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Continuum thermomechanics of nonlinear micromorphic, strain and stress gradient media

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A comprehensive constitutive theory for the thermo–mechanical behaviour of generalized continua is established within the framework of continuum thermodynamics of irreversible processes. It represents an extension of the class of generalized standard materials to higher order and higher grade continuum theories. It reconciles most existing frameworks and proposes some new extensions for micromorphic and strain gradient media. The special case of strain gradient plasticity is also included as a contribution to the current debate on the consideration of energetic and dissipative mechanisms. Finally, the stress gradient continuum theory emerges as a new research field for which an elastic-viscoplastic theory at finite deformations is provided for the first time.

1. Introduction

The objective of the present work is to provide a comprehensive thermodynamical framework for the development of nonlinear constitutive laws for general continua including micromorphic, strain gradient and recently developed stress gradient material theories. *Nonlinear* means here finite deformations, on the one hand, and nonlinear elastic-viscoplastic material response, on the other hand. The work presents extensions of existing frameworks as proposed in [6,26,28] that rely on continuum thermodynamical concepts introduced by Germain for the development of material theory in engineering [30], see also [45,46].

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The need for a thermodynamically consistent formulation of generalized continuum thermomechanics originates from the many couplings arising between classical thermomechanical terms, new degrees of freedom, internal variables and their gradients. The recent controversy on strain gradient plasticity theories [35] pleads for a more systematic consideration of thermodynamic principles in the elaboration of higher order constitutive laws.

The mechanics of generalized continua is the subject of growing interest in the structural and materials mechanics community, even though first theories go back to Cauchy, Piola, the Cosserat brothers, Mindlin and Eringen. The reason is that the combination of advanced strain field measurements techniques and computational tools makes if possible to handle such sophisticated models involving large numbers of degrees of freedom and many material parameters including characteristic lengths to be calibrated. The most promising physical and engineering applications deal with size effects in the mechanical behaviour of metals and composites [68], on the one hand, and strain and damage localization in materials and structures, on the other hand. In the latter case, many generalized continua exhibit a regularization power restoring the well-posedness of instability and bifurcation problems [20].

Micromorphic and strain gradient models are generally developed by distinct groups in the mechanics community so that it is useful to show them together for a comparison of their respective capabilities. It is often hard to recognize general guidelines in the wealth of existing constitutive models for both approaches although it is claimed in the present work that a unifying treatment exists leaving as many possibilities as material classes for specific constitutive laws. In contrast, stress gradient models have been addressed only very recently [9,24,59] and arouse a lot of confusion on the actual status of a stress gradient continuum. It is shown in the present work that the emerging stress gradient continuum fundamentally differs from the strain gradient model as advocated in [24]. An elastic-plastic stress gradient continuum theory at finite deformations is proposed in the present work for the first time and illustrated in the case of tensile loading with unusual boundary conditions.

The concepts of internal variables and internal degrees of freedom play an essential rôle in the presented theories in particular w.r.t. reversible and irreversible mechanisms, containing, or not, gradient contributions in the free energy or dissipation potentials [47]. Several aspects could be treated using the concept of dual internal variables introduced in [65]. In particular, strain gradient plasticity modelling has developed as a quasi-independent field of research in engineering [68]. It is presented here as a special case of micromorphic model with internal constraint so as to accommodate the effect of gradient of internal variables.

Notations

Throughout this work boldface letters denote tensors of various orders as indicated in the text. The reference and current configurations of the body, $\mathcal{B}_0$ and $\mathcal{B}$, are respectively equipped with material (Lagrange) and spatial (Euler) coordinates of the form

$$\boldsymbol{X} = X_I \boldsymbol{E}_I, \quad \boldsymbol{x} = x_i \boldsymbol{e}_i \tag{1.1}$$

assumed to form Cartesian orthonormal coordinate systems. Simple, double and triple contractions are denoted by:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i, \quad \boldsymbol{\sigma} : \boldsymbol{D} = \sigma_{ij} D_{ij}, \quad \boldsymbol{m} \vdots \boldsymbol{k} = m_{ijk} k_{ijk} \tag{1.2}$$

The nabla operators with respect to Lagrange and Euler coordinates are respectively denoted by $\boldsymbol{\nabla}_0 := \partial / \partial \boldsymbol{X}$ and $\boldsymbol{\nabla} := \partial / \partial \boldsymbol{x}$. The deformation gradient is related to the gradient of the displacement field and denoted by:

$$\boldsymbol{F} := \boldsymbol{1} + \boldsymbol{u} \otimes \boldsymbol{\nabla}_0 = (\delta_{iJ} + u_{i,J})\, \boldsymbol{e}_i \otimes \boldsymbol{E}_J \tag{1.3}$$

The Jacobian of the transformation is

$$J := \det \boldsymbol{F} > 0 \tag{1.4}$$

The divergence of first, second and third order tensors is written in that way:

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} = b_{i,i}, \quad \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} = \sigma_{ij,j}\, \boldsymbol{e}_i, \quad \boldsymbol{\Phi} \cdot \boldsymbol{\nabla} = \Phi_{ijk,k}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.5}$$

General thermodynamic setting

The proposed theories comply with the general continuum thermomechanical setting of [30], i.e. a local formulation of the thermodynamics of irreversible processes including the first or higher order gradients of degrees of freedom together with internal variables driven by o.d.e. evolution laws. They represent extensions of the class of generalized standard materials well–established in continuum thermodynamics [37,50,56]. The main features are presented in this introduction because they are common to all micromorphic, strain and stress gradient discussed in the work.

It starts with the general form of the energy balance in its local Eulerian form:

$$\dot{e} = p^{(i)} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + r \tag{1.6}$$

where e is the volume density of internal energy, $\boldsymbol{q}$ is the heat flux and r represents possible external source terms. The actual form of the power of internal forces $p^{(i)}$ is central to the formulation of each theory.

The general form of the local dissipation rate inequality is written in its Eulerian format as

$$d = \dot{\eta} + \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{q}}{T}\right) - \frac{r}{T} \geq 0 \tag{1.7}$$

where η is the current volume density of entropy, r some external heat source and T is the absolute temperature. Combination of the first and second principles leads to the Clausius–Duhem inequality

$$d = p^{(i)} - \dot{e} + T\dot{\eta} - \boldsymbol{q} \cdot \frac{\boldsymbol{\nabla} T}{T} = p^{(i)} - (\dot{\psi} + \dot{T}\eta) - \boldsymbol{q} \cdot \frac{\boldsymbol{\nabla} T}{T} \geq 0 \tag{1.8}$$

the latter form involving the volume Helmholtz free energy density $\psi = e - T\eta$. The fulfilment of the entropy imbalance is assumed for all processes satisfying the energy balance (1.6). Other thermodynamic frameworks restrict its application to processes also fulfilling the mechanical and other balance laws like in the Liu procedure [57,66] introducing additional constraints and associated Lagrange multipliers.

The various generalized continuum approaches developed in the following differ by the number and nature of state variables identified as the arguments of the free energy density function, and by the explicit form of the internal power where the classical Cauchy stress tensor will be complemented by higher order contributions.

2. Thermo-elastoviscoplasticity of micromorphic media

The kinematics and statics of micromorphic media are recalled briefly following Eringen's original work in notations from [49]. The main purpose is then to develop the nonlinear constitutive theory for such media.

(a) Kinematics and statics

The material point $\boldsymbol{X}$ is endowed with translational degrees of freedom, namely the displacement vector $\boldsymbol{u}$, and with rotation and stretch of a triad of directors[1] represented by the generally incompatible microdeformation second rank tensor $\boldsymbol{\chi}(\boldsymbol{X}, t)$. The micromorphic theory by Eringen and Mindlin [15,52] also incorporates the effect of the gradient of these variables.

[1]Three independent lattice vectors represent directors for a crystalline solid.

