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COMPLEXITIES OF DIFFERENTIABLE DYNAMICAL SYSTEMS

PIERRE BERGER

Abstract. We define the notion of localizable property for a dynamical system. Then we survey three properties of complexity and relate how they are known to be typical among differentiable dynamical systems. These notions are the fast growth of the number of periodic points, the positive entropy and the high emergence. We finally propose a dictionary between the previously explained theory on entropy and the ongoing one on emergence.

Introduction

A differentiable dynamical systems is a map $f \in C^r(M, M)$ of a manifold M, for $1 \leq r \leq \infty$. Our aim is to understand the properties of the orbits $(f^n(x))_{n\geq 0}$ of the points $x \in M$, where $f^n(x) = f \circ \cdots \circ f(x)$. Let us start from the abstract viewpoint. A property $(\mathscr{P})$ is a proposition involving the orbits of a dynamics which may be true, wrong or undecidable. This defines a function:

$$\mathscr{P} : C^r(M, M) \to \{\text{true}, \text{wrong}, \text{undecidable}\} .$$

We would like to study properties which, roughly speaking, occur for all systems but some in a negligible set in some rigorous sense. Such properties will be said to be *typical*. A similar way of description already occurred in singularity theory of differentiable mappings: Whitney [Whi34] showed that every closed set of a manifold M is the zero set of a function. That is why Thom and then Mather restricted their study of singular set to differentiable maps in an open and dense set.

In this article we will see that *typically* the behavior of a dynamical system is much more complex than expected, for properties involving the growth of the number of periodic points, the value of entropy, or a recent notion of emergence. This complexity will regard mostly localizable properties.

Definition 0.1. *A property $(\mathscr{P})$ on $\mathrm{Diff}^r(M, M)$ is* localizable*, if the following condition holds true: for every open set $U \subset M$, $N \geq 0$ and $f \in \mathrm{Diff}^r(M)$ such that $f^N|_U = id|_U$, there exists a C^r-perturbation $\tilde{f}$ of f which satisfies $(\mathscr{P})$.*

A property $(\mathscr{P})$ on $\mathrm{Diff}^r(M)$ is openly localizable*, if the perturbation $\tilde{f}$ can be chosen in a C^r-open set.*

A property $(\mathscr{P})$ is genericaly localizable*, if it is implied by the conjunction of a countable number of openly localizable properties $\{(\mathscr{P}_i) : i \geq 0\}$:*

$$\bigwedge_i (\mathscr{P}_i) \Rightarrow (\mathscr{P}) .$$

This definition is inspired by the following result:

Theorem 0.2 (Turaev [Tur15]). *For every surface M and $r \geq 2$, there is a non-empty open set $\mathcal{N}^r \subset \mathrm{Diff}^r(M)$ such that for a dense set $\mathcal{D} \subset \mathcal{N}^r$, for every $f \in \mathcal{D}$, there exist $U \subset M$ and $N \geq 1$ such that $f^N|_U = id|_U$.*

An immediate consequence of this result is:

Corollary 0.3. *Among surface C^r-diffeomorphisms, $r \geq 2$, a generically localizable property $(\mathcal{P})$ is locally generic: there exists $\mathcal{R} \subset \mathcal{N}^r$ topologically generic (as defined in the sequel) such that every $f \in \mathcal{R}$ satisfies $(\mathcal{P})$.*

We will discuss in the next sections about the following localizable properties:

(1) displaying a fast growth of the number of periodic points (which is generically localizable),

(2) having positive metric entropy (which is localizable),

(3) having a high emergence (which is generically localizable).

As a matter of fact, these properties are valid on sets which are typical for different notions.

1. Typicality

In this section we recall and develop some notions of typicality. This will enable better interpretations of Problem 2.4 and also Conjectures 3.2 and 4.7 which are part of the dictionary given Section 4.3.

For the sake of simplicity, let us assume that M is compact. Let $1 \leq r \leq \infty$ and let $\mathcal{U}$ be a non-empty, open subset of $C^r(M, M)$ endowed with its canonical topology.

1.1. Topological notions of typicallity.

From the topological viewpoint, there are the following notions of typicality:

Definition 1.1. *A property is* open and dense *in $\mathcal{U}$ if it holds true at an open and dense subset of $\mathcal{U}$. A property is* topologically generic *in $\mathcal{U}$ if it holds true at a countable intersection of open and dense subsets of $\mathcal{U}$. A property is* dense *in $\mathcal{U}$ if it holds true at a dense subset of $\mathcal{U}$*

As $\mathcal{U}$ is a Baire space (a countable intersection of open and dense sets is a dense set), the above notions of typicality are linked as follows, where $(\neg\mathcal{P})$ is the negation of $(\mathcal{P})$:

$$\begin{cases} (\mathcal{P}) \text{ is open and dense} \Rightarrow (\mathcal{P}) \text{ is topologically generic} \Rightarrow (\mathcal{P}) \text{ is dense} \\ (\neg\mathcal{P}) \text{ is dense} \Rightarrow (\mathcal{P}) \text{ is } \textit{not} \text{ open and dense} \\ (\neg\mathcal{P}) \text{ is topologically generic} \Rightarrow (\mathcal{P}) \text{ is } \textit{not} \text{ topologically generic} \end{cases}$$

The main failure of density as a notion of typicality is that the intersection of two dense sets might be empty (and so not dense), whereas it is reasonable to ask for the typicality of the conjunction of two typical properties.

On the other hand, a finite intersection of open and dense sets is still open and dense, and a countable intersection of topologically generic sets is topologically generic as well. However, these notions of typicality display others problems. The notion of open and dense property as typicality is very natural, but in practical it is often too restrictive: for instance having all periodic points

with eigenvalues $\neq 1$ is not an open and dense property (this has been proved first in [New74], and is also a consequence of Theorem 0.2).

