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LATTICE OF DOMINANT WEIGHTS OF AFFINE KAC-MOODY ALGEBRAS

KRISHANU ROY

ABSTRACT. The dual space of the Cartan subalgebra in a Kac-Moody algebra has a partial ordering defined by the rule that two elements are related if and only if their difference is a non-negative integer linear combination of simple roots. In this paper we study the subposet formed by dominant weights in affine Kac-Moody algebras. We give a more explicit description of the covering relations in this poset. We also study the structure of basic cells in this poset of dominant weights for untwisted affine Kac-Moody algebras of type A .

1. INTRODUCTION

Let \mathfrak{g} denote a Kac-Moody algebra with cartan subalgebra \mathfrak{h} and set of roots Φ . The dual space \mathfrak{h}^* has a partial ordering defined as follows: $\lambda > \mu$ if and only if $\lambda - \mu \in \mathbb{N}\Phi^+$ where Φ^+ denote the set of positive roots. The subposet of dominant weights is particularly of great interest because of its connection to the representation theory of integrable highest weight module over \mathfrak{g} . To give one example of this, consider a finite, affine or strictly hyperbolic Lie algebra \mathfrak{g} and an integrable highest weight module $L(\lambda)$ over \mathfrak{g} . The integrability of $L(\lambda)$ implies that λ is a dominant weight. The module $L(\lambda)$ is \mathfrak{h} -diagonalizable and has a weight space decomposition $L(\lambda) = \bigoplus_{\mu \in P(\lambda)} L(\lambda)_\mu$ where $P(\lambda)$ is the set of weights. The set $P(\lambda)$ is completely determined upto Weyl conjugacy by the dominant weights contained in it, i.e. $P(\lambda) = W(P(\lambda) \cap \Lambda^+)$ where W denote the Weyl group and Λ^+ denote the set of dominant weights corresponding to \mathfrak{g} . The set $P(\lambda) \cap \Lambda^+$ can be described by the partial ordering on Λ^+ as follows: $\mu \in P(\lambda) \cap \Lambda^+$ if and only if $\mu \leq \lambda$.

In fact, the original motivation of Stembridge to study this partial order in [8] was to compute the weight multiplicities in the finite dimensional highest weight modules over finite type Kac-Moody algebras by using Freudenthal's algorithm. It is useful to know the explicit descriptions of the covering relations in the poset of dominant weights towards calculating this multiplicity. The covering relations in this poset of dominant weights for finite type Lie algebras were described explicitly in [8] (See also [1]).

Another motivation to study this partial order for the affine Kac-Moody algebras arose while studying the atomic decomposition of the characters of integrable highest weight modules over affine Kac-Moody algebras. In [6], Lusztig defined a t -analogue $K_{\lambda,\mu}(t)$ of the weight multiplicity of μ in the irreducible highest weight representation with highest weight λ over finite type Kac-Moody algebras generalizing the Kostka-Foulkes polynomials which are just t -analogue

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of Kostka-Foulkes numbers. In [4], Lascoux proved the decomposition of Kostka-Foulkes polynomials into *atomic polynomials* (See also [7]). Later in [5], the atomic decomposition was formulated for all finite type Kac-Moody algebras and was proved for a large number of cases while conjecturing that it holds more generally. The explicit description of the covering relations and the basic cell structure in the poset of dominant weights played an important role in the proof.

In this paper we study the poset of dominant weights for all affine type Lie algebras. We prove several results about the structure of the poset (Λ^+, \leq) for affine Lie algebras, some of which can be realized as generalization of the poset structure of dominant weights in complex semisimple Lie algebras. For example, we prove that connected components of (Λ^+, \leq) are lattice (Lemma 2). We also prove that λ covers μ in this ordering only if $\lambda - \mu$ is either the canonical imaginary root δ or it belongs to a distinguished subset of positive real roots (Theorem 1).

The paper is organized as follows. In section 2, we set up the notations, recall some basic facts and deduce some basic results about dominant weights of affine Lie algebras. In section 3, we present a detailed analysis of the covering relations in (Λ^+, \leq) . In particular, we prove that a dominant weight λ is covered by the canonical imaginary root δ only if λ is a fundamental weight with only one possible exception (Lemma 8). In section 4, we analyze the basic cell structure of (Λ^+, \leq) for the untwisted affine type A . We prove that the basic cells are of shape either diamond or pentagon (Theorem 3). The basic cell structures for other affine Lie algebras can be studied as well using Theorem 2, and we hope to present the results elsewhere.

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2. PRELIMINARIES

We denote the set of complex numbers by \mathbb{C} and the set of integers, non-negative integers, and positive integers by \mathbb{Z} , $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{> 0}$, respectively. We refer to [3] for the general theory of affine Lie algebras and affine root systems.

Through out this article, $A = (a_{ij})_{i,j \geq 0}^n$ will denote an affine generalized Cartan matrix (GCM) of order $n + 1$, $\mathfrak{g}(A)$ corresponding Kac-Moody algebra with set of roots Φ , simple roots $\alpha_0, \alpha_1, \dots, \alpha_n$, co-roots $\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee$, Dynkin diagram $S(A)$ and Cartan subalgebra \mathfrak{h} . So $\alpha_j(a_i^\vee) = a_{ij}$ is non-positive for $i \neq j$ and equal to 2 for $i = j$. Let \mathring{A} be the matrix obtained from A by deleting the 0th row and column. We will call \mathring{A} to be the finite part of A .

Let $(\ , \)$ denote the normalized invariant form [[3], Page 81] on $\mathfrak{g}(A)$. Notice that its restriction on the root lattice induces a positive semi-definite bilinear form [[3], Proposition 4.7]. In a simply laced diagram all roots are considered both short and long roots. Let

$$\Lambda = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for all } 0 \leq i \leq n\}$$

denote the set of integral weights, where \mathfrak{h} is the Cartan subalgebra of $\mathfrak{g}(A)$. Let

$$\Lambda^+ = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } 0 \leq i \leq n\}$$

denote the set of dominant integral weights. There is a partial order \leq on Λ given by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a nonnegative integral sum of simple roots. Our goal is to study the covering relations of this partial order restricted to the set of dominant weights.

Let c denote the canonical central element and δ denote the canonical imaginary root of $\mathfrak{g}(A)$. Then $\delta = \sum_{i=0}^n a_i \alpha_i$ and $c = \sum_{i=0}^n a_i^\vee \alpha_i^\vee$ where a_0, a_1, \dots, a_n (resp. $a_0^\vee, a_1^\vee, \dots, a_n^\vee$) be the positive numerical labels of $S(A)$ (resp. $S(A^T)$) in Table Aff. in [[3], Page (54,55)]. Let

$$\Lambda_m^+ = \{\lambda \in \Lambda^+ : \lambda(c) = m\}$$

denote the set of dominant weights of level m . Here, we make our first observation which follows immediately from the fact that $\alpha_i(c) = 0$ for all $0 \leq i \leq n$.