The power of internal forces is a linear form with respect to the gradient of the velocity and microdeformation fields:

$$p^{(i)} = \boldsymbol{\sigma} : \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + \boldsymbol{s} : (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} - \dot{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}^{-1}) + \boldsymbol{m} \vdots (\dot{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}^{-1}) \otimes \boldsymbol{\nabla} \tag{2.1}$$

The simple stress tensor $\boldsymbol{\sigma}$ is symmetric in contrast to the relative stress $\boldsymbol{s}$ expanding work with the difference between the macro and microdeformation rates. The third rank stress tensor $\boldsymbol{m}$ is conjugate to the microdeformation rate gradient. The stress tensors fulfil the following balance equations for the momentum and the generalized moment of momentum in the current configuration:

$$(\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \boldsymbol{\nabla} = 0, \quad \boldsymbol{m} \cdot \boldsymbol{\nabla} + \boldsymbol{s} = 0, \quad \forall \boldsymbol{x} \in \mathcal{B} \tag{2.2}$$

given here in the absence of volume or inertia forces for the sake of conciseness[2]. The derivation of these balance laws based on the method of virtual power can be found in [49]. The associated Neumann boundary conditions on the body surface with normal $\boldsymbol{n}$ read

$$\boldsymbol{t} = (\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \boldsymbol{n}, \quad \boldsymbol{m}_n = \boldsymbol{m} \cdot \boldsymbol{n}, \quad \forall \boldsymbol{x} \in \partial\mathcal{B} \tag{2.3}$$

where surface densities of simple (vector $\boldsymbol{t}$) and double (second order tensor $\boldsymbol{m}_n$) contact forces are introduced. Recent approaches consider the curl of the microdeformation tensor instead of the full gradient. The *microcurl* model in [11] makes the link between the dislocation density tensor in plasticity and the curl of the microdeformation. This model is also called *relaxed micromorphic* model in [54].

(b) Constitutive equations

The Lagrangian strain measures selected in this presentation of the constitutive micromorphic theory are the classical right Cauchy–Green tensor $\boldsymbol{C}$, the second rank relative deformation tensor $\boldsymbol{\Upsilon}$ and the third rank microdeformation gradient tensor defined as:

$$\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F}, \quad \boldsymbol{\Upsilon} = \boldsymbol{\chi}^{-1} \cdot \boldsymbol{F}, \quad \boldsymbol{K} := \boldsymbol{\chi}^{-1} \cdot (\boldsymbol{\chi} \otimes \boldsymbol{\nabla}_0) \tag{2.4}$$

The latter tensor satisfies the following remarkable relation written in intrinsic and index notations:

$$(\dot{\boldsymbol{\chi}} \cdot \boldsymbol{\chi}^{-1}) \otimes \boldsymbol{\nabla} = \boldsymbol{\chi} \cdot \dot{\boldsymbol{K}} : (\boldsymbol{\chi}^{-1} \boxtimes \boldsymbol{F}^{-1}), \quad (\dot{\chi}_{iL}\chi^{-1}_{Lj})_{,k} = \chi_{iP}\dot{K}_{PQR}\chi^{-1}_{Qj}F^{-1}_{Rk} \tag{2.5}$$

The power of internal forces (2.1) then takes the following Lagrangian form:

$$Jp^{(i)} = \boldsymbol{\Pi} : \frac{\dot{\boldsymbol{C}}}{2} + \boldsymbol{S} : \dot{\boldsymbol{\Upsilon}} + \boldsymbol{M} \vdots \dot{\boldsymbol{K}} \tag{2.6}$$

with the following definitions of the generalized Piola stress tensors:

$$\boldsymbol{\Pi} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}, \quad \boldsymbol{S} = J\boldsymbol{\chi}^T \cdot \boldsymbol{s} \cdot \boldsymbol{F}^{-T}, \quad \boldsymbol{M} = J\boldsymbol{\chi}^T \cdot \boldsymbol{M} : (\boldsymbol{\chi}^{-T} \boxtimes \boldsymbol{F}^{-T}) \tag{2.7}$$

The decomposition of the three strain measures into elastic and plastic parts is now discussed based on the concept of local intermediate isoclinic configuration of the material point [26,44] or, equivalently, of elastic isomorphy [3] extended to gradient media [4]. The latter approach starts with the consideration of arbitrary change of reference configuration $\boldsymbol{P}$ transforming the deformation gradient into $\boldsymbol{F} \to \boldsymbol{F} \cdot \boldsymbol{P}$. In the case of micromorphic media, a distinct change of microstructure local configuration, $\boldsymbol{P}_\chi$, is possible: $\boldsymbol{\chi} \longrightarrow \boldsymbol{\chi}\boldsymbol{P}_\chi$. This leads to the following

[2]The analysis is limited to the static case for the sake of brevity. This is not a restriction of the apporach and it is not a necessary assumption for the considerations that follow. There are interesting and important issues related to higher order inertia terms in micromorphic, strain gradient and stress gradient media. The reader is referred to the discussion in the reference [25].

transformation rules for the three previous Lagrangian strain measures:

$$\boldsymbol{C} \to \boldsymbol{P}^T \cdot \boldsymbol{C} \cdot \boldsymbol{P}, \quad \boldsymbol{\Upsilon} \to \boldsymbol{P}_\chi^{-1} \cdot \boldsymbol{\Upsilon} \cdot \boldsymbol{P}, \quad \boldsymbol{K} \to \boldsymbol{P}_\chi^{-1} \cdot \boldsymbol{K} : (\boldsymbol{P}_\chi \boxtimes \boldsymbol{P}) + \boldsymbol{P}_\chi^{-1} \cdot (\boldsymbol{P}_\chi \otimes \boldsymbol{\nabla}_0) \cdot \boldsymbol{P} \tag{2.8}$$

According to the hypothesis of isomorphic elastic ranges [4], it is postulated that the strain potential, W_p with respect to any reference configuration can be computed from a single reference potential, W_0, by means of appropriate plastic transformations $\boldsymbol{P}, \boldsymbol{P}_\chi$ and the third rank tensor $\boldsymbol{K}^p$:

$$W_p(\boldsymbol{C}, \boldsymbol{\Upsilon}, \boldsymbol{K}) = W_0(\boldsymbol{P}^T \cdot \boldsymbol{C} \cdot \boldsymbol{P}, \boldsymbol{P}_\chi^{-1} \cdot \boldsymbol{\Upsilon} \cdot \boldsymbol{P}, \boldsymbol{P}_\chi^{-1} \cdot \boldsymbol{K} : (\boldsymbol{P}_\chi \boxtimes \boldsymbol{P}) - \boldsymbol{K}^p) \tag{2.9}$$

Introducing the notations

$$\boldsymbol{F}^p := \boldsymbol{P}^{-1}, \quad \boldsymbol{F}^e := \boldsymbol{F} \cdot \boldsymbol{P}, \quad \boldsymbol{\chi}_p := \boldsymbol{P}_\chi^{-1}, \quad \boldsymbol{\chi}_e := \boldsymbol{\chi} \cdot \boldsymbol{P}_\chi, \quad \boldsymbol{K}^e := \boldsymbol{P}_\chi^{-1} \cdot \boldsymbol{K} : (\boldsymbol{P}_\chi \boxtimes \boldsymbol{P})$$

the following elastic-plastic decompositions are derived:

$$\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^p, \quad \boldsymbol{\chi} = \boldsymbol{\chi}^e \cdot \boldsymbol{\chi}^p, \quad \boldsymbol{K} = \boldsymbol{\chi}^{p-1} \cdot \boldsymbol{K}^e : (\boldsymbol{\chi}^p \boxtimes \boldsymbol{F}^p) + \boldsymbol{K}^p \tag{2.10}$$

as initially proposed in [26,27]. The contributions $\boldsymbol{F}^e$ and $\boldsymbol{\chi}^e$ in the multiplicative decompositions of the deformation gradient and of the microdeformation are interpreted as their thermo-elastic parts. Alternative decompositions for the microdeformation gradient tensor are possible, see for instance [63].

