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CLASSICAL AND QUANTUM RESONANCES FOR HYPERBOLIC SURFACES

COLIN GUILLARMOU, JOACHIM HILGERT, AND TOBIAS WEICH

ABSTRACT. For compact and for convex co-compact oriented hyperbolic surfaces, we prove an explicit correspondence between classical Ruelle resonant states and quantum resonant states, except at negative integers where the correspondence involves holomorphic sections of line bundles.

1. INTRODUCTION

It is a classical result that on compact surfaces with constant negative curvature, Selberg's trace formula allows to identify the eigenvalues of the Laplace operator Δ and certain zeros of the Selberg zeta function [Se], which can be entirely expressed in terms of closed geodesics. Later the same result has been established for convex-co-compact hyperbolic surfaces by Patterson-Perry [PaPe] (see also [BJP, GuZw3]), where the correspondence is between certain zeros of the Selberg zeta function and the resonances of the Laplacian (recall that convex co-compact hyperbolic surfaces are complete non-compact smooth Riemannian surfaces of constant negative curvature with infinite volume). Both results show that on hyperbolic surfaces there is a deep connection between the spectral properties of the Laplacian (quantum mechanics) and the properties of the geodesic flow (classical mechanics). However, the above results do not establish a link between the spectra of the Laplacian and a transfer operator associated to the geodesic flow, nor do they establish a relation between the associated resonant states. The aim of this article is to prove such an explicit correspondence. The previously known relation to the zeta zeros is a direct consequence of this correspondence.

Let us now introduce the concept of Ruelle resonances for the transfer operator associated to the geodesic flow on M . Let M be a compact or convex co-compact hyperbolic surface and let X be the vector field generating the geodesic flow φ_t on the unit tangent bundle SM of M . The linear operator

$$\mathcal{L}_t : \begin{cases} C_c^\infty(SM) & \rightarrow C_c^\infty(SM) \\ f & \mapsto f \circ \varphi_{-t} \end{cases}$$

is called the *transfer operator* of the geodesic flow and the vector field $-X$ is its generator. The geodesic flow has the Anosov property, i.e. the tangent bundle of SM

splits into a direct sum

$$T(SM) = \mathbb{R}X \oplus E_s \oplus E_u$$

where $d\varphi_t$ is exponentially contracting in forward time (resp. backward time) on E_s (resp. on E_u), and this decomposition is φ_t -invariant. The bundles E_u and E_s are smooth and there are two smooth non-vanishing vector fields U_\pm on SM so that $E_s = \mathbb{R}U_+$, $E_u = \mathbb{R}U_-$, and $[X, U_\pm] = \pm U_\pm$. The fields U_\pm generate the horocyclic flows.

For $f_1, f_2 \in C_c^\infty(SM)$ we can define the correlation functions

$$C_X(t; f_1, f_2) := \int_{SM} \mathcal{L}_t f_1 \cdot f_2 d\mu_L$$

where μ_L is the Liouville measure (invariant by φ_t). By [BuLi, FaSj, DyZw] in the compact case and [DyGu] in the convex co-compact case, the Laplace transform

$$R_X(\lambda; f_1, f_2) := - \int_0^\infty e^{-\lambda t} C_X(t; f_1, f_2) dt$$

extends meromorphically from $\operatorname{Re}(\lambda) > 0$ to \mathbb{C}^1 . Notice that for $\operatorname{Re}(\lambda) > 0$ we have $R_X(\lambda; f_1, f_2) = \langle (-X - \lambda)^{-1} f_1, f_2 \rangle$ and $R_X(\lambda)$ gives a meromorphic extension of the Schwartz kernel of the resolvent of $-X$. The poles are called *Ruelle resonances* and the residue operator $\Pi_{\lambda_0}^X : C_c^\infty(SM) \rightarrow \mathcal{D}'(SM)$ defined by

$$\langle \Pi_{\lambda_0}^X f_1, f_2 \rangle := \operatorname{Res}_{\lambda_0} R_X(\lambda; f_1, f_2), \quad \forall f_1, f_2 \in C_c^\infty(SM)$$

has finite rank, commutes with X , and $(-X - \lambda_0)$ is nilpotent on its range. The elements in the range of $\Pi_{\lambda_0}^X$ are called *generalized Ruelle resonant states*. Note that by the results in [BuLi, FaSj, DyZw, DyGu] the poles can be identified with the discrete spectrum of X in certain Hilbert spaces and the generalized resonant states with generalized eigenfunctions.

The quantum resonances on M can be introduced in a quite similar fashion, except that we have to work with the wave flow. Let Δ_M be the non-negative Laplacian and $U(t) := \cos(t\sqrt{\Delta_M - 1/4})$ the wave operator on M . For $f_1, f_2 \in C_c^\infty(M)$, we define the correlation function

$$C_\Delta(t; f_1, f_2) := \int_M U(t) f_1 \cdot f_2 \operatorname{dvol}.$$

Then, by standard spectral theory in the compact case, and by [MaMe, GuZw1] in the convex co-compact case, the Laplace transform

$$R_\Delta(\lambda; f_1, f_2) := \frac{1}{1/2 - \lambda} \int_0^\infty e^{(-\lambda+1/2)t} C_\Delta(t; f_1, f_2) dt$$

extends meromorphically from $\operatorname{Re}(\lambda) > 1$ to $\lambda \in \mathbb{C}$. Notice that $R_\Delta(\lambda; f_1, f_2) = \langle (\Delta_M - \lambda(1 - \lambda))^{-1} f_1, f_2 \rangle$ for $\operatorname{Re}(\lambda) > 1$ and $R_\Delta(\lambda)$ is a meromorphic extension of the

¹The extension in a strip has been proved in [Po, Ru] before

resolvent of Δ_M . The poles are called *quantum resonances* and the residue operator $\Pi_{\lambda_0}^\Delta : C_c^\infty(M) \rightarrow C^\infty(M)$ defined by

$$\langle \Pi_{\lambda_0}^\Delta f_1, f_2 \rangle := \text{Res}_{\lambda_0} R_\Delta(\lambda; f_1, f_2), \quad \forall f_1, f_2 \in C_c^\infty(M)$$

has finite rank, commutes with Δ_M , and $(\Delta_M - \lambda_0(1 - \lambda_0))$ is nilpotent on its range. The elements in the range of $\Pi_{\lambda_0}^\Delta$ are called *generalized quantum resonant states*.

Note that the Ruelle generalized resonant states are distributions on SM while the quantum resonant states are functions on M . In order to formulate the explicit correspondence between them, we consider the projection $\pi_0 : SM \rightarrow M$ on the base and $\pi_{0*} : \mathcal{D}'(SM) \rightarrow \mathcal{D}'(M)$ the operator dual to the pull-back $\pi_0^* : C_c^\infty(M) \rightarrow C_c^\infty(SM)$ (note that π_{0*} corresponds to integration in the fibers of SM). In order to state the theorems, we also need to introduce the canonical line bundle $\mathcal{K} := (T^*M)^{1,0}$ and its dual $\mathcal{K}^{-1} := (T^*M)^{0,1}$, and we denote their tensor powers by $\mathcal{K}^n := \mathcal{K}^{\otimes n}$ and $\mathcal{K}^{-n} := (\mathcal{K}^{-1})^{\otimes n}$. Here the Riemann surface is oriented and thus inherits a complex structure. Then there is a natural map

$$\pi_n^* : C_c^\infty(M; \mathcal{K}^n) \rightarrow C_c^\infty(SM), \quad \pi_n^* f(x, v) := f(x)(\otimes^n v)$$

and we consider its dual operator $\pi_{n*} : \mathcal{D}'(SM) \rightarrow \mathcal{D}'(M; \mathcal{K}^n)$ which can be viewed as the n -th Fourier component in the fibers of SM (SM is equipped with the Liouville measure and M with the Riemannian volume form to define the pairings).

