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# AN ELEMENTARY QUESTION ON TRIGONOMETRIC POLYNOMIALS WITH COEFFICIENTS $\pm 1$ WHICH IMPLIES RIEMANN HYPOTHESIS

EL HOUCEIN EL ABDALAOUI

ABSTRACT. We present an elementary problem on analytic polynomials with coefficients  $\pm 1$  or in  $\{0, \pm 1\}$  which implies Riemann hypothesis. It turns out that this problem is a particular case of the weak form of a flat polynomials problem asked by Erdős in his 1957's paper. We provide also under an extra-condition the converse.

## 1. INTRODUCTION

The purpose of this article is to present an elementary question on the norms of some polynomials with coefficients  $\pm 1$  on the circle which implies Riemann hypothesis. This question can be seen as a special case in the weak form of the Erdős-Newman problem on the existence of the so called flat polynomials.

The sequence of  $L^2$ - normalized polynomials  $(P_n)$  (that is,  $\|P_n\| = 1$  is said to be flat if it converge to 1 in some sense. The sequence is said to be ultraflat if the convergence holds for the uniform norm.

We warn the reader that this notion should not be confused with the notion of flatness introduced by Littlewood in his 1966's paper [10]. For this later notion (which we attribute to Littlewood), the sequence of  $L^2$ -normalized polynomials  $(P_n)$  is flat in the sense of Littlewood if there is a constant  $c, C > 0$  such that  $c \leq |P_n(z)| < C$ , for all  $z$  with modulus 1. Therein, Littlewood asked if there exist a sequence of flat polynomials with coefficients of modulus 1 or  $\pm 1$  which are flat in his sense. He further stated three conjecture on flatness [10] and [11, Problem 19]. Very recently, using Rudin-Shapiro polynomials combined with Spencer's six deviations lemma, P. Balister and al. constructed a flat polynomials in the Littlewood sense [4]. However, it is easy to see that those polynomials are not  $L^\alpha$ -flat, for any  $\alpha \geq 0$ <sup>1</sup>. We further notice that Littlewood addressed Erdős-Newman

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<sup>1</sup>This is follows directly from Theorem 2.4 and Lemma 3.3 in [4]. Indeed, by (2) Theorem 4.3 and Lemma 3.3 in [4], we get that those polynomials are not almost every flat. For this later notion and its applications in ergodic theory and dynamical systems, we refer to [1].

question in several papers [13], [12], [14]. Therein, he established Erdős's result which says that the cosines polynomials with coefficients  $\pm 1$  are not ultraflat. In fact, Littlewood established that those polynomials are not  $L^\alpha$ -flat, for any  $\alpha \geq 0$ . He also provided a condition on the coefficients of a real trigonometric polynomials to insure that those polynomials are not  $L^\alpha$ -flat. But, it seems that Littlewood had conflicting feelings about the existence of ultraflat polynomials. He further stated in [12, p.334] that the sequence of trigonometric polynomials

$$P_n(\theta) = \sum_{n=1}^N \cos(2^{2^n} \theta),$$

is  $L^\alpha$ -flat for any  $\alpha > 0$ . But, according to the central limit theorem for the Hadamard lacunary trigonometric series due to Salem-Zygmund [18, Vol 2, p. 264] combined with the technics in [1, p.176], it can be seen that for any  $1 \leq \alpha < 2$ ,

$$\|P_n\|_\alpha \xrightarrow{n \rightarrow +\infty} \Gamma\left(\frac{\alpha}{2} + 1\right).$$

For more recent results on Erdős-Newman conjecture, we refer to [2]<sup>2</sup>.

Here, we are interested in the semi-flatness of an analytic trigonometric polynomials with Liouville or Möbius coefficients. A sequence of  $L^2$ -normalized polynomials  $(P_n)$  is semi-flat if  $|P_n(z)|$  is bounded above by some absolutely constant in some sense. Precisely, we consider  $L^\alpha$ -semi-flat polynomials with  $\alpha > 2$ , that is, a sequence of analytic trigonometric polynomials  $(P_n)$  with coefficients in  $\{\pm 1, 0\}$  such that  $\|P_n(z)\|_\alpha$  is bounded above by some absolutely constant which may depend on  $\alpha$ .