The total Helmholtz free energy density is taken as a function of the elastic strain measures, temperature and internal variables α accounting for work-hardening phenomena or other physical processes:

$$\psi_0(\boldsymbol{C}^e, \boldsymbol{\Upsilon}^e, \boldsymbol{K}^e, T, \alpha)$$

where $\boldsymbol{\Upsilon}^e := \boldsymbol{\chi}^{e-1} \cdot \boldsymbol{F}^e$ is the elastic relative deformation tensor.
The local entropy inequality (1.8) in its Lagrange form reads:

$$(\frac{\boldsymbol{\Pi}^e}{2} - \frac{\partial \psi_0}{\partial \boldsymbol{C}^e}) : \dot{\boldsymbol{C}}^e + (\boldsymbol{S}^e - \frac{\partial \psi_0}{\partial \boldsymbol{\Upsilon}^e}) : \dot{\boldsymbol{\Upsilon}}^e + (\boldsymbol{M}^e - \frac{\partial \psi_0}{\partial \boldsymbol{K}^e}) \vdots \dot{\boldsymbol{K}}^e - (\frac{\partial \psi_0}{\partial T} + J\eta)\dot{T} + D^{res} \geq 0 \tag{2.11}$$

where D^{res} denotes the residual dissipation containing all contributions to the plastic power, evolution of internal variables and thermal power. The lefthand side of the entropy inequality (2.11) contains a linear form with respect to the rates $\dot{\boldsymbol{C}}^e, \dot{\boldsymbol{\Upsilon}}^e, \dot{\boldsymbol{K}}^e, \dot{T}$. Assuming that the corresponding cofactors and the residual dissipation D^{res} do not depend on these rates, the positivity of dissipation rate for all evolutions of these variables requires the vanishing of these cofactors [10]. Generalized Piola stresses are defined with respect to the intermediate configuration[3] that must therefore satisfy the following hyperelastic state laws:

$$\boldsymbol{\Pi}^e := J\boldsymbol{F}^{e-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{e-T} = 2\frac{\partial \psi_0}{\partial \boldsymbol{C}^e}, \quad \boldsymbol{S}^e := J\,\boldsymbol{\chi}^{eT} \cdot \boldsymbol{s} \cdot \boldsymbol{F}^{e-T} = \frac{\partial \psi_0}{\partial \boldsymbol{\Upsilon}^e}, \tag{2.12}$$

$$\boldsymbol{M}^e := J\boldsymbol{\chi}^{eT} \cdot \boldsymbol{m} : (\boldsymbol{\chi}^{e-T} \boxtimes \boldsymbol{F}^{e-T}) = \frac{\partial \psi_0}{\partial \boldsymbol{K}^e} \tag{2.13}$$

$$X = \frac{\partial \psi_0}{\partial \alpha}, \quad J\eta = -\frac{\partial \psi_0}{\partial T} \tag{2.14}$$

The attention is now drawn on the residual dissipation cast in the form

$$D^{res} = \boldsymbol{\Pi}^M : (\dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p-1}) + \boldsymbol{S}^M : (\dot{\boldsymbol{\chi}}^p \cdot \boldsymbol{\chi}^{p-1}) + \boldsymbol{M} \vdots \dot{\boldsymbol{K}}^p - X\dot{\alpha} - \boldsymbol{Q} \cdot \frac{\boldsymbol{\nabla}_0 T}{T} \geq 0 \tag{2.15}$$

where generalized Mandel stress tensors are defined as the driving forces for plastic flow:

$$\boldsymbol{\Pi}^M = J\boldsymbol{F}^{eT} \cdot (\boldsymbol{\sigma} + \boldsymbol{s}) \cdot \boldsymbol{F}^{e-T}, \quad \boldsymbol{S}^M = -\boldsymbol{\chi}^{eT} \cdot \boldsymbol{s} \cdot \boldsymbol{\chi}^{e-T} \tag{2.16}$$

A convenient framework for the expression of the plastic flow rule and the evolution equations for hardening variables is the introduction of a viscoplastic potential function of the previous

[3]When using the free energy potential $J_p\psi_0$ ($J_p = \det \boldsymbol{F}^p$) instead of ψ_0, the Jacobian J should be replaced by $J_e = \det \boldsymbol{F}^e$ in the definitions.

driving forces: $\Omega(\boldsymbol{\Pi}^M, \boldsymbol{S}^M, \boldsymbol{M}, X, \boldsymbol{\nabla}_0 T)$ such that

$$\dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p-1} = \frac{\partial \Omega}{\partial \boldsymbol{\Pi}^M}, \quad \dot{\boldsymbol{\chi}}^p \cdot \boldsymbol{\chi}^{p-1} = \frac{\partial \Omega}{\partial \boldsymbol{S}^M}, \quad \dot{\boldsymbol{K}}^p = \frac{\partial \Omega}{\partial \boldsymbol{M}}, \quad \dot{\alpha} = -\frac{\partial \Omega}{\partial X}, \quad \frac{\boldsymbol{Q}}{T} = -\frac{\partial \Omega}{\partial \boldsymbol{\nabla}_0 T} \tag{2.17}$$

If the potential is a convex function of its arguments $\boldsymbol{\Pi}^M, \boldsymbol{S}^M, \boldsymbol{M}, -X, -\boldsymbol{\nabla}_0 T$, the positivity of the dissipation rate is ensured at any instant for all processes[4].

(c) General micromorphic approach

The micromorphic approach proposed in [19,20] is a generalization of the previous micromorphic medium to strain-like additional degrees of freedom distinct from Eringen's microdeformation tensor, for example related to plasticity or damage variables. A single example, taken from [20], is provided here in the case of a scalar microplastic variable χ as it will be useful for the connection to strain gradient plasticity in the next section. The following specific form of the (isothermal) free energy potential was chosen in [20] (see eq. (7.12) there) to extend classical finite plasticity models:

$$\psi_0(\boldsymbol{C}^e, p, \chi, \boldsymbol{\nabla}_0 \chi) = \psi_0^e(\boldsymbol{C}^e) + \psi_0^p(p) + \frac{1}{2} H_\chi (p - \chi)^2 + \frac{1}{2} \boldsymbol{\nabla}_0 \chi \cdot \boldsymbol{A} \cdot \boldsymbol{\nabla}_0 \chi \tag{2.18}$$

where ψ_0^e denotes any hyperelastic potential of the reader's choice, p is a hardening variable with corresponding stored energy function ψ_0^p and $\boldsymbol{A}$ is the second order tensor of higher order moduli penalizing the gradient of microplastic variable χ. The choice of a quadratic potential w.r.t. the Lagrangian gradient of χ is sufficient for regularization purposes in softening plasticity for instance. A penalty modulus H_χ is introduced to force χ to remain close to p, and therefore their respective gradients will also remain close. It should be noted that the internal variable field p is generally not continuous whereas the differentiable field χ necessarily is. This explains the smoothing (regularizing) rôle played by the micromorphic variable with respect to the initial classical theory. The higher order stresses are derived from this potential:

$$s = \frac{\partial \psi_0}{\partial \chi} = -H_\chi (p - \chi), \quad \boldsymbol{M} = \frac{\partial \psi_0}{\partial \boldsymbol{\nabla}_0 \chi} = \boldsymbol{A} \cdot \boldsymbol{\nabla}_0 \chi \tag{2.19}$$

The model provides an enhancement of the hardening law in the form:

$$X = \frac{\partial \psi_0}{\partial p} = \frac{\partial \psi_0^p}{\partial p} + H_\chi (p - \chi) \tag{2.20}$$

The first term represents any choice of hardening function and the second one is the micromorphic enhanced contribution to hardening. The model is now illustrated for two choices of the hardening variable p. In the first case, a classical plasticity flow rule in the form

$$\dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p-1} = \lambda \frac{\partial f}{\partial \boldsymbol{\Pi}^M} \tag{2.21}$$

is adopted, where $f(\boldsymbol{\Pi}^M, X)$ is the yield function for the Mandel stress tensor and λ is the plastic multiplier. The yield function is linear w.r.t. an equivalent stress measure Π_{eq}^M. The hardening variable is taken as the cumulative plastic strain

$$p \equiv p_{cum} \quad \text{with} \quad \dot{p}_{cum} = \lambda \tag{2.22}$$

Plastic deformation is not in general a proper state variable since it depends on the choice of the reference configuration. However it is often used as such in engineering applications for

[4] If the dissipation potential is not convex (softening behaviour for instance) or if the evolution equations for internal variables do not derive from a potential (the existence of a dissipation potential is not required in contrast to the existence of the internal energy potential), it must be checked that the proposed evolution laws are such that the dissipation inequality holds. In this cannot be proved analytically for all processes, it must be checked pointwise during the simulation all along the thermodynamic process.

simplicity in order to avoid more sophisticated physical quantities like dislocation densities [33,42]. In that case the isothermal residual dissipation rate reads

$$D^p := \boldsymbol{\Pi}^M : \dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p-1} - X\dot{p}_{cum} = (\boldsymbol{\Pi}^M : \frac{\partial f}{\partial \boldsymbol{\Pi}^M} - X)\dot{p}_{cum} = (\Pi^M_{eq} - X)\dot{p}_{cum} \geq 0 \tag{2.23}$$

since the yield function f is taken as a homogeneous function of order 1. It suggests to take the yield function as $f(\boldsymbol{\Pi}^M) = \Pi^M_{eq} - X$ and to consider X as the yield stress. Within a viscoplastic theory, f is positive when $\dot{p}_{cum}$ is non-zero. As a consequence the first term in Eq. (2.20) is an isotropic hardening function in the sense of Lemaitre and Chaboche [42]. This isotropic hardening law is enhanced by the micromorphic contribution in Eq. (2.20).