Observation 1. *If $\mu \leq \lambda$ in (Λ^+, \leq) , then $\lambda(c) = \mu(c)$ i.e. all dominant weights of a connected component of (Λ^+, \leq) lie in Λ_m^+ for some fixed m .*

Here, we recall some basic facts and deduce some basic results related to affine root system.

Definition 1. A weight λ is called a fundamental weight corresponding to a vertex i in $S(A)$ if $\lambda(\alpha_j^\vee) = \delta_{ji}$ for all j .

We define an element $\omega_0 \in \mathfrak{h}^*$ determined uniquely by

$$\omega_0(\alpha_0^\vee) = 1 \text{ and } \omega_0(h) = 0 \text{ for } h \in \mathfrak{h} \setminus \mathbb{C}\alpha_0^\vee.$$

Recall that we have $\dim \mathfrak{h}^* = 2(n+1) - \text{rank}(A) = n+2$, as $\text{rank}(A) = n$. The following proposition gives us a basis of \mathfrak{h}^* .

Proposition 1 ([2], Proposition 17.4). $\omega_0, \alpha_0, \alpha_1, \dots, \alpha_n$ form a basis of \mathfrak{h}^* .

Given an element $\lambda \in \Lambda_m^+$, we look at the coefficient of ω_0 when λ is expressed in terms of the above basis. The following lemma shows that the coefficient of ω_0 is same as the level of λ . This lemma also simultaneously justifies Observation 1.

Lemma 1 ([2], Lemma 20.1). *Let $\lambda \in \Lambda_m^+$. Then $\lambda = m\omega_0 + \sum_{i=0}^n \lambda_i \alpha_i$ for some λ_i .*

Now we look at the poset (Λ^+, \leq) which is of our primary interest. First notice that Observation 1 proves that (Λ^+, \leq) has infinitely many connected components. The fact that $\lambda \in \Lambda^+$ if and only if $\lambda + n\delta \in \Lambda^+$ for all $n \in \mathbb{Z}$, proves that none of these connected components has either a maximum or a minimum element. Nevertheless the next lemma shows that each connected component of (Λ^+, \leq) has a lattice structure.

Lemma 2. *Each connected component of (Λ^+, \leq) is a lattice.*

Proof. Let λ and μ belong to a connected component of (Λ^+, \leq) say, (Λ_i^+, \leq) . Therefore, we have $\lambda - \mu = \sum_{i=0}^n k_i \alpha_i$ for some $k_i \in \mathbb{Z}$. Recall that $\delta = \sum_{i=0}^n a_i \alpha_i$ for some $a_i \in \mathbb{Z}_{>0}$. Now choose $n = \max\{k_i : 0 \leq i \leq n\}$. Then we obtain $\lambda, \mu \geq \lambda - n\delta$. So each pair in (Λ_i^+, \leq) has a lower bound. Similarly by choosing $n' = \min\{k_i : 0 \leq i \leq n\}$, we get $\lambda, \mu \leq \lambda + n'\delta$. So each pair in (Λ_i^+, \leq) also has an upper bound.

As λ and μ belong to a connected component of (Λ^+, \leq) , recall that by Observation 1 we have $\lambda(c) = \mu(c) = m$. Suppose $\lambda = m\omega_0 + \sum_{i=0}^n \lambda_i \alpha_i$ and $\mu = m\omega_0 + \sum_{i=0}^n \mu_i \alpha_i$. Now let us define

$$\lambda \wedge \mu = m\omega_0 + \sum_{i=0}^n \min\{\lambda_i, \mu_i\} \alpha_i.$$

By arguments similar to Lemma 1.2 and subsection 1.1 in [8], we obtain $\lambda \wedge \mu \in (\Lambda_i^+, \leq)$. Clearly $\lambda \wedge \mu$ is the greatest lower bound of λ and μ . Now to prove (Λ_i^+, \leq) is a lattice, all we need to show is that for any two elements λ, μ in (Λ_i^+, \leq) , their lowest upper bound exists. Let $U(\lambda, \mu)$ be the set of all upper bounds of λ and μ . Observe that $U(\lambda, \mu)$ is non-empty. Let $\eta = m\omega_0 + \sum_{i=0}^n \eta_i \alpha_i$ be an arbitrary element in $U(\lambda, \mu)$. Note that η_i is bounded below by both λ_i and μ_i . Choose $\nu_i \in U(\lambda, \mu)$ such that it minimizes the coefficient of α_i in $U(\lambda, \mu)$ for each $0 \leq i \leq n$. Now clearly $\nu_0 \wedge \nu_1 \wedge \cdots \wedge \nu_n$ is the lowest upper bound of λ and μ . This finishes the proof. \square

3. COVERING RELATION

From here on we will deal only with dominant weights of positive level i.e. we will consider only those dominant weights μ such that there exists some i for which $\mu(\alpha_i^\vee) \neq 0$. We recall here an important Lemma and a Proposition from [8], which are crucial for the study of the covering relations on (Λ^+, \leq) . For the sake of completeness we will include the proof of Lemma here.

Let K be a subdiagram of $S(A)$. For $\beta = \sum_{i=0}^n k_i \alpha_i$, we define $Supp(\beta) = \{i : k_i \neq 0\}$ and $\beta|_K := \sum_{i \in K} k_i \alpha_i$. If K is a proper connected subdiagram then by [[3], Proposition 4.7], K must be of finite type. We denote the short highest root corresponding to the subdiagram K by α_K .

Proposition 2 (Proposition 2.1, [8]). *Let Ψ be an irreducible finite root system and λ be a dominant weight. If $\lambda (\neq 0)$ is a non negative integral sum of simple roots then $\lambda - \alpha_\Psi$ is also a non negative integral sum of simple roots where α_Ψ is the short highest root of Ψ .*

Lemma 3 (Lemma 2.5, [8]). *Suppose $\mu < \mu + \beta$ in (Λ^+, \leq) , $I = Supp \beta$, $J = \{i \in I : \mu(\alpha_i^\vee) = 0\}$ and $K \subset I$ be a proper connected subdiagram of $S(A)$.*

- (1) *If $\beta|_K(\alpha_i^\vee) \geq 0$ for all $i \in K - J$, then $\beta \geq \alpha_K$.*
- (2) *If in addition $(\mu + \alpha_K)(\alpha_i^\vee) \geq 0$ for all $i \in I - K$, then $\mu + \alpha_K$ is dominant.*

Proof. For $i \in J$, $\beta(\alpha_i^\vee) = (\mu + \beta)(\alpha_i^\vee) \geq 0$ as $\mu + \beta \in \Lambda^+$. So if $i \in K \cap J$, then

$$\beta|_K(\alpha_i^\vee) = \beta(\alpha_i^\vee) - (\beta - \beta|_K)(\alpha_i^\vee) \geq \beta(\alpha_i^\vee) \geq 0$$

since $i \notin \text{Supp}(\beta - \beta|_K)$. Combining this with the hypothesis (1), we get $\beta|_K - \alpha_K = \sum_{j \in K} m_j \alpha_j$ for some $m_j \in \mathbb{Z}_{\geq 0}$ by applying Proposition 2 on the irreducible finite root system Φ_K corresponding to the subdiagram K . Hence we get $\beta \geq \beta|_K \geq \alpha_K$. This proves (1).

