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**FORMAL ASYMPTOTIC ANALYSIS OF ELASTIC BEAMS AND  
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ANISOTROPIC HETEROGENEOUS CASE**

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**ABSTRACT.** The modelling of ordinary beams and thin-walled beams is rigorously obtained from a formal asymptotic analysis of three-dimensional linear elasticity. In the case of isotropic homogeneous elasticity, ordinary beams yield the Navier-Bernoulli beam model, thin-walled beams with open profile yield the Vlassov beam model and thin-walled beams with closed profile the Navier-Bernoulli beam model. The formal asymptotic analysis is also extensively performed in the case of the most general anisotropic transversely heterogeneous material (meaning the heterogeneity is the same in every cross-section), delivering the same qualitative results. We prove, in particular, the non-intuitive fact that the warping function appearing in the Vlassov model for general anisotropic transversely heterogeneous material, is the same as the one appearing in the isotropic homogeneous case. In the general case of anisotropic transversely heterogeneous material, the analysis provides a rigorous and systematic constructive procedure for calculating the reduced elastic moduli, both in Navier-Bernoulli and Vlassov theories.

## 1. Introduction.

**1.1. Asymptotic analysis in elastic thin structures.** Lower-dimensional theories for elastic thin structures (such as elastic plates or beams) have been derived, historically, on the basis of *a priori* assumptions made on the three-dimensional elastic displacement field in thin domains. This is only recently (starting around 1980) that the systematic and rational derivation of those lower-dimensional theories by asymptotic analysis of three-dimensional elasticity was undertaken. The aim was twofold.

- A theoretical concern of rationally proving the relevancy of the *a priori* assumptions on which the venerable theories of plates and beams were successfully based and, also, of identifying a systematic method for obtaining a reduced model in more intricate situations.

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- A practical concern of calculating the elastic moduli appearing in the lower-dimensional theories. This issue cannot be avoided since the elastic moduli appearing in the elastic beam theory, for example, depend not only on the nature of the constitutive material but, also, on the geometry of the cross-sections. Hence, there is no workaround to a theoretical approach of elastic constitutive laws for reduced theories from three-dimensional elasticity. Asymptotic analysis enables a systematic computation of those reduced moduli and provides a means to arbitrate between the many (and often contradictory) approximate formulas that were proposed in the past.

The asymptotic analysis of elastic thin structures in order to derive rationally lower-dimensional (or, synonymously, reduced) theories, is now a mature subject. It comprises three stages.

1. First, *one* dimensionless parameter which is numerically small, must be distinguished. It is usually denoted by  $\varepsilon$ . The asymptotic analysis consists in finding an asymptotics of the solution  $\mathbf{v}^\varepsilon$  of (nonlinear or linearized) three-dimensional elasticity, as  $\varepsilon$  goes to zero. Sometimes, the choice of such a small parameter is natural and obvious (as in a plate where it is the ratio of the thickness over the width, or in a beam where it is the ratio of the diameter of a cross-section over the length), and sometimes the identification of the appropriate small parameter is not obvious at all and requires a preliminary investigation. One such example is the case of a thin-walled beam which is a slender cylinder whose cross-section is also slender. A natural approach in that case would be to introduce *two* slenderness small parameters  $\varepsilon_1$  and  $\varepsilon_2$ . Such an approach turns out to be inappropriate, as no (asymptotic) limit exist as  $(\varepsilon_1, \varepsilon_2)$  goes to  $(0, 0)$ . Indeed, setting  $\varepsilon_2 = \varepsilon_1^\alpha$ , distinct asymptotic limits are obtained depending on the choice of  $\alpha$ . As will be seen in the sequel, in the case of a thin-walled beam, the appropriate choice (giving rise to the richest asymptotic limit, namely the Vlassov model) is to set  $\alpha = 1$ . It corresponds to the situation where the slenderness of the beam and the slenderness of the cross-section are of the same order of magnitude.
2. After one small dimensionless parameter  $\varepsilon$  has been chosen, as described in step 1, a *formal* asymptotic analysis is carried out. After proper rescaling of the space coordinates to work on a fixed domain (independent of  $\varepsilon$ ), it relies on the postulate that the solution  $\mathbf{v}^\varepsilon$  of the three-dimensional elastic problem admits an expansion of the form:

$$\mathbf{v}^\varepsilon = \varepsilon^m \mathbf{v}^m + \varepsilon^{m+1} \mathbf{v}^{m+1} + \dots \quad (1)$$

where  $m$  is an unknown (possibly negative) integer. In the particular case of linearized elasticity, it is always possible to assume  $m = 0$  by appropriately rescaling the load with respect to  $\varepsilon$ , but this is not true in the general case of nonlinear elasticity. The formal asymptotic analysis consists in injecting expansion (1) into the three-dimensional equilibrium equations to compute the first terms in the asymptotic expansion. In three-dimensional elasticity, the equilibrium equations come under three forms (which are proved to be strictly equivalent in linearized elasticity): the strong formulation, the weak formulation and the variational formulation (namely, the associated minimization problem). In general, the strong formulation cannot be used because one faces the problem that the leading term in the asymptotic expansion cannot satisfy pointwisely any Neumann boundary condition on a thin edge of the boundary.

The workaround of this issue is to inject expansion (1) into the weak formulation instead, and seems to have been first suggested by Jacques-Louis Lions in [10]. This technique was successfully implemented in [11], for example, to show that the asymptotic analysis of linearly elastic shells gives rise to two distinct two-dimensional reduced models, namely the membrane and flexural shell models, according to geometrical properties. In elasticity theory, where the weak formulation is the optimality condition of an underlying minimization problem, an alternative approach is to inject the asymptotic expansion into the total energy (variational formulation). When possible, this alternative approach shows several benefits. First, the subsequent algebra is generally lighter, as hopefully illustrated by this paper. Second, it turns out that the energy of the limit model then appears naturally in the formal asymptotic analysis as the leading term in the expansion of the three-dimensional energy. This fact provides the information of how many terms in expansion (1) need to be calculated: they are the terms that are needed to evaluate the leading term in the expansion of the three-dimensional energy. The formal asymptotic analysis based on the variational formulation will be precisely described in the sequel and systematically used throughout this paper.

3. The final step is to prove a convergence result of  $\mathbf{v}^\varepsilon$  towards the leading term of the asymptotic expansion, as  $\varepsilon$  goes to zero, in the sense of an appropriate topology. In the case of an underlying minimization problem, a general framework has been developed by the Italian School of Analysis under the name of ‘ $\Gamma$ -convergence’. Two main ingredients are needed. First, a preliminary complete knowledge of the limit model (that is, the energy functional of the lower-dimensional model) is necessary. This is the reason why the formal asymptotic analysis (step 2) is needed in the analysis. Naturally, any convergence theorem supersedes the formal asymptotic analysis, but the formal asymptotic analysis is generally a necessary preliminary to prove any convergence theorem. A second ingredient is also needed to establish the convergence theorem: some compactness must be available. In return, the available compactness determines the topology in which the convergence result holds. In elasticity, compactness is generally provided by a scaled Korn-like inequality, with a constant that depends explicitly on the small parameter  $\varepsilon$ . The needed compactness can be troublesome to prove in some cases and this can constitute an obstacle on the road to pass from the formal asymptotic analysis to the convergence result.

In short, step 2 tackles the problem of identifying the asymptotic limit model, whereas step 3 deals with the problem of making precise the sense in which the convergence towards that limit holds. In this paper, only the formal asymptotic analysis will be performed (step 2), and a convergence result (step 3) is postponed to a later article.

**1.2. Asymptotic analysis in ordinary beams and thin-walled beams.** In this section, we briefly review the state of the art of asymptotic analysis in ordinary beams and thin-walled beams, within the framework of linearized elasticity.

1.2.1. *Ordinary beams.* Geometrically, ordinary beams are slender cylinders. In coordinates, such a cylinder will be denoted by  $[0, L] \times \mathcal{S}^\varepsilon$ , where  $L$  denotes the length of the cylinder and  $\mathcal{S}^\varepsilon = \varepsilon \tilde{\mathcal{S}} \subset \mathbb{R}^2$  is the current cross-section, where  $\tilde{\mathcal{S}}$  is a fixed reference cross-section.

Assuming that the cylinder is made of an elastic material, a three-dimensional equilibrium problem is considered, with equilibrium displacement field  $\mathbf{v}^\varepsilon$ . After proper rescaling of the space variables to have  $\mathbf{v}^\varepsilon$  defined on  $[0, L] \times \tilde{\mathcal{S}}$ , the (asymptotic) limit  $\varepsilon \rightarrow 0+$  is studied.

In isotropic homogeneous linearized elasticity, this asymptotic analysis is now fully understood. The limit (synonymously, reduced) model is that of Navier-Bernoulli (or, Euler-Bernoulli) and has been known for more than a century. Many theorems about the convergence of  $\mathbf{v}^\varepsilon$  towards the unique solution of the (linearized) Navier-Bernoulli theory have been proved. We can cite, for example, the proof given by Le Dret [9] for the torsion-free case, and that given by Percivale [12] for the case with torsion. The proof of Le Dret is direct and that of Percivale is in line with the framework of  $\Gamma$ -convergence. Both rely on the Korn-like inequality by Kondrat'ev and Oleinik [8].

Considerably less effort has been devoted to the case of anisotropic transversely heterogeneous material (meaning the heterogeneity is the same in every cross-section). Starting from the first half of the twentieth century, there has long been a hope that an explicit formula in terms of an integral over the cross-section should be obtained to calculate the elastic moduli of the Navier-Bernoulli beam model systematically from those of three-dimensional elasticity. It is now established that this hope is vain and that the reduced elastic moduli can only be computed by solving (in general, numerically) a two-dimensional linearly elastic problem over the cross-section. In the most general case of anisotropic transversely heterogeneous elasticity in cylinders, the only reference we are aware of, in which formal asymptotic analysis of three-dimensional linearized elasticity yields the Navier-Bernoulli beam theory and the rigorous calculation of the corresponding reduced moduli, is the book in French by Sanchez-Hubert and Sanchez-Palencia [13]. In this paper, we recover the same result as theirs, with lighter algebra. To the best of our knowledge, no convergence theorem is available yet in the anisotropic transversely heterogeneous case, but it is highly likely that the proof of Percivale [12] for the isotropic homogeneous case could be straightforwardly adapted to yield a theorem of convergence towards the reduced model arising from our formal asymptotic analysis of anisotropic heterogeneous ordinary beams.

1.2.2. *Thin-walled beams.* Geometrically, thin-walled beams are slender cylinders with slender cross-section. It was recognized very early [15, 17] that the classical Navier-Bernoulli beam theory may prove irrelevant in the case of thin-walled beams with open cross-section. In the case of a thin-walled beam made of an isotropic homogeneous elastic material, the equations of the so-called Vlassov beam theory were known just after World War II, based on *a priori* assumptions on the solutions of three-dimensional elasticity.

The first attempt to make an asymptotic analysis of thin-walled beams was that of Traub and Viaño [16]. They introduce two small slenderness parameters (one for the slenderness of the cylinder, and one for the slenderness of the cross-section) which are successively made to converge towards zero, but they are unable to yield any convincing result in this framework. Careful examination of the Saint-Venant solution (see for example [14]), which is a quasi-explicit exact solution for the equilibrium of a three-dimensional isotropic homogeneous linearly elastic cylinder loaded only at the two extremities, shows that the richest asymptotic limit should be only encountered in the case where the two small slenderness parameters are of the same order of magnitude. Here, ‘richest asymptotic limit’ means that the ordinary beam

displacement field and the warping displacement have same orders of magnitude, as, otherwise, one is dominated by the other. This strongly suggests undertaking the asymptotic analysis by taking the two small slenderness parameters proportional, that is, by making them converge simultaneously towards 0, while their ratio is kept constant. This was the starting point of most subsequent studies, and, in particular, the starting point of Hamdouni and Millet in [6, 7]. However, they tackle the asymptotic analysis within the strong formulation, which forces them to relax the free boundary condition on the thin part of the lateral surface. This arbitrary relaxation jeopardizes uniqueness of solution in the three-dimensional problem, and they do not obtain the expected Vlassov model. The next progress comes from the Italian School. They skip the formal asymptotic analysis to attack the problem directly of the convergence within the framework of  $\Gamma$ -convergence [3]. Their analysis is restricted to the case of the rectangular cross-section (that is, a plate with edges of respective lengths 1,  $\varepsilon$ ,  $\varepsilon^2$ ). But the case of the rectangular cross-section is precisely the case where the Vlassov theory degenerates into the Navier-Bernoulli one (see section 1.3). Accordingly, they prove in [3] that the three-dimensional energy  $\Gamma$ -converges, in an appropriate topology, towards the Navier-Bernoulli energy. In a later paper [4], they take advantage of their results in [3] to treat the case where the cross-section is an assembly of two or three orthogonal thin rectangles, encompassing the cases of ‘T’, ‘I’, ‘U’-shaped cross-section. In that case, they are able to prove rigorously that the three-dimensional energy  $\Gamma$ -converges towards the Vlassov energy. In another paper [5], they extend the result of [3] to the case of an anisotropic heterogeneous material, but still in the case of the rectangular cross-section. Once again, they prove that the three-dimensional energy  $\Gamma$ -converges, in an appropriate topology, towards the Navier-Bernoulli energy. Their results seem to be the best available up to now. They address the issue of convergence, but are restricted to the rectangular cross-section (except for [4]), which is precisely the situation where the Vlassov theory degenerates into the Navier-Bernoulli theory.

In this paper, we address the general situation of an arbitrary slender cross-section (in both cases of a closed and open profile), in the case of an isotropic homogeneous material, but also in the most general case of any anisotropic transversely heterogeneous material, in the framework of linearized elasticity. We develop a rigorous formal asymptotic analysis. It shows that the limit model is that of Navier-Bernoulli in the case of a closed profile, and that of Vlassov (or its appropriate generalization in the anisotropic heterogeneous case) in the case of an open profile. Hence, the limit model is completely exhibited in all situations (within the linearized theory). A convergence theorem towards that limit is work under progress.

**1.3. The structure of the Vlassov equations of thin-walled beams.** In this section, we review the equations of the classical (linear) Vlassov theory of elastic thin-walled beams, without any attempt of asymptotic analysis yet. These equations originate in the old paper by Timoshenko [15] and the fundamental contribution of Vlassov [17]. They were based on *a priori* assumptions about the solutions of three-dimensional linear elasticity in thin-walled cylinders. This section is devoted to replacing these classical equations in the modern perspective of the Virtual Power Principle, which yields, in particular, the underlying elastic energy.

In the usual three-dimensional Euclidean space with orthonormal Cartesian coordinate system  $Oxyz$ , we consider a connected cylinder with axis along  $Ox$  and length  $L$ . Let  $\mathcal{S}$  denote the current cross-section (a smooth domain in  $\mathbb{R}^2$ ), and  $\mathcal{S}_0$ ,

$\mathcal{S}_L$  be the two extreme cross-sections located respectively in the planes  $x = 0$  and  $x = L$ .

1.3.1. *The general structure of beam theories with warping.* The usual (linearized) theory of (Timoshenko) beams is based on a reduced displacement field of the form:

$$\mathbf{u}(x) + \boldsymbol{\omega}(x) \times (y \mathbf{e}_y + z \mathbf{e}_z),$$

where  $\times$  denotes the cross-product and:

- the vector  $\mathbf{u}(x)$  is the displacement of the middle line,
- the vector  $\boldsymbol{\omega}(x)$  is the infinitesimal rotation of the current cross-section whose motion is supposed to be *rigid*.

In the Navier-Bernoulli theory, cross-sections must remain orthogonal to the middle line in the deformation, which reads as:

$$u'_y - \omega_z = 0, \quad u'_z + \omega_y = 0,$$

where  $'$  denotes the derivative with respect to  $x$ .

In thin-walled beams with open cross-section (meaning that the cross-section is simply connected), the assumption of a rigid motion of the cross-sections must be dropped. The Vlassov theory of beam is a theory whose kinematics is enriched with a nonuniform axial displacement, called *warping*. More precisely, the reduced displacement field in Vlassov (linearized) theory is of the form:

$$\mathbf{u}(x) + \boldsymbol{\omega}(x) \times (y \mathbf{e}_y + z \mathbf{e}_z) + a(x) \psi(y, z) \mathbf{e}_x,$$

where  $\psi$  is a *torsional warping function* that is supposed to be given and fixed, as part of the postulated reduced kinematics. On the contrary, the amplitude of warping  $a(x)$  of the current cross-section is a new kinematical unknown.

As the kinematical unknowns depend only on the space variable  $x$ , the Vlassov theory is a beam theory (one space variable) and it is expected that the equilibrium equation should be an ordinary differential equation (and not a partial differential equation).

Virtual velocities in Vlassov theory take the form:

$$\mathbf{w} = \dot{\mathbf{u}}(x) + \dot{\boldsymbol{\omega}}(x) \times (y \mathbf{e}_y + z \mathbf{e}_z) + \dot{a}(x) \psi(y, z) \mathbf{e}_x. \quad (2)$$

Here, the dot does not really refer to a time derivative. Rather, it is the velocity field corresponding to an imaginary time-dependent motion. The gradient with respect to the three space variables of that virtual velocity field is:

$$\begin{aligned} \nabla \mathbf{w} &= \left( \dot{u}'_x + z \dot{\omega}'_y - y \dot{\omega}'_z + \dot{a}' \psi \right) \mathbf{e}_x \otimes \mathbf{e}_x - \dot{\omega}'_x \mathbf{e}_y \otimes \mathbf{e}_z + \dot{\omega}'_x \mathbf{e}_z \otimes \mathbf{e}_y \\ &+ \left( \dot{a} \frac{\partial \psi}{\partial y} - \dot{\omega}_z \right) \mathbf{e}_x \otimes \mathbf{e}_y + \left( \dot{u}'_y - z \dot{\omega}'_x \right) \mathbf{e}_y \otimes \mathbf{e}_x \\ &+ \left( \dot{a} \frac{\partial \psi}{\partial z} + \dot{\omega}_y \right) \mathbf{e}_x \otimes \mathbf{e}_z + \left( \dot{u}'_z + y \dot{\omega}'_x \right) \mathbf{e}_z \otimes \mathbf{e}_x, \end{aligned}$$

where  $'$  still denotes the derivative with respect to  $x$ .

Considering body forces  $\mathbf{f}$  in the cylinder and surface traction  $\mathbf{t}$ , the equilibrium can be expressed in terms of the Cauchy stress field  $\boldsymbol{\sigma}$  under the following weak form (Principle of Virtual Power):

$$\int_0^L \int_{\mathcal{S}} \boldsymbol{\sigma} : \nabla \mathbf{w} = \int_0^L \int_{\mathcal{S}} \mathbf{f} \cdot \mathbf{w} + \int_0^L \int_{\partial \mathcal{S}} \mathbf{t} \cdot \mathbf{w} + \int_{\mathcal{S}_0} \mathbf{t} \cdot \mathbf{w} + \int_{\mathcal{S}_L} \mathbf{t} \cdot \mathbf{w} \quad (3)$$

for all smooth virtual velocity field  $\mathbf{w} : [0, L] \times \mathcal{S} \rightarrow \mathbb{R}^3$ . Here, the notation ‘ $\cdot$ ’ stands for the doubly contracted product of Euclidean tensors. Picking  $\mathbf{w} = \theta(x) \psi(y, z) \mathbf{e}_x$ , with  $\theta$  arbitrary, we obtain, for all  $x \in ]0, L[$ :

$$\int_{\mathcal{S}} \sigma_{xy} \frac{\partial \psi}{\partial y} + \sigma_{xz} \frac{\partial \psi}{\partial z} = \int_{\mathcal{S}} f_x \psi + \int_{\partial \mathcal{S}} t_x \psi + \frac{d}{dx} \int_{\mathcal{S}} \sigma_{xx} \psi.$$

Hence, injecting an arbitrary Vlassov virtual velocity of the form (2) into the Principle of Virtual Power (3), we get:

$$\begin{aligned} & \int_0^L F_x \dot{u}'_x + F_y (\dot{u}'_y - \dot{\omega}_z) + F_z (\dot{u}'_z + \dot{\omega}_y) + M_x \dot{\omega}'_x + M_y \dot{\omega}'_y + M_z \dot{\omega}'_z + B \dot{a}' + B' \dot{a} \\ &= \int_0^L \left\{ \dot{\mathbf{u}} \cdot \left( \int_{\mathcal{S}} \mathbf{f} + \int_{\partial \mathcal{S}} \mathbf{t} \right) + \dot{\boldsymbol{\omega}} \cdot \left( \int_{\mathcal{S}} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{f} + \int_{\partial \mathcal{S}} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{t} \right) \right\} \\ & \quad + \dot{\mathbf{u}}(0) \cdot \left( \int_{\mathcal{S}_0} \mathbf{t} \right) + \dot{\boldsymbol{\omega}}(0) \cdot \left( \int_{\mathcal{S}_0} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{t} \right) + \dot{a}(0) \int_{\mathcal{S}_0} t_x \psi \\ & \quad + \dot{\mathbf{u}}(L) \cdot \left( \int_{\mathcal{S}_L} \mathbf{t} \right) + \dot{\boldsymbol{\omega}}(L) \cdot \left( \int_{\mathcal{S}_L} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{t} \right) + \dot{a}(L) \int_{\mathcal{S}_L} t_x \psi, \quad (4) \end{aligned}$$

for all smooth functions  $\dot{\mathbf{u}}, \dot{\boldsymbol{\omega}}, \dot{a}$  defined on  $[0, L]$ , and where:

- $\mathbf{F}(x) = \int_{\mathcal{S}} \boldsymbol{\sigma} \cdot \mathbf{e}_x$ , is the internal force in the cross-section of abscissa  $x$ ,
- $\mathbf{M}(x) = \int_{\mathcal{S}} (y \mathbf{e}_y + z \mathbf{e}_z) \times (\boldsymbol{\sigma} \cdot \mathbf{e}_x)$ , is the internal moment in the cross-section of abscissa  $x$ ,
- $B(x) = \int_{\mathcal{S}} \sigma_{xx} \psi$ , is the internal *bimoment*.

The bimoment is a generalized internal force associated with restrained warping and introduced by Vlassov. Hence, the virtual power approach shows that the appropriate representation of internal forces within a beam theory with warping is the triple  $(\mathbf{F}, \mathbf{M}, B)$ . The weak form (4) of the equilibrium equation in Vlassov theory is equivalent to:

$$\left\{ \begin{array}{l} \mathbf{F}'(x) + \int_{\mathcal{S}} \mathbf{f} + \int_{\partial \mathcal{S}} \mathbf{t} = \mathbf{0}, \\ \mathbf{M}'(x) + \mathbf{e}_x \times \mathbf{F}(x) + \int_{\mathcal{S}} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{f} + \int_{\partial \mathcal{S}} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{t} = \mathbf{0}, \\ \mathbf{F}(0) = - \int_{\mathcal{S}_0} \mathbf{t}, \quad \mathbf{F}(L) = \int_{\mathcal{S}_L} \mathbf{t}, \\ \mathbf{M}(0) = - \int_{\mathcal{S}_0} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{t}, \quad \mathbf{M}(L) = \int_{\mathcal{S}_L} (y \mathbf{e}_y + z \mathbf{e}_z) \times \mathbf{t}, \\ B(0) = - \int_{\mathcal{S}_0} t_x \psi, \quad B(L) = \int_{\mathcal{S}_L} t_x \psi. \end{array} \right.$$

In the case of Vlassov theory, this general framework is complemented with the following internal constraints.

