



HAL
open science

Steady sliding frictional contact problem for a 2d elastic half-space with a discontinuous friction coefficient and related stress singularities

Patrick Ballard

► **To cite this version:**

Patrick Ballard. Steady sliding frictional contact problem for a 2d elastic half-space with a discontinuous friction coefficient and related stress singularities. *Journal of the Mechanics and Physics of Solids*, 2016, 97, pp.225-259. 10.1016/j.jmps.2016.03.021 . hal-02326962

HAL Id: hal-02326962

<https://hal.science/hal-02326962>

Submitted on 22 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Steady sliding frictional contact problem for a 2d elastic half-space with a discontinuous friction coefficient and related stress singularities

Patrick Ballard

*Laboratoire de Mécanique et d'Acoustique, CNRS,
31, chemin Joseph Aiguier,
13402 Marseille Cedex 20, France.*

Abstract

The steady sliding frictional contact problem between a moving rigid indenter of arbitrary shape and an isotropic homogeneous elastic half-space in plane strain is extensively analysed. The case where the friction coefficient is a step function (with respect to the space variable), that is, where there are jumps in the friction coefficient, is considered. The problem is put under the form of a variational inequality which is proved to always have a solution which, in addition, is unique in some cases. The solutions exhibit different kinds of universal singularities that are explicitly given. In particular, it is shown that the nature of the universal stress singularity at a jump of the friction coefficient is different depending on the sign of the jump.

Keywords: elasticity, contact, friction, singularity, variational inequality, uniqueness

2010 MSC: 74B05, 74M10, 74M15

1. Background: steady sliding frictional contact in linear elasticity

The problem governing the equilibrium of a deformable body that is pressed against a frictionless obstacle in the framework of linearised elasticity is the so-called *Signorini* problem. Formally, the problem consists in finding a displace-

Email address: ballard@lma.cnrs-mrs.fr (Patrick Ballard)

5 ment field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ ($d = 2, 3$) such that:

$$\left. \begin{array}{l}
 \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f}^P = \mathbf{0}, \\
 \mathbf{u} = \mathbf{u}^P, \\
 \mathbf{t} \stackrel{\text{def}}{=} \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^P, \\
 u_n - g^P \leq 0, \quad t_n \leq 0, \quad (u_n - g^P)t_n = 0, \\
 \mathbf{t}_t = \mathbf{0},
 \end{array} \right| \begin{array}{l}
 \text{in } \Omega, \\
 \text{on } \Gamma_u, \\
 \text{on } \Gamma_t, \\
 \text{on } \Gamma_c,
 \end{array} \quad (1)$$

where Ω denotes some smooth bounded open set in \mathbb{R}^d (the so-called stress-free reference configuration), $\Gamma_u \cup \Gamma_t \cup \Gamma_c = \partial\Omega$ denotes a splitting of the boundary into three disjoint parts, and \mathbf{n} is the outward unit normal. As usual, \mathbf{u} is the (unknown) displacement, $\boldsymbol{\sigma}(\mathbf{u})$ is the Cauchy stress associated with this displacement by the linear elastic constitutive law, and $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ denotes the surface traction. Any vector field \mathbf{v} defined on part of the boundary can be split into its normal and tangential parts: $\mathbf{v} = v_n \mathbf{n} + \mathbf{v}_t$. The loading conditions are defined by \mathbf{u}^P (the surface displacement prescribed on Γ_u), \mathbf{t}^P (the surface tractions prescribed on Γ_t), \mathbf{f}^P (the prescribed body forces), and g^P (the initial gap with the obstacle measured algebraically along the outward unit normal \mathbf{n} to $\partial\Omega$).

The existence and uniqueness of a solution was first proved in 1964 by Gaetano Fichera who inspired the theory of variational inequalities which was subsequently developed in the celebrated article [1] of Jacques-Louis Lions and Guido Stampacchia. Naturally, the need to integrate the handling of dry friction in the theory immediately emerged [2]. But, the so-called Signorini problem with Coulomb friction faced huge mathematical difficulties and the mathematical structure of the problem has not been completely elucidated yet. A good account of today's state of art is to be found in [3].

An easier contact problem involving dry friction is the steady sliding frictional contact problem which was first studied in [4]. It is the problem of the equilibrium of a (linearly) elastic body against a moving obstacle, the geometry of which remaining invariable with respect to time. In \mathbb{R}^2 , such a situation is met with a straight obstacle moving along the direction of its boundary or with an obstacle whose boundary is a circle rotating around its center (in that case, the obstacle can be either the disk or the region outside the circle). In \mathbb{R}^3 , this situation encompasses the case where the boundary of the obstacle is any surface of revolution rotating around its revolution axis but there are other cases such as the case of a rotating infinite screw. Dry friction between the moving obstacle and the elastic body is assumed and a displacement field in the elastic body, that is independent of time, is sought. This situation is far simpler than that of the Signorini problem with Coulomb friction because of the motion of the obstacle. The Signorini problem with Coulomb friction is doubly a free boundary problem because the contact zone (the points of the boundary that are in contact with the obstacle) is unknown and also within the contact zone, the sticking zone (the points of the contact zone that adhere to the obstacle) is also unknown. In the

steady sliding frictional contact problem, all the points in the contact zone are slipping along the obstacle since it is moving and the sliding velocity is known in advance. Considering the intersection of the outward normal to the body with
45 the obstacle, the motion of the body results in a given slip velocity field \mathbf{w} on Γ_c and the steady sliding frictional contact problem is formally that of finding a displacement field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ ($d = 2, 3$) such that:

$$\left\{ \begin{array}{ll} \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f}^P = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}^P, & \text{on } \Gamma_u, \\ \mathbf{t} \stackrel{\text{def}}{=} \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^P, & \text{on } \Gamma_t, \\ \begin{array}{l} u_n - g^P \leq 0, \quad t_n \leq 0, \quad (u_n - g^P)t_n = 0, \\ \mathbf{t}_t = -f t_n \mathbf{w}_t / |\mathbf{w}_t|, \end{array} & \text{on } \Gamma_c, \end{array} \right. \quad (2)$$

where $f \geq 0$ is the given friction coefficient. In the case $f = 0$, the Signorini problem is recovered. Therefore, the steady sliding frictional contact problem
50 can formally be seen as a generalization of the Signorini problem in which the unknown reaction force is no longer directed along the normal to the boundary but along a given direction which is possibly slanted with respect to the normal (see figure 1).

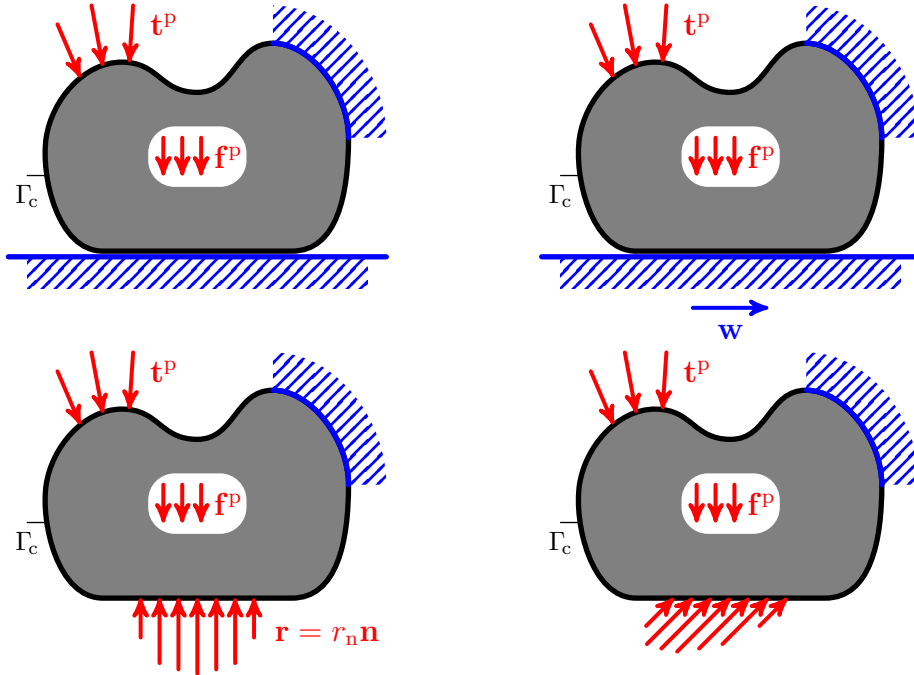


Figure 1: The Signorini problem (left) and the steady sliding frictional contact problem (right)

This situation is very common in practice, and any machine which is supposed
55 to run steadily provides an example. However, the situation of, say, a rubber
wiper blade sliding on a glass windshield, suggests that although steady motion is
generally encountered, unsteady motion, often called ‘stick-slip’ motion can also
happen. The understanding and therefore ultimately the prediction of that ‘stick-
slip’ phenomenon is still a challenge nowadays. In [4], the structure of the set of
60 solution for the steady sliding frictional contact problem was studied for the first
time. Under appropriate regularity assumptions, it was proved that the problem
has a unique solution for small values of the friction coefficient. It was also
proved that in the case of a bounded body, multiple solutions can be encountered
for large values of the friction coefficient. Hence, the set of equilibrium solutions
65 for the steady sliding frictional contact problem may exhibit a bifurcation when
the friction coefficient is taken as a parameter and this bifurcation can possibly
account for the occurrence of the ‘stick-slip’ phenomenon. Let us also mention
that the static bifurcation is only one possible mechanism to account for ‘stick-
slip’ and a flutter-type mechanism can also be possible and the latter should
70 certainly be considered to account for brake squeal for example.

In [4], the particular case where the elastic body is an isotropic elastic half-
space in plane strain was also considered. A variational inequality governing
the steady sliding frictional contact problem was derived and studied. It was
proved to be uniquely solvable for all values of the friction coefficient. Therefore,
75 the above-mentioned bifurcation does not occur in the case of the half-space.
Actually, the situation of the half-space appears to be a favourable framework to
study more thoroughly the mathematical structure of the steady sliding frictional
contact problem.

In [4], the analysis of the steady sliding frictional contact problem was
80 restricted to the situation where the friction coefficient is a given constant.
However, in the case of a composite elastic body, one should expect to have the
friction coefficient which is only a step (that is, piecewise constant) function with
respect to the space variable. But, the extension of the analysis in [4] to this new
circumstance is not straightforward at all and raises many technical difficulties.
85 It is the reason why the analysis of the steady sliding frictional contact problem
with piecewise constant friction coefficient is first performed in this article in the
favourable case where the elastic body is an isotropic elastic half-space in plane
strain.

2. Summary of the results that will be proved in this article

90 The results obtained from the analysis performed in this article convey strong
(and sometimes rather nonintuitive) mechanical information, which certainly
has practical consequences. However, proving them precisely requires many
technicalities and sophisticated mathematical tools. The aim of this section is
to have the statement of these results made as simple as possible with a lot of
95 the mathematics disarmed, for the convenience of the reader.

We consider an isotropic homogeneous linearly elastic two-dimensional half-
space defined by $z > 0$. The Poisson ratio is denoted by $\nu \in]-1, 1/2[$ and the

force unit is chosen so that the Young modulus $E = 1$. We denote by x the space variable along the boundary and by $\mathbf{t}(x)$ the surface traction distribution on the boundary and the normal and tangential components will be addressed as $t_n(x)$ and $t_t(x)$. Prescribing a vanishing stress field at infinity and setting:

$$\bar{\mathbf{u}} = \frac{\mathbf{u}}{2(1-\nu^2)}, \quad \text{and} \quad \gamma = \frac{1-2\nu}{2(1-\nu)} \in]0, 3/4[,$$

the surface displacement resulting from a given surface traction distribution $\mathbf{t}(x)$ is given (see for example [5] for a proof) by:

$$\begin{aligned} \bar{u}'_n(x) + \Omega &= \frac{1}{\pi} \oint_{-1}^1 \frac{t_n(x')}{x' - x} dx' - \gamma t_t(x), \\ \bar{u}'_t(x) &= \frac{1}{\pi} \oint_{-1}^1 \frac{t_t(x')}{x' - x} dx' + \gamma t_n(x), \end{aligned}$$

where the sign \oint recalls that the integral should be understood in terms of the Cauchy principal value, and where the real constant Ω represents an arbitrary overall rotation, which will be taken as 0 in the sequel. The surface displacement is obtained up to an arbitrary additive constant, which is interpreted as being a rigid motion. This arbitrary rigid motion cannot be fixed by prescribing appropriate conditions at infinity since the displacement field is generally infinite at infinity.

Consider some rigid obstacle, the geometry of which is defined by the equation $-z = \psi(x)$ ($x \in]-1, 1[$), moving at a constant velocity $w > 0$ along x , which is assumed to be parallel to the boundary of the half-space (see figure 2). Set:

$$\bar{\psi} = \frac{\psi}{2(1-\nu^2)}.$$

The steady sliding frictional contact problem was introduced and studied in [4]. In the reference frame moving with the indenter, it is formally that of finding $\mathbf{t}(x), \bar{\mathbf{u}}(x) :]-1, 1[\rightarrow \mathbb{R}$ such that:

- $\frac{1}{\pi} \oint_{-1}^1 \frac{t_n(x')}{x' - x} dx' - \gamma t_t(x) = \bar{u}'_n(x),$
- $\frac{1}{\pi} \oint_{-1}^1 \frac{t_t(x')}{x' - x} dx' + \gamma t_n(x) = \bar{u}'_t(x),$
- $\bar{u}_n \leq \bar{\psi}, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n \equiv 0,$
- $t_t = -f t_n,$
- $\int_{-1}^1 t_n(x') dx' = -P,$

where $P > 0$ is the given normal component of the prescribed total force exerted on the moving obstacle, $f \geq 0$ is a given friction coefficient. Note that if $\bar{\psi}$ is changed into $\bar{\psi} + C$, then we get a solution for the new problem by just changing \bar{u}_n into $\bar{u}_n + C$ in the solution. This means that the penetration of the indenter into the half-space is undefined and this is due to the fact that the displacement field is infinite at infinity. The problem can be parametrized by the total force P only, and not by the height of the moving obstacle, because it is undetermined. This fact is intimately connected with the fact that the stress field in the half-space is not square integrable: the elastic energy of the solution is infinite and this is the reason why the problem has to be brought to the boundary by use of the fundamental solution of the Neumann problem for the half-space (the so-called Boussinesq solution). Focusing on the normal components, this

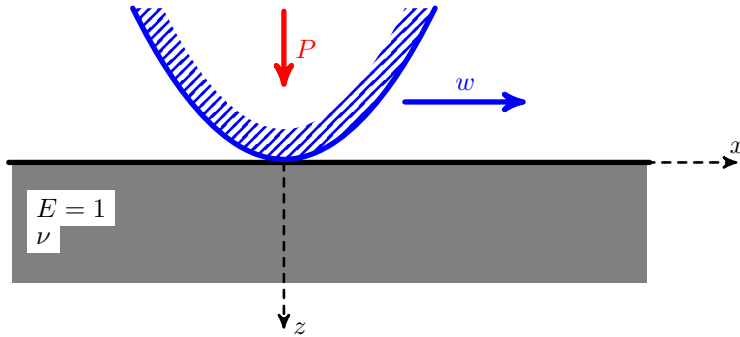


Figure 2: Geometry of the problem.

formal problem reduces to that of finding $t_n(x), \bar{u}_n(x) :]-1, 1[\rightarrow \mathbb{R}$ such that:

- $\frac{1}{\pi} \oint_{-1}^1 \frac{t_n(x')}{x' - x} dx' + \gamma f(x) t_n(x) = \bar{u}'_n(x),$
- $\bar{u}_n \leq \bar{\psi}, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n \equiv 0, \quad (3)$
- $\int_{-1}^1 t_n(x') dx' = -P.$

In this article, the following theorem will be proved. It shows the unconditional existence of a solution of the steady sliding frictional contact problem for any piecewise constant friction coefficient.

Theorem 1. *Let $P > 0$ be a positive real constant, $\bar{\psi} :]-1, 1[\rightarrow \mathbb{R}$ be a piecewise Lipschitz-continuous function and $f :]-1, 1[\rightarrow \mathbb{R}$ be a piecewise constant function. There exist a Radon measure $t_n \in \mathcal{M}([-1, 1])$ and a function*

$\bar{u}_n \in H^{1/2}(-1, 1)$ such that:

- $\frac{1}{\pi} \oint_{-1}^1 \frac{t_n(x')}{x' - x} + \gamma f t_n = \bar{u}'_n, \quad \text{in }]-1, 1[,$
- $\bar{u}_n \leq \bar{\psi}, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n \equiv 0,$
- $\int_{-1}^1 t_n = -P.$

In addition, if f is nondecreasing, t_n and \bar{u}_n are unique.

To be precise, the first equation in theorem 1 holds in the sense of distributions over $] -1, 1[$ and the Cauchy principal value integral is defined to be the distributional derivative of the convolution product $-t_n * \log$ (where the measure t_n is extended by zero on $\mathbb{R} \setminus [-1, 1]$).

Proof. This a straightforward adaptation of the proofs of theorems 16 and 23. \square

In the case where $\bar{\psi} \equiv 0$ (moving rigid flat punch of finite width), it is possible to derive an explicit formula for t_n , for any step function f , by use of theorem 36 in appendix C. We shall now give this formula in two cases.

1. The friction coefficient is a constant denoted by f . In that case, the explicit solution of the above problem seems to have been first given by Galin [6]. It reads as:

$$t_n(x) = -\frac{P \cos \pi \alpha}{\pi} \frac{1}{(1+x)^{\frac{1}{2}+\alpha}(1-x)^{\frac{1}{2}-\alpha}},$$

where:

$$\alpha \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f) = \frac{1}{\pi} \arctan\left(\frac{(1-2\nu)f}{2(1-\nu)}\right) \quad (\in [0, 1/2[). \quad (4)$$

This distribution of the normal component of the surface traction is represented in figure 3. It has power singularities at both edges. The rear part of the punch is more loaded than the front part, and consistently, the rear singularity is stronger than the front one.

2. The friction coefficient has the constant value f_- on $] -1, 0[$ and the constant value f_+ on $] 0, 1[$. The explicit formula in that case is derived by use of theorem 36 in appendix C and seems to be new. It reads as:

$$t_n(x) = -\frac{P}{\pi \sqrt{1 + \gamma^2 f^2(x)}} \frac{|x|^\beta}{(1+x)^{\frac{1}{2}+\alpha_-}(1-x)^{\frac{1}{2}-\alpha_+}},$$

where:

$$\alpha_- \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f_-) = \frac{1}{\pi} \arctan\left(\frac{(1-2\nu)f_-}{2(1-\nu)}\right) \quad (\in [0, 1/2[) \quad (5)$$

$$\alpha_+ \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f_+) = \frac{1}{\pi} \arctan\left(\frac{(1-2\nu)f_+}{2(1-\nu)}\right) \quad (\in [0, 1/2[) \quad (6)$$

$$\beta \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f_-) - \frac{1}{\pi} \arctan(\gamma f_+) \quad (\in]-1/2, 1/2[). \quad (7)$$

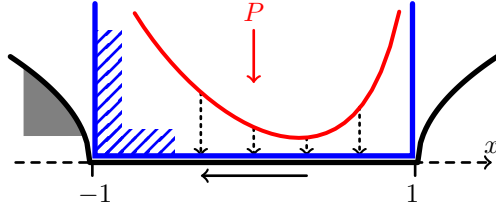


Figure 3: Normal component of the surface stress under a moving rigid flat punch with homogeneous friction coefficient.

This distribution of the normal component of the surface traction is represented in figure 4. Its shape is very different whether the larger friction coefficient is front (f_+) or rear (f_-). In the former case, β is negative and the surface traction goes to infinity at the discontinuity ($x = 0$).

We have therefore discovered a fact of engineering importance. In the situation where a rigid punch with two different coatings (producing two different friction coefficients with a jump at some point) is steadily sliding along the boundary of some linearly elastic body, the way the elastic body is loaded in the vicinity of the friction coefficient discontinuity is strongly influenced by the direction along which the indenter is moving.

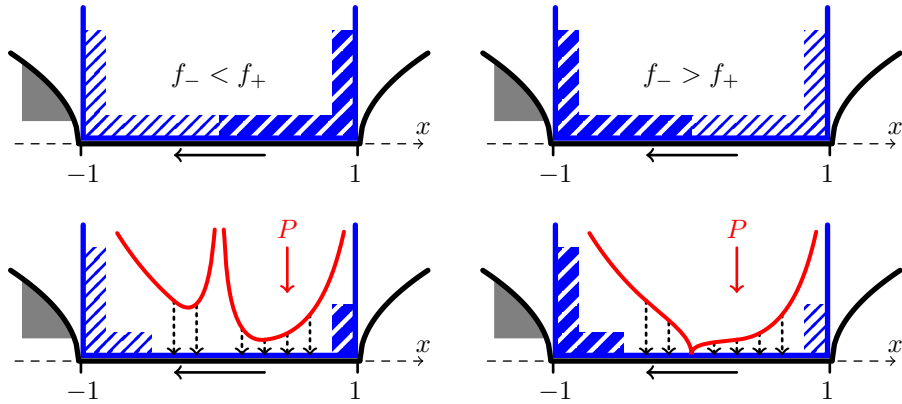


Figure 4: Normal component of the surface stress when the larger friction coefficient is front (left) or rear (right).