Let us formulate the first main result:

Theorem 1.1. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a smooth oriented compact hyperbolic surface and let SM be its unit tangent bundle. Then*

1) *for each $\lambda_0 \in \mathbb{C} \setminus (-\frac{1}{2} \cup -\mathbb{N})$ the pushforward map π_{0*} restricts to a linear isomorphism of complex vector spaces*

$$\pi_{0*} : \text{Ran}(\Pi_{\lambda_0}^X) \cap \ker U_- \rightarrow \text{Ran} \Pi_{\lambda_0+1}^\Delta = \ker(\Delta_M + \lambda_0(1 + \lambda_0))$$

2) *for $\lambda_0 = -\frac{1}{2}$, the map*

$$\pi_{0*} : \text{Ran}(\Pi_{\lambda_0}^X) \cap \ker U_- \rightarrow \ker(\Delta_M - \frac{1}{4})$$

is surjective and has a kernel of complex dimension $\dim \ker(\Delta_M - \frac{1}{4})$. In other words, the multiplicity of $-1/2$ as a Ruelle resonances is equal to twice the multiplicity of $1/4$ as eigenvalue of Δ_M .

3) *For $\lambda_0 = -n \in -\mathbb{N}$, the following map is an isomorphism of complex vector spaces*

$$\pi_{n*} \oplus \pi_{-n*} : \text{Ran}(\Pi_{-n}^X) \cap \ker U_- \rightarrow H_n(M) \oplus H_{-n}(M)$$

with $H_n(M) := \{u \in C^\infty(M; \mathcal{K}^n); \bar{\partial}u = 0\}$, $H_{-n}(M) := \{u \in C^\infty(M; \mathcal{K}^{-n}); \partial u = 0\}$. Consequently, the complex dimension of $\text{Ran}(\Pi_{-n}^X) \cap \ker U_-$ is $(2n - 1)|\chi(M)|$ if $n > 1$ and $|\chi(M)| + 2$ if $n = 1$, where $\chi(M)$ is the Euler characteristic.

Theorem 1.1 gives a full characterization of the Ruelle resonant states, that are invariant under the horocyclic flow. We call these resonances *the first band* of Ruelle resonances. Part 1) of Theorem 1.1 has been proved in [DFG] in any dimension. We slightly simplify the argument and characterize all first band resonant states including the particular points $\lambda_0 \in -1/2 - \mathbb{N}_0/2$ which were left out in [DFG]. Theorem 1.1 is also related to the classification of horocyclic invariant distributions by [FIFo] but we use a different approach avoiding the Plancherel decomposition into unitary irreducible representations. This has the advantage that our approach gives a more geometric description of the resonant states and that it extends to the convex co-compact setting. In Theorem 3.3 we give slightly more precise statements including the description of possible Jordan blocks in the Ruelle spectrum.

To state our result in the convex co-compact setting, let us first recall some geometric definitions. A convex co-compact hyperbolic surface M can be realized as a quotient $M = \Gamma \backslash \mathbb{H}^2$ of the hyperbolic plane where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ a discrete subgroup whose non-trivial elements are hyperbolic transformations. Viewing Γ as a subgroup of $\mathrm{PSL}_2(\mathbb{C})$, it also acts by conformal transformations on the Riemann sphere $\overline{\mathbb{C}} \simeq \mathbb{S}^2$, and the action is free and properly discontinuous on the complement of the limit set $\Lambda_\Gamma \subset \mathbb{R}$, which can be defined as the closure of the set of fixed points of non-trivial elements $\gamma \in \Gamma$. The quotient $M_2 := \Gamma \backslash (\overline{\mathbb{C}} \setminus \Lambda_\Gamma)$ is a compact Riemann surface containing two copies M_\pm of M corresponding to $M_\pm := \Gamma \backslash \{\pm \mathrm{Im}(z) > 0\}$, and $\overline{M} := \Gamma \backslash (\{ \mathrm{Im}(z) \geq 0 \} \setminus \Lambda_\Gamma)$ provides a smooth conformal compactification to M in which $\partial \overline{M}$ represents the geometric infinity of M . The surface M_2 has an involution \mathcal{I} induced by $z \mapsto \bar{z}$ and fixing $\partial \overline{M}$.

Theorem 1.2. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a smooth oriented convex co-compact hyperbolic surface and let SM be its unit tangent bundle. Then*

1) *for each $\lambda_0 \in \mathbb{C} \setminus -\mathbb{N}$ the pushforward map π_{0*} restricts to a linear isomorphism of complex vector spaces*

$$\pi_{0*} : \mathrm{Ran}(\Pi_{\lambda_0}^X) \cap \ker U_- \rightarrow \mathrm{Ran}(\Pi_{\lambda_0+1}^\Delta).$$

If $\lambda_0 \in -1/2 - \mathbb{N}$, $\mathrm{Ran}(\Pi_{\lambda_0}^X) \cap \ker U_- = 0$ and $\mathrm{Ran}(\Pi_{\lambda_0+1}^\Delta) = 0$.

2) *For $\lambda_0 = -n \in -\mathbb{N}$, the following map is an isomorphism of real vector spaces if Γ is not cyclic:*

$$\pi_{n*} : \mathrm{Ran}(\Pi_{-n}^X) \cap \ker U_- \rightarrow H_n(M).$$

Here $H_n(M) := \{f|_{\overline{M}}; f \in C^\infty(M_2; \mathcal{K}^n), \bar{\partial} f = 0, \mathcal{I}^ f = \bar{f}\}$ and $\mathcal{I} : M_2 \rightarrow M_2$ is the natural involution fixing $\partial \overline{M}$ in M_2 . Consequently the real dimension of $\mathrm{Ran}(\Pi_{-n}^X) \cap \ker U_-$ is $(2n-1)|\chi(M)|$ if $n > 1$ and $|\chi(M)| + 1$ if $n = 1$.*

As in the compact case we provide more precise information on the correspondence between Jordan blocks, see Theorem 4.5. In 2), we note that $\pi_{0*}(\mathrm{Ran}(\Pi_{-n}^X)) = 0$ for non-cyclic groups, $\mathrm{Ran}(\Pi_{-n+1}^\Delta) = 0$ for $n > 1$ and $\mathrm{Ran}(\Pi_0^\Delta)$ is the space of constant functions (see the comments after Corollary 5.2). In the case of a cyclic group, the

classical resonances at $-n$ have dimension 2 and no Jordan blocks, and we get that $\pi_{0*} : \text{Ran}(\Pi_{-n}^X) \mapsto \text{Ran}(\Pi_{-n+1}^\Delta)$ has a kernel of dimension 1 and image of dimension 1, while $\text{Ran}(\Pi_{-n+1}^\Delta)$ has dimension 2 and there is a Jordan block for the quantum resonance. The map $\pi_{n*} : \text{Ran}(\Pi_{-n}^X) \cap \ker U_- \cap \ker \pi_{0*} \rightarrow H_n(M)$ is one-to-one in that case.

