X,2} = C^{\mathfrak{g},X} - B^*(\rho^{\mathfrak{g}}, \rho^{\mathfrak{g}}) + B^*\left(\rho^{\mathfrak{k}} + \lambda, \rho^{\mathfrak{k}} + \lambda\right). \tag{10.7}$$

By (6.29), we can rewrite (10.7) in the form

$$D^{X,2} = C^{\mathfrak{g},X} - \phi_{\mathrm{HC}} C^{\mathfrak{g}}\left(\rho^{\mathfrak{k}} + \lambda\right). \tag{10.8}$$

10.2. **The case where** $\dim \mathfrak{b} = 0$**.** In this Subsection, we assume that $\dim \mathfrak{b} = 0$. Then $\mathfrak{h} = \mathfrak{t}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$, and

$$R = R^{\mathrm{im}}. \tag{10.9}$$

Also $R^{\mathrm{im}}_{\mathfrak{k}}$ is just the root system associated with the pair $(\mathfrak{t}, \mathfrak{k})$. We fix a positive root system $R^{\mathrm{im}}_{\mathfrak{k},+} \subset R^{\mathrm{im}}_{\mathfrak{k}}$, and a positive weight system $R^{\mathrm{im}}_{\mathfrak{p},+} \subset R^{\mathrm{im}}_{\mathfrak{p}}$ which is compatible with the orientation of $\mathfrak{p}$, so that $R^{\mathrm{im}}_{+} = R^{\mathrm{im}}_{\mathfrak{p},+} \cup R^{\mathrm{im}}_{\mathfrak{k},+}$ is a positive root system for the pair $(\mathfrak{t}, \mathfrak{g})$.

The functions $\pi^{\mathfrak{t},\mathfrak{g}}, \pi^{\mathfrak{t},\mathfrak{k}}$ on $\mathfrak{t}$ are given by

$$\pi^{\mathfrak{t},\mathfrak{g}}(h) = \prod_{\alpha \in R^{\mathrm{im}}_{+}} \langle \alpha, h \rangle, \qquad \pi^{\mathfrak{t},\mathfrak{k}}(h) = \prod_{\alpha \in R^{\mathrm{im}}_{\mathfrak{k},+}} \langle \alpha, h \rangle. \tag{10.10}$$

Here, $\rho^{\mathfrak{k}}, \lambda \in i\mathfrak{t}^*$ are calculated with this choice of $R^{\mathrm{im}}_{\mathfrak{k},+}$. We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by the quadratic form $B|_{\mathfrak{t}}$. In particular, $\pi^{\mathfrak{t},\mathfrak{k}}\left(\frac{\rho^{\mathfrak{k}}}{2\pi}\right)$ and $\pi^{\mathfrak{t},\mathfrak{k}}\left(\frac{\rho^{\mathfrak{k}}+\lambda}{2\pi}\right)$ are well-defined, and $\frac{\pi^{\mathfrak{t},\mathfrak{g}}\left(\frac{\rho^{\mathfrak{k}}+\lambda}{2\pi}\right)}{\pi^{\mathfrak{t},\mathfrak{k}}\left(\frac{\rho^{\mathfrak{k}}}{2\pi}\right)}$ only depends on the orientation of $\mathfrak{p}$. Also $\phi_{\mathrm{HC}} L$ is a polynomial on $\mathfrak{t}^*$, and so $\phi_{\mathrm{HC}} L\left(-\rho^{\mathfrak{k}} - \lambda\right)$ is well-defined.

10.3. **Orbital integrals and the index theorem: the case of the identity.** Take $L \in Z(\mathfrak{g})$. For $t > 0$, $L \exp\left(-tD^{X,2}\right)$ acts on $C^\infty\left(X, S^{TX} \otimes F\right)$.

In the sequel, $\mathrm{Tr_s}$ is our notation for the supertrace.[13]

We will extend [B11, Theorem 7.4.1]. As in [B11, Section 7.1], $\widehat{A}\left(TX, \nabla^{TX}\right), \mathrm{ch}\left(F, \nabla^F\right)$ denote the obvious characteristic forms on X. Let $\eta \in \Lambda^m(T^*X)$ be the canonical volume form on X that defines its orientation. If $\alpha \in \Lambda^{\cdot}(T^*X)$, let $\alpha^{(p)}$ denote its component in $\Lambda^p(T^*X)$. Let $\alpha^{\max} \in \mathbf{R}$ be such that

$$a^{(m)} = \alpha^{\max} \eta. \tag{10.11}$$

Let $\gamma \in G$ be semisimple.

Theorem 10.1. *If γ is nonelliptic, for any $t > 0$,*

$$\mathrm{Tr_s}^{[\gamma]}\left[L \exp\left(-tD^{X,2}\right)\right] = 0. \tag{10.12}$$

[13] If $V = V_+ \oplus V_-$ is a $\mathbf{Z}_2$-graded vector space, if $\tau = \pm 1$ is the involution defining the grading, if $A \in \mathrm{End}(V)$, the supertrace of A is defined to be $\mathrm{Tr_s}[A] = \mathrm{Tr}[\tau A]$.