The second example considers the equivalent plastic strain measure

$$p \equiv p_{eq} := (\boldsymbol{F}^p - \boldsymbol{1}) : (\boldsymbol{F}^p - \boldsymbol{1}) \tag{2.24}$$

i.e. the square of the norm of the plastic deformation, as a hardening variable. In that case,

$$\begin{aligned} D^p &:= \boldsymbol{\Pi}^M : \dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p-1} - X\dot{p}_{eq} = \boldsymbol{\Pi}^M : \dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p-1} - 2X\dot{\boldsymbol{F}}^p : (\boldsymbol{F}^p - \boldsymbol{1}) \\ &= (\boldsymbol{\Pi}^M \cdot \boldsymbol{F}^{p-T} - 2X(\boldsymbol{F}^p - \boldsymbol{1})) : \dot{\boldsymbol{F}}^p \geq 0 \end{aligned} \tag{2.25}$$

This suggests to consider the tensor $(\boldsymbol{\Pi}^M \cdot \boldsymbol{F}^{p-T} - \boldsymbol{X})$ as the argument of the viscoplastic potential where $\boldsymbol{X} = 2X(\boldsymbol{F}^p - \boldsymbol{1})$, with X given by the enhanced function (2.20), can be interpreted as a kinematic hardening variable, also called back–stress.

These two examples show that the micromorphic approach can be applied to scalar micromorphic variables, thus simplifying the original framework, and that the micromorphic enhancement can affect isotropic (modification of the yield stress) or kinematic (evolution of the yield surface centre) hardening laws from conventional plasticity. Limiting the micromorphic approach to scalar plasticity variables instead of second order tensors is advantageous from the point of view of computational analysis. It may however have some drawbacks in special situations described in [58,69]. This kind of scalar microplastic variable has been used for instance for the simulation of the propagation of Lüders-like shear bands characterising the deformation of shape memory alloys in [36].

3. Thermo-elastoviscoplasticity of gradient media

The strain gradient continuum theory is a well–established model since Mindlin's original work for small strain elasticity [53], Germain's hyperelastic formulation [29] up to the most recent elastoplastic finite deformation setting [5,26]. The coupling with temperature with the strain gradient raises the question of a possible dependence of the free energy potential on the temperature gradient as recently discussed in [43,55]. It seems more appropriate to consider a dependence of the internal energy function of the entropy gradient as proposed for fluids [32] and solids [22,40,67].

(a) Strain gradient theory

The strain gradient continuum model is based on the following generalized form of the power of internal forces:

$$p^{(i)} = \boldsymbol{\sigma} : \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + \boldsymbol{m} \vdots (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}) \otimes \boldsymbol{\nabla} + a\dot{\eta} + \boldsymbol{b} \cdot \boldsymbol{\nabla}\dot{\eta} \tag{3.1}$$

which is a linear form w.r.t. the Eulerian velocity first and second gradients and also w.r.t. entropy rate and its gradient. The third rank tensor conjugate to the second gradient of the velocity field is called the double stress or hyperstress tensor. Generalized stresses a and $\boldsymbol{b}$ are introduced following [22,23] in order to accommodate entropy gradient contributions in the theoretical framework. The method of virtual power has been used in [13,14,29] to derive the generalized

balance laws:

$$(\boldsymbol{\sigma} - \boldsymbol{m}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{\nabla} = 0, \quad \boldsymbol{b}\cdot\boldsymbol{\nabla} - a = 0, \quad \forall \boldsymbol{x}\in\mathcal{B} \tag{3.2}$$

in the static case and in the absence of volume forces. The corresponding boundary conditions are not reported for the sake of conciseness. Their complex structure and detailed derivations can be found in the latter references and in [12] for a recent discussion. The complementing generalized entropy balance law was postulated in [23].

The Lagrangian strain measures of the strain gradient continuum can be obtained from the micromorphic theory by setting the internal constraint $\boldsymbol{\chi}\equiv\boldsymbol{F}$ in the relations (2.4) and (2.10), i.e. coincidence of the macro and micro–deformations. The microdeformation gradient then coincides with the second gradient of the displacement field. The generalised strain measures can be decomposed into elastic and plastic parts as deduced from the micromorphic model:

$$\boldsymbol{F} = \boldsymbol{F}^e\cdot\boldsymbol{F}^p, \quad \boldsymbol{K} = \boldsymbol{F}^{-1}\cdot(\boldsymbol{F}\otimes\boldsymbol{\nabla}_0) = \boldsymbol{F}^{p-1}\cdot\boldsymbol{K}^e : (\boldsymbol{F}^p\boxtimes\boldsymbol{F}^p) + \boldsymbol{K}^p \tag{3.3}$$

The latter decomposition for finite strain second gradient theory was first proposed in [4,26]. The constitutive theory for thermo-elastoviscoplastic materials is then based on the internal energy state potential depending on the following Lagrangian arguments:

$$e_0(\boldsymbol{C}^e, \boldsymbol{K}^e, \alpha, \eta, \boldsymbol{\nabla}_0\eta) \tag{3.4}$$

The local entropy inequality (1.8) in its Lagrange form then reads:

$$(\frac{\boldsymbol{\Pi}^e}{2} - \frac{\partial e_0}{\partial \boldsymbol{C}^e}) : \dot{\boldsymbol{C}}^{\boldsymbol{e}} + (\boldsymbol{M}^e - \frac{\partial e_0}{\partial \boldsymbol{K}^e}) \vdots \dot{\boldsymbol{K}}^e - (\frac{\partial e_0}{\partial \eta} - Ja - JT)\dot{\eta} - (\frac{\partial e_0}{\partial \boldsymbol{\nabla}_0\eta} - J\boldsymbol{b}) : \boldsymbol{\nabla}_0\dot{\eta} + D^{res} \geq 0 \tag{3.5}$$

where the residual dissipation D^{res} contains the plastic power, the contributions of internal variables and the thermal power. The entropy inequality (3.5) contains a linear form with respect to the rates $\dot{\boldsymbol{C}}^{\boldsymbol{e}}, \dot{\boldsymbol{K}}^e, \dot{\eta}$ and $\boldsymbol{\nabla}_0\dot{\eta}$. Assuming that the corresponding cofactors and the residual dissipation D^{res} do not depend on these rates, the positivity of dissipation rate for all evolutions of these variables requires the vanishing of the cofactors [10]. This leads to the following state laws:

$$\boldsymbol{\Pi}^e = 2\frac{\partial e_0}{\partial \boldsymbol{C}^e}, \quad \boldsymbol{M}^e := J\boldsymbol{F}^{eT}\cdot\boldsymbol{m} : (\boldsymbol{F}^{e-T}\boxtimes\boldsymbol{F}^{e-T}) = \frac{\partial e_0}{\partial \boldsymbol{K}^e} \tag{3.6}$$

$$T = J^{-1}\frac{\partial e_0}{\partial \eta} + a, \quad \boldsymbol{b} = J^{-1}\frac{\partial e_0}{\partial \boldsymbol{\nabla}_0\eta} \tag{3.7}$$

The usual Gibbs relation for the temperature is therefore enhanced by the term $a = \boldsymbol{\nabla}\cdot\boldsymbol{b}$. The impact of this fundamental modification on the heat equation has been studied in [22,43].
The attention is drawn now to the residual dissipation[5]:

$$D^{res} = \boldsymbol{\Pi}^M : (\dot{\boldsymbol{F}}^p\boldsymbol{F}^{p-1}) + \boldsymbol{M}\vdots\dot{\boldsymbol{K}}^p - X\dot{\alpha} - \boldsymbol{Q}\cdot\frac{\boldsymbol{\nabla}_0 T}{T} \geq 0 \tag{3.8}$$

This suggests the introduction of a viscoplastic dissipation potential $\Omega(\boldsymbol{\Pi}^M, \boldsymbol{M}, X, \boldsymbol{\nabla}_0 T)$ from which the flow rules for plastic deformation and plastic strain gradient[6], the evolution equations for the internal variables and generalization of the Fourier law can be derived in the spirit of Eq. (2.17).

