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# ON CLASSES OF WELL-POSEDNESS FOR QUASILINEAR DIFFUSION EQUATIONS IN THE WHOLE SPACE.

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ABSTRACT. Well-posedness classes for degenerate elliptic problems in  $\mathbb{R}^N$  under the form  $u = \Delta\varphi(x, u) + f(x)$ , with locally (in  $u$ ) uniformly continuous nonlinearities, are explored. While we are particularly interested in the  $L^\infty$  setting, we also investigate about solutions in  $L^1_{loc}$  and in weighted  $L^1$  spaces. We give some sufficient conditions in order that the uniqueness and comparison properties hold for the associated solutions; these conditions are expressed in terms of the moduli of continuity of  $u \mapsto \varphi(x, u)$ . Under additional restrictions on the dependency of  $\varphi$  on  $x$ , we deduce the existence results for the corresponding classes of solutions and data. Moreover, continuous dependence results follow readily from the existence claim and the comparison property. In particular, we show that for a general continuous non-decreasing nonlinearity  $\varphi : \mathbb{R} \mapsto \mathbb{R}$ , the space  $L^\infty$  (endowed with the  $L^1_{loc}$  topology) is a well-posedness class for the problem  $u = \Delta\varphi(u) + f(x)$ .

## 1. INTRODUCTION

Motivated by our exploration [7] of uniqueness of entropy solutions to parabolic-hyperbolic convection-diffusion PDEs  $u_t + \operatorname{div} F(u) - \Delta\varphi(u) = 0$  with non-Lipschitz nonlinearities (cf. [37, 40]; see also [35, 36, 17, 4] for the underlying motivations in the pure hyperbolic case), we got interested in the analysis of well-posedness of merely bounded solutions in the whole space for degenerate diffusion equations  $u_t - \Delta\varphi(u) = 0$  of the generalized porous medium/fast diffusion kind. In this paper, we thoroughly investigate the stationary case, i.e., the elliptic problems of the kind  $u - \Delta\varphi(x, u) = f(x)$  in  $\mathbb{R}^N$ . Our main assumption is the locally uniform continuity in  $u$  of the nonlinearity  $\varphi$ ; in addition,  $\varphi$  should be of the Carathéodory type, non-decreasing in  $u$ .

Actually, instead of working with pure  $L^\infty$  solutions, we extend our investigation to solutions in  $L^1_{loc}$  or in weighted  $L^1$  spaces. Note that weighted contraction estimates have been a trend in the recent research on convection-diffusion equations, see Endal and Jakobsen [28], Alibaud, Endal and Jakobsen [1].

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In our contribution, three different frameworks are explored. First, the classical results of Brézis [20] on locally integrable solutions to the equation  $u = \Delta(|u|^{m-1}u) + f(x)$ ,  $0 < m < 1$ , are extended. We give a complete well-posedness theory for  $L^1_{loc}$  solutions under a generalized Keller-Osserman condition, extending also the results of Gallouët and Morel [30]. Further, for uniformly continuous in  $u$  nonlinearities  $\varphi$ , an original quantitative continuous dependence estimate is obtained in the setting of weighted  $L^1(\mathbb{R}^N, \rho)$  spaces with exponentially decaying weights  $\rho$ . Finally, for  $N \geq 3$ , the well-posedness results of Bénilan and Crandall [14] in the spaces  $L^1(\mathbb{R}^N, \rho)$  with super-harmonic weights are re-visited.

**1.1. A brief account on classical results.** Although we are mainly concerned with the stationary case, let us provide a brief bibliography on well-posedness classes for both stationary and evolution nonlinear diffusion equations; we specifically highlight theories allowing for study of uniqueness of merely  $L^\infty$  solutions.

An extensive account on mathematical theory (or rather theories) of generalized porous medium and fast diffusion equations can be found in the book by Vazquez [42]. The reader can also consult the surveys by Aronson [10], Kalashnikov [32], the books by DiBenedetto [27], and Wu et al. [43]. A powerful method for uniqueness analysis of very weak solutions was put forward by Brézis and Crandall [21], further classical results include Bénilan, Crandall, Pierre [16], Dahlberg and Kenig [23]. Recent results in this direction, both in the classical local case and in the nonlocal (fractional) case, were obtained by del Teso, Endal and Jakobsen [25, 26], who also provide an extensive bibliography. Close to our interests, let us mention that Herrero and Pierre [31] proved uniqueness for merely locally integrable distributional solutions of the fast diffusion equation  $u_t = \Delta(|u|^{m-1}u)$ ,  $0 < m < 1$ , under a regularity assumption on  $u_t$  which is verified in many interesting cases. Bénilan and Crandall [14] obtained well-posedness and structural stability results on the general evolution equation  $u_t - \Delta\varphi(u) = 0$  in the setting of weighted  $L^1(\mathbb{R}^N; \rho)$  spaces with  $\rho$  decaying as  $|x|^{2-N}$  as  $|x| \rightarrow \infty$ , which gets quite close to including  $L^\infty$  solutions. Later, several classes of polynomially growing solutions were investigated by many authors under assumptions of specific behavior of  $\varphi$  for  $u \sim 0$  and  $u \sim \infty$ ; we refer to the book by Daskalopoulos and Kenig [24] for an extensive account on these results and further references.

Turning back to the stationary case, first note that the problem under the form  $\beta(w)_t - \Delta w \ni 0$  was thoroughly investigated in the  $L^1(\mathbb{R}^N)$  setting by Bénilan, Brézis and Crandall in [13]. Additional information relevant to  $L^1(\mathbb{R}^N, \rho)$  setting with  $\rho(x) = (1 + |x|^2)^{-\alpha}$ ,  $0 < \alpha \leq (N - 2)/2$ , is given in the aforementioned reference [14]. Note however that these results do not cover the case of general  $L^\infty(\mathbb{R}^N)$  solutions, although they get quite close to doing so. Next, uniqueness of  $L^1_{loc}$  distributional solutions (which covers our target setting  $L^\infty(\mathbb{R}^N)$ ) to the equation  $u = \Delta\varphi(u) + f$  for  $\varphi(z) = |z|^{m-1}z$ ,  $0 < m < 1$ , was shown by Brézis in [20]. The same year, similar results were obtained by Bokalo [19] for the associated evolution problem on  $\mathbb{R} \times D$ ,  $D \subset \mathbb{R}^N$ , without conditions at  $t = \pm\infty$ . Paper [20] also contains the existence result in  $L^1_{loc}(\mathbb{R}^N)$ , based on the unusual localization property of Baras and Pierre [11]. The Brézis results were generalized by Gallouët and Morel [30] to an arbitrary increasing nonlinearity  $\varphi$  which is concave on  $\mathbb{R}^+$ , odd, and satisfies the Keller-Osserman condition

$$\int_1^{+\infty} \frac{dz}{\sqrt{\int_0^z \varphi^{-1}(s) ds}} < +\infty$$

(see Keller [34], Osserman [38]). This condition ensures that the diffusion is so fast (for large values of  $u$ ) that the requirement that a solution is defined globally on  $\mathbb{R}^N$  becomes a sharp restriction; for instance, the only globally defined solution of  $u = \Delta(|u|^{m-1}u)$ ,  $0 < m < 1$ , is identically zero.

The above references [14] and [20, 30] are most relevant to our investigation.

**1.2. The scope and the outline of the paper.** We are concerned with the nonlinear diffusion equation

$$(1) \quad u = \Delta\varphi(x, u) + f(x);$$

in Section 6, we briefly discuss implications for the associated evolution problem

$$(2) \quad u_t = \Delta\varphi(x, u) + f(t, x), \quad u|_{t=0} = u_0.$$

Our primary focus is on the comparison principle for solutions. Clearly, comparison principle implies uniqueness. Furthermore, following Wittbold et al. [2, 9] and using classical existence results e.g. in the  $L^1(\mathbb{R}^N)$  framework ([13]) we also use the comparison principle as the main tool (along with some rather weak estimates) in order to deduce existence of solutions and their continuous dependence on the data.

Let us give an outline of the paper.

In Section 2, we give the definitions of solutions and the associated Kato [33, 20] inequalities. Note that our results all come from construction of appropriate families of test functions in order to exploit these Kato inequalities. We also introduce some notation, give further assumptions on  $\varphi$  and recall the notion of a modulus of continuity and related technical objects that are central for our techniques.

In Section 3, we give two kinds of comparison results for solutions of (1) with nonlinearity  $\varphi$  uniformly continuous in the second variable. First we revisit the  $L^1_{loc}$  uniqueness results of Brézis [20], Gallouët and Morel [30] under a generalized Keller-Osserman condition on the modulus of continuity of  $\varphi$ . We argue that most of the restrictions on  $\varphi$  imposed in [30] can be dropped. To the authors' knowledge, such results were not yet published in the full generality. Moreover, a very simple continuous dependence result can be formulated with the help of the comparison technique; we refer to the work [9] of Wittbold and the first author for a similar investigation in the setting of elliptic-parabolic convection-diffusion problems. Next, a different approach to (1) with sublinear nonlinearities  $\varphi(x, \cdot)$  yields the comparison property in the weighted spaces  $L^1(\mathbb{R}^N, \rho)$  with  $\rho(x) = \exp(-c|x|)$ ,  $c > 0$ . Here, the possibility to choose a wide class of nonlinearities for the Kato inequalities is instrumental. We obtain quantitative continuous dependence estimates of solutions in terms of the data.

In Section 4 we give the associated existence results. These results exploit extensively the comparison principles of Section 3. More precisely, following the idea put forward by Ammar and Wittbold in [2] we construct monotone sequences of approximate solutions, and use *ad hoc* estimates to control their pointwise limit. Existence, together with the uniqueness and continuous dependence shown in Section 3, permits to state well-posedness results in several classes of solutions, namely, in  $L^1_{loc}(\mathbb{R}^N)$ , in  $L^1(\mathbb{R}^N, \exp(-c|x|))$ , and (for the case of an  $x$ -independent nonlinearity  $\varphi$ ) in the class  $L^\infty(\mathbb{R}^N)$ . For the case where  $\varphi$  depends on  $x$ , the  $L^\infty$  framework is not natural. Therefore we introduce the adequate class  $L^\infty_\varphi(\mathbb{R}^N)$  for the problem (1) with nonlinearity  $\varphi$ , and show well-posedness in this class under an assumption of “local” uniform continuity of  $\varphi$  in the second variable.

Before turning to the next line of analysis, let us remind that the fundamental solution of the laplacian operator played an important role for well-posedness theories in  $L^1(\mathbb{R}^N)$  for (1) (see [13, 29]) and the associated evolution problem ([14, 41]). In Section 5 we obtain uniqueness, contraction and comparison results for (1) and (2) in the weighted spaces  $L^1(\mathbb{R}^N, \rho)$ , with  $N \geq 3$  and weights  $\rho(x)$  given by truncations of the fundamental solution profile  $|x|^{2-N}$ . Existence of solutions with  $L^1(\mathbb{R}^N, \rho) \cap L^\infty(\mathbb{R}^N)$  data for the stationary problem (1) is also shown. More generally, a wide class of supersolutions to  $-\Delta\rho = 0$  can be used as weights (see Theorem 5.2 and Corollary 2 for the exact statement). Our results in Section 5 are very close to those obtained by Bénilan and Crandall in [14]. Compared to this classical reference, we work in the setting lacking space translation invariance, we detail the technical arguments based on accurate use of moduli of continuity, and finally devise a uniqueness proof suitable for the full evolution convection-diffusion problem which motivated our study (see [7]).

Investigation of related classes of well-posedness for the evolution problem (2) is mainly postponed to further work. We only briefly mention in Section 6 some classical or recent results on the porous medium / fast diffusion equations; we also indicate the possibility to combine the contraction claims in  $L^1(\mathbb{R}^N, \rho)$  in Section 5 for the resolvent equation with the tool of integral solution ([12, 15]) of the underlying abstract evolution equation, similarly to what was done in [6] by Igbida and the first author.

For the convenience of the reader, let us indicate the main results for the case of an  $x$ -independent nonlinearity  $\varphi$ . Such results are formulated in Corollary 1, Theorem 4.2, Theorem 4.3, and Theorem 5.4 (with Remark 14).

## 2. MAIN DEFINITIONS AND TOOLS

In § 2.1, we give some notation used throughout the paper. In § 2.2, we give the Kato inequalities for very weak solutions which provide the starting point for the uniqueness analysis. Many of our results, and in particular, a version of the Keller-Osserman condition (see [34, 38, 30]) are stated in terms of the moduli of continuity of the nonlinearity  $\varphi$  which we define in § 2.3.

**2.1. Basic notation.** Generically,  $\varphi$  denotes a Carathéodory function from  $\mathbb{R}^N \times \mathbb{R}$  to  $\mathbb{R}$ , non-decreasing in the second variable. In many statements, we assume that  $\varphi$  is independent of  $x$ ; this assumption will be explicitly indicated whenever it is used. In all cases, the function  $x \mapsto \varphi(x, u(x))$  (or  $(t, x) \mapsto \varphi(x, u(t, x))$ ) will be denoted by  $\varphi \circ u$ . When  $u$  is a solution to the equation considered, then  $\varphi \circ u$  will be often denoted by  $w$ . When  $u_1, u_2$  are two fixed solutions, then  $W$  will always denote the function  $(w_1 - w_2) \equiv (\varphi \circ u_1 - \varphi \circ u_2)$ .

The notation  $a \vee b$ , resp.  $a \wedge b$ , stands for the maximum, resp. for the minimum of  $a, b \in \mathbb{R}$ . The positive (resp., negative) part of a function  $g$  is denoted by  $g^+$  (resp.,  $g^-$ ); we have  $g^+ = g \vee 0$ ,  $g^- = -(g \wedge 0)$ , and  $g = g^+ - g^-$ .

For  $R > 0$ ,  $B_R$  stands for the ball of  $\mathbb{R}^N$  of radius  $R$  centered at the origin.

For a measurable set  $A \subset \mathbb{R}^m$ , we denote by  $\mathbb{1}_A(\cdot)$  the characteristic function of  $A$ . We denote by  $\text{sign}^+(\cdot)$  the characteristic function of  $(0, +\infty)$ .

For a bounded positive measurable function  $\rho$  on  $\mathbb{R}^N$ , we denote by  $L^1(\mathbb{R}^N, \rho)$  the space of measurable functions with finite norm  $\|f\|_{L^1(\rho)} := \int_{\mathbb{R}^N} |f(x)| \rho(x) dx$ . We usually drop  $dx$  in the notation for integrals.