Beyond the description of the first band of Ruelle resonances we obtain a description of the full spectrum of Ruelle resonances by applying the vector fields U_+ iteratively.

Proposition 1.3. *Let M be a compact or convex co-compact hyperbolic surface, then*

$$\text{Ran}(\Pi_{\lambda_0}^X) = \bigoplus_{0 \leq \ell \leq |\text{Re}(\lambda_0)|} U_+^\ell \left(\text{Ran}(\Pi_{\lambda_0 + \ell}^X) \cap \ker U_- \right)$$

and the map $U_+^\ell : \text{Ran}(\Pi_{\lambda_0 + \ell}^X) \cap \ker U_- \rightarrow \text{Ran}(\Pi_{\lambda_0}^X)$ is injective unless $\lambda_0 + \ell = 0$.

The case $\lambda_0 + \ell = 0$ can only occur if M is compact, in which case $\text{Ran}(\Pi_0^X)$ is the space of constant functions, killed by the differential operator U_+ .

As a consequence, in Section 5, we obtain an alternative proof of the results [PaPe, BJP] on the zeros of the Selberg zeta function in our situation.

We end the introduction by a rough outline of the proofs of Theorem 1.1 and 1.2: a central ingredient is the microlocal characterization of Ruelle resonant states in [FaSj, DyGu]. In these references it has been shown that a distribution $u \in \mathcal{D}'(SM)$ is a generalized Ruelle resonant state for a Ruelle resonance $\lambda_0 \in \mathbb{C}$ (i.e. $u \in \text{Ran} \Pi_{\lambda_0}^X$) if and only if there exists $j \geq 1$ such that $(-X - \lambda_0)^j u = 0$ and

$$\begin{aligned} \text{WF}(u) &\subset E_u^* && \text{if } M \text{ is compact} \\ \text{WF}(u) &\subset E_u^*, \text{ supp}(u) \subset \Lambda_+ && \text{if } M \text{ is convex co-compact,} \end{aligned}$$

Here $\text{WF}(u) \subset T^*(SM)$ denotes the wave-front set of u and $E_u^* \subset T^*(SM)$ is the subbundle defined by $E_u^*(\mathbb{R}X \oplus E_u) = 0$. Furthermore, on convex co-compact surfaces we use the notation

$$\Lambda_\pm := \{y \in SM; d(\pi_0(\varphi_t(y)), x_0) \not\rightarrow \infty \text{ as } t \rightarrow \mp\infty\}$$

where d denotes the Riemannian distance and $x_0 \in M$ is any fixed point. Note that the set Λ_+ (resp. Λ_-) has a clear dynamical interpretation as it corresponds to trajectories that do not escape to infinity in the past (resp. in the future).

Using this characterization we follow the general strategy of [DFG]. We consider the hyperbolic surface as a quotient $M = \Gamma \backslash \mathbb{H}^2$ of its universal cover \mathbb{H}^2 by a co-compact, respectively convex co-compact, discrete subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$. If we lift the horocyclic invariant Ruelle resonant states to \mathbb{H}^2 , we can relate them to distributions on $\mathbb{S}^1 = \partial \overline{\mathbb{H}^2}$, conformally covariant by the group Γ and supported in Λ_Γ . We then show that such distributions are in correspondence with quantum resonant states using the

bijection of the Poisson-Helgason transform at the noninteger points. While this step is straightforward for compact surfaces, the convex co-compact setting is more complicated. A central ingredient is a characterization of generalized quantum resonant states using their asymptotic behavior towards the boundary. One can show (see Proposition 4.3) that $u \in \text{Ran } \Pi_{\lambda_0}^{\Delta}$ if and only if there exists $j \geq 1$ such that $(\Delta - \lambda_0(1 - \lambda_0))^j u = 0$ and

$$\begin{aligned} u &\in C^\infty(M) && \text{if } M \text{ is compact,} \\ u &\in \bigoplus_{k=0}^{j-1} \rho^{\lambda_0} \log(\rho)^k C_{\text{ev}}^\infty(\overline{M}) && \text{if } M \text{ is convex co-compact,} \end{aligned} \tag{1.1}$$

where $C_{\text{ev}}^\infty(\overline{M})$ denotes the space of smooth functions on \overline{M} which extend smoothly to M_2 as even functions with respect to the involution \mathcal{I} , and $\rho \in C_{\text{ev}}^\infty(\overline{M})$ is a boundary defining function of $\partial\overline{M}$ in \overline{M} . We prove that the asymptotic condition (1.1) corresponds to the fact that the associated distribution $\omega \in \mathcal{D}'(\mathbb{S}^1)$ via the Poisson-Helgason transform is supported in the limit set $\Lambda_\Gamma \subset \mathbb{S}^1$. Analogously, the condition that a horocyclic invariant Ruelle resonant state is supported in Λ_+ is equivalent to the fact that its associated distribution $\omega \in \mathcal{D}'(\mathbb{S}^1)$ is again supported on the limit set.

For the negative integer points, the Poisson-Helgason transform fails to be bijective, thus there are possibly equivariant boundary distributions $\omega \in \mathcal{D}'(\mathbb{S}^1)$ and consequently Ruelle resonances that are not related to Laplace eigenfunctions, but that lie in the kernel of the Poisson-Helgason transform. In order to characterize these distributions we consider a vector valued Poisson-Helgason transformation, mapping to sections in the complex line bundles \mathcal{K}^n . Roughly speaking we prove, that a combination of these vector valued Poisson transformation yields an equivariant bijection between the kernel of the Poisson-Helgason transform at the negative integer points and the holomorphic respectively antiholomorphic sections in \mathcal{K}^n and \mathcal{K}^{-n} . Again the convex co-compact part is more complicated, as we also have to take care about the asymptotics towards, the boundary.

Note that the invariant distributions supported on the limit set which appear as an intermediate step in our proof were also studied in [BuOl1, BuOl2], and our result somehow completes the picture.

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2. GEODESIC FLOW ON HYPERBOLIC MANIFOLDS

In this Section, we recall a few facts about the geodesic flow, horocyclic derivatives and the Poisson operator on the real hyperbolic plane that are needed for the next sections. We refer to the paper [DFG] where all the material is described in full detail.

2.1. Hyperbolic space. Let \mathbb{H}^2 be the real hyperbolic space of dimension 2, which we view as the open unit ball in \mathbb{R}^2 equipped with the metric $g_{\mathbb{H}^2} := \frac{4|dx|^2}{(1-|x|^2)^2}$. The unit tangent bundle is denoted by $S\mathbb{H}^2$ and the projection is denoted by $\pi_0 : S\mathbb{H}^2 \rightarrow \mathbb{H}^2$ on the base. The hyperbolic space \mathbb{H}^2 is compactified smoothly into the closed unit ball of \mathbb{R}^2 , denoted by $\overline{\mathbb{H}}^2$ and its boundary is the unit sphere \mathbb{S}^1 .