If $\dim \mathfrak{b} > 0$, *for any* $t > 0$,

$$\mathrm{Tr}_{\mathrm{s}}^{[1]}\left[L\exp(-tD^{X,2})\right] = 0, \tag{10.13}$$
$$\left[\widehat{A}\left(TX,\nabla^{TX}\right)\mathrm{ch}\left(F,\nabla^{F}\right)\right]^{\max} = 0.$$

If $\dim \mathfrak{b} = 0$, *then*

(10.14)

$$\mathrm{Tr}_{\mathrm{s}}^{[1]}\left[L\exp\left(-tD^{X,2}\right)\right] = \phi_{\mathrm{HC}} L\left(-\rho^{\mathfrak{k}}-\lambda\right)(-1)^{m/2}\frac{\pi^{\mathfrak{t},\mathfrak{g}}\left(\frac{\rho^{\mathfrak{k}}+\lambda}{2\pi}\right)}{\pi^{\mathfrak{t},\mathfrak{k}}\left(\frac{\rho^{\mathfrak{k}}}{2\pi}\right)},$$

$$\left[\widehat{A}\left(TX,\nabla^{TX}\right)\mathrm{ch}\left(F,\nabla^{F}\right)\right]^{\max} = (-1)^{m/2}\frac{\pi^{\mathfrak{t},\mathfrak{g}}\left(\frac{\rho^{\mathfrak{k}}+\lambda}{2\pi}\right)}{\pi^{\mathfrak{t},\mathfrak{k}}\left(\frac{\rho^{\mathfrak{k}}}{2\pi}\right)}.$$

Proof. First we prove (10.12). We proceed as in [B11]. By Proposition 8.5, by Theorem 9.1 and by (10.7), we get

$$\mathrm{Tr}_{\mathrm{s}}^{[\gamma]}\left[L\exp\left(-tD^{X,2}\right)\right] = \exp\left(-tB^{*}\left(\rho^{\mathfrak{k}}+\lambda,\rho^{\mathfrak{k}}+\lambda\right)\right) \tag{10.15}$$
$$L^{\mathfrak{z}(\gamma)}\exp\left(t\Delta^{\mathfrak{z}(\gamma)}\right)\left[\mathcal{J}_{\gamma}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_a\right](0).$$

Also

$$\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right] = \mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right] \tag{10.16}$$
$$\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right].$$

It is well-known that

$$\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]$$

is a square root of $\det\left(1-\mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)|_{\mathfrak{p}}\right)$.

If γ is nonelliptic, $a \neq 0$ lies in the kernel of $1-\mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)|_{\mathfrak{p}}$, and so (10.16) vanishes. By (10.15), we get (10.12).

By [B11, Theorem 7.4.1], we get

$$\mathrm{Tr}_{\mathrm{s}}^{[1]}\left[\exp\left(-tD^{X,2}\right)\right] = \left[\widehat{A}\left(TX,\nabla^{TX}\right)\mathrm{ch}\left(F,\nabla^{F}\right)\right]^{\max}. \tag{10.17}$$

By (10.15), we get

$$\mathrm{Tr}_{\mathrm{s}}^{[1]}\left[L\exp\left(-tD^{X,2}\right)\right] = \exp\left(-tB^{*}\left(\rho^{\mathfrak{k}}+\lambda,\rho^{\mathfrak{k}}+\lambda\right)\right) \tag{10.18}$$
$$L^{\mathfrak{g}}\exp\left(t\Delta^{\mathfrak{g}}\right)\left[\mathcal{J}_{1}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_0\right](0).$$

We will use (10.16) with $k = 1$. We have the well-known identity in [B11, eq. (7.5.11)],

$$\mathrm{Tr_s}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(e^{-Y_0^{\mathfrak{k}}}\right)\right] = (-i)^{m/2}\,\mathrm{Pf}\left[\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}}\right]\widehat{A}^{-1}\left(\mathrm{ad}\left(Y_0^{\mathfrak{k}}|_{\mathfrak{p}}\right)\right). \tag{10.19}$$

In (10.19), we may and we will assume that $Y_0^{\mathfrak{k}} \in i\mathfrak{t}$. Since $\mathfrak{b} \subset \ker \mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}}$, if $\dim \mathfrak{b} > 0$, then

$$\mathrm{Pf}\left[\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}}\right] = 0, \tag{10.20}$$

and so (10.19) vanishes, which implies the vanishing of (10.18), i.e., we have established the first identity in (10.13). Combining this equation for $L = 1$ and (10.17), we obtain the second equation in (10.13).

In the sequel, we assume that $\dim \mathfrak{b} = 0$. We will use the notation and results of Subsection 10.2.

For $\gamma = 1$, we use the notation $\mathfrak{g}_i = \mathfrak{z}_i(\gamma)$, so that $\mathfrak{g}_i = \mathfrak{p} \oplus i\mathfrak{k}$. Let $L^{\mathfrak{g}} \exp\left(t\Delta^{\mathfrak{g}}\right)(f)$, $f \in \mathfrak{g}_i$ be the convolution kernel for $L^{\mathfrak{g}} \exp\left(t\Delta^{\mathfrak{g}}\right)$. Then

$$\begin{aligned} &L^{\mathfrak{g}} \exp\left(t\Delta^{\mathfrak{g}}\right)\left[\mathcal{J}_1\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_0\right](0) \\ &\quad = \int_{i\mathfrak{k}} L^{\mathfrak{g}} \exp\left(t\Delta^{\mathfrak{g}}\right)\left(-Y_0^{\mathfrak{k}}\right)\mathcal{J}_1\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-Y_0^{\mathfrak{k}}}\right)\right] dY_0^{\mathfrak{k}}. \end{aligned} \tag{10.21}$$

Let $W\left(\mathfrak{t} : \mathfrak{k}\right)$ [14] denote the Weyl group associated with the pair $(\mathfrak{t}, \mathfrak{k})$. Since the integrated function on $i\mathfrak{k}$ is K-invariant, by Weyl's integration formula, as in (8.48), from (10.21), we get

$$\begin{aligned} &L^{\mathfrak{g}} \exp\left(t\Delta^{\mathfrak{g}}/2\right)\left[\mathcal{J}_1\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_0\right](0) \\ &\quad = \frac{\mathrm{Vol}\left(K/T\right)}{\left|W\left(\mathfrak{t} : \mathfrak{k}\right)\right|}\int_{i\mathfrak{t}} L^{\mathfrak{g}} \exp\left(t\Delta^{\mathfrak{g}}\right)\left(-h_{\mathfrak{k}}\right) \\ &\qquad \mathcal{J}_1\left(h_{\mathfrak{k}}\right)\mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{t},\mathfrak{k}}\left(h_{\mathfrak{k}}\right)\right]^2 dh_{\mathfrak{k}}. \end{aligned} \tag{10.22}$$