The main problem with the notion of topological genericity is that such subsets might be negligible in the measure theoretical viewpoint. Indeed, there are topologically generic subsets of $\mathbb{R}$ whose Lebesgue measure is zero. That is why Kolmogorov [Kol57], motivated by von Neumann [vN43], proposed a notion of typicality which involves probability.

1.2. **Probabilitic notions of typicality.** In his ICM plenary talk [Kol57], Kolmogorov briefly proposed to consider deformation of a system f_0 along a parameter family $(f_a)_a$ such that

$$f_a(x) = f_0(x) + a\phi(x, a)$$

and wonder if a property persists for Lebesque almost every parameter a small. Kolmogorov was proposing an initial working frame with ϕ analytic, but with perspective in finite regularity C^r, $r < \infty$. The differentiable counterpart of this notion should involve C^r-families of mappings $(f_a)_a$ passing through the map f_0. However, the situation is here much less rigid, and one must avoid the family to lie in an exceptional region. That is why one need to re-interpret this notion. There were several interpretations of this notion of typicality [IL99, KH07]. Let us chose the following, involving the unit ball B^d of $\mathbb{R}^d$:

Definition 1.2. *A property* $(\mathscr{P})$ *is* d-Kolmogorov typical *in* $\mathcal{U}$*, if there exists a Baire residual set* $\mathcal{R} \subset \{(f_a)_a \in C^r(B^d \times M; M) : f_a \in \mathcal{U},\ \forall a \in B^d\}$ *such that for every* $(f_a)_{a\in B^d} \in \mathcal{R}$*, for Lebesgue a.e. parameter* $a \in B^d$*, the dynamics* f_a *satisfies property* $(\mathscr{P})$*.*

Note that 0-Kolmogorov typicality is the topological genericity. For $d > 0$, d-Kolmogorov typicality implies the density of a phenomena, but not necessarily its topological genericity.

We notice that a d-C^r-Kolmogorov typical subset contains Lebesgue a.e. points of (unaccountably many) submanifolds of $C^r(M, M)$ of dimension d. On the other hand, one can construct (artifical) d-C^r-Kolmogorov typical sets which intersects any $d+1$-submanifold of $C^r(M, M)$ at a set of Lebesgue measure 0 [IL99]. The latter point is the main defect of this notion. To overcome this, a way is to introduce:

Definition 1.3. *A property* $(\mathscr{P})$ *is* eventually Kolmogorov typical *in* $\mathcal{U}$*, if it is* d*-Kolmogorov typical for every* $d \geq 0$ *sufficiently large.*

Another similar notion, which is not weaker nor stronger is the following:

Definition 1.4. *A property* $(\mathscr{P})$ *is* d-C^r-abundant *in* $C^r(M; M)$*, if there exist* $d \geq 1$ *and a non-empty open subset* $\mathcal{V} \subset C^r(B^d \times M; M)$ *such that for every* $(f_a)_{a\in B^d} \in \mathcal{V}$*, there is a parameter set* $E \subset B^d$ *of positive Lebesgue measure such that for every* $a \in E$*, the map* f_a *satisfies property* $(\mathscr{P})$*.*

Note that there exist properties which are abundant and whose contrary is also abundant. This notion is related to the Kolmogorov typicality as follows:

$$\neg(\mathscr{P}) \text{ is } d\text{-}C^r\text{-Kolmogorov typical} \Rightarrow (\mathscr{P}) \text{ is } \textit{not } d\text{-}C^r\text{-abundant}$$

As the relevance of abundance is incontestable, so is the notion of Komogorov typicality.

In the next years I will be working on the following conjecture:

Conjecture 1.5. *If the dimension of M is sufficiently large, for every $2 \leq r \leq \infty$, every property which is generically localizable in $\mathrm{Diff}^r(M)$ is d-Kolmogorov typical in some open set of $\mathrm{Diff}^r(M)$, for every $d \geq 0$.*

For the sake of completeness, we shall recall the notion of prevalence [HSY92], which is another probabilistic notion of typicality. A subset of a vector space V is *prevalent* if its complement is shy. A set S is *shy* if there exists a measure μ on V which has positive and finite value on a compact set and such that $\mu(S+v) = 0$ for every $v \in V$. An important case is when μ coincides with the Lebesgue measure on a finite dimensional set $E \subset B$. Then a set is shy if at every $p \in S$, the affine line $p + E$ intersects S at a subset of Lebesgue measure 0.

A substantial problem of the notion of shy & prevalence is that the parameter space $C^r(M, M)$ does not display a canonical vector structure in general. Of course it is a Frechetic manifold and so it is locally modeled by a Frechet space, however a coordinate change of $C^r(M, M)$ nor of M does not preserve this structure. In particular, prevalence has no reason to be invariant by these coordinate changes.

2. Growth of the number of periodic points

A natural dynamically defined set is the one of its n-periodic points

$$Per_n f := \{x \in M : f^n(x) = x\} .$$

To study its cardinality, we consider also the subset $Per_n^0 f \subset Per_n f$ of its isolated points. We notice that the cardinality of $Per_n^0 f$ is invariant by conjugacy. Hence it is natural to study the growth of this cardinality with n.

Clearly, if f is a polynomial map, the cardinality $Per_n^0 f$ is bounded by the degree of f^n, which grows at most exponentially fast [FM89, DNT16].