Let X be the geodesic vector field on $S\mathbb{H}^2$ and $\varphi_t : S\mathbb{H}^2 \rightarrow S\mathbb{H}^2$ be the geodesic flow at time $t \in \mathbb{R}$. We denote by $B_{\pm} : S\mathbb{H}^2 \rightarrow \mathbb{S}^1$ the endpoint maps assigning to a vector $(x, v) \in S\mathbb{H}^2$ the endpoint on \mathbb{S}^1 of the geodesic passing through (x, v) in positive time (+) and negative time (-). These maps are submersions and allow to identify $S\mathbb{H}^2$ with $\mathbb{H}^2 \times \mathbb{S}^1$ by the map $(x, v) \mapsto (x, B_{\pm}(x, v))$. It is easy to compute B_{\pm} explicitly: using the complex coordinate $z = x_1 + ix_2 \in \mathbb{C}$ for the point $x = (x_1, x_2)$ (with $|x| < 1$) and identifying $v \in S_z\mathbb{H}^2$ with $e^{i\theta}$ through $2v/(1 - |z|^2) = \cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{x_2}$, we get

$$B_-(z, e^{i\theta}) = \frac{-e^{i\theta} + z}{-e^{i\theta}\bar{z} + 1}, \quad B_+(z, e^{i\theta}) = \frac{e^{i\theta} + z}{e^{i\theta}\bar{z} + 1}. \quad (2.1)$$

For each z , the map $B_z : e^{i\theta} \mapsto B_-(z, e^{i\theta})$ is a diffeomorphism of \mathbb{S}^1 and its inverse is given by

$$B_z^{-1}(e^{i\alpha}) = -e^{i\alpha} \frac{ze^{-i\alpha} - 1}{\bar{z}e^{i\alpha} - 1}. \quad (2.2)$$

There exists two positive functions $\Phi_{\pm} \in C^\infty(S\mathbb{H}^2)$ satisfying $X\Phi_{\pm} = \pm\Phi_{\pm}$, given by

$$\Phi_{\pm}(x, v) := P(x, B_{\pm}(x, v)) \quad (2.3)$$

where $P(x, \nu)$ is the Poisson kernel given by

$$P(x, \nu) := \frac{1 - |x|^2}{|x - \nu|^2}, \quad x \in \mathbb{H}^2, \nu \in \mathbb{S}^1. \quad (2.4)$$

The group of orientation preserving isometries of \mathbb{H}^2 is the group

$$G := \text{PSU}(1, 1) \simeq \text{PSL}_2(\mathbb{R})$$

An element $\gamma \in G \subset \text{PSL}_2(\mathbb{C})$ acts on \mathbb{C} by Möbius transformations and preserves the unit ball \mathbb{H}^2 , and this action preserves also the closure $\overline{\mathbb{H}}^2$. Furthermore the G action on \mathbb{H}^2 lifts linearly to an action on $T\mathbb{H}^2$ and as the action on the base space \mathbb{H}^2 is isometric it can be restricted to $S\mathbb{H}^2$. By abuse of notation, for $\gamma \in G$, we also denote the action of γ on $S\mathbb{H}^2$ or \mathbb{S}^1 by the same letter γ . By $|d\gamma| : \mathbb{S}^1 \rightarrow \mathbb{R}$ we denote the norm of the differential $d\gamma$ on the boundary $\mathbb{S}^1 = \partial\overline{\mathbb{H}}^2$ of the unit ball with respect

to the Euclidean norm. Note that the above defined functions Φ_{\pm} and maps B_{\pm} are compatible with respect to these G actions in the sense that one has the relations

$$\gamma^* \Phi_{\pm}(x, v) = \Phi_{\pm}(x, v) N_{\gamma}(B_{\pm}(x, v)), \quad B_{\pm}(\gamma(x, v)) = \gamma(B_{\pm}(x, v)) \quad (2.5)$$

where $N_{\gamma}(\nu) := |d\gamma(\nu)|^{-1}$.

As the G action on $S\mathbb{H}^2$ is free and transitive, we can identify $G \simeq S\mathbb{H}^2$ via the natural isomorphism

$$G \rightarrow S\mathbb{H}^2, \quad \gamma \mapsto (\gamma(0), \frac{1}{2}d\gamma(0) \cdot \partial_x). \quad (2.6)$$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ of G is spanned by

$$U_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad U_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \quad (2.7)$$

These elements can also be viewed as left invariant smooth vector fields on $G \simeq S\mathbb{H}^2$, which form at any point $(x, v) \in S\mathbb{H}^2$ a basis of $T(S\mathbb{H}^2)$, and the following commutation relations hold

$$[X, U_{\pm}] = \pm U_{\pm}, \quad [U_+, U_-] = 2X. \quad (2.8)$$

The geodesic vector field is represented by X and we call U_+ the stable derivative and U_- the unstable derivative. The vector fields X, U_{\pm} can be viewed as first order linear differential operators on $S\mathbb{H}^2$, thus acting on distributions, and by (2.8), X preserves $\ker U_{\pm}$. Another decomposition that is quite natural for $T(S\mathbb{H}^2)$ is

$$X := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad X_{\perp} := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

which satisfy $U_{\pm} = X \pm X_{\perp}$ and

$$[X, V] = X_{\perp}, \quad [X, X_{\perp}] = V, \quad [V, X_{\perp}] = X. \quad (2.9)$$

The vector field V generates the $SO(2)$ action on G and geometrically, it generates the rotation in the fibers of $S\mathbb{H}^2$ (that are circles); it is called the vertical vector field since $d\pi_0(V) = 0$. For what follows, we will always view $X, V, X_{\perp}, U_+, U_-$ as vector fields on $S\mathbb{H}^2$.

There is a smooth splitting of $T(S\mathbb{H}^2)$ into flow, stable and unstable bundles,

$$T(S\mathbb{H}^2) = \mathbb{R}X \oplus E_s \oplus E_u$$

with the property that there is $C > 0$ uniform such that

$$\|d\varphi_t(y) \cdot w\| \leq C e^{-|t|} \|w\|, \quad \forall t \geq 0, \forall w \in E_s(y) \text{ or } \forall t \leq 0, \forall w \in E_u(y). \quad (2.10)$$

Here the norm on $S\mathbb{H}^2$ is with respect to the Sasaki metric. The space E_s is generated by the vector field U_+ and E_u by the vector field U_- where U_{\pm} are the images by the map (2.6) of the left invariant vector fields in (2.7).

There are two important properties of Φ_{\pm} with respect to stable/unstable derivatives:

$$U_{\pm}\Phi_{\pm} = 0, \text{ and } dB_{\pm}.U_{\pm} = 0. \quad (2.11)$$

Let $\mathcal{Q}_{\pm} : \mathcal{D}'(\mathbb{S}^1) \rightarrow \mathcal{D}'(S\mathbb{H}^2)$ be the pull-back by B_{\pm} acting on distributions which is well defined since B_{\pm} are submersions. It is a linear isomorphism between the following spaces (see [DFG, Lemma 4.7])

$$\mathcal{Q}_{\pm} : \mathcal{D}'(\mathbb{S}^1) \rightarrow \mathcal{D}'(S\mathbb{H}^2) \cap \ker U_{\pm} \cap \ker X. \quad (2.12)$$

2.2. Poisson-Helgason transform. We say that a smooth function f on \mathbb{H}^2 is *tempered* if there exists $C > 0$ such that $|f(x)| \leq Ce^{Cd_{\mathbb{H}^2}(x,0)}$ if 0 is the center of \mathbb{H}^2 (viewed as the unit disk) and $d_{\mathbb{H}^2}(\cdot, \cdot)$ denotes the hyperbolic distance. Below, we view the space of distributions $\mathcal{D}'(\mathbb{S}^1)$ on \mathbb{S}^1 as the topological dual of $C^{\infty}(\mathbb{S}^1)$ and we embed $C^{\infty}(\mathbb{S}^1) \subset \mathcal{D}'(\mathbb{S}^1)$ by the pairing

$$\langle \omega, \chi \rangle_{\mathbb{S}^1} := \int_{\mathbb{S}^1} \omega(\nu)\chi(\nu)dS(\nu)$$

where the measure dS is the standard measure on \mathbb{S}^1 (viewed as a the unit circle in \mathbb{R}^2). Then the following result was proved² in [VBS, Corollary 11.3 and Theorem 12.2] and [OsSe, Theorem 3.15] but we follow the presentation given in [DFG, Section 6.3]:

Lemma 2.1. *For $\lambda \in \mathbb{C}$, let $\mathcal{P}_{\lambda} : \mathcal{D}'(\mathbb{S}^1) \rightarrow C^{\infty}(\mathbb{H}^2)$ be the Poisson-Helgason transform*

$$\mathcal{P}_{\lambda}(\omega)(x) := \pi_{0*}(\Phi_{-}^{\lambda}\mathcal{Q}_{-}(\omega))(x) = \langle \omega, P^{1+\lambda}(x, \cdot) \rangle_{\mathbb{S}^1}$$

where $P(x, \nu)$ is the Poisson kernel of (2.4) and π_{0*} is the adjoint of the pull-back $\pi_0^* : C_c^{\infty}(\mathbb{H}^2) \rightarrow C_c^{\infty}(S\mathbb{H}^2)$. Then \mathcal{P}_{λ} maps $\mathcal{D}'(\mathbb{S}^1)$ onto the space of tempered functions in the kernel of $(\Delta_{\mathbb{H}^2} + \lambda(1 + \lambda))$, where $\Delta_{\mathbb{H}^2} = d^*d$ is the positive Laplacian acting on functions on \mathbb{H}^2 and if $\lambda \notin -\mathbb{N}$, \mathcal{P}_{λ} is an isomorphism. Finally, if $\gamma \in G$ is an isometry of \mathbb{H}^2 , we have the relation $\gamma^*\mathcal{P}_{\lambda}(\omega) = \mathcal{P}_{\lambda}(|d\gamma|^{-\lambda}\gamma^*\omega)$ for each $\omega \in \mathcal{D}'(\mathbb{S}^1)$.

It is useful to describe the inverse of \mathcal{P}_{λ} when $\lambda \notin -\mathbb{N}$. For this purpose we can use for instance [DFG, Lemma 6.8]. First, if $\lambda \notin -\mathbb{N}$ and $\omega \in \mathcal{D}'(\mathbb{S}^1)$, for each $\chi \in C^{\infty}(\mathbb{S}^1)$ and $t \in (0, 1)$ one has³

$$\int_{\mathbb{S}^1} \mathcal{P}_{\lambda}(\omega)\left(\frac{2-t}{2+t}\nu\right)\chi(\nu)dS(\nu) = \begin{cases} t^{-\lambda}F_{\lambda}^{-}(t) + t^{\lambda+1}F_{\lambda}^{+}(t) & \text{if } \lambda \notin -1/2 + \mathbb{Z} \\ t^{-\lambda}F_{\lambda}^{-}(t) + t^{\lambda+1}\log(t)F_{\lambda}^{+}(t) & \text{if } \lambda \in -1/2 + \mathbb{N}_0 \\ t^{-\lambda}\log(t)F_{\lambda}^{-}(t) + t^{\lambda+1}F_{\lambda}^{+}(t) & \text{if } \lambda \in -1/2 - \mathbb{N} \end{cases} \quad (2.13)$$

²Note that this was proved in the setting of hyperfunctions by Helgason [He]

³The case $\lambda = -1/2$ is not really studied in [DFG, Lemma 6.8] but the analysis done there for $\lambda \in -1/2 + \mathbb{N}$ applies as well for $\lambda = -1/2$ by using the explicit expression of the modified Bessel function $K_0(z)$ as a converging series.

where $F_\lambda^\pm \in C_{\text{ev}}^\infty([0, 1))$, and $C_{\text{ev}}^\infty([0, 1))$ is the subset of $C^\infty([0, 1))$ consisting of functions with an even Taylor expansion at 0⁴. The exact expressions of $F_\lambda^\pm(0)$ can be obtained directly from the study of the Poisson operator in [GrZw] and the computation of the scattering operator $\mathcal{S}(s)$ of \mathbb{H}^2 in [GuZw1, Appendix]. The scattering operator is defined as the operator acting on $C^\infty(\mathbb{S}^1)$ given by the explicit function of the Laplacian on \mathbb{S}^1

$$\mathcal{S}(s) := \frac{\Gamma(\sqrt{\Delta_{\mathbb{S}^1}} + s)}{\Gamma(\sqrt{\Delta_{\mathbb{S}^1}} + 1 - s)}, \quad (2.14)$$

with Schwartz kernel on \mathbb{S}^1 given for $\text{Re}(s) < 1/2$ by

$$\mathcal{S}(s; \nu, \nu') = \pi^{-\frac{1}{2}} \frac{2^s \Gamma(s)}{\Gamma(-s + \frac{1}{2})} |\nu - \nu'|^{-2s}.$$

It is a holomorphic family of operators in $s \notin -\mathbb{N}_0$ with poles of order 1 at $-\mathbb{N}_0$, which is an isomorphism on $\mathcal{D}'(\mathbb{S}^1)$ outside the poles and satisfies the following functional equations

$$\mathcal{S}(s)^{-1} = \mathcal{S}(1 - s), \quad \mathcal{P}_\lambda = \frac{\Gamma(-\lambda)}{\Gamma(\lambda + 1)} \mathcal{P}_{-\lambda-1} \mathcal{S}(\lambda + 1). \quad (2.15)$$

The operator $\mathcal{S}(s)$ is an elliptic pseudo-differential operator of (complex) order $2s - 1$ on \mathbb{S}^1 , with principal symbol that of $\Delta_{\mathbb{S}^1}^{2s-1}$. This follows from the formula above but also in a more general setting by the works [JoSa, GrZw]. We remark that for $k \in \mathbb{N}$, the operator $\mathcal{S}(1/2+k)$ is a differential operator of order $2k$ (matching with the analysis of [GrZw]), and it is invertible from the expression (2.14).

We get for $\lambda \notin (-\frac{1}{2} - \mathbb{Z}) \cup -\mathbb{N}$

$$F_\lambda^-(0) = \pi^{1/2} 2^\lambda \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1)} \langle \omega, \chi \rangle_{\mathbb{S}^1}, \quad F_\lambda^+(0) = \pi^{1/2} 2^{-\lambda-1} \frac{\Gamma(-\lambda - \frac{1}{2})}{\Gamma(\lambda + 1)} \langle \mathcal{S}(\lambda + 1)\omega, \chi \rangle_{\mathbb{S}^1}. \quad (2.16)$$

At $\lambda = -1/2 + k$ with $k \in \mathbb{N}$ there is a pole of order 1 in the expression of $F_\lambda^+(0)$ and it follows from [GrZw] that, with the notation of (2.13),

$$F_{-1/2+k}^-(0) = \frac{\pi^{1/2} 2^{-1/2+k} \Gamma(k)}{\Gamma(k + 1/2)} \langle \omega, \chi \rangle_{\mathbb{S}^1}, \quad F_{-1/2+k}^+(0) = c_k \langle \mathcal{S}(1/2 + k)\omega, \chi \rangle$$

for some $c_k \neq 0$. For $\lambda = -1/2$, $F_\lambda^+(0)$, there is a constant $c_0 \in \mathbb{R}$ so that

$$F_{-1/2}^+(0) = -\sqrt{2} \langle \omega, \chi \rangle, \quad F_{-1/2}^-(0) = \sqrt{2} \langle \partial_\lambda \mathcal{S}(1/2)\omega, \chi \rangle + c_0 \langle \omega, \chi \rangle \quad (2.17)$$