By (2.70), (10.19), we get

$$\mathcal{J}_1\left(h_{\mathfrak{k}}\right)\mathrm{Tr_s}^{S^{\mathfrak{p}}}\left[e^{-h_{\mathfrak{k}}}\right] = (-i)^{m/2}\,\mathrm{Pf}\left[\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}}\right]\widehat{A}^{-1}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{k}}\right). \tag{10.23}$$

Moreover, given our choice of $R^{\mathrm{im}}_{\mathfrak{p},+}$, we have

$$\mathrm{Pf}\left[\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}}\right] = i^{m/2}\prod_{\alpha \in R^{\mathrm{im}}_{\mathfrak{p},+}}\left\langle \alpha, h_{\mathfrak{k}}\right\rangle. \tag{10.24}$$

[14] As before, we use this notation instead of $W\left(\mathfrak{t}_{\mathbf{C}} : \mathfrak{k}_{\mathbf{C}}\right)$ because this Weyl group is real.

If $w \in W(\mathfrak{t}:\mathfrak{k})$, let $\epsilon_w = \pm 1$ be the determinant of w on $\mathfrak{t}$. Using the Weyl character formula, we have the identity

$$\begin{aligned}(10.25)\quad &\left[\pi^{\mathfrak{t},\mathfrak{k}}(h_{\mathfrak{k}})\right]^2 \widehat{A}^{-1}\left(\mathrm{ad}\,(h_{\mathfrak{k}})\,|_{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(e^{-h_{\mathfrak{k}}}\right)\right]\\ &\qquad = (-1)^{\left|R^{\mathrm{im}}_{\mathfrak{t},+}\right|}\pi^{\mathfrak{t},\mathfrak{k}}(h_{\mathfrak{k}})\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_w e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),h_{\mathfrak{k}}\right\rangle}.\end{aligned}$$

By (10.10), and (10.23)–(10.25), we conclude that

$$\begin{aligned}(10.26)\quad &\mathcal{J}_1(h_{\mathfrak{k}})\,\mathrm{Tr}_{\mathrm{s}}{}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{t},\mathfrak{k}}(h_{\mathfrak{k}})\right]^2\\ &\qquad = (-1)^{\left|R^{\mathrm{im}}_{\mathfrak{t},+}\right|}\pi^{\mathfrak{t},\mathfrak{g}}(h_{\mathfrak{k}})\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_w e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),h_{\mathfrak{k}}\right\rangle}.\end{aligned}$$

By (10.26), we obtain

$$\begin{aligned}(10.27)\quad &\int_{i\mathfrak{t}} L^{\mathfrak{g}}\exp\left(t\Delta^{\mathfrak{g}}\right)(-h_{\mathfrak{k}})\,\mathcal{J}_1(h_{\mathfrak{k}})\,\mathrm{Tr}_{\mathrm{s}}{}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{t},\mathfrak{k}}(h_{\mathfrak{k}})\right]^2 dh_{\mathfrak{k}}\\ &\qquad = (-1)^{\left|R^{\mathrm{im}}_{\mathfrak{t},+}\right|}\int_{i\mathfrak{t}} L^{\mathfrak{g}}\exp\left(t\Delta^{\mathfrak{g}}\right)(-h_{\mathfrak{k}})\\ &\qquad\qquad \left(\pi^{\mathfrak{t},\mathfrak{g}}(h_{\mathfrak{k}})\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_w e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),h_{\mathfrak{k}}\right\rangle}\right)dh_{\mathfrak{k}}.\end{aligned}$$

As in (8.40), we will use Rossmann formula in (10.27) with respect to the par $(\mathfrak{t},\mathfrak{u})$. If $e \in \mathfrak{t}^*_{\mathbf{C}}$, we get

$$\begin{aligned}(10.28)\quad &\int_{i\mathfrak{t}} L^{\mathfrak{g}}\exp\left(t\Delta^{\mathfrak{g}}\right)(-h_{\mathfrak{k}})\,\pi^{\mathfrak{t},\mathfrak{g}}(h_{\mathfrak{k}})\,e^{-\langle e,h_{\mathfrak{k}}\rangle}dh_{\mathfrak{k}}\\ &\qquad = (-1)^{\left|R^{\mathrm{im}}_{+}\right|}\pi^{\mathfrak{t},\mathfrak{g}}\left(\frac{e}{2\pi}\right)L^{\mathfrak{g}}(-e)\exp\left(tB^*(e,e)\right).\end{aligned}$$

Also $W(\mathfrak{t}:\mathfrak{k}) \subset W(\mathfrak{t}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$. Moreover, if $w \in W(\mathfrak{t}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$, if ϵ_w still denotes the determinant of w on $\mathfrak{t}$, by [BrD95, Corollary V.4.6 and Lemma V.4.10],

$$(10.29)\qquad \pi^{\mathfrak{t},\mathfrak{g}}(we) = \epsilon_w\pi^{\mathfrak{t},\mathfrak{g}}(e).$$

Finally, $L^{\mathfrak{g}}|_{\mathfrak{t}}$ is $W(\mathfrak{t}_{\mathbf{C}}:\mathfrak{g}_{\mathbf{C}})$-invariant.

By (10.15), (10.22), (10.27), and (10.28), we obtain

$$\begin{aligned}(10.30)\quad &\mathrm{Tr}^{[1]}\left[L\exp\left(-tD^{X,2}\right)\right]\\ &\qquad = \mathrm{Vol}(K/T)\,(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|}\pi^{\mathfrak{t},\mathfrak{g}}\left(\frac{\rho^{\mathfrak{k}}+\lambda}{2\pi}\right)L^{\mathfrak{g}}\left(-\rho^{\mathfrak{k}}-\lambda\right).\end{aligned}$$

By construction

$$L^{\mathfrak{g}}\left(-\rho^{\mathfrak{k}}-\lambda\right)=\phi_{\mathrm{HC}}L\left(-\rho^{\mathfrak{k}}-\lambda\right). \tag{10.31}$$

By [BeGV04, Corollary 7.27], we get

$$\mathrm{Vol}\left(K/T\right)=\frac{1}{\pi^{\mathfrak{t},\mathfrak{k}}\left(\frac{\rho^{\mathfrak{k}}}{2\pi}\right)}. \tag{10.32}$$

Also

$$\left|R^{\mathrm{im}}_{\mathfrak{p},+}\right|=m/2. \tag{10.33}$$

By (10.30)–(10.33), we get the first equation in (10.14). When $L=1$, we can compare (10.17) and this first equation, and we obtain the second equation in (10.14). The proof of our theorem is completed. □

Remark 10.2. Equation (10.14) was obtained by Atiyah-Schmid [ASc77, eq. (3.10)], using Hirzebruch proportionality principle [Hi58], together with formulas like (10.32).