The first study in the C^∞-case goes back to Artin and Mazur [AM65] who proved the existence of a dense set $\mathcal{D}$ in $\mathrm{Diff}^r(M)$, $r \leq \infty$, so that for every $f \in \mathcal{D}$, the number $Card\, Per_n^0 f$ grows at most exponentially, i.e. :

$$\text{(A-M)} \qquad \limsup_{n\to\infty} \frac{1}{n} \log Card\, Per_n^0 f < \infty .$$

This leads Smale [Sma67] and Bowen [Bow78] to wonder whether (A-M) diffeomorphisms are generic. Finally Arnold asked the following problems:

Problem 2.1 (Smale 1967, Bowen 1978, Arnold Pb. 1989-2 [Arn04])**.** *Can the number of fixed points of the nth iteration of a topologically generic infinitely smooth self-mapping of a compact manifold grow, as n increases, faster than any prescribed sequence $(a_n)_n$ (for some subsequences of time values n)?*

This problem enjoys a long tradition. In dimension 1, Martens-de Melo-van Strien [MdMvS92] showed that for every $\infty \geq r \geq 2$, for an open and dense set [1] of C^r-maps, the number of periodic points grows at most exponentially fast.

1. whose complement is the infinite codimentional manifold formed by maps with at least one flat critical point.

In higher dimension, Kaloshin [Kal99] proved the existence of a dense set $\mathcal{D}$ in $\mathrm{Diff}^r(M)$, $r<\infty$, such that the cardinality of $Per_n f \supset Per_n^0 f$ is finite and grows at most exponentially fast. On the other hand, he proved in [Kal00] that whenever $2 \le r < \infty$ and $\dim M \ge 2$, *a locally topologically generic diffeomorphism displays a fast growth of the number of periodic points*: there exists an open set $\mathcal{U} \subset \mathrm{Diff}^r(M)$, such that for any sequence of integers $(a_n)_n$, a topologically generic $f \in \mathcal{U}$ satisfies:

$$(\star) \qquad \limsup_{n\to\infty} \frac{Card\, P_n^0(f)}{a_n} = \infty \ .$$

Proposition 2.2. *To display a fast growth on the number of periodic points is a generically localizable property in* $\mathrm{Diff}^r(M)$*, for every* $r \ge 1$*.*

Proof. Let $(a_j)_j \in \mathbb{N}^{\mathbb{N}}$. Let $(\mathcal{P}_j)$ be the property that f has $\ge a_j$-saddle points of period $\ge j$.

We observe that $(\mathcal{P}_j)$ is an openly localizable property. Indeed, let $U \subset M$ be an open subset and N an integer such that $f^N|_U = id|_U$. We may take N and U smaller such that $f^k(U)$ does not intersect U, and then do a small perturbation $\tilde{f}$ supported by U which satisfies $(\mathcal{P}_j)$. The property is open by hyperbolic continuation of the saddle points. Obviously the conjunction $\bigwedge_j(\mathcal{P}_j)$ implies the fast growth of the number of periodic points. □

Corollary 0.3 of Turaev's Theorem and Proposition 2.2 implies that Kaloshin's theorem is also valid in the space of C^∞-surface diffeomorphisms. In dimension ≥ 3, this result has been extended to the C^1-case and to the C^∞-case in [Ber17b, AST18]. In summary, we have the following description:

Theorem 2.3. *Let* $\infty \ge r \ge 2$ *and let* M *be a compact manifold of dimension* n*.*

- *If* $n=1$*, Property* (AM) *is satisfied by an open and dense set of* C^r*-self-mappings.*
- *If* $n \ge 2$*, there exists a (non-empty) open set* $\mathcal{U} \subset \mathrm{Diff}^r(M)$ *so that given any sequence* $(a_n)_n$ *of integers, a topologically generic* f *in* $\mathcal{U}$ *satisfies* $(\star)$*.*

The conservative counterpart of this result was established in [KS06]. For a stronger sense of typicality, it has been asked [2]:

Problem 2.4 (Arnold 1992-13 [Arn04])**.** *Prove that a typical, smooth, self-map* f *of a compact manifold satisfies that* $(Card\, Per_n f)_n$ *grows at most exponentially fast.*

To provide a positive answer to this Problem, Hunt and Kaloshin [KH07, KH] used a method described in [GHK06] to show that for $\infty \ge r > 1$, a *prevalent* C^r-diffeomorphism (for a chosen vector structure) satisfies:

$$(\Diamond) \qquad \limsup_{\infty} \frac{\log P_n(f)}{n^{1+\delta}} = 0, \quad \forall \delta > 0 \ .$$

However the latter did not completely solve Arnold's Problem 2.4 in particular because the notion of prevalence is *a priori* independent to the notion of typicality initially meant by Arnold. Indeed this problem was formulated for typicality [3] in the sense of definition 1.2. Actually following this notion, the answer to Arnold's Problem 2.4 is negative in the finitely smooth case:

2. Many other Arnold's problems are related to this question [Arn04, 1994-47, 1994-48, 1992-14].
3. See explanation below problem 1.1.5 in [KH07].

Theorem 2.5 ([Ber17b]). *Let $1 \leq r < \infty$, $0 \leq d < \infty$ and let M be a manifold of dimension ≥ 2.*

Then there exists a (non-empty) open set $\hat{\mathcal{U}}$ of C^r-families $(f_a)_{a \in B^d}$ of C^r-self-mappings f_a of M such that, for any sequence of integers $(a_n)_n$, a topologically generic $(f_a)_a \in \hat{\mathcal{U}}$ consists of maps f_a satisfying $(\star)$ for every $a \in B^d$. Moreover if $\dim M \geq 3$, the maps f_a are diffeomorphisms.

The symplectic counterpart of the above result has been proved previously in [Asa16]. Both confirm Conjecture 1.5.

3. Entropy

In his revolutionary work [Poi92], Poincaré showed the existence of an open class of systems which display an infinite, invariant, set E in which all the points have pairwise different asymptotic behaviors. His example was given by the so-called restricted 3-body problem in celestial mechanic. Mathematically, it is a Hamiltonian, analytic maps of a surface. Nowadays, the dynamics of this example (as most of the symplectic dynamics) is poorly understood. For instance, we do not know if E has zero Lebesgue measure for the Poincaré example.