⁴The evenness of the expansion at $t = 0$ comes directly from the proof in [DFG, Lemma 6.8] when acting on functions, since the special functions appearing in the argument are Bessel functions that have even expansions.

and $\partial_\lambda \mathcal{S}(1/2) = \log(\Delta_{\mathbb{S}^1}) + A$ for some pseudo-differential operator A of order 0 on \mathbb{S}^1 . To deal with the case $\lambda \in -\frac{1}{2} - \mathbb{N}$, we use the functional equation (2.15) of the scattering operator of \mathbb{H}^2 : we deduce that for $\lambda = -1/2 - k$ with $k \in \mathbb{N}$,

$$F_{-1/2-k}^-(0) = c'_k \langle \omega, \chi \rangle_{\mathbb{S}^1}, \quad F_{-1/2-k}^+(0) = c''_k \langle \mathcal{S}(1/2 - k)\omega, \chi \rangle \quad (2.18)$$

for some $c'_k \neq 0$ and $c''_k \neq 0$. This gives the expression for the inverse of \mathcal{P}_λ at those points.

To conclude, we discuss the range and kernel of \mathcal{P}_{-n} if $n \in \mathbb{N}$. Using the complex coordinate $z \in \mathbb{C}$ for the ball model of \mathbb{H}^2 , this operator is

$$\begin{aligned} \mathcal{P}_{-n}(\omega)(z) &= (1 - |z|^2)^{1-n} \int_{\mathbb{S}^1} \omega(\nu) |z\bar{\nu} - 1|^{2(n-1)} d\nu \\ &= (1 - |z|^2)^{1-n} \int_0^{2\pi} \omega(e^{i\alpha}) (|z|^2 + 1 - \bar{z}e^{i\alpha} - ze^{-i\alpha})^{(n-1)} d\alpha. \end{aligned}$$

From this we deduce that the range of \mathcal{P}_{-n} is finite and its kernel contains the space

$$W_n := \{\omega \in \mathcal{D}'(\mathbb{S}^1); \langle \omega, e^{ik\alpha} \rangle = 0, \forall k \in \mathbb{Z} \cap [-n + 1, n - 1]\}.$$

In fact, from the second functional equation (2.15) and the formula (2.14), we see that

$$\ker \mathcal{P}_{-n} = \ker(\text{Res}_{1-n} \mathcal{S}(\lambda)) = W_n \quad (2.19)$$

2.3. Co-compact and convex co-compact quotients. Below, we will consider two types of hyperbolic surfaces, the compact and the convex co-compact ones. Consider a discrete subgroup $\Gamma \subset G$ containing only hyperbolic transformations, i.e. transformations fixing two points in $\overline{\mathbb{H}^2}$. The group Γ acts properly discontinuously on \mathbb{H}^2 and the quotient $M = \Gamma \backslash \mathbb{H}^2$ is a smooth oriented hyperbolic surface. We say that Γ is *co-compact* if M is compact.

Denote by $\Lambda_\Gamma \subset \mathbb{S}^1$ the limit set of the group Γ , i.e. the set of accumulation points of the orbit $\Gamma \cdot 0 \in \mathbb{H}^2$ of $0 \in \mathbb{H}^2$ on $\mathbb{S}^1 = \partial \mathbb{H}^2$. We will call $\Omega_\Gamma = \mathbb{S}^1 \setminus \Lambda_\Gamma$ the set of discontinuity of Γ , on which Γ acts properly discontinuously.

If Γ is co-compact, then $\Lambda_\Gamma = \mathbb{S}^1$. The subgroup Γ is called *convex co-compact*, if it is not co-compact and if the action of Γ on the convex hull $\text{CH}(\Lambda_\Gamma) \subset \mathbb{H}^2$ of the limit set Λ_Γ in $\overline{\mathbb{H}^2}$ is co-compact, that is $\Gamma \backslash \text{CH}(\Lambda_\Gamma)$ is compact (see e.g. [Bo, Section 2.4]). In this case the group Γ acts totally discontinuously, freely, on \mathbb{H}^2 and more generally on $\mathbb{H}^2 \cup \Omega_\Gamma = \overline{\mathbb{H}^2} \setminus \Lambda_\Gamma$. The manifold $M = \Gamma \backslash \mathbb{H}^2$ is complete with infinite volume, and it is the interior of a smooth compact manifold with boundary $\overline{M} := \Gamma \backslash (\mathbb{H}^2 \cup \Omega_\Gamma)$. Here we notice that $\mathbb{H}^2 \cup \Omega_\Gamma$ is also a smooth manifold with boundary but it is non-compact. The boundary $\partial \overline{M} := \Gamma \backslash \Omega_\Gamma$ of \overline{M} is compact.

We now consider $M = \Gamma \backslash \mathbb{H}^2$ which is either compact or convex co-compact (here M could be as well the whole \mathbb{H}^2). The unit tangent bundle of M is $SM = \Gamma \backslash S\mathbb{H}^2 \simeq \Gamma \backslash G$, and we let $\pi_\Gamma : S\mathbb{H}^2 \rightarrow SM$ be the induced covering map. The

geodesic flow $\varphi_t : SM \rightarrow SM$ on SM lifts to the geodesic flow on $S\mathbb{H}^2$, the left invariant vector fields X, U_\pm, X_\perp, V on $T(S\mathbb{H}^2) \simeq TG$ descend to SM via $d\pi_\Gamma$; we will keep the notation X instead of $d\pi_\Gamma X$, and similarly for the vector fields U_\pm, X_\perp, V . The flow φ_t is generated by the vector field X and there is an Anosov flow-invariant smooth splitting

$$T(SM) = \mathbb{R}X \oplus E_s \oplus E_u \quad (2.20)$$

where $E_u = \mathbb{R}U_-$, $E_s = \mathbb{R}U_+$ are the stable and unstable bundles satisfying the condition (2.10). Using the Anosov splitting (2.20), we define the subbundles E_0^* , E_s^* and E_u^* of $T^*(SM)$ by

$$E_u^*(E_u \oplus \mathbb{R}X) = 0, \quad E_s^*(E_s \oplus \mathbb{R}X) = 0, \quad E_0^*(E_u \oplus E_s) = 0. \quad (2.21)$$

2.4. Complex line bundles. Note that $M = \Gamma \backslash \mathbb{H}^2$ carries a complex structure so that $\pi_\Gamma : \mathbb{H}^2 \rightarrow M$ is holomorphic and that we can thus consider the complex line bundles $\mathcal{K} := (T^*M)^{1,0}$ and $\mathcal{K}^{-1} := (T^*M)^{0,1}$. Let us consider their tensor powers: for each $k \in \mathbb{Z}$, set $\mathcal{K}^k := \otimes^{|k|} \mathcal{K}^{\text{sign}(k)}$. The bundles \mathcal{K}^k are holomorphic line bundles over M , which in addition are trivial when M is convex co-compact (see [Fo, Theorems 30.1 and Th 30.4]). A section $C_c^\infty(M; \mathcal{K}^k)$ can be viewed as a function in $C_c^\infty(SM)$ by the map

$$\pi_k^* : C_c^\infty(M; \mathcal{K}^k) \rightarrow C_c^\infty(SM), \quad \pi_k^* u(x, v) := u(x)(\otimes^k v).$$