- The Navier-Bernoulli constraint:

$$u'_y - \omega_z = 0, \quad u'_z + \omega_y = 0,$$

which expresses (within the linearized theory) that cross-sections remain orthogonal to the middle line in the deformation.

- The Vlassov constraint:

$$a = \omega'_x$$

which determines the amplitude of warping of the current cross-section in terms of the local twist.

In that case, the left-hand side in the Principle of Virtual Power (4) simplifies as:

$$\int_0^L F_x \dot{u}'_x + (M_x + B') \dot{\omega}'_x + M_y \dot{\omega}'_y + M_z \dot{\omega}'_z + B \dot{a}'.$$

In the case of an underlying elastic energy, this integral must be the virtual time-derivative of the elastic energy. The constitutive law in a beam theory enriched with warping, and obeying the Navier-Bernoulli and Vlassov internal constraints must therefore be of the form:

$$\begin{pmatrix} F_x \\ M_y \\ M_z \\ M_x + B' \\ B \end{pmatrix} = \mathbf{c} \cdot \begin{pmatrix} u'_x \\ \omega'_y \\ \omega'_z \\ \omega'_x \\ a' \end{pmatrix} = \mathbf{c} \cdot \begin{pmatrix} u''_x \\ -u''_z \\ u''_y \\ \omega''_x \\ \omega''_x \end{pmatrix} \quad (5)$$

where  $\mathbf{c}$  denotes a matrix of order 5 of elastic moduli which must be positive definite symmetric. The corresponding elastic energy reads as:

$$\mathcal{E}_{\text{el}} = \frac{1}{2} \int_0^L (u'_x, -u''_z, u''_y, \omega'_x, \omega''_x) \cdot \mathbf{c} \cdot {}^t(u'_x, -u''_z, u''_y, \omega'_x, \omega''_x).$$

1.3.2. *Vlassov equations for an isotropic homogeneous elastic material.* We now focus to the usual case where the cylinder is made of an isotropic homogeneous elastic material with Young modulus  $E$ , Poisson ratio  $\nu$ , and the section  $\mathcal{S}$  is a thin strip of thickness  $\delta$  around a smooth nonintersecting open curve of length  $l$  in the plane  $Oyz$ , with generic point  $M(s)$  parametrized by the arc-length  $s \in [0, l]$ .

The Vlassov equations for such thin-walled beams, as they were displayed in [17] fall into the general framework, presented in the previous section, of beam theories enriched with warping and obeying to the Navier-Bernoulli and Vlassov internal constraints. Hence, the Vlassov equations are going to be made completely explicit, just by providing the torsional warping function  $\psi$  in that theory, and the positive definite symmetric matrix  $\mathbf{c}$  of reduced elastic moduli.

We denote by  $(\mathbf{t}(s), \mathbf{n}(s))$  the local orthonormal Serret-Frenet basis, and we assume that the thickness is small enough to use the curvilinear coordinate system  $(s, \eta)$  defined in the strip by:

$$Om(s, \eta) = OM(s) + \eta \mathbf{n}(s) = y \mathbf{e}_y + z \mathbf{e}_z.$$

The origin  $O$  is taken at the centroid of the curve  $M(s)$  and the axes  $Oy$  and  $Oz$  are supposed to be directed along the inertia principal axis of that curve, so that:

$$\int_0^l \mathbf{e}_y \cdot OM(s) ds = \int_0^l \mathbf{e}_z \cdot OM(s) ds = 0 = \int_0^l [\mathbf{e}_y \cdot OM(s)] [\mathbf{e}_z \cdot OM(s)] ds.$$

As  $\delta \ll l$ , the inertia principal moments of  $\mathcal{S}$  can be approximated as:

$$I_y = \delta \int_0^l [\mathbf{e}_z \cdot OM(s)]^2 ds, \quad I_z = \delta \int_0^l [\mathbf{e}_y \cdot OM(s)]^2 ds.$$

The *shear center*  $C$  of the current cross-section  $\mathcal{S}$  is the point with coordinates:

$$\begin{aligned} y_c &= -\frac{\delta}{I_y} \int_0^l (\mathbf{e}_z \cdot OM(s)) \int_0^s (\mathbf{n}(s') \cdot OM(s')) ds' ds, \\ z_c &= \frac{\delta}{I_z} \int_0^l (\mathbf{e}_y \cdot OM(s)) \int_0^s (\mathbf{n}(s') \cdot OM(s')) ds' ds, \end{aligned}$$

and the *warping function*  $\psi$  is the function defined by:

$$\begin{aligned} \psi(s) &= \int_0^s (\mathbf{n}(s') \cdot OM(s')) ds' - \frac{1}{l} \int_0^l \left( \int_0^s (\mathbf{n}(s') \cdot OM(s')) \right) ds' ds \\ &\quad - z_c \mathbf{e}_y \cdot OM(s) + y_c \mathbf{e}_z \cdot OM(s), \quad (6) \end{aligned}$$

which, by construction, fulfils:

$$\int_0^l \psi(s) ds = 0 = \int_0^l \psi(s) \mathbf{e}_y \cdot OM(s) ds = \int_0^l \psi(s) \mathbf{e}_z \cdot OM(s) ds.$$

The *torsional constant*  $J$  and the *warping constant*  $J_w$  are defined by:

$$J = \frac{l\delta^3}{3}, \quad J_w = \delta \int_0^l [\psi(s)]^2 ds.$$

With these notations, the matrix  $\mathbf{c}$  of Vlassov elastic moduli (see formula (5) for the definition of  $\mathbf{c}$ ) in the case of an isotropic homogeneous elastic material reads as:

$$\mathbf{c} = \begin{pmatrix} E|\mathcal{S}| & 0 & 0 & 0 & 0 \\ 0 & EI_y & 0 & 0 & -y_c EI_y \\ 0 & 0 & EI_z & 0 & -z_c EI_z \\ 0 & 0 & 0 & \frac{EJ}{2(1+\nu)} & 0 \\ 0 & -y_c EI_y & -z_c EI_z & 0 & EJ_w + y_c^2 EI_y + z_c^2 EI_z \end{pmatrix}$$

The first four terms on the diagonal are well-known stiffness in the classical theory of elastic beams. The additional stiffness  $EJ_w$  is called the *warping stiffness*.

To sum up, the Vlassov theory is a theory of beam where the reduced displacement field is enriched with warping, and has the following form:

$$\begin{aligned} \mathbf{v}(x, y, z) &= [u_x(x) \mathbf{e}_x + u_y(x) \mathbf{e}_y + u_z(x) \mathbf{e}_z] \\ &\quad + [\omega_x(x) \mathbf{e}_x - u'_z(x) \mathbf{e}_y + u'_y(x) \mathbf{e}_z] \times [y \mathbf{e}_y + z \mathbf{e}_z] + \omega'_x(x) \psi(s(y, z)) \mathbf{e}_x, \end{aligned}$$

where the warping function  $\psi$  is defined according to formula (6). It is associated with an elastic energy of the form:

$$\begin{aligned} \frac{1}{2} \int_0^L E|\mathcal{S}| [u'_x(x)]^2 + EI_z [u''_y(x) - z_c \omega''_x(x)]^2 + EI_y [u''_z(x) + y_c \omega''_x(x)]^2 \\ + \frac{EJ}{2(1+\nu)} [\omega'_x(x)]^2 + EJ_w [\omega''_x(x)]^2. \end{aligned}$$

Then, the above reduced energy can be used to prove that the equilibrium problem of a Vlassov beam, with various line forces along, as well as various static and kinematic conditions at both extremities, is well-posed (has a unique solution in appropriate functional spaces). Needless to say, taking  $y_c = z_c = 0 \equiv \psi$ , the usual Navier-Bernoulli beam theory is recovered. This is precisely what happens in the case of the rectangular cross-section, as  $\mathbf{n}(s) \cdot OM(s) \equiv 0$  in that case.

These classical equations of Vlassov were obtained on the basis of *a priori* assumptions on the solution of the three-dimensional problem. They will be fully recovered from a formal asymptotic analysis in this paper (and we believe, for the first time). This paper also provides an answer to the following more general issue. Is the Vlassov warping function (6) appropriate only in the isotropic homogeneous case, or does it apply to the general case of anisotropic (transversely) heterogeneous elasticity? The answer is that all the thin-walled beam theories in linearized elasticity must be based on the Vlassov warping function (6), even in the most general case of anisotropic (transversely) heterogeneous elasticity, as will be proved in this paper from formal asymptotic analysis. The cases of isotropic homogeneous and anisotropic heterogeneous elasticity differ only by the corresponding matrix  $\mathbf{c}$  of reduced elastic moduli, as in the case of ordinary beams. Practical methods for calculating the entries of the matrix  $\mathbf{c}$  will also be provided.

**Remark 1.** The above equations readily extend to the case where the current cross-section is a thin strip around a middle curve which is a connected and simply-connected finite union  $\mathcal{C}$  of smooth curves. In that case, taking an arbitrary origin  $m_0 \in \mathcal{C}$ , there is a unique arc  $m_0m$  included in the middle curve, which joins the origin  $m_0$  to an arbitrary point  $m$  of the middle curve, and the arc  $m_0m$  is a finite union of smooth curves. This remark enables one to generalize the definition of  $y_c$ ,  $z_c$  and  $\psi$  as follows.

$$\begin{aligned} y_c &= -\left(\int_{\mathcal{C}} [\mathbf{e}_z \cdot Om]^2 dm\right)^{-1} \int_{\mathcal{C}} (\mathbf{e}_z \cdot Om) \int_{m_0m} (\mathbf{n} \cdot Op) dp dm, \\ z_c &= \left(\int_{\mathcal{C}} [\mathbf{e}_y \cdot Om]^2 dm\right)^{-1} \int_{\mathcal{C}} (\mathbf{e}_y \cdot Om) \int_{m_0m} (\mathbf{n} \cdot Op) dp dm, \\ \psi(m) &= \int_{m_0m} (\mathbf{n} \cdot Op) dp - \frac{1}{l} \int_{\mathcal{C}} \int_{m_0p} (\mathbf{n} \cdot Oq) dq dp - z_c \mathbf{e}_y \cdot Om + y_c \mathbf{e}_z \cdot Om, \end{aligned}$$

where  $l$  now denotes the total length of  $\mathcal{C}$ .

**1.4. Content and organization of this paper.** In this paper, the equilibrium of a slender cylinder  $\Omega^\varepsilon = [0, L] \times \mathcal{S}^\varepsilon$  within the framework of three-dimensional linearized elasticity is studied. The coordinate system  $Oxyz$  will be used, with  $Ox$  being directed along the axis of the cylinder. We treat the case where one extremity of the cylinder is clamped and the other extremity is loaded with given surface tractions  $\mathbf{t}^\varepsilon$ . The lateral surface will always be assumed to be free of external forces and the cylinder free of body forces. This choice is only a matter of lightening the algebra and is by no means an essential assumption in our analysis: the modifications to make in the case of nonvanishing prescribed body forces and/or surface tractions on the lateral surface would be straightforward.

The two-dimensional cross-section  $\mathcal{S}^\varepsilon$  depends on the small parameter  $\varepsilon$ . Two distinct types of dependency of the domain  $\mathcal{S}^\varepsilon$  with respect to  $\varepsilon$  will be considered. One is relevant in the case of ordinary beams and the other one in the case of thin-walled beams.

**1.4.1. Principle of the formal asymptotic analysis.** The formal asymptotic analysis is carried out along the following scheme.

1. New space variables in the  $yz$ -plane are introduced, so that all the functions initially defined on  $\Omega^\varepsilon$ , are now defined on a domain  $\tilde{\Omega}$  independent of  $\varepsilon$ , when expressed in terms of the new variables (rescaling of the domain). In

particular, the equilibrium displacement field is expressed in terms of the new variables and denoted by  $\mathbf{v}^\varepsilon : \tilde{\Omega} \rightarrow \mathbb{R}^3$ . The change of variable will be supposed to be smooth, and the integrals appearing in the definition of the total energy can be expressed in terms of the rescaled variables, so that the total energy reads as:

$$\mathcal{E}^\varepsilon(\mathbf{v}) = \frac{1}{2} \int_{\tilde{\Omega}} \mathbf{e}^\varepsilon(\mathbf{v}) : \mathbf{C} : \mathbf{e}^\varepsilon(\mathbf{v}) J^\varepsilon - \int_{\tilde{S}_L} \tilde{\mathbf{t}}^\varepsilon \cdot \mathbf{v} J^\varepsilon.$$

Here,  $\mathbf{C}$  denotes the fourth order tensor of elastic moduli, supposed to be independent of  $\varepsilon$ ,  $\tilde{\mathbf{t}}^\varepsilon \in L^2(\tilde{S}_L; \mathbb{R}^3)$  the prescribed surface tractions on the extremity  $x = L$ , expressed in terms of the new space variables,  $J^\varepsilon$  is the Jacobian of the change of coordinates from  $yz$  to the new ones, and  $\mathbf{e}^\varepsilon$  is the symmetric part of the gradient operator, expressed with respect to the rescaled space variables. The vector field  $\tilde{\mathbf{t}}^\varepsilon$  may also possibly have been rescaled with an appropriate power of  $\varepsilon$  (see the discussion at section 2.1). The vector field  $\mathbf{v}^\varepsilon : \tilde{\Omega} \rightarrow \mathbb{R}^3$  is therefore the unique minimizer of  $\mathcal{E}^\varepsilon(\mathbf{v})$  over the space:

$$H_{\#}^1(\tilde{\Omega}; \mathbb{R}^3) = \left\{ \mathbf{v} \in H^1 \mid \mathbf{v} = \mathbf{0}, \text{ on } x = 0 \right\}.$$

2. The following postulated expansion:

$$\mathbf{v}^\varepsilon = \mathbf{v}^0 + \varepsilon \mathbf{v}^1 + \varepsilon^2 \mathbf{v}^2 + \dots \quad (7)$$

is injected into the total energy functional  $\mathcal{E}^\varepsilon(\mathbf{v})$ . Here, it is assumed that the rescaling of  $\tilde{\mathbf{t}}^\varepsilon$  in the preceding item has been tweaked so that the leading term in the expansion of  $\mathbf{v}^\varepsilon$  is indeed of order 0 (which is always possible since the problem at stake is linear). The total energy functional is then developed and the terms in the expansion are sorted according to increasing power of  $\varepsilon$ :

$$\mathcal{E}^\varepsilon(\mathbf{v}^\varepsilon) = \varepsilon^m \mathcal{E}^m(\mathbf{v}^0) + \varepsilon^{m+1} \mathcal{E}^{m+1}(\mathbf{v}^0, \mathbf{v}^1) + \dots \quad (8)$$

The leading term depends only on  $\mathbf{v}^0$ . A minimum of that term over  $\mathbf{v}^0 \in H_{\#}^1(\tilde{\Omega}; \mathbb{R}^3)$  is then sought. The value of the minimum will always be seen to be zero, and will be seen to be achieved for  $\mathbf{v}^0$  belonging to a subspace of  $H_{\#}^1(\tilde{\Omega}; \mathbb{R}^3)$ . Such a  $\mathbf{v}^0$  is henceforth assumed, and the subsequent term in the expansion of the energy is now minimized, and so on. The process is rewound until a term having nonzero minimum is reached. As will be observed in the several examples analysed in this paper, this algorithm uniquely determines the first terms in the postulated expansion (7). Moreover, these first terms appear as the unique minimizers of a reduced energy functional, which is explicitly displayed by the algorithm as the first term having nonzero minimum in expansion (8): it is nothing but the energy functional of the lower-dimensional model.

The above method is therefore a systematic algorithm to compute the energy functional of the lower-dimensional model.

1.4.2. *Formal asymptotic analysis of ordinary beams and thin-walled beams.* Two types of slender cylinders  $\Omega^\varepsilon = [0, L] \times \mathcal{S}^\varepsilon$  will be studied in this paper. The first one is relevant in the case of an ordinary beam and the second one is relevant in the case of a thin-walled beam.

- Case of an ordinary beam. This is the case where the cross-section  $\mathcal{S}^\varepsilon$  is of the form  $\mathcal{S}^\varepsilon = \varepsilon \tilde{\mathcal{S}}$ , for some fixed subset  $\tilde{\mathcal{S}}$  in the  $yz$ -plane. The small parameter  $\varepsilon$

is simply a slenderness parameter of the cylinder. The rescaled space variables are:

$$\tilde{x} = x, \quad \tilde{y} = y/\varepsilon, \quad \tilde{z} = z/\varepsilon,$$

and wander in the rescaled domain  $\tilde{\Omega} = [0, L] \times \tilde{\mathcal{S}}$ , which is invariable with respect to  $\varepsilon$ .

- Case of a thin-walled beam. This is the case where the cross-section  $\mathcal{S}^\varepsilon$  is of the form  $\mathcal{S}^\varepsilon = \varepsilon \tilde{\mathcal{S}}^\varepsilon$ , where  $\tilde{\mathcal{S}}^\varepsilon$  is a thin strip of fixed length  $l$  and of thickness  $\varepsilon l$ . The middle line of the thin strip  $\tilde{\mathcal{S}}^\varepsilon$  is supposed to be a smooth curve with generic point  $M(\tilde{s})$  parametrized by the arc-length  $\tilde{s} \in [0, l]$ . The thin strip can be either an open profile (having two extremities) or a closed profile (loop). The cross-section of the cylinder is therefore a thin strip of length  $\varepsilon l$  of thickness  $\varepsilon^2 l$ , and the small parameter  $\varepsilon$  is a slenderness parameter both of the cylinder and of the cross-section. It will be made use of the rescaled orthogonal curvilinear coordinate system  $(\tilde{x}, \tilde{s}, \tilde{\eta}) \in [0, L] \times [0, l] \times [-l/2, l/2] = \tilde{\Omega}$ .

The formal asymptotic analysis according to the principle described in section 1.4.1 will be applied in this paper to compute the lower-dimensional energy functional in the cases of the above slender geometries. The detailed analysis will be displayed in sections 2 and 3. It yields the following results.

- In the case of an ordinary beam  $\Omega^\varepsilon = [0, L] \times \varepsilon \tilde{\mathcal{S}}$  made of an arbitrary anisotropic transversely heterogeneous elastic material, the rescaled space variables are taken so that they fulfil:

$$\int_{\mathcal{S}} \tilde{y} = \int_{\mathcal{S}} \tilde{z} = 0 = \int_{\mathcal{S}} \tilde{y}\tilde{z},$$

and surface tractions on the extremity  $\tilde{x} = L$  are taken of the form:

$$\tilde{\mathbf{t}}^\varepsilon(\tilde{y}, \tilde{z}) = \varepsilon \tilde{t}_x(\tilde{y}, \tilde{z}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{y}, \tilde{z}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{y}, \tilde{z}) \mathbf{e}_z + \varepsilon \tilde{M}_x / \tilde{I} \left[ \tilde{y} \mathbf{e}_z - \tilde{z} \mathbf{e}_y \right],$$

where  $\tilde{\mathbf{t}} : \tilde{\mathcal{S}} \rightarrow \mathbb{R}^3$  is a given function,  $\tilde{M}_x$  is a given constant and  $\tilde{I} = \int_{\tilde{\mathcal{S}}} \tilde{y}^2 + \tilde{z}^2$ . Then, the formal asymptotic analysis yields:

$$\begin{aligned} \mathbf{v}^\varepsilon(\tilde{x}, \tilde{y}, \tilde{z}) &= u_y^0(\tilde{x}) \mathbf{e}_y + u_z^0(\tilde{x}) \mathbf{e}_z \\ &+ \varepsilon u_x^1(\tilde{x}) \mathbf{e}_x + \varepsilon \left[ \omega^1(\tilde{x}) \mathbf{e}_x - u_z^{0'}(\tilde{x}) \mathbf{e}_y + u_y^{0'}(\tilde{x}) \mathbf{e}_z \right] \times \left[ \tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z \right] + \dots \end{aligned}$$

where  $u_y^0, u_z^0 \in H_{\#}^2(0, L)$  and  $u_x^1, \omega^1 \in H_{\#}^1(0, L)$  denote the unique minimizers of the lower-dimensional energy:

$$\begin{aligned} &\frac{1}{2} \int_0^L (u_x^{1'}, -u_z^{0'}, u_y^{0'}, \omega^{1'}) \cdot \mathbf{c} \cdot {}^t(u_x^{1'}, -u_z^{0'}, u_y^{0'}, \omega^{1'}) \\ &- \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

where:

$$\tilde{\mathbf{F}} = \int_{\tilde{\mathcal{S}}} \tilde{\mathbf{t}}, \quad \tilde{M}_y = \int_{\tilde{\mathcal{S}}} \tilde{z} \tilde{t}_x, \quad \tilde{M}_z = - \int_{\tilde{\mathcal{S}}} \tilde{y} \tilde{t}_x,$$

and  $\mathbf{c}$  is a positive definite symmetric matrix of order 4 (the reduced elastic moduli). The matrix  $\mathbf{c}$  depends only on the geometry  $\tilde{\mathcal{S}}$  of the cross-section and on the three-dimensional elastic moduli  $\mathbf{C}$ . Its 10 independent entries can be explicitly expressed in terms of the unique solutions of 4 two-dimensional

linear elastic problems over the cross-section  $\tilde{\mathcal{S}}$ . In the particular case of an isotropic homogeneous elastic material,  $\mathbf{c}$  reduces to:

$$\mathbf{c} = \begin{pmatrix} E|\tilde{\mathcal{S}}| & 0 & 0 & 0 \\ 0 & E\tilde{I}_y & 0 & 0 \\ 0 & 0 & E\tilde{I}_z & 0 \\ 0 & 0 & 0 & EJ/[2(1+\nu)] \end{pmatrix},$$

where  $E$  denotes the Young modulus,  $\nu$  the Poisson ratio and:

$$\tilde{I}_y = \int_{\tilde{\mathcal{S}}} \tilde{z}^2, \quad \tilde{I}_z = \int_{\tilde{\mathcal{S}}} \tilde{y}^2, \quad \tilde{J} = \min_{\varphi \in H^1(\tilde{\mathcal{S}})} \int_{\tilde{\mathcal{S}}} \left( \frac{\partial \varphi}{\partial \tilde{y}} - \tilde{z} \right)^2 + \left( \frac{\partial \varphi}{\partial \tilde{z}} + \tilde{y} \right)^2.$$

The classical Navier-Bernoulli (or Euler-Bernoulli) model is recovered.