The concept of ergodicity – introduced by Boltzmann – may enable to understand most orbits of such systems. The idea is that the statistical behavior of the orbit may be the same for "most" of the point in E. The statistical behavior of the orbit of a point x is given by the sequence of measures $\mathsf{e}_n(x)$ equidistributed on the first iterates up to time n. These are called the *empirical measures*, and are defined as follow (when the time is discrete):

$$\mathsf{e}_n(x) = \frac{1}{n} \sum_{0 \leq k < n} \delta_{f^k(x)} \ .$$

A probability measure μ is *ergodic* if for μ-a.e. every point x, the sequence $(\mathsf{e}_n(x))_n$ converges to μ. In other words, the time average of the orbit of μ-a.e. point is equal to the space average. The first example of differentiable realization of such systems was done first for billiards [Had98] and then for geodesic flow of negatively curved compact manifolds by Hedlund and Hopf [Hed39, Hop39].

To quantify the sensitivity to the initial conditions, we consider the sequence of distances:

$$d_n(x, y) := \max_{0 \leq i < n} d(f^i(x), f^i(y)) \ .$$

The concept of entropy was introduced by Kolmogorov in dynamics, and reformulated as follows by Katok [Kat80].

Definition 3.1 (Entropy). *Given an ergodic measure μ, let $\mathcal{N}_n^\mu(\epsilon)$ be the minimal number of d_n-balls of radius $\epsilon > 0$ whose union has μ-measure $\geq 1 - \epsilon$. The entropy h_μ of μ is:*

$$h_\mu := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathcal{N}_n^\mu(\epsilon) \ .$$

Hence entropy measures the complexity of the asymptotic behavior. It turns out that ergodic measures of positive entropy are well modeled. By Austin's theorem [Aus18], every ergodic automorphism of positive entropy is isomorphic to the direct product of a Bernoulli shift and a transformation of arbitrarily small entropy.

After the work of Anosov and Sinai, it has been shown that the Liouville measure under the action of a geodesic flow of a negatively curved surface has positive entropy. This example has been

generalized to discrete time, by Anosov and then Smale [Sma67] to reach the notion of uniformly hyperbolic dynamics. Such systems are such that their non-wandering set is locally maximal and hyperbolic. An invariant set K is *hyperbolic* if there is a continuous splitting $E^s \oplus E^u = TM|_K$ by two invariant bundles E^s and E^u which are respectivelly contracted and expanded by the differential of the dynamics. By the work of Sinai, Ruelle and Bowen [BR75], we know that every uniformly hyperbolic smooth dynamics displays a finite number of ergodic probability measures whose basins cover all the manifold modulo a set of Lebesgue zero. We recall that the *basin* B_μ of an ergodic probability measures μ is

$$B_\mu := \{x \in M : \mathsf{e}_n(x) \rightharpoonup \mu\} .$$

In this extend, a special interest is given ergodic measure which are *physical*: those whose basin has positive Lebesgue measure.

3.1. **The positive entropy conjecture.** A diffeomorphism is said to have *positive (metric) entropy* if it leaves invariant an ergodic, physical probability with positive entropy. Here is one of the most important and difficult question in this field:

Conjecture 3.2. *A typical symplectic, surface diffeomorphisms has positive entropy.*

Here "typical" may be view as occurring in some natural examples (from physical science): for instance the Poincaré example, or as conjectured by Sinai [Sin94, P. 144], the Chirikov standard maps. This may be seen also following one of the notions of typicality of the first section.

Herman proposed the following (weak) version of the positive entropy conjecture:

Conjecture 3.3 ([Her98])**.** *There are conservative, infinitely smooth perturbations of the identity of the* 2*-disk with positive metric entropy.*

Note that this conjecture implies that having positive entropy is a localizable property.

In the same paper, he also wondered if the set of surface, conservative maps of the disk with positive metric entropy are dense (in the C^∞-topology), or if its interior is non-empty. Herman's conjecture has been solved recently:

Theorem 3.4 ([BT19])**.** *In the open set of conservative C^∞-surface maps which display an elliptic periodic point, those with positive metric entropy form a dense subset.*

A map which robustly does not display any periodic elliptic point is called *weakly stable*. The conservative counterpart of the Mañe conjecture [Mañ82] states that weakly stable maps are uniformly hyperbolic. As a conservative, uniformly hyperbolic map has positive entropy, this theorem implies, *up to the latter conjecture*, the density of the positive metric entropy. This theorem was proved by showing that a dynamics constructed by surgery from a uniformly hyperbolic system (similar to Przytycki [Prz82] example), appears after perturbation and renormalization. We know that these examples do not appear among entire maps [Ush80]. Also these examples are not locally generic nor typical in the sense of Kolmogorov [Her83].

Hence to solve Conjecture 3.2 (if it is correct!), one needs to find new models. In order to do so, Yoccoz' strong regularity program proposes [Yoc97b, BY19] to look for geometrical and combinatorial definition of invariant sets which display an ergodic, physical measure with positive

entropy. The idea of this program is to work on paradigmatic and abundant examples of (not necessarily conservative) smooth surface maps, such that the dimension of the measure is higher and higher. The *dimension of a measure* μ is the infimum of the Hausdorff dimension of a set E such that $\mu(E) = 1$.

The first paradigmatic examples were Yoccoz' strongly regular quadratic maps [Yoc97a, Yoc19] which allowed him to give an alternative proof of Jakobson theorem [Jak81]. In [PY09], the strongly regular horseshoes were introduced and shown to be abundant. In [Ber19a], strongly regular Hénon-like attractors were introduced and shown to be abundant. This gives an alternative proof of Benedicks-Carleson theorem [BC91] and solves the second step of Yoccoz's program as stated during his first lecture at Collège de France (and the main source of inspiration of his last, unfinished lecture [Yoc16]). In [BY19], a few conjectures are stated to develop this theory.