Another case where an explicit solution is available is the case of the moving rigid parabola with radius of curvature $\bar{\mathcal{R}}/(2(1 - \nu^2))$ at the minimum and with homogeneous friction coefficient f . A derivation of this solution which seems to have been first discovered by Galin [6] can be found in [4]. The corresponding

function $\bar{\psi}$ reads as follows:

$$\bar{\psi}(x) = \frac{x^2}{2\bar{\mathcal{R}}}.$$

165 The corresponding solution for the steady sliding frictional contact problem exhibits a connected contact zone surrounded by two detached zones. The half-diameter a of the contact zone is given by:

$$a^2 = \frac{P\bar{\mathcal{R}}}{2\pi(1/4 - \alpha^2)},$$

where α is given by formula (4). Then, provided that P is small enough so that $(1 + 2\alpha)a < 1$, the function t_n associated with the unique solution of the corresponding problem in theorem 1 is given by:

$$\begin{aligned} t_n(x) &= \frac{-P \cos \pi\alpha}{2\pi a^2(1/4 - \alpha^2)} \left[(1 + 2\alpha)a + x \right]^{1/2-\alpha} \left[(1 - 2\alpha)a - x \right]^{1/2+\alpha}, \\ &= \frac{-1}{\bar{\mathcal{R}}\sqrt{1 + \gamma^2 f^2}} \left[(1 + 2\alpha)a + x \right]^{1/2-\alpha} \left[(1 - 2\alpha)a - x \right]^{1/2+\alpha}. \end{aligned}$$

The contact zone $[-(1 + 2\alpha)a, (1 - 2\alpha)a]$ is shifted towards the rear of the parabola and the maximum amplitude of the surface traction is reached at $x = -4\alpha a$, which is always in the rear half of the contact zone.

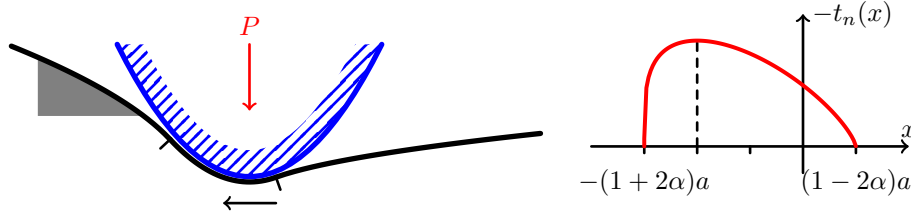


Figure 5: Rigid parabola moving at the surface of an elastic half-space.

170

It happens that the power singularities that have been exhibited in all the explicit solutions above are actually universal and apply to any shape of the indenter as expressed by the following theorems.

175 **Theorem 2.** Assume that $\bar{\psi}'$ is piecewise Lipschitz-continuous. Consider a solution of the problem in theorem 1 that achieves active contact in a neighborhood of the point $x = c \in]-1, 1[$. Then the measure t_n satisfies the estimate:

$$t_n(x) = \begin{cases} \frac{1}{\gamma} \frac{\bar{\psi}'(c+) - \bar{\psi}'(c-)}{f(c+) - f(c-)} - \frac{\theta |c - x|^\beta}{\sqrt{1 + \gamma^2 f^2(x)}}, & \text{if } f(c+) \neq f(c-), \\ \frac{\bar{\psi}'(c+) - \bar{\psi}'(c-)}{1 + ((\gamma f(c))^2)} \left(\frac{\log |x - c|}{\pi} + \gamma f(c) H(x - c) \right) + \theta, & \text{otherwise,} \end{cases}$$

for some measure θ whose restriction to a neighborhood of c is a continuous function, and where β is given by formula (7) and H is the Heaviside function.

Proof. This is a straightforward adaptation of the proofs of theorems 9, 17 and 24. \square

The stress singularity arising in the case of a jump in the friction coefficient seems to have never been investigated before. Stress singularities arising in other but similar situations of contact problems in linear elasticity were also investigated in [7], [8] and [5].

We shall express in words the conclusions of theorem 2, revealing how nonintuitive they are.

- In the case where the friction coefficient has no jump at c ($f(c+) = f(c-)$), but where the indenter has a wedge at c ($\bar{\psi}'(c+) > \bar{\psi}'(c-)$), then the surface traction has a logarithmic singularity at c . Note that the coefficient of that singularity (the so-called stress intensity factor) is fully determined by the geometry of the wedge, the Poisson ratio and the friction coefficient. This fact is in sharp contrast with the usual singularities met in linear elasticity in polyhedra (see [9]) where the stress intensity factor is indeterminate and depends actually on the global problem (and not only on the local data). Needless to say, this result applies to the Signorini problem which corresponds to the particular case where $f \equiv 0$. Note that the nonpositivity condition on $t_n(x)$ entails $\bar{\psi}'(c+) \geq \bar{\psi}'(c-)$. Therefore, in case of an obstacle with reentrant corner ($\bar{\psi}'(c+) < \bar{\psi}'(c-)$), the vertex is never an interior point of the contact zone.
- In the case where the friction coefficient has a jump at c ($f(c+) \neq f(c-)$), but where the indenter has no wedge at c ($\bar{\psi}'(c+) = \bar{\psi}'(c-)$), then the surface traction has a power singularity at c . The surface traction goes to infinity in the case $f(c+) > f(c-)$, but it always goes to zero in the case $f(c+) < f(c-)$.
- In the case where the friction coefficient has a jump at c ($f(c+) \neq f(c-)$), and the indenter has a wedge at c ($\bar{\psi}'(c+) \neq \bar{\psi}'(c-)$), then the wedge has no effect on the power singularity in the case $f(c+) > f(c-)$, but in the case $f(c+) < f(c-)$, the surface traction has now a negative limit at c , instead of zero.

Theorem 3. Let $c \in [-1, 1]$. We assume a constant friction coefficient $f_- \geq 0$ on $]c - \epsilon, c[$ and a constant friction coefficient $f_+ \geq 0$ on $]c, c + \epsilon[$ ($\epsilon > 0$), if appropriate.

We consider a solution \bar{u}_n, t_n of the problem in theorem 1 such that contact with the indenter is achieved in a right-neighbourhood of c whereas detachment occurs in a left-neighbourhood of c . More precisely, if one of the two following assumptions is satisfied:

- $c = -1$, $\bar{u}_n = \bar{\psi}$ on $] -1, -1 + \epsilon[$ ($\epsilon > 0$) and $\bar{\psi}'$ is Lipschitz-continuous on $] -1, -1 + \epsilon[$,
- $c \in] -1, 1[$, $\bar{\psi}(c-) > \bar{\psi}(c+)$, $\bar{u}_n < \bar{\psi}$ on $]c - \epsilon, c[$ ($\epsilon > 0$), $\bar{u}_n = \bar{\psi}$ on $]c, c + \epsilon[$ and $\bar{\psi}'$ is Lipschitz-continuous on $]c - \epsilon, c[$ and $]c, c + \epsilon[$,

then, the solution \bar{u}_n, t_n in theorem 1 satisfies the estimate:

$$t_n = -\frac{\theta_-}{|x - c|^{1/2+\alpha_-}},$$

where α_- is given by formula (5) and θ_- is a measure whose restriction to a right-neighbourhood of $x = c$ is a Hölder-continuous function. In addition, in the case where $\theta_-(c) > 0$, then the following estimate holds true:

$$\bar{u}'_n(x) \sim -\frac{\theta_-(c) \sqrt{1 + \gamma^2 f_-^2}}{|x - c|^{1/2+\alpha_-}}, \quad \text{as } x \rightarrow c - .$$

225 If the following assumption is satisfied:

- $c \in]-1, 1[$, $\bar{\psi}(c-) = \bar{\psi}(c+)$, $\bar{u}_n < \bar{\psi}$ on $]c - \epsilon, c[$ ($\epsilon > 0$), $\bar{u}_n = \bar{\psi}$ on $]c, c + \epsilon[$ and $\bar{\psi}'$ is Lipschitz-continuous on $]c - \epsilon, c[$ and $]c, c + \epsilon[$,

then, the solution \bar{u}_n, t_n in theorem 1 satisfies the estimate:

$$t_n = -|c - x|^{1/2-\alpha_-} \theta_-$$

230 where θ_- is a measure whose restriction to a right-neighbourhood of $x = c$ is a Hölder-continuous function.

Similarly, we consider a solution \bar{u}_n, t_n of the problem in theorem 1 such that contact with the indenter is achieved in a left-neighbourhood of c whereas detachment occurs in a right-neighbourhood of c . More precisely, if one of the two following assumptions is satisfied:

- 235
- $c = 1$, $\bar{u}_n = \bar{\psi}$ on $]1 - \epsilon, 1[$ ($\epsilon > 0$) and $\bar{\psi}'$ is Lipschitz-continuous on $]1 - \epsilon, 1[$,
 - $c \in]-1, 1[$, $\bar{\psi}(c-) < \bar{\psi}(c+)$, $\bar{u}_n = \bar{\psi}$ on $]c - \epsilon, c[$ ($\epsilon > 0$), $\bar{u}_n < \bar{\psi}$ on $]c, c + \epsilon[$ and $\bar{\psi}'$ is Lipschitz-continuous on $]c - \epsilon, c[$ and $]c, c + \epsilon[$,

then, the solution \bar{u}_n, t_n in theorem 1 satisfies the estimate:

$$t_n = -\frac{\theta_+}{|x - c|^{1/2-\alpha_+}},$$

240 where α_+ is given by formula (6) and θ_+ is a measure whose restriction to a left-neighbourhood of $x = c$ is a Hölder-continuous function. In addition, in the case where $\theta_+(c) > 0$, then the following estimate holds true:

$$\bar{u}'_n(x) \sim \frac{\theta_+(c) \sqrt{1 + \gamma^2 f_+^2}}{|x - c|^{1/2-\alpha_+}}, \quad \text{as } x \rightarrow c - .$$

If the following assumption is satisfied:

- 245
- $c \in]-1, 1[$, $\bar{\psi}(c-) = \bar{\psi}(c+)$, $\bar{u}_n = \bar{\psi}$ on $]c - \epsilon, c[$ ($\epsilon > 0$), $\bar{u}_n < \bar{\psi}$ on $]c, c + \epsilon[$ and $\bar{\psi}'$ is Lipschitz-continuous on $]c - \epsilon, c[$ and $]c, c + \epsilon[$,

then, the solution \bar{u}_n, t_n in theorem 1 satisfies the estimate:

$$t_n = -|c - x|^{1/2+\alpha_+} \theta_+$$

where θ_+ is a measure whose restriction to a left-neighbourhood of $x = c$ is a Hölder-continuous function.

Proof. This a straightforward adaptation of the proof of theorem 10. □

250 In words, theorem 3 says that the singularity of the surface traction at a discontinuity of the indenter (a jump of $\bar{\psi}$) is the same power singularity, going to infinity, as the one obtained at the edge of the moving rigid flat punch. This power singularity is different whether this discontinuity is a front or a rear boundary of the contact zone. In the case of a boundary of the contact zone at a smooth point of the indenter (a continuity point of $\bar{\psi}$), then the surface traction
 255 has a power singularity having zero as a limit, as the one obtained at the edge of the contact zone for the moving rigid parabola. This power singularity is different whether this boundary is a front or a rear boundary of the contact zone.

Remark 1. The fact that the front and the rear edges of the moving indenter experiment different stress singularities was already noted in [7]. It relies strongly
 260 on the fact that $\gamma \neq 0$ and the same is true of the existence of singularities at a jump of the friction coefficient. This highlights the fact that the usual and convenient approximation which consists in neglecting the influence of friction on the normal component of the surface traction is *not valid* in the case of a nonsmooth indenter of a nonconstant friction coefficient, except for the limiting
 265 case of incompressibility which corresponds to the situation $\gamma \rightarrow 0$. More generally, the analysis extends readily to the case where the moving indenter is elastic (instead of rigid) by replacing γ by Dundurs' bimaterial constant. Naturally, in the exceptional cases where Dundurs' bimaterial constant vanishes,
 270 there is no influence of friction on the normal component of the surface traction, and the analysis is driven back to the frictionless situation.

Remark 2. The analysis performed for the geometry of the half-space in this paper evidences the functional spaces that must be used to extend the analysis performed for a smooth bounded body in the case of a constant friction coefficient
 275 in [4] to the case of a piecewise constant friction coefficient. This extension still has to be written down, but in the light of [4], we must expect that the existence result will be obtained only under the condition that the maximum value of the friction coefficient is small enough. In that case, and provided that the bounded body is made of an homogenous isotropic linear elastic material, the solution
 280 will display the same singularities as those described in the present paper for the half-space, as is usual in linear elasticity.

3. Analysis of homogeneous friction

In this section, it is assumed that the friction coefficient f is a given real constant. The dual (that is, taking the surface traction as main unknown) weak

285 formulation of the steady sliding frictional contact problem for the half-space
with homogeneous friction coefficient was first obtained in [4] in terms of a
variational inequality. The Lions-Stampacchia theorem was applied to this weak
formulation to yield the existence and uniqueness of a solution. This will be
recalled here and, in addition, the displacement weak formulation will be derived.

290 3.1. Preliminaries

Some useful facts, detailed proof of which can be found in [5], will first be
recalled. With arbitrary $\hat{t} \in H^{-1/2}(-1, 1)$, the extension by zero on the whole
real line (still denoted by \hat{t}) defines a distribution $\hat{t} \in H^{-1/2}(\mathbb{R})$. The following
convolution products:

$$\hat{t} * \log |\cdot|, \quad \hat{t} * \operatorname{sgn}(\cdot),$$

295 (where $\operatorname{sgn}(\cdot)$ is the sign function) define distributions over \mathbb{R} whose restrictions
to the interval $] -1, 1[$ are in $H^{1/2}(-1, 1)$. In addition, the bilinear form defined
by:

$$\hat{t}_1, \hat{t}_2 \mapsto -\left\langle \hat{t}_1 * \log |x|, \hat{t}_2 \right\rangle_{H^{1/2}, H^{-1/2}},$$

is symmetric and is also positive definite. It therefore defines a scalar product on
the space $H^{-1/2}(-1, 1)$, and this scalar product induces a norm that is equivalent
300 to that of $H^{-1/2}$ (see [5] for a proof). The bilinear form:

$$\hat{t}_1, \hat{t}_2 \mapsto \left\langle \hat{t}_1 * \operatorname{sgn}(x), \hat{t}_2 \right\rangle_{H^{1/2}, H^{-1/2}},$$

can easily be seen to be skew-symmetric. It is continuous on $H^{-1/2} \times H^{-1/2}$.
These facts can be used to obtain the weak formulation of the steady sliding
frictional contact problem. The first equation in the formal problem (3) can be
rewritten under the form:

$$-\frac{1}{\pi} t_n * \log |\cdot| + \frac{\gamma f}{2} t_n * \operatorname{sgn}(\cdot) = \bar{u}_n + C, \quad \text{in }] -1, 1[,$$

305 where C is an arbitrary constant. Introducing the 1-dimensional vector space:

$$\mathcal{C} \stackrel{\text{def}}{=} \left\{ C \chi_{]-1, 1[} \mid C \in \mathbb{R} \right\},$$

where χ_S denotes the characteristic function of the set S , the normal surface
displacement is therefore seen to naturally belong to the quotient space $H_0 \stackrel{\text{def}}{=} H^{1/2}(-1, 1)/\mathcal{C}$. The dual space H'_0 is:

$$H'_0 = \left\{ \hat{t} \in H^{-1/2}(-1, 1) \mid \left\langle \hat{t}, \chi_{]-1, 1[} \right\rangle = 0 \right\},$$

and is a closed subspace of codimension 1 in $H^{-1/2}(-1, 1)$. Furthermore, taking
310 t_n equal to:

$$t_0(x) \stackrel{\text{def}}{=} \frac{\cos \pi \alpha}{\pi} \frac{1}{(1+x)^{\frac{1}{2}+\alpha} (1-x)^{\frac{1}{2}-\alpha}}, \quad (8)$$

with:

$$\alpha \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f) \quad (\in]-1/2, 1/2[),$$

in the first equation of problem (3), gives $\bar{u}'_n \equiv 0$ (see [5] or appendix C). Since $\int_{-1}^1 t_0 = 1$, any $\hat{t} \in H^{-1/2}(-1, 1)$ can be decomposed in the form:

$$t = \tilde{t} + t_0 \left\langle t, \chi_{]-1, 1[} \right\rangle_{H^{-1/2}, H^{1/2}},$$

where $\tilde{t} \in H'_0$.

315 Now, given $\bar{u}_n \in H_0 = H^{1/2}(-1, 1)/\mathcal{C}$, consider the problem of finding $t_n \in H^{-1/2}(-1, 1)$ such that:

$$\begin{aligned} -\frac{1}{\pi} t_n * \log |\cdot| + \frac{\gamma f}{2} t_n * \text{sgn}(\cdot) &= \bar{u}_n, & \text{on }]-1, 1[, \\ \left\langle t_n, \chi_{]-1, 1[} \right\rangle_{H^{-1/2}, H^{1/2}} &= -P. \end{aligned} \quad (9)$$

To obtain a weak formulation of that problem, we introduce the bilinear form defined on $H^{-1/2}(-1, 1) \times H^{-1/2}(-1, 1)$ by:

$$b(\hat{t}_1, \hat{t}_2) \stackrel{\text{def}}{=} -\frac{1}{\pi} \left\langle \hat{t}_1 * \log |x|, \hat{t}_2 \right\rangle_{H^{1/2}, H^{-1/2}} + \frac{\gamma f}{2} \left\langle \hat{t}_1 * \text{sgn}(x), \hat{t}_2 \right\rangle_{H^{1/2}, H^{-1/2}}.$$

The restriction of $b(\cdot, \cdot)$ to $H'_0 \times H'_0$ is continuous and coercive, for all $f \in \mathbb{R}$.

320 Hence, making the shift of unknown:

$$t_n = \tilde{t}_n - P t_0,$$

the above problem (9) reduces to that of finding $\tilde{t}_n \in H'_0$ such that:

$$\forall \hat{t} \in H'_0, \quad b(\tilde{t}_n, \hat{t}) = \langle \bar{u}_n, \hat{t} \rangle_{H_0, H'_0}.$$

By the Lions-Stampacchia theorem [1] (actually, only the Lax-Milgram theorem, which is a particular case of the Lions-Stampacchia theorem, is needed), this problem has a unique solution denoted by:

$$\tilde{t}_n = \mathcal{L}(\bar{u}_n),$$

325 where \mathcal{L} is a continuous linear mapping from H_0 onto H'_0 , and the unique solution t_n of problem (9) is given by:

$$t_n = \mathcal{L}(\bar{u}_n) - P t_0.$$

Taking $u \in W^{1,p}(-1, 1)$ for some $p > 1$, the problem of finding $t \in \cup_{p>1} L^p(-1, 1)$ such that:

- $\frac{1}{\pi} \oint_{-1}^1 \frac{t(x')}{x' - x} dx' + \gamma f t(x) = u'(x), \quad \text{for a.a. } x \in]-1, 1[,$
- $\int_{-1}^1 t(x') dx' = -P.$

has a unique solution given by:

$$t(x) = \frac{\gamma f u'(x)}{1 + (\gamma f)^2} - \frac{1}{1 + (\gamma f)^2} \frac{1}{\pi} \oint_{-1}^1 \frac{(1+x')^{\frac{1}{2}+\alpha}(1-x')^{\frac{1}{2}-\alpha}}{(1+x)^{\frac{1}{2}+\alpha}(1-x)^{\frac{1}{2}-\alpha}} \cdot \frac{u'(x')}{x' - x} dx' - \frac{P \cos \pi \alpha}{\pi} \frac{1}{(1+x)^{\frac{1}{2}+\alpha}(1-x)^{\frac{1}{2}-\alpha}},$$

where:

$$\alpha \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f) \quad (\in]-1/2, 1/2[),$$

(see [5] or appendix C in the case $p = +\infty$). Therefore, the restriction of \mathcal{L} to $W^{1,p}(-1, 1)/\mathcal{C}$ ($p > 1$) is given by:

$$\mathcal{L}(u) = \frac{\gamma f u'(x)}{1 + (\gamma f)^2} - \frac{1}{1 + (\gamma f)^2} \frac{1}{\pi} \oint_{-1}^1 \frac{(1+x')^{\frac{1}{2}+\alpha}(1-x')^{\frac{1}{2}-\alpha}}{(1+x)^{\frac{1}{2}+\alpha}(1-x)^{\frac{1}{2}-\alpha}} \cdot \frac{u'(x')}{x' - x} dx'. \quad (10)$$

3.2. Weak formulations of the steady sliding frictional contact problem

Following Schwartz, $\mathcal{D}]-1, 1[$ stands for the space of infinitely many differentiable real-valued functions with compact support in $] -1, 1[$. Such a function $f \in \mathcal{D}]-1, 1[$ will be said nonnegative (notation $f \geq 0$) if it takes only nonnegative real values. A distribution $t \in \mathcal{D}']-1, 1[$ is said to be nonnegative (notation $t \geq 0$), if it is nonnegative in the sense of the dual ordering:

$$t \geq 0 \text{ in } \mathcal{D}' \iff \forall f \in \mathcal{D} \text{ such that } f \geq 0 \text{ in } \mathcal{D}, \quad \langle t, f \rangle_{\mathcal{D}', \mathcal{D}} \geq 0.$$

It is well-known that a nonnegative distribution is a nonnegative measure, that is, an element of the dual space of the space $C_c^0]-1, 1[$ of all the continuous functions with compact support in $] -1, 1[$. In the particular case where the distribution t can be identified with a locally integrable function, then $t \geq 0$ in \mathcal{D}' if and only if it takes nonnegative values almost everywhere.