- In the case of a thin-walled beam  $\Omega^\varepsilon = [0, L] \times \varepsilon\tilde{\mathcal{S}}^\varepsilon$  where  $\tilde{\mathcal{S}}^\varepsilon$  is an *open* thin strip, with  $Oyz$  chosen so that:

$$\int_0^l \mathbf{e}_y \cdot OM(\tilde{s}) = 0 = \int_0^l \mathbf{e}_z \cdot OM(\tilde{s}), \quad \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})] [\mathbf{e}_z \cdot OM(\tilde{s})] = 0,$$

and made of an arbitrary anisotropic transversely heterogeneous elastic material, surface tractions on the extremity  $\tilde{x} = L$  are taken of the form:

$$\mathbf{t}^\varepsilon(\tilde{y}, \tilde{z}) = \varepsilon \tilde{t}_x(\tilde{y}, \tilde{z}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{y}, \tilde{z}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{y}, \tilde{z}) \mathbf{e}_z,$$

where  $\tilde{\mathbf{t}} : \tilde{\mathcal{S}} \rightarrow \mathbb{R}^3$  is a given function. Then, the formal asymptotic analysis yields:

$$\begin{aligned} \mathbf{v}^\varepsilon &= u_y^0(\tilde{x}) \mathbf{e}_y + u_z^0(\tilde{x}) \mathbf{e}_z + \omega^0(\tilde{x}) \mathbf{e}_x \times [\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z] \\ &+ \varepsilon u_x^1(\tilde{x}) \mathbf{e}_x + \varepsilon \omega^{0'}(\tilde{x}) \left[ \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) - \frac{1}{l} \int_0^l \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) \right] \mathbf{e}_x \\ &+ \varepsilon \left[ -u_z^{0'}(\tilde{x}) \mathbf{e}_y + u_y^{0'}(\tilde{x}) \mathbf{e}_z \right] \times [\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z] + \dots \end{aligned}$$

where  $u_y^0, u_z^0, \omega^0 \in H_{\#}^2(0, L)$  and  $u_x^1 \in H_{\#}^1(0, L)$  denote the unique minimizers of the lower-dimensional energy:

$$\begin{aligned} &\frac{1}{2} \int_0^L \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{0'}, \omega^{0''} \right) \cdot \mathbf{c} \cdot t \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{0'}, \omega^{0''} \right) \\ &- \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^0(L) \\ &+ \tilde{M}_y \left[ u_z^{0'}(L) + y_c \omega^{0'}(L) \right] - \tilde{M}_z \left[ u_y^{0'}(L) - z_c \omega^{0'}(L) \right] - \tilde{B} \omega^{0'}(L), \end{aligned}$$

where:

$$\begin{aligned} \tilde{I}_y &= l \int_0^l [\mathbf{e}_z \cdot OM(\tilde{s})]^2, \quad \tilde{I}_z = l \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})]^2, \\ y_c &= -\frac{l}{\tilde{I}_y} \int_0^l [\mathbf{e}_z \cdot OM(\tilde{s})] \int_0^{\tilde{s}} (\mathbf{n} \cdot OM), \\ z_c &= \frac{l}{\tilde{I}_z} \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})] \int_0^{\tilde{s}} (\mathbf{n} \cdot OM), \end{aligned}$$

$$\begin{aligned}\tilde{\psi} &= \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) - \frac{1}{l} \int_0^l \int_0^{\tilde{\sigma}} (\mathbf{n} \cdot OM) - z_c \mathbf{e}_y \cdot OM(\tilde{s}) + y_c \mathbf{e}_z \cdot OM(\tilde{s}), \\ \tilde{\mathbf{F}} &= \int_{\tilde{S}} \tilde{\mathbf{t}}, \quad \tilde{\mathbf{M}} = \int_{\tilde{S}} [\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z] \times \tilde{\mathbf{t}}, \quad \tilde{B} = \int_{\tilde{S}} \tilde{\psi} \tilde{t}_x\end{aligned}$$

and  $\mathbf{c}$  is a positive symmetric matrix of order 5 (the reduced elastic moduli). It is positive definite provided that the middle line  $M(\tilde{s})$  of the thin cross-section is not a line segment (degenerate case). The matrix  $\mathbf{c}$  depends only on the geometry  $\tilde{S}$  of the cross-section and on the three-dimensional elastic moduli  $\mathbf{C}$ . In the case of a *laminate* (that is,  $\mathbf{C}$  depends only on  $\tilde{\eta}$  and is independent of  $\tilde{x}$  and  $\tilde{s}$ ), its 15 independent entries can be explicitly expressed in terms of the unique solution of 5 two-dimensional linear elastic problems over the cross-section  $\tilde{S}$ . In the case of general transversely heterogeneity, the calculation of  $\mathbf{c}$  involves an additional (but computationally straightforward) minimization. In the particular case of an isotropic homogeneous elastic material,  $\mathbf{c}$  reduces to:

$$\mathbf{c} = \begin{pmatrix} El^2 & 0 & 0 & 0 & 0 \\ 0 & E\tilde{I}_y & 0 & 0 & -y_c E\tilde{I}_y \\ 0 & 0 & E\tilde{I}_z & 0 & -z_c E\tilde{I}_z \\ 0 & 0 & 0 & \frac{E\tilde{J}}{2(1+\nu)} & 0 \\ 0 & -y_c E\tilde{I}_y & -z_c E\tilde{I}_z & 0 & E\tilde{J}_w + y_c^2 E\tilde{I}_y + z_c^2 E\tilde{I}_z \end{pmatrix}$$

where:

$$\tilde{J} = \frac{l^4}{3}, \quad \tilde{J}_w = l \int_0^l [\tilde{\psi}(\tilde{s})]^2.$$

In the case of an isotropic homogeneous elastic material, the classical Vlassov model is recovered and its generalization to the case of an anisotropic heterogeneous elastic material has been derived, seemingly for the first time.

- In the case of a thin-walled beam  $\Omega^\varepsilon = [0, L] \times \varepsilon \tilde{S}^\varepsilon$  where  $\tilde{S}^\varepsilon$  is an *closed* thin strip, with  $Oyz$  chosen so that:

$$\int_0^l \mathbf{e}_y \cdot OM(\tilde{s}) = 0 = \int_0^l \mathbf{e}_z \cdot OM(\tilde{s}), \quad \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})] [\mathbf{e}_z \cdot OM(\tilde{s})] = 0,$$

and made of an arbitrary anisotropic transversely heterogeneous elastic material, surface traction on the extremity  $\tilde{x} = L$  are taken of the form:

$$\begin{aligned}\tilde{\mathbf{t}}^\varepsilon(\tilde{s}, \tilde{\eta}) &= \varepsilon \tilde{t}_x(\tilde{s}, \tilde{\eta}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{s}, \tilde{\eta}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{s}, \tilde{\eta}) \mathbf{e}_z \\ &\quad + \varepsilon \tilde{M}_x / \tilde{I} \left[ [\mathbf{e}_y \cdot OM(\tilde{s})] \mathbf{e}_z - [\mathbf{e}_z \cdot OM(\tilde{s})] \mathbf{e}_y \right],\end{aligned}$$

where  $\tilde{\mathbf{t}} : \tilde{S} \rightarrow \mathbb{R}^3$  is a given function,  $\tilde{M}_x$  is a given torsion moment and:

$$\tilde{I} = l \int_0^l |OM(\tilde{s})|^2.$$

Then, the formal asymptotic analysis yields:

$$\begin{aligned}\mathbf{v}^\varepsilon(\tilde{x}, \tilde{y}, \tilde{z}) &= u_y^0(\tilde{x}) \mathbf{e}_y + u_z^0(\tilde{x}) \mathbf{e}_z \\ &\quad + \varepsilon u_x^1(\tilde{x}) \mathbf{e}_x + \varepsilon \left[ \omega^1(\tilde{x}) \mathbf{e}_x - u_z^{0'}(\tilde{x}) \mathbf{e}_y + u_y^{0'}(\tilde{x}) \mathbf{e}_z \right] \times [\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z] + \dots\end{aligned}$$

where  $u_y^0, u_z^0 \in H_{\#}^2(0, L)$  and  $u_x^1, \omega^1 \in H_{\#}^1(0, L)$  denote the unique minimizers of the lower-dimensional energy:

$$\begin{aligned} & \frac{1}{2} \int_0^L (u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{1'}) \cdot \mathbf{c} \cdot {}^t(u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{1'}) \\ & - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

where:

$$\tilde{\mathbf{F}} = \int_{\tilde{\mathcal{S}}} \tilde{\mathbf{t}}, \quad \tilde{M}_y = \int_{\tilde{\mathcal{S}}} \tilde{z} \tilde{t}_x, \quad \tilde{M}_z = - \int_{\tilde{\mathcal{S}}} \tilde{y} \tilde{t}_x,$$

and  $\mathbf{c}$  is a positive definite symmetric matrix of order 4 (the reduced elastic moduli). The matrix  $\mathbf{c}$  depends only on the geometry  $\tilde{\mathcal{S}}$  of the cross-section and on the three-dimensional elastic moduli  $\mathbf{C}$ . In the particular case of isotropic homogeneity, one obtains:

$$\mathbf{c} = \begin{pmatrix} El^2 & 0 & 0 & 0 \\ 0 & E\tilde{I}_y & 0 & 0 \\ 0 & 0 & E\tilde{I}_z & 0 \\ 0 & 0 & 0 & E\tilde{J}/[2(1+\nu)] \end{pmatrix},$$

where:

$$\begin{aligned} \tilde{I}_y &= l \int_0^l [\mathbf{e}_z \cdot OM(\tilde{s})]^2, & \tilde{I}_z &= l \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})]^2, \\ \tilde{J} &= \left( \int_0^l [\mathbf{n}(\tilde{s}) \cdot OM(\tilde{s})] \right)^2. \end{aligned}$$

Hence, the asymptotic analysis of a thin-walled beam with *closed* profile yields the Navier-Bernoulli model. The fact that the Vlassov model pertains only to thin-walled beams with *open* profile and that the Navier-Bernoulli should be used in case of a thin-walled beam with *closed* profile was already claimed by Timoshenko and Vlassov. This is now fully justified on the basis of an asymptotic analysis.

**2. Formal asymptotic analysis of ordinary beams.** This section is devoted to a detailed account of the formal asymptotic analysis of ordinary beams.

**2.1. Position of problem and scalings.** In the usual three-dimensional Euclidean space with orthonormal Cartesian coordinate system  $Oxyz$ , we consider the connected cylinder  $\Omega^\varepsilon = [0, L] \times \mathcal{S}^\varepsilon$ . The origin  $O$  is taken as the centroid of the cross-section  $\mathcal{S}_0^\varepsilon$  at  $x = 0$ , and  $Oy$  and  $Oz$  are directed along the inertia principal axis of  $\mathcal{S}_0^\varepsilon$ , so that:

$$\int_{\mathcal{S}^\varepsilon} y = \int_{\mathcal{S}^\varepsilon} z = 0 = \int_{\mathcal{S}^\varepsilon} yz.$$

We study the following equilibrium problem within the framework of linearized elasticity theory. The extremity  $\mathcal{S}_0^\varepsilon$  is clamped, the cylinder is free of body force, and the lateral surface is free of traction. The cylinder is therefore loaded only by given surface traction  $\mathbf{t}^\varepsilon \in L^2(\mathcal{S}_L^\varepsilon; \mathbb{R}^3)$  on the extremity  $\mathcal{S}_L^\varepsilon$ . This three-dimensional linear elastic equilibrium problem admits a unique equilibrium displacement field  $\mathbf{v}^\varepsilon$ .

The cross-section  $\mathcal{S}^\varepsilon$  is supposed to be of the form  $\mathcal{S}^\varepsilon = \varepsilon \tilde{\mathcal{S}}$ , where  $\tilde{\mathcal{S}}$  denotes a fixed open bounded Lipschitzian subset of  $\mathbb{R}^2$ . Our subsequent objective will be

to study asymptotically the equilibrium displacement  $\mathbf{v}^\varepsilon$  as  $\varepsilon \rightarrow 0+$ . A change of variable is performed, so that the displacement field  $\mathbf{v}^\varepsilon$  is defined upon a domain which remains invariable as  $\varepsilon$  varies. With respect to this, we introduce the scaled variables:

$$\tilde{x} = x, \quad \tilde{y} = y/\varepsilon, \quad \tilde{z} = z/\varepsilon,$$

so that the displacement field  $\mathbf{v}^\varepsilon(\tilde{x}, \tilde{y}, \tilde{z})$  is now defined on the domain  $\tilde{\Omega} = [0, L] \times \tilde{\mathcal{S}}$  which is independent of  $\varepsilon$ .

The cylinder is made of an arbitrary anisotropic elastic material which is, in addition, allowed to be transversely heterogeneous (meaning the heterogeneity is the same in every cross-section). This is implemented by an elastic tensor  $\mathbf{C}(\tilde{y}, \tilde{z})$  of the two variables  $\tilde{y}, \tilde{z}$  only (in particular, it does not depend on  $\varepsilon$ ), having the usual symmetries and satisfying the positivity and boundedness conditions:

$$\exists K, k > 0, \quad \forall(\tilde{y}, \tilde{z}) \in \tilde{\mathcal{S}}, \quad \forall \mathbf{e} \text{ symmetric} \quad k \mathbf{e} : \mathbf{e} \leq \mathbf{e} : \mathbf{C}(\tilde{y}, \tilde{z}) : \mathbf{e} \leq K \mathbf{e} : \mathbf{e}. \quad (9)$$

We will also consider the particular case of isotropic homogeneous elasticity in which:

$$\mathbf{C} : \mathbf{e} = \frac{\nu E}{(1+\nu)(1-2\nu)} (\text{tr } \mathbf{e}) \mathbb{1} + \frac{E}{1+\nu} \mathbf{e}, \quad (10)$$

where the above positivity condition is fulfilled provided that the Poisson ratio  $\nu \in ]-1, 1/2[$  and the Young modulus  $E > 0$ .

The displacement field  $\mathbf{v}^\varepsilon$  is the unique minimizer in:

$$H_0^1(\tilde{\Omega}; \mathbb{R}^3) = \left\{ \mathbf{v} \in H^1 \mid \mathbf{v}(0, \tilde{y}, \tilde{z}) = \mathbf{0} \right\},$$

of the three-dimensional energy:

$$\mathcal{E}^\varepsilon(\mathbf{v}) = \frac{\varepsilon^2}{2} \int_0^L \int_{\tilde{\mathcal{S}}} \mathbf{e}^\varepsilon(\mathbf{v}) : \mathbf{C} : \mathbf{e}^\varepsilon(\mathbf{v}) - \varepsilon^2 \int_{\tilde{\mathcal{S}}_L} \tilde{\mathbf{t}}^\varepsilon \cdot \mathbf{v}, \quad (11)$$

where:

$$\begin{aligned} \mathbf{e}^\varepsilon(\mathbf{v}) &= \frac{\partial v_x}{\partial \tilde{x}} \mathbf{e}_x \otimes \mathbf{e}_x + \frac{1}{\varepsilon} \frac{\partial v_y}{\partial \tilde{y}} \mathbf{e}_y \otimes \mathbf{e}_y + \frac{1}{2} \left( \frac{1}{\varepsilon} \frac{\partial v_x}{\partial \tilde{y}} + \frac{\partial v_y}{\partial \tilde{x}} \right) (\mathbf{e}_y \otimes \mathbf{e}_x + \mathbf{e}_x \otimes \mathbf{e}_y) \\ &+ \frac{1}{\varepsilon} \frac{\partial v_z}{\partial \tilde{z}} \mathbf{e}_z \otimes \mathbf{e}_z + \frac{1}{2} \left( \frac{1}{\varepsilon} \frac{\partial v_x}{\partial \tilde{z}} + \frac{\partial v_z}{\partial \tilde{x}} \right) (\mathbf{e}_z \otimes \mathbf{e}_x + \mathbf{e}_x \otimes \mathbf{e}_z) \\ &+ \frac{1}{2\varepsilon} \left( \frac{\partial v_y}{\partial \tilde{z}} + \frac{\partial v_z}{\partial \tilde{y}} \right) (\mathbf{e}_y \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_y). \end{aligned} \quad (12)$$

The asymptotic analysis will be undertaken under the initial postulate that this displacement field admits a power expansion of the form:

$$\mathbf{v}^\varepsilon(\tilde{x}, \tilde{y}, \tilde{z}) = \mathbf{v}^0(\tilde{x}, \tilde{y}, \tilde{z}) + \varepsilon \mathbf{v}^1(\tilde{x}, \tilde{y}, \tilde{z}) + \dots \quad (13)$$

The expansion (13) is going to be injected into the total energy (11), so that it will be possible to calculate a power expansion of the total energy with respect to  $\varepsilon$ , each terms of that expansion involving a finite number of the unknown functions  $\mathbf{v}^i$ . Then, each term in this power expansion of the total energy will be successively minimized with respect to the unknown functions  $\mathbf{v}^i$ , starting with the lower order term. As it will be observed in the analysis, the minimum value corresponding to the first few minimization problems is always 0. The analysis is continued until the leading term in the expansion of the energy is completely identified. The above procedure is an algorithm which leaves no room for any tweak. The only choice that remains free at first sight is that of the scaling of the load  $\tilde{\mathbf{t}}^\varepsilon$  (meaning the choice

of the dependence of  $\mathbf{t}^\varepsilon$  upon  $\varepsilon$ ). Actually, this freedom is only apparent and there is essentially only one appropriate scaling, and therefore only one reduced model. The asymptotic analysis will be performed in section 2.2, based on the following rescaling of the load:

$$\tilde{\mathbf{t}}^\varepsilon(\tilde{y}, \tilde{z}) = \varepsilon \tilde{t}_x(\tilde{y}, \tilde{z}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{y}, \tilde{z}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{y}, \tilde{z}) \mathbf{e}_z + \varepsilon \tilde{M}_x / \tilde{I} [\tilde{y} \mathbf{e}_z - \tilde{z} \mathbf{e}_y], \quad (14)$$

where  $\tilde{t} \in L^2(\tilde{\mathcal{S}}; \mathbb{R}^3)$  is a given function,  $\tilde{M}_x$  is a given torsion moment and  $\tilde{I} = \int_{\tilde{\mathcal{S}}} \tilde{y}^2 + \tilde{z}^2$ .

**Justification of the choice of the rescaling (14) of the load.** The identification of the appropriate rescaling of the load must actually be performed along the asymptotic analysis itself. It runs as follows. One could start with an ansatz of the type:

$$\mathbf{t}^\varepsilon(\tilde{x}, \tilde{y}, \tilde{z}) = \mathbf{t}^0(\tilde{x}, \tilde{y}, \tilde{z}) + \varepsilon \mathbf{t}^1(\tilde{x}, \tilde{y}, \tilde{z}) + \dots$$

Then, injecting (13) into the energy (11) and sorting the terms by increasing order of  $\varepsilon$ , one obtains that the lower order term is of order  $\varepsilon^0$  and reads as:

$$\mathcal{E}^0 = \frac{1}{2} \int_0^L \int_{\tilde{\mathcal{S}}} \mathbf{e}^{-1} : \mathbf{C} : \mathbf{e}^{-1},$$

where  $\mathbf{e}^{-1}$  is given by formula (15). The minimum is zero and is achieved by the function  $\mathbf{v}^0$  of the form (16). Assuming that  $\mathbf{v}^0$  is of that form, the low order term in the expansion of the energy (11) is now of order  $\varepsilon^2$  and reads as:

$$\mathcal{E}^2 = \frac{1}{2} \int_0^L \int_{\tilde{\mathcal{S}}} \mathbf{e}^0 : \mathbf{C} : \mathbf{e}^0 - \int_{\tilde{\mathcal{S}}_L} \mathbf{t}^0 \cdot \mathbf{v}^0,$$

where  $\mathbf{e}^0$  is now given by formula (17). Then, one looks for an infimum with respect to  $\mathbf{v}^0$ ,  $\mathbf{v}^1 \in H_{\sharp}^1(\tilde{\Omega}; \mathbb{R}^3)$  and observes that  $\mathcal{E}^2$  is bounded from below only if:

$$\int_{\tilde{\mathcal{S}}_L} \mathbf{t}^0 = \mathbf{0}, \quad \int_{\tilde{\mathcal{S}}_L} [\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z] \times \mathbf{t}^0 = \mathbf{0},$$

that is, in particular, the total force on  $\tilde{\mathcal{S}}_L$  must vanish. Since such a condition cannot be expected to be fulfilled, in general, in the original three-dimensional problem, one is driven to adopt  $\mathbf{t}^0 = \mathbf{0}$  instead, and start over again, until the analysis is able to yield a result without having to make some unacceptable assumption about the form of the load. It turns out to be the case for:

$$\tilde{\mathbf{t}}^\varepsilon(\tilde{y}, \tilde{z}) = \varepsilon \tilde{t}_x(\tilde{y}, \tilde{z}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{y}, \tilde{z}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{y}, \tilde{z}) \mathbf{e}_z + \text{higher order terms},$$

for some fixed (independent of  $\varepsilon$ ) given function  $\tilde{\mathbf{t}}$  defined on  $\tilde{\mathcal{S}}$ . Based on such a rescaling of the load, the formal asymptotic procedure described in section 2.2 can then be fully completed. Note that taking  $\tilde{\mathbf{t}}^\varepsilon$  of higher order with respect to  $\varepsilon$  would only result in multiplying the corresponding displacement  $\mathbf{v}^\varepsilon$  by  $\varepsilon^p$ , for some  $p$ , as the problem under study is linear. The reduced model that the formal asymptotic procedure delivers in case of the above rescaling of the load, is that of Navier-Bernoulli *with no torsion*. The reason why torsion is missing can be gathered from the calculation of the moments (at the centroid of  $\tilde{\mathcal{S}}_L$ ) associated with the above choice of  $\mathbf{t}^\varepsilon$ . One obtains:

$$M_x = \varepsilon^5 \int_{\tilde{\mathcal{S}}_L} \tilde{y} \tilde{t}_z - \tilde{z} \tilde{t}_y, \quad M_y = \varepsilon^4 \int_{\tilde{\mathcal{S}}_L} \tilde{z} \tilde{t}_x, \quad M_z = -\varepsilon^4 \int_{\tilde{\mathcal{S}}_L} \tilde{y} \tilde{t}_x,$$

showing that the torsion moment  $M_x$  is of order higher than that of the bending moments. Hence, the above rescaling of the loading sends the torsional displacement to higher order. To bring it back at principal order, we are therefore led to enforce a torsion moment of order  $\varepsilon^4$  artificially by taking:

$$\tilde{\mathbf{t}}^\varepsilon(\tilde{y}, \tilde{z}) = \varepsilon \tilde{t}_x(\tilde{y}, \tilde{z}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{y}, \tilde{z}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{y}, \tilde{z}) \mathbf{e}_z + \varepsilon \tilde{M}_x / \tilde{I} [\tilde{y} \mathbf{e}_z - \tilde{z} \mathbf{e}_y],$$

where  $\tilde{M}_x$  is some given torsion moment.

**2.2. Formal asymptotic expansion.** Adopting the rescaling of the load defined by formula (14), the three-dimensional displacement field  $\mathbf{v}^\varepsilon$  is the unique minimizer in:

$$H_{\#}^1(\tilde{\Omega}; \mathbb{R}^3) = \left\{ \mathbf{v} \in H^1 \mid \mathbf{v}(0, \tilde{y}, \tilde{z}) = \mathbf{0} \right\},$$

of the total energy:

$$\begin{aligned} \mathcal{E}^\varepsilon(\mathbf{v}) &= \frac{\varepsilon^2}{2} \int_0^L \int_{\tilde{S}} \mathbf{e}^\varepsilon(\mathbf{v}) : \mathbf{C} : \mathbf{e}^\varepsilon(\mathbf{v}) - \varepsilon^3 \int_{\tilde{S}} t_x(\tilde{y}, \tilde{z}) v_x(L, \tilde{y}, \tilde{z}) \\ &\quad - \varepsilon^3 \int_{\tilde{S}} \left[ M_x \tilde{y} / \tilde{I} v_z(L, \tilde{y}, \tilde{z}) - M_x \tilde{z} / \tilde{I} v_y(L, \tilde{y}, \tilde{z}) \right] \\ &\quad - \varepsilon^4 \int_{\tilde{S}} \left[ \tilde{t}_y(\tilde{y}, \tilde{z}) v_y(L, \tilde{y}, \tilde{z}) + \tilde{t}_z(\tilde{y}, \tilde{z}) v_z(L, \tilde{y}, \tilde{z}) \right], \end{aligned}$$

where the operator  $\mathbf{e}^\varepsilon$  is defined by formula (12). As outlined in the previous section, we are now looking for a formal asymptotic expansion of the three-dimensional displacement field  $\mathbf{v}^\varepsilon$  of the form:

$$\mathbf{v}^\varepsilon(\tilde{x}, \tilde{y}, \tilde{z}) = \mathbf{v}^0(\tilde{x}, \tilde{y}, \tilde{z}) + \varepsilon \mathbf{v}^1(\tilde{x}, \tilde{y}, \tilde{z}) + \dots,$$

by injecting that expansion into the energy, sorting the terms by increasing power of  $\varepsilon$ :

$$\mathcal{E}^\varepsilon(\mathbf{v}^\varepsilon) = \mathcal{E}^0(\mathbf{v}^0) + \varepsilon \mathcal{E}^1(\mathbf{v}^0, \mathbf{v}^1) + \dots,$$

and minimizing successively each of them.

