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# Uniform blow-up estimates for nonlinear heat equations and applications

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We consider the following nonlinear heat equation

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u \\ u(0) &= u_0, \end{cases} \quad (1)$$

where  $u : (x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$ ,

$$1 < p, (N - 2)p < N + 2 \text{ and either } u_0 \geq 0 \text{ or } (3N - 4)p < 3N + 8. \quad (2)$$

More general vector-valued heat equations can be considered with similar results (see [MZ99] for more details) :

$$\begin{cases} u_t &= \Delta u + F(|u|)u \\ u(0) &= u_0, \end{cases} \quad (3)$$

where  $u : \Omega \times [0, T) \rightarrow \mathbb{R}^M$ ,  $p$  satisfies (2),  $\Omega = \mathbb{R}^N$  or  $\Omega$  is a smooth bounded convex domain of  $\mathbb{R}^N$ ,  $F(|u|) \sim |u|^{p-1}$  as  $|u| \rightarrow +\infty$ , and  $M \in \mathbb{N}$ .

We are interested in the blow-up phenomenon for (1). Many authors has been interested in this topic. Let us mention for instance Friedman [Fri65], Fujita [Fuj66], Ball [Bal77], Bricmont and Kupiainen [BKL94], Chen and Matano [CM89], Galaktionov and Vázquez [GV95], Giga and Kohn [GK89], [GK87], [GK85], Herrero and Velázquez [HV93], [HV92].

In the following, we consider  $u(t)$  a blow-up solution of (1) and denote its blow-up time by  $T$ . We aim at finding sharp uniform estimates at blow-up and specifying the blow-up behavior of  $u(t)$ . Such a study is done considering equation (1) in its self-similar form : for all  $a \in \mathbb{R}^N$ , we define

$$y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad w_{a,T}(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t). \quad (4)$$

Therefore,  $w_{a,T} = w$  satisfies  $\forall s \geq -\log T, \forall y \in \mathbb{R}^N$  :

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2}y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1}w. \quad (5)$$

Let us introduce the following Lyapunov functional associated with (5) :

$$E(w) = \int_{\mathbb{R}^N} \left( \frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho(y) dy$$

where  $\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}$ .

The question is to estimate  $w_{a,T}(s)$  as  $s \rightarrow +\infty$ , uniformly with respect to  $a$ , whether  $a$  is a blow-up point or not ( $a$  is called a blow-up point if there exists  $(a_n, t_n) \rightarrow (a, T)$  such that  $|u(a_n, t_n)| \rightarrow +\infty$ ).

Giga and Kohn showed that self-similar variables are convenient for describing the blow-up rate in the following sense : there exists  $\epsilon_0(u_0) > 0$  such that  $\forall s \geq s_0^*(u_0)$ ,

$$\epsilon_0 \leq |w(s)|_{L^\infty} \leq \frac{1}{\epsilon_0}. \quad (6)$$

We first aim at sharpening this result in order to obtain compactness properties in our problem.

## 1 A Liouville Theorem for equation (5)

We are interested in classifying all global and bounded solutions of (5), for all subcritical  $p$  :

$$p > 1 \text{ and } (N-2)p < N+2. \quad (7)$$

We claim the following :

**Theorem 1 (A Liouville Theorem for equation (5))** *Assume (7) and consider  $w$  a solution of (5) defined for all  $(y, s) \in \mathbb{R}^N \times \mathbb{R}$  such that  $\forall (y, s) \in \mathbb{R}^N \times \mathbb{R}, |w(y, s)| \leq C$ . Then, either  $w \equiv 0$ , or  $w \equiv \kappa$  or  $w(y, s) = \pm \varphi(s - s_0)$  where  $\kappa = (p-1)^{-\frac{1}{p-1}}$ ,  $s_0 \in \mathbb{R}$  and  $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$ .*

**Remark :**  $\varphi$  is in fact an  $L^\infty$  connection between two critical points of (5):  $\kappa$  and 0. Indeed,

$$\dot{\varphi} = -\frac{\varphi}{p-1} + \varphi^p, \quad \varphi(-\infty) = \kappa, \quad \varphi(+\infty) = 0.$$

**Remark :** A similar classification result can be obtained with a solution  $w$  defined only on  $(-\infty, s^*)$  (see [MZ98]).