We denote by $\pi_{k*} : \mathcal{D}'(SM) \rightarrow \mathcal{D}'(M; \mathcal{K}^k)$ its transpose defined by duality. Note that the operator π_k^* extends to $\mathcal{D}'(M; \mathcal{K}^k)$ and $(2\pi)^{-1} \pi_{k*} \pi_k^*$ is the identity map on $\mathcal{D}'(M; \mathcal{K}^k)$. Each smooth function $f \in C_c^\infty(SM)$ can be decomposed into Fourier modes in the fibers of SM by using the eigenvectors of the vector field V :

$$f = \sum_{k \in \mathbb{Z}} f_k \quad \text{with } Vf_k = ikf_k \text{ and } f_k = \frac{1}{2\pi} \pi_k^* \pi_{k*} f.$$

It is easy to see that for each $f \in C_c^\infty(SM)$, $s \geq 0$, and $N > 0$

$$\|f_k\|_{H^s(SM)} \leq C_{f,N,s} \langle k \rangle^{-N} \quad (2.22)$$

for some constant $C_{f,N,s}$ independent of k . A distribution $u \in \mathcal{D}'(SM)$ can also be decomposed as a sum

$$u = \sum_{k \in \mathbb{Z}} u_k \quad \text{with } Vu_k = iku_k, \quad u_k = \frac{1}{2\pi} \pi_k^* \pi_{k*} u$$

which converges in the distribution sense. In order to see this recall that any distribution $u \in \mathcal{D}'(SM)$ restricted to a precompact open set $A \subset SM$ is of finite order, i.e. there is $s > 0$, $C > 0$ with

$$|\langle u, \varphi \rangle| \leq C \|\varphi\|_{H^s(A)}, \quad \forall \varphi \in C_c^\infty(A). \quad (2.23)$$

Now for $f \in C_c^\infty(SM)$, we can write $f = \sum_{k \in \mathbb{Z}} f_k$. Then (2.22) and (2.23) imply that

$$\langle u, f \rangle = \sum_{k \in \mathbb{Z}} \langle u, f_k \rangle = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \langle \pi_k^* \pi_{k*} u, f \rangle$$

is absolutely convergent. For convenience of notations, we avoid the π_k^*, π_{k*} operators and we will view u_k as an element in $\mathcal{D}'(M; \mathcal{K}^k)$ or as an element in $\{w \in \mathcal{D}'(SM) \text{ with } Vw = ikw\}$ depending on which point of view is more appropriate in a given situation. First of all V acts on $\mathcal{D}'(M, \mathcal{K}^k)$ by

$$V : \mathcal{D}'(M, \mathcal{K}^k) \rightarrow \mathcal{D}'(M, \mathcal{K}^k), \quad u_k \mapsto Vu_k = iku_k.$$

Furthermore, if we define the complex valued vector fields

$$\eta_\pm := \frac{1}{2}(X \pm iX_\perp), \tag{2.24}$$

they fulfill the commutation relations

$$[V, \eta_\pm] = \pm \eta_\pm.$$

They are called the raising/lowering operators as they shift the vertical Fourier components by ± 1 and when restricted to sections of \mathcal{K}^k through π_k^* , they define operators

$$\eta_\pm : \mathcal{D}'(M; \mathcal{K}^k) \mapsto \mathcal{D}'(M; \mathcal{K}^{k \pm 1}). \tag{2.25}$$

If $z = x + iy$ are local isothermal coordinates, the hyperbolic metric can be written as $g = e^{2\alpha(z)}|dz|^2$ and for $k \geq 0$ the operators η_\pm acting on a section $u \in C_c^\infty(M; \mathcal{K}^k)$ of the form $u = f(z)dz^k$ is given by

$$\eta_- u = e^{-2\alpha}(\partial_{\bar{z}} f)dz^{k-1}, \quad \eta_+ u = (e^{2k\alpha} \partial_z (e^{-2k\alpha} f))dz^{k+1}. \tag{2.26}$$

A similar expression holds for $k \leq 0$ and we directly deduce

$$\forall k \geq 0, \eta_- u_k = 0 \iff \bar{\partial} u_k = 0, \quad \forall k < 0, \eta_+ u_k = 0 \iff \partial u_k = 0. \tag{2.27}$$

We notice that these operators η_\pm , as well as the operators V, X and $X_\perp = [X, V]$, have nothing to do with constant curvature and Lie groups, they are well defined for any oriented compact Riemannian surface, see Guillemin-Kazhdan [GuKa]. The *Casimir operator* is defined as the second order operator on SM by

$$\begin{aligned} \Omega &= X^2 + X_\perp^2 - V^2 = X^2 - V^2 + (2U_- + V)^2 = X^2 + 4U_-^2 + 2U_-V + 2VU_- \\ &= X^2 - X + 4U_- + 4VU_-. \end{aligned} \tag{2.28}$$

It satisfies $(\Omega u)_k = \Omega u_k$ since Ω commutes with V .

For later purpose, we will need a few lemmas which follow for algebraic reasons and Fourier decomposition in the fibers.

Lemma 2.2. *Let $\lambda \in \mathbb{C}$ and $u, v \in \mathcal{D}'(SM)$ be two distributions, then they fulfill the set of differential equations*

$$(X + \lambda)u = v \text{ and } U_-u = 0$$

if and only if their Fourier modes fulfill the recursion relations

$$2\eta_{\pm}u_{k\mp 1} = (-\lambda \mp k)u_k + v_k. \quad (2.29)$$

Proof. Recall that $X = \eta_+ + \eta_-$ and assume $(X + \lambda)u = v$ then, taking the Fourier components of this equation and using (2.25) we get for any $k \in \mathbb{Z}$

$$\eta_+u_{k-1} + \eta_-u_{k+1} + \lambda u_k = v_k. \quad (2.30)$$

Similarly we can express $U_- = -i(\eta_+ - \eta_-) - V$ and the condition $U_-u = 0$ becomes

$$-i(\eta_+u_{k-1} - \eta_-u_{k+1}) - ik u_k = 0 \quad \forall k \in \mathbb{Z}. \quad (2.31)$$

Now inserting one equation into the other, one deduces that (2.30) and (2.31) are equivalent to (2.29) and this completes the proof. \square

Lemma 2.3. *Let $\lambda \in \mathbb{C}$ and $u \in \mathcal{D}'(SM)$ satisfies $U_-u = 0$, $(X + \lambda)^j u = 0$ for some $j \geq 1$. Set $u^{(\ell)} := (X + \lambda)^\ell u$ for each $\ell \leq j$ and assume that $u^{(j-1)} \neq 0$. Then for each $\ell \leq j - 2$*

$$(\Delta_M + \lambda(\lambda + 1))u_0^{(j-1)} = 0, \quad (\Delta_M + \lambda(\lambda + 1))^{j-1-\ell}u_0^{(\ell)} = (2\lambda + 1)^{j-1-\ell}u_0^{(j-1)}.$$