(b) Strain gradient plasticity models

Strain gradient plasticity as reviewed for instance in [33,68] is a very active field of research to address size effects in the mechanical behaviour of generally metallic materials. These models are different from the previous strain gradient continua since they do not rely on the consideration of the strain gradient but, instead, on the gradient of plastic deformation only. This subtle difference amounts to neglecting the effect of the gradient of elastic deformation. It can be seen as an

[5]The Piola hyperstress tensor is defined as in (2.7), $\boldsymbol{M} = J\,\boldsymbol{F}^T\cdot\boldsymbol{m} : (\boldsymbol{F}^{-T}\boxtimes\boldsymbol{F}^{-T})$.
[6]Note that the presented theory is such that the plastic part of the strain gradient generally differs from the gradient of the plastic deformation as done and discussed in [17,26,31].

approximation of the full strain gradient model. However the strain gradient plasticity models are generally closer to micromorphic models than the genuine strain gradient theory, especially regarding the resulting type of boundary conditions, because the field variables are not limited to the displacement but also include the plastic deformation itself regarded as a degree of freedom of the theory. The objective of this section is not to review those numerous models but to propose a unifying formulation of the many often contradictory constitutive equations found in the literature. For the sake of conciseness, the presentation is given at small strains, in the isothermal case, and limited to the effect of the gradient of the cumulative plastic strain, $p \equiv p_{cum}$, a simple fruitful theory initiated by Aifantis [1]. It starts, as usual, from the linear form of the power of internal forces for such a class of media:

$$p^{(i)} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + s\dot{p} + \boldsymbol{m} \cdot \boldsymbol{\nabla}\dot{p} \tag{3.9}$$

where s and $\boldsymbol{m}$ are scalar and vector generalized stresses conjugate to plastic strain rate and its gradient, p being treated as an internal degree of freedom. The application of the method of virtual power for displacement and plastic degrees of freedom considered as independent leads to the static mechanical balance laws:

$$\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} = 0, \quad \boldsymbol{m} \cdot \boldsymbol{\nabla} - s = 0 \tag{3.10}$$

in the absence of volume forces. The second equation is called *microforce balance* by Gurtin [34]. After splitting the infinitesimal strain into elastic, $\boldsymbol{\varepsilon}^e$, and plastic part, $\boldsymbol{\varepsilon}^p$, it continues with the following state laws:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}, \quad \eta = -\frac{\partial \psi}{\partial T}, \quad X = \frac{\partial \psi}{\partial p}, \quad \boldsymbol{m} = \frac{\partial \psi}{\partial \boldsymbol{\nabla} p} \tag{3.11}$$

The latter equation assumes that the whole energy due to plastic strain gradient is stored in the Helmholtz potential. This simple assumption leads to the following form of the residual dissipation rate:

$$D^{res} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + s\dot{p} - X\dot{p} = (\sigma_{eq} - X + s)\dot{p} \geq 0 \tag{3.12}$$

where σ_{eq} is an equivalent stress measure. This prompts us to introduce a yield function of the shape:

$$f(\boldsymbol{\sigma}, s, X) = \sigma_{eq} - X + s \tag{3.13}$$

where $X(p)$ is the usual yield radius enhanced here by the new generalized stress $s = \boldsymbol{\nabla} \cdot \boldsymbol{m}$. If an isotropic quadratic potential is chosen w.r.t. to $\boldsymbol{\nabla} p$, s will be proportional to the Laplacian of the plastic field distribution. In that way, the celebrated Aifantis model is recovered from a solid thermodynamic formulation [19]. Following Gurtin's notations [2], the contribution $X - s$ is denoted by the so–called microstress $\pi = \pi_{NR} + \pi_{dis}$ where $\pi_{NR} = X$ is called the *energetic nonrecoverable generalized stress*. We prefer to interpret the latter as the thermodynamic force associated with the internal variable p, as usual in engineering continuum thermodynamics [42]. The quantity $\pi_{dis} = -s$ is called dissipative (micro)stress in [2]. The presented strain gradient plasticity model can be viewed as a micromorphic model (2.18) combined with the internal constraint $\chi \equiv p$, the penalty term $H_\chi(p - \chi)$ being replaced by a Lagrange multiplier. Strain gradient plasticity models can be implemented efficiently in that way in finite element programs [70].

A recent debate in the literature deals with the consideration of a dissipative part associated with the plastic strain gradient which requires a modification of the last state law in Eq. (3.11). This has been done along two main lines which are presented here together for the first time, groups of authors using either one or the other.

(i) Model in parallel

Like in a Kelvin–Voigt rheological model (parallel branches), the authors following [2] decompose the higher order stress into elastic (recoverable/stored) and plastic (dissipative) parts:

$$\boldsymbol{m} = \boldsymbol{m}^e + \boldsymbol{m}^p, \quad \text{with} \quad \boldsymbol{m}^e = \frac{\partial \psi}{\partial \boldsymbol{\nabla} p} \tag{3.14}$$

As a result, the residual dissipation (3.12) is modified into

$$D^{res} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + s\dot{p} - X\dot{p} + \boldsymbol{m}^p \cdot \boldsymbol{\nabla}\dot{p} \geq 0 \tag{3.15}$$

The last term is analogous to a viscous contribution in a fluid or to a generalized plastic power. This suggests the writing of constitutive laws based on a dissipation potential $\Omega(\boldsymbol{\sigma}, s, X, \boldsymbol{m}^p)$, for instance through a yield function $f = \sigma_{eq} - X + s$.

(ii) Model in series

According to rheological models in series like Maxwell or Saint-Venant assemblies, it is proposed instead to split all strain measures into elastic (reversible) and plastic (irreversible) parts. This has been done already for the classical strain tensor $\boldsymbol{\varepsilon}$ and remains to be done for the plastic strain gradient in the following way

$$\boldsymbol{k} := \boldsymbol{\nabla} p = \boldsymbol{k}^e + \boldsymbol{k}^p \tag{3.16}$$

which leads to the following modified state law and residual dissipation:

$$\boldsymbol{m} = \frac{\partial \psi}{\partial \boldsymbol{k}^e}, \quad D^{res} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + s\dot{p} - X\dot{p} + \boldsymbol{m} \cdot \dot{\boldsymbol{k}}^p \geq 0 \tag{3.17}$$

This prompts us to propose a yield function of the form:

$$f(\boldsymbol{\sigma}, \boldsymbol{m}, s, X) = \sigma_{eq}(\boldsymbol{\sigma}, \boldsymbol{m}) - X + s \tag{3.18}$$

where $\sigma_{eq}(\boldsymbol{\sigma}, \boldsymbol{m})$ is an equivalent stress measure combining the two stress tensors. For instance, a generalization of the von Mises criterion could be based on the following equivalent stress measure:

$$\sigma_{eq} = \left(\frac{3}{2} \boldsymbol{\sigma}^{\text{dev}} : \boldsymbol{\sigma}^{\text{dev}} + \ell_p^{-2} \boldsymbol{m} \cdot \boldsymbol{m} \right)^{\frac{1}{2}} \tag{3.19}$$

where $\boldsymbol{\sigma}^{\text{dev}}$ is the deviatoric part of the stress tensor and ℓ_p a material characteristic length. Assuming a generalized normality rule, the flow rules are obtained as

$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \lambda \frac{\boldsymbol{\sigma}^{\text{dev}}}{\sigma_{eq}}, \quad \dot{\boldsymbol{k}}^p = \lambda \frac{\partial f}{\partial \boldsymbol{m}} = \lambda \ell_p^{-2} \frac{\boldsymbol{m}}{\sigma_{eq}} \tag{3.20}$$

The plastic multiplier λ can then be computed from the consistency condition $\dot{f} = 0$ in the rate–independent case or from an appropriate viscoplastic potential otherwise.

