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Dramatic acceleration of wave condensation mediated by disorder in multimode fibers

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Classical nonlinear waves exhibit a phenomenon of condensation that results from the natural irreversible process of thermalization, in analogy with the quantum Bose-Einstein condensation. Wave condensation originates in the divergence of the thermodynamic equilibrium Rayleigh-Jeans distribution, which is responsible for the macroscopic population of the fundamental mode of the system. However, achieving complete thermalization and condensation of incoherent waves through nonlinear optical propagation is known to require prohibitive large interaction lengths. Here, we derive a discrete kinetic equation describing the nonequilibrium evolution of the random wave in the presence of a structural disorder of the medium. Our theory reveals that a weak disorder accelerates the rate of thermalization and condensation by several order of magnitudes. Such a counterintuitive dramatic acceleration of condensation can provide an explanation for the recently discovered phenomenon of optical beam self-cleaning. We report experiments in multimode optical fibers that highlight the transition from an incoherent thermal distribution to wave condensation, with a condensate fraction of up to 60% in the fundamental mode of the waveguide trapping potential.

Introduction.— The observation of the phenomenon of Bose-Einstein condensation has been reported in a variety of genuine quantum systems, such as ultracold atoms and molecules [1], exciton polaritons [2] and photons [3]. On the other hand, recent studies on wave turbulence revealed that a purely classical system of random waves can exhibit a process of condensation with thermodynamic properties analogous to those of Bose-Einstein condensation [4–14]. Classical wave condensation finds its origin in the natural thermalization of the wave system toward the thermodynamic Rayleigh-Jeans equilibrium distribution, whose divergence is responsible for the macroscopic occupation of the fundamental mode of the system [4–7, 15–19]. This self-organization process takes place in a formally reversible system: The formation of the large scale coherent structure (‘condensate’) remains immersed in a sea of small-scale fluctuations (‘uncondensed particles’), which store the information for time reversal.

There is a current surge of interest in studying quantum properties of fluids with light waves, such as superfluidity and the generation of Bogoliubov sound waves [2, 20, 21]. Along this way, different forms of condensation processes have been reported in optical cavity systems, which are inherently nonequilibrium forced-dissipative systems [18, 22–24]. On the other hand, the irreversible process of condensation is predicted for purely *conservative and formally reversible* (Hamiltonian) systems of random waves. Unfortunately, however, the experimental study of condensation in a conservative (cavity-less) configuration constitutes a major challenge, because of the prohibitive large propagation lengths required to achieve thermalization [14, 17]. In marked contrast with this commonly accepted opinion, an astonishing phenomenon of beam self-cleaning has been recently discovered in multimode optical fibers (MMFs) [25–27].

This phenomenon is due to a purely conservative Kerr nonlinearity [27] and, so far, its underlying mechanism remains unexplained.

As a matter of fact, light propagation in MMFs is known to be affected by a structural disorder of the material due to inherent imperfections and external perturbations [28], a feature of interest, e.g. in image formation [29] or to study integrable nonlinear Manakov systems [30–33]. The remarkable result of our work is to show that a (‘time’-dependent) structural disorder is responsible for a dramatic acceleration of the process of wave condensation. On the basis of the wave turbulence theory [4, 5, 15], we develop a nonequilibrium kinetic formulation of the random waves that accounts for the impact of disorder. The theory reveals that a *conservative* disorder introduces an effective dissipation in the system, which is shown to deeply modify the regularization of resonant wave interactions. We derive a *discrete* kinetic equation revealing that a weak disorder accelerates the rate of thermalization and condensation by several order of magnitudes. Note that at variance with the notion of prethermalization to out of equilibrium states [14, 34, 35], here the system achieves a fast relaxation to a fully thermalized equilibrium state. The counterintuitive mechanism of condensation acceleration can provide a natural explanation for the effect of optical beam self-cleaning. We report experiments in MMFs that highlight the transition to condensation with clear evidence of the macroscopic population of the fundamental mode of the fiber.

The present work contributes to the challenging question of spontaneous organization of coherent states in nonlinear disordered (turbulent) systems [36–43]. Furthermore, MMFs are attracting for telecommunication applications [28] and novel fiber laser sources [44].