The maximal monotone graph which is the inverse of  $\varphi(x, \cdot)$  will be denoted by  $\beta(x, \cdot)$ . Its minimal section is the function

$$\beta_0 : (x, z) \in \mathbb{R}^N \times \mathbb{R} \mapsto \{ b \mid \forall c \in \beta(x, z), |b| \leq |c| \}.$$

**Remark 1.** Notice that for all measurable  $w : \mathbb{R}^N \rightarrow \mathbb{R}$ , the composition  $\beta_0(\cdot, w(\cdot))$  is measurable. Indeed, e.g. for  $\lambda \geq 0$ , we have

$$\{x \in \mathbb{R}^N \mid \beta_0(x, w(x)) \leq \lambda\} = \{x \in \mathbb{R}^N \mid \varphi(x, \lambda) \leq w(x)\};$$

this set is measurable, by the assumptions on  $\varphi$  and  $w$ .

**2.2. Very weak solutions and Kato inequalities.** In what follows, for the stationary case we always consider the very weak (distributional) solutions, which we call  $\mathcal{D}'$  solutions for the sake of conciseness.

**Definition 2.1.** Let  $f \in L^1_{loc}(\mathbb{R}^N)$ . A function  $u \in L^1_{loc}(\mathbb{R}^N)$  such that  $w := \varphi \circ u \in L^1_{loc}(\mathbb{R}^N)$  is called a  $\mathcal{D}'$  solution to Problem (1) if for all  $\xi \in \mathcal{D}(\mathbb{R}^N)$ ,

$$\int_{\mathbb{R}^N} (u\xi - w\Delta\xi - f\xi) = 0.$$

In the sequel, we will consider two solutions  $u_i$ ,  $i = 1, 2$  ( $\mathcal{D}'$  solutions to the same problem) corresponding to different data; recall that we write  $w_i$  for the functions  $\varphi \circ u_i := \varphi(\cdot, u_i(\cdot))$ . The function  $w_1 - w_2$  will be denoted by  $W$ . In the following proposition, we recall (and slightly generalize) the Kato inequalities [33, 20] that hold for solutions  $u_1, u_2$  in the sense of Definition 2.1. Let us introduce the set

$$\mathcal{S} := \{S \in W^{1,\infty}_{loc}(\mathbb{R}) \mid S(0) = 0, \\ S' \text{ is non-decreasing, bounded, piecewise continuous} \}.$$

We denote  $\mathcal{S}_0 = \{S \in \mathcal{S} \mid zS'(z) \geq 0\}$ . In particular,  $S : z \mapsto z^+$  belongs to  $\mathcal{S}_0$ .

**Proposition 1.** Assume that  $u_i$  are  $\mathcal{D}'$  solutions of Problem (1) corresponding to the data  $f_i$ , respectively,  $i = 1, 2$ . Then for all  $S \in \mathcal{S}$ , for all  $\xi \in \mathcal{D}(\mathbb{R}^N)^+$ , we have

$$(3) \quad \int_{\mathbb{R}^N} ((u_1 - u_2) S'(W)\xi - S(W)\Delta\xi) \leq \int_{\mathbb{R}^N} (f_1 - f_2) S'(W)\xi.$$

Furthermore,

$$(4) \quad \int_{\mathbb{R}^N} ((u_1 - u_2)^+\xi - W^+\Delta\xi) \leq \int_{\mathbb{R}^N} (f_1 - f_2) \text{sign}^+(u_1 - u_2)\xi.$$

*Proof.* Since the distributional laplacian  $-\Delta W$  of  $W$  equals to  $f_1 - f_2 - u_1 + u_2 \in L^1_{loc}(\mathbb{R}^N)$ , inequality (3) follows directly from the generalized Kato inequality (see Brézis [20, Lemma A.2]). In order to get (4), we follow the idea of Blanchard and Porretta [18]. We choose  $\pi \in \mathcal{D}(\mathbb{R}^N)$  and consider the relation

$$-\Delta(W + \varepsilon\pi) = f_1 - f_2 - u_1 + u_2 - \varepsilon\Delta\pi \in L^1_{loc}(\mathbb{R}^N).$$

Let  $S'_\varepsilon(z) = \min\{\frac{z^+}{\varepsilon}, 1\}$ ,  $S_\varepsilon(z) = \int_0^z S'_\varepsilon(s) ds$ , for  $\varepsilon > 0$ . We use [20, Lemma A.2] for the function  $W + \varepsilon\pi$  and the “test function”  $S'(W + \varepsilon\pi)$ . We obtain

$$-\Delta S_\varepsilon(W + \varepsilon\pi) \leq (f_1 - f_2 - u_1 + u_2)S'_\varepsilon(W + \varepsilon\pi) - \varepsilon\Delta\pi S'_\varepsilon(W + \varepsilon\pi) \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

As  $\varepsilon \downarrow 0$ , the last term vanishes and the first term converges to  $-\Delta W^+$  in  $\mathcal{D}'(\mathbb{R}^N)$ . Further,  $S'_\varepsilon(W + \varepsilon\pi)$  converges in  $L^1_{loc}(\mathbb{R}^N)$  to the function  $\text{sign}^+(W) + \pi \mathbb{1}_{W=0}$ . Letting  $\pi \in \mathcal{D}(\mathbb{R}^N)$  converge to  $\text{sign}^+(u_1 - u_2)$  in  $L^1_{loc}(\mathbb{R}^N)$ , we infer (4).  $\square$

**2.3. Assumptions on  $\varphi$ , moduli of continuity and generalized Keller-Osserman condition.** Let us precise the assumptions on the nonlinearity  $\varphi$ . We usually assume

$$(H_0) \quad \left\{ \begin{array}{l} \varphi : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a Carathéodory function} \\ \text{non-decreasing in the second variable;} \\ \varphi \text{ is uniformly continuous in the second variable on } \mathbb{R}^N \times \mathbb{R}; \end{array} \right.$$

(when speaking about  $L^\infty$  solutions, we relax the uniform continuity assumption to the suitable local uniform continuity assumption; see e.g. Remark 11 in § 4.3). In addition, for the  $L^1_{loc}$  theory, the generalized Keller-Osserman condition (see Definition 2.3 below) will be required.

It is usual to assume the normalization  $\varphi(x, 0) = 0$  a.e., and  $\beta_0(x, k) \in L^1_{loc}(\mathbb{R}^N)$  for all  $k \in \mathbb{R}$  (cf. the existence results of [20, 29]). Our framework only differs by translation by some given solution  $u_*$ . We do not assume  $\varphi(x, 0) = 0$ , but show that, under a kind of growth assumption on  $\beta_0(x, \cdot)$ , problem (1) either admits no solution whatever be the datum, or it admits a solution in the affine space  $u_* + E := \{u = u_* + \tilde{u}, \tilde{u} \in E\}$  for data in  $f_* + E$ , for the appropriately chosen vector spaces  $E$ . For the sake of simplicity, we require that either

$$(H_{aut}) \quad \varphi \text{ is independent of } x, \text{ normalized by } \varphi(0) = 0$$

(in which case  $f_* \equiv 0, u_* \equiv 0$ ); or

$$(H_{surj}) \quad \left\{ \begin{array}{l} \text{for a.e. } x \in \mathbb{R}^N, \varphi(x, \cdot) \text{ is surjective;} \\ \text{there exist } f_*, u_* \in L^1_{loc}(\mathbb{R}^N) \\ \text{such that } u_* \text{ is a } \mathcal{D}' \text{ solution of (1) with datum } f_*; \\ \text{and for all } k \in \mathbb{R}, \beta_0(\cdot, w_*(\cdot) + k) \in L^1_{loc}(\mathbb{R}^N), \text{ where } w_* = \varphi \circ u_*. \end{array} \right.$$

Let us point out that  $(H_{surj})$  does hold under the following simple assumption:

$$(H'_{surj}) \quad \left\{ \begin{array}{l} \text{for all } k \in \mathbb{R}, \text{ for a.e. } x \in \mathbb{R}^N \varphi(x, \cdot) \text{ is surjective,} \\ \text{and } \beta_0(\cdot, k) \in L^1_{loc}(\mathbb{R}^N). \end{array} \right.$$

This assumption corresponds to the case where  $\beta_0(x, 0) \in L^1_{loc}(\mathbb{R}^N)$ , so that  $(H_{surj})$  holds true, with the choice  $u_* \equiv f_* = \beta_0(x, 0)$  and  $w_* \equiv 0$ .

Also notice that, because there holds  $|\beta_0(\cdot, w_*(\cdot))| \leq |u_*| \in L^1_{loc}(\mathbb{R}^N)$ , a growth assumption on  $\beta_0(x, \cdot)$  of the kind  $|\beta_0(x, r + k)| \leq C(k)|\beta_0(r)|$  guarantees that  $\beta_0(\cdot, w_*(\cdot) + k) \in L^1_{loc}(\mathbb{R}^N)$ . This allows for a (uniform in  $x$ ) polynomial of exponential growth of  $\beta_0(x, \cdot)$ .

The surjectivity assumption on  $\varphi(x, \cdot)$  in  $(H_{surj})$  can be relaxed (cf. assumption  $(H_{aut})$ ). However, let us stress that some restrictions on the dependence of the domain of  $\beta_0(x, \cdot)$  on  $x$  are needed in order to achieve existence (cf. [8, Example 3.1], which can be interpreted as an example of non-existence for our formulation).

Now we make precise what we mean by a modulus of continuity of  $\varphi$ , by its inverse, and by sublinear or strictly sublinear nonlinearity.

**Definition 2.2.** Let  $\varphi : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function.

- (i) Assume  $\varphi$  is uniformly continuous in the second variable on  $\mathbb{R}^N \times \mathbb{R}$ . We then say that  $\varphi$  admits a modulus of continuity, and call the function

$$(5) \quad r \in \mathbb{R}^+ \mapsto \sup \{ |\varphi(x, z) - \varphi(x, \hat{z})| \mid x \in \mathbb{R}^N, z, \hat{z} \in \mathbb{R}, |z - \hat{z}| \leq r \} \in \mathbb{R}^+$$

the best modulus of continuity of  $\varphi$ . In the sequel, the function  $\mathbb{R}^+ \mapsto \mathbb{R}^+$  given by (5) will be denoted by  $\omega_0(\cdot)$

More generally, any continuous sub-additive non-decreasing function  $\omega : \mathbb{R}^+ \mapsto \mathbb{R}^+$  with  $\omega(0) = 0$  such that  $\omega$  dominates  $\omega_0$  is called a *modulus of continuity of  $\varphi$* .

- (ii) Given a modulus of continuity  $\omega$ , we define its inverse (in the generalized sense)  $\Omega$  by Remark (v) below. In particular, the inverse of the best modulus of continuity  $\omega_0$  will be denoted by  $\Omega_0$ .
- (iii) When  $\varphi$  is uniformly continuous in the second variable, the quantity

$$|\varphi(x, z + y) - \varphi(x, y)|$$

is dominated by an affine function of  $z \in \mathbb{R}^+$  independent of  $x \in \mathbb{R}^N$  and  $y \in \mathbb{R}$  (in this case, we say that  $\varphi$  is *sublinear*). If, in addition, the limit  $\lim_{z \rightarrow \infty} \frac{\omega_0(z)}{z}$  exists and equals zero, we say that  $\varphi$  is *strictly sublinear*.

**Remark 2.**

- (i) The best modulus of continuity  $\omega_0$  of  $\varphi$  is a modulus of continuity of  $\varphi$ .
- (ii) In the sequel, we will always assume that  $\varphi$  actually depends on its second variable; then for all modulus of continuity  $\omega$  of  $\varphi$  we have  $\omega(r) \geq \omega_0(r) > 0$  for  $r > 0$ .
- (iii) If  $\varphi$  is independent of  $x$ , the truncated function  $z \mapsto (\varphi(z) \vee (\varphi(-M))) \wedge \varphi(M)$  admits a modulus of continuity, for all  $M > 0$ .
- (iv) Being sub-additive, the function  $\omega_0$  is sublinear; therefore its concave hull

$$\widehat{\omega} : r \in \mathbb{R}^+ \mapsto \inf \{ g(r) \mid g \text{ is concave on } \mathbb{R}^+, g \geq \omega_0 \text{ on } \mathbb{R}^+ \}$$

is well defined. The function  $\widehat{\omega}$  is the smallest concave modulus of continuity of  $\varphi$ . If  $\varphi$  is strictly sublinear, then its smallest concave modulus of continuity  $\widehat{\omega}$  is also strictly sublinear.

- (v) If  $\omega$  is a modulus of continuity of  $\varphi$ , we can inverse it; the procedure is trivial if  $\omega$  is strictly increasing and onto. In the general case, let  $\Omega : \omega(\mathbb{R}^+ \cup \{+\infty\}) \mapsto \mathbb{R}^+ \cup \{+\infty\}$  denote the greatest function such that  $\omega \circ \Omega$  is the identity mapping on  $\omega(\mathbb{R}^+ \cup \{+\infty\})$ . If necessary, we extend  $\Omega$  by  $+\infty$  on  $\mathbb{R}^+ \setminus \omega(\mathbb{R}^+ \cup \{+\infty\})$ .
- (vi) The so defined inverse  $\Omega$  of  $\omega$  is a super-additive non-decreasing function with  $\Omega(0) = 0$ .
- (vii) If  $\varphi$  admits a modulus of continuity, it can be chosen concave and bijective. In this case,  $\Omega$  is convex and strictly increasing.

**Definition 2.3.** We say that  $\varphi$  satisfies the generalized Keller-Osserman condition, if  $\varphi$  admits a bijective modulus of continuity  $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that the inverse function  $\Omega$  of  $\omega$  satisfies

$$(H_{KO}) \quad \int_1^{+\infty} \frac{dz}{\sqrt{z\Omega(z)}} < +\infty.$$

**Remark 3.** For technical reasons, it is convenient to assume  $\omega$  bijective in the above definition. But the so defined generalized Keller-Osserman condition is equivalent to the requirement that  $(H_{KO})$  hold with  $\Omega = \Omega_0$ , with the convention that  $\frac{1}{+\infty} = 0$ .

Indeed, assume  $(H_{KO})$  holds with  $\Omega = \Omega_0$ . If  $\lim_{r \rightarrow +\infty} \omega_0(r) < +\infty$ , we set  $\omega(r) = \omega_0(r) + \sqrt{r}$ , which is a bijective modulus of continuity of  $\varphi$ ; since  $\Omega(z) = \omega^{-1}(z)$  is equivalent to  $z^2$  at  $+\infty$ , we still have  $(H_{KO})$  with  $\Omega$ . Now assume that  $\omega_0$  is surjective, but not strictly increasing. Then we extend  $\Omega_0$  by zero on  $\mathbb{R}^-$  and use the convolution with a standard mollification kernel  $\theta$  supported in  $[-1, 1]$  to define



$\Omega(r) = (\Omega_0 * \theta)(r-1)$ . By construction,  $\Omega(0) = 0$ ,  $\Omega$  is continuous and it inherits the super-additivity of  $\Omega_0$ ; thus it is a continuous bijective function on  $\mathbb{R}^+$ . Since we also have  $\Omega \leq \Omega_0$ , we get  $\omega := \Omega^{-1} \geq \omega_0$ . The so constructed  $\omega$  verifies  $(H_{KO})$ , because  $\Omega(z) \geq \Omega_0(z-2)$  by the definition of  $\Omega$ .