10.4. **The case where $\gamma=k^{-1},\dim\mathfrak{b}=0$.** In this Subsection, we assume that γ is elliptic, i.e., $\gamma=k^{-1},k\in K$. Recall that the orientation of $\mathfrak{p}$ is fixed.

Let $T\subset K$ be a maximal torus, and let $\mathfrak{t}\subset\mathfrak{k}$ be the corresponding Lie algebra. We may and we will assume that $k\in T$, so that $\mathfrak{t}\subset\mathfrak{k}\left(k\right)$. Then T is a maximal torus in $K^{0}\left(k\right)$, and $\mathfrak{t}\subset\mathfrak{k}\left(k\right)$. Let $\kappa\in\mathfrak{t}$ be such that

$$k=e^{\kappa}. \tag{10.34}$$

Then κ is well-defined up to the lattice of integral elements in $\mathfrak{t}$ associated with K. Since k is in the center of $K^{0}\left(k\right)$, if $w\in W\left(\mathfrak{t}:\mathfrak{k}\left(k\right)\right)$, $w\kappa-\kappa$ is integral in $\mathfrak{t}$.

Let $\mathfrak{h}=\mathfrak{b}\oplus\mathfrak{t}$ be the associated fundamental θ-stable Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a θ-stable fundamental Cartan subalgebra of $\mathfrak{z}\left(k\right)$.

In this Subsection, we assume that $\dim\mathfrak{b}=0$. Then $\mathfrak{t}$ is a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}$ and of $\mathfrak{z}\left(k\right)$. As in (10.9), we have

$$R=R^{\mathrm{im}},\qquad\qquad R\left(k\right)=R^{\mathrm{im}}\left(k\right). \tag{10.35}$$

We make the same choice of $R^{\mathrm{im}}_{\mathfrak{t},+},R^{\mathrm{im}}_{\mathfrak{p},+}$ as in Subsection 10.2. Set

$$R^{\mathrm{im}}_{+}\left(k\right)=R^{\mathrm{im}}_{+}\cap R^{\mathrm{im}}\left(k\right). \tag{10.36}$$

Then $R^{\mathrm{im}}_{+}\left(k\right)$ is a positive root system associated with the pair $\left(\mathfrak{t},\mathfrak{z}\left(k\right)\right)$, and we still have

$$R^{\mathrm{im}}_{+}\left(k\right)=R^{\mathrm{im}}_{\mathfrak{p},+}\left(k\right)\cup R^{\mathrm{im}}_{\mathfrak{t},+}\left(k\right). \tag{10.37}$$

The choice of $R^{\rm im}_{\mathfrak{p},+}(k)$ defines an orientation on $\mathfrak{p}(k)$.

The functions $\pi^{\mathfrak{t},\mathfrak{z}(k)}, \pi^{\mathfrak{t},\mathfrak{k}(k)}$ on $\mathfrak{t}$ are given by

$$\pi^{\mathfrak{t},\mathfrak{z}(k)} = \prod_{\alpha\in R^{\rm im}_{+}(k)} \langle \alpha, h\rangle\,, \qquad \pi^{\mathfrak{t},\mathfrak{k}(k)} = \prod_{\alpha\in R^{\rm im}_{\mathfrak{k},+}(k)} \langle \alpha, h\rangle\,. \tag{10.38}$$

Again $\rho^{\mathfrak{k}}, \lambda \in i\mathfrak{t}^*$ are calculated with respect to $R^{\rm im}_{\mathfrak{k},+}$, and $\rho^{\mathfrak{k}(k)} \in i\mathfrak{t}^*$ is obtained via $R^{\rm im}_{\mathfrak{k},+}(k)$. We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by the quadratic form $B|_{\mathfrak{t}}$. In particular $\pi^{\mathfrak{t},\mathfrak{k}(k)}\left(\frac{\rho^{\mathfrak{k}(k)}}{2\pi}\right), \pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{\rho^{\mathfrak{k}}+\lambda}{2\pi}\right)$ are well-defined as well as $\phi_{\rm HC} L\left(-\rho^{\mathfrak{k}}-\lambda\right)$.

Note that $W(\mathfrak{t}:\mathfrak{k}(k)) \subset W(\mathfrak{t}:\mathfrak{k})$. If $w \in W(\mathfrak{t}:\mathfrak{k})$, $e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa\right\rangle}$ depends only on the image of w in $W(\mathfrak{t}:\mathfrak{k}(k)) \setminus W(\mathfrak{t}:\mathfrak{k})$. The same is true for $\epsilon_w \pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{w\left(\rho^{\mathfrak{k}}+\lambda\right)}{2\pi}\right)$.

10.5. **Orbital integrals and index theory: the case of elliptic elements.** We use the same notation as in Subsection 10.4. In particular, $\gamma = k^{-1}, k \in K$.

Let $X(\gamma)$ be the fixed point set of γ in X. Let

$$\widehat{A}^{\gamma}\left(TX|_{X(\gamma)}, \nabla^{TX|_{X(\gamma)}}\right), \mathrm{ch}^{\gamma}\left(F, \nabla^{F}\right)$$

denote the corresponding Atiyah-Bott characteristic forms on $X(\gamma)$, that are defined as in [B11, eqs. (7.7.2) and (7.7.4)].