This definition will apply in particular to distributions belonging to the Sobolev spaces $H^s]-1, 1[$ with $s \in \mathbb{R}$: an element $t \in H^s]-1, 1[$ will be said nonnegative (notation $t \geq 0$) if the distribution is nonnegative. For $s \geq 0$, this definition reduces to that of functions that are nonnegative almost everywhere. For $s \leq 0$, this definition is consistent with that of the dual ordering for the duality $\langle H_0^s, H^{-s} \rangle$. Hence, no ambiguity will arise when writing $t \geq 0$ for an arbitrary $t \in H^s]-1, 1[$. In the sequel, we will frequently meet quotient spaces of type H^s / \mathcal{L} where \mathcal{L} is a finite-dimensional subspace of $H^s]-1, 1[$. We recall that an element $t \in H^s / \mathcal{L}$ satisfies $t \geq 0$ in the sense of the quotient ordering if there exists at least one nonnegative element in the equivalence class of v :

$$t \geq 0 \text{ in } H^s / \mathcal{L} \iff \exists l \in \mathcal{L}, \quad t + l \geq 0 \text{ in } H^s.$$

Finally, a distribution $t \in \mathcal{D}']-1, 1[$ is said to be nonpositive (notation $t \leq 0$), if $-t$ is nonnegative. We will also use the notations $t_1 \leq t_2$ or $t_2 \geq t_1$ to mean $t_2 - t_1 \geq 0$.

Whenever not confusing, we will frequently use the same notation for the equivalence class $\bar{\psi} \in H_0 = H^{1/2}(-1, 1)/\mathcal{C}$ and one of its arbitrary element $\tilde{\psi} \in H^{1/2}(-1, 1)$. Recalling the splitting $t_n = \tilde{t}_n - Pt_0$, with $\tilde{t}_n \in H'_0$, this abuse helps in making the contact conditions explicit:

$$\begin{aligned} \exists C \in \mathbb{R}, \quad \bar{u}_n - \bar{\psi} + C \leq 0, \quad \tilde{t}_n - Pt_0 \leq 0, \\ 0 = (\tilde{t}_n - Pt_0)(\bar{u}_n - \bar{\psi} + C) \in \mathcal{D}'(]-1, 1[), \end{aligned} \quad (11)$$

355 which entails that $\langle \tilde{t}_n - Pt_0, \bar{u}_n - \bar{\psi} + C \rangle_{H^{-1/2}, H^{1/2}} = 0$.

Given $\bar{\psi} \in H_0$, we set:

$$\begin{aligned} K_0 &\stackrel{\text{def}}{=} \left\{ \hat{u} \in H_0 \mid \hat{u} - \bar{\psi} \leq 0 \right\}, \\ K'_0 &\stackrel{\text{def}}{=} \left\{ \hat{t} \in H'_0 \mid \hat{t} - Pt_0 \leq 0 \right\}, \end{aligned} \quad (12)$$

and:

$$\begin{aligned} \varphi(\hat{u}) &\stackrel{\text{def}}{=} \begin{cases} \inf_{C \mid \hat{u} - \bar{\psi} + C \leq 0} \left\langle -Pt_0, \hat{u} - \bar{\psi} + C \right\rangle_{H^{-1/2}, H^{1/2}}, & \text{if } \hat{u} \in K_0, \\ +\infty, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \left\langle -Pt_0, \bar{u}_n - \bar{\psi} - \operatorname{ess\,sup}_{]-1, 1[}(\hat{u} - \bar{\psi}) \right\rangle_{H^{-1/2}, H^{1/2}}, & \text{if } \hat{u} \in K_0, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned} \quad (13)$$

where the fact $P \geq 0$ was used.

Lemma 4. *For $P \geq 0$, the functional φ is proper, lower semicontinuous on H_0 and convex.*

Proof. Since $\varphi(\hat{u}) = \langle \operatorname{ess\,sup}(\hat{u} - \bar{\psi}) - (\hat{u} - \bar{\psi}), Pt_0 \rangle$, it is obvious that φ is
360 convex. If a sequence $(u_n)_{n \in \mathbb{N}}$ converges strongly in $H^{1/2}(-1, 1)$ (or even in $L^2(-1, 1)$) towards u and, if:

$$\forall n \in \mathbb{N}, \quad \text{for a.a. } x \in]-1, 1[, \quad u_n(x) \leq M,$$

for some $M \in \mathbb{R}$, then:

$$\text{for a.a. } x \in]-1, 1[, \quad u(x) \leq M.$$

Therefore:

$$\operatorname{ess\,sup}_{]-1, 1[} u \leq \liminf_{n \rightarrow +\infty} \operatorname{ess\,sup}_{]-1, 1[} u_n,$$

and the functional $\hat{u} \mapsto \operatorname{ess\,sup}_{]-1, 1[} \hat{u}$ is lower semicontinuous on $H^{1/2}(-1, 1)$.

365 This entails the lower semicontinuity of φ on H_0 . \square

Proposition 5. Given $P > 0$, $\bar{u}_n \in H_0$ and $\tilde{t}_n \in H'_0$, the contact conditions (11) can be equivalently rewritten as (i) or (ii) with:

$$(i) \quad \tilde{t}_n \in K'_0 \quad \text{and} \quad \forall \hat{t} \in K'_0, \quad \left\langle \hat{t} - \tilde{t}_n, \bar{u}_n - \bar{\psi} \right\rangle_{H'_0, H_0} \geq 0,$$

$$(ii) \quad \bar{u}_n \in K_0 \quad \text{and} \quad \forall \hat{u} \in K_0, \quad \left\langle \tilde{t}_n, \hat{u} - \bar{u}_n \right\rangle_{H'_0, H_0} + \varphi(\hat{u}) - \varphi(\bar{u}_n) \geq 0.$$

370 **Proof.**

The fact that the contact conditions (11) imply (i) is obvious. Reciprocally, assume that (i) is true. Since $\tilde{t}_n \in K'_0$, $\tilde{t}_n - Pt_0$ is an element of $H^{-1/2}(-1, 1)$ which is also a nonpositive measure. Pick $\bar{u}_n, \bar{\psi} \in H^{1/2}(-1, 1)$ in the respective equivalence classes. Since $t_0 \in H^{-1/2}(-1, 1)$, there exists a unique $C \in \mathbb{R}$ such that:

$$PC = \left\langle \tilde{t}_n - Pt_0, \bar{u}_n - \bar{\psi} \right\rangle_{H^{-1/2}, H^{1/2}},$$

that is:

$$\left\langle \tilde{t}_n - Pt_0, \bar{u}_n - \bar{\psi} + C \right\rangle_{H^{-1/2}, H^{1/2}} = 0. \quad (14)$$

Therefore:

$$\forall \hat{t} \in K'_0, \quad \left\langle \hat{t} - Pt_0, \bar{u}_n - \bar{\psi} + C \right\rangle_{H^{-1/2}, H^{1/2}} \geq 0. \quad (15)$$

Set:

$$S \stackrel{\text{def}}{=} \text{Supp} \langle \bar{u}_n(x) - \bar{\psi}(x) + C \rangle^+,$$

where $\langle x \rangle^+ = \max\{x, 0\}$ stands for the positive part. If the Lebesgue measure of S was not equal to zero, then, choosing:

$$\hat{t}(x) = \begin{cases} Pt_0(x), & \text{if } x \in]-1, 1[\setminus S, \\ -\frac{1}{\text{meas } S} \int_{]-1, 1[\setminus S} Pt_0, & \text{if } x \in S, \end{cases}$$

in formula (15) would lead to contradiction. Therefore $\text{meas } S = 0$, that is $\bar{u}_n - \bar{\psi} + C \leq 0$ almost everywhere in $]-1, 1[$. Finally, $\bar{u}_n - \bar{\psi} + C$ is integrable with respect to the measure $\tilde{t}_n - Pt_0$, thanks to formula (14), and $(\tilde{t}_n - Pt_0)(\bar{u}_n - \bar{\psi} + C)$ is nothing but the null measure.

385 Let us now prove the equivalence between the contact condition (11) and statement (ii). First note that $\tilde{t} = Pt_0$ being impossible (since $P > 0$), the constant C appearing in (11) is unique and must equal $-\text{ess sup}(\bar{u}_n - \bar{\psi})$. It is therefore obvious that the contact condition (11) implies statement (ii). Reciprocally, let us assume that statement (ii) holds true. Applying it with the two choices $\hat{u} = \bar{\psi}$ and $\hat{u} = \bar{\psi} + 2(\bar{u}_n - \bar{\psi})$ gives:

$$\left\langle \tilde{t}_n - Pt_0, \bar{u}_n - \bar{\psi} - \text{ess sup}(\bar{u}_n - \bar{\psi}) \right\rangle_{H^{-1/2}, H^{1/2}} = 0.$$

Therefore:

$$\forall \hat{u} \in K_0, \quad \left\langle \tilde{t}_n - Pt_0, \hat{u} - \bar{\psi} - \text{ess sup}(\hat{u} - \bar{\psi}) \right\rangle_{H^{-1/2}, H^{1/2}} \geq 0,$$

and, since $\tilde{t}_n \in H'_0$, $P > 0$:

$$\forall \hat{u} \in K_0, \quad \forall C \in \mathbb{R}^+ \quad \left\langle \tilde{t}_n - Pt_0, \hat{u} - \bar{\psi} - \text{ess sup}(\hat{u} - \bar{\psi}) - C \right\rangle_{H^{-1/2}, H^{1/2}} \geq 0,$$

which can be summarized as:

$$\forall \hat{u} \in H^{1/2}(-1, 1) \text{ such that: } \hat{u} \leq 0, \quad \left\langle \tilde{t}_n - Pt_0, \hat{u} \right\rangle_{H^{-1/2}, H^{1/2}} \geq 0.$$

This is nothing but $\tilde{t}_n - Pt_0 \leq 0$. Finally, take an arbitrary function $\rho \in \mathcal{D}(-1, 1]$. The function:

$$\hat{u} = \bar{\psi} + ((\max \rho)\chi_{]-1, 1[} - \rho)(\bar{u}_n - \bar{\psi} - \text{ess sup}(\bar{u}_n - \bar{\psi})),$$

belongs to K_0 and statement (ii) entails:

$$\left\langle \tilde{t}_n - Pt_0, (\bar{u}_n - \bar{\psi} - \text{ess sup}(\bar{u}_n - \bar{\psi}))\rho \right\rangle_{H^{-1/2}, H^{1/2}} \leq 0,$$

and since this is true also for $-\rho$, this inequality is actually an identity. The last statement in the contact condition (11) is therefore proved. \square

Remark. Since φ is a proper, convex, lower semicontinuous function on H_0 (lemma 4), statement (ii) in proposition 5 can be expressed in the language of subdifferentials as:

$$-\tilde{t}_n \in \partial\varphi[\bar{u}_n].$$

Similarly, setting:

$$\varphi^*(-\hat{t}) \stackrel{\text{def}}{=} -\left\langle \bar{\psi}, \hat{t} \right\rangle_{H_0, H'_0} + I_{K'_0}(\hat{t}), \quad (16)$$

(where I_S denotes the indicator function of S , that is, taking the value 0 on S and $+\infty$ outside), statement (i) reads as:

$$\bar{u}_n \in \partial\varphi^*[-\tilde{t}_n],$$

showing that φ is nothing but the Legendre-Fenchel conjugate of φ^* (up to some additive constant), and reciprocally.

Making use of the above definition and results, we have the following weak formulation for the formal problem (3) governing the normal component t_n (or equivalently \hat{t}_n) of the surface traction exerted by the moving obstacle.

Problem I'. Let $f \in \mathbb{R}$, P be positive and $\bar{\psi} \in H_0$. Find $\tilde{t}_n \in K'_0$ such that:

$$\forall \hat{t} \in K'_0, \quad b(\tilde{t}_n, \hat{t} - \tilde{t}_n) \geq \left\langle \bar{\psi}, \hat{t} - \tilde{t}_n \right\rangle_{H_0, H'_0}.$$

Straightforwardly applying the Lions-Stampacchia theorem (see [1] or theorem 25 of which the Lions-Stampacchia theorem is a particular case) gives:

Proposition 6. *Problem I' has a unique solution.*

To obtain the dual formulation of problem I', we first construct the dual bilinear form defined by:

$$a(u_1, u_2) \stackrel{\text{def}}{=} \langle u_1, \mathcal{L}(u_2) \rangle = b(\mathcal{L}(u_1), \mathcal{L}(u_2)).$$

Since $\mathcal{L} : H_0 \rightarrow H'_0$ is continuous and one-to-one, a is continuous and coercive on H_0 , thanks to the open mapping theorem. This definition associated with the dual form of the contact conditions based on proposition 5, makes it possible to give the dual formulation of problem I':

Problem I. Let $f \in \mathbb{R}$, P be positive and $\bar{\psi} \in H_0$. Find $\bar{u}_n \in K_0$ such that:

$$\forall \hat{u} \in K_0, \quad a(\hat{u} - \bar{u}_n, \bar{u}_n) + \varphi(\hat{u}) - \varphi(\bar{u}_n) \geq 0,$$

where $t_0(x)$ is the function defined by formula (8).

Proposition 7. *Problem I has a unique solution. The unique solutions \tilde{t}_n and \bar{u}_n of problems I' and I are linked by the relationship:*

$$\tilde{t}_n = \mathcal{L}(\bar{u}_n).$$

Proof. Note that the mapping $\hat{u} \mapsto \langle \text{ess sup}(\hat{u}) - \hat{u}, Pt_0 \rangle$ is not linear, so that the Lions-Stampacchia theorem cannot be directly applied to yield a solution of problem I. However, the extension of the Lions-Stampacchia theorem that is provided in appendix A (theorem 25) together with lemma 4 yields a unique solution of problem I. Another proof can also be proposed. Indeed, taking \tilde{t}_n the unique solution of problem I' provided by proposition 6 and setting:

$$\bar{u}_n = \mathcal{L}^{-1}(\tilde{t}_n),$$

we get a solution of problem I, thanks to proposition 5. This solution is the only one, thanks to the coercivity of the bilinear form a . \square

Summing up, denoting by $\text{pv } 1/x$ the distributional derivative of the locally integrable function $\log|x|$, we have proved the following theorem.

Theorem 8. *Let $f \in \mathbb{R}$, P be positive and $\bar{\psi} \in H^{1/2}(-1, 1)$. There exists a unique $(\bar{u}_n, t_n) \in H^{1/2}(-1, 1) \times H^{-1/2}(-1, 1)$ such that:*

$$\begin{aligned} -\frac{1}{\pi} t_n * \text{pv } 1/x + \gamma f t_n &= \bar{u}'_n, & \text{in }]-1, 1[, \\ \bar{u}_n - \bar{\psi} &\leq 0, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n = 0, \\ \left\langle t_n, \chi_{]-1, 1[} \right\rangle_{H^{-1/2}, H^{1/2}} &= -P. \end{aligned}$$

Finally, if $\bar{\psi}$ is replaced by $\bar{\psi} + C$ ($C \in \mathbb{R}$), then the corresponding solution is $(\bar{u}_n + C, t_n)$.

To conclude this section about homogeneous friction, we establish the form of the universal singularities that should be expected in the solutions of problems I and I', in the spirit of the singularities of the solutions of linear elasticity in polyhedra, as discussed by Grisvard [9].

The following theorem shows that a logarithmic singularity must be expected at a wedge of the moving indenter.

Theorem 9. *Assume that $\bar{\psi}'$ is piecewise Lipschitz-continuous. If the unique solution of the problem in theorem 8 is such that contact is achieved in a neighbourhood of $c \in]-1, 1[$, then:*

$$t_n = \frac{\bar{\psi}'(c+) - \bar{\psi}'(c-)}{1 + (\gamma f)^2} \left(\frac{\log |x - c|}{\pi} + \gamma f H(x - c) \right) + \theta,$$

where H is the Heaviside function and θ is a measure whose restriction to a neighbourhood of c is a continuous function.

Proof. Set:

$$\bar{\phi}(x) = \bar{\psi}'(c-) (x - c) \chi_{]-1, c[}(x) + \bar{\psi}'(c+) (x - c) \chi_{]c, 1[}(x),$$

where χ_S is the indicator function of the set S . We have the decomposition:

$$t_n = \mathcal{L}(\bar{\phi}) + \mathcal{L}(\bar{\psi} - \bar{\phi}) + \mathcal{L}(\bar{u}_n - \bar{\psi}) - P t_0.$$

Since there exists $\epsilon > 0$ such that $]c - \epsilon, c + \epsilon[$ does not meet the support of $\bar{u}_n - \bar{\psi}$, the restriction of $\mathcal{L}(\bar{u}_n - \bar{\psi})$ to $]c - \epsilon, c + \epsilon[$ is a C^∞ function. Besides, the function $\bar{\psi}' - \bar{\phi}'$ is Lipschitz-continuous in a neighbourhood of c which entails that the function $\mathcal{L}(\bar{\psi} - \bar{\phi})$ is continuous in a neighbourhood of c , thanks to formula (10) and proposition 7 of [5]. Therefore:

$$t_n = \mathcal{L}(\bar{\phi}) + \tilde{\theta},$$

for some measure $\tilde{\theta}$ whose restriction to a neighbourhood of c is a continuous function. The claim in the theorem is now a straightforward consequence of formula (10) and propositions 7 and 8 of [5]. \square

Remark. Since t_n is a nonpositive measure, proposition 9 entails in particular that, if active contact is achieved around a wedge of the indenter, then this wedge must satisfy:

$$\bar{\psi}'(c+) \geq \bar{\psi}'(c-),$$

otherwise (reentrant corner of the obstacle), it is impossible for active contact to occur all over a neighbourhood of c .

The following theorem describes the singularities that should be expected at the boundary of the contact zone and shows in particular that the front and rear power singularities encountered at the edges of the rigid flat punch are universal.

Theorem 10. Assume that $\bar{\psi}'$ is piecewise Lipschitz-continuous. If the unique solution of problem I is such that contact is achieved in a right-neighbourhood of $c \in [-1, 1[$, that is, $\bar{u}_n = \bar{\psi}$ in $]c, c + \epsilon[$, then the following estimates at c hold true.

- In the case where $c = -1$, we have:

$$t_n = -\frac{\theta_-}{(1+x)^{1/2+\alpha}},$$

where

$$\alpha \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(\gamma f) \quad (\in]-1/2, 1/2[),$$

and θ_- is a measure whose restriction to a right-neighbourhood of $x = -1$ is a Hölder-continuous function. In the case where $\theta_-(-1) > 0$, then the following estimate holds true:

$$\bar{u}'_n(x) \sim -\frac{\theta_-(-1) \sqrt{1 + \gamma^2 f^2}}{|1+x|^{1/2+\alpha}}, \quad \text{as } x \rightarrow -1 - .$$

- In the case where $c > -1$ and detachment occurs in a left-neighbourhood of c , that is $\bar{u}_n < \bar{\psi}$ in $]c - \epsilon, c[$, we have:

$$t_n = -|c-x|^{1/2-\alpha} \theta_-$$

where θ_- is a measure whose restriction to a right-neighbourhood of $x = c$ is a Hölder-continuous function.