The plan of the paper is as follows. In Section 2, we recall a basic notions and the results that we need in the sequel. In section 3, we state and prove our main result.

## 2. SET-UP AND TOOLS

Let us denote by the circle  $S^1 = \{z \in \mathbb{C}, |z| = 1\}$  and for any  $\alpha > 1$ , we define the  $L^\alpha$ -norm of any trigonometric polynomials  $P$  by

$$\|P\|_\alpha = \left( \int |P(z)|^\alpha dz \right)^{\frac{1}{\alpha}},$$

where  $dz$  is the Lebesgue measure on the circle. For any  $N \in \mathbb{N}^*$ , we denote by  $(\xi_{N,j})_{j=0}^{N-1}$ , the the  $N$ th roots of unity given by

$$\xi_{N,j} = e^{\frac{2i\pi j}{N}}, \quad j = 0, \dots, N-1.$$

The sequence of analytic trigonometric polynomials  $(P_n)$  is said to be  $L^\alpha$ -semi-flat polynomials with  $\alpha \geq 1$  if  $\|P_n(z)\|_\alpha$  is bounded above by some

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<sup>2</sup>Notice that it is not known if there exist almost everywhere flat polynomials with coefficients  $\pm 1$ .

absolutely constant which may depend on  $\alpha$ .

The analytic trigonometric polynomials with Liouville or Möbius coefficients are defined as follows

$$P_n(z) = \sum_{j=1}^n \lambda(j)z^j, \quad |z| = 1, \quad (2.1)$$

$$Q_n(z) = \sum_{j=1}^n \mu(j)z^j, \quad |z| = 1 \quad (2.2)$$

where  $\lambda$  and  $\mu$  are respectively the Liouville function and the Möbius function. The Liouville function is given by

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ not necessarily distinct prime numbers;} \end{cases}$$

The Liouville function is related to another famous functions in number theory called the Möbius function. Indeed, the Möbius function is defined for the positive integers  $n$  by

$$\mu(n) = \begin{cases} \lambda(n) & \text{if } n \text{ is not divisible by the square of any prime;} \\ 0 & \text{if not} \end{cases}$$

Those two functions are related to Riemann  $\zeta$ -function via the formulae

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad \sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \text{ with } \operatorname{Re}(s) > 1,$$

and

$$\sum_{n=1}^{+\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \text{ with } \operatorname{Re}(s) > 1.$$

Let us further notice that the Dirichlet inverse of the Liouville function is the absolute value of the Möbius function.

For the reader's convenience, we briefly recall some useful well-known results on the Riemann  $\zeta$ -function. The Riemann  $\zeta$ -function is defined, for  $s \in \mathbb{C}$ ,  $\operatorname{Re}(s) > 1$  by

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},$$

or by the Euler formula

$$\zeta(s) = \prod_{\substack{p \\ \text{prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

It is easy to check that  $\zeta$  is analytic for  $\operatorname{Re}(s) > 1$ . Moreover, it is well-known that  $\zeta$  is regular for all values of  $s$  except  $s = 1$ , where there is a

simple pole with residue 1. Thanks to the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s),$$

where  $\Gamma$  is the gamma function given by

$$\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} dx, \quad \operatorname{Re}(z) > 0.$$

We notice that the gamma function never vanishes and it is analytic everywhere except at  $z = 0, -1, -2, \dots$ , with the residue at  $z = -k$  is equal to  $\frac{(-1)^k}{k!}$ . We further have the following formula (useful in the proof of the functional equation)

$$\Gamma(s) \sin\left(\frac{\pi s}{2}\right) = \int_0^{+\infty} y^{s-1} \sin(y) dy.$$

For the proof of it we refer to [16, p.88]. Changing  $s$  to  $1-s$ , we obtain

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s).$$

Putting

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

and

$$E(s) = \xi\left(\frac{1}{2} + is\right).$$

It follows that

$$\xi(s) = \xi(1-s), \quad \text{and} \quad E(z) = E(-z).$$

We further remind that we have

$$\zeta(s) \Gamma(s) = \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx, \quad \operatorname{Re}(s) > 1.$$