**Remark 4.** It is more usual to formulate the Keller-Osserman condition by requiring that

$$(6) \quad \int_1^{+\infty} \frac{dz}{\sqrt{\int_0^z \Omega(s) ds}} < +\infty.$$

It is shown by Gallouët and Morel that whenever  $\limsup_{z \rightarrow +\infty} z/\Omega(z)$  is finite, the two conditions  $(H_{KO})$  and (6) are equivalent (see [30, p.904]). In our framework  $\Omega$  is a superlinear function, hence this equivalence holds true.

### 3. COMPARISON RESULTS FOR PROBLEM (1)

The following comparison results (together with the “basic” existence result of § 4.1 and with adequate *a priori* estimates) are the basis of the whole well-posedness theory for growing solutions of (1). Note that the two kinds of results presented below are not directly related. Indeed, the generalized Keller-Osserman condition is required in § 3.1 but not in § 3.2; while in § 3.2, additional growth assumptions are imposed on the solutions and data.

**3.1. Comparison of  $L^1_{loc}$  solutions.** The following theorem is a rather straightforward generalization of the Gallouët and Morel result in [30].

**Theorem 3.1.** *Assume  $\varphi$  satisfies  $(H_0)$  and the generalized Keller-Osserman condition  $(H_{KO})$  of Definition 2.3.*

*Then for all  $f \in L^1_{loc}(\mathbb{R}^N)$  there exists at most one  $\mathcal{D}'$  solution to Problem (1). Furthermore, if  $u_i$  are  $\mathcal{D}'$  solutions of (1) corresponding to  $f_i$ ,  $i = 1, 2$ , then  $f_1 \leq f_2$  implies  $u_1 \leq u_2$ .*

**Remark 5.** Here and in the subsequent results, the comparison properties persist if we assume that  $u_1$  is a subsolution, and  $u_2$  is a supersolution of the corresponding equations, with standard definition of sub- and super- very weak solutions.

*Proof.* Recall that  $W = w_1 - w_2 = \varphi \circ u_1 - \varphi \circ u_2$ . Let  $\omega$  denote the (bijective) modulus of continuity of  $\varphi$  such that  $\Omega = \omega^{-1}$  satisfies  $(H_{KO})$ . Choose  $f_1 \leq f_2$  in  $L^1_{loc}(\mathbb{R}^N)$ .

Step 1. Using the Kato inequality (4), we find  $(u_1 - u_2)^+ - \Delta W^+ \leq (f_1 - f_2)^+ \leq 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ . But

$$(7) \quad W^+(x) = (\varphi(x, u_1(x)) - \varphi(x, u_2(x)))^+ \leq \omega((u_1(x) - u_2(x))^+)$$

thanks to the monotonicity of  $\varphi(x, \cdot)$ ; therefore we have  $\Omega(W^+) - \Delta W^+ \leq 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ , with  $\Omega$  satisfying the Keller-Osserman condition.

If  $\Omega$  were convex, the proof of [30] would apply and yield  $W^+ \leq 0$  a.e. in  $\mathbb{R}^N$ . In fact, the super-additivity of  $\Omega$  and the observation that  $\Omega(z) \leq \text{const} \times z$  in a neighbourhood of zero are sufficient for the proof to work. We give it here for the sake of completeness.

Step 2. One constructs a family of barrier functions  $w_R \geq 0$  defined on the open cubes  $\Pi_R$  of  $\mathbb{R}^N$  with side  $2R > 0$ , centered at the origin. The following properties are needed:

- (i)  $w_R(x) \rightarrow +\infty$  as  $x$  approaches the boundary  $\partial\Pi_R$ ;
- (ii) as  $R \rightarrow +\infty$ ,  $w_R$  converges to zero uniformly on compact subsets of  $\mathbb{R}^N$ ;
- (iii) the inequality  $\Omega(w_R) - \Delta w_R \geq 0$  holds in  $\mathcal{D}'(\Pi_R)$ .

An adequate candidate is the sum  $w_R(x) := \sum_{i=1}^N \tilde{w}_R(|x_i|)$ , where  $\tilde{w}_R : [0, R) \mapsto \mathbb{R}^+$  is given by

$$\tilde{w}_R(y) = (H_{w_0})^{-1}(\sqrt{2}y), \quad H_{w_0}(u) = \int_{w_0}^u \frac{dz}{\sqrt{G(z) - G(w_0)}}, \quad G(z) = \int_0^z \Omega(s) ds,$$

with  $w_0 > 0$ . This construction corresponds to the value

$$R = R(w_0) = \int_{w_0}^{+\infty} \frac{dz}{\sqrt{G(z) - G(w_0)}}.$$

This value is finite, thanks to the Keller-Osserman condition (see Remark 4) and the fact that, by the monotonicity of  $\Omega$ ,  $G(z) - G(w_0) \geq \Omega(w_0)(z - w_0)$  for  $z > w_0$ .

One checks easily from the definition of  $\tilde{w}_R$  that  $\tilde{w}_R \in C^1([0, R)) \cap C^2((0, R))$ , and for  $y > 0$ ,  $\tilde{w}_R''(y) = \Omega(w_R(y))$ ; moreover,  $\tilde{w}_R(y) = w_0 + \frac{\Omega(w_0)}{2} y^2 + \bar{o}(y^2)$  as  $y \rightarrow 0$ . Thus the function  $y \mapsto \tilde{w}_R(|y|)$  is in  $C^2((-R, R))$ , and one has pointwise on  $\Pi_R$ ,

$$\begin{aligned} 0 &= \sum_{i=1}^N \left( \Omega(\tilde{w}_R(|x_i|)) - \Delta \tilde{w}_R(|x_i|) \right) \leq \Omega\left(\sum_{i=1}^N w_R(|x_i|)\right) - \Delta\left(\sum_{i=1}^N \tilde{w}_R(|x_i|)\right) \\ &= \Omega(w_R(x)) - \Delta w_R(x), \end{aligned}$$

because  $\Omega$  is super-additive. Thus (iii) holds. Further,  $\Omega$  being super-additive, we necessarily have  $\limsup_{z \downarrow 0} \Omega(z)/z < +\infty$ , therefore the integral of  $1/\sqrt{G(z)}$  diverges at zero. This implies property (i). Finally, note that  $R = R(w_0) \rightarrow \infty$  if  $w_0 \rightarrow 0$ . For all  $i = 1, \dots, N$ , we have

$$|x_i| = \frac{1}{\sqrt{2}} \int_{w_0}^{\tilde{w}_R(|x_i|)} \frac{dz}{\sqrt{G(z) - G(w_0)}}.$$

Thus the divergence of the integral of  $1/\sqrt{G(z)}$  at zero implies that  $\tilde{w}_R(|x_i|)$  tends to zero uniformly when  $|x_i|$  stays bounded and  $w_0$  converges to zero. The property (ii) follows.

**Step 3.** By Step 1, we have  $-\Delta W^+ \leq 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ . Being non-negative and subharmonic,  $W^+$  is, in particular, locally bounded on  $\mathbb{R}^N$ . Combining Step 1 with the (iii) of Step 2, we get

$$\Omega(W^+) - \Omega(w_R) - \Delta(W^+ - w_R) \leq 0 \quad \text{in } \mathcal{D}'(\Pi_R).$$

Using once more the Kato inequality, from the monotonicity of  $\Omega$  we get  $-\Delta(W^+ - w_R)^+ \leq 0$  in  $\mathcal{D}'(\Pi_R)$ . By (i) of Step 2, for all sufficiently small  $\varepsilon$ ,  $w_R$  is greater than the finite quantity  $\text{ess sup}_{x \in \Pi_R} W^+(x)$  on the boundary of the subdomain  $\Pi_{R-\varepsilon}$ . As  $\varepsilon \rightarrow 0$ , the comparison principle for the laplacian yields the inequality  $W^+ \leq w_R$  in  $\Pi_R$ . As  $R \rightarrow +\infty$ , the property (ii) of Step 2 implies  $W^+ \leq 0$  a.e. on  $\mathbb{R}^N$ . This means that  $u_1 \leq u_2$ , and the proof is complete.  $\square$

**3.2. Comparison in weighted  $L^1$  spaces with exponential growth.** Now we establish quantitative continuous dependence estimates for a subclass of  $\mathcal{D}'$  solutions

of problem (1). Notice that the estimate (7) together with the generalized Kato inequality (3) yield

$$(8) \quad \int_{\mathbb{R}^N} \Omega(W) S'(W) \xi \leq \int_{\mathbb{R}^N} S(W) \Delta \xi + \int_{\mathbb{R}^N} (f_1 - f_2) S'(W) \xi.$$

In order to exploit (8), we will restrict the class of admissible data  $f$  and solutions  $u$  to functions integrable on  $\mathbb{R}^N$  with some well-chosen weight  $\rho$  decaying at infinity.

The approach of this paragraph is based upon a linearization of (8) (a different approach will be presented in Section 5). Roughly speaking, here we choose for  $S$  a function in  $\mathcal{S}_0$  satisfying the relation

$$(9) \quad S'(z) \Omega(z) = \text{const} \times S(z);$$

because we need  $S'$  to be non-decreasing, we have to truncate the solutions of (9) in the way described in (11) below. In order to carry out this program, according to Remark 2(iii) we choose a concave modulus of continuity  $\omega$  of  $\varphi$ , which we further regularize in Lemma 3.2 below; but we don't require the generalized Keller-Osserman condition any more.

Let  $\varphi$  satisfy  $(H_0)$  and  $\omega_0$  be its best modulus of continuity given by (5). Then  $\varphi$  is sublinear, i.e.,  $\limsup_{z \rightarrow +\infty} \omega_0(z)/z$  is finite. We define

$$(10) \quad m_\infty := 1 / \left( \limsup_{z \rightarrow +\infty} \omega_0(z)/z \right) \in (0, +\infty]$$

(the case  $m_\infty = +\infty$  corresponds to the case a strictly sublinear  $\varphi$ ). First, we regularize  $\omega_0$ .

**Lemma 3.2.** *Let  $\varphi$  satisfy  $(H_0)$ . Then  $\varphi$  admits a modulus of continuity  $\omega$  such that  $\Omega = \omega^{-1}$  is a convex  $C^2$  function on  $\mathbb{R}^+$  with  $\Omega'(k) \uparrow m_\infty$  as  $k \rightarrow +\infty$ .*

*Proof.* The proof reproduces the arguments used to justify Remark 3, but we start with the smallest concave modulus of continuity  $\bar{\omega}$  of Remark 2(iii) instead of  $\omega_0$ . Reasoning by contradiction, from the definition of  $\bar{\omega}$  we easily get

$$\limsup_{z \rightarrow +\infty} \bar{\omega}(z)/z = \limsup_{z \rightarrow +\infty} \omega_0(z)/z.$$

Denote  $(\bar{\omega})^{-1}$  by  $\bar{\Omega}$ ; by construction,  $\bar{\Omega}(k-2) \leq \Omega(k) \leq \bar{\Omega}(k)$ . Therefore thanks to the convexity of  $\Omega$  we have in the case  $m_\infty < +\infty$ ,

$$\lim_{k \rightarrow \infty} \Omega'(k) = m_\infty \Leftrightarrow \lim_{k \rightarrow \infty} \Omega(k)/k = m_\infty \Leftrightarrow \lim_{k \rightarrow \infty} \bar{\Omega}(k)/k = \lim_{r \rightarrow \infty} r / \bar{\omega}(r) = m_\infty.$$

In the case  $m_\infty = +\infty$ , we just start by replacing  $\bar{\omega}(\cdot)$  by  $\bar{\omega}(\cdot) + \sqrt{\cdot}$ .  $\square$

We now construct a family of functions in  $\mathcal{S}_0$  to be used in (8). Fix  $k > 0$ . Consider

$$(11) \quad S_k(z) = \begin{cases} \frac{\Omega(k)}{\Omega'(k)} \times \exp\left(\Omega'(k) \int_k^z \frac{ds}{\Omega(s)}\right) & , \quad 0 < k \leq z \\ z - k + \frac{\Omega(k)}{\Omega'(k)} & , \quad z \geq k, \end{cases}$$

where  $\Omega$  is constructed in Lemma 3.2.