For $i \notin I$,

$$(\mu + \alpha_K)(\alpha_i^\vee) = [(\mu + \beta) - (\beta - \alpha_K)](\alpha_i^\vee) \geq 0$$

since $\mu + \beta$ is dominant and $i \notin \text{Supp}(\beta - \alpha_K)$. For $i \in I \cap K$,

$$(\mu + \alpha_K)(\alpha_i^\vee) \geq 0$$

as $\mu \in \Lambda^+$ and α_K dominant with respect to Φ_K . This combined with the stated hypothesis proves that $\mu + \alpha_K \in \Lambda_+$. \square

Here we recall a general version of Cauchy-Schwarz inequality for a real vector space with a positive semi-definite symmetric bilinear form. We will use it to establish some basic facts about affine root systems.

Proposition 3. *Let V be a real vector space with a positive semi-definite symmetric bilinear form (\cdot, \cdot) . Then for $u, v \in V$, $(u, v)^2 \leq |u|^2 |v|^2$. If the equality holds then either $|v|^2 = 0$ or $|u - \frac{(u, v)}{|v|^2} v| = 0$.*

Lemma 4. *Consider α and β two real roots of $\mathfrak{g}(A)$. Then $\alpha(\beta^\vee)\beta(\alpha^\vee) \leq 4$.*

Proof. By Proposition 3, $(\alpha, \beta)^2 \leq |\alpha|^2 |\beta|^2$. So, $\alpha(\beta^\vee)\beta(\alpha^\vee) = \frac{2(\alpha, \beta)}{|\beta|^2} \frac{2(\alpha, \beta)}{|\alpha|^2} \leq 4$. \square

Lemma 5. *Let $\alpha \in \Phi$ be a real root and α_i be a simple root such that $\alpha(\alpha_i^\vee)\alpha_i(\alpha^\vee) = 4$ and $\alpha_i \notin \text{Supp}(\alpha)$. Then $\alpha - \frac{(\alpha, \alpha_i)}{|\alpha_i|^2} \alpha_i = -\frac{(\alpha, \alpha_i)}{a_i |\alpha_i|^2} \delta$.*

Proof. Since the bilinear form (\cdot, \cdot) is positive semi-definite [Proposition 4.7, [3]], by applying Proposition 3,

$$(\alpha, \alpha_i)^2 \leq |\alpha|^2 |\alpha_i|^2.$$

By hypothesis $\alpha(\alpha_i^\vee)\alpha_i(\alpha^\vee) = 4$. therefore we get $\frac{2(\alpha, \alpha_i)}{|\alpha|^2} \frac{2(\alpha, \alpha_i)}{|\alpha_i|^2} = 4$. Hence the above equality holds. So,

$$|\alpha - \frac{(\alpha, \alpha_i)}{|\alpha_i|^2} \alpha_i|^2 = 0$$

As the radical of the bilinear form (\cdot, \cdot) is 1-dimensional and it is generated by δ , we get $\alpha - \frac{(\alpha, \alpha_i)}{|\alpha_i|^2} \alpha_i = n\delta$ for some n . Since $\alpha_i \notin \text{Supp}(\alpha)$, $n = -\frac{(\alpha, \alpha_i)}{a_i |\alpha_i|^2}$ where a_i is the coefficient of α_i in the expansion of δ . Hence the result. \square

Remark 1. Let α and α_i be as in Lemma 5. If $\frac{(\alpha, \alpha_i)}{|\alpha_i|^2} = -1$, then we have $\alpha + \alpha_i = \delta$.

Lemma 6. *Let $\beta \in \mathbb{Z}\Phi$ be an element of the affine root lattice corresponding to the affine GCM A with the property that $\beta(\alpha_i^\vee) \geq 0$ for all $0 \leq i \leq n$. Then $\beta = n\delta$ for some $n \in \mathbb{Z}$.*

Proof. Let $\beta = \sum_{i=0}^n m_i \alpha_i$ for some $m_i \in \mathbb{Z}$. Since $\beta(\alpha_i^\vee) \geq 0$ for all $0 \leq i \leq n$, we have

$$Am \geq 0$$

where $m = [m_0, m_1, \dots, m_n]^T$. By [Theorem 4.3, [3]], $Am = 0$ and $\beta = n\delta$ for some $n \in \mathbb{Z}$. \square

Now we state our first main theorem which describes the covering relations in the poset (Λ^+, \leq) for any affine GCM A .

Theorem 1. *If λ covers μ in (Λ_m^+, \leq) for some $m \in \mathbb{Z}_{>0}$. Then one of the following is true:*

- (1) $\lambda - \mu = \alpha_K$ for some proper connected subdiagram K of $S(A)$
- (2) $\lambda - \mu = \delta$ where δ is the canonical imaginary root of $\mathfrak{g}(A)$.
- (3) $A = D_4^{(3)}$ or $A = G_2^{(1)}$, μ is not a fundamental weight and $\lambda - \mu = \alpha_1 + \alpha_2$, where α_1 and α_2 are the simple roots generating a root system of type G_2 .
- (4) $A = G_2^{(1)}$, μ is a multiple of the fundamental weight corresponding to the unique short simple root and $\lambda - \mu = \alpha_1 + \alpha_2 + \alpha_3$, where α_1, α_2 and α_3 are the simple roots of A .

Proof. This theorem is clearly true for the case $A = A_1^{(1)}$. Since $\delta(A_1^{(1)})$ is just the sum of two simple roots, $\lambda - \mu$ is either a simple root or $\lambda - \mu = \delta$. So for the proof let us assume $A \neq A_1^{(1)}$.

Let $\beta = \lambda - \mu$, $I = \text{Supp } \beta$ and $J = \{i \in I : \mu(\alpha_i^\vee) = 0\}$. If there exists a proper connected subdiagram K in I satisfying the conditions of Lemma 3, then $\mu + \alpha_K$ is dominant. Since λ covers μ . we get $\lambda - \mu = \alpha_K$. Otherwise, we claim that either $\lambda - \mu = \delta$ or it falls in the three exceptional cases i.e. (3), (4) and (5).