Theorem 1 has the following corollary :

**Corollary 1 (A Liouville Theorem for equation (1))** *Assume (7) and consider  $u$  a solution of (1) defined for all  $(x, t) \in \mathbb{R}^N \times (-\infty, 0)$  such that  $\forall (x, t) \in \mathbb{R}^N \times (-\infty, 0)$ ,  $|u(x, t)| \leq C(-t)^{-\frac{1}{p-1}}$ . Then, either  $u \equiv 0$ , or  $u(x, t) = \pm \kappa(T^* - t)^{-\frac{1}{p-1}}$  for some  $T^* \geq 0$ .*

The proofs can be found in [MZ99] and [MZ98]. The key tools in the proof are the following :

- i) A classification of all possible linear behaviors of  $w(s)$  as  $s \rightarrow -\infty$  in  $L^2_\rho(\mathbb{R}^N)$  ( $L^\infty_{loc}(\mathbb{R}^N)$ ),
- ii) The following geometric transformations which keeps (5) invariant :

$$w(y, s) \rightarrow w_{a,b}(y, s) = w(y + ae^{\frac{s}{2}}, s + b),$$

where  $a \in \mathbb{R}^N$  and  $b \in \mathbb{R}$ ,

- iii) A blow-up criterion for (5) used for solutions close to the constant point  $\kappa$  (This criterion is also a blow-up criterion for (1) via the transformation (4)) :

*If for some  $s_0 \in \mathbb{R}$ ,  $I(w(s_0)) > 0$  where*

$$I(w) = -2E(w) + \frac{p-1}{p+1} \left( \int_{\mathbb{R}^N} |w(y)|^2 \rho(y) dy \right)^{\frac{p+1}{2}},$$

*then  $w(s)$  blows up in finite time.*

**Remark :** This criterion is sharp for solutions near constants. Indeed, if  $w(s_0) \equiv C_0$ , then

$$w \text{ blows-up in finite time} \Leftrightarrow |C_0| > \kappa \Leftrightarrow I(C_0) > 0.$$

**Remark :** The proof of the Liouville Theorem strongly relies on the existence of a Lyapunov functional for equation (5) and can not be extended to other systems where the nonlinearity is not a gradient. In [Zaa], we go beyond this restriction and introduce new tools to prove a Liouville Theorem of the same type for the following system

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q.$$

## 2 Localization at blow-up

We assume again (2). The estimate (6) of Giga and Kohn gives compactness in the problem. Using a compactness procedure in the singular zone of  $\mathbb{R}^N$  (which is, say  $\{y \mid |w(y, s)| \geq \frac{\kappa}{2}\}$ ), we find a solution satisfying the hypotheses of Theorem 1. Therefore,  $\Delta w$  is small with respect to  $|w|^p$  in this singular zone (or equivalently,  $\Delta u$  is small with respect to  $|u|^p$ ). A

subcritical localization procedure introduced by Zaag [Zaa98] (under the level of the constant  $\kappa$ ) allows us to propagate this estimate towards the intermediate zone between the singular and the regular one. We claim the following :

**Theorem 2 (Comparison with the associated ordinary differential equation)** *Assume (2) and consider  $T \leq T_0$  and  $\|u_0\|_{C^2(\mathbb{R}^N)} \leq C_0$ . Then,  $\forall \epsilon > 0$ , there is  $C(\epsilon, C_0, T_0)$  such that  $\forall (x, t) \in \mathbb{R}^N \times [0, T)$ ,*

$$|u_t - |u|^{p-1}u| \leq \epsilon |u|^p + C.$$

**Remark :** This way, we prove that the solution of the PDE (1) can be uniformly and globally in space-time compared to a solution of an ODE (localized by definition). Note that the condition  $u(0) \in C^2(\mathbb{R}^N)$  is not restrictive because of the regularizing effect of the Laplacian.

**Remark :** Many striking corollaries can be derived from this theorem. It implies in particular that no oscillation is possible near a blow-up point  $a$ , and that  $|u(x, t)| \rightarrow +\infty$  as  $(x, t) \rightarrow (a, T)$ . Moreover,  $\forall \epsilon_0 > 0$ , there exists  $t_0(\epsilon_0) < T$  such that for all  $b \in \mathbb{R}^N$ , if  $|u(b, t)| \leq (1 - \epsilon_0)\kappa(T - t)^{-\frac{1}{p-1}}$  for some  $t \in [t_0, T)$ , then  $b$  is not a blow-up point (this specifies more precisely a former result by Giga and Kohn where  $t_0 = t_0(\epsilon_0, a)$ ).