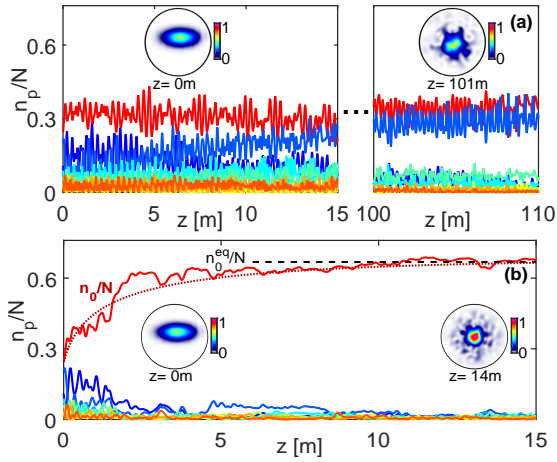


FIG. 1: **Disorder-induced condensation:** (a) Numerical simulations of the modal NLS Eq.(1) in the absence of disorder (a), and in the presence of disorder (b) [NLS Eq.(2)], starting from the same *initial coherent condition*. Evolutions of the modal populations n_p/N : fundamental mode $p = 0$ (red), $p = 1$ (dark blue), $p = 2$ (blue), $p = 3$ (light blue), $p = 4$ (cyan), $p = 5$ (light green), $p = 6$ (green), $p = 7$ (yellow), $p = 8$ (orange). Disorder breaks the coherent modal interaction (a) and induces condensation in the fundamental mode n_0 (b), which relaxes to the theoretical value $n_0^{eq}/N = 0.67$ at thermal equilibrium (dashed black). The dotted red line in (b) reports $n_0(z)$ from the simulation of the kinetic Eq.(3). The insets show the corresponding intensity patterns $|\psi|^2(\mathbf{r}, z)$ ($N_* = 120$ modes without including polarization degeneracy).

Nonlinear Schrödinger model.- We consider the standard scalar (2+1)D nonlinear Schrödinger (NLS) equation, which is known to describe the propagation (along z) of a polarized optical beam in a waveguide modelled by a confining potential $V(\mathbf{r})$ (with $\mathbf{r} = (x, y)$) [45]. The potential $V(\mathbf{r})$ is parabolic-shaped, which models graded-index MMFs [18, 25–28], or trapped Bose-Einstein condensates [1]. We expand the random wave into the basis of the linear eigenmodes $u_p(\mathbf{r})$, $\psi(\mathbf{r}, z) = \sum_p A_p(z) u_p(\mathbf{r}) \exp(-i\beta_p z)$, β_p being the eigenvalues of the N_* modes of the fiber. The modal components $A_p(z)$ of the NLS equation verify

$$i\partial_z A_p = \beta_p A_p - \gamma F_p(A), \quad (1)$$

where the modal expansion of the cubic Kerr nonlinearity reads $F_p(A) = \sum_{q,l,m} S_{pqlm} A_q A_l A_m^*$ and the tensor S_{pqlm} accounts for the spatial overlap among the eigenmodes [18, 45].

We report in Fig. 1(a) a typical evolution of the modal components $A_p(z)$ and corresponding intensity pattern $|\psi|^2(\mathbf{r}, z)$, obtained by solving the NLS Eq.(1) with typical experimental parameters [27], i.e., a MMF with $N_* = 120$ modes (core radius $R = 26\mu\text{m}$, $n_2 = 3.2 \times 10^{-20} \text{m}^2/\text{W}$) and injected power $N = 47.5\text{kW}$. At variance with usual simulations of wave turbulence