Case 1: Suppose $J = \emptyset$. Let us choose $K = \{i\}$ for some $i \in I$ which is short related to I . The condition (1) of Lemma 3 is trivial. Since $J = \emptyset$, for $j \in I \setminus \{i\}$, $\mu(\alpha_j^\vee) \geq 1$ and we get $\alpha_i(\alpha_j^\vee) \geq -1$ as $A \neq A_1^{(1)}$. Hence

$$(\mu + \alpha_i)(\alpha_j^\vee) \geq 0$$

Therefore, the condition (2) of Lemma 3 is also satisfied. Hence we get $\lambda - \mu = \alpha_K$, where K contains the only vertex α_i .

Case 2: Suppose $J \neq \emptyset$ and let K be a connected component of J containing a short root relative to J . So α_K is short relative to J . Since $m \in \mathbb{Z}_{>0}$, there exists at least on $i \in S(A)$ such that $\mu(\alpha_i^\vee) \neq 0$. So, J is a proper subdiagram of $S(A)$ and hence so is K .

Subcase 2.1: Suppose $(\mu + \alpha_K)(\alpha_i^\vee) \geq 0$ for all $i \in I - J$. The condition (1) of lemma 3 is vacuous here. For $i \in J - K$, $\alpha_K(\alpha_i^\vee) = 0$ since K is a connected component of J and hence $(\mu + \alpha_K)(\alpha_i^\vee) = 0$. So the condition (2) of lemma 3 is also satisfied.

Subcase 2.2: Now let us assume there exist $i \in I - J$, such that $(\mu + \alpha_K)(\alpha_i^\vee) < 0$. Since $i \notin J$, $\mu(\alpha_i^\vee) \in \mathbb{Z}_{>0}$. So

$$\alpha_K(\alpha_i^\vee) \leq -2$$

By Lemma 4, $\alpha_K(\alpha_i^\vee)$ could be -2, -3 or -4. We will now study each cases one by one. Let L be the connected component of $J \cup \{i\}$ containing α_i .

Subcase 2.2.1: Suppose $\alpha_K(\alpha_i^\vee) = -2$. Lemma 4 shows that the root length of α_K can not be strictly smaller than α_i . If $|\alpha_K|^2 = |\alpha_i|^2$, then α_K and α_i satisfy the conditions of Lemma 5. So, $\alpha_K + \alpha_i = \delta$ and hence

$$\mu + \delta = \mu + \alpha_K + \alpha_i \leq \mu + \beta$$

This implies $\lambda - \mu = \delta$.

Suppose that the root length of α_K is strictly bigger than α_i . This means that the root length of α_i is strictly smaller than that of the short root of J . Then $\alpha_i(\alpha_K^\vee)$ must be -1 . Hence $|\alpha_K|^2 = 2|\alpha_i|^2$. If all roots in J have same length, then α_i is the only simple root in L which is shorter than all other roots. Hence $\alpha_j(\alpha_i^\vee) \in 2\mathbb{Z}$ for all $j \in L$. If J has roots of two different lengths, then Φ has roots of 3 different lengths. So, Φ is of type $A_{2l}^{(2)}$ and α_i is the unique short simple root. Hence, $\alpha_j(\alpha_i^\vee) \in 2\mathbb{Z}$ for all j .

So in any case,

$$\gamma(\alpha_i^\vee) \in 2\mathbb{Z} \text{ for all } \gamma \in \mathbb{Z}\Phi_L.$$

Since $(\mu + \alpha_K)(\alpha_i^\vee) < 0$, $\alpha_K(\alpha_i^\vee) = -2$ and $i \notin J$, $\mu(\alpha_i^\vee)$ must be equal to 1. Therefore,

$$\beta|_L(\alpha_i^\vee) \geq \beta(\alpha_i^\vee) = \lambda(\alpha_i^\vee) - \mu(\alpha_i^\vee) \geq -1$$

As $\beta|_L(\alpha_i^\vee)$ is even, $\beta|_L(\alpha_i^\vee) \geq 0$. So the condition (1) of the Lemma 3 is satisfied for L .

If $L = S(A)$ then by Lemma 6, $\beta = n\delta$ for some $n \in \mathbb{Z}_{\geq 0}$. Hence

$$\mu + \delta \leq \mu + \beta.$$

This implies $\lambda - \mu = \delta$.

Now suppose L is a proper subdiagram of $S(A)$. If L contains a short root relative to $S(A)$, then α_L is short and hence $\alpha_L(\alpha_j^\vee) \geq -1$ for all $j \in I \setminus J$. If L does not contain a short simple root relative to $S(A)$ then $A = A_{2n}^{(2)}$, α_i must be an intermediate root and K just contains the unique long simple root. Hence $\alpha_L(\alpha_j^\vee) \geq -1$ for all $j \in I \setminus J$. As L is a Connected component of $J \cup \{i\}$, we have $\alpha_L(\alpha_j^\vee) = 0$ for $j \in J \setminus L$. So for $j \in I \setminus L$,

$$(\mu + \alpha_L)(\alpha_j^\vee) \geq 0.$$

Therefore, the condition (2) of Lemma 3 is also satisfied for L .

Subcase 2.2.2: Suppose $\alpha_K(\alpha_i^\vee) = -3$. Then Φ must be of type $G_2^{(1)}$ or $D_4^{(3)}$ by Table Aff. in [[3], Page (54,55)]. First suppose Φ is of type $G_2^{(1)}$ and let $\alpha_i, \alpha_j, \alpha_k$ be the simple roots. Since $\alpha_K(\alpha_i^\vee) = -3$, α_i must be the unique short simple root. Hence $K = J$, as J is connected and K is a connected component of J . Let α_j be the vertex connected to α_i . So, $j \in K$ as $\alpha_K(\alpha_i^\vee) \neq 0$. We claim $\mu + \alpha_K + \alpha_i$ is dominant.

$$(\mu + \alpha_K + \alpha_i)(\alpha_i^\vee) \geq 1 - 3 + 2 = 0,$$

$$(\mu + \alpha_K + \alpha_i)(\alpha_j^\vee) \geq 0 + 1 - 1 = 0.$$

If $k \in K$, then μ is a multiple of the fundamental weight corresponding to the unique short simple root α_i and $(\mu + \alpha_K + \alpha_i)(\alpha_k^\vee) \geq 0 + 1 + 0 = 1$.