Since  $\Omega$  is convex,  $S_k(z)$  tends to zero as  $z \downarrow 0$ . Thus we can extend  $S_k$  to a continuous function on  $\mathbb{R}$  with  $S_k(z) = 0$  for  $z \leq 0$ . In fact,  $S_k$  is differentiable on  $\mathbb{R} \setminus \{0, k\}$ : we have  $S'_k(z) = 1$  for  $z > k$ , and

$$S'_k(z) = \frac{\Omega(k)}{\Omega(z)} \exp\left(\Omega'(k) \int_k^z \frac{ds}{\Omega(s)}\right) = \frac{\Omega'(k)}{\Omega(z)} S_k(z),$$

for  $z \in (0, k)$ . By convexity,  $\Omega(z) \geq \Omega(k) + \Omega'(k)(z - k) \equiv \Omega'(k)S_k(z)$  for  $z \geq k$ . A direct calculation gives that for  $r \in (0, k)$ ,  $S_k''(r)$  has the same sign as  $\Omega'(k) - \Omega'(r)$ ; since  $\Omega$  is convex, so is  $S_k$  on  $(0, k)$ . Thus  $S_k \in \mathcal{S}_0$ , more exactly, it verifies the following properties :

$$(12) \quad \left\{ \begin{array}{l} S_k' \text{ is a bounded Lipschitz continuous function, } S_k(z) = 0 \ \forall z \leq 0, \\ S_k' \text{ is strictly increasing on } [0, k], \ S_k'(z) = 1 \text{ for } z \geq k, \\ \text{and } S_k(z) \text{ verifies } m_k S_k(z) \leq S_k'(z)\Omega(z^+) \ \forall z \in \mathbb{R}, \text{ where } m_k = \Omega'(k+). \end{array} \right.$$

**Theorem 3.3.** *Let  $\varphi$  satisfy  $(H_0)$  and  $m_\infty$  be given by (10). Let  $\Omega$  be the convex  $C^2$  function on  $\mathbb{R}^+$  given by Lemma 3.2. Let  $k > 0$ . Set  $m_k := \Omega'(k)$  (we have  $m_k \uparrow m_\infty$  as  $k \rightarrow +\infty$ ).*

*Let  $u_i, i = 1, 2$ , be  $\mathcal{D}'$  solutions of (1) corresponding to data  $f_i \in L^1_{loc}(\mathbb{R}^N)$ , and  $w_i = \varphi \circ u_i$ .*

(i) *Let  $\rho(x) = e^{-c|x|}$  with  $0 < c < \sqrt{m_k}$ . If  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N, \rho)$  and  $(w_1 - w_2)^+ \in L^1(\mathbb{R}^N, \rho)$ , then  $(u_1 - u_2)^+ \in L^1(\mathbb{R}^N, \rho)$  and*

$$(13) \quad \int S_k((w_1 - w_2)^+) \rho \leq \frac{1}{m_k - c^2} \int (f_1 - f_2) S_k'((w_1 - w_2)^+) \rho,$$

$$(14) \quad \int (u_1 - u_2)^+ S_k'((w_1 - w_2)^+) \rho \leq \frac{m_k}{m_k - c^2} \int (f_1 - f_2) S_k'((w_1 - w_2)^+) \rho.$$

(ii) *Let  $\rho(x) = e^{-\sqrt{m_k}|x|}$ . If  $f_1 \leq f_2$  and  $(w_1 - w_2)^+ \in L^1(\mathbb{R}^N, \rho)$ , then  $u_1 \leq u_2$ .*

**Remark 6.**

- (i) In case  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N)$ , letting  $c \rightarrow 0$  and  $k \rightarrow 0$  in (14) we deduce the classical  $T$ -contraction result in  $L^1(\mathbb{R}^N)$  of Bénilan et al. [13]. Our version is different because we need the uniform continuity of  $\varphi$ ; on the other hand, we only require that  $u_i, i = 1, 2$ , be  $\mathcal{D}'$  solutions of (1) satisfying  $(\varphi \circ u_1 - \varphi \circ u_2)^+ \in L^1(\mathbb{R}^N)$ , while in [13]  $\varphi$  is independent of  $x$  and higher integrability (in Marcinkiewicz spaces) of each of the functions  $\varphi \circ u_{1,2}$  is assumed.
- (ii) Greater is the parameter  $k$ , larger is the uniqueness class  $L^1(\mathbb{R}^N, e^{-\sqrt{m_k}|x|})$ . If  $\varphi$  is in addition, strictly sublinear, then  $m_k \uparrow +\infty$  as  $k \rightarrow +\infty$ , so that uniqueness in the class  $L^1(\mathbb{R}^N, e^{-c|x|})$  is true for all  $c > 0$ .
- (iii) Uniqueness may fail in  $L^1(\mathbb{R}^N, e^{-c|x|})$  for  $c > \sqrt{m_\infty}$ . An easy example is provided by the linear case  $\varphi = \frac{1}{m} Id$ . Here we have  $m_\infty = m$ , and for  $N = 1$  and  $f = 0$ ,  $u_1(x) \equiv 0$  and  $u_2(x) = e^{\sqrt{m}x}$  are both solutions to (1) integrable with any weight  $\rho(x) = e^{-(\sqrt{m}+\varepsilon)|x|}$ ,  $\varepsilon > 0$ .

*Proof.* of Th. 3.3. While using radial weight and test functions, we will denote by the same letter an even function on  $\mathbb{R}$  and the corresponding radial function.

Let us apply inequality (8) with the function  $S_k$  and with radial test functions  $\xi$ . For a nonnegative even function  $\xi \in \mathcal{D}(\mathbb{R})$  non-increasing on  $\mathbb{R}^+$ , dropping the second (negative) term in  $\Delta \xi(|x|) = \xi''(|x|) + \frac{N-1}{|x|} \xi'(|x|)$ , from (12) we deduce

$$\forall \xi \in \mathcal{D}^+(\mathbb{R}^N)$$

$$(15) \quad \int_{\mathbb{R}^N} m_k S_k(W^+) \xi(|x|) \leq \int_{\mathbb{R}^N} S_k(W^+) \xi''(|x|) + \int_{\mathbb{R}^N} (f_1 - f_2)^+ S_k'(W^+) \xi(|x|).$$

Let  $R \geq 0$  and  $0 < c \leq \sqrt{m_k}$ . The fact that  $S_k'$  is bounded (in particular,  $S_k$  has a linear growth at infinity) and the integrability assumptions on  $W^+ = (w_1 - w_2)^+$

permit to approximate the weight  $\rho(|x|) := e^{-c(|x|-R)^+}$  by functions  $\xi \in \mathcal{D}(\mathbb{R}^N)^+$  and pass to the limit in (15). Indeed, for  $L > 0$ , set

$$\rho_L(|x|) = \begin{cases} \exp(-c|x|), & 0 \leq |x| \leq L \\ c \exp(-cL)(|x| - L - \frac{1}{2c})^2, & L < |x| < L + \frac{1}{2c} \\ 0, & |x| > L + \frac{1}{2c}. \end{cases}$$

(for the sake of simplicity, we only describe the case  $R = 0$ );  $\rho_L$  is  $C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ , and it is concave at the origin. Therefore  $\rho_L$  can be approximated uniformly on  $\mathbb{R}$  by functions  $\xi_{h,L} \in \mathcal{D}((-L - \frac{1}{c}, L + \frac{1}{c}))$  so that  $\lim_{h \rightarrow 0} (\xi_{h,L}'') = \rho_L''$  pointwise on  $\mathbb{R} \setminus \{0\}$ , and

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}^N} S_k(W^+) \xi_{h,L}''(|x|) \leq \int_{\mathbb{R}^N \setminus \{0\}} S_k(W^+) \rho_L''(|x|).$$

Thus (15) still holds with  $\xi = \rho_L$ , where  $\rho_L''$  is taken in the pointwise sense (i.e., on  $\mathbb{R} \setminus \{0\}$ ).

Now we pass to the limit as  $L \rightarrow \infty$ , using the fact that  $\rho_L \rightarrow \rho$  and  $\rho_L'' \rightarrow \rho''$  pointwise on  $(0, +\infty)$ . Moreover, for all  $L$ ,  $\rho_L$  is dominated by  $\rho$  and  $\rho_L''$  is dominated by  $(2c\sqrt{e})\rho$ , as an explicit calculation shows. Therefore  $S_k(W^+) \rho_L(|x|)$ ,  $(f_1 - f_2)^+ S_k'(W^+) \rho_L(|x|)$  and  $S_k(W^+) \rho_L''(|x|)$  are dominated by an  $L^1(\mathbb{R}^N)$  function independent of  $L$ , and we pass to the limit using the Lebesgue dominated convergence theorem.

(i) With  $R = 0$ , we readily deduce (13). Then we come back to (3); with the same approximation as above, we deduce

$$\int_{\mathbb{R}^N} (u_1 - u_2) S_k'(W^+) \rho(|x|) \leq \int_{\mathbb{R}^N} S_k(W^+) c^2 \rho(|x|) + \int_{\mathbb{R}^N} (f_1 - f_2) S_k'(W) \rho(|x|).$$

Now (13) implies (14), because  $w^+ = 0$  as soon as  $u_1 - u_2 < 0$ . The same regularization technique applied to (4) permits to deduce that  $(u_1 - u_2)^+ \in L^1(\mathbb{R}^N, \rho)$  from the fact that  $(w_1 - w_2)^+ \in L^1(\mathbb{R}^N, \rho)$ .

(ii) Passing to the limit in (15) with  $\xi(x)$  approximating  $\rho(x) = e^{-\sqrt{m_k}(|x|-R)^+}$ ,  $R > 0$ , we get

$$\int_{\mathbb{R}^N} m_k S_k(W^+) \rho \leq \int_{\{|x| > R\}} m_k S_k(W^+) \rho.$$

It follows that  $S_k(W^+)$  equals zero a.e. on  $\{|x| < R\}$ ; because  $R$  is arbitrary, we infer  $W^+ = 0$  a.e. on  $\mathbb{R}^N$ . Now the Kato inequality (4) yields  $u_1 \leq u_2$  a.e. on  $\mathbb{R}^N$ .  $\square$

**3.3. Comparison property implies continuous dependence.** A very simple continuous dependence result follows from the comparison principle of Theorem 3.1, under the a priori existence assumption.

**Proposition 2.** *Assume that for all data  $f \in L^1_{loc}(\mathbb{R}^N)$  there exists a  $\mathcal{D}'$  solution to Problem (1). Assume that the conclusion of Theorem 3.1 holds.*

*Then the solutions depend continuously on the data in  $L^1_{loc}$ . More exactly, let  $f \in L^1_{loc}(\mathbb{R}^N)$ ; let  $(f_n)_{n \in \mathbb{N}}$  be a sequence converging to  $f$  in  $L^1_{loc}(\mathbb{R}^N)$ . Denote by  $u$ , resp. by  $u_n$ , the (unique)  $\mathcal{D}'$  solution of (1) associated with the datum  $f$ , resp.  $f_n$ . Then  $(u_n)_{n \in \mathbb{N}}$  converges to  $u$  in  $L^1_{loc}(\mathbb{R}^N)$ .*

In the next section, we will justify the existence assumption in Proposition 2.

*Proof.* Because of the uniqueness of  $u$ , it is sufficient to show that all subsequence of  $(u_n)_n$  possesses itself a subsequence converging to  $u$  in  $L^1_{loc}(\mathbb{R}^N)$ .

In the sequel, we work with an extracted (not relabelled) subsequence such that  $(f_n)_n$  is dominated by a function  $F \in L^1_{loc}(\mathbb{R}^N)$  and converges a.e. on  $\mathbb{R}^N$ . We can define pointwise (a.e. on  $\mathbb{R}^N$ ) the functions  $\underline{m}_n := \inf_{k \geq n} f_k$  and  $\overline{m}_n := \sup_{k \geq n} f_k$ ; these functions are a.e. finite and belong to  $L^1_{loc}(\mathbb{R}^N)$  because of the inequality  $-F \leq \underline{m}_n \leq f_n \leq \overline{m}_n \leq F$  a.e. on  $\mathbb{R}^N$ . By the assumption, there exist the associated  $\mathcal{D}'$  solutions with the data  $\underline{m}_n$ , resp.  $\overline{m}_n$ , which we denote by  $\underline{u}_n$ , resp.  $\overline{u}_n$ . Also denote by  $\underline{U}$ , resp. by  $\overline{U}$ , the  $\mathcal{D}'$  solutions associated with the data  $-F$ , resp.  $F$ .

By construction,  $(\underline{m}_n)_n$  (resp.,  $(\overline{m}_n)_n$ ) is a non-decreasing sequence (resp., a non-increasing sequence); both sequences converge to  $f$  in  $L^1_{loc}(\mathbb{R}^N)$ , as  $n \rightarrow \infty$ .

By the comparison principle,  $(\underline{u}_n)_n$  (resp.,  $(\overline{u}_n)_n$ ) is also non-decreasing (resp., non-increasing). Similarly, both sequences  $(\underline{u}_n)_n$  and  $(\overline{u}_n)_n$  are lower bounded by  $\underline{U} \in L^1_{loc}(\mathbb{R}^N)$ , and upper bounded by  $\overline{U} \in L^1_{loc}(\mathbb{R}^N)$ . Therefore  $(\underline{u}_n)_n$  (resp.,  $(\overline{u}_n)_n$ ) converges in  $L^1_{loc}(\mathbb{R}^N)$  and a.e. on  $\mathbb{R}^N$  to some limit  $\underline{u}$  (resp., to some limit  $\overline{u}$ ), as  $n \rightarrow \infty$ . Also  $\underline{w}_n =: \varphi \circ \underline{u}_n$  and  $\overline{w}_n =: \varphi \circ \overline{u}_n$  are both bounded from above (resp., bounded from below) by the  $L^1_{loc}(\mathbb{R}^N)$  function  $\overline{W} =: \varphi \circ \overline{U}$  (resp., by  $\underline{W} =: \varphi \circ \underline{U}$ ). The notion of  $\mathcal{D}'$  solution being stable with respect to the convergence of data and solutions in the sense obtained above, we deduce that both  $\underline{u}, \overline{u}$  are  $\mathcal{D}'$  solutions to Problem (1) with the same datum  $f$ . Therefore both of them coincide with  $u$ . The comparison principle also yields  $\underline{u}_n \leq u_n \leq \overline{u}_n$ ; thus the convergence of  $u_n$  to  $u$  follows.  $\square$

**Remark 7.** The same arguments show that also  $w = \varphi \circ u$  depends continuously on  $f$  in  $L^1_{loc}$ .

**Remark 8.** The  $L^1_{loc}(\mathbb{R}^N)$  continuous dependence result of Proposition 2 is purely qualitative; indeed, it is based upon the qualitative comparison property.

Analogous (conditional, and qualitative) continuous dependence result can be obtained for the  $L^1(\mathbb{R}^N, e^{-c|x|})$  framework of § 3.2, starting from the comparison property of Theorem 3.3(ii). But we also have quantitative continuous dependence estimates (13), (14) for this framework. For instance, estimate (14) states a “quasi-Lipschitz” dependence of  $u$  on  $f$  in  $L^1(\mathbb{R}^N, e^{-c|x|})$ .

For the sake of completeness, let us state the  $L^1(\mathbb{R}^N, e^{-c|x|})$  continuous dependence result.

**Proposition 3.** *Assume  $(H_0)$ . Assume that there exist  $f_*, u_* \in L^1_{loc}(\mathbb{R}^N)$  such that  $u_*$  is a  $\mathcal{D}'$  solution of (1) with the datum  $f_*$ .*

*Let  $c \in [0, \sqrt{m_\infty})$  and  $\rho_c(x) = e^{-c|x|}$ ; let  $k$  be chosen so that  $m_k > c^2$  (see Lemma 3.2).*

*Let  $f_n - f \rightarrow 0$  in  $L^1(\mathbb{R}^N, \rho_c)$  as  $n \rightarrow \infty$ . Assume that there exist  $\mathcal{D}'$  solutions  $u_n, u$  of (1) with the data  $f_n, f$ , respectively, such that  $w_n, w \in w_* + L^1(\mathbb{R}^N, \rho_c)$ , where  $w_n = \varphi \circ u_n, w = \varphi \circ u$ .*

*Then  $u_n - u$  and  $w_n - w$  tend to zero in  $L^1(\mathbb{R}^N, \rho_c)$ ; moreover, the estimates (13), (14) hold. Further, if  $c \neq 0$ , then for all  $\alpha > 0$  there exists a constant  $d(\alpha)$  which only depends on  $\varphi$  and  $c$  such that*

$$(16) \quad \int |u_1 - u_2| \rho_c \leq \alpha + d(\alpha) \int |f_1 - f_2| \rho_c.$$

The corresponding result for  $c = 0$  is the  $L^1(\mathbb{R}^N)$  contraction mentioned in Remark 6(i).