Similarly, if the unique solution of problem I is such that contact is achieved in a left-neighbourhood of $c \in]-1, 1]$, that is, $\bar{u}_n = \bar{\psi}$ in $]c - \epsilon, c[$, then the following estimates at c hold true.

- In the case where $c = 1$, we have:

$$t_n = -\frac{\theta_+}{(1-x)^{1/2-\alpha}},$$

where θ_+ is a measure whose restriction to a left-neighbourhood of $x = 1$ is a Hölder-continuous function. In the case where $\theta_+(1) > 0$, then the following estimate holds true:

$$\bar{u}'_n(x) \sim \frac{\theta_+(1) \sqrt{1 + \gamma^2 f^2}}{|1-x|^{1/2-\alpha}}, \quad \text{as } x \rightarrow 1 + .$$

- In the case where $c < 1$ and detachment occurs in a right-neighborhood of c , that is $\bar{u}_n < \bar{\psi}$ in $]c, c + \epsilon[$, we have:

$$t_n = -|x-c|^{1/2+\alpha} \theta_+$$

where θ_+ is a measure whose restriction to a left-neighbourhood of $x = c$ is a Hölder-continuous function.

Proof. The proof will be developed only for the case where contact occurs in a right-neighbourhood of c . We first consider the case where $c = -1$. Set:

$$\bar{\phi}(x) \equiv (x+1)\bar{\psi}'(-1+).$$

We have the decomposition:

$$t_n = \mathcal{L}(\bar{\phi}) + \mathcal{L}(\bar{\psi} - \bar{\phi}) + \mathcal{L}(\bar{u}_n - \bar{\psi}) - P t_0.$$

Since, $\bar{u}_n - \bar{\psi}$ is supported in $[-1 + \epsilon, 1]$, the restriction to $] -1, -1 + \epsilon[$ of the term $\mathcal{L}(\bar{u}_n - \bar{\psi})$ is readily seen to equal the product of t_0 with a C^∞ function. Therefore, the claim in the proposition holds true for that term. The term $\mathcal{L}(\bar{\psi} - \bar{\phi})$ can be developed by means of formula (10). The Cauchy integral in that expression is seen to be the product of t_0 with some Cauchy integral of the form:

$$\oint_{-1}^1 \frac{(1+x')^{1+\beta} \eta(x')}{x' - x} dx',$$

for some positive β and some Hölder-continuous function η . Splitting

$$(1+x')^{1+\beta} = (1+x)(1+x')^\beta + (x'-x)(1+x')^\beta,$$

it entails:

$$\mathcal{L}(\bar{\psi} - \bar{\phi}) = \left(a' + (1+x)\tilde{\theta} \right) t_0,$$

for some constant $a' \in \mathbb{R}$ and some Hölder-continuous function $\tilde{\theta}$, thanks to proposition 7 of [5]. Besides, the term $\mathcal{L}(\bar{\phi})$ is:

$$\begin{aligned} \mathcal{L}(\bar{\phi}) &= \frac{\bar{\psi}'(-1+)}{1 + (\gamma f)^2} \left\{ \gamma f - \frac{1}{\pi} \oint_{-1}^1 \frac{(1+x')^{\frac{1}{2}+\alpha} (1-x')^{\frac{1}{2}-\alpha}}{(1+x)^{\frac{1}{2}+\alpha} (1-x)^{\frac{1}{2}-\alpha}} \cdot \frac{dx'}{x' - x} \right\}, \\ &= \left(a'' + b''(1+x) \right) t_0(x), \end{aligned}$$

thanks to formula (19) in [5]. Finally, it has been proved a little bit more than claimed:

$$t_n = - \frac{a + (1+x)\theta}{(1+x)^{1/2+\alpha}},$$

for some constant $a \in \mathbb{R}$ and some measure θ whose restriction to a right-neighbourhood of $c = -1$ is a Hölder-continuous function. This stronger result will be of use to prove the claim in the case $c > -1$. Since, for $x < -1$:

$$\bar{u}'_n(x) = \frac{1}{\pi} \int_{-1}^1 \frac{t_n(x')}{x' - x},$$

the estimate for $\bar{u}'_n(x)$ is now a straightforward consequence of proposition 8 in [5], and:

$$\sin \pi(1/2 + \alpha) = \frac{1}{\sqrt{1 + \gamma^2 f^2}}.$$

It remains to prove the claim for the case $c > -1$. By hypothesis, t_n vanishes
 510 on $]c - \epsilon, c[$ (taking ϵ smaller if necessary). We denote by \tilde{t}_n the restriction of t_n
 to $] -1, c - \epsilon[$. Then, we denote by $\tilde{u}_n \in C^\infty([c, 1])$, the function defined by:

$$\tilde{u}_n(x) = \frac{1}{\pi} \int_{-1}^{c-\epsilon} \frac{\tilde{t}_n(x')}{x' - x},$$

so that the function $\bar{u}_n - \tilde{u}_n$ defined on $[c, 1]$ is the unique solution of the problem I
 where $\bar{\psi}$ is replaced by $\bar{\psi} - \tilde{u}_n$, P is replaced by $P + \int_{-1}^{c-\epsilon} \tilde{t}_n$ and $] -1, 1[$ is replaced
 515 by $]c, 1[$. Hence, c is the left edge for this new contact problem and the results
 of the first part of the proof can be applied, which gives, for $x \in]c, 1[$:

$$t_n = -\frac{a + (x - c)\theta}{(x - c)^{1/2+\alpha}},$$

for some constant $a \in \mathbb{R}$ and some measure θ whose restriction to a right-
 neighbourhood of c is a Hölder-continuous function. We must have $a \geq 0$. If
 $a \neq 0$, then:

$$\bar{u}'_n(x) \sim -\frac{a \sqrt{1 + \gamma^2 f^2}}{(c - x)^{1/2+\alpha}}, \quad \text{as } x \rightarrow c - .$$

But this is incompatible with $\bar{u}_n - \bar{\psi} \leq 0$ on $] -1, 1[$. Therefore $a = 0$ and the
 520 expected claim is proved. \square

4. Analysis of the case where $f(x) = f_- \chi_{]-1, 0[}(x) + f_+ \chi_{]0, 1[}(x)$

In this section, we focus on the case where the friction coefficient has only
 one jump located at $x = 0$. The complete understanding of that situation will
 provide all the necessary tools to handle the more general situation where the
 525 friction coefficient $f(x)$ is an arbitrary step function (synonymously, piecewise
 constant function). From now on, we set:

$$s(x) \stackrel{\text{def}}{=} \gamma f(x),$$

so that the steady sliding frictional contact problem to solve now reads formally
 as:

- $\frac{1}{\pi} \oint_{-1}^1 \frac{t_n(t)}{t - x} dt + s(x) t_n(x) = \bar{u}'_n(x),$
- $\exists C \in \mathbb{R}, \quad \bar{u}_n + C \leq \bar{\psi}, \quad t_n \leq 0, \quad (\bar{u}_n + C - \bar{\psi}) t_n \equiv 0,$
- $\int_{-1}^1 t_n(t) dt = -P,$

and we focus on the case where:

$$s(x) = \begin{cases} s_- & \text{if } x \in] -1, 0[, \\ s_+ & \text{if } x \in]0, 1[, \end{cases}$$

for two real constants $s_-, s_+ \in \mathbb{R}$. Since the product $s\hat{t}$ is generally undefined for $\hat{t} \in H^{-1/2}(-1, 1)$, an appropriate functional framework must be designed.
 530 This is the object of the next section.

4.1. A functional framework

To be able to formulate the corresponding problem, we introduce the space \overline{H}' defined as the completion of $L^2(-1, 1)$ (or $C^\infty([-1, 1])$) in the norm:

$$\|\varphi\|_{\overline{H}'} \stackrel{\text{def}}{=} \left(\|\varphi\|_{H^{-1/2}}^2 + \left(\int_0^1 \varphi \right)^2 \right)^{1/2}.$$

It is readily checked that \overline{H}' is a Hilbert space which is contained in $H^{-1/2}$, and
 535 that the two linear forms defined on $C^\infty([-1, 1])$ by:

$$\varphi \mapsto \int_{-1}^0 \varphi, \quad \varphi \mapsto \int_0^1 \varphi,$$

extend into continuous linear forms over \overline{H}' (this last result is not true if \overline{H}' is replaced with $H^{-1/2}$). In the sequel, we will commit the abuse of keeping the above integral notation for general $\varphi \in \overline{H}'$. The dual space \overline{H} of \overline{H}' is:

$$\overline{H} = \left\{ u + C \chi_{]0,1[} \mid u \in H^{1/2}(-1, 1) \text{ and } C \in \mathbb{R} \right\},$$

where $\chi_{]0,1[}$ is the function that takes identically the value 1 on $]0, 1[$ and 0
 540 outside.

Proposition 11. *The following embeddings are continuous:*

$$H^{-1/2}(-1, 0) \times H^{-1/2}(0, 1) \subset \overline{H}' \subset H^{-1/2}(-1, 1).$$

In addition, these three spaces are distinct.

Proof. The restriction operator from $H^{1/2}(-1, 1)$ into $H^{1/2}(0, 1)$ is obviously continuous. Going to the dual side, it gives the existence of a real constant C such that:

$$\begin{aligned} \forall \varphi \in H^{-1/2}(-1, 0) \times H^{-1/2}(0, 1), \\ \|\varphi\|_{H^{-1/2}(-1, 1)} \leq C \|\varphi\|_{H^{-1/2}(-1, 0) \times H^{-1/2}(0, 1)}. \end{aligned}$$

The continuity of the second embedding is trivial and that of the first one is easily deduced from the latter estimate.

545 For all $C \in \mathbb{R}$, the function $C + \log|\log|x/2||$ is in $H^{1/2}(-1, 1)$ (thanks to the trace theorem and the fact that $C + \log|\log|\sqrt{x^2 + y^2}/2||$ is in H^1 of the unit disk). However, based on theorem 31, the function defined by:

$$\begin{cases} C + \log|\log|x/2||, & \text{if } x \in]-1, 0[, \\ 0, & \text{if } x \in]0, 1[, \end{cases}$$

is not in $H^{1/2}(-1, 1)$. Therefore, the function:

$$\begin{cases} \log|\log|x/2||, & \text{if } x \in]-1, 0[, \\ 0, & \text{if } x \in]0, 1[, \end{cases}$$

is not in \overline{H} , whereas it is obviously in $H^{1/2}(-1, 0) \times H^{1/2}(0, 1)$. These two spaces are therefore distinct and the same is true of their dual spaces \overline{H}' and $H^{-1/2}(-1, 0) \times H^{-1/2}(0, 1)$. For the second embedding, the function u defined by:

$$u(x) = \begin{cases} \log|\log|x/2|| - \log|\log|1/2||, & \text{if } x \in]-1, 1[, \\ 0, & \text{if } x \notin]-1, 1[, \end{cases}$$

is in $H^{1/2}(\mathbb{R})$. The sequence (u_n) defined by:

$$u_n(x) = \begin{cases} 0, & \text{if } x \notin]-1, 1[, \\ \log|\log|1/(2(n+1))|| - \log|\log|1/2||, & \text{if } x \in \left] -\frac{1}{(n+1)}, \frac{1}{(n+1)} \right[, \\ \log|\log|x/2|| - \log|\log|1/2||, & \text{otherwise,} \end{cases}$$

converges towards u in $H^{1/2}(\mathbb{R})$. Therefore, the sequence (u'_n) converges towards u' in $H^{-1/2}(\mathbb{R})$. Since all the u'_n are supported in $[-1, 1]$, the sequence of the restrictions to $[-1, 1]$ of each u'_n converges in $H^{-1/2}(-1, 1)$ towards the restriction of u' . However:

$$\int_0^1 u'_n = \log|\log|1/2|| - \log|\log|1/(2(n+1))||,$$

is unbounded. Finally, u' is in $H^{-1/2}(-1, 1)$, but not in \overline{H}' . \square

Theorem 12. We denote by $H_{00}^{1/2}(]0, 1[)$ the Hilbert space (see theorem 31) of all the functions in $H^{1/2}(]0, 1[)$ such that the extension by zero is in $H^{1/2}(]-1, 1[)$. Let $t \in \overline{H}'$ be arbitrary. We consider, first, the restriction $t|_{]0, 1[}$ of t to $]0, 1[$ and then, the extension \bar{t} of this restriction, by zero to the whole real line \mathbb{R} . Then, \bar{t} is a distribution on \mathbb{R} (such a result fails to be true for $t \in H^{-1/2}(-1, 1)$ in general). Therefore, denoting by $H(\cdot)$ the Heaviside function, the convolution product $H * \bar{t}$ is well-defined in the sense of Schwartz's distributions. In addition, the restriction of $H * \bar{t}$ to the interval $]0, 1[$ is in $H_{00}^{1/2}(]0, 1[)$, and the continuity estimate:

$$\forall t_1, t_2 \in \overline{H}', \quad \langle \bar{t}_1, H * \bar{t}_2 \rangle \leq M \|t_1\|_{\overline{H}'} \|t_2\|_{\overline{H}'},$$

holds true for some real constant M independent of t_1 and t_2 .

Proof. Let $t \in \overline{H}'$ and $\varphi \in C_c^\infty(\mathbb{R})$ be arbitrary. The restriction of $\varphi(\cdot) - \varphi(0)$ to $]0, 1[$ is in $H_{00}^{1/2}(]0, 1[)$ and we therefore have:

$$\begin{aligned} \langle \bar{t}, \varphi - \varphi(0) \rangle &\leq C \|\varphi - \varphi(0)\|_{H_{00}^{1/2}(]0, 1[)}, \\ &\leq C' \|\varphi - \varphi(0)\|_{C^1([-1, 1])}, \end{aligned}$$

for some constants $C, C' \in \mathbb{R}$. Since:

$$\langle \bar{t}, \varphi(0) \rangle = \langle \bar{t}, \varphi(0) \chi_{]0,1[} \rangle = \varphi(0) \int_0^1 \bar{t} = \varphi(0) \int_0^1 t,$$

570 it is readily seen that \bar{t} defines a Schwartz distribution (of order 1) on \mathbb{R} . Note that if t is only in $H^{-1/2}(-1, 1)$ (instead of \overline{H}'), then the restriction $t|_{]0,1[}$ of t to $]0, 1[$ is only in $H_{00}^{1/2}'(]0, 1[)$ which does not make it possible to define $\langle t|_{]0,1[}, \chi_{]0,1[} \rangle$ in general.

Now, take $t \in \overline{H}'$ arbitrary and φ_n a sequence in $C_c^\infty(]-1, 1[)$ that converges 575 strongly towards t in \overline{H}' . The sequence $H * \varphi_n = \int_{-1}^x \varphi_n$ converges towards $H * t$ in the sense of distributions. Similarly, $H * \bar{\varphi}_n$ converges towards $H * \bar{t}$ in the sense of distributions. Thanks to the use of the Fourier transform, the continuity property:

$$\forall t \in H^{-1/2}(-1, 1), \quad \|H * t\|_{H^{1/2}(-1,1)} \leq M \|t\|_{H^{-1/2}(-1,1)},$$

holds true, for some real constant independent of t . But, $H * \bar{\varphi}_n$ is nothing but 580 the C^∞ function equalling 0 on $] -1, 0[$ and $H * \varphi_n - \int_{-1}^0 \varphi_n$ on $]0, 1[$. Therefore,

$$\|H * \bar{\varphi}_n\|_{H^{1/2}(-1,1)} \leq M \|\varphi_n\|_{H^{-1/2}(-1,1)} + \left| \int_{-1}^0 \varphi_n \right| \leq M' \|\varphi_n\|_{\overline{H}'},$$

for some constant M' independent of n . Hence, $H * \bar{\varphi}_n$ converges to some function in $H^{1/2}(-1, 1)$ which is supported in $[0, 1[$. This limit must equal $H * \bar{t}$. We therefore have proved:

$$\forall t \in \overline{H}', \quad \|H * \bar{t}\|_{H_{00}^{1/2}(0,1)} \leq M \|t\|_{\overline{H}'},$$

for some (other) real constant M independent of t . This is enough to get the 585 whole claim in the theorem. \square

Theorem 13. *The bilinear mapping defined for $\varphi_1, \varphi_2 \in C^\infty([-1, 1])$ by:*

$$(\varphi_1, \varphi_2) \mapsto \int_{-1}^1 \varphi_1(x) \int_{-1}^x s(x') \varphi_2(x') dx' dx,$$

defines a continuous bilinear mapping on $\overline{H}' \times \overline{H}'$.

Proof.

Step 1. *Case where $s(x)$ is constant.*

590 Let us note that in the particular case where $s(x)$ is constant, the linear mapping under consideration is nothing but:

$$s \langle \varphi_1, \varphi_2 * H \rangle_{H^{-1/2}(-1,1), H^{1/2}(-1,1)}$$

where $H(x)$ denotes the Heaviside function and $\varphi_2 * H$ denotes the convolution product of H with the extension by 0 on the whole line of φ_2 . The extension of an arbitrary distribution in $H^{-1/2}(-1, 1)$ by 0 outside $] -1, 1[$ is in $H^{-1/2}(\mathbb{R})$. The convolution with $H(x)$ is in $H_{\text{loc}}^{1/2}(\mathbb{R})$ (because its derivative is in $H^{-1/2}(\mathbb{R})$) and the restriction of this convolution product to $] -1, 1[$ is therefore in $H^{1/2}(] -1, 1[)$. This bilinear mapping is therefore continuous on $H^{-1/2}(-1, 1) \times H^{-1/2}(-1, 1)$. By proposition 11, it is therefore continuous on $\overline{H}' \times \overline{H}'$. Hence, it is now sufficient to prove theorem 13 in the particular case where $s_- = 0$ and $s_+ = 1$ which will be assumed from now on.

Step 2. *Case where $s_- = 0$ and $s_+ = 1$.*

Take $t_1, t_2 \in \overline{H}'$. With the same notations as the ones in theorem 12, the bilinear form under consideration is nothing but:

$$\langle \bar{t}_1, H * \bar{t}_2 \rangle \leq M \|t_1\|_{\overline{H}'} \|t_2\|_{\overline{H}'},$$

which is well-defined and continuous, thanks to theorem 12. \square

Theorem 13 shows that the bilinear form:

$$\bar{b}(t_1, t_2) \stackrel{\text{def}}{=} -\frac{1}{\pi} \langle t_1 * \log |x|, t_2 \rangle_{H^{1/2}, H^{-1/2}} + \left\langle \int_{-1}^x st_1, t_2 \right\rangle_{\overline{H}, \overline{H}'},$$

is continuous on \overline{H}' . As previously, we focus on the closed subspace \overline{H}'_0 of \overline{H}' defined by:

$$\overline{H}'_0 \stackrel{\text{def}}{=} \left\{ \hat{t} \in \overline{H}' \mid \langle \hat{t}, \chi_{]-1, 1[} \rangle_{\overline{H}', \overline{H}} = 0 \right\}.$$

In order to analyse the coercivity of \bar{b} on \overline{H}'_0 , we are led to perform the following calculation, for arbitrary $p \in C^\infty([-1, 1])$:

$$\begin{aligned} & -\frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 p(x) p(t) \log |x-t| dt dx + \int_{-1}^1 \int_{-1}^x s(t) p(t) p(x) dt dx + \\ C \left(\int_{-1}^1 sp \right) \left(\int_{-1}^1 p \right) &= -\frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 p(x) p(t) \log |x-t| dt dx + \frac{s_+ - s_-}{2} \left(\int_0^1 p \right)^2 \\ &+ \frac{s_-(1+2C)}{2} \left(\int_{-1}^1 p \right)^2 + (s_+ - s_-) C \left(\int_{-1}^1 p \right) \left(\int_0^1 p \right). \quad (17) \end{aligned}$$

A surprising fact happens: under the condition that $s_+ > s_-$, the bilinear form is coercive on \overline{H}'_0 , but it is not even nonnegative on \overline{H}'_0 in the case $s_+ < s_-$ (otherwise, one would have $\overline{H}' = H^{-1/2}$, which is not true thanks to proposition 11). We must therefore study these two cases separately.

4.2. Analysis of the case $s_+ > s_-$

Based on theorem 36 in appendix C, the function

$$t_0(x) = \frac{e^{\tau(x)}}{\pi(1-x)\sqrt{1+s^2(x)}},$$

where:

$$\tau(x) = \frac{1}{\pi} \oint_{-1}^1 \arctan(-1/s(x')) \frac{dx'}{x' - x},$$

615 is in $L^p(-1, 1)$ for some $p > 1$, is positive and satisfies:

$$\forall \hat{t} \in \overline{H}'_0, \quad \bar{b}(t_0, \hat{t}) = 0.$$

In addition,

$$\int_{-1}^1 t_0(x') dx' = 1$$

thanks to the identity proved in the step 1 of the proof of theorem 14 of [5] (used with the choices $\lambda = -1$ and $x = 1$). As previously, the function t_0 is going to be used to make a shift of the unknown in the steady sliding frictional contact problem. We set:

$$\overline{K}'_0 \stackrel{\text{def}}{=} \left\{ \hat{t} \in \overline{H}'_0 \mid \hat{t} - Pt_0 \leq 0 \right\},$$

Since t_0 is positive, \overline{K}'_0 is a nonempty closed convex subset of \overline{H}'_0 . The dual space \overline{H}_0 of \overline{H}'_0 is the quotient space \overline{H}/\mathcal{C} (where \mathcal{C} is the line of all the constant functions on $] -1, 1[$).