If $k \notin K$, then $J = K = \{j\}$ and hence μ is not a fundamental weight. In this case $\alpha_K(\alpha_k^\vee) = -1$ and $\mu(\alpha_k^\vee) \geq 1$. So, $(\mu + \alpha_K + \alpha_i)(\alpha_k^\vee) \geq 0$. Hence $\lambda - \mu = \alpha_K + \alpha_i$

Suppose Φ is of type $D_4^{(3)}$. Since $\alpha_K(\alpha_i^\vee) = -3$, α_K is a long root with respect to Φ . As α_K is short corresponding to J , J can contain only the unique long simple root say α_k . Therefore, we have $K = J$ and μ is not a fundamental weight. Let α_i be the short simple root connected to α_k and let α_j be the other short simple root. Then

$$\begin{aligned}(\mu + \alpha_k + \alpha_i)(\alpha_i^\vee) &\geq 1 - 3 + 2 = 0, \\(\mu + \alpha_k + \alpha_i)(\alpha_j^\vee) &\geq 1 + 0 - 1 = 0, \\(\mu + \alpha_k + \alpha_i)(\alpha_k^\vee) &= 0 + 2 - 1 = 1.\end{aligned}$$

Hence $\mu + \alpha_k + \alpha_i$ is dominant and hence $\lambda - \mu = \alpha_k + \alpha_i$.

Subcase 2.2.3: Suppose $\alpha_K(\alpha_i^\vee) = -4$. This means α_K and α_i generate a root system of type $A_2^{(2)}$. So Φ is of type $A_{2l}^{(2)}$.

Since $\alpha_K(\alpha_i^\vee) = -4$, α_K is a long root and α_i is the short simple root with respect to Φ . As α_K is short corresponding to J , J can contain only the unique long simple root say α_k and hence $\alpha_K = \alpha_k$. The fact $\alpha_k(\alpha_i^\vee) \neq 0$ implies Φ must be of type $A_2^{(2)}$. Since $(\mu + \alpha_K)(\alpha_i^\vee) < 0$, $i \notin J$ and $\alpha_K(\alpha_i^\vee) = -4$, we get $\mu(\alpha_i^\vee) \in \{1, 2, 3\}$. Therefore

$$\begin{aligned}(\mu + \alpha_K + \alpha_i)(\alpha_i^\vee) &= \mu(\alpha_i^\vee) - 4 + 2, \\(\mu + \alpha_K + \alpha_i)(\alpha_K^\vee) &= 0 + 2 - 1 = 1.\end{aligned}$$

If $\mu(\alpha_i^\vee) \geq 2$, then $\mu + \alpha_K + \alpha_i$ is dominant. Since $\mu + \alpha_K + \alpha_i \leq \mu + \beta$, we have $\lambda - \mu = \alpha_K + \alpha_i$. If $\mu(\alpha_i^\vee) = 1$, then $\mu + n\alpha_K + \alpha_i$ is not dominant for any $n \in \mathbb{Z}_{\geq 0}$. But $\mu + \alpha_K + 2\alpha_i$ is dominant. By Lemma 5 $\alpha_K + 2\alpha_i = \delta$. So, $\lambda - \mu = \delta$.

This finishes the proof. □

Let $CR(A)$ denote the set of all roots appearing in the statement of Theorem 1. It is clear from theorem 1 that λ covers μ in (Λ_m^+, \leq) implies $\lambda - \mu \in CR(A)$. The following results strengthen this theorem.

Definition 2. A vertex i of the Dynkin diagram $S(A)$ is called special if $L := S(A) \setminus \{i\}$ is connected and $\delta = \alpha_L + \alpha_i$.

Lemma 7. A simple root corresponding to a special vertex is always a short root.

Proof. Let i be a special vertex in $S(A)$. Then we have $\alpha_L = \delta - \alpha_i$ and hence $|\alpha_L|^2 = |\alpha_i|^2$. Suppose α_i is not a short root. Then the subdiagram L must contain a short root say, α_j . Then we have $|\alpha_L|^2 = |\alpha_j|^2 < |\alpha_i|^2$ which is a contradiction. Hence α_i must be a short root in $S(A)$. □

Lemma 8. Let μ covers $\mu - \delta$ in (Λ_m^+, \leq) . Then one of the following is true:

- (1) μ is a fundamental weight corresponding to a special vertex of $S(A)$.
- (2) $S(A)$ is a non triply laced Dynkin diagram containing a unique short simple root and μ is a fundamental weight corresponding to that unique short simple root.
- (3) $A = D_{n+1}^{(2)}$ and $\mu = \omega_0 + \omega_n$.

(4) $A = A_1^{(1)}$ and $\mu = \omega_0 + \omega_1$.

Proof. First notice that $\mu(\alpha_i^\vee) \leq 1$ for all i . Otherwise, $\mu - \alpha_i$ would be dominant. This would contradict the fact μ covers $\mu - \delta$.

Now let $\mu(\alpha_i^\vee) = \mu(\alpha_j^\vee) = 1$ for some $i \neq j$. Choose the smallest connected subdiagram K of $S(A)$ with endpoints i and j . For $q \notin K$,

$$(\mu - \sum_{p \in K} \alpha_p)(\alpha_q^\vee) \geq 0.$$

For $q \in K - \{i, j\}$, $\sum_{p \in K} \alpha_p(\alpha_q^\vee) \leq 0$ as α_q has at least 2 neighbours. Therefore,

$$(\mu - \sum_{p \in K} \alpha_p)(\alpha_q^\vee) \geq \mu(\alpha_q^\vee) \geq 0.$$

And $(\sum_{p \in K} \alpha_p)(\alpha_i^\vee) \leq 1$, $(\sum_{p \in K} \alpha_p)(\alpha_j^\vee) \leq 1$ as both α_i and α_j have 1 neighbour. So, we have

$$(\mu - \sum_{p \in K} \alpha_p)(\alpha_i^\vee) \geq 0 \text{ and } (\mu - \sum_{p \in K} \alpha_p)(\alpha_j^\vee) \geq 0.$$

This contradicts the fact μ covers $\mu - \delta$ unless $\sum_{p \in K} \alpha_p = \delta$. By Table Aff. in [Page (54,55), [3]] , this is possible when either $A = D_{n+1}^{(2)}$ and $\mu = \omega_0 + \omega_n$ or $A = A_1^{(1)}$ and $\mu = \omega_0 + \omega_1$. Thus, except these two special cases, μ is a fundamental weight corresponding to a vertex, say i .

Let L be a connected component of $S(A) \setminus \{i\}$. Note that for $j \in L$

$$\delta|_L(\alpha_j^\vee) = \delta(\alpha_j^\vee) - (\delta - \delta|_L)(\alpha_j^\vee) \geq 0.$$

Therefore, by Proposition 2, $\alpha_L \leq \delta|_L$ and hence $\alpha_L \leq \delta$. As μ covers $\mu - \delta$, $\mu - \delta + \alpha_L$ is not dominant. For $j \notin L$ and $j \neq i$,

$$(\mu - \delta + \alpha_L)(\alpha_j^\vee) = \mu(\alpha_j^\vee) \geq 0$$

since L is a connected component of $S(A) \setminus \{i\}$. For $j \in L$

$$(\mu - \delta + \alpha_L)(\alpha_j^\vee) \geq 0$$

as α_L is dominant on L . Hence we get

$$(\mu - \delta + \alpha_L)(\alpha_i^\vee) < 0$$

Therefore, $\alpha_L(\alpha_i^\vee) \leq -2$.