### 3 Optimal $L^\infty$ estimates at blow-up

We still assume (2). Using estimate (6) and the Liouville Theorem, we make a compactness argument to get the following sharp estimates :

**Theorem 3 ( $L^\infty$  refined estimates for  $w(s)$ )** *Assume that (2) holds. Then, there exist positive constants  $C_i$  for  $i = 1, 2, 3$  such that if  $u$  is a solution of (1) which blows-up at time  $T$  and satisfies  $u(0) \in C^3(\mathbb{R}^N)$ , then  $\forall \epsilon > 0$ , there exists  $s_1(\epsilon) \geq -\log T$  such that  $\forall s \geq s_1, \forall a \in \mathbb{R}^N$ ,*

$$\begin{aligned} \|w_{a,T}(s)\|_{L^\infty} &\leq \kappa + \left(\frac{N\kappa}{2p} + \epsilon\right)\frac{1}{s}, & \|\nabla w_{a,T}(s)\|_{L^\infty} &\leq \frac{C_1}{\sqrt{s}}, \\ \|\nabla^2 w_{a,T}(s)\|_{L^\infty} &\leq \frac{C_2}{s}, & \|\nabla^3 w_{a,T}(s)\|_{L^\infty} &\leq \frac{C_3}{s^{3/2}}. \end{aligned}$$

**Remark :** In the case  $N = 1$ , Herrero and Velázquez (Filippas and Kohn also) proved some related estimates, using a Sturm property introduced in particular by Chen et Matano (the number of space oscillations is a decreasing function of time).

**Remark :** The constant  $\frac{N\kappa}{2p}$  is optimal (see Herrero and Velázquez, Brimont and Kupiainen, Merle and Zaag).

## 4 Different notions of blow-up profiles and the stability question

We assume (2). We consider  $a \in \mathbb{R}^N$ , a blow-up point of  $u(t)$ , solution of (1). From translation invariance, we can assume  $a = 0$ . We would like to know whether  $u(t)$  (or  $w_{0,T}(s)$  defined in (4)) has a universal behavior or not, as  $t \rightarrow T$  (or  $s \rightarrow +\infty$ ).

Filippas, Kohn, Liu, Herrero et Velázquez prove that  $w$  behaves in two distinct ways :

- either  $\forall R > 0$ ,  $\sup_{|y| \leq R} \left| w(Qy, s) - \left[ \kappa + \frac{\kappa}{2ps} \left( l - \frac{1}{2} \sum_{k=1}^l y_k^2 \right) \right] \right| = O\left(\frac{1}{s^{1+\delta}}\right)$  as  $s \rightarrow +\infty$ , for some  $\delta > 0$  where  $l \in \{1, \dots, N\}$ ,  $Q$  is a  $N \times N$  orthogonal matrix and  $I_l$  is the  $l \times l$  identity matrix.
- or  $\forall R > 0$ ,  $\sup_{|y| \leq R} |w(y, s) - \kappa| \leq C(R)e^{-\lambda_0 s}$  for some  $\lambda_0 > 0$ .

From a physical point of view, these results do not tell us much about the transition between the singular zone ( $w \geq \alpha$  where  $\alpha > 0$ ) and the regular one ( $w \simeq 0$ ). In [MZ99], we specify this transition by proving the existence of a profile in the variable  $z = \frac{y}{\sqrt{s}}$ .

### Theorem 4 (Existence of a blow-up profile for equation (1))

*Assume (2) holds. There exists  $l \in \{0, 1, \dots, N\}$  and a  $N \times N$  orthogonal matrix  $Q$  such that  $w(Q(z)\sqrt{s}, s) \rightarrow f_l(z)$  uniformly on compact sets  $|z| \leq C$ ,*

*where  $f_l(z) = (p-1 + \frac{(p-1)^2}{4p} \sum_{i=1}^l |z_i|^2)^{-\frac{1}{p-1}}$  if  $l \geq 1$  and  $f_0(z) = \kappa = (p-1)^{-\frac{1}{p-1}}$ .*

This result has been proved by Velázquez in [Vel92]. However, the convergence speed depends on the considered blow-up point in [Vel92], whereas they are uniform in [MZ99]. This uniformity allows us to derive the stability of the profile  $f_N$  in [FKMZ].

Using renormalization theory, Bricmont and Kupiainen prove in [BK94] the existence of a solution of (5) such that

$$\forall s \geq s_0, \forall y \in \mathbb{R}^N, |w(y, s) - f_N\left(\frac{y}{\sqrt{s}}\right)| \leq \frac{C}{\sqrt{s}}.$$

Merle and Zaag prove the same result in [MZ97], thanks to a technique of finite-dimension reduction. They also prove the stability of such a behavior with respect to initial data, in a neighborhood of the constructed solution.

In [Zaa98] and [Vel92], it is proved that in this case,  $u(x, t) \rightarrow u^*(x)$  as  $t \rightarrow T$  uniformly on  $\mathbb{R}^N \setminus \{0\}$  and that  $u^*(x) \sim \left[ \frac{8p|\log|x||}{(p-1)^2|x|^2} \right]^{\frac{1}{p-1}}$  as  $x \rightarrow 0$ .