The first term is of order 0 and given by:

$$\mathcal{E}^0 = \frac{1}{2} \int_0^L \int_{\tilde{S}} \mathbf{e}^{-1} : \mathbf{C} : \mathbf{e}^{-1},$$

where:

$$\mathbf{e}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{y}} & \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{z}} \\ \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{y}} & \frac{\partial v_y^0}{\partial \tilde{y}} & \frac{1}{2} \left( \frac{\partial v_y^0}{\partial \tilde{z}} + \frac{\partial v_z^0}{\partial \tilde{y}} \right) \\ \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{z}} & \frac{1}{2} \left( \frac{\partial v_y^0}{\partial \tilde{z}} + \frac{\partial v_z^0}{\partial \tilde{y}} \right) & \frac{\partial v_z^0}{\partial \tilde{z}} \end{pmatrix}. \quad (15)$$

The minimum of  $\mathcal{E}^0$  with respect to  $\mathbf{v}^0 \in H_{\#}^1$  is 0 and all the minimizers are of the form:

$$\mathbf{v}^0(\tilde{x}, \tilde{y}, \tilde{z}) = \mathbf{u}^0(\tilde{x}) + \omega^0(\tilde{x}) \mathbf{e}_x \times [\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z], \quad (16)$$

where  $u_x^0$ ,  $u_y^0$ ,  $u_z^0$  and  $\omega^0$  are still arbitrary functions in:

$$H_{\#}^1(0, L) = \left\{ u \in H^1(0, L) \mid u(0) = 0 \right\}.$$

From now on, we assume that  $\mathbf{v}^0$  is of the above form, and we calculate the next lower order term  $\varepsilon^2$  in the energy. It reads as:

$$\mathcal{E}^2 = \frac{1}{2} \int_0^L \int_{\tilde{S}} \mathbf{e}^0 : \mathbf{C} : \mathbf{e}^0,$$

with:

$$\mathbf{e}^0 = \begin{pmatrix} u_x^{0'} & \frac{1}{2} \left( \frac{\partial v_x^1}{\partial \tilde{y}} + u_y^{0'} - \tilde{z} \omega^{0'} \right) & \frac{1}{2} \left( \frac{\partial v_x^1}{\partial \tilde{z}} + u_z^{0'} + \tilde{y} \omega^{0'} \right) \\ \frac{1}{2} \left( \frac{\partial v_x^1}{\partial \tilde{y}} + u_y^{0'} - \tilde{z} \omega^{0'} \right) & \frac{\partial v_y^1}{\partial \tilde{y}} & \frac{1}{2} \left( \frac{\partial v_y^1}{\partial \tilde{z}} + \frac{\partial v_z^1}{\partial \tilde{y}} \right) \\ \frac{1}{2} \left( \frac{\partial v_x^1}{\partial \tilde{z}} + u_z^{0'} + \tilde{y} \omega^{0'} \right) & \frac{1}{2} \left( \frac{\partial v_y^1}{\partial \tilde{z}} + \frac{\partial v_z^1}{\partial \tilde{y}} \right) & \frac{\partial v_z^1}{\partial \tilde{z}} \end{pmatrix}, \quad (17)$$

where the prime  $'$  denotes the derivative of a function of  $\tilde{x}$  only. Once again, the minimum of  $\mathcal{E}^2$  with respect to  $\mathbf{u}^0, \omega^0, \mathbf{v}^1$  is 0 and the minimizers are  $u_x^0 \equiv 0 \equiv \omega^0$  and functions  $\mathbf{v}^1$  of the form:

$$\mathbf{v}^1(\tilde{x}, \tilde{y}, \tilde{z}) = \mathbf{u}^1(\tilde{x}) + \left[ \omega^1(\tilde{x}) \mathbf{e}_x - u_z^{0'}(\tilde{x}) \mathbf{e}_y + u_y^{0'}(\tilde{x}) \mathbf{e}_z \right] \times \left[ \tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z \right],$$

where  $\mathbf{u}^1(\tilde{x})$  and  $\omega^1(\tilde{x})$  are still arbitrary functions in  $H_{\#}^1(0, L)$ , and where  $u_y^0(\tilde{x}), u_z^0(\tilde{x})$  must henceforth belong to the smaller space:

$$H_{\#}^2(0, L) = \left\{ u \in H^2(0, L) \mid u(0) = u'(0) = 0 \right\}.$$

The subsequent term in the expansion of the energy is:

$$\begin{aligned} \mathcal{E}^4 = \frac{1}{2} \int_0^L \int_{\tilde{S}} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1 - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) \\ - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

with:

$$\mathbf{e}^1 = \begin{pmatrix} u_x^{1'} - \tilde{y} u_y^{0''} - \tilde{z} u_z^{0''} & \frac{1}{2} \left( \frac{\partial v_x^2}{\partial \tilde{y}} + u_y^{1'} - \tilde{z} \omega^{1'} \right) & \frac{1}{2} \left( \frac{\partial v_x^2}{\partial \tilde{z}} + u_z^{1'} + \tilde{y} \omega^{1'} \right) \\ \frac{1}{2} \left( \frac{\partial v_x^2}{\partial \tilde{y}} + u_y^{1'} - \tilde{z} \omega^{1'} \right) & \frac{\partial v_y^2}{\partial \tilde{y}} & \frac{1}{2} \left( \frac{\partial v_y^2}{\partial \tilde{z}} + \frac{\partial v_z^2}{\partial \tilde{y}} \right) \\ \frac{1}{2} \left( \frac{\partial v_x^2}{\partial \tilde{z}} + u_z^{1'} + \tilde{y} \omega^{1'} \right) & \frac{1}{2} \left( \frac{\partial v_y^2}{\partial \tilde{z}} + \frac{\partial v_z^2}{\partial \tilde{y}} \right) & \frac{\partial v_z^2}{\partial \tilde{z}} \end{pmatrix}, \quad (18)$$

and where we have set:

$$\tilde{\mathbf{F}} = \int_{\tilde{S}_L} \tilde{\mathbf{t}}, \quad \tilde{M}_y = \int_{\tilde{S}_L} \tilde{z} \tilde{t}_x, \quad \tilde{M}_z = - \int_{\tilde{S}_L} \tilde{y} \tilde{t}_x, \quad (19)$$

whereas  $\tilde{M}_x$  was already introduced in formula (14).

There remains only to take the infimum of  $\mathcal{E}^4$  with respect to  $u_y^0, u_z^0, \mathbf{u}^1, \omega^1$  and  $\mathbf{v}^2$ . This can be achieved in two steps.

1. We fix  $u_y^0, u_z^0, u_x^1$  and  $\omega^1$  temporarily. Then, we look for an infimum with respect to  $u_y^1, u_z^1$  and  $\mathbf{v}^2$ . The value of that infimum (which depends on  $u_y^0, u_z^0, u_x^1$  and  $\omega^1$ ) will provide the total energy of the reduced model.
2. Then, we take the minimum with respect to  $u_y^0, u_z^0, u_x^1$  and  $\omega^1$ . This amounts to solving the equilibrium equations of the reduced model (namely, the Navier-Bernoulli model, as will be observed in the sequel).

These two steps will be first performed in the particular case of isotropic homogeneous elasticity (10) (due to its ubiquity in the applications and also because the calculation of the reduced elastic constants turns out to be explicit in that case) in the next section and the general anisotropic transversely heterogeneous case will be tackled only in section 2.4.

**2.3. The isotropic homogeneous case.** In the isotropic homogeneous case, the fourth-order tensor  $\mathbf{C}$  takes the particular form (10) and the energy to be minimized reads as:

$$\begin{aligned} \mathcal{E}^4 &= \frac{E}{2(1+\nu)} \int_0^L \int_{\tilde{S}} \frac{\nu}{1-2\nu} \left[ \text{tr} \mathbf{e}^1 \right]^2 + \text{tr} \left[ \mathbf{e}^1 \right]^2 \\ &\quad - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

where  $\mathbf{e}^1$  is given by formula (18), and  $\tilde{\mathbf{F}}, \tilde{\mathbf{M}}$ , by formulae (19).

We now look for the infimum with respect to  $u_y^0, u_z^0 \in H_{\#}^2$ , and  $\mathbf{u}^1, \omega^1, \mathbf{v}^2 \in H_{\#}^1$ . First, note that  $\mathcal{E}^4$  is the sum of two independent functionals:

$$\begin{aligned} \mathcal{E}^{4,1} &= \frac{E}{2(1+\nu)} \int_0^L \int_{\tilde{S}} \left\{ \frac{\nu}{1-2\nu} \left[ u_x^{1'} - \tilde{y} u_y^{0''} - \tilde{z} u_z^{0''} + \frac{\partial v_y^2}{\partial \tilde{y}} + \frac{\partial v_z^2}{\partial \tilde{z}} \right]^2 \right. \\ &\quad \left. + \left[ u_x^{1'} - \tilde{y} u_y^{0''} - \tilde{z} u_z^{0''} \right]^2 + \left[ \frac{\partial v_y^2}{\partial \tilde{y}} \right]^2 + \left[ \frac{\partial v_z^2}{\partial \tilde{z}} \right]^2 + \frac{1}{2} \left[ \frac{\partial v_y^2}{\partial \tilde{z}} + \frac{\partial v_z^2}{\partial \tilde{y}} \right]^2 \right\} \\ &\quad - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

and

$$\mathcal{E}^{4,2} = \frac{E}{4(1+\nu)} \int_0^L \int_{\tilde{S}} \left\{ \left( \frac{\partial v_x^2}{\partial \tilde{y}} + u_y^{1'} - \tilde{z} \omega^{1'} \right)^2 + \left( \frac{\partial v_x^2}{\partial \tilde{z}} + u_z^{1'} + \tilde{y} \omega^{1'} \right)^2 \right\} - \tilde{M}_x \omega^1(L).$$

Let us start with  $\mathcal{E}^{4,1}$  in which  $u_x^1 \in H_{\#}^1$  and  $u_y^0, u_z^0 \in H_{\#}^2$  are kept fixed temporarily. In the particular case where  $u_x^1 \in H_{\#}^2$  and:

$$u_y^0, u_z^0 \in H_{\#}^3(0, L) = \left\{ u \in H^3(0, L) \mid u(0) = u'(0) = u''(0) = 0 \right\},$$

the minimizers  $\bar{v}_y^2, \bar{v}_z^2 \in H_{\#}^1(\tilde{\Omega})$  are given by:

$$\begin{aligned} \bar{v}_y^2(\tilde{x}, \tilde{y}, \tilde{z}) &= -\nu u_x^{1'}(\tilde{x}) \tilde{y} + \nu u_y^{0''}(\tilde{x}) \frac{\tilde{y}^2 - \tilde{z}^2}{2} + \nu u_z^{0''}(\tilde{x}) \tilde{y} \tilde{z} + u_y^2(\tilde{x}) - \tilde{z} \omega^2(\tilde{x}), \\ \bar{v}_z^2(\tilde{x}, \tilde{y}, \tilde{z}) &= -\nu u_x^{1'}(\tilde{x}) \tilde{z} + \nu u_y^{0''}(\tilde{x}) \tilde{y} \tilde{z} - \nu u_z^{0''}(\tilde{x}) \frac{\tilde{y}^2 - \tilde{z}^2}{2} + u_z^2(\tilde{x}) + \tilde{y} \omega^2(\tilde{x}), \end{aligned} \quad (20)$$

where  $u_y^2, u_z^2$ , and  $\omega^2$  are arbitrary functions in  $H_{\#}^1(0, L)$ . Hence, in the particular case where  $u_x^1 \in H_{\#}^2$  and  $u_y^0, u_z^0 \in H_{\#}^3$ , the value of the minimum of  $\mathcal{E}^{4,1}$  for fixed  $u_x^1, u_y^0$  and  $u_z^0$  is therefore given by:

$$\begin{aligned} \min_{v_y^2, v_z^2 \in H_{\#}^1} \mathcal{E}^{4,1} &= \frac{E}{2} \int_0^L \int_{\tilde{S}} \left[ \tilde{S} \left[ u_x^{1'} \right]^2 + \tilde{I}_y \left[ u_z^{0''} \right]^2 + \tilde{I}_z \left[ u_y^{0''} \right]^2 \right. \\ &\quad \left. - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L) \right], \quad (21) \end{aligned}$$

where we have set:

$$\tilde{I}_y = \int_{\tilde{S}} \tilde{z}^2, \quad \tilde{I}_z = \int_{\tilde{S}} \tilde{y}^2.$$

In the general case where we only have  $u_x^1 \in H_{\sharp}^1$  and  $u_y^0, u_z^0 \in H_{\sharp}^2$ , we can find sequences  $u_{x,n}^1 \in H_{\sharp}^2$  converging to  $u_x^1$  in  $H_{\sharp}^1$ , and  $u_{y,n}^0, u_{z,n}^0 \in H_{\sharp}^3$  converging to  $u_y^0, u_z^0$  in  $H_{\sharp}^2$ , as  $H_{\sharp}^2$  and  $H_{\sharp}^3$  are dense in  $H_{\sharp}^1$  and  $H_{\sharp}^2$ , respectively. Now, let  $\bar{v}_{y,n}^2, \bar{v}_{z,n}^2$  be minimizers associated with  $u_{x,n}^1, u_{y,n}^0, u_{z,n}^0$  along formulas (20). We have:

$$\forall v_y^2, v_z^2 \in H_{\sharp}^1, \quad \mathcal{E}^{4,1}(u_{x,n}^1, u_{y,n}^0, u_{z,n}^0, v_y^2, v_z^2) \geq \mathcal{E}^{4,1}(u_{x,n}^1, u_{y,n}^0, u_{z,n}^0, \bar{v}_{y,n}^2, \bar{v}_{z,n}^2),$$

for all  $n \in \mathbb{N}$ . Taking the limit  $n \rightarrow \infty$ , we have proved that formula (21) still provides a bound from below of  $\mathcal{E}^{4,1}$  in the general case where  $u_x^1 \in H_{\sharp}^1$  and  $u_y^0, u_z^0 \in H_{\sharp}^2$ . In addition, this bound from below is obtained as the following limit

$$\lim_{n \rightarrow \infty} \mathcal{E}^{4,1}(u_x^1, u_y^0, u_z^0, \bar{v}_{y,n}^2, \bar{v}_{z,n}^2),$$

so that, it is an infimum. Hence, in the general case where  $u_x^1 \in H_{\sharp}^1$  and  $u_y^0, u_z^0 \in H_{\sharp}^2$ , formula (21) is no longer the minimum of  $\mathcal{E}^{4,1}$  with respect to  $v_y^2, v_z^2 \in H_{\sharp}^1$  but is still the infimum. The infimum (21) is nothing but the total energy of an untwisted Navier-Bernoulli beam. Its minimizers  $u_x^1 \in H_{\sharp}^1$  and  $u_y^0, u_z^0 \in H_{\sharp}^2$  are characterized by:

$$\begin{aligned} E|\tilde{\mathcal{S}}|u_x^{1''}(\tilde{x}) &= 0, & u_x^1(0) &= 0, & E|\tilde{\mathcal{S}}|u_x^{1'}(L) &= \tilde{F}_x, \\ E\tilde{I}_z u_y^{0''''}(\tilde{x}) &= 0, & u_y^0(0) = u_y^{0'}(0) &= 0, & E\tilde{I}_z u_y^{0''}(L) &= \tilde{M}_z & E\tilde{I}_z u_y^{0'''}(L) &= -\tilde{F}_y \\ E\tilde{I}_y u_z^{0''''}(\tilde{x}) &= 0, & u_z^0(0) = u_z^{0'}(0) &= 0, & E\tilde{I}_y u_z^{0''}(L) &= -\tilde{M}_y & E\tilde{I}_y u_z^{0'''}(L) &= -\tilde{F}_z \end{aligned}$$

and can be made explicit as:

$$\begin{aligned} u_x^1(\tilde{x}) &= \frac{\tilde{F}_x}{E|\tilde{\mathcal{S}}|} \tilde{x}, \\ u_y^0(\tilde{x}) &= \frac{\tilde{F}_y}{E\tilde{I}_z} \frac{\tilde{x}^2}{6} (3L - \tilde{x}) + \frac{\tilde{M}_z}{E\tilde{I}_z} \frac{\tilde{x}^2}{2}, \\ u_z^0(\tilde{x}) &= \frac{\tilde{F}_z}{E\tilde{I}_y} \frac{\tilde{x}^2}{6} (3L - \tilde{x}) - \frac{\tilde{M}_y}{E\tilde{I}_y} \frac{\tilde{x}^2}{2}, \end{aligned} \quad (22)$$

The displacement field of an untwisted Navier-Bernoulli beam is recovered.

We now look for the infimum of  $\mathcal{E}^{4,2}$  and, as previously, keep the function  $\omega^1 \in H_{\sharp}^1(0, L)$  fixed temporarily. In the particular case where  $\omega^1 \in H_{\sharp}^2$ , the minimizers  $\bar{v}_x^2 \in H_{\sharp}^1(\tilde{\Omega}; \mathbb{R})$  of  $\mathcal{E}^{4,2}$  are characterized by:

$$\bar{v}_x^2(\tilde{x}, \tilde{y}, \tilde{z}) + \tilde{y} u_y^{1'}(\tilde{x}) + \tilde{z} u_z^{1'}(\tilde{x}) = \omega^{1'}(\tilde{x}) \tilde{\psi}(\tilde{y}, \tilde{z}),$$

where  $\tilde{\psi}$  denotes the unique solution in  $H^1(\tilde{\mathcal{S}})/\mathbb{R}$  of the Neumann problem:

$$\left\{ \begin{array}{ll} \Delta \tilde{\psi} = 0, & \text{in } \tilde{\mathcal{S}}, \\ \nabla \tilde{\psi} \cdot \mathbf{n} = \tilde{z} n_y - \tilde{y} n_z, & \text{on } \partial \tilde{\mathcal{S}}. \end{array} \right.$$

Taking  $u_y^1 \equiv 0 \equiv u_z^1$ , there is always such a minimizer that vanishes at  $\tilde{x} = 0$ , thanks to the restrictive assumption initially made on  $\omega^1$ . The corresponding minimum value of  $\mathcal{E}^{4,2}$  is:

$$\min_{v_x^2 \in H_{\sharp}^1} \mathcal{E}^{4,1} = \frac{E}{4(1+\nu)} \int_0^L \tilde{J}[\omega^{1'}]^2 - \tilde{M}_x \omega^1(L), \quad (23)$$

where we have set:

$$\tilde{J} = \int_{\tilde{\mathcal{S}}} \left( \frac{\partial \tilde{\psi}}{\partial \tilde{y}} - \tilde{z} \right)^2 + \left( \frac{\partial \tilde{\psi}}{\partial \tilde{z}} + \tilde{y} \right)^2 = \min_{\varphi \in H^1(\tilde{\mathcal{S}})} \int_{\tilde{\mathcal{S}}} \left( \frac{\partial \varphi}{\partial \tilde{y}} - \tilde{z} \right)^2 + \left( \frac{\partial \varphi}{\partial \tilde{z}} + \tilde{y} \right)^2.$$

Again, in the general case where  $\omega^1 \in H_{\#}^1 \supset H_{\#}^2$ , the above value is no longer a minimum in general, but it is still an infimum.

Taking the minimum with respect to  $\omega^1 \in H_{\#}^1$  of that infimum, the familiar torsion of a Navier-Bernoulli beam is recovered:

$$\omega^1(\tilde{x}) = \frac{2(1+\nu)}{E\tilde{J}} \tilde{M}_x \tilde{x}. \quad (24)$$

Finally, the infimum of  $\mathcal{E}^4$  is also the minimum of the reduced energy:

$$\begin{aligned} & \frac{E}{2} \int_0^L |\tilde{\mathcal{S}}| [u_x^{1'}]^2 + \tilde{I}_z [u_y^{0''}]^2 + \tilde{I}_y [u_z^{0''}]^2 + \frac{E\tilde{J}}{2(1+\nu)} \int_0^L [\omega^{1'}]^2 \\ & - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L). \end{aligned}$$

The minimizers of the reduced energy are given by formulae (22) and (24), and correspond to the Navier-Bernoulli displacement field:

$$u_x^1(\tilde{x}) \mathbf{e}_x + u_y^0(\tilde{x}) \mathbf{e}_y + u_z^0(\tilde{x}) \mathbf{e}_z + \left[ \omega^1(\tilde{x}) \mathbf{e}_x - u_z^{0'}(\tilde{x}) \mathbf{e}_y + u_y^{0'}(\tilde{x}) \mathbf{e}_z \right] \times \left[ \tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z \right]. \quad (25)$$

**2.4. The general anisotropic heterogeneous case.** We now go back to the general expression of  $\mathcal{E}^4$  in the anisotropic transversely heterogeneous case:

$$\begin{aligned} \mathcal{E}^4 = & \frac{1}{2} \int_0^L \int_{\tilde{\mathcal{S}}} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1 - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) \\ & - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

where  $\mathbf{C}(\tilde{y}, \tilde{z})$  is only required to fulfil conditions (9), and where  $\mathbf{e}^1$  is given by formula (18), and  $\tilde{\mathbf{F}}, \tilde{\mathbf{M}}$ , by formulae (19).

Again, we fix  $u_y^0 \in H_{\#}^2(0, L)$ ,  $u_z^0 \in H_{\#}^2(0, L)$ ,  $u_x^1 \in H_{\#}^1(0, L)$ ,  $\omega^1 \in H_{\#}^1(0, L)$  temporarily and compute the infimum of:

$$\frac{1}{2} \int_0^L \int_{\tilde{\mathcal{S}}} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1, \quad (26)$$

with respect to  $u_y^1, u_z^1 \in H_{\#}^1(0, L)$ ,  $\mathbf{v}_2 \in H_{\#}^1(\tilde{\Omega}, \mathbb{R}^3)$ . Actually, in the particular case where  $u_y^1, u_z^1 \in H_{\#}^2(0, L)$ , we can always suppose  $u_y^1 \equiv 0 \equiv u_z^1$  since it amounts to replacing  $v_x^2$  by:

$$v_x^2 + \tilde{y} u_y^{1'} + \tilde{z} u_z^{1'}.$$

In the general case where  $u_y^1, u_z^1 \in H_{\#}^1(0, L)$ , the same is true since  $H_{\#}^2(0, L)$  is dense in  $H_{\#}^1(0, L)$ . Finally, we need only to compute the infimum of the functional (26) with respect to  $\mathbf{v}_2 \in H_{\#}^1(\tilde{\Omega}, \mathbb{R}^3)$ , for fixed  $u_y^0 \in H_{\#}^2(0, L)$ ,  $u_z^0 \in H_{\#}^2(0, L)$ ,  $u_x^1 \in H_{\#}^1(0, L)$ ,  $\omega^1 \in H_{\#}^1(0, L)$ .