Theorem 10.3. *If* $\dim \mathfrak{b} > 0$, *then*

$$\begin{aligned} &\mathrm{Tr_s}^{[\gamma]}\left[L\exp\left(-tD^{X,2}\right)\right] = 0,\\ &\left[\widehat{A}^{\gamma}\left(TX|_{X(\gamma)}, \nabla^{TX|_{X(\gamma)}}\right)\mathrm{ch}^{\gamma}\left(F, \nabla^{F}\right)\right]^{\max} = 0. \end{aligned} \tag{10.39}$$

If $\dim \mathfrak{b} = 0$, *then*

$$\begin{aligned} \mathrm{Tr_s}^{[\gamma]}\left[L\exp\left(-tD^{X,2}\right)\right] &= \phi_{\rm HC} L\left(-\rho^{\mathfrak{k}}-\lambda\right)\\ (-1)^{\dim \mathfrak{p}(k)/2} &\frac{1}{\pi^{\mathfrak{t},\mathfrak{k}(k)}\left(\frac{\rho^{\mathfrak{k}(k)}}{2\pi}\right)} \frac{1}{\prod_{\alpha\in R^{\rm im}_{+}\setminus R^{\rm im}_{+}(k)} 2\sinh\left(-\langle\alpha,\kappa\rangle/2\right)}\\ &\sum_{w\in W(\mathfrak{t}:\mathfrak{k}(k))\setminus W(\mathfrak{t}:\mathfrak{k})} \epsilon_w \pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{w\left(\rho^{\mathfrak{k}}+\lambda\right)}{2\pi}\right) e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa\right\rangle}, \end{aligned} \tag{10.40}$$

and

$$\begin{aligned}(10.41)\quad &\left[\widehat{A}^{\gamma}\left(TX|_{X(\gamma)},\nabla^{TX|_{X(\gamma)}}\right)\mathrm{ch}^{\gamma}\left(F,\nabla^{F}\right)\right]^{\max}\\ &=(-1)^{\dim\mathfrak{p}(k)/2}\frac{1}{\pi^{\mathfrak{t},\mathfrak{k}(k)}\left(\frac{\rho^{\mathfrak{k}(k)}}{2\pi}\right)}\frac{1}{\prod_{\alpha\in R_{+}^{\mathrm{im}}\setminus R_{+}^{\mathrm{im}}(k)}2\sinh\left(-\langle\alpha,\kappa\rangle/2\right)}\\ &\qquad\sum_{w\in W(\mathfrak{t}:\mathfrak{k}(k))\setminus W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{w\left(\rho^{\mathfrak{k}}+\lambda\right)}{2\pi}\right)e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa\right\rangle}.\end{aligned}$$

Proof. By [B11, Theorem 7.7.1], for $t>0$, we get

$$\begin{aligned}(10.42)\quad &\mathrm{Tr}_{\mathrm{s}}^{[\gamma]}\left[\exp\left(-tD^{X,2}\right)\right]\\ &\qquad=\left[\widehat{A}^{\gamma}\left(TX|_{X(\gamma)},\nabla^{TX|_{X(\gamma)}}\right),\mathrm{ch}^{\gamma}\left(F,\nabla^{F}\right)\right]^{\max}.\end{aligned}$$

Equation (10.15) still holds. We claim that if $\dim\mathfrak{b}>0$,

$$(10.43)\qquad \mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(k^{-1}e^{-Y_{0}^{\mathfrak{k}}}\right)\right]=0.$$

If $Y_0^{\mathfrak{k}}\in i\mathfrak{k}(k)$, after conjugation by an element of $K^0(k)$, we may assume that $Y_0^{\mathfrak{k}}\in i\mathfrak{t}$. If $\dim\mathfrak{b}>0$, then $1-\mathrm{Ad}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)$ vanishes on $\mathfrak{b}$. The argument we gave after (10.16) shows that (10.43) vanishes. This proves the first equation in (10.39). Combining this equation for $L=1$ and (10.42), we get the second equation in (10.39).

In the sequel, we assume that $\dim\mathfrak{b}=0$. We will use the notation and results of Subsection 10.4, and also equation (10.15). As in (10.21), and with a similar notation, we get

$$\begin{aligned}(10.44)\quad &L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left[\mathcal{J}_{k^{-1}}\left(Y_{0}^{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-Y_{0}^{\mathfrak{k}}}\right)\right]\delta_{0}\right](0)\\ &\qquad=\int_{i\mathfrak{k}(k)}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left(-Y_{0}^{\mathfrak{k}}\right)\\ &\qquad\qquad\mathcal{J}_{k^{-1}}\left(Y_{0}^{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-Y_{0}^{\mathfrak{k}}}\right)\right]dY_{0}^{\mathfrak{k}}.\end{aligned}$$

Using Weyl integration as in (10.22), we deduce from (10.44) that

(10.45)
$$L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left[\mathcal{J}_{k^{-1}}\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-Y_0^{\mathfrak{k}}}\right)\right]\delta_0\right](0)$$
$$=\frac{\mathrm{Vol}\left(K^0(k)/T\right)}{\left|W\left(\mathfrak{t}:\mathfrak{k}(k)\right)\right|}\int_{i\mathfrak{t}}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)(-h_{\mathfrak{k}})$$
$$\mathcal{J}_{k^{-1}}(h_{\mathfrak{k}})\,\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{t},\mathfrak{k}(k)}(h_{\mathfrak{k}})\right]^2 dh_{\mathfrak{k}}.$$

By [B11, eq. (7.7.7)] and using the corresponding notation, if $h_{\mathfrak{k}}\in i\mathfrak{t}$, we have the identity

$$\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]=(-i)^{\dim\mathfrak{p}(k)/2}\,\mathrm{Pf}\left[\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}(k)}\right]$$
$$\widehat{A}^{-1}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}(k)}\right)\left(\widehat{A}^{ke^{h_{\mathfrak{k}}}|_{\mathfrak{p}^{\perp}(k)}}(0)\right)^{-1}. \tag{10.46}$$