Making the shift of unknown:

$$t_n = \tilde{t}_n - Pt_0,$$

625 the steady sliding frictional contact problem now reads as follows.

Problem II'. Let $\bar{\psi} \in \overline{H}_0 \supset H^{1/2}(-1, 1)/\mathcal{C}$. Find $\tilde{t}_n \in \overline{K}'_0$ such that:

$$\forall \hat{t} \in \overline{K}'_0, \quad \bar{b}(\tilde{t}_n, \hat{t} - \tilde{t}_n) \geq \left\langle \bar{\psi}, \hat{t} - \tilde{t}_n \right\rangle_{\overline{H}_0, \overline{H}'_0}.$$

Since in the case $s_+ > s_-$, \bar{b} is continuous and coercive on \overline{H}'_0 , we have already proved the

Proposition 14. *Assume that $s_+ > s_-$, then problem II' has a unique solution.*

630 For arbitrary $u \in \overline{H}_0$, there exists a unique $\tilde{t} \in \overline{H}'_0$ such that:

$$\forall \hat{t} \in \overline{H}'_0, \quad \bar{b}(\tilde{t}, \hat{t}) = \left\langle u, \hat{t} \right\rangle_{\overline{H}_0, \overline{H}'_0}.$$

Setting $\tilde{t} = \overline{\mathcal{L}}(u)$, we obtain a continuous linear mapping from \overline{H}_0 onto \overline{H}'_0 (take $\tilde{t} \in \overline{H}'_0$ arbitrary, then $\hat{t} \mapsto \bar{b}(\tilde{t}, \hat{t})$ defines a continuous linear form $u \in \overline{H}'_0 = \overline{H}_0$ and we have $\tilde{t} = \overline{\mathcal{L}}(u)$). The space $H_0 = H^{1/2}/\mathcal{C}$ is identified to a closed subspace (with codimension 1) of \overline{H}_0 . Thanks to theorem 36, the restriction of

635 $\overline{\mathcal{L}}$ to $W^{1,\infty}(-1, 1)/\mathcal{C}$ is given by:

$$\overline{\mathcal{L}}(u) = \frac{s(x) u'(x)}{1 + s^2(x)} - \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x') (1-x') e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \frac{dx'}{x' - x}, \quad (18)$$

where:

$$\tau(x) = \frac{1}{\pi} \oint_{-1}^1 \arctan(-1/s(x')) \frac{dx'}{x' - x}.$$

Recalling:

$$t_0(x) = \frac{e^{\tau(x)}}{\pi(1-x)\sqrt{1+s^2(x)}}, \quad (19)$$

any $t \in H^{-1/2}(-1, 1)$ can be decomposed into:

$$t = \overline{\mathcal{L}}(u) + t_0 \left\langle t, \chi_{]-1, 1[} \right\rangle_{H^{-1/2}, H^{1/2}},$$

for some $u \in \overline{H}_0$.

640 Remark 1. The step function $\chi_{]0, 1[} + C$ is in \overline{H}_0 and the question of what $\tau \stackrel{\text{def}}{=} \overline{\mathcal{L}}(\chi_{]0, 1[} + C) \in \overline{H}'_0$ is arises. It satisfies:

$$-\frac{1}{\pi} \tau * \text{vp} \frac{1}{x} + s\tau = \delta, \quad \text{in }]-1, 1[,$$

where $\text{vp} 1/x$ is the distributional derivative of $\log|x|$ and δ the Dirac measure. Since the Hilbert transform maps $L^p(-1, 1)$ into $L^p(-1, 1)$ for all $p \in]1, \infty[$, it is readily checked that:

$$\forall p \in]1, \infty[, \quad \tau \notin L^p(-1, 1).$$

645 I do not know any explicit expression for τ .

Remark 2. In problem II' , the function $\bar{\psi}$ describing the geometry of the obstacle is allowed to be not only in $H^{1/2}(-1, 1)/\mathcal{C}$ but in \overline{H}_0 . In particular, $\bar{\psi}$ may have a jump at 0, which is not allowed in problem I' . Hence, the new functional framework displayed in section 4.1 is appropriate to deal with
650 discontinuities not only in the friction coefficient, but also in the shape of the obstacle.

The bilinear form:

$$\bar{a}(u_1, u_2) \stackrel{\text{def}}{=} \left\langle u_1, \overline{\mathcal{L}}(u_2) \right\rangle = \bar{b}\left(\overline{\mathcal{L}}(u_1), \overline{\mathcal{L}}(u_2)\right),$$

is continuous and coercive on \overline{H}_0 (we recall that \overline{H}_0 is the quotient space \overline{H}/\mathcal{C} where \mathcal{C} is the line of all the constant functions on $]-1, 1[$). It is recalled that an
655 element $\hat{u} \in \overline{H}_0$ will be said nonpositive (notation $\hat{u} \leq 0$) if one (and therefore all) element of its equivalence class is (essentially) bounded by above. We define:

$$\overline{K}_0 \stackrel{\text{def}}{=} \left\{ \hat{u} \in \overline{H}_0 \mid \hat{u} - \bar{\psi} \leq 0 \right\},$$

which is a nonempty closed convex subset of \overline{H}_0 and:

$$\begin{aligned} \varphi(\hat{u}) &\stackrel{\text{def}}{=} \begin{cases} \inf_{C|\hat{u}-\bar{\psi}+C\leq 0} \left\langle -Pt_0, \hat{u} - \bar{\psi} + C \right\rangle_{\overline{H}', \overline{H}}, & \text{if } \hat{u} \in \overline{K}_0, \\ +\infty, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \left\langle -Pt_0, \bar{u}_n - \bar{\psi} - \text{ess sup}_{]-1,1[}(\hat{u} - \bar{\psi}) \right\rangle_{\overline{H}', \overline{H}}, & \text{if } \hat{u} \in \overline{K}_0, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

where $P \geq 0$ is assumed and where $t_0(x)$ is the function defined by formula (19). Note that the functional φ is well-defined over the space \overline{H}_0 . Lemma 4 is readily seen to extend to this new definition of φ , so that φ is proper, lower
660 semicontinuous on \overline{H}_0 and convex. Making use of proposition 5 (which can be readily seen to extend to the case where $\bar{\psi} \in \overline{H}_0$), we obtain the dual formulation of problem II':

Problem II. Let $P \in \mathbb{R}$ be positive, and $\bar{\psi} \in \overline{H}_0 \supset H^{1/2}(-1, 1)/\mathcal{C}$. Find $\bar{u}_n \in \overline{K}_0$ such that:

$$\forall \hat{u} \in \overline{K}_0, \quad \bar{a}(\hat{u} - \bar{u}_n, \bar{u}_n) + \varphi(\hat{u}) - \varphi(\bar{u}_n) \geq 0.$$

665 **Proposition 15.** Assume that $s_+ > s_-$, then problem II has a unique solution. The solution $\bar{u}_n \in \overline{H}_0$ of problem II belongs actually to $H^{1/2}(-1, 1)/\mathcal{C}$. The solutions t_n and \bar{u}_n of problems II' and II are linked by the relationship:

$$\tilde{t}_n = \overline{\mathcal{L}}(\bar{u}_n).$$

Proof. The bilinear form \bar{a} is continuous and coercive on \overline{H}_0 . The extension of the Lions-Stampacchia theorem that is provided in appendix A (theorem 25)
670 yields a unique solution of problem II. By proposition 5 (which can be readily seen to extend to the case where $\bar{\psi} \in \overline{H}_0$), we obtain easily:

$$\tilde{t}_n = \overline{\mathcal{L}}(\bar{u}_n).$$

Since $\bar{u}_n \in \overline{H}_0 = (H^{1/2} + \mathbb{R}\chi_{]0,1[})/\mathcal{C}$, we have $\bar{u}'_n = \tilde{u}'_n + C\delta$, for some $\tilde{u}_n \in H^{1/2}/\mathcal{C}$ and $C \in \mathbb{R}$. Therefore:

$$-\frac{1}{\pi} \tilde{t}_n * \text{vp} \frac{1}{x} - \tilde{u}'_n = C\delta - s \tilde{t}_n, \quad \text{in }]-1, 1[,$$

where $\text{vp} 1/x$ is the distributional derivative of $\log|x|$ and δ the Dirac measure.
675 But, \tilde{t}_n being in $H^{-1/2}(-1, 1)$ and \tilde{u}_n being in $H^{1/2}(-1, 1)/\mathcal{C}$, the left member of this equality belongs to $H_{00}^{1/2}'(-1, 1)$. Since t_n is a nonpositive distribution

and is therefore a measure on $] -1, 1[$ which has finite total mass, \tilde{t}_n is a Radon measure which belongs to $H^{-1/2}(-1, 1)$. Hence the right member of the equality is a Radon measure. Thanks to proposition 35, this measure has no atom. Since
680 \tilde{t}_n and therefore $s\tilde{t}_n$ have no atoms, we necessarily have $C = 0$ and the expected conclusion follows. \square

Summing up, we have proved the following theorem.

Theorem 16. *Let $P \in \mathbb{R}$ be positive, $s :] -1, 1[\rightarrow \mathbb{R}$ the function defined by $s = s_- \chi_{]-1, 0[} + s_+ \chi_{]0, 1[}$ with $s_- < s_+$ and $\bar{\psi} \in H^{1/2}(-1, 1) + \mathbb{R}\chi_{]0, 1[}$. Then,
685 there exists a unique $(\bar{u}_n, t_n) \in H^{1/2}(-1, 1) \times \overline{H}'$ such that:*

$$\begin{aligned} -\frac{1}{\pi} t_n * \text{pv } 1/x + s t_n &= \bar{u}'_n, & \text{in }] -1, 1[, \\ \bar{u}_n - \bar{\psi} &\leq 0, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n = 0, \\ \left\langle t_n, \chi_{]-1, 1[} \right\rangle_{H^{-1/2}, H^{1/2}} &= -P. \end{aligned}$$

Finally, if $\bar{\psi}$ is replaced by $\bar{\psi} + C$ ($C \in \mathbb{R}$), then the corresponding solution is $(\bar{u}_n + C, t_n)$.

The solution of problem II' is explicit in the case where the moving obstacle is a rigid flat punch ($\bar{\psi} \equiv 0$). Based on theorem 36, the explicit solution in that
690 case, is given by

$$t_n(x) = -Pt_0(x),$$

where $t_0(x)$ is the function defined by formula (19). This distribution of surface traction under the punch exhibits a singularity at the place where a jump in the friction coefficient occurs. More precisely, the function $t_0(x)$ behaves in the neighbourhood of $x = 0$ as:

$$\begin{aligned} t_0(x) &\sim \frac{1}{\pi|x|^{|\beta|} \sqrt{1+s_-^2}}, & \text{as } x \rightarrow 0-, \\ t_0(x) &\sim \frac{1}{\pi|x|^{|\beta|} \sqrt{1+s_+^2}}, & \text{as } x \rightarrow 0+, \end{aligned}$$

695 where

$$\beta \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(-1/s_-) - \frac{1}{\pi} \arctan(-1/s_+),$$

is in $] -1, 0[$ because of the condition $s_+ > s_-$. This asymptotic behaviour is actually universal with respect to the shape of the obstacle, as seen in the next theorem.

Theorem 17. *Assume that $\bar{\psi}'$ is piecewise Lipschitz-continuous. In the case
700 $s_+ > s_-$, if the unique solution of problem II is such that the point $x = 0$ (where*

is the jump in the friction coefficient) is an interior point of the contact set (the support of t_n), then:

$$t_n = -\frac{\theta}{|x|^{|\beta|} \sqrt{1+s^2(x)}},$$

where

$$\beta \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(-1/s_-) - \frac{1}{\pi} \arctan(-1/s_+),$$

is in $] -1, 0[$ because of the condition $s_+ > s_-$, and θ is a nonnegative measure on $] -1, 1[$ whose restriction to a neighbourhood of $x = 0$ is a continuous continuous function.

Proof. Set:

$$\bar{\phi}(x) = \bar{\psi}'(0-) x \chi_{]-1,0[}(x) + \bar{\psi}'(0+) x \chi_{]0,1[}(x),$$

where χ_S is the indicator function of the set S . We have the decomposition:

$$t_n = \overline{\mathcal{L}}(\bar{\phi}) + \overline{\mathcal{L}}(\bar{\psi} - \bar{\phi}) + \overline{\mathcal{L}}(\bar{u}_n - \bar{\psi}) - P t_0.$$

Since there exists $\epsilon > 0$ such that $]c - \epsilon, c + \epsilon[$ does not meet the support of $\bar{u}_n - \bar{\psi}$, the restriction of $\overline{\mathcal{L}}(\bar{u}_n - \bar{\psi})$ to $]c - \epsilon, c + \epsilon[$ is the product of t_0 with some C^∞ function. Besides, the function $\bar{\psi}' - \bar{\phi}'$ is Lipschitz-continuous in a neighbourhood of 0 which entails that the restriction of $\overline{\mathcal{L}}(\bar{\psi} - \bar{\phi})$ to a neighbourhood of 0, is the product of t_0 with a continuous function, thanks to formula (18) and proposition 7 of [5]. Therefore, the claim in the theorem holds true for the last three terms in the above expression of $t_n(x)$. It remains only to prove that it also holds true for $\overline{\mathcal{L}}(\bar{\phi})$. But, this is a straightforward consequence of formula (18) and proposition 7 of [5]. \square

4.3. Analysis of the case $s_+ = s_-$

In this section, we analyse the case where s is a constant function as in section 3, but we now admit $\bar{\psi} \in \overline{H}_0 \supset H_0 = H^{1/2}(-1, 1)/\mathcal{C}$. The function $\bar{\psi}$ may therefore have a jump at $x = 0$, which was precluded in the case $\bar{\psi} \in H^{1/2}(-1, 1)/\mathcal{C}$ analysed in section 3. The necessary adaptation is going to be described.

First, note that the definition (12) of K_0 still makes sense in the case $\bar{\psi} \in \overline{H}_0$ and K_0 is still a closed convex subset on H_0 . To check that it is nonempty, just set $\bar{\psi} = \tilde{\psi} + C \chi_{]0,1[}$ with $\tilde{\psi} \in H_0$ and note that $\tilde{\psi} \in K_0$. Based on this simple observation, the formulation of the problem governing the normal surface displacement remains unchanged.

Problem III. Let $s, P \in \mathbb{R}$, P be positive, and $\bar{\psi} \in \overline{H}_0$. Find $\bar{u}_n \in K_0$ such that:

$$\forall \hat{u} \in K_0, \quad a(\hat{u} - \bar{u}_n, \bar{u}_n) + \varphi(\hat{u}) - \varphi(\bar{u}_n) \geq 0,$$

where φ is the functional defined by formula (13).

Problem III has a unique solution, thanks to theorem 25 in appendix A. Let us turn now to the dual problem. Note that K'_0 contains only Radon measures with no atom so that actually $\overline{K}'_0 = K'_0$, and the problem governing the normal surface traction remains unchanged.

Problem III'. Let $\bar{\psi} \in \overline{H}_0 \supset H^{1/2}(-1, 1)/\mathcal{C}$. Find $\tilde{t}_n \in \overline{K}'_0$ such that:

$$\forall \hat{t} \in \overline{K}'_0, \quad b(\tilde{t}_n, \hat{t} - \tilde{t}_n) \geq \langle \bar{\psi}, \hat{t} - \tilde{t}_n \rangle_{\overline{H}_0, \overline{H}_0}.$$

In this case, b is coercive on H'_0 , but it is not coercive on \overline{H}'_0 , and $\bar{\psi} \in \overline{H}_0$, but $\bar{\psi} \notin H_0$ in general. Problem III' can therefore not be solved by use of the Lions-Stampacchia theorem. However, Problem III' has a unique solution thanks to theorem 25 and the following lemma.

Lemma 18. *The functional defined on H'_0 by:*

$$\varphi^*(-\hat{t}) \stackrel{\text{def}}{=} \begin{cases} \langle -\hat{t}, \bar{\psi} \rangle_{\overline{H}'_0, \overline{H}_0}, & \text{if } \hat{t} \in \overline{K}'_0, \\ +\infty, & \text{otherwise,} \end{cases}$$

is proper, convex and lower semicontinuous on H'_0 .

Proof. First, note that the functional is obviously proper and convex. Now, pick $t_\infty \in \overline{K}'_0$ and a sequence (t_n) in \overline{K}'_0 that converges strongly in H'_0 towards t_∞ . The sequence (t_n) must converge weakly-* in the space of measures towards t_∞ . Hence, the sequence $(\int_0^1 t_n)$ converges towards $(\int_0^1 t_\infty)$. This entails that the sequence (t_n) converges weakly in \overline{H}'_0 towards t_∞ , and therefore $\lim \varphi^*(t_n) = \varphi^*(t_\infty)$. This is enough to get the lower semicontinuity of φ^* on H'_0 . \square

The solutions \bar{u}_n and \tilde{t}_n of problems III and III' are connected by:

$$\tilde{t}_n = \mathcal{L}(\bar{u}_n),$$

thanks to proposition 5 (which can be readily seen to extend to the case where $\bar{\psi} \in \overline{H}_0$).

Summing up, theorem 8 holds true not only for $\bar{\psi} \in H^{1/2}(-1, 1)$, but also for $\bar{\psi} \in H^{1/2}(-1, 1) + \mathbb{R}\chi_{]0,1[}$.

4.4. Analysis of the case $s_+ < s_-$

In that case, the bilinear form \bar{b} is not coercive in \overline{H}'_0 .

To get insight into this case, let us examine the situation of the particular case of the rigid flat punch, that is, $\bar{\psi} \equiv 0$. Reproducing the same reasoning as that in the proof of proposition 2 of [5], it can be proved that any solution of the

steady sliding frictional contact problem such that $t_n \in \cup_{p>1} L^p(-1, 1)$ achieves active contact everywhere below the punch: $\bar{u}_n \equiv 0$. Therefore, t_n must solve the following problem.

- $\frac{1}{\pi} \oint_{-1}^1 \frac{t_n(x')}{x' - x} dx' + s(x) t_n(x) = 0$, for a.a. $x \in]-1, 1[$,
- $\int_{-1}^1 t_n(x') dx' = -P$.

Thanks to theorem 36 in appendix C, all the solutions in $\cup_{p>1} L^p(-1, 1)$ of this singular integral equation are:

$$t_n(x) = -\frac{P e^{\tau(x)}}{(1-x)\sqrt{s^2(x)+1}} \left(\int_{-1}^1 \frac{e^{\tau(x') dx'}{(1-x')\sqrt{1+s^2(x')}}} \right)^{-1} + \frac{C e^{\tau(x)}}{\pi x(1-x)\sqrt{s^2(x)+1}},$$

where $C \in \mathbb{R}$ is an arbitrary constant. Note that the identity proved in the step 1 of the proof of theorem 14 of [5] (with the choices $\lambda = -1$ and $x = 1$), gives:

$$\int_{-1}^1 \frac{e^{\tau(x')} dx'}{(1-x')\sqrt{1+s^2(x')}} = \pi.$$

The fact that the above linear problem admits infinitely many solutions is another proof of the fact that the bilinear form \bar{b} is not coercive in \overline{H}_0 . An interesting fact arises: although the linear problem has infinitely many solutions, there is only exactly one of these solutions which is a nonpositive function: that corresponding to the choice $C = 0$. Therefore, the steady sliding contact problem has one and only one solution in $\cup_{p>1} L^p(-1, 1)$. It reads as:

$$t_n(x) = -\frac{P e^{\tau(x)}}{\pi(1-x)\sqrt{s^2(x)+1}}.$$

It is interesting to note that the asymptotic behaviour of that solution in the neighbourhood of $x = 0$ is now given by:

$$\begin{aligned} t_n(x) &\sim \frac{P |x|^\beta}{\pi \sqrt{1+s_-^2}}, & \text{as } x \rightarrow 0-, \\ t_n(x) &\sim \frac{P |x|^\beta}{\pi \sqrt{1+s_+^2}}, & \text{as } x \rightarrow 0+, \end{aligned} \tag{20}$$

where

$$\beta \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(-1/s_-) - \frac{1}{\pi} \arctan(-1/s_+),$$

is in $]0, 1[$ because of the condition $s_+ < s_-$. Therefore, $t_n(x)$ is now locally bounded around $x = 0$ (see figure 4).