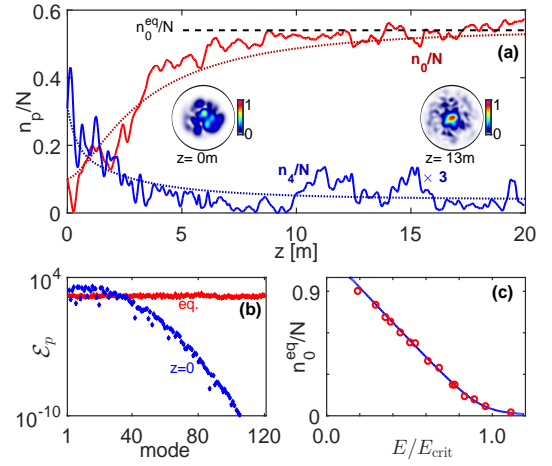


FIG. 2: **Condensation to equilibrium.** (a) Simulation of the modal NLS Eq.(2) showing the evolution of $n_p(z)$ for the fundamental mode $p = 0$ (red), and $p = 4$ (blue), and corresponding simulation of the kinetic Eq.(3) (dotted red and blue lines) starting from the same *initial incoherent condition*: $n_0(z)$ reaches the theoretical equilibrium value $n_0^{eq}/N \simeq 0.57$ (dashed black). (b) Corresponding energy per mode $\mathcal{E}_p(z) = (\beta_p - \beta_0)n_p(z)$ at $z = 0$ (blue dots), and at $z = 20\text{m}$ (red dots) showing that the beam reaches thermal equilibrium with *energy equipartition* $\mathcal{E}_p^{eq} = T$. (c) Condensate fraction at equilibrium n_0^{eq}/N vs energy E (keeping constant the power N): For $E \leq E_{\text{crit}}$ the system undergoes a phase transition to condensation. The NLS simulations of Eq.(2) (red circles) are in quantitative agreement with the theory (blue line) without adjustable parameters.

[5, 13, 16, 18], we did not impose a random phase among the initial modes $A_p(z = 0)$, which is consistent with the experimental conditions where a laser beam featured by a coherent transverse phase front is launched into the optical fiber. The simulations of Eq.(1) show that a strong phase-correlation among the modes is preserved during the propagation, thus leading to a phase-sensitive *coherent regime* of modal interaction. The modes $A_p(z)$ then experience a quasi-reversible exchange of power with each other (Fig. 1(a)), which leads to an oscillatory dynamics of the intensity pattern $|\psi|^2(\mathbf{r}, z)$ (see the movie in [46]). Such a multimode beam does not exhibit an enhanced brightness that characterizes a stable self-cleaning effect. This coherent regime of mode interactions then freezes the thermalization process, a feature of growing interest that is analyzed in the framework of finite size effects in discrete or mesoscopic wave turbulence [47–51].

Kinetic equation with weak disorder.- Light propagation in MMFs is known to be affected by disordered random fluctuations. Such a disorder breaks the coherent modal regime discussed in Fig. 1(a) and leads to a turbulent incoherent regime with uncorrelated random phases fluctuations of the modes [46]. We stress that the acceleration of thermalization predicted by our theory is *not*

simply due to a breaking of the coherent regime by disorder, but solely results from the interplay of disorder and the incoherent modal interaction. Note that our theory goes beyond the mean-field approximation reported in [27], which is a formally *reversible* theory that does not explain the irreversible process of beam self-cleaning.

We first consider the dominant contribution of a *weak disorder* that originates in polarization random coupling [28, 32, 33]. Accordingly, the vector modal components in the linear polarization basis $\mathbf{A}_p^T(z) = (A_{p,x}(z), A_{p,y}(z))^T$ are governed by a straightforward generalization of the scalar NLS Eq.(1):

$$i\partial_z \mathbf{A}_p = \beta_p \mathbf{A}_p + \mathbf{D}_p(z) \mathbf{A}_p - \gamma \mathbf{F}_p(\mathbf{A}), \quad (2)$$

where $\mathbf{F}_p(\mathbf{A}) = \sum_{q,l,m} S_{pqlm} (\frac{1}{3} \mathbf{A}_q^T \mathbf{A}_l \mathbf{A}_m^* + \frac{2}{3} \mathbf{A}_m^\dagger \mathbf{A}_l \mathbf{A}_q)$ and the superscript T (\dagger) denotes the transpose (conjugate transpose) operation. Note that each mode \mathbf{A}_p can be represented as a Stokes vector on the Poincaré sphere, so that the NLS Eq.(2) can model a system of N_* -coupled spins, in relation with spinor Bose-Einstein condensates. Our model of disorder is general because the random matrices $\mathbf{D}_p(z) = \sum_{j=1}^3 \nu_{p,j}(z) \boldsymbol{\sigma}_j$ are expanded on the Pauli spin matrices $\boldsymbol{\sigma}_j$, which constitute a complete basis of 2×2 Hermitian matrices. The functions $\nu_{p,j}(z)$ are zero mean identically distributed random real-valued processes, with variance σ_β^2 and correlation length l_β [46]. The effective strength of disorder is controlled by the parameter $\Delta\beta = \sigma_\beta^2 l_\beta$. This model of disorder is *conservative*, in the sense that the NLS Eq.(2) conserves the power $N = \sum_p |\mathbf{A}_p|^2(z)$ and the linear contribution to the energy $E = \sum_p \beta_p |\mathbf{A}_p|^2(z)$, which dominates the nonlinear contribution. Indeed, in the experiments of beam self-cleaning the system evolves in the weakly nonlinear regime where linear dispersion (diffraction) effects dominate nonlinear effects $L_{lin} = \beta_0^{-1} \ll L_{nl}$, and where disorder is a perturbation with respect to linear propagation $L_{lin} \ll L_{disor} = 1/\Delta\beta$ ($\Delta\beta \ll \beta_0$). Notice that since disorder is ('time') z -dependent, our system is of different nature than those studying the interplay of thermalization and Anderson localization [41, 52].