By (2.68), (10.46), and by proceeding as in [B11, eq. (7.7.8)], we get

$$\mathcal{J}_{k^{-1}}(h_{\mathfrak{k}})\,\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]=(-i)^{\dim\mathfrak{p}(k)/2}$$
$$\mathrm{Pf}\left[\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}(k)}\right]\widehat{A}^{-1}\left(\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{k}(k)}\right)\widehat{A}^{k^{-1}|_{\mathfrak{p}^{\perp}(k)}}(0)$$
$$\left[\frac{\det\left(1-\mathrm{Ad}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}^{\perp}(k)}}{\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}^{\perp}(k)}}\right]^{1/2}. \tag{10.47}$$

Also we have the identity,

$$\left[\frac{\det\left(1-\mathrm{Ad}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right)|_{\mathfrak{k}^{\perp}(k)}}{\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}^{\perp}(k)}}\right]^{1/2}=\frac{\widehat{A}^{k^{-1}|_{\mathfrak{k}^{\perp}(k)}}(0)}{\widehat{A}^{k^{-1}e^{-h_{\mathfrak{k}}}|_{\mathfrak{k}^{\perp}(k)}}(0)}. \tag{10.48}$$

Using (10.48), we can rewrite (10.47) in the form,

$$\mathcal{J}_{k^{-1}}(h_{\mathfrak{k}})\,\mathrm{Tr}_{\mathrm{s}}^{S^{\mathfrak{p}}}\left[\rho^{S^{\mathfrak{p}}}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]$$
$$=(-i)^{\dim\mathfrak{p}(k)/2}\,\mathrm{Pf}\left[\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}(k)}\right]\widehat{A}^{k^{-1}}(0)\left(\widehat{A}^{k^{-1}}\right)^{-1}\left(-\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{k}}\right). \tag{10.49}$$

As in (10.24), we get

$$\mathrm{Pf}\left[\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{p}(k)}\right]=i^{\dim\mathfrak{p}(k)/2}\prod_{\alpha\in R^{\mathrm{im}}_{\mathfrak{p},+}(k)}\langle\alpha,h_{\mathfrak{k}}\rangle. \tag{10.50}$$

By Weyl's character formula, we obtain

$$\left[\pi^{\mathfrak{t},\mathfrak{k}(k)}(h_{\mathfrak{k}})\right]^2\widehat{A}^{k^{-1}}\left(-\mathrm{ad}\left(h_{\mathfrak{k}}\right)|_{\mathfrak{k}}\right)\mathrm{Tr}^{E}\left[\rho^{E}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]$$
$$=(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right|}\,\pi^{\mathfrak{t},\mathfrak{k}(k)}(h_{\mathfrak{k}})\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_w e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa+h_{\mathfrak{k}}\right\rangle}. \tag{10.51}$$

By (10.49)–(10.51), we find that

$$\begin{aligned}(10.52)\quad &\mathcal{J}_{k^{-1}}\left(h_{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}{}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]\left[\pi^{\mathfrak{t},\mathfrak{k}(k)}\left(h_{\mathfrak{k}}\right)\right]^{2}\\ &=(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right|}\,\widehat{A}^{k^{-1}}\left(0\right)\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(h_{\mathfrak{k}}\right)\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa+h_{\mathfrak{k}}\right\rangle}.\end{aligned}$$

By (10.52), we obtain

$$\begin{aligned}(10.53)\quad &\int_{i\mathfrak{t}}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left(-h_{\mathfrak{k}}\right)\mathcal{J}_{k^{-1}}\left(h_{\mathfrak{k}}\right)\mathrm{Tr}_{\mathrm{s}}{}^{S^{\mathfrak{p}}\otimes E}\left[\rho^{S^{\mathfrak{p}}\otimes E}\left(k^{-1}e^{-h_{\mathfrak{k}}}\right)\right]\\ &\left[\pi^{\mathfrak{t},\mathfrak{k}(k)}\left(h_{\mathfrak{k}}\right)\right]^{2}dh_{\mathfrak{k}}=(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{k},+}(k)\right|}\,\widehat{A}^{k^{-1}}\left(0\right)\\ &\int_{i\mathfrak{t}}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left(-h_{\mathfrak{k}}\right)\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(h_{\mathfrak{k}}\right)\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa+h_{\mathfrak{k}}\right\rangle}dh_{\mathfrak{k}}.\end{aligned}$$

Using Rossmann's formula as in (10.28), if $e\in\mathfrak{t}^{*}_{\mathbf{C}}$, we find that

$$\begin{aligned}(10.54)\quad &\int_{i\mathfrak{t}}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left(-h_{\mathfrak{k}}\right)\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(h_{\mathfrak{k}}\right)e^{-\langle e,h_{\mathfrak{k}}\rangle}dh_{\mathfrak{k}}\\ &=(-1)^{\left|R^{\mathrm{im}}_{+}(k)\right|}\,\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{e}{2\pi}\right)L^{\mathfrak{g}}\left(-e\right)\exp\left(tB^{*}\left(e,e\right)\right).\end{aligned}$$

By (10.54), we get

$$\begin{aligned}(10.55)\quad &\int_{i\mathfrak{t}}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left(-h_{\mathfrak{k}}\right)\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(h_{\mathfrak{k}}\right)\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa+h_{\mathfrak{k}}\right\rangle}dh_{\mathfrak{k}}\\ &=(-1)^{\left|R^{\mathrm{im}}_{+}(k)\right|}\,L^{\mathfrak{g}}\left(-\rho^{\mathfrak{k}}-\lambda\right)\exp\left(tB^{*}\left(\rho^{\mathfrak{k}}+\lambda,\rho^{\mathfrak{k}}+\lambda\right)\right)\\ &\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{w\left(\rho^{\mathfrak{k}}+\lambda\right)}{2\pi}\right)e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa\right\rangle}.\end{aligned}$$