*Proof.* This result is essentially contained in Theorem 3.3(i). We only have to notice that (16) follows from the Kato inequality (4), the estimate (13) and the fact that  $z \leq \alpha + \text{const}(\alpha, k, \varphi)S_k(z)$  for all  $z > 0$ ; the constant  $\text{const}(\alpha, k, \varphi)$  can be computed from the definition (11) of  $S_k(\cdot)$ .  $\square$

#### 4. DEDUCING WELL-POSEDNESS RESULTS FOR (1)

In this section, we complement the previous uniqueness and comparison results with the corresponding existence results.

Our existence results are not the most general possible with respect to the assumptions on  $\varphi$  (cf. e.g. the comments to the existence results of [20, 29]). But they fit precisely the framework in which uniqueness and continuous dependence of solutions take place. Indeed, in the proofs we use extensively the comparison results of the previous sections; then monotone convergence arguments (in the spirit of [2]) allow to build very simple existence proofs on the top of a “basic” existence result.

The first paragraph aims at giving such basic result. We mainly focus on the case of a non-autonomous nonlinearity  $\varphi$ , under the assumption  $(H_{surj})$ . Under the assumption  $(H_{aut})$ , we simply exploit the Bénéilan, Brézis and Crandall existence theory (see [13]).

**4.1. A “basic” existence result and related comparison properties.** We will treat separately the cases  $(H_{aut})$  and  $(H_{surj})$ .

- Under hypothesis  $(H_{aut})$  and for  $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , existence of a  $\mathcal{D}'$  solution  $u$  such that  $\|u\|_{L^1} \leq \|f\|_{L^1}$ ,  $\text{ess inf } f \leq \text{ess inf } u \leq \text{ess sup } u \leq \text{ess sup } f$  is well known (see [13]). These solutions satisfy additional integrability properties in Marcinkiewicz spaces; one consequence is that the so constructed solutions verify the comparison principle. Note that, under assumption  $(H_{aut})$ , for all  $f \in L^1_{loc}(\mathbb{R}^N)$ , the truncated functions

$$(17) \quad f^{m,n} := (f^+ \wedge n)\mathbb{1}_{B_n} - (f^- \wedge m)\mathbb{1}_{B_m}$$

form a bimonotone sequence of compactly supported  $L^\infty$  functions converging to  $f$  pointwise, i.e.,  $f^{n,m} \downarrow_{m \rightarrow +\infty} \uparrow_{n \rightarrow +\infty} f$ . By the comparison principle, the corresponding solutions  $u^{n,m}$  provided by the result of [13] also form a sequence non-increasing in  $m$ , non-decreasing in  $n$ .

- Now we construct similar bimonotone sequence of truncated functions under the hypothesis  $(H_{surj})$  in such a way that the comparison principle of Theorem 3.3(ii) can be used. Let  $u_*$  be a  $\mathcal{D}'$  solution of (1) with datum  $f_*$ , and  $w_* = \varphi \circ u_*$ . For  $\varepsilon \in [0, 1]$ , set

$$(18) \quad \varphi_\varepsilon(x, z) := \varphi(x, z) + \varepsilon(z - u_*(x)).$$

For  $\varepsilon > 0$  and for a.e.  $x \in \mathbb{R}$ ,  $\varphi_\varepsilon(x, \cdot)$  is bijective; we denote by  $\beta_\varepsilon(x, \cdot)$  its inverse function. Recall that  $\beta_0(x, \cdot)$  is the minimal section of the inverse graph to  $\varphi(x, \cdot)$ . Now for  $\varepsilon > 0$  and  $k \in \mathbb{R}$ , set  $u_\varepsilon^k(x) := \beta_\varepsilon(x, w_*(x) + k)$  and  $f_\varepsilon^k := f_* - u_* + u_\varepsilon^k$ . For  $\varepsilon = 0$  and  $k \neq 0$ , we replace  $\beta_\varepsilon$  by  $\beta_0$  in the above definitions; the assumption of surjectivity of  $\varphi(x, \cdot)$  in  $(H_{surj})$  is used to justify the definition of  $u_0^k$ . Finally, setting  $u_0^0 := u_*$ , we have the inclusion  $u_0^0 \in \beta_0(x, w_*(x) + 0)$ .

For  $n, m \in \mathbb{N}$ , set

$$(19) \quad f^{n,m} := f_* + ((f - f_*)^+ \wedge (f_1^n - f_*) \wedge n)\mathbb{1}_{B_n} - ((f - f_*)^- \wedge (f_* - f_1^{-m}) \wedge m)\mathbb{1}_{B_m}.$$



**Proposition 4.** *Assume  $\varphi$  satisfies  $(H_0)$  and  $(H_{surj})$ . With the notation introduced hereabove, the following properties hold.*

- (i) *For all  $\varepsilon \in [0, 1]$ ,  $u_*$  is a  $\mathcal{D}'$  solution of the equation  $u - \Delta\varphi_\varepsilon(x, u) = f_*$ .*
- (ii) *For all  $\varepsilon > 0$ , for all  $k \in \mathbb{R}$ , the function  $u_\varepsilon^k$  is a  $\mathcal{D}'$  solution of  $u - \Delta\varphi_\varepsilon(x, u) = f_\varepsilon^k$ ; moreover,  $w_\varepsilon^k := \varphi_\varepsilon \circ u_\varepsilon^k = w_* + k$ . In particular,  $w_\varepsilon^k - w_* \in L^\infty(\mathbb{R}^N) \subset L^1(\mathbb{R}^N, \rho)$  with weight  $\rho(x) = e^{-c|x|}$ , for any  $c > 0$ .*
- (iii) *For all  $\varepsilon \in [0, 1]$ , the map  $k \mapsto f_\varepsilon^k$  is non-decreasing. Moreover, as  $k \rightarrow \pm\infty$ , we have  $f_\varepsilon^k \rightarrow \pm\infty$  a.e. on  $\mathbb{R}^N$ . In particular, for all  $f \in L^1_{loc}(\mathbb{R}^N)$ , we have  $f^{n,m} \downarrow_{m \rightarrow \infty} \uparrow_{n \rightarrow \infty} f$ .*
- (iv) *For  $k > 0$ , the maps  $\varepsilon \mapsto (f_\varepsilon^k - f_*)$  and  $\varepsilon \mapsto (u_\varepsilon^k - u_*)$  are non-increasing. For  $k < 0$ , the same maps are non-decreasing. In particular, if  $f_1^{-k} \leq f \leq f_1^k$  for some  $k > 0$ , then  $f_\varepsilon^{-k} \leq f \leq f_\varepsilon^k$  for all  $\varepsilon \in [0, 1]$ . Similarly, if  $u_\varepsilon^{-k} \leq u \leq u_\varepsilon^k$  for some  $k > 0$  and  $\varepsilon > 0$ , then  $u_0^{-k} \leq u \leq u_0^k$ .*

All the properties are easily deduced from the construction of  $f_\varepsilon^k$ .

The statement below plays a role analogous to the aforementioned  $L^1 \cap L^\infty$  existence result for (1), when  $(H_{aut})$  is replaced with  $(H_{surj})$ .

**Proposition 5.** *Assume  $\varphi$  satisfies  $(H_0)$  and  $(H_{surj})$ .*

- (i) *Assume that  $f \in L^1_{loc}(\mathbb{R}^N)$  is such that  $f_1^{-k} \leq f \leq f_1^k$  for some  $k > 0$ . Then there exists a  $\mathcal{D}'$  solution  $u$  of problem (1) satisfying the inequalities  $w_* - k \leq \varphi \circ u \leq w_* + k$ .  
Moreover, the comparison principle holds for the so constructed solutions: if  $f, \hat{f}$  are two data satisfying  $f_1^{-k} \leq f \leq \hat{f} \leq f_1^k$ , then the associated solutions  $u, \hat{u}$  satisfy  $u \leq \hat{u}$ .*
- (ii) *Let  $f \in L^1_{loc}(\mathbb{R}^N)$ . Let  $f^{n,m}$  be the truncates of  $f$  defined by (19). Then there exists a sequence  $(u^{n,m})_{n,m}$  of  $\mathcal{D}'$  solutions of (1) associated with the data  $(f^{n,m})_{n,m}$  and such that  $u^{n,m}$  is non-increasing in  $m$ , non-decreasing in  $n$ .*

*Proof.*

(i) Step 1 : First assume that  $(f - f_*) \in L^2(\mathbb{R}^N)$  and consider  $\varphi_\varepsilon$  defined by (18). We construct solutions to the regularized and translated problem (here we use Proposition 4(i)):

$$(20) \quad (u - u_*) - \Delta(\varphi_\varepsilon(x, u) - \varphi_\varepsilon(x, u_*)) = f - f_*.$$

Rewriting the problem in terms of the new unknown  $\tilde{w} := \varphi_\varepsilon(x, u) - \varphi_\varepsilon(x, u_*)$ , we reduce the above problem to

$$(21) \quad \tilde{\beta}_\varepsilon(x, w) - \Delta w = \tilde{f}.$$

with  $\tilde{f} \in L^2(\mathbb{R})$  and  $\tilde{\beta}_\varepsilon : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  a Carathéodory function such that  $\tilde{\beta}_\varepsilon(x, 0) = 0$  and  $|\tilde{\beta}_\varepsilon(x, k)| \leq \frac{1}{\varepsilon}|k|$  for a.e.  $x \in \mathbb{R}^N$ , for all  $k \in \mathbb{R}$ . Therefore a variational solution  $\tilde{w}_\varepsilon \in H^1_0(\mathbb{R}^N)$  to (21) can be constructed through minimization of the convex coercive functional

$$J_\varepsilon : \tilde{w} \in H^1(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla \tilde{w}| + \int_0^{\tilde{w}} \tilde{\beta}_\varepsilon(x, s) ds \right);$$

this solution is, in particular, a solution in the sense of distributions. Then  $u_\varepsilon = u_* + \tilde{\beta}_\varepsilon \circ \tilde{w}_\varepsilon$  is a  $\mathcal{D}'$  solution of (20).

Step 2 : Notice that by construction,  $\tilde{w}_\varepsilon \in L^2(\mathbb{R}^N)$ ; therefore for all  $c > 0$ ,  $\tilde{w}_\varepsilon \in L^1(\mathbb{R}^N, e^{-c|x|})$ . Thus  $u_\varepsilon$  is a  $\mathcal{D}'$  solution of  $u - \Delta\varphi_\varepsilon(x, u) = f$  which satisfies



$\varphi_\varepsilon \circ u_\varepsilon \in w_* + L^1(\mathbb{R}^N, e^{-c|x|})$ . By Proposition 4(ii),  $u_\varepsilon^{\pm k}$  are  $\mathcal{D}'$  solutions of the equation  $u - \Delta\varphi_\varepsilon(x, u) = f_\varepsilon^{\pm k}$  such that

$$w_\varepsilon^{\pm k} \equiv \varphi_\varepsilon \circ u_\varepsilon^{\pm k} \equiv w_* \pm k \in w_* + L^\infty(\mathbb{R}^N) \subset w_* + L^1(\mathbb{R}^N, e^{-c|x|}).$$

Notice that by the assumption and by Proposition 4(iv),  $f_\varepsilon^{-k} \leq f \leq f_\varepsilon^k$ . Because  $\varphi_\varepsilon$  is uniformly continuous in the second variable and  $w_\varepsilon - w_\varepsilon^{\pm k} \in L^1(\mathbb{R}^N, e^{-c|x|})$ , by the comparison principle of Theorem 3.3(ii) we deduce that  $u_\varepsilon^{-k} \leq u_\varepsilon \leq u_\varepsilon^k$ . By the monotonicity of  $\varphi_\varepsilon(x, \cdot)$ , we deduce the uniform bound  $w_* - k \leq w_\varepsilon \leq w_* + k$  on  $w_\varepsilon \equiv \varphi_\varepsilon \circ u_\varepsilon$ . Using Proposition 4(iv), we deduce in addition the uniform bound  $u_0^{-k} \leq u_\varepsilon \leq u_0^k$  on  $u_\varepsilon$ .

Step 3 : From the uniform bounds of Step 2, it follows that the sets  $(u_\varepsilon)_\varepsilon$  and  $(w_\varepsilon)_\varepsilon$  are relatively compact in  $L^1_{loc}(\mathbb{R}^N)$  weakly and in  $w_* + L^\infty(\mathbb{R}^N)$  weakly-\*. We can extract a (not labelled) sequence such that  $u_\varepsilon$  converges in  $\mathcal{D}'$  to  $u \in L^1_{loc}(\mathbb{R}^N)$ , and  $w_\varepsilon$  converges in  $\mathcal{D}'$  to  $w \in w_* + L^\infty(\mathbb{R}^N)$ . Because the sequences are linked by the relation  $w_\varepsilon = \varphi(x, u_\varepsilon) + \varepsilon(u_\varepsilon - u_*)$  with a monotone function  $\varphi(x, \cdot)$ , by the Minty argument we deduce that  $w = \varphi(x, u)$ . Thus we have constructed a  $\mathcal{D}'$  solution  $u$  of problem (1) for  $f \in f_* + L^2(\mathbb{R}^N)$  such that  $f_1^{-k} \leq f \leq f_1^k$  for some  $k > 0$ . In addition, this solution satisfies

$$(22) \quad u_0^{-k} \leq u \leq u_0^k \quad \text{and} \quad w_* - k \leq \varphi \circ u \leq w_* + k.$$

Step 4 : Now we can drop the assumption  $f \in f_* + L^2(\mathbb{R}^N)$ . Indeed, replace  $f$  with the truncates  $f^{n,m}$  defined in (19); by construction,  $f^{n,m} \in f_* + L^2(\mathbb{R}^N)$ . By the previous steps, there exist corresponding solutions  $u^{n,m}$  satisfying (22). Using Proposition 4(iii), the uniform bounds (22) and repeating the compactness arguments of Step 3, we deduce the existence of a  $\mathcal{D}'$  solution  $u$  with datum  $f$ . Moreover,  $u$  satisfies (22); in particular, (22) allows to use the comparison principle of Theorem 3.3(ii).

This ends the proof of (i).