Random mode coupling in MMFs has been widely studied in recent years. In the limit of rapid disordered fluctuations, the generalized NLS equation has been remarkably reduced to the integrable Manakov equation [30–33]. However, the Manakov equation does not describe power exchange among the modes. To describe a process of beam self-cleaning, here we go beyond the Manakov limit. We have derived the following kinetic equation governing the evolution of the averaged modal components $n_p(z) = \langle |\mathbf{A}_p(z)|^2 \rangle$ [46]:

$$\begin{aligned} \partial_z n_p(z) = & \frac{2\gamma^2}{9\Delta\beta} \sum_{q,l,m} \delta_{\beta_q+\beta_l-\beta_m-\beta_p} |S_{pqlm}|^2 M_{pqlm}(\mathbf{n}) \\ & + \frac{16\gamma^2}{27\Delta\beta} \sum_q \delta_{\beta_q-\beta_p}^K |s_{pq}(\mathbf{n})|^2 (n_q - n_p), \quad (3) \end{aligned}$$

with $s_{pq}(\mathbf{n}) = \sum_{m'} S_{pqm'm'} n_{m'}$, and $M_{pqlm}(\mathbf{n}) = n_q n_l n_p + n_q n_l n_m - n_m n_p n_l - n_m n_p n_q$, where ' n_m ' stands for ' $n_m(z)$ ', while the Kronecker symbol reflects energy conservation of wave resonances. **The discrete nature of the kinetic equation originates in finite size effects related to the relative small number of modes of the trapping potential (~ 100 modes with $\beta_0 \gg 1/L_{nl}$) [46].**

Acceleration of thermalization.— Our derivation of the kinetic Eq.(3) differs from the conventional wave turbulence approach [5] in that it accounts for the presence of disorder. The theory reveals that disorder deeply affects the evolution of the moments equations. Specifically, the dynamics of a fourth-order moment of the random wave is governed by an effective forced-damped oscillator equation, in which dissipation originates from the phase-mixing dynamics induced by the conservative disorder. It turns out that the singularity associated to a resonance is regularized by the dissipation due to disorder: *The lower the magnitude of disorder $\Delta\beta$, the stronger the efficiency of the wave resonance.* Clearly, the amount of disorder $\Delta\beta$ cannot decrease arbitrarily since $\Delta\beta L_{nl} \gtrsim 1$ – in the opposite regime $\Delta\beta \ll 1/L_{nl}$ the regularization due to disorder is negligible and the kinetic equation recovers the standard continuous form [5, 18]. More precisely, the characteristic lengths ('times') of thermalization (ζ_{th}) in the presence and the absence of disorder scale as

$$\zeta_{th}^{disor} / \zeta_{th}^{ord} \sim \Delta\beta / \beta_0. \quad (4)$$

This shows that thermalization is significantly accelerated by the perturbative disorder $\Delta\beta / \beta_0 \ll 1$. Considering typical experimental parameters [27] used in the simulations of Figs. 1-2, $\beta_0 \simeq 5 \times 10^3 \text{m}^{-1}$, $L_{disor} \simeq 0.4 \text{m}$ (beat length $2\pi/\sigma_\beta \simeq 2.14 \text{m}$, $l_\beta \simeq 30 \text{cm}$), one obtains $\Delta\beta / \beta_0 \lesssim 5 \times 10^{-4}$: The rate of thermalization is increased by several orders of magnitude by the disorder. Although the experimental parameters of disorder are not precisely known, the effect of disorder induces beam cleaning shown in Figs. 1-2 is robust and has been observed over a *wide range of parameters* (e.g., $2\pi/\sigma_\beta$ and l_β of several meters) [46].