We first consider the particular case where  $u_y^0 \in H_{\#}^3(0, L)$ ,  $u_z^0 \in H_{\#}^3(0, L)$ ,  $u_x^1 \in H_{\#}^2(0, L)$ ,  $\omega^1 \in H_{\#}^2(0, L)$ . For every  $\tilde{x} \in [0, L]$ , we are therefore driven to minimize:

$$\mathbf{v}_2 \in H^1(\tilde{\mathcal{S}}, \mathbb{R}^3) \mapsto \int_{\tilde{\mathcal{S}}} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1. \quad (27)$$

Denoting by  $\mathcal{R}$ , the four-dimensional vector space of infinitesimal rigid displacement of  $\tilde{\mathcal{S}}$  of the form:

$$u_x^2 \mathbf{e}_x + u_y^2 \mathbf{e}_y + u_z^2 \mathbf{e}_z + \omega^2 \mathbf{e}_x \times (\tilde{y} \mathbf{e}_y + \tilde{z} \mathbf{e}_z), \quad (u_x^2, u_y^2, u_z^2, \omega^2) \in \mathbb{R}^4,$$

the functional to minimize is convex, continuous and coercive on  $H^1(\tilde{\mathcal{S}}, \mathbb{R}^3)/\mathcal{R}$ , thanks to conditions (9) and Korn inequality. It therefore has a unique minimizer  $\mathbf{v}_2 \in H^1(\tilde{\mathcal{S}}, \mathbb{R}^3)/\mathcal{R}$ . This minimizer is nothing but the solution  $\mathbf{v}_2 \in H^1(\tilde{\mathcal{S}}, \mathbb{R}^3)/\mathcal{R}$  of the following two-dimensional elastic problem in the cross-section  $\tilde{\mathcal{S}}$ , with unknowns  $\mathbf{v}_2, \boldsymbol{\sigma}$ :

$$\begin{cases} \operatorname{div}_{(\tilde{y}, \tilde{z})} \boldsymbol{\sigma} = \mathbf{0}, & \text{in } \tilde{\mathcal{S}}, \\ \boldsymbol{\sigma} = \mathbf{C}(\tilde{y}, \tilde{z}) : \left[ \boldsymbol{\varepsilon}_{(\tilde{y}, \tilde{z})}(\mathbf{v}_2) + u_x^{1'} \boldsymbol{\varepsilon}_0^{(1)} + u_y^{0''} \boldsymbol{\varepsilon}_0^{(2)} + u_z^{0''} \boldsymbol{\varepsilon}_0^{(3)} + \omega^{1'} \boldsymbol{\varepsilon}_0^{(4)} \right], & \text{in } \tilde{\mathcal{S}}, \\ \boldsymbol{\sigma} \cdot \tilde{\mathbf{n}} = \mathbf{0}, & \text{on } \partial \tilde{\mathcal{S}}, \end{cases} \quad (28)$$

where  $\tilde{\mathbf{n}}$  denotes the outward unit normal, and:

$$\begin{aligned} \boldsymbol{\varepsilon}_0^{(1)} &= \mathbf{e}_x \otimes \mathbf{e}_x, \\ \boldsymbol{\varepsilon}_0^{(2)} &= -\tilde{y} \mathbf{e}_x \otimes \mathbf{e}_x, \\ \boldsymbol{\varepsilon}_0^{(3)} &= -\tilde{z} \mathbf{e}_x \otimes \mathbf{e}_x, \\ \boldsymbol{\varepsilon}_0^{(4)} &= -\frac{\tilde{z}}{2} (\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x) + \frac{\tilde{y}}{2} (\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x). \end{aligned} \quad (29)$$

We cannot expect the solution of the elastic problem (28) to be explicit, in general, although it is, in the isotropic homogeneous case. However, the general solution can be expressed in terms of four particular solutions  $\mathbf{v}_2^{(1)}, \mathbf{v}_2^{(2)}, \mathbf{v}_2^{(3)}$  and  $\mathbf{v}_2^{(4)}$  as:

$$\mathbf{v}_2 = u_x^{1'} \mathbf{v}_2^{(1)} + u_y^{0''} \mathbf{v}_2^{(2)} + u_z^{0''} \mathbf{v}_2^{(3)} + \omega^{1'} \mathbf{v}_2^{(4)}, \quad (30)$$

where the  $\mathbf{v}_2^{(i)}$  are the four particular solutions of the problem (28), corresponding to the case where all the entries of  $(u_x^{1'}, u_y^{0''}, u_z^{0''}, \omega^{1'})$  vanish except for one of them which equals 1.

Injecting expression (30) into the minimization problem (27) yields:

$$\min_{\mathbf{v}_2 \in H^1(\tilde{\mathcal{S}})/\mathcal{R}} \int_{\tilde{\mathcal{S}}} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1 = (u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{1'}) \cdot \mathbf{c} \cdot \begin{pmatrix} u_x^{1'} \\ -u_z^{0''} \\ u_y^{0''} \\ \omega^{1'} \end{pmatrix},$$

for some matrix  $\mathbf{c}$  of order 4 whose entries can be expressed in terms of an integral over the cross-section  $\tilde{\mathcal{S}}$  involving the  $\mathbf{v}_2^{(i)}$  and  $\mathbf{C}$ , only. The following properties of the matrix  $\mathbf{c}$  are readily established.

- The matrix  $\mathbf{c}$  is positive symmetric, since the functional to minimize is non-negative.
- The matrix  $\mathbf{c}$  is actually positive definite symmetric. Indeed, if the minimum were zero, then it would require that  $\mathbf{e}^1$  itself should vanish. But, in that case, it is readily checked that it entails that  $u_x^{1'} = u_z^{0''} = u_y^{0''} = \omega^{1'} = 0$ .
- The matrix  $\mathbf{c}$  depends only on  $\tilde{\mathcal{S}}$  and  $\mathbf{C}(\tilde{y}, \tilde{z})$ , and, in particular, is independent of  $\tilde{x}$ .

- The entries of the matrix  $\mathbf{c}$  can be easily expressed in terms of the  $\mathbf{v}_2^{(i)}$  ( $i = 1, 2, 3, 4$ ). Since the  $\mathbf{v}_2^{(i)}$  can be computed, as precisely as desired, by means, for example, of four finite element computations, the same is true for the 10 independent entries of the matrix  $\mathbf{c}$  which can be computed, once for all, by means of four finite element computations of four elastic problems on the cross-section  $\tilde{\mathcal{S}}$ .

Finally, in the particular case where  $u_y^0 \in H_{\sharp}^3(0, L)$ ,  $u_z^0 \in H_{\sharp}^3(0, L)$ ,  $u_x^2 \in H_{\sharp}^2(0, L)$ ,  $\omega^1 \in H_{\sharp}^2(0, L)$ , the minimum of the functional (26) for  $\mathbf{v}_2 \in H_{\sharp}^1(\tilde{\mathcal{S}}; \mathbb{R}^3)$  is given by:

$$\frac{1}{2} \int_0^L (u_x^{1'} - u_z^{0''}, u_y^{0''}, \omega^{1'}) \cdot \mathbf{c} \cdot {}^t(u_x^{1'} - u_z^{0''}, u_y^{0''}, \omega^{1'}).$$

In the general case where  $u_y^0 \in H_{\sharp}^2(0, L)$ ,  $u_z^0 \in H_{\sharp}^2(0, L)$ ,  $u_x^2 \in H_{\sharp}^1(0, L)$ ,  $\omega^1 \in H_{\sharp}^1(0, L)$ , the above value is still an infimum as  $H_{\sharp}^3(0, L)$  and  $H_{\sharp}^2(0, L)$  are dense in  $H_{\sharp}^2(0, L)$  and  $H_{\sharp}^1(0, L)$ , respectively.

The total energy of the reduced model in the anisotropic heterogeneous case is the infimum of  $\mathcal{E}^4$  with respect to  $u_y^1, u_z^1 \in H_{\sharp}^1(0, L)$ ,  $\mathbf{v}_2 \in H_{\sharp}^1(\tilde{\Omega}, \mathbb{R}^3)$ , and is therefore given by:

$$\begin{aligned} & \frac{1}{2} \int_0^L (u_x^{1'} - u_z^{0''}, u_y^{0''}, \omega^{1'}) \cdot \mathbf{c} \cdot {}^t(u_x^{1'} - u_z^{0''}, u_y^{0''}, \omega^{1'}) \\ & - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L). \end{aligned}$$

It is the total energy of the general Navier-Bernoulli model, which has unique minimizers  $u_y^0 \in H_{\sharp}^2(0, L)$ ,  $u_z^0 \in H_{\sharp}^2(0, L)$ ,  $u_x^1 \in H_{\sharp}^1(0, L)$ ,  $\omega^1 \in H_{\sharp}^1(0, L)$ . These minimizers define a Navier-Bernoulli three-dimensional displacement field of the form (25).

In the particular case of an isotropic homogeneous material, the matrix  $\mathbf{c}$  of so-called reduced elastic moduli was seen, in section 2.3, to be diagonal:

$$\mathbf{c} = \begin{pmatrix} E|\tilde{\mathcal{S}}| & 0 & 0 & 0 \\ 0 & E\tilde{I}_y & 0 & 0 \\ 0 & 0 & E\tilde{I}_z & 0 \\ 0 & 0 & 0 & EJ/[2(1+\nu)] \end{pmatrix}.$$

In the general case, the non-diagonal entries should not be expected to vanish, therefore inducing couplings between extension, bending and torsion.

**3. Formal asymptotic analysis of thin-walled beams.** This section is devoted to a detailed account of the formal asymptotic analysis of thin-walled beams.

**3.1. Position of problem.** In the usual three-dimensional Euclidean space with orthonormal Cartesian coordinate system  $Oxyz$ , we consider the connected cylinder  $\Omega^\varepsilon = [0, L] \times \mathcal{S}^\varepsilon$ . The origin  $O$  is taken in the extremal section  $\mathcal{S}_0^\varepsilon$ .

The connected cylinder  $\Omega^\varepsilon$  is supposed to be a thin-walled beam, in the sense that it is a slender cylinder having slender cross-section, the two small slenderness parameters having the same order of magnitude. This is implemented by considering the case where  $\mathcal{S}^\varepsilon$  is a thin strip of length  $\varepsilon l$  and of thickness  $\varepsilon^2 l$ . Hence, we take  $\mathcal{S}^\varepsilon = \varepsilon \tilde{\mathcal{S}}^\varepsilon$ , where  $\tilde{\mathcal{S}}^\varepsilon$  is a thin strip of fixed length  $l$  and of thickness  $\varepsilon l$ . More precisely, we are given a regular smooth (at least of class  $C^3$ ) curve of finite length  $l$  in the  $Oyz$  plane. It will be parametrized by its arc-length  $\tilde{s} \in [0, l]$  and a generic point of this curve will be referred to as  $M(\tilde{s})$ . This curve is supposed

to be nonintersecting (that is, the mapping  $\tilde{s} \in [0, l[ \mapsto M(\tilde{s})$  is one-to-one), but our setting will encompass both the cases where the curve has two extremities and where it has no extremity at all (closed loop). The classical Frenet-Serret formulas read as:

$$\frac{dM(\tilde{s})}{d\tilde{s}} = \mathbf{t}(\tilde{s}), \quad \frac{d\mathbf{t}(\tilde{s})}{d\tilde{s}} = \mathcal{C}(\tilde{s}) \mathbf{n}(\tilde{s}), \quad \frac{d\mathbf{n}(\tilde{s})}{d\tilde{s}} = -\mathcal{C}(\tilde{s}) \mathbf{t}(\tilde{s}).$$

Here,  $\mathbf{t}$ ,  $\mathbf{n}$  denote unit vectors, respectively tangent and normal to the curve. The moving frame  $(\mathbf{t}(\tilde{s}), \mathbf{n}(\tilde{s}))$  is supposed to be positively oriented and  $\mathcal{C}(\tilde{s})$  denotes the current algebraic curvature (inverse of the curvature radius) of the curve. A thin strip is constructed around the curve as follows. An orthogonal line segment of length  $\varepsilon l$  is attached at each point of the curve, in such a way that the middle-point of each segment lies on the smooth curve of generic point  $M(\tilde{s})$ . The abscissa on such a segment will be denoted by  $\eta \in [-\varepsilon l/2, \varepsilon l/2]$ . The subset  $\tilde{\mathcal{S}}^\varepsilon$  is simply taken as the union of all the attached segments. The parameter  $\varepsilon$  is supposed to be small enough so that  $(\tilde{s}, \eta) \in [0, l] \times [-\varepsilon l/2, \varepsilon l/2]$  defines a curvilinear coordinate system in  $\tilde{\mathcal{S}}^\varepsilon$  (this requires in particular that  $\varepsilon l |\mathcal{C}(\tilde{s})|/2 < 1$ ). By construction, this curvilinear coordinate system is orthogonal. The current point in  $\tilde{\mathcal{S}}^\varepsilon$  can now be denoted by  $m(\tilde{s}, \eta)$  with  $m(\tilde{s}, \eta) = M(\tilde{s}) + \eta \mathbf{n}(\tilde{s})$  and we have:

$$\frac{\partial m(\tilde{s}, \eta)}{\partial \tilde{s}} = [1 - \eta \mathcal{C}(\tilde{s})] \mathbf{t}(\tilde{s}), \quad \frac{\partial m(\tilde{s}, \eta)}{\partial \eta} = \mathbf{n}(\tilde{s}).$$

It is therefore natural to consider the rescaled variables:

$$\tilde{x} = x, \quad \tilde{y} = y/\varepsilon, \quad \tilde{z}/\varepsilon = z/\varepsilon, \quad \tilde{s} = s/\varepsilon, \quad \tilde{\eta} = \eta/\varepsilon,$$

so that  $(\tilde{x}, \tilde{s}, \tilde{\eta}) \in [0, L] \times [0, l] \times [-l/2, l/2] = \tilde{\Omega}$ .

The thin-walled beam is supposed to be made of an arbitrary anisotropic elastic material which is allowed, in addition, to be transversely heterogeneous (the heterogeneity is the same for every cross-section). This is implemented by an elastic tensor  $\mathbf{C}(\tilde{s}, \tilde{\eta})$  depending on the variables  $(\tilde{s}, \tilde{\eta})$  only (in particular, it is independent of  $\varepsilon$ ), and satisfying the usual positivity and boundedness conditions (9).

Again, we consider the three-dimensional linear equilibrium problem for this elastic cylinder, in which the extremity  $\mathcal{S}_0^\varepsilon$  is clamped, the cylinder is free of body force, and the lateral surface free of traction. The cylinder will therefore be loaded only by given surface traction  $\mathbf{t}^\varepsilon$  on the extremity  $\mathcal{S}_L^\varepsilon$ . This equilibrium problem in three-dimensional linear elasticity admits a unique equilibrium displacement field  $\mathbf{v}^\varepsilon$ . Again, our subsequent objective will be to study asymptotically the three-dimensional equilibrium displacement field  $\mathbf{v}^\varepsilon$  in the elastic cylinder, as  $\varepsilon \rightarrow 0+$ .

As in the previously considered case of ordinary beams, we have to make a choice upon the way the given surface traction  $\mathbf{t}^\varepsilon$  on the extremity  $\mathcal{S}_L^\varepsilon$  depends on  $\varepsilon$ . This choice is made along the same considerations as the ones that led to the scaling (14). Once again, the general idea that prevails in the search for the appropriate scaling of the load is that we are going to inject the ansatz (13) into the three-dimensional total energy, sort the terms by increasing order of  $\varepsilon$  and minimize successively each term. The scaling of the load will then be adjusted in such a way that each term to minimize is bounded by below without having to make some unacceptable assumption on the load (see the similar discussion in the case of ordinary beams). Here, the appropriate choice turns out to be:

$$\mathbf{t}^\varepsilon(s, \varepsilon \eta) = \varepsilon \tilde{t}_x(s/\varepsilon, \eta/\varepsilon) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(s/\varepsilon, \eta/\varepsilon) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(s/\varepsilon, \eta/\varepsilon) \mathbf{e}_z, \quad (31)$$

for some given, independent of  $\varepsilon$ , function  $\tilde{\mathbf{t}} \in L^2(]0, l[ \times ]-l/2, l/2[; \mathbb{R}^3)$ .

**3.2. Formal asymptotic expansion.** Adopting the rescaling of the load defined by formula (31), the three-dimensional displacement field  $\mathbf{v}^\varepsilon$  is the unique minimizer in:

$$H_{\sharp}^1(\tilde{\Omega}; \mathbb{R}^3) = \left\{ \mathbf{v} \in H^1 \mid \mathbf{v}(0, \tilde{s}, \tilde{\eta}) = \mathbf{0} \right\},$$

of the total energy:

$$\begin{aligned} \mathcal{E}^\varepsilon(\mathbf{v}) &= \frac{\varepsilon^3}{2} \int_0^L \int_0^l \int_{-l/2}^{l/2} \left\{ \mathbf{e}^\varepsilon(\mathbf{v}) : \mathbf{C} : \mathbf{e}^\varepsilon(\mathbf{v}) \right\} \left\{ 1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s}) \right\} d\tilde{x} d\tilde{s} d\tilde{\eta} \\ &\quad - \int_0^l \int_{-l/2}^{l/2} \left[ \varepsilon^4 \tilde{t}_x(\tilde{s}, \tilde{\eta}) v_x(L, \tilde{s}, \tilde{\eta}) + \varepsilon^5 \tilde{t}_y(\tilde{s}, \tilde{\eta}) v_y(L, \tilde{s}, \tilde{\eta}) \right. \\ &\quad \left. + \varepsilon^5 \tilde{t}_z(\tilde{s}, \tilde{\eta}) v_z(L, \tilde{s}, \tilde{\eta}) \right] \left\{ 1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s}) \right\} d\tilde{s} d\tilde{\eta}, \end{aligned}$$

where  $\mathbf{e}^\varepsilon$  now denotes the symmetric part of the gradient operator  $\nabla^\varepsilon$  derived in the appendix, that is:

$$\begin{aligned} \mathbf{e}^\varepsilon(\mathbf{v}) &= \frac{\partial v_x}{\partial \tilde{x}} \mathbf{e}_x \otimes \mathbf{e}_x + \frac{1}{\varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})]} \left( \frac{\partial v_t}{\partial \tilde{s}} - \mathcal{C}(\tilde{s}) v_n \right) \mathbf{t} \otimes \mathbf{t} + \frac{1}{\varepsilon^2} \frac{\partial v_n}{\partial \tilde{\eta}} \mathbf{n} \otimes \mathbf{n} \\ &\quad + \frac{1}{2} \left[ \frac{1}{\varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})]} \left( \frac{\partial v_n}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) v_t \right) + \frac{1}{\varepsilon^2} \frac{\partial v_t}{\partial \tilde{\eta}} \right] (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}) \\ &\quad + \frac{1}{2} \left[ \frac{\partial v_t}{\partial \tilde{x}} + \frac{1}{\varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})]} \frac{\partial v_x}{\partial \tilde{s}} \right] (\mathbf{t} \otimes \mathbf{e}_x + \mathbf{e}_x \otimes \mathbf{t}) \\ &\quad + \frac{1}{2} \left[ \frac{\partial v_n}{\partial \tilde{x}} + \frac{1}{\varepsilon^2} \frac{\partial v_x}{\partial \tilde{\eta}} \right] (\mathbf{n} \otimes \mathbf{e}_x + \mathbf{e}_x \otimes \mathbf{n}). \end{aligned}$$

Looking for a formal asymptotic expansion of the three-dimensional displacement field  $\mathbf{v}^\varepsilon$  of the form:

$$\mathbf{v}^\varepsilon(\tilde{x}, \tilde{s}, \tilde{\eta}) = \mathbf{v}^0(\tilde{x}, \tilde{s}, \tilde{\eta}) + \varepsilon \mathbf{v}^1(\tilde{x}, \tilde{s}, \tilde{\eta}) + \dots,$$

we inject that expansion into the energy  $\mathcal{E}^\varepsilon$  and sort the terms by increasing power of  $\varepsilon$ :

$$\mathcal{E}^\varepsilon(\mathbf{v}^\varepsilon) = \frac{1}{\varepsilon} \mathcal{E}^{-1}(\mathbf{v}^0) + \mathcal{E}^0(\mathbf{v}^0) + \varepsilon \mathcal{E}^1(\mathbf{v}^0, \mathbf{v}^1) + \dots.$$

The first term in the expansion of the energy is of order  $\varepsilon^{-1}$  and given by:

$$\mathcal{E}^{-1} = \frac{1}{2} \int_0^L \int_0^l \int_{-l/2}^{l/2} \mathbf{e}^{-2} : \mathbf{C} : \mathbf{e}^{-2},$$

where:

$$\mathbf{e}^{-2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{\eta}} \\ 0 & 0 & \frac{1}{2} \frac{\partial v_t^0}{\partial \tilde{\eta}} \\ \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{\eta}} & \frac{1}{2} \frac{\partial v_t^0}{\partial \tilde{\eta}} & \frac{\partial v_n^0}{\partial \tilde{\eta}} \end{pmatrix}.$$

The minimum of  $\mathcal{E}^{-1}$  with respect to  $\mathbf{v}^0 \in H_{\sharp}^1(\tilde{\Omega}; \mathbb{R}^3)$  is 0 and the minimizers are all the  $\mathbf{v}^0$  that are independent of  $\tilde{\eta}$ .