(ii) Taking  $k = \max\{m, n\} + 1$ , we have

$$f_1^{-k} \leq f^{n,m+1} \leq f^{n,m} \leq f^{n+1,m} \leq f_1^k.$$

Thus the claim of (ii) is an immediate consequence of the result of (i).  $\square$

**Remark 9.** Notice that another existence proof was given by Gallouët and Morel in [29], for all  $L^1(\mathbb{R}^N)$  data  $f$ . In the place of (22), an equi-integrability estimate was deduced from the Kato inequality (3). The uniform continuity of  $\varphi$  was not used in [29], and more integrability (in Marcinkiewicz spaces) was shown for the so constructed solutions, as in [13].

In conclusion, in this paragraph we have derived existence under the assumption  $f_1^{-k} \leq f \leq f_1^k$  made in Proposition 5(i). Proposition 5(ii) is a step towards dropping this restriction; but we need bounds replacing (22) in order to ensure that the bimonotone sequence  $(u^{n,m})_{n,m}$  converges to some a.e. finite limit. Such *ad hoc* bounds are obtained in the three paragraphs below, under the additional assumptions on the nonlinearity  $\varphi$  and/or on the datum  $f$ .

#### 4.2. Well-posedness in $L^1(\mathbb{R}^N, e^{-c|x|})$ .

**Theorem 4.1.** *Assume that  $\varphi$  satisfies either  $(H_{aut})$  or  $(H_{surj})$ . Assume that  $(H_0)$  holds; let  $\rho_c(x) = e^{-c|x|}$  with  $c^2 \in [0, m_\infty)$  (see (10) for the definition of  $m_\infty$ ). Then  $L^1(\mathbb{R}^N, \rho_c)$  is a well-posedness class for problem (1).*

More exactly, for all  $f \in f_* + L^1(\mathbb{R}^N, \rho_c)$  there exists a unique  $\mathcal{D}'$  solution  $u_f$  of (1) such that  $w_f = \varphi \circ u_f \in w_* + L^1(\mathbb{R}^N, \rho_c)$ ; we also have  $u_f \in u_* + L^1(\mathbb{R}^N, \rho_c)$ ; both  $u_f - u_*$  and  $w_f - w_*$  depend continuously in  $L^1(\mathbb{R}^N, \rho_c)$  on  $f - f_* \in L^1(\mathbb{R}^N, \rho_c)$ ; and the map  $f \mapsto u_f$  is order-preserving on  $f_* + L^1(\mathbb{R}^N, \rho_c)$ .

Notice that under the assumption  $(H_0)$ , we have  $|\varphi(x, y+z) - \varphi(x, y)| \leq \text{const}(1+|z|)$ . Therefore the implication

$$u - u_* \in L^1(\mathbb{R}^N, e^{-c|x|}) \implies w - w_* \in L^1(\mathbb{R}^N, e^{-c|x|})$$

holds for  $c > 0$ . Thus for  $c > 0$  and under any of the assumptions  $(H_{aut}), (H'_{surj})$ , Theorem 4.1 means in particular existence, uniqueness and continuous dependence of a solution  $u_f \in L^1(\mathbb{R}^N, e^{-c|x|})$  for all datum  $f \in L^1(\mathbb{R}^N, e^{-c|x|})$ .

By Remark 6(ii), as soon as Theorem 4.1 is justified we readily deduce

**Corollary 1.** *Let  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  be a nondecreasing uniformly continuous on  $\mathbb{R}$  function. Assume that  $\varphi$  is strictly sublinear in the sense that*

$$\lim_{z \rightarrow +\infty} \frac{1}{z} \sup_{y \in \mathbb{R}} |\varphi(y+z) - \varphi(y)| = 0.$$

*Then the problem  $u - \Delta \varphi(u) = f$  is well posed, in the sense of Theorem 4.1, in the space  $L^1(\mathbb{R}^N, \rho_c)$  with  $\rho_c = e^{-c|x|}$ , for all  $c \in [0, +\infty)$ .*

*Proof.* of Th. 4.1. In view of Theorem 3.3 and Proposition 3, it only remains to prove existence.

Under assumption  $(H_{surj})$ , we use Proposition 5(ii). It yields a bimonotone sequence of  $\mathcal{D}'$  solutions  $u^{n,m}$  corresponding to the truncated data  $f^{n,m}$ ; recall that  $f^{n,m}$  converge to  $f$  a.e., by Proposition 4(iii). The case of the assumption  $(H_{aut})$  is similar, with  $f^{n,m}$  given by (17).

By the monotone or by the dominated convergence theorems,  $f^{n,m}$  converge to  $f$  in  $L^1(\mathbb{R}^N, \rho_c)$  as  $m \rightarrow \infty, n \rightarrow \infty$ . Thus  $\int_{\mathbb{R}^N} |f^{n,m} - f_*| \rho_c$  is bounded by a constant independent of  $n, m$ . From the estimates of Proposition 3 applied to solutions  $u^{n,m}$  and  $u_*$ , it follows that the quantities

$$(23) \quad \int_{\mathbb{R}^N} |w^{n,m} - w_*| \rho_c \quad \text{and} \quad \int_{\mathbb{R}^N} |u^{n,m} - u_*| \rho_c$$

are bounded by a constant independent of  $n, m$ . Thanks to the monotone convergence theorem, we conclude that there exists  $u \in u_* + L^1(\mathbb{R}^N, \rho_c)$  such that  $u^{n,m} \downarrow_{m \rightarrow +\infty} \uparrow_{n \rightarrow +\infty} u$ ; moreover,  $w^{n,m}$  converge to  $w = \varphi \circ u$  in the same sense, and  $w \in w_* + L^1(\mathbb{R}^N, \rho_c)$ . It follows that  $u$  is the required solution.  $\square$

**Remark 10.** Let us briefly discuss the finer case where  $m_k \equiv m_\infty$  for  $k$  large enough; one example is the case of a linear autonomous  $\varphi$  (see Remark 6(iii)). Basing ourselves on the comparison result of Theorem 3.3(ii), we can get a conditional continuous dependence result (similar to the one of Proposition 2) for data in the critical space  $L^1(\mathbb{R}^N, e^{-\sqrt{m_\infty}|x|})$ . But we do not know whether the existence of a solution holds for these data.

**4.3. Well-posedness in  $L^\infty(\mathbb{R}^N)$ .** For the sake of simplicity, let us restrict our attention to the autonomous case  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ . Corollary 1 readily yields the following result (which can also be deduced from the results of § 4.4 or of § 5.2 below).

**Theorem 4.2.** *Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous non-decreasing function. Then  $L^\infty(\mathbb{R}^N)$  is a well-posedness class for problem (1).*

*More exactly, for all  $f \in L^\infty(\mathbb{R}^N)$  there exists a unique  $\mathcal{D}'$  solution  $u_f$  to problem (1) within the class  $L^\infty(\mathbb{R}^N)$ ; the map  $f \mapsto u_f$  is order-preserving (in particular, the maximum principle holds, i.e.,  $\text{ess inf } f \leq u_f \leq \text{ess sup } f$ ); and, in case  $(f_n)_{n \in \mathbb{N}}$  is a uniformly bounded sequence of functions that converges to  $f$  a.e. on  $\mathbb{R}^N$ , the corresponding solutions  $u_{f_n}$  converge to  $u_f$  a.e. on  $\mathbb{R}^N$ .*

*Proof.* It suffices to apply e.g. Theorem 4.1 to the problem (1) with  $\varphi$  replaced by the truncated function

$$\varphi_M : z \mapsto (\varphi(z) \vee (\varphi(-M))) \wedge \varphi(M),$$

with  $M = \|f\|_{L^\infty}$ . The function  $\varphi_M$  is uniformly continuous, and the constants  $M^- := \text{ess inf } f$  and  $M^+ := \text{ess sup } f$  are, obviously, constant  $\mathcal{D}'$  solutions of problem  $u - \Delta \varphi_M(u) = f$  with the right-hand sides  $M^-$  and  $M^+$ , respectively. Thus the existence of a unique associated solution  $u_f^M$  (together with the comparison principle) follows by Corollary 1. Due to the comparison principle, we have  $M^- \leq u_f^M \leq M^+$ , so that  $\varphi(u_f^M) \equiv \varphi_M(u_f^M)$ , thus  $u_f^M$  also solves the original problem (1).

For the uniqueness and comparison result, it suffices to consider two solutions  $u_1, u_2 \in L^\infty(\mathbb{R}^N)$  with data  $f_1, f_2 \in L^\infty(\mathbb{R}^N)$ ,  $f_1 \leq f_2$ . Set

$$M' = \max\{\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty}, \|u_1\|_{L^\infty}, \|u_2\|_{L^\infty}\}$$

and truncate  $\varphi$  at the level  $M'$ . By Corollary 1, it follows that  $u_1 = u_{f_1}^{M'} \leq u_{f_2}^{M'} = u_2$ .

Finally, the continuous dependence claim of the theorem follows from the existence and comparison properties, in the same way as in Proposition 2.  $\square$

**Remark 11.** Let us indicate a framework which replaces  $L^\infty$  in the case of a non-autonomous nonlinearity  $\varphi$ . For the sake of simplicity, assume that  $\varphi$  satisfies  $(H'_{surj})$  and  $(H_0)$  (in the place of the uniform continuity restriction in  $(H_0)$ , it is sufficient to assume that for all  $k > 0$ , the truncated function  $(x, z) \mapsto (\varphi(x, z) \vee (-k)) \wedge k$  is uniformly continuous in the second variable).

For  $k \in \mathbb{R}$ , set  $u_0^k = \beta_0(x, k)$ . Introduce the set  $L_\varphi^\infty(\mathbb{R}^N)$  as the set of all measurable functions  $f$  on  $\mathbb{R}^N$  such that

$$\exists k \in \mathbb{R} \quad u_0^{-k} \leq f \leq u_0^k.$$

Then  $L_\varphi^\infty$  is a well-posedness class for problem (1), in the same sense as in Theorem 4.2. The corresponding maximum principle reads :

$$u_0^l \leq f \leq u_0^k \quad \implies \quad u_0^l \leq u_f \leq u_0^k.$$

The proof of the remark follows the lines of the proof of Theorem 4.2, with  $M^-, M^+$  replaced by  $u_0^l, u_0^k$  and with truncations  $\varphi_M$  replaced by truncations of the form  $(\varphi \vee (-k)) \wedge k$ .

**4.4. Well-posedness in  $L^1_{loc}(\mathbb{R}^N)$ .** Following [20, 30] let us study the existence of solutions of (1) without growth restrictions at infinity, under the generalized Keller-Osserman condition on the nonlinearity  $\varphi$ . The argument, based on the ‘‘local estimate’’ of [11, 30], is somewhat simplified thanks to the use of the comparison property, as e.g. in [31], and of the bi-monotone data approximations of Ammar and Wittbold [2].

Notice that, unlike in [20, 29, 30] we have chosen to write (1) in terms of the unknown  $u$  rather than in terms of  $w = \varphi \circ u$ . The two frameworks are not exactly equivalent, unless for a.e.  $x \in \mathbb{R}^N$  the function  $\varphi(x, \cdot)$  is strictly increasing.

**Theorem 4.3.** *Assume that  $\varphi$  satisfies  $(H_0)$  and, moreover, the generalized Keller-Osserman condition  $(H_{KO})$  holds. Assume that either  $(H_{aut})$  or  $(H_{surj})$  holds. Then problem (1) is well-posed in  $L^1_{loc}(\mathbb{R}^N)$ .*

*More exactly, for all datum  $f \in L^1_{loc}(\mathbb{R}^N)$  there exists a unique  $\mathcal{D}'$  solution  $u_f$  of (1), and the mappings  $f \in L^1_{loc}(\mathbb{R}^N) \mapsto u_f \in L^1_{loc}(\mathbb{R}^N)$ ,  $f \in L^1_{loc}(\mathbb{R}^N) \mapsto w_f := \varphi \circ u_f \in L^1_{loc}(\mathbb{R}^N)$  are continuous and order-preserving.*

*Proof.* Thanks to Theorem 3.1 and Proposition 2, it is sufficient to prove the existence of a  $\mathcal{D}'$  solution for all datum  $f$  in  $L^1_{loc}(\mathbb{R}^N)$ . Making the change  $f \rightsquigarrow f - f_*$ ,  $u \rightsquigarrow u - u_*$ ,  $\varphi(\cdot, z) \rightsquigarrow \varphi(\cdot, z + u_*(\cdot)) - \varphi(\cdot, u_*(\cdot))$ , we can assume that  $\varphi(x, 0) = 0$ .

Step 1. Assume  $u$  is a  $\mathcal{D}'$  solution of (1) with some datum  $f$ , and  $w = \varphi \circ u$ .

By assumption, there exists a bijective modulus of continuity  $\omega$  of  $\varphi$  such that  $\Omega = \omega^{-1}$  satisfies  $(H_{KO})$ . Being super-additive, positive and non-decreasing,  $\Omega$  satisfies the inequality

$$(24) \quad \forall r \geq t > 0, \quad \frac{\Omega(r)}{r} \geq \frac{\Omega([r/t]t)}{([r/t] + 1)t} \geq \frac{[r/t]t}{([r/t] + 1)t} \frac{\Omega([r/t]t)}{[r/t]t} \geq \frac{1}{2} \frac{\Omega(t)}{t},$$

where  $[r/t]$  stands for the smallest  $n \in \mathbb{N}$  such that  $r/t \geq n$ .

Inequality (24) successfully replaces the monotonicity of  $\Omega(t)/t$  in the proof of [30, Lemma 3]. Slightly modifying the proof of Lemma 2 in the same reference, we get for all  $R > 0$ ,  $R' > R$  the estimate

$$\int_{B_R} \Omega(Z) \leq C(R, R') \left( 1 + \int_{B_{R'}} |f| \right)$$

valid for all nonnegative function  $Z$  satisfying

$$(25) \quad \Omega(Z) - \Delta Z \leq |f| \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Because  $\varphi(x, 0) = 0$ , as in the proof of Theorem 3.1, Step 1, we deduce that (25) holds with  $Z = w^\pm$ . By the convexity of  $\Omega$ ,  $Z \leq \text{const}(1 + \Omega(Z))$ . It follows that  $\int_{B_R} |w|$  is estimated by a constant that only depends on  $R$ , on  $R' > R$  and on  $\int_{B_{R'}} |f|$ .

Step 2. Now fix  $f \in L^1_{loc}(\mathbb{R}^N)$ . Assuming  $(H_{surj})$ , we define  $f^{n,m}$  by (19) and use Proposition 5. Let  $(u^{m,n})_{m,n}$  be the corresponding bimonotone sequence of solutions. Under the assumption  $(H_{aut})$ , following [2] we define  $f^{n,m}$  by (17), and use the results of [13] (see § 4.1).