The kinetic Eq.(3) has the same structure as the conventional wave turbulence equation, hence it describes wave condensation [5, 7] (irrespective of the sign of the nonlinearity γ [18]). It conserves the 'number of particles' N , the energy E and exhibits a H -theorem of entropy growth for the nonequilibrium entropy $\mathcal{S}(z) = \sum_p n_p(z) \log(n_p(z))$, so that it describes an irreversible evolution to the Rayleigh-Jeans equilibrium distribution realizing the maximum of entropy $n_p^{eq} = T/(\beta_p - \mu)$ (the finite number of modes N_* regularizing the ultraviolet catastrophe of classical waves [18]). **Proceeding as in Refs.[5, 7, 9, 18], the system exhibits a phase transition to condensation: For $E \leq E_{crit} \simeq N\beta_0 \sqrt{N_*/2}$, $\mu \rightarrow \beta_0$ and the condensate amplitude increases as the energy decreases, $n_0/N \simeq 1 - (E - E_0)/(E_{crit} - E_0)$, where $E_0 = N\beta_0$ is the lower energy of the system. The**

fundamental mode then gets macroscopically populated $n_0 \gg n_p$, while the higher-order modes exhibit energy equipartition $\mathcal{E}_p^{eq} = (\beta_p - \beta_0)n_p^{eq} = T$. Note that E_{crit} only depends on the geometry of the waveguide potential. These theoretical predictions have been confirmed by the simulations of the NLS Eq.(2) and kinetic Eq.(3): A quantitative agreement has been obtained without adjustable parameters even beyond the validity regime of the kinetic equation, see Fig. 2 and [46] – we have checked that even the low-order modes evolve in the weakly nonlinear regime through a scale by scale analysis [53]. As a consequence of the macroscopic population of the fundamental mode $u_0(\mathbf{r})$, the intensity pattern of the random wave $|\psi|^2(\mathbf{r})$ exhibits a *stable self-cleaned shape* (see the movie in [46]).

Impact of strong disorder. – For relatively large propagation lengths, a strong coupling among different modes can no longer be neglected [28, 30–33]. We have extended the above theory by considering a general form of the $N_* \times N_*$ matrix $\mathbf{D}(z)$ modelling random mode coupling. The theory shows that strong disorder introduces an additional term in the kinetic Eq.(3):

$$\partial_z n_p = \Delta\beta_{sd} \sum_q \Gamma_{pq} \hat{\mathcal{R}}((\beta_p - \beta_q)l_\beta)(n_q(z) - n_p(z)), \quad (5)$$

where the matrix Γ_{pq} describes mode coupling and $\Delta\beta_{sd}$ the corresponding amount of strong disorder [46]. Note that Eq.(5) has a form similar to power coupling models [28]. In principle, the extra term (5) breaks the conservation of energy E , so that a H –theorem of the complete kinetic Eq.(3) and (5) would describe an irreversible thermalization toward an equilibrium state of power equipartition *among all the modes*. Strong disorder would then deteriorate the condensation process. However, the key observation is that mode coupling among non-degenerate modes is quenched by the Fourier transform of the correlation function of the fluctuations $\hat{\mathcal{R}}(\beta_0 l_\beta) \simeq 0$ [28], because $\beta_0 l_\beta \gg 1$ in the experiments of beam self-cleaning. Mode coupling is then restricted to degenerate modes ($\hat{\mathcal{R}}(0) = 1$), and Eq.(5) describes an exponential relaxation to an *equipartition of power within groups of degenerate modes*. Interestingly, this is a property of the Rayleigh-Jeans distribution (n_p^{eq} only depends on β_p). This reveals that strong disorder does not deteriorate condensation, but instead *enforces the thermalization* to the Rayleigh-Jeans equilibrium distribution. However, given the short fiber lengths considered in beam cleaning experiments (~ 10 – 20 m) [27], such an acceleration of thermalization is negligible with respect to the dramatic acceleration due to weak disorder, see Eq.(4).

Experiments. – We performed experiments in a MMF to evidence the transition to light condensation by varying the coherence of the input beam. The energy E provides a measure of the ‘coherence’ of the beam in the sense that E increases as the beam populates higher order modes: By increasing the coherence (keeping constant