770 We have therefore discovered a fact of engineering importance. In the situation, where a rigid punch with two different coatings (producing two different friction coefficients with a jump at some point) is steadily sliding along the boundary of some linearly elastic body, there is a singularity of the surface traction at the jump in the case where the largest friction coefficient is front, 775 whereas the surface traction is locally bounded at the jump when the largest friction coefficient is rear.

So, in the case $s_+ < s_-$, the bilinear form \bar{b} is coercive neither in \overline{H}' nor in \overline{H}'_0 . A glance at theorem 36 in appendix C shows that, in the case $s_+ < s_-$, the functions

$$C_0 t_0(x) + C_1 t_1(x) = \frac{(C_0 x + C_1) e^{\tau(x)}}{\pi x(1-x) \sqrt{s^2(x) + 1}}, \quad (21)$$

780 where:

$$\tau(x) = \frac{1}{\pi} \oint_{-1}^1 \arctan(-1/s(x')) \frac{dx'}{x' - x},$$

and $C_0, C_1 \in \mathbb{R}$, are all in $L^p(-1, 1)$ for some $p > 1$, and satisfy:

$$\forall C_0, C_1 \in \mathbb{R}, \quad \forall \hat{t} \in \overline{H}'_0, \quad \bar{b}(C_0 t_0 + C_1 t_1, \hat{t}) = 0.$$

In addition, t_0 is positive, the restriction of t_1 to $] -1, 0[$ is negative and the restriction of t_1 to $]0, 1[$ is positive. Furthermore, we have:

$$\int_{-1}^1 t_0(x) dx = 1, \quad \int_{-1}^1 t_1(x) dx = 0.$$

785 Formula (17) shows that \bar{b} is not coercive in \overline{H}'_0 . However, it is coercive in the space:

$$\overline{H}'_{00} \stackrel{\text{def}}{=} \left\{ t \in \overline{H}' \mid \int_{-1}^0 \tilde{t} = \int_0^1 \tilde{t} = 0 \right\},$$

which is clearly a subspace of codimension 2 in \overline{H}' and a subspace of codimension 1 in \overline{H}'_0 . The restriction of \bar{b} to \overline{H}'_{00} is continuous and coercive. The dual space of \overline{H}'_{00} is $\overline{H}_{00} = \overline{H}_0 / (\mathbb{R}\chi_{]0,1[})$ which can be identified to $H_0 = H^{1/2}(-1, 1) / \mathcal{C}$. Therefore, for arbitrary $u \in \overline{H}_{00} \simeq H_0$, there exists a unique $t \in \overline{H}'_{00}$ such that:

$$\forall \hat{t} \in \overline{H}'_{00}, \quad \bar{b}(t, \hat{t}) = \langle u, \hat{t} \rangle.$$

790 Setting $t = \overline{\mathcal{L}}'(u)$, we obtain a continuous linear mapping from \overline{H}_{00} onto \overline{H}'_{00} . The distribution t satisfies:

$$-\frac{1}{\pi} t * \text{pv } 1/x + s t = u' + C \delta, \quad \text{in }]-1, 1[,$$

for some $C \in \mathbb{R}$. In the sequel, it will be desirable to ensure that $C = 0$. This fact is equivalent to:

$$\forall \hat{t} \in \overline{H}'_0, \quad \bar{b}(t, \hat{t}) = \langle u, \hat{t} \rangle,$$

which is itself equivalent to $\bar{b}(t, t_1) = \langle u, t_1 \rangle$, due to the fact that $\overline{H}'_0 = \overline{H}'_{00} + \mathbb{R}t_1$.

795 Proposition 19. *The unique solution $t = \overline{\mathcal{L}}'(u)$ in \overline{H}'_{00} of the variational equality:*

$$\forall \hat{t} \in \overline{H}'_{00}, \quad \bar{b}(t, \hat{t}) = \langle u, \hat{t} \rangle,$$

actually satisfies:

$$\forall \hat{t} \in \overline{H}'_0, \quad \bar{b}(t, \hat{t}) = \langle u, \hat{t} \rangle.$$

Furthermore, for $u \in W^{1,\infty}(-1,1)/\mathcal{C} \subset H^{1/2}(-1,1)/\mathcal{C} = H_0$, $\overline{\mathcal{L}}'(u) \in \overline{H}'_{00}$ is given by:

$$\begin{aligned} \overline{\mathcal{L}}'(u) &= \frac{s(x)u'(x)}{1+s^2(x)} \\ &\quad - \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x')(1-x')e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} \\ &\quad + C_1(u)t_1(x), \end{aligned}$$

where:

$$t_1(x) = \frac{e^{\tau(x)}}{\pi x(1-x)\sqrt{s^2(x)+1}}, \quad \tau(x) = \frac{1}{\pi} \oint_{-1}^1 \arctan(-1/s(x')) \frac{dx'}{x'-x},$$

and $C_1(u)$ is a real constant which is uniquely determined by the condition **800** $\overline{\mathcal{L}}'(u) \in \overline{H}'_{00}$, that is, $\int_0^1 \overline{\mathcal{L}}'(u) = 0$.

Proof. Taking u in $W^{1,\infty}(-1,1)$, the general solution $\tau \in \cup_{q>1} L^q(-1,1)$ of the singular integral equation:

$$\frac{1}{\pi} \oint_{-1}^1 \frac{\tau(x')}{x'-x} dx' + s(x)\tau(x) = u'(x), \quad \text{for a.a. } x \in]-1,1[,$$

is given by:

$$\begin{aligned} \tau &= \frac{s(x)u'(x)}{1+s^2(x)} - \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x')(1-x')e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} \\ &\quad + \frac{[xC_0 + C_1]e^{\tau(x)}}{\pi x(1-x)\sqrt{s^2(x)+1}}, \\ &= \frac{s(x)u'(x)}{1+s^2(x)} - \frac{e^{\tau(x)}}{x(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x')x'(1-x')e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} \\ &\quad + \frac{[xC_0 + C_1]e^{\tau(x)}}{\pi x(1-x)\sqrt{s^2(x)+1}}, \end{aligned}$$

thanks to theorem 36 of appendix C. Invoking theorem 11 (i) of [5], it can be readily checked that:

$$\int_{-1}^1 \left\{ \frac{s(x) u'(x)}{1 + s^2(x)} - \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x') (1-x') e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} \right\} dx = 0,$$

$$\int_{-1}^1 \left\{ \frac{s(x) u'(x)}{1 + s^2(x)} - \frac{e^{\tau(x)}}{x(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x') x'(1-x') e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} \right\} dx = 0.$$

Therefore:

$$\begin{aligned} \bar{t} &\stackrel{\text{def}}{=} \tau - t_0 \int_{-1}^1 \tau \\ &= \frac{s(x) u'(x)}{1 + s^2(x)} - \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x') (1-x') e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} + C_1 t_1, \\ &= \frac{s(x) u'(x)}{1 + s^2(x)} - \frac{e^{\tau(x)}}{x(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{u'(x') x'(1-x') e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x'-x} \\ &\quad + C_1' t_1, \end{aligned}$$

belongs to \overline{H}'_0 . There is only one value of the constant C_1 such that it actually belongs to \overline{H}'_{00} . Making that choice, we adopt the notation $\tilde{t} \in \overline{H}'_{00}$ for \bar{t} . It satisfies:

805

$$-\frac{1}{\pi} \int_{-1}^1 \tilde{t}(x') \log |x' - x| dx' + \int_{-1}^x s(x') \tilde{t}(x') dx' = u(x) + C,$$

for all $x \in]-1, 1[$ and some constant $C \in \mathbb{R}$. Therefore:

$$\forall \hat{t} \in \overline{H}'_0, \quad \bar{b}(\bar{t}, \hat{t}) = \langle u, \hat{t} \rangle,$$

which is the expected conclusion $\tilde{t} = \overline{\mathcal{L}}'(u)$.

Next, take $u \in H_0 = H^{1/2}(-1, 1)/\mathcal{C}$ arbitrary and let $t = \overline{\mathcal{L}}'(u) \in \overline{H}'_{00}$ be the unique solution of the variational equality:

$$\forall \hat{t} \in \overline{H}'_{00}, \quad \bar{b}(t, \hat{t}) = \langle u, \hat{t} \rangle.$$

810 We have to prove:

$$\bar{b}(t, t_1) = \langle u, t_1 \rangle.$$

We define:

$$l(u) = \bar{b}(\overline{\mathcal{L}}'(u), t_1) - \langle u, t_1 \rangle,$$

which is clearly a continuous linear form on $H_0 = H^{1/2}(-1, 1)/\mathcal{C}$. Thanks to the first part of the proof, this continuous linear form vanishes identically on $W^{1,\infty}(-1, 1)/\mathcal{C}$. Since $W^{1,\infty}(-1, 1)$ is dense in $H^{1/2}(-1, 1)$, the continuous linear form $l(u)$ vanishes identically on $H_0 = H^{1/2}(-1, 1)/\mathcal{C}$. This is the expected conclusion. \square

The set:

$$\overline{K}'_{00} \stackrel{\text{def}}{=} \left\{ \hat{t} \in \overline{H}'_{00} \mid \exists F \in \mathbb{R}, \quad \hat{t} + Ft_1 - Pt_0 \leq 0 \right\},$$

is clearly a closed convex subset of \overline{H}'_{00} . In the case where $P \geq 0$, it is nonempty because it contains 0. Set:

$$\phi^*(\hat{t}) \stackrel{\text{def}}{=} \begin{cases} \inf_{F \mid \hat{t} + Ft_1 \leq Pt_0} \left\langle -\bar{\psi}, \hat{t} + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0}, & \text{if } \hat{t} \in \overline{K}'_{00}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 20. *The functional ϕ^* is proper, lower semicontinuous on \overline{H}'_{00} and convex.*

Proof. Recalling that t_0 is positive on $] -1, 1[$, t_1 is negative on $] -1, 0[$ and positive on $] 0, 1[$, we have:

$$\hat{t} + Ft_1 - Pt_0 \leq 0, \quad \text{with } \hat{t} \in \overline{H}'_{00} \implies -P \left(\int_{-1}^0 t_0 \right) / \left(\int_0^1 t_1 \right) \leq F \leq P \left(\int_0^1 t_0 \right) / \left(\int_0^1 t_1 \right).$$

Therefore, the infimum in the definition of ϕ^* is taken over a closed subset of a bounded real interval and is therefore achieved so that this infimum is actually a minimum. Taking \hat{t}_1, \hat{t}_2 arbitrary in \overline{K}'_{00} , we therefore have:

$$\begin{aligned} \left\langle -\bar{\psi}, \hat{t}_1 + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} &= \inf_{F \mid \hat{t}_1 + Ft_1 \leq Pt_0} \left\langle -\bar{\psi}, \hat{t}_1 + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0}, \\ \left\langle -\bar{\psi}, \hat{t}_2 + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} &= \inf_{F \mid \hat{t}_2 + Ft_1 \leq Pt_0} \left\langle -\bar{\psi}, \hat{t}_2 + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0}, \end{aligned}$$

so that:

$$\forall \lambda \in [0, 1], \quad \lambda \hat{t}_1 + (1 - \lambda) \hat{t}_2 + [\lambda F_1 + (1 - \lambda) F_2] t_1 \leq Pt_0,$$

and therefore:

$$\forall \lambda \in [0, 1], \quad \phi^* \left(\lambda \hat{t}_1 + (1 - \lambda) \hat{t}_2 \right) \leq \lambda \phi^*(\hat{t}_1) + (1 - \lambda) \phi^*(\hat{t}_2),$$

that is, the functional ϕ^* is convex. We will now prove that it is also lower
825 semicontinuous. Let $\hat{t}_\infty \in \overline{K}'_{00}$ and a sequence $(\hat{t}_k)_{k \in \mathbb{N}}$ in \overline{K}'_{00} converging
strongly towards \hat{t}_∞ . Set:

$$\left\langle -\bar{\psi}, \hat{t}_k + F_k t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} = \inf_{F | \hat{t}_k + F t_1 \leq P t_0} \left\langle -\bar{\psi}, \hat{t}_k + F t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0}.$$

We have:

$$\liminf_{k \rightarrow +\infty} \left\langle -\bar{\psi}, \hat{t}_k + F_k t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} = \left\langle -\bar{\psi}, \hat{t}_\infty + F_\infty t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0},$$

for some F_∞ satisfying:

$$\hat{t}_\infty + F_\infty t_1 \leq P t_0.$$

Therefore:

$$\phi^*(\hat{t}_\infty) \leq \liminf_{k \rightarrow +\infty} \phi^*(\hat{t}_k),$$

830 that is, ϕ^* is lower semicontinuous. □

Problem IV'. Let $\bar{\psi} \in \overline{H}_0 \supset H^{1/2}(-1, 1)/\mathcal{C}$. Find $\tilde{t}_n \in \overline{K}'_{00}$ such that:

$$\forall \hat{t} \in \overline{K}'_{00}, \quad \bar{b}(\tilde{t}_n, \hat{t} - \tilde{t}_n) + \phi^*(\hat{t}) - \phi^*(\tilde{t}_n) \geq 0.$$

Problem IV' has a unique solution, thanks to proposition 20 and theorem 25.
The following proposition provides a pointwise interpretation of the solution of
problem IV'.

835 **Proposition 21.** Let $P \in \mathbb{R}$ be positive, $\bar{u}_n \in H_0 = H^{1/2}(-1, 1)/\mathbb{R}$ and $\tilde{t}_n \in \overline{H}'_{00}$. Then, the two following statements are equivalent.

$$(i) \quad \tilde{t}_n \in \overline{K}'_{00}, \quad \text{and} \quad \forall \hat{t} \in \overline{K}'_{00}, \quad \langle \bar{u}_n, \hat{t} - \tilde{t}_n \rangle_{\overline{H}_{00}, \overline{H}'_{00}} + \phi^*(\hat{t}) - \phi^*(\tilde{t}_n) \geq 0,$$

(ii) The function $\bar{u}_n - \bar{\psi}$ is essentially bounded by above and there exists $\tilde{F} \in \mathbb{R}$
such that the three following statements hold true in $] -1, 1[$:

$$\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} - \operatorname{ess\,sup}_{]-1,1[} \left(\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} \right) \leq 0,$$

$$\tilde{t}_n + \tilde{F} t_1 - P t_0 \leq 0,$$

$$\left[\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} - \operatorname{ess\,sup}_{]-1,1[} \left(\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} \right) \right] \times \\ \times \left[\tilde{t}_n + \tilde{F} t_1 - P t_0 \right] = 0.$$

Proof.

(ii) \Rightarrow (i). Pick $\hat{t} \in \overline{K}'_{00}$ arbitrarily. Since the infimum is taken over a closed subset of a bounded interval, there exists $\hat{F} \in \mathbb{R}$ such that:

$$\begin{aligned} \left\langle -\bar{\psi}, \hat{t} + \hat{F}t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} &= \inf_{F \mid \hat{t} + Ft_1 \leq Pt_0} \left\langle -\bar{\psi}, \hat{t} + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0}, \\ \hat{t} + \hat{F}t_1 &\leq Pt_0. \end{aligned}$$

Therefore:

$$\begin{aligned} \left[\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} - \operatorname{ess\,sup}_{]-1,1[} \left(\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} \right) \right] \times \\ \times \left[\hat{t} + \hat{F}t_1 - Pt_0 - \tilde{t}_n - \tilde{F}t_1 + Pt_0 \right] \geq 0. \end{aligned}$$

840 Integrating over $] -1, 1[$, we get:

$$\left\langle \bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi}, \hat{t} + \hat{F}t_1 - \tilde{t}_n - \tilde{F}t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} \geq 0,$$

that is:

$$\left\langle \bar{u}_n, \hat{t} - \tilde{t} \right\rangle + \phi^*(\hat{t}) \geq \left\langle -\bar{\psi}, \hat{t} + \hat{F}t_1 \right\rangle \geq \inf_{F \mid \hat{t} + Ft_1 \leq Pt_0} \left\langle -\bar{\psi}, \tilde{t}_n + Ft_1 \right\rangle = \phi^*(\tilde{t}_n),$$

which is nothing but (i).

(i) \Rightarrow (ii). So we assume that (i) is true. There exists $\tilde{F} \in \mathbb{R}$ such that:

$$\begin{aligned} \left\langle -\bar{\psi}, \tilde{t}_n + \tilde{F}t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} &= \inf_{F \mid \tilde{t}_n + Ft_1 \leq Pt_0} \left\langle -\bar{\psi}, \tilde{t}_n + Ft_1 \right\rangle_{\overline{H}_0, \overline{H}'_0}, \\ \tilde{t}_n + \tilde{F}t_1 &\leq Pt_0, \end{aligned}$$

and we obtain:

$$\begin{aligned} \forall \hat{t} \in \overline{K}'_{00}, \quad \forall \hat{F} \text{ such that } \hat{t} + \hat{F}t_1 \leq Pt_0, \\ \left\langle \bar{u}_n, \hat{t} - \tilde{t}_n \right\rangle_{\overline{H}_0, \overline{H}'_0} - \left\langle \bar{\psi}, \hat{t} + \hat{F}t_1 - \tilde{t}_n - \tilde{F}t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} \geq 0. \end{aligned}$$

Recalling the notation:

$$\overline{K}'_0 = \left\{ \hat{t} \in \overline{H}'_0 \mid \hat{t} \leq Pt_0 \right\},$$

we actually have:

$$\forall \hat{t} \in \overline{K}'_0, \quad \left\langle \bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi}, \hat{t} - \tilde{t}_n - \tilde{F}t_1 \right\rangle_{\overline{H}_0, \overline{H}'_0} \geq 0.$$

Thanks to proposition 5 (which can be readily seen to extend to the situation
 845 where H_0 is replaced by \overline{H}_0), this entails that (ii) holds true. \square

As previously, problem IV' can be given an equivalent formulation in terms
 of displacement (primal formulation). We will systematically identify \overline{H}_{00} with
 $H_0 = H^{1/2}(-1, 1)/\mathbb{R}$. The bilinear form:

$$\bar{a}(u_1, u_2) \stackrel{\text{def}}{=} \left\langle u_1, \overline{\mathcal{L}}'(u_2) \right\rangle_{\overline{H}_{00}, \overline{H}'_{00}} = \bar{b}\left(\overline{\mathcal{L}}'(u_1), \overline{\mathcal{L}}'(u_2)\right),$$

is continuous and coercive on \overline{H}_{00} . The conjugate of ϕ^* by the Legendre-Fenchel
 850 transform is defined by:

$$\phi(\hat{u}) = \sup_{\hat{t} \in \overline{H}'_{00}} \left\{ \langle \hat{t}, \hat{u} \rangle - \phi^*(\hat{t}) \right\}$$

Since ϕ^* is proper, convex and lower semicontinuous on \overline{H}'_{00} (proposition 20), the
 functional ϕ is proper, convex and lower semicontinuous on $\overline{H}_{00} \simeq H_0$. Hence,
 theorem 25 yields a unique solution of the following problem.

Problem IV. Let $P \in \mathbb{R}$ be positive, and $\bar{\psi} \in \overline{H}_0 \supset H^{1/2}(-1, 1)/\mathcal{C}$. Find
 855 $\bar{u}_n \in H_0$ such that:

$$\forall \hat{u} \in H_0, \quad \bar{a}(\hat{u} - \bar{u}_n, \bar{u}_n) + \phi(-\hat{u}) - \phi(-\bar{u}_n) \geq 0.$$

Setting $\tilde{t}_n = \overline{\mathcal{L}}'(\bar{u}_n)$, the equivalence:

$$-\bar{u}_n \in \partial\phi^*[\tilde{t}_n] \iff \tilde{t}_n \in \partial\phi[-\bar{u}_n],$$

shows the equivalence of problems IV' and IV. Thanks to propositions 19 and 21,
 the solutions $\bar{u}_n \in H_0$ and $\tilde{t}_n \in \overline{H}'_{00}$ of problems IV and IV' provide the unique
 $(\bar{u}_n, \tilde{t}_n) \in H_0 \times \overline{H}'_{00}$ such that there exists $\tilde{F} \in \mathbb{R}$ such that:

$$\begin{aligned} & -\frac{1}{\pi} \tilde{t}_n * \text{pv } 1/x + s \tilde{t}_n = \bar{u}'_n, \\ & \bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} - \text{ess sup}_{]-1,1[} \left(\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} \right) \leq 0, \\ & \tilde{t}_n + \tilde{F} t_1 - P t_0 \leq 0, \\ & \left[\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} - \text{ess sup}_{]-1,1[} \left(\bar{u}_n - \frac{\int_{-1}^1 t_1 \bar{u}_n}{\int_0^1 t_1} \chi_{]0,1[} - \bar{\psi} \right) \right] \times \\ & \quad \times [\tilde{t}_n + \tilde{F} t_1 - P t_0] = 0, \end{aligned}$$

860 in $]-1, 1[$. We now must get rid of the unwanted term containing $\chi_{]0,1[}$. This is
 going to be achieved in the next proposition by replacing first $\bar{\psi}$ by $\bar{\psi} - C\chi_{]0,1[}$,
 and then, performing a fixed point strategy.