By the considerations that follow (10.34) and by (10.55), we obtain

$$\begin{aligned}(10.56)\quad &\int_{i\mathfrak{t}}L^{\mathfrak{z}(k)}\exp\left(t\Delta^{\mathfrak{z}(k)}\right)\left(-h_{\mathfrak{k}}\right)\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(h_{\mathfrak{k}}\right)\sum_{w\in W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa+h_{\mathfrak{k}}\right\rangle}dh_{\mathfrak{k}}\\ &=(-1)^{\left|R^{\mathrm{im}}_{+}(k)\right|}\left|W\left(\mathfrak{t}:\mathfrak{k}\left(k\right)\right)\right|L^{\mathfrak{g}}\left(-\rho^{\mathfrak{k}}-\lambda\right)\exp\left(tB^{*}\left(\rho^{\mathfrak{k}}+\lambda,\rho^{\mathfrak{k}}+\lambda\right)\right)\\ &\sum_{w\in W(\mathfrak{t}:\mathfrak{k}(k))\backslash W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{w\left(\rho^{\mathfrak{k}}+\lambda\right)}{2\pi}\right)e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa\right\rangle}.\end{aligned}$$

By (10.15), (10.45), (10.53), and (10.56), we get

(10.57)
$$\mathrm{Tr}_{\mathrm{s}}^{[\gamma]}\left[L\exp\left(-tD^{X,2}\right)\right]=\mathrm{Vol}\left(K^{0}\left(k\right)/T\right)(-1)^{\left|R^{\mathrm{im}}_{\mathfrak{p},+}(k)\right|}\,\widehat{A}^{k-1}\left(0\right)$$
$$\phi_{\mathrm{HC}}L\left(-\rho^{\mathfrak{k}}-\lambda\right)\sum_{w\in W(\mathfrak{t}:\mathfrak{k}(k))\backslash W(\mathfrak{t}:\mathfrak{k})}\epsilon_{w}\pi^{\mathfrak{t},\mathfrak{z}(k)}\left(\frac{w\left(\rho^{\mathfrak{k}}+\lambda\right)}{2\pi}\right)e^{-\left\langle w\left(\rho^{\mathfrak{k}}+\lambda\right),\kappa\right\rangle}.$$

As in (10.32), we obtain

$$\mathrm{Vol}\left(K^{0}\left(k\right)/T\right)=\frac{1}{\pi^{\mathfrak{t},\mathfrak{k}(k)}\left(\frac{\rho^{\mathfrak{k}(k)}}{2\pi}\right)}. \tag{10.58}$$

As in (10.33), we get

$$\left|R^{\mathrm{im}}_{\mathfrak{p},+}\left(k\right)\right|=\dim\mathfrak{p}\left(k\right)/2. \tag{10.59}$$

Moreover,

$$\widehat{A}^{k-1}\left(0\right)=\frac{1}{\prod_{\alpha\in R^{\mathrm{im}}_{+}\backslash R^{\mathrm{im}}_{+}(k)}2\sinh\left(-\left\langle\alpha,\kappa\right\rangle/2\right)}. \tag{10.60}$$

By (10.57)–(10.60), we get (10.40), which combined with (10.42) gives (10.41). The proof of our theorem is completed. □

Remark 10.4. Equation (10.41) can also be obtained by a suitable application of Hirzebruch proportionality principle [Hi58] similar to what was done by Atiyah-Schmid [ASc77].

10.6. **Orbital integrals and a conjecture by Vogan.** Let π be an irreducible unitary representation of G acting on a Hilbert space V_{π}. By [Kn86, p. 205], a vector $v\in V_{\pi}$ is called K-finite if the vector subspace generated by the vectors $kv|_{k\in K}$ has finite dimension. Let $V_{\pi,K}\subset V_{\pi}$ be the vector subspace of the K-finite vectors in V_{π}. By [Kn86, Proposition 8.5], $U(\mathfrak{g}_{\mathbf{C}})$ acts on $V_{\pi,K}$, so that $V_{\pi,K}$ is a $(\mathfrak{g}_{\mathbf{C}},K)$-module.

Let $e_1,\ldots,e_m$ be an orthonormal basis of $\mathfrak{p}$. Let $D\in U(\mathfrak{g})\otimes c(\mathfrak{p})$ be the Dirac operator,

$$D=\sum_{i=1}^{m}c(e_i)e_i. \tag{10.61}$$

We denote by $D|_{V_{\pi,K}\otimes S^{\mathfrak{p}}}$ the restriction of D to $V_{\pi,K}\otimes S^{\mathfrak{p}}$. By [HuP02, p. 189], the Dirac cohomology of $V_{\pi,K}$ is the K-module defined by

$$H_{D}(V_{\pi,K})=\ker D|_{V_{\pi,K}\otimes S^{\mathfrak{p}}}. \tag{10.62}$$

By [Kn86, Theorem 8.1], each K-type in $H_D(V_{\pi,K})$ has finite multiplicity.

The Vogan conjecture, solved by Huang-Pandžić [HuP02, Corollary 2.4] states the following.

Theorem 10.5. *If the Dirac cohomology $H_D(V_{\pi,K})$ contains a K-type of highest weight $\lambda \in i\mathfrak{t}^*$, the infinitesimal character of $V_{\pi,K}$ is $\rho^{\mathfrak{k}} + \lambda$.*

An equivalent formulation of Theorem 10.5 says that if E is an irreducible K representation of highest weight $\lambda \in i\mathfrak{t}^*$, if $D^{S^{\mathfrak{p}}\otimes E}$ denotes the restriction of D to $(V_{\pi,K} \otimes S^{\mathfrak{p}} \otimes E)^K$, if $\ker D^{S^{\mathfrak{p}}\otimes E} \neq 0$, then $L \in Z(\mathfrak{g})$ acts on $V_{\pi,K}$ as the scalar $\phi_{\mathrm{HC}} L\left(-\rho^{\mathfrak{k}} - \lambda\right)$.