3.2. **Description of measure with positive entropy.** Pesin theory describes the ergodic measures of positive entropy of a (possibly not conservative) diffeomorphism of class C^r, with $r > 1$.

We recall that by Oseledets' theorem, for any ergodic measure μ, there exists a Df-invariant splitting $T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$ depending measurably of μ at a.e. x and such that:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n | E^s\| < 0 \quad \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n | E^c\| = 0 \quad \text{and} \quad \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n | E^u\| > 0 \ .$$

Furthermore, with $u = \dim E^u$, there exists a flag $0 \subset E^1 \subset E^2 \subset \cdots \subset E^{u-1} \subset E^u$ associated to values $0 < \lambda_1 \leq \cdots \leq \lambda_u$ such that any vector of $E_{i+1} \setminus E_i$ is asymptotically expanded by a factor $\exp(\lambda_i)$. The numbers $0 < \lambda_1 \leq \cdots \leq \lambda_u$ are called the *positive Lyapunov exponents* and their sum is denoted by Σ_+.

Theorem 3.5 (Ruelle inequality [Rue78])**.** *It holds:* $h_\nu \leq \Sigma_+ \leq \dim M \cdot \int \log \|Df\| d\nu$.

Hence a measure with positive entropy has necessarily a positive Lyapunov exponent. By the Pesin theorem [Pes76, FHY83], for μ-a.e. x, there exist immersed manifolds $W^u(x)$ and $W^s(x)$ tangent to respectively $E^u(x)$ and $E^s(x)$ such that

$$W^s(x; f) := \left\{ y \in M : \overline{\lim_{n\to+\infty}} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\} \quad \text{and} \quad W^u(x; f) := W^s(x; f^{-1}) \ .$$

Similarly for every i, the following is an immersed manifold $W^i(x) \subset W^u$ tangent to $E^i(x)$:

$$W^i(x; f) := \left\{ y \in M : \overline{\lim_{n\to\infty}} \frac{1}{n} \log d(f^{-n}(x), f^{-i}(y)) \leq -\lambda_i \right\} \ .$$

Given $\epsilon > 0$, let $W^i_\epsilon(x; f)$ be the connected component of x in the intersection of $W^i_\epsilon(x; f)$ with the ϵ-neighborhood of x. These subsets vary measurably in function of the point x, and so for every i, there is a compact set T_i such that $(W^i_\epsilon(x))_{x\in T_i}$ is a continuous disjoint family of C^2-subsets, whose union has positive μ-measure. Then there exists a family of measure μ^i_x and a measure ν_i on T_i such that for every $A \subset \bigcup_{x\in T_i} W^i_\epsilon(x)$, it holds:

$$\mu(A) = \int_{T_i} \mu^i_x(A \cap W^i_\epsilon(x)) d\nu_i(x) \ .$$

The measure μ is SRB (Sinai-Ruelle-Bowen) if μ^u_x is absolutely continuous w.r.t. the Lebesgue measure of $W^u_\epsilon(x; f)$ for μ-a.e. x. If the measure is furthermore *hyperbolic* (i.e. $E^c = 0$), then the

measure μ is physical. Also by the Ledrappier-Strelcyn Theorem, its entropy is positive and equal to $h_\mu = \sum_i (dim(E_i) - \dim(E_{i-1}))\lambda_i$.

More generaly, Young defined the dimension of the measure μ_i^x as:

$$d_i := \lim_{\epsilon \to 0} \frac{\log \mu_i^x(W_\epsilon^i(x; f(x))}{\log \epsilon} .$$

This is invariant by the dynamics, and so by ergodicity, this does not depend on x. We have:

Theorem 3.6 (Ledrappier-Young entropy formula [LY85])**.** $h_\mu = \int \sum_i \lambda_i (d_i - d_{-i-1}) d\mu(x)$.

The proof of this theorem sheds light on a certain fractal structure of the support of the measure restricted to the submanifold $W_\epsilon^i(x; f)$. This will be a source of inspiration for the next section.

3.3. **Measure of maximal entropy.** Given a C^1-map f of a compact manifold. For every $n \geq 0$, let d_n be the distance defined Page 6. Let $\mathcal{N}_n(\epsilon)$ be the minimal number of d_n-balls of radius $\epsilon > 0$ covering M.

Definition 3.7. *The topological entropy of f is:*

$$h_{\text{top}} := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathcal{N}_n(\epsilon) .$$

Let us recall the following:

Theorem 3.8 (Variational principle [Din71])**.** *The topological entropy is the supremum of the metric entropy among all the ergodic probability measures.*

For many transitive sets [Mañ83, Lyu83, BLS93, Par64, Ber19b, BCS18], the maximal entropy measure exists and is unique. Interestingly, the maximal entropy measure μ_h is often equidistributed on the periodic points:

$$\frac{1}{\text{Card}\,\text{Fix}(f^n)} \sum_{x \in Fix(f^n)} \delta_x \to \mu_h .$$

This property has been shown for any rational function (of positive degree) of the Riemanian sphere by [Mañ83, Lyu83], any complex polynomial automorphisms (of positive dynamical degree) by [BLS93], for strongly regular Hénon map in [Ber19b], and recently [Bur] for smooth diffeomorphisms whose ergodic measures are all hyperbolic with Lyapunov exponents uniformly non-zero.

4. Emergence

Metric entropy describes very well the complexity of a dynamics of a.e. point of an ergodic probability measure with positive entropy. This would be sufficient if Bolzman ergodic hypothesis would be correct. In modern language [BK32], this hypothesis states that typical Hamiltonian systems are ergodic on [Leb. almost] every [component] of energy level. However, this hypothesis turned out to be wrong after a development of the KAM theory [Mos62]. This theory states that any perturbation of some integrable systems displays an invariant set of positive Lebesgue measure which is laminated by invariant Lagragian tori on which the dynamics acts as an irrational rotation.