This class of strain gradient plasticity models based on a kinematic decomposition of the gradient like (3.16) was proposed in [19,26]. It generalizes similar decompositions for Cosserat [7], micromorphic [60] and strain gradient media [17,31]. The particular case of a purely dissipative contribution of $\boldsymbol{\nabla}\dot{p}$ (i.e. $\boldsymbol{k}^e = 0$) was put forward by Fleck and Willis [18].

4. Stress gradient plasticity

The stress gradient continuum theory was first introduced in the reference [24] and later in [59]. It fundamentally differs from the strain gradient model because it requires the introduction of new kinematic degrees of freedom $\boldsymbol{\Phi}$, a third rank tensor, in addition to the usual displacement $\boldsymbol{u}$ of the material point. The stress gradient elasticity theory was developed and discussed in [24,62,64] in the small deformation case. The objective of the present section is to propose an extension to plasticity.

(a) Kinematics, balance of momentum and boundary conditions

The symmetric stress tensor $\boldsymbol{\sigma}$ and its gradient fulfil the following field equations

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}+\boldsymbol{f}=0,\quad \boldsymbol{\sigma}\otimes\boldsymbol{\nabla}-\boldsymbol{R}+\boldsymbol{F}=0 \tag{4.1}$$

where $\boldsymbol{f}$ (unit $\mathrm{N\,m^{-3}}$) and $\boldsymbol{F}$ (unit $\mathrm{N\,m^{-4}}$) are given volume simple and triple forces. The second equation defines the third rank tensor $\boldsymbol{R}$ as the difference between the stress gradient and prescribed triple forces. Note that its trace with respect to the last two indices is not free but prescribed by external forces

$$R_{ijj}=f_i-F_{ijj} \tag{4.2}$$

as a consequence of the first balance law (4.1). In this subsection, all involved third order tensors are taken symmetric w.r.t. the first two indices. The weak form is obtained by multiplying the first equation by the virtual displacement vector $\boldsymbol{u}$ and the second one by the third rank tensor[7] $\boldsymbol{\Phi}$, and summing up yields

$$\int_{\mathcal{B}}\left(\boldsymbol{\sigma}:(\boldsymbol{\varepsilon}+\boldsymbol{\Phi}\cdot\boldsymbol{\nabla})+\boldsymbol{R}\vdots\boldsymbol{\Phi}\right)dv=\int_{\mathcal{B}}(\boldsymbol{f}\cdot\boldsymbol{u}+\boldsymbol{F}\vdots\boldsymbol{\Phi})\,dv+\int_{\partial\mathcal{B}}\boldsymbol{\sigma}:(\boldsymbol{u}\overset{s}{\otimes}\boldsymbol{n}+\boldsymbol{\Phi}\cdot\boldsymbol{n})\,da \tag{4.3}$$

after integration by parts, where $\overset{s}{\otimes}$ denotes the symmetrized tensor product.

Conversely the theory can be built upon the following form of the virtual work of internal forces

$$p^{(i)}(\dot{\boldsymbol{u}},\dot{\boldsymbol{\Phi}})=\boldsymbol{\sigma}:\boldsymbol{e}(\dot{\boldsymbol{u}},\dot{\boldsymbol{\Phi}})+\boldsymbol{R}\vdots\dot{\boldsymbol{\Phi}} \tag{4.4}$$

with the generalized strain measure

$$\boldsymbol{e}:=\boldsymbol{\varepsilon}+\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}\quad\text{with}\quad\boldsymbol{\varepsilon}=\boldsymbol{u}\overset{s}{\otimes}\boldsymbol{\nabla} \tag{4.5}$$

The previous field equations together with the following boundary conditions

- 6 Dirichlet boundary conditions: $\boldsymbol{u}\overset{s}{\otimes}\boldsymbol{n}+\boldsymbol{\Phi}\cdot\boldsymbol{n}$ given at a part of the boundary $\partial\mathcal{B}$
- or 6 Neumann conditions: The full tensor $\boldsymbol{\sigma}$ given at the complementing part of the boundary

provide a well-posed boundary value problem as proved mathematically by [62] in the case of linear elasticity. Alternative mixed b.c. were also considered in the latter reference. It is remarkable that, in contrast to Cauchy's classical theory, all the components of the stress tensor can be prescribed at the outer boundary and not only the usual traction vector $\boldsymbol{\sigma}\cdot\boldsymbol{n}$. This is a stunning feature of the stress gradient model. Note that new kinematic degrees of freedom Φ_{ijk} are symmetric with respect to the first two indices. They have the physical dimension of length and are called *microdisplacements* in [24]. The micromechanical interpretation of these microdisplacements is still an open question. Two tentative interpretations can be mentioned. The first one stems from the composite plate theory, more precisely the *bending gradient* plate theory designed by [61] and justified by an asymptotic homogenization method. In this theory there is a similar microdisplacement variable which is defined as a suitable weighted average of the local discplacement in a periodic unit cell of the composite plate. The second interpretation makes use of Eringen's averaging procedure from the microcontinuum to the macrocontinuum. In [38] an explicit relation is derived between the $\boldsymbol{\Phi}$ and the weighted strain average in the material representative volume element.

[7] It is sufficient to consider fields $\boldsymbol{\Phi}$ with vanishing right-hand trace, $\Phi_{ijj}=0$.

(b) Elasto–plasticity of stress gradient media

A possible extension of classical plasticity to stress gradient media can be based on the additive decomposition of the generalized strain measures into elastic and plastic parts, in the form:

$$\boldsymbol{e} = \boldsymbol{e}^e + \boldsymbol{e}^p, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}^e + \boldsymbol{\Phi}^p \tag{4.6}$$

In contrast to the usual displacement vector which cannot be uniquely split into elastic and plastic parts due to the absence of translation invariance, the microdisplacements are assumed to be objective quantities which allows for such a decomposition. The generalized strain tensor $\boldsymbol{e}$ is decomposed as a whole[8]. The Helmholtz free energy density function is taken as a function

$$\psi(\boldsymbol{e}^e, \boldsymbol{\Phi}^e, T, \alpha)$$

where α denotes generic internal (hardening) variables. The Clausius–Duhem inequality (1.8) now takes the form:

$$d = \boldsymbol{\sigma} : \dot{\boldsymbol{e}} + \boldsymbol{R} \vdots \dot{\boldsymbol{\Phi}} - \dot{\psi} - \dot{T}\eta - \boldsymbol{q} \cdot \frac{\boldsymbol{\nabla} T}{T} \geq 0 \tag{4.7}$$

$$(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{e}^e}) : \dot{\boldsymbol{e}}^e + (\boldsymbol{R} - \frac{\partial \psi}{\partial \boldsymbol{\Phi}^e}) : \dot{\boldsymbol{\Phi}}^e - (\eta + \frac{\partial \psi}{\partial T})\dot{T} + \boldsymbol{\sigma} : \dot{\boldsymbol{e}}^p + \boldsymbol{R} \vdots \dot{\boldsymbol{\Phi}}^p - \frac{\partial \psi}{\partial \alpha}\dot{\alpha} - \boldsymbol{q} \cdot \frac{\boldsymbol{\nabla} T}{T} \geq 0 \tag{4.8}$$

The following state laws are then adopted:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{e}^e}, \quad \boldsymbol{R} = \frac{\partial \psi}{\partial \boldsymbol{\Phi}^e}, \quad \eta = -\frac{\partial \psi}{\partial T}, \quad X = \frac{\partial \psi}{\partial \alpha} \tag{4.9}$$

This leads to the following form of the residual dissipation

$$\boldsymbol{\sigma} : \dot{\boldsymbol{e}}^p + \boldsymbol{R} \vdots \dot{\boldsymbol{\Phi}}^p - X\dot{\alpha} \geq 0 \tag{4.10}$$

A yield potential $f(\boldsymbol{\sigma}, \boldsymbol{R}, X)$ depending on the stress and stress gradient tensors can now be introduced as an extension of the traditional plasticity yield function, from which the flow rules are derived as:

$$\dot{\boldsymbol{e}}^p = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\Phi}}^p = \lambda \frac{\partial f}{\partial \boldsymbol{R}} \tag{4.11}$$

These relations involve a (visco)plastic multiplier λ which is determined from a consistency condition in the case of rate-independent plasticity, and by an explicit evolution equation in the viscoplastic case. Within the class of generalized standard materials according to [30], the evolution rules for internal variables should also follow from the derivation of the yield potential:

$$\dot{\alpha} = -\lambda \frac{\partial f}{\partial X} \tag{4.12}$$

(c) Example: Simple tension of a stress gradient material

The previous framework is applied to the case of an isotropic yield function of the form:

$$f(\boldsymbol{\sigma}, \boldsymbol{R}) = J_2(\boldsymbol{\sigma}) + \ell_p \|\boldsymbol{R}\| - \sigma_Y \text{ with } J_2(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^{\text{dev}} : \boldsymbol{\sigma}^{\text{dev}}} \quad \text{and } \|\boldsymbol{R}\| = \sqrt{R_{ijk}R_{ijk}} \tag{4.13}$$

The yield function represents one possible extension of classical J_2–plasticity. Quadratic forms could also be used, as done for example by [26] in the case of strain gradient plasticity.