Proposition 22. Let $P \in \mathbb{R}$ be positive and $\bar{\psi} \in \bar{H}_0 = H^{1/2}(-1, 1)/\mathbb{R} + \mathbb{R}\chi_{]0,1[}$. Then, there exist $(\bar{u}_n, \tilde{t}_n) \in H_0 \times \bar{H}'_{00}$ and $\tilde{F} \in \mathbb{R}$ such that:

$$\begin{aligned} -\frac{1}{\pi} \tilde{t}_n * \text{pv } 1/x + s \tilde{t}_n &= \bar{u}'_n, \\ \bar{u}_n - \bar{\psi} - \text{ess sup}_{]-1,1[}(\bar{u}_n - \bar{\psi}) &\leq 0, \\ \tilde{t}_n + \tilde{F}t_1 - Pt_0 &\leq 0, \\ \left[\bar{u}_n - \bar{\psi} - \text{ess sup}_{]-1,1[}(\bar{u}_n - \bar{\psi}) \right] \left[\tilde{t}_n + \tilde{F}t_1 - Pt_0 \right] &= 0, \end{aligned}$$

865 in $]-1, 1[$.

Proof. In all the proof, we denote problems $IV'(C)$ and $IV(C)$, the problems IV' and IV , in which the data $\bar{\psi} \in \bar{H}_0$ has been replaced by $\bar{\psi} - C\chi_{]0,1[}$.

Step 1. Let $u \in H^{1/2}/\mathbb{R}$ and $\tilde{t} = \bar{\mathcal{L}}'(u)$. We have the identity:

$$\int_{-1}^1 t_1 u = -\frac{2}{\pi} \int_{-1}^1 t_1 (\tilde{t} * \log |\cdot|).$$

870 We shall establish the result only in the case where \tilde{t} happens to be an integrable function, the general case being easily deduced from this particular case by a density argument. Using proposition 19, we have:

$$\int_{-1}^1 t_1 u = \bar{b}(\tilde{t}, t_1) = -\frac{1}{\pi} \int_{-1}^1 t_1 (\tilde{t} * \log |\cdot|) + \int_{-1}^1 t_1 \int_{-1}^x s \tilde{t},$$

where the integral notation makes sense since \tilde{t} has been supposed to be an integrable function. Since $\tilde{t} \in \bar{H}'_{00}$ and s is constant over $]-1, 0[$ and $]0, 1[$, we have:

$$\int_{-1}^x s \tilde{t} = s(x) \int_{-1}^x \tilde{t},$$

875 so that:

$$\int_{-1}^1 t_1 \int_{-1}^x s \tilde{t} = \int_{-1}^1 s t_1 \int_{-1}^x \tilde{t} = - \int_{-1}^1 \tilde{t} \int_{-1}^x s t_1 = -\frac{1}{\pi} \int_{-1}^1 t_1 (\tilde{t} * \log |\cdot|),$$

where the last equality comes from $\bar{b}(t_1, \tilde{t}) = 0$. Hence:

$$\int_{-1}^1 t_1 u = -\frac{2}{\pi} \int_{-1}^1 t_1 (\tilde{t} * \log |\cdot|).$$

Step 2. Let $C_1, C_2 \in \mathbb{R}$, arbitrary, and $u_1, u_2 \in H^{1/2}/\mathbb{R}$ denote the solutions of problems $IV(C_1)$ and $IV(C_2)$, respectively. Then, the following estimate holds true:

$$\frac{\left| \int_{-1}^1 t_1 (u_2 - u_1) \right|}{\int_0^1 t_1} \leq \sqrt{2P(s_- - s_+)} \sqrt{|C_2 - C_1|}.$$

Set $\tilde{t}_1 = \overline{\mathcal{L}'}(u_1)$ and $\tilde{t}_2 = \overline{\mathcal{L}'}(u_2)$. Thanks to step 1 and the Cauchy-Schwarz inequality, we have:

$$\left| \int_{-1}^1 t_1(u_2 - u_1) \right| \leq 2 \sqrt{-\frac{1}{\pi} \int_{-1}^1 (\tilde{t}_2 - \tilde{t}_1) ((\tilde{t}_2 - \tilde{t}_1) * \log |\cdot|)} \sqrt{-\frac{1}{\pi} \int_{-1}^1 t_1 (t_1 * \log |\cdot|)}.$$

880 But:

$$-\frac{1}{\pi} \int_{-1}^1 (\tilde{t}_2 - \tilde{t}_1) ((\tilde{t}_2 - \tilde{t}_1) * \log |\cdot|) = \bar{b}(\tilde{t}_2 - \tilde{t}_1, \tilde{t}_2 - \tilde{t}_1),$$

and:

$$-\frac{1}{\pi} \int_{-1}^1 t_1 (t_1 * \log |\cdot|) = - \int_{-1}^1 t_1 \int_{-1}^x s t_1 = \frac{s_- - s_+}{2} \left(\int_0^1 t_1 \right)^2,$$

so that:

$$\frac{\left| \int_{-1}^1 t_1(u_2 - u_1) \right|}{\int_0^1 t_1} \leq \sqrt{2(s_- - s_+) \bar{b}(\tilde{t}_2 - \tilde{t}_1, \tilde{t}_2 - \tilde{t}_1)},$$

and there remains only to estimate $\bar{b}(\tilde{t}_2 - \tilde{t}_1, \tilde{t}_2 - \tilde{t}_1)$. To obtain it, we start with:

$$\begin{aligned} & \left[u_2 - \frac{\int_{-1}^1 t_1 u_2}{\int_0^1 t_1} \chi_{]0,1[- \bar{\psi} + C_2 \chi_{]0,1[- \text{ess sup}_{]-1,1[} \left(u_2 - \frac{\int_{-1}^1 t_1 u_2}{\int_0^1 t_1} \chi_{]0,1[- \bar{\psi} + C_2 \chi_{]0,1[} \right) \right. \\ & \left. - u_1 + \frac{\int_{-1}^1 t_1 u_1}{\int_0^1 t_1} \chi_{]0,1[+ \bar{\psi} - C_1 \chi_{]0,1[+ \text{ess sup}_{]-1,1[} \left(u_1 - \frac{\int_{-1}^1 t_1 u_1}{\int_0^1 t_1} \chi_{]0,1[- \bar{\psi} + C_1 \chi_{]0,1[} \right) \right] \times \\ & \quad \times \left[\tilde{t}_2 + \tilde{F}_2 t_1 - P t_0 - \tilde{t}_1 - \tilde{F}_1 t_1 + P t_0 \right] \leq 0, \end{aligned}$$

in $] -1, 1[$. By integration:

$$\int_{-1}^1 \left[(u_2 - u_1) - \frac{\int_{-1}^1 t_1 (u_2 - u_1)}{\int_0^1 t_1} \chi_{]0,1[+ (C_2 - C_1) \chi_{]0,1[} \right] \left[\tilde{t}_2 - \tilde{t}_1 + (\tilde{F}_2 - \tilde{F}_1) t_1 \right] \leq 0,$$

which is nothing but:

$$\bar{b}(\tilde{t}_2 - \tilde{t}_1, \tilde{t}_2 - \tilde{t}_1) + (C_2 - C_1)(\tilde{F}_2 - \tilde{F}_1) \int_0^1 t_1 \leq 0.$$

885 Recalling that, for $i = 1, 2$:

$$-P \left(\int_{-1}^0 t_0 \right) \leq \tilde{F}_i \left(\int_0^1 t_1 \right) \leq P \left(\int_0^1 t_0 \right),$$

the expected conclusion follows.

Step 3. For arbitrary $C \in \mathbb{R}$, consider problem $IV(C)$, its unique solution $u_C \in \overline{H}_{00} \simeq H_0$. Then:

$$\exists M \in \mathbb{R}, \quad \forall C \in \mathbb{R}, \quad \frac{\left| \int_{-1}^1 t_1 u_C \right|}{\int_0^1 t_1} \leq M.$$

Let $C \in \mathbb{R}$ and $u_C \in H^{1/2}(-1, 1)/\mathbb{R}$ the solution of problem $IV(C)$. Setting $\tilde{t}_C = \overline{\mathcal{L}}'(u_C) \in \overline{H}'_{00}$, there exist $\delta \in \mathbb{R}$ and F_C in the interval:

$$-P \left(\int_{-1}^0 t_0 \right) / \left(\int_0^1 t_1 \right) \leq F_C \leq P \left(\int_0^1 t_0 \right) / \left(\int_0^1 t_1 \right),$$

satisfying:

$$\begin{aligned} -\frac{1}{\pi} \tilde{t}_C * \text{pv } 1/x + s \tilde{t}_C &= u'_C, \\ u_C - \bar{\psi} - \delta \chi_{]0,1[} &\leq 0, \\ \tilde{t}_C + F_C t_1 - P t_0 &\leq 0, \\ [u_C - \bar{\psi} - \delta \chi_{]0,1[}][\tilde{t}_C - F_C t_1 - P t_0] &= 0, \end{aligned}$$

in $] -1, 1[$. Setting:

$$\overline{\overline{K}}'_0(F_C) \stackrel{\text{def}}{=} \left\{ \hat{t} \in \overline{H}'_{00} \mid \hat{t} + F_C t_1 - P t_0 \leq 0 \right\},$$

$\tilde{t}_C \in \overline{\overline{K}}'_0(F_C)$ is the unique solution of the variational inequality:

$$\forall \hat{t} \in \overline{\overline{K}}'_0(F_C), \quad \bar{b}(\tilde{t}_C, \hat{t} - \tilde{t}_C) \geq \langle \bar{\psi}, \hat{t} - \tilde{t}_C \rangle_{\overline{H}_{00}, \overline{H}'_{00}}.$$

The test function:

$$\hat{t}_0 \stackrel{\text{def}}{=} \begin{cases} P t_0 + t_1 \left(\int_{-1}^0 t_0 \right) / \left(\int_0^1 t_1 \right), & \text{in }] -1, 0[, \\ P t_0 - t_1 \left(\int_0^1 t_0 \right) / \left(\int_0^1 t_1 \right), & \text{in }] 0, 1[, \end{cases}$$

is independant of C and belongs to $\overline{\overline{K}}'_0(F_C)$. Using it in the variational inequality gives:

$$\bar{b}(\tilde{t}_C, \tilde{t}_C) \leq \bar{b}(\tilde{t}_C, \hat{t}_0) + \langle \bar{\psi}, \hat{t}_0 - \tilde{t}_C \rangle_{\overline{H}_{00}, \overline{H}'_{00}},$$

which entails that $\|\tilde{t}_C\|_{\overline{H}'_{00}}$ is bounded by a constant independant of C . The same is therefore true of $\|u_C\|_{\overline{H}_{00}}$. This entails the expected result.

Step 4. The mapping:

$$C \mapsto \frac{\int_{-1}^1 t_1 u_C}{\int_0^1 t_1},$$

defined for arbitrary $C \in \mathbb{R}$, has at least one fixed point which entails the proposition.

900 This function maps continuously $[-M, M]$ into itself so the conclusion follows from the Brouwer¹ fixed point theorem and proposition 21. \square

As a corollary of proposition 22, we get the following theorem.

Theorem 23. Let $P \in \mathbb{R}$ be positive, $s :]-1, 1[\rightarrow \mathbb{R}$ the function defined by $s = s_- \chi_{]-1, 0[} + s_+ \chi_{]0, 1[}$ with $s_- > s_+$ and $\bar{\psi} \in H^{1/2}(-1, 1) + \mathbb{R}\chi_{]0, 1[}$. Then,
905 there exist $\bar{u}_n \in H^{1/2}(-1, 1)$ and $t_n \in \bar{H}'$ such that:

$$\begin{aligned} -\frac{1}{\pi} t_n * \text{pv } 1/x + s t_n &= \bar{u}'_n, & \text{in }]-1, 1[, \\ \bar{u}_n - \bar{\psi} \leq 0, \quad t_n \leq 0, \quad (\bar{u}_n - \bar{\psi}) t_n &= 0, \\ \left\langle t_n, \chi_{]-1, 1[} \right\rangle_{H^{-1/2}, H^{1/2}} &= -P. \end{aligned}$$

Finally, if $\bar{\psi}$ is replaced by $\bar{\psi} + C$ ($C \in \mathbb{R}$), then the corresponding solution is $(\bar{u}_n + C, t_n)$.

Remark. The example of the rigid flat punch (that is, the case $\bar{\psi} \equiv 0$) studied in the beginning of this section made it plausible that some kind of uniqueness should be expected. In the case $s_- < s_+$, uniqueness was a straightforward consequence of the coercivity of \bar{b} . In the case $s_- > s_+$, the situation is more involved since coercivity does not hold. Let us give an example of nonuniqueness of t_n . Set:

$$\begin{aligned} t_n(x) &= -P \frac{|t_1(x)|}{\int_{-1}^1 |t_1(u)| du}, \\ \bar{u}'_n &= -\frac{1}{\pi} t_n * \text{pv } 1/x + s t_n, & \text{in }]-1, 1[. \end{aligned}$$

Choose $\bar{\psi} \equiv \bar{u}_n$, so that the above defined $(\bar{u}_n, t_n) \in W^{1,p} \times L^p$ for all p such that:

$$\frac{1}{p} > \alpha = \frac{1}{\pi} \arctan(-1/s_-) - \frac{1}{\pi} \arctan(-1/s_+),$$

910 provides a solution of the corresponding problem. But, it can be readily checked that $(\bar{u}_n, t_n + \beta t_1 / \int_{-1}^1 |t_1|)$ is also a solution, for all $\beta \in [-P, P]$.

The above construction fails in the case $\bar{\psi}$ is Lipschitz-continuous. It seems plausible that \bar{u}_n is generally unique, and that the same is true of t_n in the case of a Lipschitz-continuous obstacle. But I have not been able to prove it so far.

¹of course, it was not necessary to invoke Brouwer's theorem here, but the choice has been made to make no reference to the ordering on the real line, so that the proof can be readily extended to the situation of an arbitrary finite number of jumps of the function s .

915 The aim of the following theorem is to show that the asymptotic behaviour of t_n at $x = 0$ encountered in the example at the beginning of the section, is universal. It is remarkable that the surface traction goes to zero at the jump of the friction coefficient whatever the shape of the (smooth) indenter is.

920 **Theorem 24.** *Assume that $\bar{\psi}'$ is piecewise Lipschitz-continuous. Let (\bar{u}_n, t_n) be a solution of the problem considered in theorem 23, such that the point $x = 0$ where is the jump in the friction coefficient, is an interior point of the contact set (the support of t_n). Then, the measure t_n behaves in the neighbourhood of $x = 0$ as:*

$$t_n = \frac{\bar{\psi}'(0+) - \bar{\psi}'(0-)}{s_+ - s_-} - \frac{\theta |x|^\beta}{\sqrt{1 + s^2(x)}},$$

where

$$\beta \stackrel{\text{def}}{=} \frac{1}{\pi} \arctan(-1/s_-) - \frac{1}{\pi} \arctan(-1/s_+),$$

925 is in $]0, 1[$ because of the condition $s_+ < s_-$, and θ is a measure whose restriction to a neighborhood of $x = 0$ is a function which is Hölder-continuous of exponent $1 - \beta$.

Proof. Set:

$$\bar{\phi}(x) = \bar{\psi}'(0-) x \chi_{]-1, 0[}(x) + \bar{\psi}'(0+) x \chi_{]0, 1[}(x),$$

where χ_S is the indicator function of the set S . We have the decomposition:

$$t_n = \overline{\mathcal{L}'}(\bar{\phi}) + \overline{\mathcal{L}'}(\bar{\psi} - \bar{\phi}) + \overline{\mathcal{L}'}(\bar{u}_n - \bar{\psi}) - P t_0 + C t_1, \quad (22)$$

for some $C \in \mathbb{R}$. By hypothesis, $[-\epsilon, \epsilon]$ ($0 < \epsilon < 1$) is contained in the interior of the contact set, and $\bar{\psi}'$ is Lipschitz-continuous in $]-\epsilon, 0[$ and in $]0, \epsilon[$. The distribution $\bar{u}_n - \bar{\psi}$ is therefore supported in $]-1, 1[\setminus]-\epsilon, \epsilon[$. We first focus on the first term of expression (22) and set:

$$\begin{aligned} \tau(x) &\stackrel{\text{def}}{=} \overline{\mathcal{L}'}(\bar{\phi})(x) \\ &= \frac{s(x) \bar{\phi}'(x)}{1 + s^2(x)} - \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{\bar{\phi}'(x') (1-x') e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x' - x}, \end{aligned}$$

930 (up to an additive term of the form $C t_1$ that we bring together with the last term of (22)), thanks to proposition 19. The left limit and the right limit at $x = 0$ of $\tau(x)$ can be computed by use of proposition 8 in [5]. They are found to have the same value:

$$\lim_{x \rightarrow 0} \tau(x) = \frac{\bar{\psi}'(0+) - \bar{\psi}'(0-)}{s_+ - s_-}.$$

935 Therefore, the function τ is continuous at 0. It is readily checked to have also finite limits at $x = -1$ and $x = 1$. Since its restriction to $]-1, 0[$ and to $]0, 1[$ is of class C^∞ , the distributional derivative τ' is therefore an integrable function.

Thanks to theorem 36 in appendix C, the function τ (extended by 0 outside $] -1, 1[$) satisfies:

$$-\frac{1}{\pi} \tau * \text{pv } 1/x + s \tau = \bar{\phi}', \quad \text{in }] -1, 1[.$$

Taking the derivative (in the sense of distributions), we obtain:

$$-\frac{1}{\pi} \tau' * \text{pv } 1/x + s \tau' + \tau(0)(s_+ - s_-) \delta = (\bar{\psi}'(0+) - \bar{\psi}'(0-)) \delta, \quad \text{in }] -1, 1[,$$

940 that is:

$$-\frac{1}{\pi} \tau' * \text{pv } 1/x + s \tau' = 0, \quad \text{in }] -1, 1[.$$

Making use² of theorem 36 in appendix C, we have:

$$\tau'(x) = C_0 t_0 + C_1 t_1,$$

for some constants $C_0, C_1 \in \mathbb{R}$, and therefore:

$$\tau(x) = \tau(0) + \int_0^x (C_0 t_0(x') + C_1 t_1(x')) dx'.$$

Hence, the claim in proposition 24 holds true at least for the term $\tau = \overline{\mathcal{L}}'(\bar{\phi})$ of expression (22). Since $\bar{\psi}' - \bar{\phi}'$ is Lipschitz-continuous, $\overline{\mathcal{L}}'(\bar{\psi} - \bar{\phi})$ is
 945 product of t_0 and some Hölder-continuous function of exponent $1 - \beta$, thanks to proposition 19 in this article and proposition 7 in [5]. Hence, the claim in proposition 24 holds true at least for the sum of the first two terms of expression (22).

In the case where $\bar{u}_n - \bar{\psi} \in W^{1,\infty}(-1, 1)$, $\overline{\mathcal{L}}(\bar{u}_n - \bar{\psi})$ is given by:

$$\begin{aligned} \overline{\mathcal{L}}(\bar{u}_n - \bar{\psi})(x) &= \frac{s(x)(u'(x) - \bar{\psi}(x))}{1 + s^2(x)} \\ &- \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \oint_{-1}^1 \frac{(u'(x') - \bar{\psi}(x'))(1-x')e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x' - x} + C' t_1, \end{aligned}$$

thanks to theorem 36. In that case:

$$\begin{aligned} \forall x \in] -\varepsilon, \varepsilon[, \quad \overline{\mathcal{L}}(\bar{u}_n - \bar{\psi})(x) &= \\ &- \frac{e^{\tau(x)}}{(1-x)\sqrt{1+s^2(x)}} \frac{1}{\pi} \int_{-1}^1 \frac{(u'(x') - \bar{\psi}(x'))(1-x')e^{-\tau(x')}}{\sqrt{1+s^2(x')}} \cdot \frac{dx'}{x' - x} + C' t_1. \end{aligned}$$

²we use actually a slightly stronger result, which is that $C_0 t_0 + C_1 t_1$ are not only all the solutions of the homogeneous equation in $\cup_{p>1} L^p(-1, 1)$ but also in the larger space of all the L^1 functions whose convolution with $\text{vp}1/x$ is in L^1 (Hardy space). The proof of this stronger result is straightforward from the proof of theorem 36.