By construction of  $f^{m,n}$ , we have the uniform bound  $\int_K |f^{m,n}| \leq \text{const}(K)$  for all  $K \in \mathbb{R}^N$ . By Step 1, the functions  $w^{m,n} = \varphi \circ u^{m,n}$  are locally equi-integrable.

Fix  $m$  and let  $n \rightarrow \infty$ . By the comparison principle of Theorem 3.1, both  $(u^{m,n})_n$  and  $(w^{m,n})_n$  are non-decreasing sequences. Thus  $w^{m,n}$  converges in  $L^1_{loc}(\mathbb{R}^N)$  to some function  $w^m$  as  $n \rightarrow \infty$ ; furthermore,  $u^{m,n}$  converges to some measurable function  $u^m$  taking values in  $(-\infty, +\infty]$ , and  $w^m = \varphi \circ u^m$ . Passing to the limit in the integral identities satisfied by  $u^{m,n}$ , by the monotone convergence theorem we infer that for all  $\xi \in \mathcal{D}(\mathbb{R}^N)$ , there exists the limit

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} u^{m,n} \xi = \int_{\mathbb{R}^N} f^m \xi + \int_{\mathbb{R}^N} w^m \Delta \xi < +\infty,$$

where  $f^m = \lim_{n \rightarrow +\infty} f^{n,m}$ . By the monotone convergence theorem,  $u^{m,n}\xi$  converges to  $u^m\xi$  in  $L^1(\mathbb{R}^N)$  for an arbitrary nonnegative  $\xi \in \mathcal{D}(\mathbb{R}^N)$ . We infer that  $u^m \in L^1_{loc}(\mathbb{R}^N)$ . Hence  $u^m$  is a  $\mathcal{D}'$  solution of (1) with the datum  $f^m$ .

We then pass to the limit as  $m \rightarrow +\infty$  in the same way. This ends the existence proof.  $\square$

## 5. A CONTRACTION THEORY IN WEIGHTED $L^1$ SPACES FOR (1)

The results of this section bear much resemblance to those of Bénylan and Crandall in [14, Section 2], although we use quite different techniques. We are mainly concerned with the case of dimension  $N \geq 3$  (analogous results for  $N = 2$  are indicated). The Kato inequalities are exploited in a way different from the previous sections: to be short, we just pick weights satisfying  $-\Delta\rho \geq 0$ . Instead of working directly with arbitrary locally integrable super-harmonic weights (see Corollary 2), we first use the simplest weights which are the truncated fundamental solutions of the laplacian operator. Note that in their context, Bénylan and Crandall [14] use the superharmonic weights  $x \mapsto (1 + |x|^2)^{-\alpha}$  with  $\alpha \leq \frac{N-2}{2}$ , inspired as well by the fundamental solution of the laplacian. Let us mention in passing that a different use of the fundamental solution was made by Pierre in [41], in the context of very weak solutions of the associated evolution problem. The choice made in the present paper is instrumental in the study [7] of the convection-diffusion problem  $u_t + \operatorname{div}f(u) - \Delta\varphi(u) = 0$ ; note that it further extends to fractional convection-diffusion integrodifferential equations ([5]).

In § 5.1 we show the  $L^1(\mathbb{R}^N, \rho)$  contraction result for “ $L^\infty$  solutions”. This turns out to be another well-posedness class (see § 5.2). Recalling the main focus of our work, note that this result does not allow to claim uniqueness of merely  $L^\infty(\mathbb{R}^N)$  solutions (claim already proved in Theorem 4.2). However this claim can be eventually deduced from the estimates of § 5.1 following the method of [7] outlined in Remark 13.

Finally, notice that, extending by density the solution operator, one can construct a pure  $L^1(\mathbb{R}^N, \rho)$  abstract theory for the problem (1), which can be seen as the resolvent problem for the associated abstract evolution problem (cf. [14]; see Section 6 for further comments in this direction).

**5.1. The contraction and comparison property.** First, note the following elementary observation.

**Lemma 5.1.** *Let  $\Omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a convex function such that  $\Omega(r) = 0$  iff  $r = 0$ . Let  $\Omega^* : z \in \mathbb{R}^+ \rightarrow \sup_{r \in \mathbb{R}^+} (rz - \Omega(r))$  be its Legendre (Fenchel) transform. Then  $\operatorname{ess\,lim}_{z \downarrow 0} (\Omega^*)'(z) = 0$ .*

*Proof.* It is well known that  $\Omega^*$  is convex (and thus, its derivative is a.e. defined on  $\mathbb{R}^+$ ), and  $\Omega$  is, in turn, the Legendre (Fenchel) transform of  $\Omega^*$ . In addition,  $\Omega^*(0) = 0$ . We argue by contradiction. If  $\operatorname{ess\,lim}_{z \downarrow 0} (\Omega^*)'(z)$  is not zero, then  $(\Omega^*)'$  is lower bounded by some  $c > 0$ . Then  $\Omega^*(z) \geq cz$  for all  $z$ . Then  $\Omega(r) = \sup_{z \in \mathbb{R}^+} (rz - \Omega^*(z)) \leq \sup_{z \in \mathbb{R}^+} (rz - cz)$ , which is zero for all  $r \leq c$ . This contradicts the assumption  $\Omega(r) > 0$  for  $r > 0$ .  $\square$

**Theorem 5.2.** *Assume  $N \geq 3$ . Assume  $(H_0)$ . Let  $u_i$  be  $\mathcal{D}'$  solutions of (1) corresponding to data  $f_i \in L^1_{loc}(\mathbb{R}^N)$ ,  $i = 1, 2$ . Assume that  $W^+ = (w_1 - w_2)^+ \in L^\infty(\mathbb{R}^N)$ , where  $w_i = \varphi \circ u_i$ .*

Let  $R > 0$ ,  $\rho(x) = (|x| \vee R)^{2-N}$ . Assume  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N, \rho)$ . Then  $(u_1 - u_2)^+ \in L^1(\mathbb{R}^N, \rho)$  and

$$(26) \quad \int_{\mathbb{R}^N} (u_1 - u_2)^+ \rho \leq \int_{\mathbb{R}^N} (f_1 - f_2) \text{sign}^+(u_1 - u_2) \rho.$$

In particular, if  $f_1 \leq f_2$ , then  $u_1 \leq u_2$ .

**Remark 12.** With the same arguments applied to the case  $N = 2$ , we deduce (26) with  $\rho \equiv 1$ . We thus get, in particular, the  $L^1(\mathbb{R}^2)$  contraction result for  $L^\infty$  solutions of  $u - \Delta\varphi(u) = f$ , result which is already known. Indeed, it is contained in Theorem 3.3 (see Remark 6(i)), because the hypothesis  $u_1, u_2 \in L^\infty(\mathbb{R}^N)$  permits to use a local modulus of continuity in Theorem 3.3. Also note that this case is essentially – up to an adaptation of the Kato inequalities from weak to very weak solutions of the stationary equation, see Proposition 1 – contained the work of Maliki and Touré in [37] (see also Ouédraogo [40] and references therein, where anisotropic diffusion case is analyzed). Also the case  $N = 1$ , with the constant weight  $\rho \equiv 1$ , is covered by the result of [37].

*Proof.* We are intended to use (4) with the test functions  $\rho_\varepsilon(x) = (|x| \vee R)^{2-N-\varepsilon}$ ,  $\varepsilon > 0$ , and let  $\varepsilon$  decrease to zero.

To this end, let  $\rho_{\varepsilon,L}(x) = \left( (|x| \vee R)^{2-N-\varepsilon} - (L)^{2-N-\varepsilon} \right)^+$ ,  $L > R$ . Regularizing  $\rho_{\varepsilon,L}$  by convolution, dropping the negative measure part of the distribution  $\Delta\rho_{\varepsilon,L}$  concentrated of  $\{x \in \mathbb{R}^N \mid |x| = R\}$ , from (8) we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} (u_1 - u_2)^+ \rho_{\varepsilon,L} &\leq \int_{\mathbb{R}^N} W^+ \Delta\rho_{\varepsilon,L} + \int_{\{|x|=L\}} W^+ (N-2+\varepsilon) L^{1-N-\varepsilon} \\ &\quad + \int_{\mathbb{R}^N} (f_1 - f_2) \text{sign}^+(u_1 - u_2) \rho_{\varepsilon,L} \end{aligned}$$

for a.e.  $L > R'$ , where  $\Delta\rho_{\varepsilon,L}$  is taken in the pointwise (a.e. on  $\mathbb{R}^N$ ) sense. Because  $W^+$  is bounded, the integral on the sphere  $\{|x| = L\}$  is upper bounded by  $\text{const } L^{N-1} L^{1-N-\varepsilon} = \text{const } L^{-\varepsilon}$ . We calculate  $\Delta\rho_{\varepsilon,L} = \varepsilon(N-2+\varepsilon)|x|^{-N-\varepsilon} \mathbb{1}_{R < |x| < L}$  and notice that  $|x|^{-N-\varepsilon} \mathbb{1}_{|x| > R} = |x|^{-2} \rho_\varepsilon \mathbb{1}_{|x| > R}$  is integrable. Letting  $L \rightarrow \infty$ , thanks to the boundedness assumption on  $W^+$  and the integrability assumption on  $(f_1 - f_2)^+$  we deduce that  $(u_1 - u_2)^+ \in L^1(\mathbb{R}^N, \rho_\varepsilon)$  and

$$(27) \quad \begin{aligned} \int_{\mathbb{R}^N} (u_1 - u_2)^+ \rho_\varepsilon &\leq \varepsilon(N-2+\varepsilon) \int_{\{|x| > R\}} W^+ |x|^{-2} \rho_\varepsilon \\ &\quad + \int_{\mathbb{R}^N} (f_1 - f_2) \text{sign}^+(u_1 - u_2) \rho_\varepsilon. \end{aligned}$$

Now fix  $\delta > 0$ . By assumption and Remark 2(viii),  $\varphi$  admits a concave modulus of continuity  $\omega$ . Using (7) and setting  $\Omega = \omega^{-1}$ , from (27) we get

$$(28) \quad \begin{aligned} \int_{\mathbb{R}^N} \left( (1-\delta)(u_1 - u_2)^+ + \delta\Omega(W^+) \right) \rho_\varepsilon &\leq c\varepsilon \int_{\{|x| > R\}} W^+ |x|^{-2} \rho_\varepsilon \\ &\quad + \int_{\mathbb{R}^N} (f_1 - f_2) \text{sign}^+(u_1 - u_2) \rho_\varepsilon. \end{aligned}$$

Let us show that

$$(29) \quad c\varepsilon \int_{\{|x| > R\}} W^+ |x|^{-2} \rho_\varepsilon \leq \delta \int_{\mathbb{R}^N} \Omega(W^+) \rho_\varepsilon + r^\delta(\varepsilon)$$

with  $r^\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon$  decreases to zero. Indeed, let  $\Omega^*$  be the Legendre (Fenchel) transform of  $\Omega$ . By definition, we have

$$c\varepsilon W^+ |x|^{-2} = \delta \left( W^+ \frac{c\varepsilon}{\delta} |x|^{-2} \right) \leq \delta \Omega(W^+) + \delta \Omega^* \left( \frac{c\varepsilon}{\delta} |x|^{-2} \right);$$

$$c\varepsilon \int_{\{|x|>R\}} W^+ |x|^{-2} \rho_\varepsilon \leq \delta \int_{\mathbb{R}^N} \Omega(W^+) \rho_\varepsilon + \delta \int_{\{|x|>R\}} \Omega^* \left( \frac{c\varepsilon}{\delta} |x|^{-2} \right) \rho_\varepsilon.$$

By the convexity of  $\Omega^*$ , we have  $\Omega^* \left( \frac{c\varepsilon}{\delta} |x|^{-2} \right) \leq (\Omega^*)' \left( \frac{c\varepsilon}{\delta} R^{-2} \right) \frac{c\varepsilon}{\delta} |x|^{-2}$  whenever  $|x| \geq R$ . We can use Lemma 5.1 to deduce that  $(\Omega^*)' \left( \frac{c\varepsilon}{\delta} R^{-2} \right) =: \bar{o}_\varepsilon$  tends to zero as  $\varepsilon \rightarrow 0$ . Therefore

$$(30) \quad 0 \leq r^\delta(\varepsilon) := \delta \int_{\{|x|>R\}} \Omega^* \left( \frac{c\varepsilon}{\delta} |x|^{-2} \right) \rho_\varepsilon \leq \bar{o}_\varepsilon \int_{\{|x|>R\}} c\varepsilon |x|^{-2} \rho_\varepsilon$$

$$= c\varepsilon \bar{o}_\varepsilon \int_R^{+\infty} r^{-2+2-N-\varepsilon} r^{N-1} dr = c R^{-\varepsilon} \bar{o}_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Thus letting  $\varepsilon \rightarrow 0$  and then  $\delta \rightarrow 0$ , from (28), (29), (30) we infer that  $(u_1 - u_2)^+ \in L^1(\mathbb{R}^N, \rho)$  and (26) holds.  $\square$

Now, the latter result implies the analogous one in the  $L^1(\mathbb{R}^N, \rho)$  setting for quite general superharmonic weights  $\rho$ . Indeed, recall that the convolution of a source term  $\mu$  with the fundamental solution of the laplacian yields solution to the problem  $-\Delta \rho = \mu$  (cf. e.g. [14, Prop. 8] for one precise statement of the kind). The case  $\mu \geq 0$  corresponds to superharmonic functions. Then, exploiting the truncation of the fundamental solution with the parameter  $R > 0$ , the discretization of the measure  $\mu$  on a fine grid, and the the monotone convergence theorem in the context of  $R \rightarrow 0$ , we deduce the following generalization.

**Corollary 2.** *In the assumptions of Theorem 5.2, replace the weight  $\rho$  by the superharmonic weight*

$$\rho^\mu : x \mapsto \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} d\mu(y),$$

where  $\mu$  is a Radon measure such that  $\rho^\mu$  is well defined as an element of  $L^1_{loc}(\mathbb{R}^N)$ . Then the claim (26) of Theorem 5.2 remains true.