The case of simple tension is particularly straightforward and illustrative of the main features of the new model. In the idealized two-dimensional case of a bar (width $2H$) in tension along the Cartesian direction 1, the single non vanishing component of the stress tensor is $\sigma_{11}(x_2)$ taken as a function of the transverse coordinate x_2 along direction 2. The stress tensor cannot be homogeneous since it must vanish at the free lateral surface according to the generalized Neumann boundary conditions introduced in section (a). It follows that the single non vanishing stress gradient component is $R_{112} = \sigma_{11,2}$. In that case, the stress gradient tensor fulfils the

[8] An alternative formulation is to split the strain tensor $\boldsymbol{\varepsilon}$ as usual and to define $\boldsymbol{e}^e = \boldsymbol{\varepsilon}^e + \boldsymbol{\Phi}^e \cdot \boldsymbol{\nabla}$. It is essentially equivalent in the small strain setting.

condition $R_{ijj} = 0$ as required by the balance of momentum equation.
The axial strain $\varepsilon_{11} = \bar{\varepsilon} = u_{1,1}$ is homogeneous and prescribed as $\bar{\varepsilon}$.

(i) Solution in the elastic case

The following simplified linear elastic law is assumed in the following:

$$\boldsymbol{\Phi} = E\ell_e^2 \boldsymbol{R} \tag{4.14}$$

where E the Young modulus and ℓ_e is an intrinsic length arising in the stress gradient elasticity model. In the investigated tensile case, this gives

$$\Phi_{112} = E\ell_e^2 R_{112} = E\ell_e^2 \sigma_{11,2} \tag{4.15}$$

The generalized strain measure (4.5) writes

$$e_{11} = \bar{\varepsilon} + \Phi_{112,2} = \sigma_{11}/E \tag{4.16}$$

The last equation results from the application of the isotropic elasticity relation between the stress tensor and the generalized strain measure. The combination of the latter equation and the generalized Hooke law (4.14) leads to the following differential equation

$$\sigma_{11} - E\ell_e^2 \sigma_{11,22} = E\bar{\varepsilon} \tag{4.17}$$

to be solved for the stress σ_{11}. Its solution is of the form

$$\sigma_{11} = E\bar{\varepsilon} + A\cosh\frac{x_2}{\ell_e} + B\sinh\frac{x_2}{\ell_e} \tag{4.18}$$

The integration constants A and B are determined from the following boundary conditions:

$$\sigma_{11}(x_2 = \pm H) = 0 \tag{4.19}$$

Finally, we obtain:

$$\sigma_{11} = E\bar{\varepsilon}\left(1 - \frac{\cosh(x_2/\ell_e)}{\cosh(H/\ell_e)}\right) \tag{4.20}$$

The corresponding stress profile in a special case ($E = 200$ GPa, $\bar{\varepsilon} = 2.10^{-4}$) is represented by the bottom curve of Fig. 1(a).

(ii) Solution in the elastic-plastic case with $\ell_p = 0$

The yield criterion (4.13) is considered first in the case $\ell_p = 0$. The stress component reaches the yield stress σ_Y at the middle of the sample $x_2 = 0$ for the prescribed strain value:

$$\bar{\varepsilon} = \frac{\sigma_Y}{E}\left(1 - \frac{1}{\cosh(H/\ell_e)}\right)^{-1} \tag{4.21}$$

For higher values of the prescribed strain, a plastic zone of size $2a$ expands in the sample.

In the elastic domain $|x_2| \geq a$, the stress distribution still has the form (4.18).

In the plastic domain $|x_2| \leq a$, the stress is uniform

$$\sigma_{11} = \sigma_Y, \quad R_{112} = 0, \quad \Phi_{112} = E\ell_e^2 R_{112} = 0 \tag{4.22}$$

The problem therefore has 3 unknowns a, A, B which are determined from 3 appropriate boundary and interface conditions:

- Vanishing stress at $x_2 = \pm H$ (here written at $x_2 = H$):

$$E\bar{\varepsilon} + A\cosh\frac{H}{\ell_e} + B\sinh\frac{H}{\ell_e} = 0 \tag{4.23}$$

- Stress continuity at $x_2 = \pm a$ (here written at $x_2 = a$):

$$E\bar{\varepsilon} + A\cosh\frac{a}{\ell_e} + B\sinh\frac{a}{\ell_e} = \sigma_Y \tag{4.24}$$

- Microdisplacement continuity at $x_2 = \pm a$ (here written at $x_2 = a$):

$$\Phi_{112}(a) = 0 \Longrightarrow \sinh \frac{a}{\ell_e} + B \cosh \frac{a}{\ell_e} = 0 \tag{4.25}$$

The corresponding stress profile in a special case ($E = 200$ GPa, $\bar{\varepsilon} = 5.10^{-4}, \sigma_Y = 100$ MPa) is represented by the top curve of Fig. 1(a). The plastic zone is growing when increasing the prescribed strain value as shown in Fig. 1(b). According to this model in the special case $\ell_p = 0$, it is found that an elastic boundary layer always remains close to the free boundary with a very strong stress gradient.

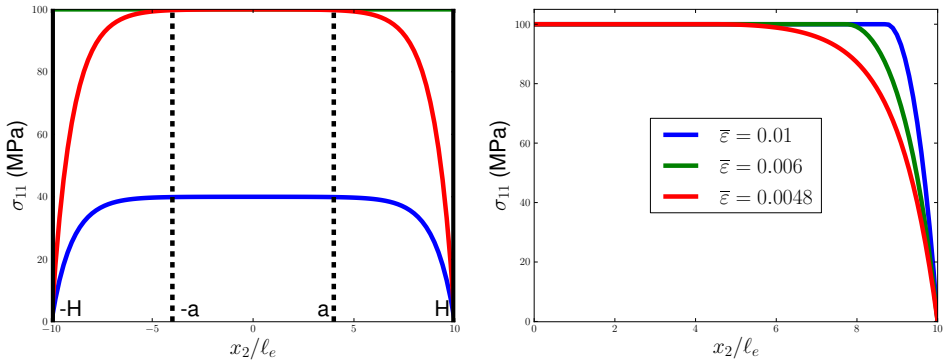

Figure 1. Tensile test in stress gradient elastoplasticity: (a) elastic (bottom curve) and elastic-plastic (top curve) stress profiles, (b) extension of the plastic zone with increasing applied uniaxial strain. The width of the bar and of the plastic zone respectively are $2H$ and $2a$.

(iii) Solution in the fully plastic case

Fully plastic solutions can be worked out when the plastic intrinsic length in 4.13 has a finite value. In that case, the yield condition is met everywhere in the sample:

$$|\sigma_{11}| + \ell_p |\sigma_{11,2}| = \sigma_Y \tag{4.26}$$

The tensile stress is expected to take larger values in the centre than at the free boundary. In the region $x_2 \geq 0$, we have therefore:

$$\sigma_{11} - \ell_p \sigma_{11,2} = \sigma_Y \tag{4.27}$$

which can be solved by an exponential function. The boundary condition $\sigma_{11}(\pm H) = 0$ is used to fix the constants. Finally, the stress profile is given by:

$$\sigma_{11}(x_2) = \sigma_Y \left(1 - \exp\left(\frac{|x_2| - H}{\ell_p}\right)\right) \tag{4.28}$$

which exhibits a cusp in the middle of the section. In this case no elastic layer remains at the free boundaries.