To get rid of this obstruction, Smale proposed to simplify this study, by considering differentiable systems of surface which do not leave invariant a volume form. At some point he conjectured that

an open and dense set of such systems should be uniformly hyperbolic. As we saw above, by [BR75], this would have implied that an open and dense set of differentiable dynamical systems satisfy the following property: there exists a finite number of ergodic probability measures such that the union of their basins contains Lebesgue a.e. point of the manifold. This scenario (and Smale's conjecture) was disproved by his student Newhouse [New74] who discovered that the following phenomenon appears topologically generically in an open set of C^r-surface diffeomorphisms, $2 \leq r \leq \infty$:

Definition 4.1 (Newhouse phenomenon)**.** *There exists an infinite, transitive hyperbolic compact set which is accumulated by infinitely many sinks or totally elliptic periodic points.*

Interestingly this phenomenon appears in many fields of dynamical systems:

	Newhouse phenomenon is topologically generic in an open set of:
Newhouse [New74]	C^r-surface diffeomorphisms for every $r \geq 2$.
Duarte [Dua99]	C^r-surface, conservative diffeomorphisms for every $r \geq 2$.
Buzzard [Buz97]	Polynomial automorphisms of $\mathbb{C}^2$ of some large degree.
Bonatti-Diaz [BD99]	C^1-diffeomorphisms of manifold of dimension ≥ 3.
Biebler [Bie16]	Polynomial automorphisms of $\mathbb{C}^3$ of any degree ≥ 2.

We will see below that the set of invariant probability measures is huge. In this extend, this phenomenon is very much in opposition with Bolzman ergodic hypothesis.

Several conjectures from the 90's [TLY86, PT93, PS96, Pal00, Pal05, Pal08] gave some (philosophical) hope that Newhouse phenomenon should be negligible in some sense of Kolmogorov. But this turns out to be *not* the case in finite differentiability $1 \leq r < \infty$ and dimension ≥ 2:

Theorem 4.2 ([Ber16, Ber17a])**.** *Let $\infty > r \geq 1$ and $0 \leq d < \infty$, let M be a manifold of dimension ≥ 2. Then there exists a (non-empty) open set $\hat{\mathcal{U}}$ of C^r-families $(f_a)_{a \in B^d}$ of C^r-self-mappings f_a of M so that, a topologically generic $(f_a)_a \in \hat{\mathcal{U}}$ satisfies that for every $\|a\| \leq 1$, the dynamics f_a displays Newhouse phenomenon.*

Moreover if $\dim M \geq 3$, we can chose $\hat{\mathcal{U}}$ to be formed by families of diffeomorphisms.

Still one can argue that in Newhouse phenomenon, the Lebesgue measure of the basins of the sinks of period n might decrease very fast, and a typical mapping (including those displaying Newhouse phenomenon) might have their statistical behavior which is well approximated by a finite number of probability measures. Let us study this possibility.

4.1. **Quantitative approach.** The aim of the notion of emergence is to quantify how far a system is to be ergodic. Let us start with the abstract setting where the dynamics f is a continuous map of a compact metric space X with well defined box dimension and endowed with a reference probability measure μ. The important cases will be when X is a compact manifold and μ a Lebesgue measure. We endow the space of probability measures $\mathcal{M}(X)$ of X with the following distance:

Definition 4.3 (Kantorovich-Rubinstein distance $\mathbf{d}$)**.** *For every $\nu_1, \nu_2 \in \mathcal{M}(X)$, put:*

$$\mathbf{d}(\nu_1, \nu_2) = \sup_{\phi \in Lip^1(X)} \int \phi \, d(\nu_1 - \nu_2),$$

with $Lip^1(X)$ the space of real functions on X which are 1-Lipschitz.

We recall that the statistical behavior of the system is given by the empirical measures $\mathsf{e}_n(x) = \frac{1}{n}\sum_{0\le k<n}\delta_{f^k(x)}$, for $x\in X$ and $n\ge 0$. Roughly speaking, the emergence $\mathscr{E}_{\mathrm{Leb}}(\epsilon)$ at scale ϵ is the number of probability measures needed to approximate the statistical behavior of the system with precision ϵ:

Definition 4.4 (Metric emergence [Ber17a])**.** *The* metric emergence $\mathscr{E}_\mu(f)$ *of* f *at scale* $\epsilon>0$ *is the minimal number* N *of probability measures* $\{\nu_i\}_{i\le N}$ *so that*

$$(\diamondsuit)\qquad \limsup_n \int_M \mathbf{d}(\mathsf{e}_n(x),\{\nu_i\}_{i\le N})d\mu<\epsilon\ .$$

We recall that if f leaves invariant μ, then by the Birkhoff ergodic theorem, for μ-a.e. x, the sequence $(\mathsf{e}_n(x))_n$ converges to a measure $\mathsf{e}(x)$. Then $(\diamondsuit)$ is equivalent to:

$$\int_M \mathbf{d}(\mathsf{e}(x),\{\nu_i\}_{i\le N})d\mu<\epsilon\ .$$

Let us recall the following examples:

Example 4.5 ([Ber17a, BB19])**.** *Let* X *be a compact manifold of dimension* d *and* $\mu=\mathrm{Leb}$ *:*

- *If* Leb *is ergodic for* f *(e.g.* f *is an Anosov map), then* $\mathscr{E}_{\mathrm{Leb}}(f)=1$.
- *A dynamics displaying Newhouse phenomenon has not finite emergence.*
- *The identity satisfies* $\lim_{\epsilon\to 0}\frac{\log\mathscr{E}_{\mathrm{Leb}}(id)}{-\log\epsilon}=d$.
- *If* M *is symplectic and* f *displays KAM phenomenon, then* $\frac{\log\mathscr{E}_{\mathrm{Leb}}(f)}{-\log\epsilon}\ge d/2$.