We will show that (10.14) and (10.40) are compatible with Theorem 10.5. Let Γ be a discrete cocompact subgroup of G. By [GGP90, Theorem, p.23], we have

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\widehat{G}_u}^{\mathrm{Hil}} n_\Gamma(\pi) V_\pi, \tag{10.63}$$

with $n_\Gamma(\pi) \in \mathbf{N}$.

We use the notation of Subsection 10.1. In particular, we assume that ρ^E is an irreducible representation of K with highest weight $\lambda \in i\mathfrak{t}^*$. Let Z be the compact orbifold $Z = \Gamma \setminus X$. The vector bundle F on X descends to an orbifold vector bundle on Z, which we still denote F. Also D^X descends to the orbifold Dirac operator D^Z. By (10.63), we have

$$\ker D^Z = \bigoplus_{\pi\in\widehat{G}_u} n_\Gamma(\pi)(H_D(V_{\pi,K}) \otimes E)^K. \tag{10.64}$$

Since $\ker D^Z$ is finite dimensional, the sum in the right-hand side only contains finitely many nonzero terms. By Theorem 10.5, $L \in Z(\mathfrak{g})$ acts on $\ker D^Z$ as $\phi_{\mathrm{HC}} L\left(-\rho^{\mathfrak{k}} - \lambda\right)$.

Using the McKean-Singer formula [MS67] and the above, for $t > 0$, we get

$$\mathrm{Tr}_{\mathrm{s}}\left[L \exp\left(-tD^{Z,2}\right)\right] = \phi_{\mathrm{HC}} L\left(-\rho^{\mathfrak{k}} - \lambda\right) \mathrm{Tr}_{\mathrm{s}}\left[\exp\left(-tD^{Z,2}\right)\right]. \tag{10.65}$$

Also $\mathrm{Tr}_{\mathrm{s}}\left[L \exp\left(-tD^{Z,2}\right)\right]$ can be evaluated in terms of corresponding orbital integrals using Selberg's trace formula.

Assume first that Γ is torsion free. By equation (10.12) in Theorem 10.1, only the identity element contributes to the above supertrace. Then equation (10.14) can be viewed as a consequence of (10.17), (10.65).

When Γ is not torsion free, only the finite number of conjugacy classes of elliptic elements in Γ contribute to (10.65). Then equations (10.40), (10.42) are compatible to (10.65).

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References

[ABo67] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407, 1967.

[ABo68] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491, 1968.

[ASc77] M. F. Atiyah and W. Schmid. A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math.*, 42:1–62, 1977.

[AS68a] M. F. Atiyah and I. M. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968.

[AS68b] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[BaGS85] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of nonpositive curvature*, volume 61 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[BeGV04] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[B11] J.-M. Bismut. *Hypoelliptic Laplacian and orbital integrals*, volume 177 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2011.

[BL99] J.-M. Bismut and F. Labourie. Symplectic geometry and the Verlinde formulas. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 97–311. Int. Press, Boston, MA, 1999.

[BrD95] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[CP81] J. Chazarain and A. Piriou. *Introduction à la théorie des équations aux dérivées partielles linéaires*. Gauthier-Villars, Paris, 1981.

[Du70] M. Duflo. Caractères des groupes et des algèbres de Lie résolubles. *Ann. Sci. École Norm. Sup. (4)*, 3:23–74, 1970.

[E96] P. B. Eberlein. *Geometry of nonpositively curved manifolds*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[GGP90] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro. *Representation theory and automorphic functions*, volume 6 of *Generalized Functions*. Academic Press, Inc., Boston, MA, 1990. Translated from the Russian by K. A. Hirsch, Reprint of the 1969 edition.

[HC56] Harish-Chandra. The characters of semisimple Lie groups. *Trans. Amer. Math. Soc.*, 83:98–163, 1956.

[HC57a] Harish-Chandra. Differential operators on a semisimple Lie algebra. *Amer. J. Math.*, 79:87–120, 1957.

[HC57b] Harish-Chandra. A formula for semisimple Lie groups. *Amer. J. Math.*, 79:733–760, 1957.

[HC64] Harish-Chandra. Some results on an invariant integral on a semisimple Lie algebra. *Ann. of Math. (2)*, 80:551–593, 1964.

[HC65] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.

[HC66] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966.

[HC75] Harish-Chandra. Harmonic analysis on real reductive groups. I. The theory of the constant term. *J. Functional Analysis*, 19:104–204, 1975.

[Hi58] F. Hirzebruch. Automorphe Formen und der Satz von Riemann-Roch. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 129–144. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[Hö83] L. Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[Hö85] L. Hörmander. *The analysis of linear partial differential operators. III*. Grundl. Math. Wiss. Band 274. Springer-Verlag, Berlin, 1985. Pseudo-differential operators.

[HuP02] J.-S. Huang and P. Pandžić. Dirac cohomology, unitary representations and a proof of a conjecture of Vogan. *J. Amer. Math. Soc.*, 15(1):185–202, 2002.

[K79] T. Kawasaki. The Riemann-Roch theorem for complex V-manifolds. *Osaka J. Math.*, 16(1):151–159, 1979.

[Kn86] A. W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[Kn02] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[Ko73] B. Kostant. On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sci. École Norm. Sup. (4)*, 6:413–455 (1974), 1973.

[Ko76] B. Kostant. On Macdonald's η-function formula, the Laplacian and generalized exponents. *Advances in Math.*, 20(2):179–212, 1976.

[MS67] H. P. McKean, Jr. and I. M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.

[R78] W. Rossmann. Kirillov's character formula for reductive Lie groups. *Invent. Math.*, 48(3):207–220, 1978.

[Se56] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956.

[T81] M. E. Taylor. *Pseudodifferential operators*, volume 34 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1981.

[Va77] V. S. Varadarajan. *Harmonic analysis on real reductive groups*. Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977.

[V79] M. Vergne. On Rossmann's character formula for discrete series. *Invent. Math.*, 54(1):11–14, 1979.

[W88] N. R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

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