Therefore, it is easy to check that  $\zeta$  has no zeros for  $\operatorname{Re}(s) > 1$ . It follows also from the functional equation that  $\zeta$  has no zeros for  $\operatorname{Re}(s) < 0$  except for simple zeros at  $s = -2, -4, \dots$ . Indeed,  $\zeta(1-s)$  has no zeros for  $\operatorname{Re}(s) < 0$ ,  $\sin\left(\frac{s\pi}{2}\right)$  has simple zeros at  $s = -2, -4, \dots$ . It is also a simple matter to see that  $\xi(s)$  has no zeros for  $\operatorname{Re}(s) > 1$  or  $\operatorname{Re}(s) < 0$ . Hence its zeros which are also the zeros of  $\zeta$  lie in the strip  $0 \leq \operatorname{Re}(s) \leq 1$ . Hardy point out that for  $\operatorname{Re}(s) > 1$ , it is easily seen that

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} &= \sum_{n=1}^{+\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{+\infty} \frac{1}{2^n n^s} \\ &= (1 - 2^{1-s}) \zeta(s), \end{aligned}$$

This formula allows us to continue  $\zeta$  analytically to half-plan  $\operatorname{Re}(s) > 0$  with simple pole at  $s = 1$ . We further have  $\zeta(s) \neq 0$  for all  $s > 0$  since  $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} > 0$ .

We thus conclude that all zeros of  $\zeta$  are complex. The functional equation allows us also to see that if  $z$  is a zero then  $1 - z$  and  $1 - \bar{z}$  are also a zeros. Whence, the zeros of  $\zeta$  lie on the vertical line  $\operatorname{Re}(s) = \frac{1}{2}$  or occur in pairs symmetrical about this line.

**Conjecture** (Riemann hypothesis (RH)). *All nontrivial zeros of  $\zeta$  lie on the critical line  $\operatorname{Re}(s) = \frac{1}{2}$ .*

Here, we need the following characterization of Riemann hypothesis due to Littlewood [8].

**Lemma 2.1.** *The Riemann hypothesis is equivalent to*

$$\left| \sum_{n=1}^x \lambda(n) \right| = o\left(x^{\frac{1}{2}+\varepsilon}\right) \quad \text{as } x \rightarrow +\infty, \quad \forall \varepsilon > 0 \quad (2.3)$$

For the proof of Littlewood, we refer also to [17, p.371], [5, p.261]. Notice that by applying Bateman-Chowla trick, the same conclusion can be drawn for  $\mu$ .

We will further need the following fundamental inequalities from the interpolation theory due to Marcinkiewz & Zygmund [18, Theorem 7.10, Chapter X, p.30].

**Lemma 2.2.** *For  $\alpha > 1$ ,  $n \geq 1$ , and any analytic trigonometric polynomial  $P$  of degree  $\leq n$ ,*

$$\frac{A_\alpha}{n} \sum_{j=0}^{n-1} |P(\xi_{n,j})|^\alpha \leq \int_{S^1} |P(z)|^\alpha dz \leq \frac{B_\alpha}{n} \sum_{j=0}^{n-1} |P(\xi_{n,j})|^\alpha, \quad (2.4)$$

where  $A_\alpha$  and  $B_\alpha$  are independent of  $n$  and  $P$ .

For the trigonometric polynomials, Marcinkiewz-Zygmund interpolation inequalities can be stated as follows [18, Theorem 7.5, Chapter X, p.28].

**Lemma 2.3.** *For  $\alpha > 1$ ,  $n \geq 1$ , and any trigonometric polynomial  $P$  of degree  $\leq n$ ,*

$$\frac{A_\alpha}{2n+1} \sum_{j=0}^{2n} |P(\xi_{2n+1,j})|^\alpha \leq \int_{S^1} |P(z)|^\alpha dz \leq \frac{B_\alpha}{2n+1} \sum_{j=0}^{2n} |P(\xi_{2n+1,j})|^\alpha, \quad (2.5)$$

where  $A_\alpha$  and  $B_\alpha$  are independent of  $n$  and  $P$ .