All the latter bounds from below are at most polynomial (ϵ^{-d}). We proposed the following notion to describe dynamics whose statistical behavior is poorly described by finitely many probability measures:

Definition 4.6. *A differential dynamics is with* high emergence *if:*

$$\text{(Super Polynomial Emergence)}\qquad \limsup_{\epsilon\to 0}\frac{\log\mathscr{E}_{\mathrm{Leb}}(\epsilon)}{-\log\epsilon}=\infty\ .$$

The first aim of our program on Emergence is to prove the following conjecture:

Conjecture 4.7 ([Ber17a])**.** *Dynamics of high emergence are typical in many senses and contexts.*

The second aim of our program on emergence is to build a theory describing differentiable dynamics with high emergence.

4.2. **Results on typicality of dynamics with high emergence.** There are basically two ways to obtain high emergence:

(a) For every x in a set of positive Lebesgue measure, the sequence of empirical measures $(\mathsf{e}_n(x))_n$ does not converge and its cluster values form an infinite dimensional set,

(b) At a.e. x the empirical measures $(\mathsf{e}_n(x))_n$ converges to a measure $\mathsf{e}(x)$, but the ergodic decomposition $\mathsf{e}_*\mathrm{Leb}$ is poorly approximated by a finitely supported measure on $\mathcal{M}(M)$.

In our bounds on emergence, the scale will be often the following:

Definition 4.8 (Order)**.** *Given a function $\phi : (0,\infty) \to (0,\infty)$, its* order *is:*

$$\mathcal{O}\phi := \limsup_{\epsilon\to 0} \frac{\log\log\phi(\epsilon)}{-\log\epsilon}.$$

Let us observe that the emergence is high if the order of the emergence is positive:

$$\mathcal{O}\mathscr{E}_{\mathrm{Leb}} := \limsup_{\epsilon\to 0} \frac{\log\log\mathscr{E}_{\mathrm{Leb}}(\epsilon)}{-\log\epsilon} > 0 \Rightarrow \lim_{\epsilon\to 0} \frac{\log\mathscr{E}_{\mathrm{Leb}}(\epsilon)}{-\log\epsilon} = \infty\ .$$

We observe that the metric emergence is at most the covering number [4] of the space of probability measures, which is bounded by the following:

Theorem 4.9 (Bolley-Guillin-Villani [BGV07], Kloeckner [Klo12], Berger-Bochi-Peyre [BB19])**.** *If X is a compact space with well defined box dimension, then the covering number $H(\epsilon)$ of $(\mathcal{M}(X), \mathbf{d})$ has order $\mathcal{O}H = \dim X$.*

Hence we obtain the following counterpart of the Ruelle inequality formula for emergence:

Corollary 4.10. *If f is a continuous map of a compact metric space X with box dimension d, and μ a probability measure on X, it holds:*

$$\mathcal{O}\mathscr{E}_{\mu}(f) \leq d.$$

This maximum is attained in generic differentiable examples:

Theorem 4.11 ([BB19])**.** *A topologically generic $f \in \mathrm{Diff}^{\infty}_{\mathrm{Leb}}(M^2)$ which displays an elliptic periodic point has its metric emergence of maximal order $\dim M^2 = 2$:*

$$\mathcal{O}\mathcal{E}_{\mathrm{Leb}} = \dim M^2\ .$$

Let us recall that maps with robustly no elliptic points (which are called *weakly stable*) are conjecturally uniformly hyperbolic (Mañé-Like conjecture [Mañ82]) and ergodic. Hence conjecturally, the emergence of a generic conservative maps is either minimal [5] or maximal.

The latter theorem, and the next one were proved using that maximal order of emergence is a generically localizable property. Then Corollary 0.3 implies the dissipative counterpart of Theorem 4.11:

Theorem 4.12 ([BB19])**.** *For every $1 \leq r \leq \infty$, a topologically generic $f \in \mathcal{N}^r$ has its metric emergence of maximal order 2: $\mathcal{O}\mathcal{E}_{\mathrm{Leb}} = \dim M^2$.*

Remark 4.13. *Using that high emergence is a generically localizable property, we observe that Conjecture 1.5 implies in particular Conjecture 4.7.*

For the second kind of examples, we will use the following notion:

Definition 4.14 ([BB19])**.** *The* topological emergence $\mathscr{E}_{top}$ *is the covering number of the space of ergodic probabilities supported by K.*

We showed:

4. The *covering number* at scale ϵ of a metric space is the least number of ϵ-balls which cover it.
5. The minimal emergence equal 1, in this conservative setting, this means that the Lebesgue measure is ergodic.

Theorem 4.15 ([BB19]). *Let $r > 1$ and $f \in C^r(M, M)$ which leaves invariant a hyperbolic, locally maximal, compact set K. Then $\mathcal{OE}_{top} = \dim K$ if one of the two following conditions holds true:*
- *the restriction $f|_K$ is conformal.*
- *the restriction $f|_K$ is invertible, the differential has determinant 1 and $\dim M = 2$.*

Let us apply this result to the following:

Theorem 4.16 (Hofbauer-Keller [HK95]). *There exists uncountably many unimodal maps f of the interval such that for any of its invariant measure μ, for Lebesgue almost any point x, a subsequence of its empirical measures $(\mathsf{e}_n(x))_n$ converges to μ.*

Actually such dynamics turn out to have a hyperbolic, locally maximal, compact set K of box dimension arbitrarily close to 1. So we deduce:

Corollary 4.17. *There exists uncountably many unimodal maps f such that $\mathcal{OE}_{\mathrm{Leb}}(f) = 1$.*

In his PhD Thesis Aminosadat Talebi is showing the complex counterpart of this result, and moreover that this phenomena is topologically generic in the total bifurcation locus.