From now on, we assume that  $\mathbf{v}^0$  is independent of  $\tilde{\eta}$ , which entails  $\mathcal{E}^0 = 0$ , and we calculate the subsequent lower order term in the energy. It reads as:

$$\mathcal{E}^1 = \frac{1}{2} \int_0^L \int_0^l \int_{-l/2}^{l/2} \mathbf{e}^{-1} : \mathbf{C} : \mathbf{e}^{-1},$$

where:

$$\mathbf{e}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{s}} & \frac{1}{2} \frac{\partial v_x^1}{\partial \tilde{\eta}} \\ \frac{1}{2} \frac{\partial v_x^0}{\partial \tilde{s}} & \frac{\partial v_t^0}{\partial \tilde{s}} - \mathcal{C} v_n^0 & \frac{1}{2} \left( \frac{\partial v_n^0}{\partial \tilde{s}} + \mathcal{C} v_t^0 + \frac{\partial v_t^1}{\partial \tilde{\eta}} \right) \\ \frac{1}{2} \frac{\partial v_x^1}{\partial \tilde{\eta}} & \frac{1}{2} \left( \frac{\partial v_n^0}{\partial \tilde{s}} + \mathcal{C} v_t^0 + \frac{\partial v_t^1}{\partial \tilde{\eta}} \right) & \frac{\partial v_n^1}{\partial \tilde{\eta}} \end{pmatrix}.$$

The minimum of  $\mathcal{E}^1$  in  $H_{\sharp}^1(\tilde{\Omega}; \mathbb{R}^3)$  is 0 again, and the minimizers are given by the  $\mathbf{v}^0$  and  $\mathbf{v}^1$  of the form:

$$\begin{aligned} \mathbf{v}^0(\tilde{x}, \tilde{s}) &= v_x^0(\tilde{x}) \mathbf{e}_x + v_t^0(\tilde{x}, \tilde{s}) \mathbf{t}(\tilde{s}) + v_n^0(\tilde{x}, \tilde{s}) \mathbf{n}(\tilde{s}), \\ \mathbf{v}^1(\tilde{x}, \tilde{s}, \tilde{\eta}) &= v_x^1(\tilde{x}, \tilde{s}) \mathbf{e}_x + \left[ f_t^1(\tilde{x}, \tilde{s}) - \tilde{\eta} \left[ \frac{\partial v_n^0}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) v_t^0(\tilde{x}, \tilde{s}) \right] \right] \mathbf{t}(\tilde{s}) + v_n^1(\tilde{x}, \tilde{s}) \mathbf{n}(\tilde{s}), \end{aligned} \quad (32)$$

where:

$$\frac{\partial v_t^0}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) = \mathcal{C}(\tilde{s}) v_n^0(\tilde{x}, \tilde{s}). \quad (33)$$

From now on, we assume that  $\mathbf{v}^0$  and  $\mathbf{v}^1$  take the above form, where the coordinate functions are still unknown functions. This entails  $\mathcal{E}^2 = 0$ . The next higher order term in the energy reads as:

$$\mathcal{E}^3 = \frac{1}{2} \int_0^L \int_0^l \int_{-l/2}^{l/2} \mathbf{e}^0 : \mathbf{C} : \mathbf{e}^0,$$

where:

$$\mathbf{e}^0 = \begin{pmatrix} \frac{\partial v_x^0}{\partial \tilde{x}} & \frac{1}{2} \left( \frac{\partial v_x^0}{\partial \tilde{x}} + \frac{\partial v_x^1}{\partial \tilde{s}} \right) & \frac{1}{2} \left( \frac{\partial v_n^0}{\partial \tilde{x}} + \frac{\partial v_x^2}{\partial \tilde{\eta}} \right) \\ " & \frac{\partial v_t^1}{\partial \tilde{s}} - \mathcal{C} v_n^1 & \frac{1}{2} \left[ \tilde{\eta} \mathcal{C} \left( \frac{\partial v_n^0}{\partial \tilde{s}} + \mathcal{C} v_t^0 \right) + \left( \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C} v_t^1 \right) + \frac{\partial v_t^2}{\partial \tilde{\eta}} \right] \\ " & " & \frac{\partial v_n^2}{\partial \tilde{\eta}} \end{pmatrix}.$$

It is going to be proved that the minimum of  $\mathcal{E}^3$  is 0 again, and we are going to describe all the minimizers  $\mathbf{v}^0$ ,  $\mathbf{v}^1$  and  $\mathbf{v}^2$ .

Injecting the expression of  $v_t^1$  from formula (32) into:

$$\frac{\partial v_t^1}{\partial \tilde{s}} = \mathcal{C} v_n^1,$$

(where we recall that  $f_t^1$  and  $v_n^1$  are independent of  $\tilde{\eta}$ ), we obtain:

$$\frac{\partial}{\partial \tilde{s}} \left[ \frac{\partial v_n^0}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) + \mathcal{C}(\tilde{s}) v_t^0(\tilde{x}, \tilde{s}) \right] = 0,$$

so that we can define:

$$\omega^0(\tilde{x}) = \frac{\partial v_n^0}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) + \mathcal{C}(\tilde{s}) v_t^0(\tilde{x}, \tilde{s}).$$

Recalling identity (33), the above equation can be integrated with respect to  $\tilde{s}$ , yielding:

$$\begin{aligned} v_t^0(\tilde{x}, \tilde{s}) &= u_y^0(\tilde{x}) \cos\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + u_z^0(\tilde{x}) \sin\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) - \omega^0(\tilde{x}) \left[ \mathbf{n}(\tilde{s}) \cdot OM(\tilde{s}) \right], \\ v_n^0(\tilde{x}, \tilde{s}) &= -u_y^0(\tilde{x}) \sin\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + u_z^0(\tilde{x}) \cos\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + \omega^0(\tilde{x}) \left[ \mathbf{t}(\tilde{s}) \cdot OM(\tilde{s}) \right], \end{aligned}$$

for three, yet arbitrary, functions  $u_y^0, u_z^0, \omega^0$  of  $\tilde{x}$  only, the constant  $\alpha_0$  being defined by:

$$\cos \alpha_0 = \mathbf{t}(0) \cdot \mathbf{e}_y, \quad \sin \alpha_0 = \mathbf{t}(0) \cdot \mathbf{e}_z. \quad (34)$$

Using these notations, it is readily checked that  $\mathbf{e}^0 = \mathbf{0}$ , if and only if  $\mathbf{v}^0, \mathbf{v}^1$  and  $\mathbf{v}^2$  take the form:

$$\begin{aligned} \mathbf{v}^0(\tilde{x}, \tilde{s}) &= u_y^0(\tilde{x}) \mathbf{e}_y + u_z^0(\tilde{x}) \mathbf{e}_z + \omega_0(\tilde{x}) \mathbf{e}_x \times OM(\tilde{s}), \\ \mathbf{v}^1(\tilde{x}, \tilde{s}, \tilde{\eta}) &= \left[ g_x^1(\tilde{x}) - \mathbf{u}^{0'}(\tilde{x}) \cdot OM(\tilde{s}) + \omega^{0'}(\tilde{x}) \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) \right] \mathbf{e}_x \\ &\quad + \left[ f_t^1(\tilde{x}, \tilde{s}) - \tilde{\eta} \omega^0(\tilde{x}) \right] \mathbf{t}(\tilde{s}) + v_n^1(\tilde{x}, \tilde{s}) \mathbf{n}(\tilde{s}), \\ \mathbf{v}^2(\tilde{x}, \tilde{s}, \tilde{\eta}) &= \left[ f_x^2(\tilde{x}, \tilde{s}) - \tilde{\eta} \left( \mathbf{u}^{0'}(\tilde{x}) \cdot \mathbf{n}(\tilde{s}) + \omega^{0'}(\tilde{x}) \left[ \mathbf{t}(\tilde{s}) \cdot OM(\tilde{s}) \right] \right) \right] \mathbf{e}_x \\ &\quad + \left[ f_t^2(\tilde{x}, \tilde{s}) - \tilde{\eta} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] \right] \mathbf{t}(\tilde{s}) + v_n^2(\tilde{x}, \tilde{s}) \mathbf{n}(\tilde{s}), \end{aligned} \quad (35)$$

for some functions  $g_x^1(\tilde{x}) \in H_{\#}^1(0, L)$  and  $f_t^1(\tilde{x}, \tilde{s}), f_x^2(\tilde{x}, \tilde{s}), f_t^2(\tilde{x}, \tilde{s}) \in H^1$  satisfying:

$$\frac{\partial f_t^1}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) = \mathcal{C}(\tilde{s}) v_n^1(\tilde{x}, \tilde{s}). \quad (36)$$

Note that the fact that  $\mathbf{v}^0, \mathbf{v}^1, \mathbf{v}^2$  belong to  $H_{\#}^1(\tilde{\Omega})$ , entails that  $u_y^0, u_z^0, \omega^0 \in H_{\#}^2(0, L)$ . In the particular case where the curve  $M(\tilde{s})$  has no extremity (closed loop), an additional periodicity condition on  $\mathbf{v}^1$  and  $\mathbf{v}^2$  with respect to the variable  $\tilde{s}$ , must be enforced. In particular, the periodicity of  $\mathbf{v}^1$  requires:

$$\omega^{0'}(\tilde{x}) \int_0^l (\mathbf{n} \cdot OM) \equiv 0,$$

where the absolute value of the integral is nothing but twice the area enclosed by the curve  $M(\tilde{s})$ , which shows that the integral is not zero. Therefore, in the case of a closed profile, one must have  $\omega_0 \equiv 0$ . The case of an open and a closed profile should therefore be discussed separately.

The remaining part of the asymptotic analysis will therefore be split as follows.

- Case of an open profile and an isotropic homogeneous material in section 3.3,
- Case of an open profile and an anisotropic heterogeneous material in section 3.4,
- Case of a closed profile and an isotropic homogeneous material in section 3.5,
- Case of a closed profile and an anisotropic heterogeneous material in section 3.6,

**3.3. Case of an open profile and an isotropic homogeneous material.** The first nonzero term in the expansion of the energy is now of order  $\varepsilon^5$ . In the isotropic homogeneous case, it reads as:

$$\begin{aligned} \mathcal{E}^5 &= \frac{E}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} \frac{\nu}{1-2\nu} \left[ \text{tr} \mathbf{e}^1 \right]^2 + \text{tr} \left[ \mathbf{e}^1 \right]^2 \\ &\quad - \tilde{F}_x g_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^0(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L) \\ &\quad - \omega^{0'}(L) \int_0^l \int_{-l/2}^{l/2} \tilde{t}_x \int_0^{\tilde{s}} (\mathbf{n} \cdot OM), \end{aligned}$$

where:

$$\mathbf{e}^1 = \begin{pmatrix} \frac{\partial v_x^1}{\partial \tilde{x}} & \frac{1}{2} \left( \frac{\partial v_t^1}{\partial \tilde{x}} + \tilde{\eta} \mathcal{C} \frac{\partial v_x^1}{\partial \tilde{s}} + \frac{\partial v_x^2}{\partial \tilde{s}} \right) & \frac{1}{2} \left( \frac{\partial v_n^1}{\partial \tilde{x}} + \frac{\partial v_x^3}{\partial \tilde{\eta}} \right) \\ " & \frac{\partial v_n^2}{\partial \tilde{s}} - \mathcal{C} v_n^2 & \frac{1}{2} \left[ \sum_{i=0}^2 \tilde{\eta}^{2-i} \mathcal{C}^{2-i} \left( \frac{\partial v_n^i}{\partial \tilde{s}} + \mathcal{C} v_t^i \right) + \frac{\partial v_t^3}{\partial \tilde{\eta}} \right] \\ " & " & \frac{\partial v_n^3}{\partial \tilde{\eta}} \end{pmatrix},$$

$$= \begin{pmatrix} \frac{\partial v_x^1}{\partial \tilde{x}} & \frac{1}{2} \left( \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \tilde{s}} - 2\tilde{\eta} \omega^{0'} \right) & \frac{1}{2} \left( \frac{\partial v_n^1}{\partial \tilde{x}} + \frac{\partial v_x^3}{\partial \tilde{\eta}} \right) \\ " & \frac{\partial f_t^2}{\partial \tilde{s}} - \mathcal{C} v_n^2 - \tilde{\eta} \frac{\partial}{\partial \tilde{s}} \left( \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C} f_t^1 \right) & \frac{1}{2} \left( \frac{\partial v_n^2}{\partial \tilde{s}} + \mathcal{C} f_t^2 + \frac{\partial v_t^3}{\partial \tilde{\eta}} \right) \\ " & " & \frac{\partial v_n^3}{\partial \tilde{\eta}} \end{pmatrix}, \quad (37)$$

the vector  $\tilde{\mathbf{F}}$  denotes the reduced total force defined by:

$$\tilde{F}_x = \int_0^l \int_{-l/2}^{l/2} \tilde{t}_x d\tilde{\eta} d\tilde{s}, \quad \tilde{F}_y = \int_0^l \int_{-l/2}^{l/2} \tilde{t}_y d\tilde{\eta} d\tilde{s}, \quad \tilde{F}_z = \int_0^l \int_{-l/2}^{l/2} \tilde{t}_z d\tilde{\eta} d\tilde{s}, \quad (38)$$

and  $\tilde{\mathbf{M}}$  denotes the reduced moment at the extremity point  $(0, 0, L)$ :

$$\begin{aligned} \tilde{M}_x &= \int_0^l \int_{-l/2}^{l/2} [\tilde{t}_z \mathbf{e}_y - \tilde{t}_y \mathbf{e}_z] \cdot OM(\tilde{s}) d\tilde{\eta} d\tilde{s}, \\ \tilde{M}_y &= \int_0^l \int_{-l/2}^{l/2} \tilde{t}_x \mathbf{e}_z \cdot OM(\tilde{s}) d\tilde{\eta} d\tilde{s}, \\ \tilde{M}_z &= \int_0^l \int_{-l/2}^{l/2} -\tilde{t}_x \mathbf{e}_y \cdot OM(\tilde{s}) d\tilde{\eta} d\tilde{s}, \end{aligned} \quad (39)$$

Hence, to find the infimum of  $\mathcal{E}^5$ , we fix the unknown functions  $g_x^1(\tilde{x})$ ,  $u_y^0(\tilde{x})$ ,  $u_z^0(\tilde{x})$ ,  $\omega^0(\tilde{x})$  temporarily and compute the infimum of the elastic energy:

$$\mathcal{E}_{\text{el}} = \frac{E}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} \frac{\nu}{1-2\nu} [\text{tr} \mathbf{e}^1]^2 + \text{tr} [\mathbf{e}^1]^2,$$

with respect to the unknowns  $f_t^1(\tilde{x}, \tilde{s})$ ,  $v_n^1(\tilde{x}, \tilde{s})$ ,  $f_x^2(\tilde{x}, \tilde{s})$ ,  $f_t^2(\tilde{x}, \tilde{s})$ ,  $v_n^2(\tilde{x}, \tilde{s})$ ,  $v_x^3(\tilde{x}, \tilde{s}, \tilde{\eta})$ ,  $v_t^3(\tilde{x}, \tilde{s}, \tilde{\eta})$  and  $v_n^3(\tilde{x}, \tilde{s}, \tilde{\eta})$ . We recall that  $v_n^1$  and  $f_t^1$  are not independent but have to fulfil identity (36).

In the particular case where  $v_n^1$  belongs to:

$$H_{\#}^2(\tilde{\Omega}) = \left\{ v(\tilde{x}, \tilde{s}, \tilde{\eta}) \in H^2 \mid v(0, \tilde{s}, \tilde{\eta}) = \frac{\partial v}{\partial \tilde{x}}(0, \tilde{s}, \tilde{\eta}) = \mathbf{0} \right\},$$

the minimum with respect to  $v_x^3(\tilde{x}, \tilde{s}, \tilde{\eta})$  and  $v_t^3(\tilde{x}, \tilde{s}, \tilde{\eta})$  is achieved by taking respectively:

$$\begin{aligned} \frac{\partial v_x^3}{\partial \tilde{\eta}} &= -\frac{\partial v_n^1}{\partial \tilde{x}}, \\ \frac{\partial v_t^3}{\partial \tilde{\eta}} &= -\left( \frac{\partial v_n^2}{\partial \tilde{s}} + \mathcal{C} f_t^2 \right). \end{aligned}$$

The corresponding detailed expression of  $\mathcal{E}_{\text{el}}$  is readily seen to be given by:

$$\mathcal{E}_{\text{el}} = \frac{E}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} \frac{\nu}{1-2\nu} \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\tilde{s}) + \omega^{0''} \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) \right]$$

$$\begin{aligned}
& + \frac{\partial f_t^2}{\partial \bar{s}} - \mathcal{C}v_n^2 + \frac{\partial v_n^3}{\partial \tilde{\eta}} - \tilde{\eta} \frac{\partial}{\partial \bar{s}} \left[ \frac{\partial v_n^1}{\partial \bar{s}} + \mathcal{C}(\bar{s}) f_t^1(\tilde{x}, \bar{s}) \right]^2 + \left[ \frac{\partial v_n^3}{\partial \tilde{\eta}} \right]^2 \\
& + \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\bar{s}) + \omega^{0''} \int_0^{\bar{s}} (\mathbf{n} \cdot OM) \right]^2 + \frac{1}{2} \left[ \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \bar{s}} - 2\tilde{\eta} \omega^{0'} \right]^2 \\
& + \left[ \frac{\partial f_t^2}{\partial \bar{s}} - \mathcal{C}v_n^2 - \tilde{\eta} \frac{\partial}{\partial \bar{s}} \left[ \frac{\partial v_n^1}{\partial \bar{s}} + \mathcal{C}(\bar{s}) f_t^1(\tilde{x}, \bar{s}) \right] \right]^2.
\end{aligned}$$

In the general case where  $v_n^1 \in H_{\sharp}^1$  but  $v_n^1 \notin H_{\sharp}^2$ , the above minimum is still an infimum, since  $H_{\sharp}^2$  is dense in  $H_{\sharp}^1$ .

Likewise, in the particular case where  $g_x^1 \in H_{\sharp}^2$ ,  $u_y^0, u_z^0, \omega^0 \in H_{\sharp}^3$ , the minimum with respect to  $v_n^3(\tilde{x}, \bar{s}, \tilde{\eta})$  is achieved by taking:

$$\begin{aligned}
\frac{\partial v_n^3}{\partial \tilde{\eta}} = -\frac{\nu}{1-\nu} \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\bar{s}) + \omega^{0''} \int_0^{\bar{s}} (\mathbf{n} \cdot OM) + \frac{\partial f_t^2}{\partial \bar{s}} - \mathcal{C}v_n^2 \right. \\
\left. - \tilde{\eta} \frac{\partial}{\partial \bar{s}} \left[ \frac{\partial v_n^1}{\partial \bar{s}} + \mathcal{C}(\bar{s}) f_t^1(\tilde{x}, \bar{s}) \right] \right],
\end{aligned}$$

and the corresponding minimum is:

$$\begin{aligned}
\mathcal{E}_{\text{el}} = \frac{E}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} \frac{\nu}{1-\nu} \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\bar{s}) + \omega^{0''} \int_0^{\bar{s}} (\mathbf{n} \cdot OM) \right. \\
+ \left. \frac{\partial f_t^2}{\partial \bar{s}} - \mathcal{C}v_n^2 - \tilde{\eta} \frac{\partial}{\partial \bar{s}} \left[ \frac{\partial v_n^1}{\partial \bar{s}} + \mathcal{C}(\bar{s}) f_t^1(\tilde{x}, \bar{s}) \right] \right]^2 + \frac{1}{2} \left[ \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \bar{s}} \right]^2 + 2\tilde{\eta}^2 [\omega^{0'}]^2 \\
+ \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\bar{s}) + \omega^{0''} \int_0^{\bar{s}} (\mathbf{n} \cdot OM) \right]^2 \\
+ \left[ \frac{\partial f_t^2}{\partial \bar{s}} - \mathcal{C}v_n^2 - \tilde{\eta} \frac{\partial}{\partial \bar{s}} \left[ \frac{\partial v_n^1}{\partial \bar{s}} + \mathcal{C}(\bar{s}) f_t^1(\tilde{x}, \bar{s}) \right] \right]^2.
\end{aligned}$$

In the general case where  $g_x^1 \in H_{\sharp}^1$ ,  $u_y^0, u_z^0, \omega^0 \in H_{\sharp}^2$ , only, the above minimum is still an infimum, since  $H_{\sharp}^2$  and  $H_{\sharp}^3$  are dense in  $H_{\sharp}^1$  and  $H_{\sharp}^2$ , respectively. There remains only to look for the infimum of this reduced elastic energy with respect to the four unknown functions  $f_t^1(\tilde{x}, \bar{s})$ ,  $f_t^2(\tilde{x}, \bar{s})$ ,  $f_x^2(\tilde{x}, \bar{s})$ ,  $v_n^2(\tilde{x}, \bar{s})$ . We first take the infimum of  $\mathcal{E}_{\text{el}}$  with respect to  $\partial f_t^2 / \partial \bar{s} - \mathcal{C}v_n^2$ . It is achieved by taking:

$$\frac{\partial f_t^2}{\partial \bar{s}} - \mathcal{C}v_n^2 = -\nu \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\bar{s}) + \omega^{0''} \int_0^{\bar{s}} (\mathbf{n} \cdot OM) \right],$$

in the particular case where  $g_x^1 \in H_{\sharp}^2$ ,  $u_y^0, u_z^0, \omega_0 \in H_{\sharp}^3$ , and the corresponding infimum is given by:

$$\begin{aligned}
\mathcal{E}_{\text{el}} = \frac{E}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} (1+\nu) \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\bar{s}) + \omega^{0''} \int_0^{\bar{s}} (\mathbf{n} \cdot OM) \right]^2 \\
+ 2\tilde{\eta}^2 [\omega^{0'}]^2 + \frac{1}{2} \left[ \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \bar{s}} \right]^2 + \frac{\tilde{\eta}^2}{1-\nu} \left[ \frac{\partial}{\partial \bar{s}} \left[ \frac{\partial v_n^1}{\partial \bar{s}} + \mathcal{C}(\bar{s}) f_t^1(\tilde{x}, \bar{s}) \right] \right]^2, \quad (40)
\end{aligned}$$

whose minimum with respect to  $f_t^1(\tilde{x}, \tilde{s})$ ,  $v_n^1(\tilde{x}, \tilde{s})$  is readily seen to be achieved for  $f_t^1 \equiv 0$ ,  $v_n^1 \equiv 0$  and  $f_x^2 \equiv 0$  giving the reduced elastic energy:

$$\mathcal{E}_{\text{el}} = \frac{El}{2} \int_0^L \int_0^l \left[ g_x^{1'}(\tilde{x}) - \mathbf{u}^{0''}(\tilde{x}) \cdot OM(\tilde{s}) + \omega^{0''}(\tilde{x}) \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) \right]^2 + \frac{El^4}{12(1+\nu)} \int_0^L [\omega^{0'}]^2.$$

There remains only to determine  $g_x^1(\tilde{x})$ ,  $u_y^0(\tilde{x})$ ,  $u_z^0(\tilde{x})$  and  $\omega_0(\tilde{x})$  by minimizing the reduced total energy:

$$\begin{aligned} & \frac{El}{2} \int_0^L \int_0^l \left[ g_x^{1'}(\tilde{x}) - \mathbf{u}^{0''}(\tilde{x}) \cdot OM(\tilde{s}) + \omega^{0''}(\tilde{x}) \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) \right]^2 + \frac{El^4}{12(1+\nu)} \int_0^L [\omega^{0'}]^2 \\ & - \varepsilon^5 \tilde{F}_x g_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^0(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L) \\ & - \omega^{0'}(L) \int_0^l \int_{-l/2}^{l/2} \tilde{t}_x \int_0^{\tilde{s}} (\mathbf{n} \cdot OM). \end{aligned}$$

The origin  $O$  has not been fixed yet. It is convenient to take it as the centroid of the curve  $M(\tilde{s})$ , so that:

$$\int_0^l OM(\tilde{s}) = \mathbf{0},$$

and to take  $Oy$  and  $Oz$  along the principal inertia directions of that curve, so that:

$$\int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})] [\mathbf{e}_z \cdot OM(\tilde{s})] = 0.$$

The corresponding principal inertia moments are denoted by:

$$\tilde{I}_y = l \int_0^l [\mathbf{e}_z \cdot OM(\tilde{s})]^2, \quad \tilde{I}_z = l \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})]^2. \quad (41)$$

They are both strictly positive, provided that the whole curve  $M(\tilde{s})$  is not a line segment. The coordinates  $(y_c, z_c)$  of the *reduced shear center*  $C$  are defined by:

$$\begin{aligned} y_c &= -\frac{l}{\tilde{I}_y} \int_0^l [\mathbf{e}_z \cdot OM(\tilde{s})] \int_0^{\tilde{s}} (\mathbf{n} \cdot OM), \\ z_c &= \frac{l}{\tilde{I}_z} \int_0^l [\mathbf{e}_y \cdot OM(\tilde{s})] \int_0^{\tilde{s}} (\mathbf{n} \cdot OM). \end{aligned} \quad (42)$$

We also define the *reduced warping function*:

$$\tilde{\psi}(\tilde{s}) = \int_0^{\tilde{s}} (\mathbf{n} \cdot OM) - \frac{1}{l} \int_0^l \int_0^{\tilde{\sigma}} (\mathbf{n} \cdot OM) - z_c \mathbf{e}_y \cdot OM(\tilde{s}) + y_c \mathbf{e}_z \cdot OM(\tilde{s}), \quad (43)$$

which, by construction, fulfils:

$$\int_0^l \tilde{\psi}(\tilde{s}) = 0 = \int_0^l \tilde{\psi}(\tilde{s}) \mathbf{e}_y \cdot OM(\tilde{s}) = \int_0^l \tilde{\psi}(\tilde{s}) \mathbf{e}_z \cdot OM(\tilde{s}).$$

Roughly speaking, these conditions are meant to ensure that  $\tilde{\psi}(\tilde{s})$  reduces to zero in any rigid motion of the cross-section, so that it captures only the warping of the cross-section.