The first term in the second member is the product of t_0 with some function whose restriction to $] -\epsilon, \epsilon[$ is of class C^∞ . Therefore, $C' = -C$ since t_n is a nonpositive measure. Hence, the restriction to $] -\epsilon, \epsilon[$ of the last three terms in the second member of (22) equals the product of t_0 with some C^∞ function. If $\bar{u}_n - \bar{\psi} \notin W^{1,\infty}(-1, 1)$, a straightforward limiting procedure shows that the same conclusion holds true. This is enough to conclude the proof. \square

955 Appendix A: a generalization of the Lions-Stampacchia theorem

Let H be a Hilbert space with scalar product (\cdot, \cdot) and norm $\|\cdot\|$. We are given a bilinear form $a(\cdot, \cdot)$ on $H \times H$ which is assumed to be:

- coercive: $\exists \alpha > 0, \quad \forall v \in H, \quad a(v, v) \geq \alpha \|v\|^2,$
- continuous: $\exists M > 0, \quad \forall u, v \in H, \quad a(u, v) \leq M \|u\| \|v\|.$

960 We are also given a function $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ which is assumed to be:

- proper: $\exists u \in H, \quad \phi(u) \neq +\infty,$
- convex: $\forall \lambda \in [0, 1], \quad \forall u, v \in H, \quad \phi(\lambda u + (1 - \lambda)v) \leq \lambda \phi(u) + (1 - \lambda)\phi(v).$
- lower-semicontinuous: $\forall u \in H, \quad \phi(u) \leq \liminf_{v \rightarrow u} \phi(v).$

The following theorem is in the line of [10], chapter 2, section 8.5 or in [11], chapter II, section 3. Since the precise statement that is needed in this paper is not exactly found in these references, a proof is provided here for easy reference, but it brings nothing newer than the above references.

Theorem 25. *There exists a unique $u \in H$ such that:*

$$\forall v \in H, \quad a(u, v - u) + \phi(v) - \phi(u) \geq 0.$$

970 *Furthermore, in the particular case where $a(\cdot, \cdot)$ is symmetric, u is characterised by:*

$$\frac{1}{2}a(u, u) + \phi(u) = \min_{v \in H} \left\{ \frac{1}{2}a(v, v) + \phi(v) \right\}.$$

Remark. Let K be a nonempty closed convex subset of H . Its indicator function I_K (that is, the function that takes identically the value 0 in K and $+\infty$ outside K) is proper, lower-semicontinuous and convex. Let also $l \in H'$ be a continuous linear form on H . In the particular case where $\phi = I_K - l$, theorem 25 is nothing but the well-known Lions-Stampacchia theorem [1], and in the case where $K = H$, one recovers the Lax-Milgram theorem.

Proof. Note that the uniqueness of u is obvious due to the coercivity of a . Let us first examine the particular situation where a is symmetric. The epigraph:

$$\text{epi } \phi = \left\{ (v, \lambda) \in H \times \mathbb{R} \mid \phi(v) \leq \lambda \right\},$$

is a nonempty (since ϕ is proper), closed (since ϕ is lower semicontinuous), convex
 980 (since ϕ is convex) subset of $H \times \mathbb{R}$. Pick $\lambda_0 < \phi(v_0) < +\infty$. By Hahn-Banach
 theorem, there exists a closed hyperplane that separates strictly the singleton
 $\{(v_0, \lambda_0)\}$ and $\text{epi } \phi$. In particular, there exists $\gamma \in \mathbb{R}$ and a continuous linear
 form $l \in H'$ such that:

$$\forall v \in H, \quad \phi(v) \geq \langle l, v \rangle_{H', H} \geq \beta \|v\| + \gamma,$$

where $\beta = -\|l\|_{H'}$. Therefore:

$$\forall v \in H, \quad \frac{1}{2}a(v, v) + \phi(v) \geq \alpha \|v\|^2 + \beta \|v\| + \gamma.$$

985 Since $\alpha > 0$:

$$\lim_{\|v\| \rightarrow +\infty} \left\{ \frac{1}{2}a(v, v) + \phi(v) \right\} = +\infty,$$

and that function, being convex and lower semicontinuous attains its minimum,
 that is, there exists $u \in H$ such that:

$$\frac{1}{2}a(u, u) + \phi(u) = \min_{v \in H} \left\{ \frac{1}{2}a(v, v) + \phi(v) \right\}.$$

This classically entails that 0 is in the subdifferential of that function at u :

$$\exists g \in \partial\phi[u], \quad \forall v \in H, \quad a(u, v - u) + (g, v - u) = 0,$$

where $\partial\phi[u]$ is the subdifferential of ϕ at u (we recall that the subdifferential of
 990 the sum of a convex function and an everywhere continuous convex function is
 the sum of the subdifferentials, [12], corollary 2.63). But, from the definition of
 the subdifferential:

$$\forall v \in H, \quad \phi(v) - \phi(u) \geq (g, v - u),$$

and therefore:

$$\forall v \in H, \quad a(u, v - u) + \phi(v) - \phi(u) \geq 0.$$

Hence the theorem is proved in the particular case where $a(\cdot, \cdot)$ is symmetric.

995 Let us turn to the general case. There exists a continuous linear operator
 $A : H \mapsto H$ such that:

$$\forall u, v \in H, \quad a(u, v) = (Au, v).$$

It satisfies $\|Au\| \leq M\|u\|$. Let $\rho > 0$ be a real constant to be fixed in the
 sequel, and $l_1, l_2 \in H$. Thanks to the first part of the proof, there exists uniquely
 $u_1, u_2 \in H$ such that:

$$\forall v \in H, \quad (u_i, v - u_i) + (l_i, v) + \rho\phi(v) - (l_i, u_i) - \rho\phi(u_i) \geq 0,$$

1000 where $i = 1, 2$. We easily obtain:

$$\|u_2 - u_1\|^2 \leq (l_1 - l_2, u_2 - u_1),$$

which entails:

$$\|u_2 - u_1\| \leq \|l_2 - l_1\|,$$

by the Cauchy-Schwarz inequality. Now, given an arbitrary $w \in H$, we denote by $u = Pw$ the unique solution of:

$$\forall v \in H, \quad (u, v - u) + (\rho Aw - w, v) + \rho\phi(v) - (\rho Aw - w, u) - \rho\phi(u) \geq 0,$$

The nonlinear mapping $P : H \mapsto H$ satisfies the estimate:

$$\|Pw_2 - Pw_1\| \leq \|w_2 - w_1 - \rho(Aw_2 - Aw_1)\|,$$

and therefore:

$$\begin{aligned} \|Pw_2 - Pw_1\|^2 &\leq \|w_2 - w_1\|^2 - 2\rho(w_2 - w_1, Aw_2 - Aw_1) + \rho^2 \|Aw_2 - Aw_1\|^2, \\ &\leq (1 - 2\rho\alpha + \rho^2 M^2) \|w_2 - w_1\|^2. \end{aligned}$$

1005 The nonlinear mapping $P : H \mapsto H$ is therefore Lipschitz-continuous with modulus:

$$k = \sqrt{1 - 2\rho\alpha + \rho^2 M^2}.$$

Choosing $\rho \in]0, 2\alpha/M^2[$, we have $k < 1$ and P therefore has a unique fixed point u , thanks to the Banach fixed point theorem. It satisfies:

$$\forall v \in H, \quad \rho(Au, v) + \rho\phi(v) - \rho(Au, u) - \rho\phi(u) \geq 0,$$

and is therefore the expected solution. \square

1010 Appendix B: the spaces $H^{1/2}$ and $H^{-1/2}$

We recall here for convenience some basic definitions and facts about the spaces $H^{1/2}$ and $H^{-1/2}$. The proofs of all the results (except for the last two ones) that are stated here are to be found in [13], chapter 1. Incidentally, the subtleties related to the use of the space $H_{00}^{1/2}$ seems to have escaped the attention
1015 of the contact mechanics community with the exception of [14].

Definition 26. For $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, the space:

$$H^{1/2}(a, b) \stackrel{\text{def}}{=} \left\{ u \in L^2(a, b) \mid \int_a^b \int_a^b \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy < \infty \right\},$$

endowed with the natural norm:

$$\|u\|_{H^{1/2}(a, b)} \stackrel{\text{def}}{=} \left(\int_a^b |u(x)|^2 dx + \int_a^b \int_a^b \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy \right)^{1/2}$$

is a Hilbert space.

The step function:

$$H(x) = \begin{cases} 0 & \text{if } x \in [-1, 0[, \\ 1 & \text{if } x \in [0, 1], \end{cases}$$

1020 is not in $H^{1/2}(-1, 1)$. Therefore, the extension by zero of some $u \in H^{1/2}(-1, 1)$ will not be, in general, in $H^{1/2}(\mathbb{R})$. Besides, we have:

$$\log \log |x/2| \in H^{1/2}(-1, 1),$$

which shows that $H^{1/2}(-1, 1)$ contains noncontinuous or even unbounded functions.

With $u \in L^1 \cap L^2$, the following definition:

$$\mathcal{F}[u](t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x) e^{ixt} dx, \quad (23)$$

1025 is adopted for the Fourier transform, which can be extended, as usual, to all the tempered distributions.

Proposition 27. For $u \in H^{1/2}(\mathbb{R})$, the expression:

$$\|u\|_{H^{1/2}(\mathbb{R})} \stackrel{\text{def}}{=} \left(\int_{-\infty}^{+\infty} (1 + |t|) |\mathcal{F}[u](t)|^2 dt \right)^{1/2},$$

defines a norm on the space $H^{1/2}(\mathbb{R})$ which is equivalent to that of $H^{1/2}(\mathbb{R})$.

1030 **Theorem 28.** The space $H^{1/2}(a, b)$ is (algebraically) the space of the restrictions to $]a, b[$ of the functions in $H^{1/2}(\mathbb{R})$:

$$H^{1/2}(a, b) = \left\{ U_{]a, b[} \mid U \in H^{1/2}(\mathbb{R}) \right\}.$$

For $u \in H^{1/2}(a, b)$, the expression:

$$\|u\|_{H^{1/2}(a, b)} \stackrel{\text{def}}{=} \inf \left\{ \|U\|_{H^{1/2}(\mathbb{R})} \mid U_{]a, b[} = u \right\}$$

defines a norm on the space $H^{1/2}(a, b)$ which is equivalent to that of $H^{1/2}(a, b)$.

Theorem 29. The set $C_0^\infty(]a, b[)$ of the functions with infinitely many continuous derivatives and compact support in $]a, b[$ is dense in $H^{1/2}(a, b)$:

$$H_0^{1/2}(a, b) = H^{1/2}(a, b).$$

1035 **Definition 30.** For $u \in H^{1/2}(a, b)$, we will denote by \bar{u} , the extension of u by 0 outside $]a, b[$. We set:

$$H_{00}^{1/2}(a, b) \stackrel{\text{def}}{=} \left\{ u \in H^{1/2}(a, b) \mid \bar{u} \in H^{1/2}(\mathbb{R}) \right\}.$$

Theorem 31. *The following identity holds true:*

$$H_{00}^{1/2}(a, b) = \left\{ u \in H^{1/2}(a, b) \mid u(x) \frac{\sqrt{b-a}}{\sqrt{b-x}\sqrt{x-a}} \in L^2(a, b) \right\},$$

and the space $H_{00}^{1/2}(a, b)$ endowed with the norm:

$$\|u\|_{H_{00}^{1/2}(a, b)} = \left(\|u\|_{H^{1/2}(a, b)}^2 + \left\| \frac{u\sqrt{b-a}}{\sqrt{b-x}\sqrt{x-a}} \right\|_{L^2(a, b)}^2 \right)^{1/2}.$$

is a Hilbert space.

1040 Definition 32. *We denote by $H^{-1/2}(a, b)$ the dual space of $H^{1/2}(a, b)$.*

Since the restriction to $]a, b[$ of some function in $H^{1/2}(\mathbb{R})$ is in $H^{1/2}(a, b)$, then, on the dual side, the extension \bar{u} of some $u \in H^{-1/2}(a, b)$ by 0 outside $]a, b[$ defines a distribution in $H^{-1/2}(\mathbb{R})$.

Proposition 33. *For $u \in H^{-1/2}(a, b)$, the expression:*

$$\|u\|_{H^{-1/2}(a, b)} \stackrel{\text{def}}{=} \left(\int_{-\infty}^{+\infty} \frac{|\mathcal{F}[\bar{u}](t)|^2}{1+|t|} dt \right)^{1/2},$$

1045 *defines a norm on the space $H^{-1/2}(a, b)$ which is equivalent to that of $H^{-1/2}(a, b)$.*

Based on theorem 31, the restriction to $]a, b[$ of some distribution in $H^{-1/2}(\mathbb{R})$ is not in $H^{-1/2}(a, b)$, in general, but in the larger space $H_{00}^{1/2'}(a, b)$. Any $u \in H_{00}^{1/2'}(a, b)$ can be written in the form:

$$u = u_0 + f_1,$$

1050 with $u_0 \in H^{-1/2}(a, b)$ and f_1 is a function such that $f_1\sqrt{b-x}\sqrt{x-a}/\sqrt{b-a} \in L^2(a, b)$.

Theorem 34 (Sobolev embeddings). *Assume that a and b are finite. Then:*

$$\forall p \in [1, \infty[, \quad H^{1/2}(a, b) \subset L^p(a, b),$$

with continuous embeddings. On the dual side:

$$\forall p \in]1, \infty], \quad L^p(a, b) \subset H^{-1/2}(a, b),$$

with continuous embeddings.

1055 Proposition 35. *Assume that a and b are finite. Let $u \in \mathcal{M}([a, b])$ be a Radon measure (that is, an element of the dual space of $C^0([a, b])$). Suppose that in addition, $u \in H^{-1/2}(a, b)$ or $u \in H_{00}^{1/2'}(a, b)$. Then the measure u has no atom in $]a, b[$.*

Proof. Let $u \in \mathcal{M}([a, b]) \cap H^{-1/2}(a, b)$. The function f defined by:

$$f(x) = \begin{cases} 0, & \text{if } x < a, \\ \int_{[a,x]} u - \frac{x-a}{b-a} \int_{[a,b]} u, & \text{if } a < x < b, \\ 0, & \text{if } x > b, \end{cases}$$

is supported in $[a, b]$, has bounded variation and its distributional derivative equals the extension of u by zero outside $[a, b]$. In addition, the function f is in $H^{1/2}(a, b)$ (consider the Fourier transform of f). Having bounded variation, the function f admits a left limit and a right limit for all $x_0 \in [a, b]$. But, the integrability of the function:

$$(x, y) \rightarrow \left(\frac{f(x) - f(y)}{x - y} \right)^2,$$

on a neighbourhood of (x_0, x_0) requires that these left and right limits must be equal. The function f is therefore a continuous function with bounded variation. Hence, its distributional derivative u is a measure with no atom.

In the case where we have only $u \in \mathcal{M}([a, b]) \cap H_{00}^{1/2'}(a, b)$, it is an easy matter to build an extension of u (still denoted by u) such that $u \in \mathcal{M}([a-1, b+1]) \cap H^{-1/2}(a-1, b+1)$. Then, the expected conclusion follows from the first part of the proof. \square

Appendix C: the Carleman singular integral equation with discontinuous coefficient.

Given $g \in L^\infty(-1, 1)$ and $a \in W^{1,\infty}(-1, 1)$, the Carleman singular integral equation is the problem of finding $f \in \cup_{p>1} L^p(-1, 1)$ such that:

$$\text{for a.a. } x \in]-1, 1[, \quad a(x) f(x) - \frac{1}{\pi} \oint_{-1}^1 \frac{f(t)}{t-x} dt = g(x),$$

This linear equation was solved very elegantly by Carleman in 1922. The set of solutions is of dimension 1, and the solution is unique when the total mass $\int_{-1}^1 f$ has a prescribed value.

The extension of Carleman's analysis to the case where $a(x)$ is supposed only *piecewise* Lipschitz-continuous was performed in [5]. Surprisingly, the kernel (the vector space of solutions to the homogeneous equation), although still finite-dimensional needs not to be of dimension 1 anymore. A precise proof of the following theorem is to be found in [5].

Theorem 36. *Let $g \in L^\infty(-1, 1; \mathbb{R})$ and $a : [-1, 1] \rightarrow \mathbb{R}$ be some piecewise Lipschitz-continuous function. Take:*

$$\theta(x) = \begin{cases} \arctan \frac{-1}{]0, \pi[a(x)}, & \text{if } x \in [-1, 1], \\ 0, & \text{if } |x| > 1. \end{cases}$$

1085 The function $\theta(x)$ is piecewise Lipschitz-continuous in $[-1, 1]$ and we take:

$$-1 = \tilde{x}_1 < \tilde{x}_2 < \cdots < \tilde{x}_m = 1,$$

to denote all its discontinuity points, and:

$$-1 < x_1 < x_2 < \cdots < x_n = 1,$$

to denote all the discontinuity points \tilde{x}_i which in addition satisfy the condition:

$$\theta(\tilde{x}_i - 0) > \theta(\tilde{x}_i + 0).$$

Then, all the solutions $f \in \cup_{p>1} L^p(-1, 1; \mathbb{R})$ of the singular integral equation:

$$\text{for a.a. } x \in]-1, 1[, \quad \frac{1}{\pi} \oint_{-1}^1 \frac{f(t)}{t-x} dt + a(x) f(x) = g(x),$$

are given by:

$$f(x) = \frac{a(x)g(x)}{a^2(x)+1} - \frac{e^{\tau(x)}}{\sqrt{a^2(x)+1}} \frac{1}{\pi} \oint_{-1}^1 \frac{g(t)e^{-\tau(t)}}{\sqrt{a^2(t)+1}} \cdot \frac{dt}{t-x} + \frac{P(x)e^{\tau(x)}}{\prod_{i=1}^n (x_i - x)\sqrt{a^2(x)+1}},$$

where:

$$\tau(x) = \frac{1}{\pi} \oint_{-1}^1 \frac{\theta(t)}{t-x} dt,$$

1090 and, $P \in \mathbb{R}_{n-1}[X]$ is some arbitrary real polynomial the degree of which is at most $n-1$.

Careful examination of the proof of theorem 36 in [5] shows that we also have:

$$\forall k \in \{1, 2, \dots, n-1\}, \quad \int_{-1}^1 \frac{e^{\tau(x)}}{\prod_{i=k}^n (x_i - x)\sqrt{a^2(x)+1}} = 0.$$

References

- 1095 [1] J. L. Lions, G. Stampacchia, Variational inequalities, Communications on Pure and Applied Mathematics 20 (1967) 493–519.
- [2] G. Duvaut, J. L. Lions, Les Inéquations en Mécanique et en Physique, Dunod, Paris, 1972.
- 1100 [3] C. Eck, J. Jarušek, M. Krbec, Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems., Monographs & Textbooks in Pure & Appl. Math. No. 270 (ISBN 1-57444-629-0). Chapman & Hall/CRC, Boca Raton, 2005.

- [4] P. Ballard, Steady sliding frictional contact problems in linear elasticity, *Journal of Elasticity* 110 (1) (2013) 33–61.
- 1105 [5] P. Ballard, J. Jarušek, Indentation of an elastic half-space by a rigid flat punch as a model problem for analyzing contact problems with coulomb friction, *Journal of Elasticity* 103 (2011) 15–52.
- [6] L. A. Galin, Contact problems in the theory of elasticity, Moscow, 1953. English Translation by H. Moss, North Carolina State College, Department of Mathematics, 1961.
- 1110 [7] M. Comninou, Stress singularity at a sharp edge in contact problems with friction, *Zeitschrift für Angewandte Mathematik und Physik* 27 (1976) 493–499.
- [8] J. Dundurs, M. Comninou, Some consequences of the inequality conditions in contact and crack problems, *Journal of Elasticity* 9 (1) (1979) 71–82.
- 1115 [9] P. Grisvard, Singularités en élasticité, *Archive for Rational Mechanics and Analysis* 107 (2) (1989) 157–180.
- [10] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.
- 1120 [11] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems*, SIAM, Philadelphia, 1999.
- [12] V. Barbu, T. Precupanu, *Convexity and Optimisation in Banach Spaces*, Editura Academiei, București and D. Reidel Publishing Company, Dordrecht/Boston/Lancaster (Third Edition, 1986), 1975.
- 1125 [13] J. L. Lions, E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Volume 1, Dunod, Paris, 1968.
- [14] C. Naéjus, A. Cimetière, A. Léger, A functional framework for the Signorini problem with Coulomb friction, in: M. Raous, M. Jean, J. J. Moreau (Eds.), *Contact Mechanics*, Plenum Press, New York and London, 1995, pp. 87–94.