If A is simply laced, then by Lemma 5, $\alpha_L + \alpha_i = \delta$. Hence i is a special vertex.

Now assume A is non simply laced. We claim that in this case α_i must be a short root of $S(A)$. Suppose α_i is not short and L be the connected component of $S(A) \setminus \{i\}$ containing a short root. Then $\alpha_i(\alpha_L^\vee) < \alpha_L(\alpha_i^\vee) \leq -2$ contradicting the Lemma 4.

Suppose α_i is not the unique short simple root in $S(A)$ and let α_j be a short root different from α_i . Let L be the connected component of $S(A) \setminus \{\alpha_i\}$ containing α_j . Then as before we get $\alpha_L(\alpha_i^\vee) \leq -2$. Since α_L and α_i have same root length, $\alpha_i(\alpha_L^\vee) \leq -2$. Therefore, $\alpha_L(\alpha_i^\vee) = \alpha_i(\alpha_L^\vee) = -2$. Now by Lemma 5, $\alpha_L + \alpha_i = \delta$.

Let α_i be the unique short simple root in $S(A)$. Then by Theorem 1, A can not be $G_2^{(1)}$. Hence the claim. \square

The following theorem describes the covering relations in (Λ_m^+, \leq) more explicitly.

Theorem 2. *If $\mu < \lambda$ in (Λ_m^+, \leq) , $I = \text{Supp}(\lambda - \mu)$ and $J = \{i \in I : \mu(\alpha_i^\vee) = 0\}$, then λ covers μ if and only if I is a connected subdiagram of $S(A)$ and one of the following holds.*

- (a) $\lambda - \mu$ is a simple root.
- (b) $I = J$ is a proper subdiagram of $S(A)$ and $\lambda - \mu = \alpha_I$.
- (c) $I = J \cup \{i\}$, Φ_I is of type B_l , α_i is short, $\mu(\alpha_i^\vee) = 1$ and $\lambda - \mu = \alpha_I$.
- (d) $I = J \cup \{i\}$, Φ_I is of type G_2 , α_i is short, $\mu(\alpha_i^\vee) = \{1, 2\}$ and $\lambda - \mu = \sum_{i \in I} \alpha_i$.
- (e) $I = J \cup \{i\} = S(A)$, Φ_I is of type $G_2^{(1)}$, α_i is short, $\mu(\alpha_i^\vee) \in \{1, 2\}$ and $\lambda - \mu = \sum_{i \in I} \alpha_i$.
- (f) $\lambda - \mu = \delta$ and λ, μ both are fundamental weights corresponding to a special vertex.
- (g) $\lambda - \mu = \delta$. $S(A)$ is a Dynkin diagram which is not triply laced and contains a unique short simple root. λ, μ both are fundamental weights corresponding to that unique short simple root.
- (h) $A = D_{n+1}^{(2)}$, $\lambda = \omega_0 + \omega_n$ and $\lambda - \mu = \delta$.
- (i) $A = A_1^{(1)}$, $\lambda = \omega_0 + \omega_1$ and $\lambda - \mu = \delta$.

Proof. If λ covers μ , then Theorem 1 and Lemma 8 imply one of the above mentioned cases holds.

Now we will prove the converse i.e. each of the cases (a)-(i) gives rise to a covering relation. If $\lambda - \mu$ is a simple root, then this is immediate. For case (b), assume $I = J$ be a proper subdiagram of $S(A)$, $\lambda - \mu = \alpha_I$ and λ does not cover μ . Since $\mu + \alpha_I$ does not cover μ , by Theorem 1, there exists a proper connected subdiagram K of I such that $\mu + \alpha_K$ covers μ . As K is a proper subdiagram of I , there exists $i \in I$ which is connected to K . The fact

$$\alpha_K(\alpha_i^\vee) < 0$$

implies $\mu(\alpha_i^\vee) > 0$, contradicting the fact $I = J$. Therefore, λ covers μ . This proves the case (b).

For the cases (c) and (d), observe that $\lambda - \mu \in CR(A)$; say $\lambda - \mu = \eta$. If $\mu + \eta$ does not cover μ , then there exist some $\xi < \eta$ in $CR(A)$ such that $\mu + \xi$ is dominant. but $\xi < \eta$ implies that ξ is not dominant in Φ_I . Therefore, there exists $j \in I$ such that $\xi(\alpha_j^\vee) < 0$. As $\mu + \xi$ is dominant, we get, $\mu(\alpha_j^\vee) > 0$. Hence j must be same as i . As α_i is the only short root in Φ_I , $\xi(\alpha_j^\vee) < 0$ implies that $\xi(\alpha_j^\vee) = -2$ (in type B) or $\xi(\alpha_j^\vee) = -3$ (in type G_2). So for dominance of $\mu + \xi$ it requires $\mu(\alpha_i^\vee) \geq 2$ and $\mu(\alpha_i^\vee) \geq 3$ respectively, a contradiction. This proves the cases (c) and (d).

For the cases (e), notice that μ is either a fundamental weight or twice a fundamental weight corresponding to the short simple root α_i . A direct check shows that both cases give a covering relation.

Let λ be a fundamental weight corresponding to the special vertex i , i.e. $\lambda(\alpha_i^\vee) = 1$ and $\lambda(\alpha_j^\vee) = 0$ for $j \neq i$. We claim $\lambda + \delta$ covers λ .

Assume $\lambda + \sum_{j=0}^n k_j \alpha_j$ covers λ in (Λ_m^+, \leq) and $k_l = 0$ for some $l \neq i$. As

$$\left(\lambda + \sum_{j=0}^n k_j \alpha_j\right)(\alpha_l^\vee) = \sum_{j \neq l} k_j \alpha_j(\alpha_l^\vee) \geq 0$$

$k_j = 0$ if j is connected to l . Since $L := S(A) \setminus \{i\}$ is connected, $k_j = 0$ for all $j \neq i$. But $\lambda + k_i \alpha_i$ is not dominant, therefore $k_l \neq 0$ for all $l \neq i$. Now suppose $k_i = 0$. Then by Theorem 1, $\lambda + \alpha_L$ covers λ . Hence,

$$(\lambda + \alpha_L)(\alpha_i^\vee) \geq 0 \Rightarrow \alpha_L(\alpha_i^\vee) \geq -1.$$

Since $i \notin L$ and i is connected to L , $\alpha_L(\alpha_i^\vee) \leq -1$ and therefore $\alpha_L(\alpha_i^\vee) = -1$. This means $|\alpha_L + \alpha_i|^2 \neq 0$, contradicting the fact that $\alpha_L + \alpha_i = \delta$. So, $k_i \neq 0$. Again by applying Theorem 1, we get $\lambda + \delta$ covers λ . This proves the case (f).