*Proof.* Introduce for  $R > 0$

$$(31) \quad \rho_R^\mu : x \mapsto \int_{\mathbb{R}^N} \frac{1}{(|x - y| \vee R)^{N-2}} d\mu(y).$$

Since  $\rho_R^\mu \leq \rho^\mu$  by construction, then the assumption  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N, \rho^\mu)$  yields  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N, \rho_R^\mu)$ . Assuming for a moment that the analogue of Theorem 5.2 is proved with weight  $\rho_R^\mu$  – which is deduced by linearity from Theorem 5.2 itself, because  $\rho$  in its statement can be replaced by any translation  $\rho(\cdot - y)$ ,  $y \in \mathbb{R}^N$ , – we can let  $R \rightarrow 0$  in the inequality

$$(32) \quad \int_{\mathbb{R}^N} (u_1 - u_2)^+ \rho_R^\mu \leq \int_{\mathbb{R}^N} (f_1 - f_2) \text{sign}^+(u_1 - u_2) \rho_R^\mu.$$

Separating  $(f_1 - f_2)$  into the positive and the negative parts, using the assumed integrability of  $(f_1 - f_2)^+$  with the weight  $\rho^\mu$ , we deduce the claim from the monotone convergence theorem as  $R \rightarrow 0$ .

It remains to assess the claim (32). To do so, we will exploit Theorem 5.2 in the context of a two-step approximation procedure. First, for  $L > 0$  we set  $\mu_L = \mu \mathbb{1}_{[-L, L]^N}$  and introduce the corresponding weight  $\rho_R^{\mu_L}$  with the formula



analogous to (31). If the weighted contraction property (32) is achieved with weights  $\rho_R^{\mu_L}$  in the place of  $\rho_R^\mu$ , at the limit  $L \rightarrow \infty$ , using the monotone convergence theorem as above we deduce the claimed property (32) itself. Now, to assess the analogue of (32) with weights  $\rho_R^{\mu_L}$ , for  $R, L$  fixed and  $k \in \mathbb{N}$  we set

$$\mu_{L,k}(\cdot) = \sum_{-Lk \leq i_1, \dots, i_N < Lk} \mu(\Pi_{i_1 \dots i_N}) \delta(\cdot - y_{i_1 \dots i_N})$$

with parallelepipeds  $\Pi_{i_1 \dots i_N}$  and their centers  $y_{i_1 \dots i_N}$  given by

$$\Pi_{i_1 \dots i_N} = \left[ \frac{i_1}{k}, \frac{i_1 + 1}{k} \right) \times \dots \times \left[ \frac{i_N}{k}, \frac{i_N + 1}{k} \right), \quad y_{i_1 \dots i_N} = \left( \frac{i_1 + 1/2}{k}, \dots, \frac{i_N + 1/2}{k} \right);$$

this is the discretization of  $\mu_L$  with a linear combination of Dirac masses on a uniform cartesian grid. Introduce the associated weights following the recipe of (31). Since  $\rho_R^{\mu_{L,k}}$  is a linear combination of shifted weights appearing in inequalities (26), the claim (32) with weight  $\rho_R^\mu$  replaced by  $\rho_R^{\mu_{L,k}}$  is immediate. Now, it is not difficult to check that the weights  $\rho_R^{\mu_{L,k}}$  and  $\rho_R^{\mu_L}$  are equivalent up to the multiplicative constant  $(1 + \frac{1}{2kR})^{N-2}$ ; this verification is postponed to Lemma 5.3 below. In particular, by construction if  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N, \rho^\mu)$  we also have  $(f_1 - f_2)^+ \in L^1(\mathbb{R}^N, \rho)$  for  $\rho = \rho_R^\mu$ ,  $\rho = \rho_R^{\mu_L}$  (because  $\rho_R^{\mu_L} \leq \rho_R^\mu \leq \rho^\mu$ ) and for  $\rho = \rho_R^{\mu_{L,k}}$  (by the above stated weights' equivalence). Since  $(1 + \frac{1}{2kR})^{N-2} \rightarrow 1$  as  $k \rightarrow \infty$ , passage to the limit  $k \rightarrow \infty$  in (32) with weight  $\rho_R^{\mu_{L,k}}$  is immediate. This ends the proof of the Corollary.  $\square$

**Lemma 5.3.** *With the notation of the above proof, there holds for a.e.  $x \in \mathbb{R}^N$*

$$\left(1 + \frac{1}{2kR}\right)^{-(N-2)} \rho_R^{\mu_L} \leq \rho_R^{\mu_{L,k}} \leq \left(1 + \frac{1}{2kR}\right)^{N-2} \rho_R^{\mu_L}.$$

*Proof.* For  $x, y \in \mathbb{R}^N$ , set  $d_R(x, y) = \frac{1}{(|x-y|\sqrt{R})^{N-2}}$ . By construction of  $\rho_R^{\mu_{L,k}}$  and  $\rho_R^{\mu_L}$ , it is enough to prove that whenever  $|y - y_*| \leq \frac{1}{2k}$  there holds

$$\left(1 + \frac{1}{2kR}\right)^{-1} d_R(x, y) \leq d_R(x, y_*) \leq \left(1 + \frac{1}{2kR}\right) d_R(x, y).$$

The latter follows by a straightforward case study. Indeed,

- (a) If  $|x - y|, |x - y_*| \leq R$  then  $d_R(x, y) = R^{2-N} = d_R(x, y_*)$ .
- (b) If  $|x - y|, |x - y_*| \geq R$  then

$$\frac{d_R(x, y)}{d_R(x, y_*)} = \left(\frac{|x - y_*|}{|x - y|}\right)^{N-2} \leq \left(\frac{|x - y| + |y - y_*|}{|x - y|}\right)^{N-2} \leq \left(1 + \frac{1/(2k)}{R}\right)^{N-2};$$

the reciprocal bound is obtained by simply exchanging the roles of  $y, y_*$ .

- (c) If  $|x - y_*| \leq R \leq |x - y|$  then

$$\frac{d_R(x, y)}{d_R(x, y_*)} = \left(\frac{R}{|x - y|}\right)^{N-2} \leq 1$$

while

$$\frac{d_R(x, y_*)}{d_R(x, y)} = \left(\frac{|x - y|}{R}\right)^{N-2} \leq \left(\frac{|x - y_*| + |y_* - y|}{R}\right)^{N-2} \leq \left(1 + \frac{1/(2k)}{R}\right)^{N-2}.$$

- (d) Finally, if  $|x - y| \leq R \leq |x - y_*|$  then the argument of (c) applies by merely exchanging the roles of  $y, y_*$ .  $\square$



**Remark 13.** Notice that instead of letting  $R \rightarrow 0$  as in the above proof, we can also rescale the weights of Theorem 5.2, so that we can take  $(\frac{|x|}{R} \vee 1)^{2-N}$ ; then we can let  $R \rightarrow +\infty$  and reach the weight  $\rho = 1$ . This yields another technique for proving uniqueness of  $L^\infty$  solutions of (1); we refer to our work [7] for an application of the above techniques to convection-diffusion problems.

**5.2. Well-posedness in  $L^1(\mathbb{R}^N, \rho)$  for  $L^\infty$  solutions.** Now let us turn to the well-posedness issue for (1) in  $L^1(\mathbb{R}^N, (|x| \vee R)^{2-N})$ ,  $N \geq 3$ . Notice that analogous results for  $N = 2$  are already contained in § 4.3.

**Theorem 5.4.** *Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous non-decreasing function. Let  $N \geq 3$ . Let  $R > 0$  and  $\rho(x) = (|x| \vee R)^{2-N}$ . Then  $L^1(\mathbb{R}^N, \rho) \cap L^\infty(\mathbb{R}^N)$  is a well-posedness class for problem (1).*

*More exactly, for all  $f \in L^1(\mathbb{R}^N, \rho) \cap L^\infty(\mathbb{R}^N)$  there exists a unique  $\mathcal{D}'$  solution  $u_f$  of (1) such that  $w_f := \varphi \circ u_f \in L^\infty(\mathbb{R}^N)$ . Moreover,  $u_f \in L^1(\mathbb{R}^N, \rho) \cap L^\infty(\mathbb{R}^N)$ ; the mapping  $f \in L^1(\mathbb{R}^N, \rho) \cap L^\infty(\mathbb{R}^N) \mapsto u_f$  is an order-preserving contraction in  $L^1(\mathbb{R}^N, \rho)$ ; and the maximum principle holds, i.e.,  $\text{ess inf } f \leq u_f \leq \text{ess sup } f$ .*

**Remark 14.** With  $\rho$  replaced by a super-harmonic weight  $\rho^\mu$  as in Corollary 2, in the same way we obtain the well-posedness results for the data in  $L^1(\mathbb{R}^N, \rho^\mu) \cap L^\infty(\mathbb{R}^N)$ .

*Proof.* of Th. 5.4 For the existence, the arguments are the same as the ones used for the proof of Theorem 4.1. We consider the truncated nonlinearities  $z \mapsto (\varphi(z) \vee \varphi(-M)) \wedge \varphi(M)$ , where  $M = \|f\|_{L^\infty}$ . We consider truncated data  $f^{n,m}$  given by (17) and the corresponding bimonotone sequence of solutions  $(u^{n,m})_{n,m}$  (given by the results of [13], see § 4.1), which also satisfies  $\|u^{n,m}\|_{L^\infty} \leq M$ . Instead of (23) we use the uniform estimates of  $\|w^{n,m} - w_*\|_{L^\infty}$  and  $\int_{\mathbb{R}^N} |u^{n,m} - u_*| \rho$  (with  $u_* \equiv 0, w_* \equiv 0$ ) deduced from Theorem 5.2 (see (26)). It follows that  $u^{n,m}$  (resp.,  $w^{n,m}$ ) converge in  $\mathcal{D}'(\mathbb{R}^N)$  to a function  $u_f$  in  $L^1(\mathbb{R}^N, \rho) \cap L^\infty(\mathbb{R}^N)$  (resp., to the function  $w_f := \varphi \circ u_f \in L^\infty(\mathbb{R}^N)$ ). The function  $u_f$  is the required solution.

The uniqueness and comparison are obtained as in the proof of Theorem 4.2; the continuous dependence follows from the comparison principle, in the way of Proposition 2.  $\square$

As in Remark 11, the result of Theorem 5.4 can be generalized to the case of a non-autonomous nonlinearity  $\varphi$ ; it suffices to replace the space  $L^\infty$  for  $f, u$  by the space  $L^\infty_\varphi$ .

**Remark 15.** Assume the nonlinearity  $\varphi$  satisfies  $(H'_{surj})$ . Assume that for all  $k > 0$ , the truncated function  $(x, z) \mapsto (\varphi(x, z) \vee (-k)) \wedge k$  is uniformly continuous in the second variable (the assumption  $(H_0)$  is a stronger sufficient condition).

Let  $N \geq 3$ . Let  $R > 0$  and  $\rho(x) = (|x| \vee R)^{2-N}$ .

With the notation of Remark 11,  $L^1(\mathbb{R}^N, \rho) \cap L^\infty_\varphi$  is a well-posedness class for problem (1).

More exactly, for all  $f \in L^1(\mathbb{R}^N, \rho) \cap L^\infty_\varphi(\mathbb{R}^N)$  there exists a unique  $\mathcal{D}'$  solution  $u_f$  of (1) such that  $w_f := \varphi \circ u_f \in L^\infty(\mathbb{R}^N)$ . Moreover,  $u_f$  belongs to  $L^1(\mathbb{R}^N, \rho) \cap L^\infty_\varphi(\mathbb{R}^N)$ ; the mapping  $f \in L^1(\mathbb{R}^N, \rho) \cap L^\infty_\varphi(\mathbb{R}^N) \mapsto u_f$  is an order-preserving contraction in  $L^1(\mathbb{R}^N, \rho)$ ; and the following maximum principle holds:

$$u_0^l \leq f \leq u_0^k \quad \implies \quad u_0^l \leq u_f \leq u_0^k.$$

*Proof.* In view of the uniform continuity in  $z$  of the function  $(\varphi(x, z) \vee (-k)) \wedge k$ , Proposition 5(ii) yields a bimonotone sequence  $u^{n,m}$  of solutions with data  $f^{n,m}$  in (19). In order to show that the limit  $u_f$  of  $u^{n,m}$  belongs to  $L^1(\mathbb{R}^N, \rho)$ , we use the estimate (26). Indeed, one crucial point of the construction of  $L^\infty_\varphi$  is that  $w = \varphi \circ u \in L^\infty(\mathbb{R}^N)$  when  $u \in L^\infty_\varphi(\mathbb{R}^N)$ . This allows us to apply Theorem 5.4.

Thus the existence follows; the other points are shown as in the proof of Theorem 5.4.  $\square$

## 6. ON THE EVOLUTION PROBLEM (2)

The evolution problem corresponding to the stationary equation (1) was the object of intense study over years: in the space-homogeneous case, it corresponds to the celebrated porous medium / fast diffusion equations. We refer to the books [24, 42] and references therein for extensive account on well-posedness theories and well-posedness classes for these equations.

In this section, we only briefly comment on the application of the techniques and ideas of the present paper to the evolution case. First, in what concerns the setting of  $L^\infty$  solutions, the technique of Section 5 works for entropy solutions of the evolution problem (2). It was demonstrated in the previous work [7] of the authors that it yields the uniqueness and comparison principle.

Second, because our study is based upon Kato inequalities, we are not able to address very weak solutions and have to limit the scope to either solutions having  $u_t \in L^1_{loc}$ , or those having  $\nabla w := \nabla \varphi(x, u) \in L^2_{loc}$ . Concerning the first case, note that Herrero and Pierre [31] proved, in many interesting situations, that the  $L^1$  regularity assumption on  $u_t$  is fulfilled (and then derived uniqueness of merely locally integrable distributional solutions) for the fast diffusion equation  $u_t = \Delta(|u|^{m-1}u)$ ,  $0 < m < 1$ . Concerning the second case, Kato and entropy inequalities were addressed in [39, 22, 18] and many other contributions (this is the basis of the aforementioned result of [7]).

It should be stressed that even under the additional regularity assumptions on  $\nabla w$  or on  $u_t$  the case where  $\varphi(x, \cdot)$  may degenerate, the Blanchard-Porretta trick used in Proposition 1 is difficult to adapt to the evolution setting. Different results in this direction are contained in [18, 3, 6]. The results of [3, 6] both concern obtention of the Kato inequality between one solution  $u$  with  $\nabla w \in L^2_{loc}$  and a time-independent solution  $\hat{u}$  with the same  $\nabla \hat{w}$  integrability. These results can be naturally interpreted within the nonlinear semigroup theory, in the way expolited, e.g., in [6] by Igbida and the first author. The uniqueness for the evolution problem is related to the uniqueness of the integral solutions in the sense of Bénilan [12, 15] to the abstract evolution problem associated with the resolvent equation of the kind (1). This approach, combined with the results of Section 5, eventually leads to well-posedness results in weighted  $L^1(\mathbb{R}^N, \rho)$  spaces with superharmonic weights  $\rho$ . However, the detailed analysis of this problem (closely related to classical results of Bénilan and Crandall [14] where specific weights and  $x$ -independent  $\varphi$  were addressed) is beyond the scope of the present paper.

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