(d) Finite deformation stress gradient plasticity

The previous elastoplastic model for stress gradient media is now justified by proper linearisation of the stress gradient theory at finite deformations. Such a general framework has been proposed in the reference [25] and it is extended here to the elastoplastic case.

The stress gradient continuum is endowed with displacement and microdisplacement degrees of freedom. The latter are assumed to take the form

$$\boldsymbol{\Phi} = \Phi_{iJK}\boldsymbol{e}_i \otimes \boldsymbol{E}_J \otimes \boldsymbol{E}_K \tag{4.29}$$

The microdisplacements $\boldsymbol{\Phi}$ are assumed to transform in the following way by superimposing a rigid body motion: $\boldsymbol{\Phi} \longrightarrow \boldsymbol{Q} \cdot \boldsymbol{\Phi}$, thus transforming like the deformation gradient $\boldsymbol{F}$. The following Lagrangian generalized strain measures emerge in the theory:

$$\boldsymbol{E} := \frac{\boldsymbol{C}}{2} + \boldsymbol{\Upsilon} \cdot \boldsymbol{\nabla}_0, \quad \boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F}, \quad \boldsymbol{\Upsilon} := \boldsymbol{F}^T \cdot \boldsymbol{\Phi} \tag{4.30}$$

The Lagrangian power of internal forces takes the form

$$Jp^{(i)} = \boldsymbol{S} : \dot{\boldsymbol{E}} + \boldsymbol{T} \vdots \dot{\boldsymbol{\Upsilon}} \tag{4.31}$$

where $\boldsymbol{S}$ is the usual Piola (Lagrangian) stress tensor and T_{IJK} is the conjugate stress to the pulled-back microdisplacement Υ_{IJK}.
The method of virtual power was applied in [25] to derive the following balance equations:

$$\left(\boldsymbol{P} + (\boldsymbol{\Phi} \cdot \boldsymbol{\nabla}_0) \cdot \boldsymbol{S} + \boldsymbol{\Phi} : \boldsymbol{T}^T\right) \cdot \boldsymbol{\nabla}_0 = 0, \quad \boldsymbol{F} \cdot \boldsymbol{T} - \boldsymbol{P} \otimes \boldsymbol{\nabla}_0 = 0 \tag{4.32}$$

which is the finite version of Eq. (4.1). Volume and inertia forces were excluded for simplicity and can be found in [25]. The usual Boussinesq stress tensor is $\boldsymbol{P} = \boldsymbol{F} \cdot \boldsymbol{S}$. Transposition of a third rank tensor is taken as $T^T_{JKI} = T_{IJK}$. It appears that $\boldsymbol{F} \cdot \boldsymbol{T}$ is nothing but the gradient of the Boussinesq stress tensor, hence the name *stress gradient* continuum. These field equations are complemented by the following boundary conditions

$$\boldsymbol{t} = \left((\boldsymbol{F} + \boldsymbol{\Phi} \cdot \boldsymbol{\nabla}_0) \cdot \boldsymbol{S} + \boldsymbol{\Phi} : \boldsymbol{T}^T\right) \cdot \boldsymbol{N}, \quad \overset{3}{\boldsymbol{t}} = \boldsymbol{P} \otimes \boldsymbol{N} \tag{4.33}$$

where first and third order tractions are applied on the boundary. The latter amounts to the unusual situation of prescribing the full Boussinesq stress tensor at the boundary.

The Lagrangian generalized strain measures can be split additively:

$$\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^p, \quad \boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^e + \boldsymbol{\Upsilon}^e \tag{4.34}$$

in the spirit of Green and Naghdi's celebrated decomposition of the Green-Lagrange strain discussed in [41] and recently revisited for splitting other strain measures like the logarithmic strain in [51]. The choice of free energy density function $\psi(\boldsymbol{E}^e, \boldsymbol{\Phi}^e, T, \alpha)$ is associated with the following state laws

$$\boldsymbol{S} = \frac{\partial \psi}{\partial \boldsymbol{E}^e}, \quad \boldsymbol{T} = \frac{\partial \psi}{\partial \boldsymbol{\Upsilon}^e}, \quad \eta = -\frac{\partial \psi}{\partial T} \tag{4.35}$$

and the following residual dissipation rate

$$\boldsymbol{S} : \dot{\boldsymbol{E}}^p + \boldsymbol{T} \vdots \dot{\boldsymbol{\Phi}}^p - X\dot{\alpha} \geq 0 \tag{4.36}$$

This pleads for the introduction of a dissipation potential $\Omega(\boldsymbol{S}, \boldsymbol{T}, X)$ from which flow and hardening rules are derived:

$$\dot{\boldsymbol{E}}^p = \frac{\partial \Omega}{\partial \boldsymbol{S}}, \quad \dot{\boldsymbol{\Phi}}^p = \frac{\partial \Omega}{\partial \boldsymbol{T}}, \quad \dot{\alpha} = -\frac{\partial \psi}{\partial X} \tag{4.37}$$

The linearisation technique of [25] has been applied in order to check that all the balance, boundary and constitutive equations introduced in the previous section are retrieved in the small deformation case.

The proposed formulation based on an additive decomposition of the symmetric tensor $\boldsymbol{E}$ suffers from the same drawbacks as in classical finite strain elastoplasticity, namely the choice of a privileged reference configuration for the additive decomposition, and the absence of independent plastic spin for anisotropic media. An alternative would be to employ the multiplicative decomposition of the deformation gradient and decomposition of the divergence

of the microdisplacement in the spirit of relations (3.3). This will lead however to a necessary reformulation of the stress gradient model by abandoning the use of the gradient of the Boussinesq stress and using instead the gradient of another stress measure for instance the one pulled-back to the intermediate configuration (Mandel or Piola forms). This is left open at this stage of the stress gradient theory.

5. Conclusion

A unifying presentation of micromorphic, strain and stress gradient continuum theories has been proposed reconciling many available and new constitutive frameworks under the umbrella of continuum thermodynamics with internal variables and internal degrees of freedom. These models accounting for size effects in the thermomechanical behaviour of materials and structures share common features but also have clear distinct capabilities depending on the detailed description of the underlying material microstructure. The novel aspects of the presented theories are the following:

- A concise formulation of micromorphic and strain gradient media at large thermo-elasto-viscoplastic deformations including temperature effects; Special attention has been given to the consideration of entropy gradients in the constitutive theory of nonlinear strain gradient media;
- A thorough discussion of so-called energetic and dissipative contributions to hardening in strain gradient plasticity, encompassing most recent contributions from literature;
- A comparison of work-hardening types (isotropic vs. kinematic hardening) depending of the choice of enhanced kinematic variables or gradient contribution;
- The first formulation of stress gradient plasticity theory, an extension of the recently developed stress gradient continuum [24,59]. The proposed yield function depends on the stress and stress gradient tensors. The finite deformation formulation was provided together with its linearization. The example of simple tension shows that the stress gradient model predicts a boundary layer close to free boundaries where all components of the stress tensor are assumed to vanish in contrast to all other generalized continuum theories.

Strain and stress gradient media were shown to fundamentally differ in their respective kinematic degrees of freedom essentially. The existence of microtemperature or microentropy field variables in the thermo–micromorphic model was not discussed in the present work, see [21,39] for such investigations. Instead the entropy gradient effects were introduced in the strain gradient theories as a necessary counterpart of the strain gradient term.

There exist a wealth of available and future possible applications of generalized continua to many engineering materials and structures, especially regarding the consideration of combined damage and plasticity [8,16]. Crystal plasticity also is a major realm of application of generalized continuum mechanics in order to describe grain and inclusion size effects in metallic polycrystals [17,48,68]. Gradient damage models, also called phase field models of fracture, and used for the simulation of crack initiation and propagation, fall in the class of micromorphic or gradient of internal variable theories and can be handled along the lines of this work [50]. Applications of the stress gradient continuum remain to be found. First attempts deal with size effects in composites and enhancement of homogenization theory [64].

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