For case (g), suppose $S(A)$ is a Dynkin diagram which is not triply laced, and contains a unique short simple root say, α_i . Suppose λ is a fundamental weight corresponding to α_i . Assume $\lambda + \sum_{j=0}^n k_j \alpha_j$ covers λ in (Λ_m^+, \leq) . Note that $\alpha_j(\alpha_i^\vee)$ is even for all $j \neq i$. We have,

$$\left(\lambda + \sum_{j=0}^n k_j \alpha_j\right)(\alpha_i^\vee) = 1 + \sum_{j \neq i} k_j \alpha_j(\alpha_i^\vee) + 2k_i \geq 0.$$

Therefore we get, $k_i \neq 0$. And hence $k_j \neq 0$ for all j by similar arguments as before. Now by Theorem 1, we get $\lambda + \delta$ covers λ . This proves the case (g).

For the case (h), we have $A = D_{n+1}^{(2)}$ and $\lambda = \omega_0 + \omega_n$. Suppose $\lambda + \sum_{j=0}^n k_j \alpha_j$ covers λ in (Λ_m^+, \leq) . Assume $k_j \neq 0$ for some $0 \leq j \leq n$. Since for any $0 < l < n$,

$$\left(\lambda + \sum_{j=0}^n k_j \alpha_j\right)(\alpha_l^\vee) = \sum_{j \neq l} k_j \alpha_j(\alpha_l^\vee) + 2k_l \geq 0$$

we get $k_l \neq 0$ for any l connected to j . Therefore $k_l \neq 0$ for all $0 < l < n$. Since $\alpha_1(\alpha_0^\vee) = \alpha_{n-1}(\alpha_n^\vee) = -2$, $k_0, k_n \neq 0$. As $\delta(D_{n+1}^{(2)})$ is just the sum of all simple roots, we get $\lambda + \delta$ covers λ . An easy check also proves the case (i). This finishes the proof. \square

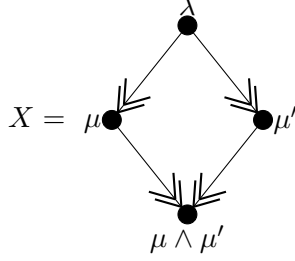
4. BASIC CELLS OF THE LATTICE OF DOMINANT WEIGHTS

If λ covers μ , we call μ a cocover of λ and it is denoted by $\lambda \twoheadrightarrow \mu$. By Theorem 2, covering relations in (Λ_m^+, \leq) for any simply laced Dynkin diagram $S(A)$ are given either by a proper subdiagram K of $S(A)$ or by δ . Observe that if λ has two distinct cocovers μ and μ' , then by Theorem 2 both μ and μ' correspond to two different proper subdiagrams of $S(A)$. Now we give an explicit description of the basic cell structure in (Λ_m^+, \leq) for type $A_{n+1}^{(1)}$.

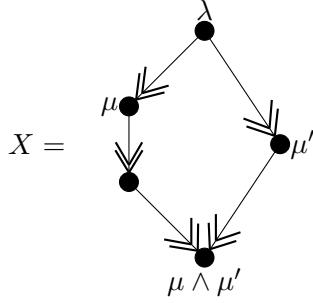
Theorem 3. *If μ and μ' are two distinct cocovers of λ corresponding to the proper subdiagrams K and K' respectively, then the interval $X = [\mu \wedge \mu', \lambda]$ has one of the following structures:*

- (1) *If K and K' satisfy one of the following conditions:*
 - (a) *K and K' both are singletons.*
 - (b) *$K \cup K'$ is a disconnected subdiagram of $S(A)$*

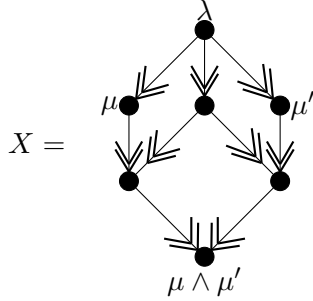
(c) $K \cup K'$ is connected and $K \cap K' \neq \emptyset$,
then,



(2) If $K \cup K'$ is connected, $K \cap K' = \emptyset$, $|K| = 1$ and $|K'| > 1$ then,



(3) If $K \cup K'$ is connected, $K \cap K' = \emptyset$, $|K| > 1$ and $|K'| > 1$ then,



Proof. We investigate each cases observing that they are exhaustive. In case (1) (a), $\mu = \lambda - \alpha_i$ and $\mu' = \lambda - \alpha_j$ for some $i \neq j$ and hence we get $\mu \wedge \mu' = \lambda - \alpha_i - \alpha_j$. Clearly $\mu \wedge \mu'$ is a cocover of both μ and μ' with no other elements in the interval.

In case (1) (b), $\mu = \lambda - \alpha_K$ and $\mu' = \lambda - \alpha'_{K'}$ such that $K \cup K'$ is a disconnected subdiagram of $S(A)$. Therefore we get $\mu \wedge \mu' = \lambda - \alpha_K - \alpha_{K'}$. First assume both K and K' are not singletons. As $\lambda \twoheadrightarrow \lambda - \alpha_K$, by Theorem 2 (b), we obtain $(\lambda - \alpha_K)(\alpha_i^\vee) = 0$ for all $i \in K$. Similarly we get $(\lambda - \alpha_{K'})(\alpha_i^\vee) = 0$ for all $i \in K'$. Hence,

$$(\lambda - \alpha_K - \alpha_{K'})(\alpha_i^\vee) = 0 \quad \text{for all } i \in K'$$

$$(\lambda - \alpha_K - \alpha_{K'})(\alpha_i^\vee) = 0 \quad \text{for all } i \in K$$

since $K \cup K'$ is disconnected. Again by applying Theorem 2 (b), we get $\lambda - \alpha_K \twoheadrightarrow \lambda - \alpha_K - \alpha_{K'}$ and $\lambda - \alpha'_{K'} \twoheadrightarrow \lambda - \alpha_K - \alpha_{K'}$. If one of the two diagrams K and K' is singleton, then same arguments as above show that X has the same diamond structure.