We change the unknown  $g_x^1$  into the new  $u_x^1$  defined by:

$$u_x^1(\tilde{x}) = g_x^1(\tilde{x}) + \frac{\omega^{0'}(\tilde{x})}{l} \int_0^l \int_0^{\tilde{\sigma}} (\mathbf{n} \cdot OM). \quad (44)$$

With these notations, the reduced total energy, up to the multiplication scaling  $\varepsilon^5$ , takes the form:

$$\begin{aligned} & \frac{El}{2} \int_0^L \int_0^l \left\{ \left[ u_x^{1'}(\tilde{x}) \right]^2 + \left[ OM(\tilde{s}) \cdot \left[ \mathbf{u}^{0''}(\tilde{x}) + \omega^{0''}(\tilde{x}) (-z_c \mathbf{e}_y + y_c \mathbf{e}_z) \right] \right]^2 \right. \\ & \left. + \left[ \tilde{\psi}(\tilde{s}) \omega^{0''}(\tilde{x}) \right]^2 \right\} + \frac{El^4}{12(1+\nu)} \int_0^L \left[ \omega^{0'} \right]^2 - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) \\ & - \tilde{M}_x \omega^0(L) - \tilde{M}_y \left[ -u_z^{0'}(L) - y_c \omega^{0'}(L) \right] - \tilde{M}_z \left[ u_y^{0'}(L) - z_c \omega^{0'}(L) \right] \\ & - \omega^{0'}(L) \int_0^l \int_{-l/2}^{l/2} \tilde{t}_x \tilde{\psi}(\tilde{s}). \quad (45) \end{aligned}$$

Finally, we introduce the reduced torsional constant:

$$\tilde{J} = \frac{l^4}{3}$$

reduced warping stiffness:

$$\tilde{J}_w = l \int_0^l \left[ \tilde{\psi}(\tilde{s}) \right]^2,$$

and the reduced bimoment:

$$\tilde{B} = \int_0^l \int_{-l/2}^{l/2} \tilde{t}_x \tilde{\psi}(\tilde{s}), \quad (46)$$

so that the reduced total energy, up to the multiplication scaling  $\varepsilon^5$ , simplifies as:

$$\begin{aligned} & \frac{E}{2} \int_0^L \left\{ l^2 \left[ u_x^{1'}(\tilde{x}) \right]^2 + \tilde{I}_z \left[ u_y^{0''} - z_c \omega^{0''} \right]^2 + \tilde{I}_y \left[ u_z^{0''} + y_c \omega^{0''} \right]^2 + \tilde{J}_w \left[ \omega^{0''} \right]^2 \right. \\ & \left. + \frac{\tilde{J}}{2(1+\nu)} \left[ \omega^{0'} \right]^2 \right\} - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^0(L) \\ & - \tilde{M}_y \left[ -u_z^{0'}(L) - y_c \omega^{0'}(L) \right] - \tilde{M}_z \left[ u_y^{0'}(L) - z_c \omega^{0'}(L) \right] - \tilde{B} \omega^{0'}(L). \end{aligned}$$

The reduced moduli  $El^2$ ,  $E\tilde{I}_y$ ,  $E\tilde{I}_z$ ,  $E\tilde{J}/2(1+\nu)$  and  $E\tilde{J}_w$  are all strictly positive if and only if the curve  $M(\tilde{s})$  is not a line segment. In that latter case,  $\tilde{J}_w$  vanishes as well as one among  $\tilde{I}_y$  and  $\tilde{I}_z$ . Excluding that degenerate case, the above reduced total energy has unique minimizers  $u_x^1 \in H_{\#}^1$ ,  $u_y^0 \in H_{\#}^2$ ,  $u_z^0 \in H_{\#}^2$  and  $\omega^0 \in H_{\#}^2$  given by:

$$\begin{aligned} u_x^1(\tilde{x}) &= \frac{\tilde{F}_x}{El^2} \tilde{x}, \\ u_y^0(\tilde{x}) - z_c \omega^0(\tilde{x}) &= \frac{\tilde{F}_y}{E\tilde{I}_z} \frac{\tilde{x}^2}{6} (3L - \tilde{x}) + \frac{\tilde{M}_z}{E\tilde{I}_z} \frac{\tilde{x}^2}{2}, \\ u_z^0(\tilde{x}) + y_c \omega^0(\tilde{x}) &= \frac{\tilde{F}_z}{E\tilde{I}_y} \frac{\tilde{x}^2}{6} (3L - \tilde{x}) - \frac{\tilde{M}_y}{E\tilde{I}_y} \frac{\tilde{x}^2}{2}, \end{aligned}$$

where  $\omega^0(\tilde{x})$  is the unique solution of the boundary value problem:

$$\begin{cases} \tilde{J}_w \omega^{0''''} - \frac{\tilde{J}}{2(1+\nu)} \omega^{0''} \equiv 0, \\ \omega^0(0) = 0 = \omega^{0'}(0), \\ E\tilde{J}_w \omega^{0''}(L) = \tilde{B}, \\ \frac{E\tilde{J}}{2(1+\nu)} \omega^{0'}(L) - E\tilde{J}_w \omega^{0'''}(L) = \tilde{M}_x + z_c \tilde{F}_y - y_c \tilde{F}_z. \end{cases}$$

That corresponds to the equilibrium displacement in the Vlassov theory of beams.

### 3.4. Case of an open profile and an anisotropic heterogeneous material.

We now go back to the general expression of  $\mathcal{E}^5$  in the anisotropic transversely heterogeneous case:

$$\begin{aligned} \mathcal{E}^5 = & \frac{1}{2} \int_0^L \int_0^l \int_{-l/2}^{l/2} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1 - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^0(L) \\ & + \tilde{M}_y \left[ u_z^{0'}(L) + y_c \omega^{0'}(L) \right] - \tilde{M}_z \left[ u_y^{0'}(L) - z_c \omega^{0'}(L) \right] - \tilde{B} \omega^{0'}(L), \end{aligned}$$

where  $\tilde{\mathbf{F}}$ ,  $\tilde{\mathbf{M}}$  and  $\tilde{B}$  are defined in formulae (38), (39) and (46),  $y_c, z_c$  in formulae (42) and  $\mathbf{e}^1$  in formula (37). Here, the unknown  $g_x^1$  has been shifted to the new one  $u_x^1$  defined by formula (44).

Again, we are going to fix the unknowns  $u_x^1, u_y^0, u_z^0$  and  $\omega^0$  temporarily and try to compute the infimum of the elastic energy:

$$\frac{1}{2} \int_0^L \int_0^l \int_{-l/2}^{l/2} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1$$

with respect to the unknowns  $f_t^1(\tilde{x}, \tilde{s}), v_n^1(\tilde{x}, \tilde{s}), f_x^2(\tilde{x}, \tilde{s}), f_t^2(\tilde{x}, \tilde{s}), v_n^2(\tilde{x}, \tilde{s}), v_x^3(\tilde{x}, \tilde{s}, \tilde{\eta}), v_t^3(\tilde{x}, \tilde{s}, \tilde{\eta})$  and  $v_n^3(\tilde{x}, \tilde{s}, \tilde{\eta})$ . We recall that  $v_n^1$  and  $f_t^1$  are not independent but have to fulfil identity (36).

In a first step, the functions  $f_t^1, v_n^1, f_x^2, f_t^2, v_n^2$  are fixed temporarily so that the minimization process run with respect to  $\mathbf{v}^3$  only. Mimicking the analysis in section 2.4, we are therefore driven to minimize:

$$\mathbf{v}_3 \in H^1(-l/2, l/2; \mathbb{R}^3) \mapsto \int_{-l/2}^{l/2} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1, \quad (47)$$

for every  $(\tilde{x}, \tilde{s}) \in [0, L] \times [0, l]$ . The functional to minimize is convex, continuous and coercive on  $H^1(-l/2, l/2; \mathbb{R}^3)/\mathbb{R}^3$ , thanks to conditions (9). It therefore has a unique minimizer  $\mathbf{v}_3 \in H^1(-l/2, l/2; \mathbb{R}^3)/\mathbb{R}^3$ . To make this minimizer more explicit, we define:

$$\begin{aligned} \boldsymbol{\varepsilon}_0^{(1)} &= \frac{1}{2} \left( \mathbf{e}_x \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{e}_x \right), \\ \boldsymbol{\varepsilon}_0^{(2)} &= \frac{1}{2} \left( \mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t} \right), \\ \boldsymbol{\varepsilon}_0^{(3)} &= \mathbf{t} \otimes \mathbf{t}, \\ \boldsymbol{\varepsilon}_0^{(4)} &= -\tilde{\eta} \mathbf{t} \otimes \mathbf{t}, \\ \boldsymbol{\varepsilon}_0^{(5)} &= \frac{1}{2} \left( \mathbf{e}_x \otimes \mathbf{t} + \mathbf{t} \otimes \mathbf{e}_x \right), \\ \boldsymbol{\varepsilon}_0^{(6)} &= \mathbf{e}_x \otimes \mathbf{e}_x, \end{aligned}$$

$$\boldsymbol{\varepsilon}_0^{(7)} = -\tilde{\eta} \left( \mathbf{e}_x \otimes \mathbf{t} + \mathbf{t} \otimes \mathbf{e}_x \right). \quad (48)$$

The function  $\mathbf{v}_3^{(i)}$  ( $i = 1, 2, \dots, 7$ ) is then defined as the unique solution in the space  $H^1(-l/2, l/2; \mathbb{R}^3)/\mathbb{R}^3$  of the elastic problem in the line segment  $[-l/2, l/2]$ :

$$\left\{ \begin{array}{ll} \operatorname{div}_{\tilde{\eta}} \boldsymbol{\sigma} = \mathbf{0}, & \text{in } [-l/2, l/2], \\ \boldsymbol{\sigma} = \mathbf{C}(\tilde{s}, \tilde{\eta}) : \left[ \boldsymbol{\varepsilon}_{\tilde{\eta}}(\mathbf{v}_3^{(i)}) + \boldsymbol{\varepsilon}_0^{(i)} \right], & \text{in } [-l/2, l/2], \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}, & \text{on } \tilde{\eta} = \pm l/2, \end{array} \right. \quad (49)$$

where the index  $\tilde{\eta}$  in  $\operatorname{div}_{\tilde{\eta}}$  and  $\boldsymbol{\varepsilon}_{\tilde{\eta}}$  denote the differential operators ‘divergence’ and ‘symmetrized gradient’ in curvilinear coordinates  $(\tilde{x}, \tilde{s}, \tilde{\eta})$ , in which the space variables  $\tilde{x}$  and  $\tilde{s}$  are frozen. Note that the stress tensors  $\boldsymbol{\sigma}^{(1)}$  and  $\boldsymbol{\sigma}^{(2)}$  associated to  $\mathbf{v}_3^{(1)}$  and  $\mathbf{v}_3^{(2)}$  vanish, whereas the other  $\boldsymbol{\sigma}^{(i)}$ , for  $i \geq 3$ , are nonzero. In general, the  $\mathbf{v}_3^{(i)}$  will depend on  $\tilde{s}$ . However, we can distinguish two cases.

1. The case of a *laminate* (meaning the components of  $\mathbf{C}$  in the local basis  $(\mathbf{e}_x, \mathbf{t}, \mathbf{n})$  depend on  $\tilde{\eta}$  only). Then, the components of the  $\mathbf{v}_3^{(i)}$  in the local basis  $(\mathbf{e}_x, \mathbf{t}, \mathbf{n})$  are independent of  $\tilde{s}$ . In that latter simpler case, the  $\mathbf{v}_3^{(i)}$  can be computed, once for all and as precisely as desired, by 7 finite element computations on the line segment  $[-l/2, l/2]$ .
2. The general case of an arbitrary *transverse heterogeneous* material ( $\mathbf{C}(\tilde{s}, \tilde{\eta}) \in L^\infty(]0, L[ \times ]-l/2, l/2[)$ ). Then,  $\mathbf{v}_3^{(i)} \in L^\infty(0, l; H^1(-l/2, l/2; \mathbb{R}^3)/\mathbb{R}^3)$ .

The functions  $\mathbf{v}_3^{(i)}$  can be used to express the solution of the minimization problem (47) under the form:

$$\begin{aligned} \mathbf{v}_3 &= \frac{\partial v_n^1(\tilde{x}, \tilde{s})}{\partial \tilde{x}} \mathbf{v}_3^{(1)} + \left( \frac{\partial v_n^2(\tilde{x}, \tilde{s})}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^2(\tilde{x}, \tilde{s}) \right) \mathbf{v}_3^{(2)} \\ &+ \left( \frac{\partial f_t^2(\tilde{x}, \tilde{s})}{\partial \tilde{s}} - \mathcal{C}(\tilde{s}) v_n^2(\tilde{x}, \tilde{s}) \right) \mathbf{v}_3^{(3)} + \frac{\partial}{\partial \tilde{s}} \left( \frac{\partial v_n^1(\tilde{x}, \tilde{s})}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right) \mathbf{v}_3^{(4)} \\ &+ \left( \frac{\partial f_t^1(\tilde{x}, \tilde{s})}{\partial \tilde{x}} + \frac{\partial f_x^2(\tilde{x}, \tilde{s})}{\partial \tilde{s}} \right) \mathbf{v}_3^{(5)} \\ &+ \left[ u_x'(\tilde{x}) + OM(\tilde{s}) \cdot \left[ \mathbf{u}^{0''}(\tilde{x}) + \omega^{0''}(\tilde{x}) (-z_c \mathbf{e}_y + y_c \mathbf{e}_z) \right] + \tilde{\psi}(\tilde{s}) \omega^{0''}(\tilde{x}) \right] \mathbf{v}_3^{(6)} \\ &+ \omega^{0'}(\tilde{x}) \mathbf{v}_3^{(7)}. \end{aligned}$$

Denoting by  $\mathbf{V}(\tilde{x}, \tilde{s}) : [0, L] \times [0, l] \rightarrow \mathbb{R}^5$  the vector whose 5 components are the coefficients of the  $\mathbf{v}_3^{(i)}$  ( $i = 3, 4, 5, 6, 7$ ) in the above definition, we can use the above form of the minimizer  $\mathbf{v}_3$  of minimization problem (47) to express the value of the minimum under the form:

$$\min_{\mathbf{v}_3 \in H^1(-l/2, l/2)} \int_{-l/2}^{l/2} \mathbf{e}^1 : \mathbf{C} : \mathbf{e}^1 = {}^t \mathbf{V}(\tilde{x}, \tilde{s}) \cdot \mathbf{b}(\tilde{s}) \cdot \mathbf{V}(\tilde{x}, \tilde{s}),$$

for some positive symmetric matrix  $\mathbf{b}(\tilde{s})$  of order 5 whose entries are in  $L^\infty(0, l)$ . The fact that the value of this minimum is independent of the coefficient of  $\mathbf{v}_3^{(1)}$  and  $\mathbf{v}_3^{(2)}$  in the above expression of the minimizer  $\mathbf{v}_3$  is due to  $\boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^{(2)} = \mathbf{0}$ .

Again, in the particular case of a laminate, the matrix  $\mathbf{b}$  is constant with respect to  $\tilde{s}$  and can be calculated explicitly in terms of the  $\mathbf{v}_3^{(i)}$ . In that case, the matrix  $\mathbf{b}$  is readily checked to be positive definite. In the general case of a transversely heterogeneous material, the matrix  $\mathbf{b}(\tilde{s})$  is positive definite, for almost all  $\tilde{s} \in ]0, l[$ .

There remains only to solve the minimization problem:

$$f_t^1, v_n^1, f_x^2, f_t^2, v_n^2 \mapsto \int_0^l {}^t \mathbf{V}(\tilde{x}, \tilde{s}) \cdot \mathbf{b}(\tilde{s}) \cdot \mathbf{V}(\tilde{x}, \tilde{s}), \quad (50)$$

for every  $\tilde{x} \in [0, L]$ . Note that, given arbitrary functions  $V_3, V_4, V_5$  in  $L^2(0, l)$ , one can always find  $f_t^1, v_n^1, f_x^2, f_t^2, v_n^2$  in  $H^1(0, l)$  satisfying:

$$\frac{df_t^2}{d\tilde{s}} - \mathcal{C}v_n^2 = V_3, \quad \frac{d}{d\tilde{s}} \left( \frac{dv_n^1}{d\tilde{s}} + \mathcal{C}f_t^1 \right) = V_4, \quad \frac{df_x^2}{d\tilde{s}} - \mathcal{C}v_n^2 = V_5.$$

The minimization problem (50) is therefore seen to be equivalent to a convex minimization problem over  $L^2(0, l) \times L^2(0, l) \times L^2(0, l)$ . Thanks to condition (9), it is coercive and therefore well-posed. Its minimum value is of the form:

$$\min_{V_3, V_4, V_5 \in L^2(0, l)} \int_0^l {}^t \mathbf{V}(\tilde{s}) \cdot \mathbf{b}(\tilde{s}) \cdot \mathbf{V}(\tilde{s}) = \\ \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{0'}, \omega^{0''} \right) \cdot \mathbf{c} \cdot {}^t \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{0'}, \omega^{0''} \right),$$

for some positive symmetric matrix  $\mathbf{c}$  whose entries are constants (independent of  $\tilde{x}$ ). As in the isotropic homogeneous case, the matrix  $\mathbf{c}$  is checked to be positive definite if and only if the curve  $M(\tilde{s})$  is not a line segment.

Finally, we have proved that the total energy of the reduced model, in the case of a thin-walled beam with open profile and general anisotropic transversely heterogeneous material is of the form:

$$\frac{1}{2} \int_0^L \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{0'}, \omega^{0''} \right) \cdot \mathbf{c} \cdot {}^t \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{0'}, \omega^{0''} \right) \\ - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^0(L) \\ + \tilde{M}_y \left[ u_z^{0'}(L) + y_c \omega^{0'}(L) \right] - \tilde{M}_z \left[ u_y^{0'}(L) - z_c \omega^{0'}(L) \right] - \tilde{B} \omega^{0'}(L),$$

for some positive definite symmetric matrix  $\mathbf{c}$  of order 5 (excluding the degenerate case where the curve  $M(\tilde{s})$  is a line segment). In addition, the above reasoning contains a constructive procedure of the matrix  $\mathbf{c}$  from the knowledge of the three-dimensional elastic tensor  $\mathbf{C}(\tilde{s}, \tilde{\eta})$ .

The reduced displacement field of that Vlassov model is still expressed by formulae (35). It is remarkable that it keeps the same form in the most general anisotropic transversely heterogeneous case as in the isotropic homogeneous case, involving, in particular, the same warping function.

### 3.5. Case of a closed profile and an isotropic homogeneous material.

In this section, we go back in the formal asymptotic analysis to formulae (35). It was noted there that in the case of a closed profile (meaning that the curve  $M(\tilde{s})$  is a closed loop, having no extremity), the periodicity condition applying on  $\mathbf{v}^1$  requires that the following additional condition:

$$\omega^{0'}(\tilde{x}) \int_0^l (\mathbf{n} \cdot OM) \equiv 0,$$

must be fulfilled. As the integral is nothing but twice the area enclosed by the curve  $M(\tilde{s})$ , it is not zero, and the function  $\omega_0(\tilde{x})$  must therefore be zero in the case of a closed profile. We now go back over the asymptotic analysis in that case.

Deleting therefore  $\omega_0$ , formulae (35) reduce to:

$$\begin{aligned} \mathbf{v}^0(\tilde{x}, \tilde{s}) &= u_y^0(\tilde{x}) \mathbf{e}_y + u_z^0(\tilde{x}) \mathbf{e}_z, \\ \mathbf{v}^1(\tilde{x}, \tilde{s}, \tilde{\eta}) &= \left[ g_x^1(\tilde{x}) - \mathbf{u}^{0'}(\tilde{x}) \cdot OM(\tilde{s}) \right] \mathbf{e}_x + f_t^1(\tilde{x}, \tilde{s}) \mathbf{t}(\tilde{s}) + v_n^1(\tilde{x}, \tilde{s}) \mathbf{n}(\tilde{s}), \\ \mathbf{v}^2(\tilde{x}, \tilde{s}, \tilde{\eta}) &= \left[ f_x^2(\tilde{x}, \tilde{s}) - \tilde{\eta} \mathbf{u}^{0'}(\tilde{x}) \cdot \mathbf{n}(\tilde{s}) \right] \mathbf{e}_x \\ &\quad + \left[ f_t^2(\tilde{x}, \tilde{s}) - \tilde{\eta} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] \right] \mathbf{t}(\tilde{s}) + v_n^2(\tilde{x}, \tilde{s}) \mathbf{n}(\tilde{s}), \end{aligned}$$

where  $\mathbf{u}^0 \in H_{\sharp}^2$  is arbitrary, the functions  $f_t^1 \in H_{\sharp}^1$ ,  $v_n^1 \in H_{\sharp}^1$  fulfils:

$$\frac{\partial f_t^1}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) = \mathcal{C}(\tilde{s}) v_n^1(\tilde{x}, \tilde{s}), \quad (51)$$

and the remaining functions  $g_x^1 \in H_{\sharp}^1$ ,  $f_x^2 \in H_{\sharp}^1$ ,  $f_t^2 \in H_{\sharp}^1$  and  $v_n^2 \in H_{\sharp}^1$  are arbitrary.