Recently it has been shown the following:

Theorem 4.18 (Kiriki-Nakano-Soma [KNS19]). *For every $2 \leq r < \infty$, if $f \in \mathrm{Diff}^r(M^2)$ displays a saddle periodic point whose local stable and unstable manifold are tangent, then there exists a C^r-perturbation $\tilde{f}$ of f which displays a non-empty domain with the following properties:*
- $\mathrm{diam}\, f^n(U) \to 0$,
- *For every $x \in U$, the set of cluster values of $(\mathsf{e}_n(x))_n$ has infinite dimension.*

In particular $(\mathsf{e}_n|_U)_n$ diverges Leb. *a.e. and the emergence of f is high.*

In a work in progress with Biebler we are strengthening this theorem to obtain any topology (including C^∞, analytic and polynomial) to obtain local density of emergence with positive order.

4.3. **Toward a theory on dynamics with high emergence.** Let X be a compact set of finite box dimension and f a continuous map of X.

Here is the emergent counterpart of the variational principle for entropy:

Theorem 4.19 ([BB19]). *The maximum of the orders of the metric emergences is attained and equal to the order of the topological emergence:*

$$\mathcal{OE}_{top} = \max_{\mu \in \mathcal{M}(X): f_*\mu = \mu} \mathcal{OE}_\mu \ .$$

We can reformulate both the metric emergence and the metric entropy in terms of *quantization number*. We recall that the quantization number $\mathcal{Q}_\mu(\epsilon)$ of a probability measure μ of a compact metric space (Y, d) is:

$$\mathcal{Q}_\mu(\epsilon) = \min \left\{ N \geq 1 : \exists (x_i)_i \in Y^N \& (a_i)_i \in \mathbb{R}^N \text{ s.t. } d_W(\mu, \sum_i a_i \delta_{x_i}) < \epsilon \right\} \ .$$

The following dictionary is under construction:

Topological entropy (def. 3.7): $\frac{1}{n}\log(\text{covering } \# \text{ of } M \text{ for } d_n)$.	Topological emergence (def. 4.14): covering # of space of erg. measures.
Metric entropy of ν ergodic [BB19, Th.A.7]: $h_\nu = \lim_{n\to\infty} \frac{1}{n}\log \mathcal{Q}_\nu^{\mathsf{d}_n}$	Metric emergence of μ invariant[BB19, Prop.3.11]: $\mathscr{E}_\mu = \mathcal{Q}_{\mathsf{e}_*\mu}$
Variational Principle 3.8: $h_{top} = \sup_\nu h_\nu$	Variational Principle 4.19: $\mathcal{O}\mathscr{E}_{top} = \max_\mu \mathcal{O}\mathscr{E}_\mu$
Margulis-Ruelle inequality 3.5: $h_\nu \leq \dim M \cdot \int \log \lVert Df \rVert d\nu$	Theorem 4.9: $\mathcal{O}\mathcal{E}_\mu \leq \dim \mu$
Positive entropy conjecture 3.2: Dynamics with positive entropy are typical	Conjecture 4.7: Dynamics with high emergence are typical
Ledrappier-Young Entropy formula 3.6.	?
Maximal entropy measure Equidistribution of periodic points	?

To define the counterpart of the uniqueness of the measure with maximal entropy, an idea is to look at a sequence $(\epsilon_n)_n \to 0$ and a sequence $(E_n)_{n\geq 0}$ of maximal ϵ_n-separated finite set in the space of ergodic measures of f, and then to look at the convergence of:

$$\hat{\mu}_n := \frac{1}{\#E_n}\sum_{\nu\in E_n}\delta_\nu \ .$$

In general, this sequence of measures does not need to converge. The same phenomenon happened for the measures equidistributed on the periodic points. For instance, a generic example having a fast growth of the number of periodic points in Theorem 2.3 satisfies that the following sequence does not converge:

$$\left(\frac{1}{\#\mathrm{Per}_n}\sum_{x\in\mathrm{Per}_n}\delta_x\right)_n \ .$$

Question 4.20. *For the map $\theta \in \mathbb{R}/\mathbb{Z} \mapsto 2\theta \in \mathbb{R}/\mathbb{Z}$, does $(\hat{\mu}_n)_n$ converge toward a measure $\hat{\mu}$ for every $(\epsilon_n)_n$ and $(E_n)_{n\geq 0}$? With $\mu := \int_{\mathcal{M}_1(X)} \nu d\hat{\mu}(\nu)$, does $\mathcal{O}\mathscr{E}_\mu = 1$ (as a measure of maximal emergence given by Theorems 4.15 and 4.19)?*

The Ledrappier-Young entropy formula follows from an autosimilarity (which is constant by ergodicity). Roughly speaking, the scaling factor is given by the Lyapunov exponent and the number of branches is given by the entropy.

Nevertheless, a dynamics with high emergence might have an ergodic decomposition $\mathsf{e}_*\mu$ which is not self similar. One might need to be more restrictive to obtain autosimilarity. In order to obtain this, it might useful to deal with a local version of the emergence.

Definition 4.21. *The* local order of emergence *at an ergodic probability ν is*

$$\mathcal{OE}_\mu^{loc}(\nu) = \limsup_{\epsilon\to 0} \frac{\log - \log \mathsf{e}_*\mu(B(\nu,\epsilon))}{-\log\epsilon} .$$

Problem 4.22. *Does the following inequality is correct?*

$$\int \mathcal{OE}_\mu^{loc}(\nu) d\mathsf{e}_*\mu(\nu) \leq \mathcal{OE}_\mu .$$

Then an idea would be to consider systems of local order of emergence $\mathsf{e}_*\mu$-a.e. constant, and then study the autosimilarity of their ergodic decomposition to deduce the analogous of the entropy formula.

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