In case (1) (c), first notice that none of the diagrams K and K' is singleton as $K \cap K' \neq \emptyset$. As $\lambda \longrightarrow \lambda - \alpha_K$, by Theorem 2 (b), we obtain $(\lambda - \alpha_K)(\alpha_i^\vee) = 0$ for all $i \in K$. So,

$$\lambda(\alpha_i^\vee) = \alpha_K(\alpha_i^\vee) = \begin{cases} 0 & \text{if } i \text{ is not an end node of } K, \\ 1 & \text{if } i \text{ is an end node of } K. \end{cases}$$

Similarly

$$\lambda(\alpha_i^\vee) = \alpha_{K'}(\alpha_i^\vee) = \begin{cases} 0 & \text{if } i \text{ is not an end node of } K', \\ 1 & \text{if } i \text{ is an end node of } K'. \end{cases}$$

Therefore $K \cap K'$ can contain only the common end nodes of K and K' and hence $|K \cap K'| \leq 2$. In this case we have $\mu \wedge \mu' = \lambda - \alpha_K - \alpha_{K'} + \sum_{j \in K \cap K'} \alpha_j$. So we get,

$$(\lambda - \alpha_K - \alpha_{K'} + \sum_{j \in K \cap K'} \alpha_j)(\alpha_i^\vee) = 0 \quad \text{for all } i \in K' \setminus K$$

$$(\lambda - \alpha_K - \alpha_{K'} + \sum_{j \in K \cap K'} \alpha_j)(\alpha_i^\vee) = 0 \quad \text{for all } i \in K \setminus K'.$$

Again by applying Theorem 2 (b), we get $\lambda - \alpha_K \longrightarrow \lambda - \alpha_K - \alpha_{K'} + \sum_{j \in K \cap K'} \alpha_j$ and $\lambda - \alpha_{K'} \longrightarrow \lambda - \alpha_K - \alpha_{K'} + \sum_{j \in K \cap K'} \alpha_j$.

In case (2), we have $\mu = \lambda - \alpha_i$, $\mu' = \lambda - \alpha_{K'}$ such that $i \notin K'$ as $K \cap K' = \emptyset$ and i is connected to some end node of K' say i_1 . In this case we get $\mu \wedge \mu' = \lambda - \alpha_i - \alpha_{K'}$. $\lambda \longrightarrow \lambda - \alpha_{K'}$ implies $(\lambda - \alpha_{K'})(\alpha_i^\vee) = 0$ for all $i \in K'$ i.e.

$$\lambda(\alpha_i^\vee) = \alpha_{K'}(\alpha_i^\vee) = \begin{cases} 0 & \text{if } i \text{ is not an end node of } K', \\ 1 & \text{if } i \text{ is an end node of } K'. \end{cases}$$

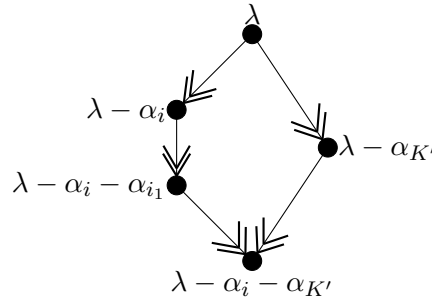
Notice that

$$(\lambda - \alpha_i - \alpha_{i_1})(\alpha_{i_1}^\vee) = 1 + 1 - 2 = 0$$

and $(\lambda - \alpha_i - \alpha_{i_1})(\alpha_i^\vee) \geq 0$ as $\lambda - \alpha_i$ is dominant. This proves $\lambda - \alpha_i - \alpha_{i_1}$ is dominant. By Theorem 2 (a); we get $\lambda - \alpha_i \longrightarrow \lambda - \alpha_i - \alpha_{i_1}$. Now the fact

$$(\lambda - \alpha_i - \alpha_{K'})(\alpha_j^\vee) = 0 \quad \text{for all } j \in K' \text{ and } j \neq i_1$$

together with Theorem 2 (b) imply $\lambda - \alpha_i - \alpha_{K'} \longrightarrow \lambda - \alpha_i - \alpha_{i_1}$. Therefore we have the following structure:



In case (3), we have $\mu = \lambda - \alpha_K$ and $\mu' = \lambda - \alpha_{K'}$ for some $K \cap K' = \emptyset$. Hence we get $\mu \wedge \mu' = \lambda - \alpha_K - \alpha_{K'}$. As explained above λ takes value 0 at mid nodes and value 1 at end nodes of K and K' . As $K \cup K'$ is connected, one end node of K say, i is connected to one end node of K' say, j . Now let us consider $\lambda - \alpha_K - \alpha_j$. This is dominant as $\lambda - \alpha_K$ is dominant and

$$(\lambda - \alpha_K - \alpha_j)(\alpha_j^\vee) = 1 + 1 - 2 = 0.$$

Similarly $\lambda - \alpha_{K'} - \alpha_i$ is also dominant. Notice that we have,

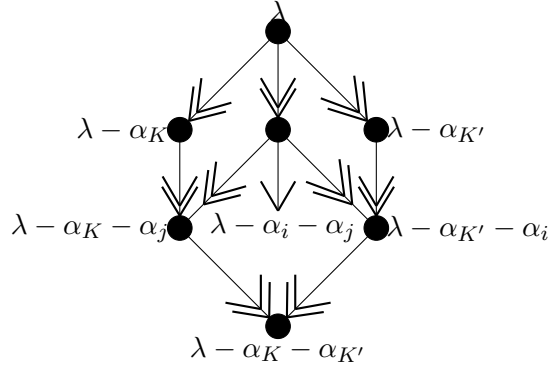
$$(\lambda - \alpha_i - \alpha_j)(\alpha_j^\vee) = 1 + 1 - 2 = 0,$$

$$(\lambda - \alpha_i - \alpha_j)(\alpha_i^\vee) = 1 - 2 + 1 = 0.$$

So, $\lambda - \alpha_i - \alpha_j$ is also dominant. Since $(\lambda - \alpha_j)(\alpha_j^\vee) = 1 - 2 < 0$ and $(\lambda - \alpha_i)(\alpha_i^\vee) = 1 - 2 < 0$, $\lambda - \alpha_i$ and $\lambda - \alpha_j$ are not dominant. Hence we obtain $\lambda \twoheadrightarrow \lambda - \alpha_i - \alpha_j$. We also have,

$$(\lambda - \alpha_K - \alpha_{K'}) (\alpha_l^\vee) = 0 \quad \text{for all } l \in K \text{ and } l \neq i$$

as $(\lambda - \alpha_K)(\alpha_l^\vee) = 0$ for all $l \in K$. So by applying Theorem 2 (b), we obtain $\lambda - \alpha_{K'} - \alpha_i \twoheadrightarrow \lambda - \alpha_K - \alpha_{K'}$. Similar arguments prove that $\lambda - \alpha_K - \alpha_j \twoheadrightarrow \lambda - \alpha_K - \alpha_{K'}$. Hence we have the following structure:



This finishes the proof. □

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