Focusing on the particular case of an isotropic homogeneous material, the analysis can then be driven along the same lines as that in section 3.3 with the only difference that  $\omega_0$  is deleted everywhere. As a consequence, the total energy of the reduced model that is obtained is given by formula (45) in which  $\omega_0$  must be deleted. This is the total energy of the Navier-Bernoulli model *with no torsion*. The reason why torsion is missing is that the chosen scaling of the load sends the torsional term to higher order, as in the asymptotic analysis of an ordinary beam (as already mentioned in the introductory discussion to the asymptotic analysis of ordinary beams in section 2.1). In order to bring back torsion at principal order in the case of a thin-walled beam with closed profile, we are therefore driven to rescale the load in the spirit of formula (14), instead of the choice (31) which was relevant only for the case of an open profile. More precisely, the scaling of the load which turns out to be appropriate in that case is:

$$\begin{aligned} \tilde{\mathbf{t}}^\varepsilon(\tilde{s}, \tilde{\eta}) &= \varepsilon \tilde{t}_x(\tilde{s}, \tilde{\eta}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{s}, \tilde{\eta}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{s}, \tilde{\eta}) \mathbf{e}_z \\ &\quad + \varepsilon \tilde{M}_x / \tilde{I} \left[ [\mathbf{e}_y \cdot OM(\tilde{s})] \mathbf{e}_z - [\mathbf{e}_z \cdot OM(\tilde{s})] \mathbf{e}_y \right], \quad (52) \end{aligned}$$

where  $\tilde{M}_x$  is a given torsion moment and:

$$\tilde{I} = l \int_0^l |OM(\tilde{s})|^2.$$

Finally, the functional to minimize in the case of a thin-walled beam with closed profile and isotropic homogeneous elastic material is:

$$\begin{aligned} \mathcal{E}^{\varepsilon^5} &= \frac{E}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} \frac{\nu}{1-2\nu} \left[ \text{tr} \mathbf{e}^1 \right]^2 + \text{tr} \left[ \mathbf{e}^1 \right]^2 \\ &\quad - \tilde{F}_x g_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L) \\ &\quad - \frac{\tilde{M}_x}{\tilde{I}} \int_0^{\tilde{s}} \left\{ [\mathbf{t}(\tilde{s}) \cdot OM(\tilde{s})] v_n^1(L, \tilde{s}) - [\mathbf{n}(\tilde{s}) \cdot OM(\tilde{s})] f_t^1(L, \tilde{s}) \right\}, \end{aligned}$$

where  $\mathbf{e}^1$  is still given by formula (37), and  $\tilde{\mathbf{F}}$  and  $\tilde{\mathbf{M}}$  are computed from  $\tilde{\mathbf{t}}^\varepsilon$  exactly as in section 3.3, except for  $\tilde{M}_x$  which is now the constant appearing in formula (52).

Fixing the functions  $u_y^0, u_z^0, g_x^1, f_t^1$  and  $v_n^1$  temporarily, we must therefore compute the infimum of the elastic energy:

$$\mathcal{E}_{\text{el}} = \frac{E\varepsilon^5}{2(1+\nu)} \int_0^L \int_0^l \int_{-l/2}^{l/2} \frac{\nu}{1-2\nu} \left[ \text{tr} \mathbf{e}^1 \right]^2 + \text{tr} \left[ \mathbf{e}^1 \right]^2,$$

with respect to the unknowns  $f_t^2(\tilde{x}, \tilde{s}), v_n^2(\tilde{x}, \tilde{s}), v_x^3(\tilde{x}, \tilde{s}, \tilde{\eta}), v_t^3(\tilde{x}, \tilde{s}, \tilde{\eta})$  and  $v_n^3(\tilde{x}, \tilde{s}, \tilde{\eta})$ . The calculation runs exactly as in section 3.3 (the only difference being that we now have  $\omega_0 \equiv 0$ ). From formula (40), the corresponding infimum is seen to be:

$$\begin{aligned} \mathcal{E}_{\text{el}} = \frac{El}{2} \int_0^L \int_0^l \left[ g_x^{1'} - \mathbf{u}^{0''} \cdot OM(\tilde{s}) \right]^2 + \frac{1}{2(1+\nu)} \left[ \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \tilde{s}} \right]^2 \\ + \frac{l^2}{12(1-\nu^2)} \left[ \frac{\partial}{\partial \tilde{s}} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] \right]^2. \end{aligned}$$

The corresponding total energy splits into two independent parts (as for the ordinary beam made of an isotropic homogeneous material in section 2.3):

$$\begin{aligned} \mathcal{E}^{5,1} = \frac{E}{2} \int_0^L \left\{ l^2 \left[ g_x^{1'}(\tilde{x}) \right]^2 + \tilde{I}_z \left[ u_y^{0''} \right]^2 + \tilde{I}_y \left[ u_z^{0''} \right]^2 \right\} \\ - \tilde{F}_x g_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) + \tilde{M}_y u_z^0(L) - \tilde{M}_z u_y^0(L), \end{aligned}$$

where  $\tilde{I}_y$  and  $\tilde{I}_z$  are expressed by formulae (41). This is the reduced energy of a Navier-Bernoulli beam with no torsion. The other part reads as:

$$\begin{aligned} \mathcal{E}^{5,2} = \frac{El}{2} \int_0^L \int_0^l \frac{1}{2(1+\nu)} \left[ \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \tilde{s}} \right]^2 + \frac{l^2}{12(1-\nu^2)} \left[ \frac{\partial}{\partial \tilde{s}} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] \right]^2 \\ - \frac{\tilde{M}_x}{\tilde{I}} \int_0^{\tilde{s}} \left\{ [\mathbf{t}(\tilde{s}) \cdot OM(\tilde{s})] v_n^1(L, \tilde{s}) - [\mathbf{n}(\tilde{s}) \cdot OM(\tilde{s})] f_t^1(L, \tilde{s}) \right\}, \end{aligned}$$

where  $\tilde{I} = \tilde{I}_y + \tilde{I}_z$ . There remains to minimize this part with respect to  $f_t^1, v_n^1$  and  $f_x^2$ , recalling that  $f_t^1$  and  $v_n^1$  are not independent, but must satisfy condition (51). We also recall that  $f_t^1$  and  $v_n^1$  must be  $l$ -periodic with respect to the variable  $\tilde{s}$  (case of a closed profile).

The minimum with respect to  $f_x^2$  is achieved by:

$$\frac{\partial f_x^2}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) = -\frac{\partial f_t^1}{\partial \tilde{x}}(\tilde{x}, \tilde{s}) + \frac{1}{l} \int_0^l \frac{\partial f_t^1}{\partial \tilde{x}}(\tilde{x}, \tilde{\sigma}),$$

and is given by:

$$\begin{aligned} \mathcal{E}^{5,2} = \frac{El}{2} \int_0^L \int_0^l \frac{1}{2l^2(1+\nu)} \left\{ \left[ \int_0^l \frac{\partial f_t^1}{\partial \tilde{x}}(\tilde{x}, \tilde{\sigma}) \right]^2 \right. \\ \left. + \frac{l^2}{12(1-\nu^2)} \left[ \frac{\partial}{\partial \tilde{s}} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] \right]^2 \right\} \\ - \frac{\tilde{M}_x}{\tilde{I}} \int_0^{\tilde{s}} \left\{ [\mathbf{t}(\tilde{s}) \cdot OM(\tilde{s})] v_n^1(L, \tilde{s}) - [\mathbf{n}(\tilde{s}) \cdot OM(\tilde{s})] f_t^1(L, \tilde{s}) \right\}, \end{aligned}$$

The optimality condition yields:

$$\frac{\partial}{\partial \tilde{s}} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] = 0,$$

so that we can define:

$$\omega^1(\tilde{x}) = \frac{\partial v_n^1}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}).$$

Recalling identity (51), the above equation can be integrated with respect to  $\tilde{s}$ , yielding:

$$\begin{aligned} f_t^1(\tilde{x}, \tilde{s}) &= u_y^1(\tilde{x}) \cos\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + u_z^1(\tilde{x}) \sin\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) - \omega^1(\tilde{x}) \left[ \mathbf{n}(\tilde{s}) \cdot OM(\tilde{s}) \right], \\ v_n^1(\tilde{x}, \tilde{s}) &= -u_y^1(\tilde{x}) \sin\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + u_z^1(\tilde{x}) \cos\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + \omega^1(\tilde{x}) \left[ \mathbf{t}(\tilde{s}) \cdot OM(\tilde{s}) \right], \end{aligned}$$

for three, yet arbitrary, functions  $u_y^1$ ,  $u_z^1$ ,  $\omega^1$  of  $\tilde{x}$  only, the constant  $\alpha_0$  being defined by formulas (34). Taking into account those inevitable forms of the functions  $f_t^1(\tilde{x}, \tilde{s})$  and  $v_n^1(\tilde{x}, \tilde{s})$ ,  $\mathcal{E}^{5,2}$  simplify as:

$$\mathcal{E}^{5,2} = \frac{1}{2} \int_0^L \frac{E\tilde{J}}{2(1+\nu)} \left[ \omega^{1'} \right]^2 - \tilde{M}_x \omega^1(L),$$

where:

$$\tilde{J} = \left( \int_0^l \left[ \mathbf{n}(\tilde{s}) \cdot OM(\tilde{s}) \right] \right)^2,$$

is four times the square of the area enclosed by the closed loop. The function  $\omega_1 \in H_{\#}^1(0, L)$  must be the unique minimizer of  $\mathcal{E}^{5,2}$ .

Finally, the total reduced energy is that of the Navier-Bernoulli model:

$$\begin{aligned} \mathcal{E}^5 &= \frac{E}{2} \int_0^L \left\{ l^2 \left[ g_x^{1'}(\tilde{x}) \right]^2 + \tilde{I}_z \left[ u_y^{0''} \right]^2 + \tilde{I}_y \left[ u_z^{0''} \right]^2 + \frac{\tilde{J}}{2(1+\nu)} \left[ \omega^{1'} \right]^2 \right\} \\ &\quad - \tilde{F}_x g_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L). \end{aligned}$$

### 3.6. Case of a closed profile and an anisotropic heterogeneous material.

As in the previous section, we go back in the formal asymptotic analysis to formulae (35), in which  $\omega^0$  is deleted, and carry on the analysis along the same lines of the previous section, but now in the heterogeneous case. This amounts to delete  $\omega^0$ , and therefore  $\boldsymbol{\varepsilon}_0^{\textcircled{7}}$ ,  $\mathbf{v}_3^{\textcircled{7}}$ , in the analysis in the section 3.4. This yields a Navier-Bernoulli model *with no torsion* as the resulting reduced model, as in the previous section. As in the previous section, we are therefore driven to adopt the following rescaling of the load:

$$\begin{aligned} \tilde{\mathbf{t}}^\varepsilon(\tilde{s}, \tilde{\eta}) &= \varepsilon \tilde{t}_x(\tilde{s}, \tilde{\eta}) \mathbf{e}_x + \varepsilon^2 \tilde{t}_y(\tilde{s}, \tilde{\eta}) \mathbf{e}_y + \varepsilon^2 \tilde{t}_z(\tilde{s}, \tilde{\eta}) \mathbf{e}_z \\ &\quad + \varepsilon \tilde{M}_x / \tilde{I} \left[ \left[ \mathbf{e}_y \cdot OM(\tilde{s}) \right] \mathbf{e}_z - \left[ \mathbf{e}_z \cdot OM(\tilde{s}) \right] \mathbf{e}_y \right], \end{aligned}$$

in order to bring back torsion at principal order.

Rewinding the algebra in section 3.4, we introduce, as there, the four-dimensional vector  $\mathbf{V}(\tilde{x}, \tilde{s})$  with components:

$$\begin{aligned} \frac{\partial f_t^2(\tilde{x}, \tilde{s})}{\partial \tilde{s}} - \mathcal{C}(\tilde{s})v_n^2(\tilde{x}, \tilde{s}), \quad \frac{\partial}{\partial \tilde{s}} \left( \frac{\partial v_n^1(\tilde{x}, \tilde{s})}{\partial \tilde{s}} + \mathcal{C}(\tilde{s})f_t^1(\tilde{x}, \tilde{s}) \right), \\ \frac{\partial f_t^1(\tilde{x}, \tilde{s})}{\partial \tilde{x}} + \frac{\partial f_x^2(\tilde{x}, \tilde{s})}{\partial \tilde{s}}, \quad u_x^1(\tilde{x}) + OM(\tilde{s}) \cdot \mathbf{u}^{0''}(\tilde{x}), \end{aligned}$$

and, as there, consider for every  $\tilde{x} \in [0, L]$  the minimization problem:

$$\frac{\partial f_t^2}{\partial \tilde{s}} - \mathcal{C}v_n^2, \quad \frac{\partial}{\partial \tilde{s}} \left( \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}f_t^1 \right), \quad \frac{\partial f_t^1}{\partial \tilde{x}} + \frac{\partial f_x^2}{\partial \tilde{s}} \mapsto \int_0^l \mathbf{V}(\tilde{x}, \tilde{s}) \cdot \mathbf{b}(\tilde{s}) \cdot \mathbf{V}(\tilde{x}, \tilde{s}),$$

where  $\mathbf{b}(\tilde{s})$  is the positive definite symmetric matrix defined in section 3.4, with the only difference that the last row and column are deleted, due to  $\omega^0 \equiv 0$ . Note that all functions are now  $l$ -periodic with respect to  $\tilde{s}$ . We also recall that  $f_t^1$  and  $v_n^1$  are not independent, and must fulfil condition (51).

Introducing the space  $L_0^2(0, l)$  of square integrable functions with zero average value, the minimum value of the above minimization problem is the same as:

$$C + \theta_1, \theta_2, \frac{\partial f_t^1}{\partial \tilde{x}} + \theta_3 \mapsto \int_0^l \mathbf{V}(\tilde{s}) \cdot \mathbf{b}(\tilde{s}) \cdot \mathbf{V}(\tilde{s}),$$

where the first three components of  $\mathbf{V}$  are replaced by  $C + \theta_1, \theta_2, \theta_3 + \partial f_t^1 \partial \tilde{x}$ , and where  $C$  wanders in  $\mathbb{R}$  and,  $\theta_1, \theta_2, \theta_3$  wander in  $L_0^2(0, l)$ . The minimum value of this well-posed convex minimization problem is readily to be of the form:

$$\left( u_x^1, -u_z^{0''}, u_y^{0''}, - \left( \int_0^l \frac{\partial f_t^1}{\partial \tilde{x}} \right) \left( \int_0^l \mathbf{n} \cdot OM \right)^{-1} \right) \cdot \mathbf{c} \cdot \begin{pmatrix} u_x^1 \\ -u_z^{0''} \\ u_y^{0''} \\ - \left( \int_0^l \frac{\partial f_t^1}{\partial \tilde{x}} \right) \left( \int_0^l \mathbf{n} \cdot OM \right)^{-1} \end{pmatrix}$$

for some positive definite symmetric matrix  $\mathbf{c}$  of order 4. Hence, the reduced energy take the form:

$$\begin{aligned} \frac{1}{2} \int_0^L \left( u_x^1, -u_z^{0''}, u_y^{0''}, - \left( \int_0^l \frac{\partial f_t^1}{\partial \tilde{x}} \right) / \left( \int_0^l \mathbf{n} \cdot OM \right) \right) \cdot \mathbf{c} \cdot \begin{pmatrix} u_x^1 \\ -u_z^{0''} \\ u_y^{0''} \\ - \left( \int_0^l \frac{\partial f_t^1}{\partial \tilde{x}} \right) / \left( \int_0^l \mathbf{n} \cdot OM \right) \end{pmatrix} \\ - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L) \\ - \frac{\tilde{M}_x}{\tilde{I}} \int_0^{\tilde{s}} \left\{ [\mathbf{t}(\tilde{s}) \cdot OM(\tilde{s})] v_n^1(L, \tilde{s}) - [\mathbf{n}(\tilde{s}) \cdot OM(\tilde{s})] f_t^1(L, \tilde{s}) \right\}, \end{aligned}$$

where  $u_x^1 \in H_{\sharp}^1(0, L)$ ,  $u_y^0 \in H_{\sharp}^1(0, L)$ ,  $u_z^0 \in H_{\sharp}^1(0, L)$ , and the functions  $f_t^1(\tilde{x}, \tilde{s})$ ,  $v_n^1(\tilde{x}, \tilde{s})$  are in  $H_{\sharp}^1$  and satisfy:

$$\frac{\partial}{\partial \tilde{s}} \left[ \frac{\partial v_n^1}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}) \right] = 0, \quad \frac{\partial f_t^1}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) = \mathcal{C}(\tilde{s}) v_n^1(\tilde{x}, \tilde{s}),$$

so that we can define:

$$\omega^1(\tilde{x}) = \frac{\partial v_n^1}{\partial \tilde{s}}(\tilde{x}, \tilde{s}) + \mathcal{C}(\tilde{s}) f_t^1(\tilde{x}, \tilde{s}).$$

and the functions  $f_t^1, v_n^1$  take inevitably the form:

$$\begin{aligned} f_t^1(\tilde{x}, \tilde{s}) &= u_y^1(\tilde{x}) \cos\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + u_z^1(\tilde{x}) \sin\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) - \omega^1(\tilde{x}) \left[\mathbf{n}(\tilde{s}) \cdot OM(\tilde{s})\right], \\ v_n^1(\tilde{x}, \tilde{s}) &= -u_y^1(\tilde{x}) \sin\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + u_z^1(\tilde{x}) \cos\left(\alpha_0 + \int_0^{\tilde{s}} \mathcal{C}\right) + \omega^1(\tilde{x}) \left[\mathbf{t}(\tilde{s}) \cdot OM(\tilde{s})\right], \end{aligned}$$

for three, yet arbitrary, functions  $u_y^1, u_z^1, \omega^1$  of  $\tilde{x}$  only. The reduced energy takes therefore the form:

$$\begin{aligned} &\frac{1}{2} \int_0^L \left( u_x^{1'} - u_z^{0''}, u_y^{0''}, \omega^{1'} \right) \cdot \mathbf{c} \cdot \left( u_x^{1'}, -u_z^{0''}, u_y^{0''}, \omega^{1'} \right) \\ &\quad - \tilde{F}_x u_x^1(L) - \tilde{F}_y u_y^0(L) - \tilde{F}_z u_z^0(L) - \tilde{M}_x \omega^1(L) + \tilde{M}_y u_z^{0'}(L) - \tilde{M}_z u_y^{0'}(L), \end{aligned}$$

yielding the expected Navier-Bernoulli beam model.

**Appendix: Expression of the gradient in curvilinear coordinates.** Writing the current point in the scaled cross section  $\tilde{\mathcal{S}}^\varepsilon$  as  $m(\tilde{s}, \tilde{\eta}) = M(\tilde{s}) + \varepsilon \tilde{\eta} \mathbf{n}(\tilde{s})$ , we easily obtain the expression of the natural local vector basis at this point:

$$\frac{\partial m(\tilde{s}, \tilde{\eta})}{\partial \tilde{s}} = [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})] \mathbf{t}(\tilde{s}), \quad \frac{\partial m(\tilde{s}, \tilde{\eta})}{\partial \tilde{\eta}} = \varepsilon \mathbf{n}(\tilde{s}).$$

Then, any variation  $dm$  of the current point can be expressed by means of its coordinates  $(d\tilde{x}, d\tilde{s}, d\tilde{\eta}) \in \mathbb{R}^3$  in the local vector basis as:

$$dm = \mathbf{e}_x d\tilde{x} + \varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})] \mathbf{t}(\tilde{s}) d\tilde{s} + \varepsilon^2 \mathbf{n}(\tilde{s}) d\tilde{\eta}.$$

We next consider a vector field:

$$\mathbf{v}(\tilde{x}, \tilde{s}, \tilde{\eta}) = v_x(\tilde{x}, \tilde{s}, \tilde{\eta}) \mathbf{e}_x + v_t(\tilde{x}, \tilde{s}, \tilde{\eta}) \mathbf{t}(\tilde{s}) + v_n(\tilde{x}, \tilde{s}, \tilde{\eta}) \mathbf{n}(\tilde{s}),$$

whose first order variation reads as:

$$\begin{aligned} d\mathbf{v} &= \left( \frac{\partial v_x}{\partial \tilde{x}} d\tilde{x} + \frac{\partial v_x}{\partial \tilde{s}} d\tilde{s} + \frac{\partial v_x}{\partial \tilde{\eta}} d\tilde{\eta} \right) \mathbf{e}_x \\ &\quad + \left[ \frac{\partial v_t}{\partial \tilde{x}} d\tilde{x} + \left( \frac{\partial v_t}{\partial \tilde{s}} - \mathcal{C}(\tilde{s}) v_n \right) d\tilde{s} + \frac{\partial v_t}{\partial \tilde{\eta}} d\tilde{\eta} \right] \mathbf{t}(\tilde{s}) + \\ &\quad + \left[ \frac{\partial v_n}{\partial \tilde{x}} d\tilde{x} + \left( \frac{\partial v_n}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) v_t \right) d\tilde{s} + \frac{\partial v_n}{\partial \tilde{\eta}} d\tilde{\eta} \right] \mathbf{n}(\tilde{s}). \end{aligned}$$

By definition, the gradient  $\nabla^\varepsilon \mathbf{v}$  is the linear mapping (second order tensor field) that connects  $dm$  to the first order variation of  $d\mathbf{v}$  by the formula:

$$\forall (d\tilde{x}, d\tilde{s}, d\tilde{\eta}) \in \mathbb{R}^3, \quad d\mathbf{v} = \nabla^\varepsilon \mathbf{v} \cdot dm.$$

It readily yields:

$$\begin{aligned} \nabla^\varepsilon \mathbf{v} &= \frac{\partial v_x}{\partial \tilde{x}} \mathbf{e}_x \otimes \mathbf{e}_x + \frac{1}{\varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})]} \frac{\partial v_x}{\partial \tilde{s}} \mathbf{e}_x \otimes \mathbf{t} + \frac{1}{\varepsilon^2} \frac{\partial v_x}{\partial \tilde{\eta}} \mathbf{e}_x \otimes \mathbf{n} \\ &\quad + \frac{\partial v_t}{\partial \tilde{x}} \mathbf{t} \otimes \mathbf{e}_x + \frac{1}{\varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})]} \left( \frac{\partial v_t}{\partial \tilde{s}} - \mathcal{C}(\tilde{s}) v_n \right) \mathbf{t} \otimes \mathbf{t} + \frac{1}{\varepsilon^2} \frac{\partial v_t}{\partial \tilde{\eta}} \mathbf{t} \otimes \mathbf{n} \\ &\quad + \frac{\partial v_n}{\partial \tilde{x}} \mathbf{n} \otimes \mathbf{e}_x + \frac{1}{\varepsilon [1 - \varepsilon \tilde{\eta} \mathcal{C}(\tilde{s})]} \left( \frac{\partial v_n}{\partial \tilde{s}} + \mathcal{C}(\tilde{s}) v_t \right) \mathbf{n} \otimes \mathbf{t} + \frac{1}{\varepsilon^2} \frac{\partial v_n}{\partial \tilde{\eta}} \mathbf{n} \otimes \mathbf{n}. \end{aligned}$$

**Addendum.** We have discovered references [1] and [2] after that this article was accepted for publication, and we were therefore unaware of this important contribution at the time of writing the present paper. In these references, three-dimensional linear elastic equilibrium problems in heterogeneous anisotropic thin-walled beams with the same geometry as in our sections 3.4 and 3.6 are considered. Asymptotic analysis is performed in the framework of  $\Gamma$ -convergence both for the cases of the open profile and the closed profile. The  $\Gamma$ -limit is consistent with the energy of the reduced model that is obtained in our paper. Hence, the analysis in [1] and [2] provides a convergence result which is a full and rigorous mathematical justification of the formal analysis performed in this paper. The particular case of isotropic homogeneous elasticity is not specifically considered in references [1] and [2]. However, the analysis provided in our paper shows that, in that particular case of isotropic homogeneous elasticity, the  $\Gamma$ -limit obtained in [1] reduces to the classical Vlassov model for thin-walled beams with open profile and the  $\Gamma$ -limit obtained in [2] reduces to the classical Navier-Bernoulli model. Hence, the classical linear Vlassov model is now mathematically fully justified.

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