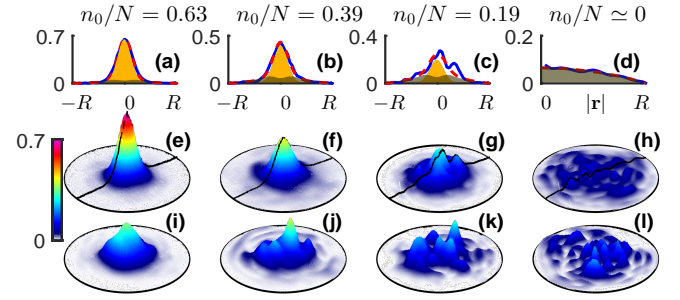


FIG. 3: **Experiments.** Intensity patterns recorded at $Z_0 = 20$ cm (bottom line), and at the output of the whole fiber length $L = 13$ m (middle line): By reducing the number of modes initially excited (i.e., by decreasing E), a transition occurs from $n_0 = 0$ to a condensate fraction of $n_0/N \simeq 0.63$. (a-d) The fundamental Gaussian mode $u_0(\mathbf{r})$ of the MMF gets macroscopically populated as evidenced by the intensity line-outs $I(x, y = 0)$ recorded experimentally at $z = L$ (blue line), and the corresponding fits of the condensate contribution $I_{\text{cond}}(\mathbf{r})$ (orange filled region), and of the incoherent contribution $I_{\text{inc}}(\mathbf{r})$ (grey filled region). The red-dashed line denotes $I_{\text{cond}}(\mathbf{r}) + I_{\text{inc}}(\mathbf{r})$. Above the transition to condensation $n_0 = 0$ (d), we computed the angle averaged distance $|\mathbf{r}|$ of the intensity recorded experimentally (blue), which is in agreement with the Rayleigh-Jeans distribution $I_{\text{inc}}(\mathbf{r})$ (red-dashed). $R = 26\mu\text{m}$ is the fiber core radius (corresponding to the circles (e-l)), $N_* = 120$ modes, $N = 19\text{kW}$.

the power N), E decreases and n_0 increases according to the condensation curve in Fig. 2(c). The sub-nanosecond pulses delivered by a Nd:YAG laser ($\lambda = 1.06\mu\text{m}$) are passed through a diffuser to generate a beam with different properties of coherence. The beam is subsequently injected into a graded-index MMF, and the near field intensity is recorded at the fiber output with a (CMOS) camera. The specific fiber launch conditions are known to significantly alter the number of modes excited at the fiber input. Hence, it is important to compare with the same launch conditions and the same power, the intensity pattern after a small propagation length (Z_0) representing the ‘input’ (initial) field, with the corresponding intensity pattern after propagation through the whole fiber length L . To this aim, the field intensity has been recorded at the output of the fiber length $L = 13$ m, which has been subsequently cut to $Z_0 = 20$ cm to record the corresponding ‘input’ field.

The input and output intensity patterns are reported in Fig. 3 for a power fixed to $N = 19\text{kW}$ (at the fiber output). The incoherent beam evolves in the weakly nonlinear regime, $L_{\text{lin}}(\simeq 0.2\text{mm}) \ll L_{\text{nl}}(\simeq 1\text{m})$. When a large number of modes are excited by the ‘initial’ beam (i.e., $E > E_{\text{crit}}$), the intensity distribution tends to relax to the equilibrium Rayleigh-Jeans distribution $I_{\text{inc}}(\mathbf{r}) = \sum_p n_p^{eq} |u_p(\mathbf{r})|^2$, see Fig. 3(d). By reducing the excitation of modes (i.e., $E < E_{\text{crit}}$), the power gradually condenses into the fundamental Gaussian mode

$u_0(\mathbf{r})$ of the parabolic potential $V(\mathbf{r})$, see the orange-filled region in Fig. 3(a-c). This allowed us to compute with accuracy the condensate contribution in the fundamental mode $I_{cond}(\mathbf{r}) = n_0 u_0^2(\mathbf{r})$ (orange region), and the incoherent contribution from all other modes $I_{inc}(\mathbf{r}) = \sum_{p \neq 0} n_p^{eq} |u_p(\mathbf{r})|^2$ (grey region). We observe a transition from a vanishing n_0 to a condensate fraction of up to $\simeq 60\%$ as the coherence of the ‘input’ beam is increased – these measurements being only weakly affected by the Raman effect [46].

Conclusion. – We have shown that the previously unrecognized process of disorder-induced acceleration of condensation can explain the phenomenon of optical beam self-cleaning. Our experiments report the first observation of light condensation into the fundamental mode of a (cavity-less) waveguide trapping potential. The mechanism of acceleration of condensation is general in the sense that the scaling of Eq.(4) does not depend on the details of the system, such as the specific forms of the considered model of disorder and of the nonlinearity [46]. The theory and the experiment can be extended to study turbulence cascades [5, 6], or spatio-temporal effects [25, 26, 54–57]. At variance with recent observations of superfluid light flows [20, 21], our experiment of light condensation could demonstrate a key manifestation of superfluidity, namely the nucleation of vortices induced by a rotating confining potential (along the ‘time’ z -variable) in manufactured fibers, in complete analogy with rotating trapped Bose-Einstein condensates [1].

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