One interesting problem is to relate all known blow-up profiles' notions : profiles for  $|y|$  bounded,  $\frac{|y|}{\sqrt{s}}$  bounded or  $x \simeq 0$ . We prove in the following that all these descriptions are equivalent, in the case of single point blow-up with a non degenerate profile (generic case). This answers many questions which arose in former works.

**Theorem 5 (Equivalence of blow-up behaviors at a blow-up point)**

Assume (2) and consider  $a$  be an isolated blow-up point of  $u(t)$  solution of (1). The following behaviors of  $u(t)$  and  $w_{a,T}(s)$  (defined in (4)) are equivalent :

- i)  $\forall R > 0$ ,  $\sup_{|y| \leq R} \left| w(y, s) - \left[ \kappa + \frac{\kappa}{2ps} \left( N - \frac{1}{2}|y|^2 \right) \right] \right| = o\left(\frac{1}{s}\right)$  as  $s \rightarrow +\infty$ ,
- ii)  $\forall R > 0$ ,  $\sup_{|z| \leq R} |w(z\sqrt{s}, s) - f_0(z)| \rightarrow 0$  as  $s \rightarrow +\infty$  with  $f_0(z) = (p-1 + \frac{(p-1)^2}{4p}|z|^2)^{-\frac{1}{p-1}}$ ,
- iii)  $\exists \epsilon_0 > 0$  such that for all  $|x-a| \leq \epsilon_0$ ,  $u(x, t) \rightarrow u^*(x)$  as  $t \rightarrow T$  and  $u^*(x) \sim \left[ \frac{8p|\log|x-a||}{(p-1)^2|x-a|^2} \right]^{\frac{1}{p-1}}$  as  $x \rightarrow a$ .

A further application of the Liouville Theorem is the stability of the behavior described in Theorem 5, with respect to perturbations in initial data. Using a dynamical system approach, we prove in [FKMZ], with Fermanian the following :

**Theorem 6 (Stability of the blow-up profile)** Assume (7) and consider  $\tilde{u}(t)$  a blow-up solution of (1) with initial data  $\tilde{u}_0$  which blows-up at  $t = \tilde{T}$  at only one point  $\tilde{a} = 0$  and satisfies (6). Assume that

$$\text{for all } |x| \geq R \text{ and } t \in [0, \tilde{T}), \quad |\tilde{u}(x, t)| \leq M.$$

and that the function  $\tilde{w}_{0, \tilde{T}}(y, s)$  defined in (4) satisfies uniformly on compact sets of  $\mathbb{R}^N$

$$\tilde{w}_{0, \tilde{T}}(y, s) - \kappa \underset{s \rightarrow +\infty}{\sim} \frac{\kappa}{2ps} \left( N - \frac{|y|^2}{2} \right). \quad (8)$$

Then, there is a neighborhood  $\mathcal{V}$  in  $L^\infty$  of  $\tilde{u}_0$  such that for all  $u_0 \in \mathcal{V}$  the solution of (1) with initial data  $u_0$  blows-up at time  $T = T(u_0)$  at a unique point  $a = a(u_0)$  and the function  $w_{a,T}(y, s)$  defined in (4) satisfies uniformly on compact sets of  $\mathbb{R}^N$

$$w_{a,T}(y, s) - \kappa \underset{s \rightarrow +\infty}{\sim} \frac{\kappa}{2ps} \left( N - \frac{|y|^2}{2} \right).$$

Moreover,  $(a(u_0), T(u_0))$  goes to  $(0, \tilde{T})$  as  $u_0$  goes to  $\tilde{u}_0$ .

**Remark :** This results generalizes the stability result of [MZ97]. Note that unlike most applications of the Liouville Theorem, this result is valid for all subcritical  $p$ . In [FKZ], the same result is proved (only under the condition (2)), by a completely different approach based on the Liouville Theorem and on [MZ97].

**Remark :** In [FKMZ], we prove the stability with respect to initial data of the blow-up behavior with the minimal speed

$$\|u(t)\|_{L^\infty} \leq C(T - t)^{-\frac{1}{p-1}} \quad (9)$$

for all subcritical  $p$  (that is under the condition (7)). Note that this result is obvious under the weaker assumption (2), for Giga and Kohn proved in [GK87] that all blow-up solutions satisfy (9). No blow-up rate estimate is known if

$$u_0 \text{ has no sign and } (3N - 4)p \geq 3N - 8.$$

Therefore, our result is meaningful in this last case.

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