### 3. MAIN RESULT AND ITS PROOF

In this section, we start by stating our main result.

**Theorem 3.1.** *Let  $\alpha > 2$  and suppose that the sequence of analytic polynomials  $(P_n)$  is  $L^\alpha$ -semi-flat, that is, for each  $N \in \mathbb{N}^*$ ,*

$$\left\| \frac{1}{\sqrt{N}} \sum_{j=1}^N \mu(j) z^j \right\|_\alpha < C_\alpha,$$

for some constant  $C_\alpha$ . Then the Riemann hypothesis holds.

We are not able to see that the converse of our main theorem holds. although, we have the following

**Theorem 3.2.** *Suppose the Riemann hypothesis holds and for  $\alpha > 2$ , the sequence*

$$\left( \frac{1}{N^{1+\frac{\alpha}{2}}} \sum_{k=1}^{N-1} \left| \sum_{n=1}^{N-1} \lambda(n) \xi_{N,k}^n \right|^\alpha \right)_{N \geq 1}$$

is bounded. Then the sequence of analytic polynomials  $(P_n)$  is  $L^\alpha$ -semi-flat.

**Proof.** [of the main theorem.] Assume that the Riemann Hypothesis does not hold. Then, according to Littlewood criterion (Lemma 2.1), there exist  $c > 0$  and  $\epsilon > 0$  such that for infinitely many positive integers  $N$ , we have  $|M(N)| \geq c.N^{\frac{1}{2}+\epsilon}$ . Let  $\alpha > 1$  such that  $\alpha\epsilon > 1$ . Then, by Marcinkiewicz-Zygmund inequalities, we have

$$\begin{aligned} \left\| \frac{1}{\sqrt{N}} \sum_{j=1}^N \lambda(j) z^j \right\|_\alpha^\alpha &\geq A_\alpha \frac{\left| \sum_{j=1}^N \lambda(j) \right|^\alpha}{N^{\frac{\alpha}{2}+1}} \\ &\geq C_\alpha \frac{N^{\frac{\alpha}{2}+\alpha\epsilon}}{N^{\frac{\alpha}{2}+1}} = N^{\alpha\epsilon-1}, \end{aligned}$$

Letting  $N \rightarrow +\infty$ , we conclude that

$$\left\| \frac{1}{\sqrt{N}} \sum_{j=1}^N \lambda(j) z^j \right\|_\alpha \xrightarrow{N \rightarrow +\infty} +\infty.$$

This accomplishes the proof of the theorem. ■

Let us point out that we can also give a direct proof. Indeed, assume that for any  $\alpha \geq 1$ , we have

$$\left\| \frac{1}{\sqrt{N}} \sum_{j=1}^N \lambda(j) z^j \right\|_\alpha < +\infty.$$

Then, by Marcinkiewicz-Zygmund inequalities, for any  $\alpha > 1$ , there exist  $A_\alpha$  such that

$$A_\alpha \frac{\left| \sum_{j=1}^N \lambda(j) \right|^\alpha}{N^{\frac{\alpha}{2}+1}} \leq c_\alpha,$$

where  $c_\alpha$  is some positive constant. This gives

$$\left| \sum_{j=1}^N \lambda(j) \right| \leq C_\alpha N^{\frac{1}{2}+\frac{1}{\alpha}}.$$

Since  $\alpha$  is arbitrary, it follows, with the help of (2.3), that RH holds.

Now, let us give the proof of Theorem 3.2.

**Proof.** [of Theorem 3.2.] Under our assumptions combined with Marcinkiewicz-Zygmund Theorem, it is easy to see that the sequence of polynomials  $(P_n)_{n \geq 1}$  is  $L^\alpha$ -semiflat for any  $\alpha \geq 0$ . ■

At this point we ask the following

**Question 3.3.** *Does the RH implies the semi-flatness. In another word, is the converse of Theorem 3.1 true?*

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