Proof. Denote by $u^{(\ell)} := (X + \lambda)^\ell u$, then $U_-u^{(\ell)} = 0$ for all ℓ by the fact that X preserves $\ker U_-$. Using (2.28) we get for each $\ell \leq j$

$$\begin{aligned} \Delta_M u_0^{(\ell)} &= -\Omega u_0^{(\ell)} = ((X - X^2)u^{(\ell)})_0 = ((X - 1)(\lambda u^{(\ell)} - u^{(\ell+1)}))_0 \\ &= -\lambda(\lambda + 1)u_0^{(\ell)} + (2\lambda + 1)u_0^{(\ell+1)} - u_0^{(\ell+2)} \end{aligned}$$

The result then follows from an easy induction. \square

Proposition 2.4. *Let $n \in \mathbb{N}$, assume that $u \in \mathcal{D}'(SM)$ satisfies $U_-u = 0$, $u_0 = 0$ and $(X - n)^j u = 0$ for some $j \in \mathbb{N}$, then $(X - n)u = 0$.*

Proof. Let $u^{(\ell)} := (X - n)^\ell u$. By Lemma 2.3, we have $u_0^\ell = 0$ for all $\ell = 0, \dots, j - 1$. Assume that $(X - n)u \neq 0$, then without loss of generality we can assume that $u^{(j-1)} \neq 0$. Then using that $(X - n)$ preserves $\ker U_-$, we get that $w := u^{(j-2)}$ and $v := u^{(j-1)}$ are non-zero distributions such that $U_-w = U_-v = 0$, $(X - n)w = v$, $(X - n)v = 0$ as well as $w_0 = v_0$. Let us next use the knowledge that $w_0 = v_0 = 0$ in order to obtain a contradiction. Lemma 2.2 applied to $(X - n)w = v$ and $(X - n)v = 0$ implies, that

$$2\eta_{\pm}w_{k\mp 1} = (n \mp k)w_k + v_k, \quad 2\eta_{\pm}v_{k\mp 1} = (n \mp k)v_k \quad (2.32)$$

From $v_0 = 0$ and the second equation of (2.32) we obtain that $v_k = 0$ for all $|k| < n$. Now this knowledge together with $w_0 = 0$ and the first recursion relation in (2.32)

leads to $w_k = 0$ for all $|k| < n$. Now let us consider the first equation of (2.32) with $k = n$

$$0 = 2\eta_+ w_{n-1} = v_n.$$

Using once more the second recursion relations (2.32) this implies $v_k = 0$ for all $k \geq n$. Analogously we obtain $v_{-n} = 0$ and using the recursion we see that $v_k = 0$ for all $k \neq 0$. Thus $v = 0$ which is the desired contradiction. \square

3. RUELLE RESONANCES FOR CO-COMPACT QUOTIENTS

In this section, we consider a co-compact discrete subgroup $\Gamma \subset G$ acting freely on \mathbb{H}^2 , so that $M := \Gamma \backslash \mathbb{H}^2$ is a smooth oriented compact hyperbolic surface, and we describe the Ruelle resonance spectrum and eigenfunctions. The characterization of the spectrum and eigenfunctions was done in [DFG] except for some special points localized at negative half-integers. Here we analyze those points as well, and we simplify the proof of the fact that the algebraic multiplicities and geometric multiplicities agree.

First we recall the result [FaSj, Theorems 1.4 and 1.7] in the case of geodesic flows (see also [BuLi, Theorem 1] or [DyZw, Propositions 3.1 and 3.3]):

Proposition 3.1. *Let (M, g) be a Riemannian manifold with Anosov geodesic flow. For each $N > 0$, the vector field X generating the flow is such that $\lambda \mapsto -X - \lambda$ is a holomorphic family of Fredholm operators of index 0*

$$-X - \lambda : \text{Dom}(X) \cap \mathcal{H}^N \rightarrow \mathcal{H}^N$$

for $\{\text{Re}(\lambda) > -N\}$, where \mathcal{H}^N is a Hilbert space such that $C^\infty(SM) \subset \mathcal{H}^N \subset \mathcal{D}'(SM)$. The spectrum of $-X$ on \mathcal{H}^N in $\{\text{Re}(\lambda) > -N\}$ is discrete, contained in $\{\text{Re}(\lambda) \leq 0\}$, and minus the residue

$$\Pi_{\lambda_0}^X := -\text{Res}_{\lambda_0}(-X - \lambda)^{-1}, \quad \text{Re}(\lambda_0) > -N$$

is a projector onto the finite dimensional space

$$\text{Res}_X(\lambda_0) := \{u \in \mathcal{H}^N; \exists j \in \mathbb{N}, (-X - \lambda_0)^j u = 0\}$$

of generalized eigenstates, satisfying $X\Pi_{\lambda_0}^X = \Pi_{\lambda_0}^X X$. The eigenvalues, generalized eigenstates and the Schwartz kernel of $\Pi_{\lambda_0}^X$ are independent of N and one has

$$\text{Res}_X(\lambda_0) = \{u \in \mathcal{D}'(SM); \text{WF}(u) \subset E_u^*, \exists j \in \mathbb{N}, (-X - \lambda_0)^j u = 0\}. \quad (3.1)$$

Notice that the description (3.1) of resonant states is not explicit in [FaSj] but it follows easily from that paper [FaSj] - a proof is given in Lemma 5.1 of [DFG]. The eigenvalues, eigenstates and generalized eigenstates are respectively called *Ruelle resonances*, *Ruelle resonant states* and *Ruelle generalized resonant states*. The existence of generalized resonant states which are not resonant states means that the algebraic

multiplicity of the eigenvalue is larger than the geometric multiplicity, in which case there are Jordan blocks in the matrix representing $(-X - \lambda_0)$ on $\text{Im}(\Pi_{\lambda_0}^X)$.

A direct corollary of Proposition 3.1 is that the L^2 -resolvent

$$\lambda \mapsto R_X(\lambda) := (-X - \lambda)^{-1}$$

of X extends from $\text{Re}(\lambda) > 0$ to \mathbb{C} as a meromorphic family of bounded operators $C^\infty(SM) \rightarrow \mathcal{D}'(SM)$, with poles the Ruelle resonances of $-X$. Indeed, Proposition 3.2 in [DyZw] shows that in the region $\text{Re}(\lambda) > C$ for some $C > 0$, the L^2 -resolvent $R_X(\lambda)$ is also bounded as a map $\mathcal{H}^N \rightarrow \mathcal{H}^N$, thus $R_X(\lambda)$ is the inverse of $-X - \lambda$ on \mathcal{H}^N in that half-space. By Proposition 3.1, $R_X(\lambda) : \mathcal{H}^N \rightarrow \mathcal{H}^N$ extends as a meromorphic family of bounded operators in $\text{Re}(\lambda) > -N$ and thus is meromorphic as a bounded map $C^\infty(SM) \rightarrow \mathcal{D}'(SM)$. Density of $C^\infty(SM)$ in \mathcal{H}^N and unique continuation in λ shows that $R_X(\lambda)$ is actually independent of N as a map $C^\infty(SM) \rightarrow \mathcal{D}'(SM)$.

We define the following spaces for $j \geq 1$

$$\text{Res}_X^j(\lambda_0) = \{u \in \mathcal{D}'(SM); \text{WF}(u) \subset E_u^*, (-X - \lambda_0)^j u = 0\}, \quad (3.2)$$

The operator $(-X - \lambda_0)$ is nilpotent on the finite dimensional space $\text{Res}_X(\lambda_0)$ and λ_0 is a Ruelle resonance if and only if $\text{Res}_X^1(\lambda_0) \neq 0$. The presence of Jordan blocks for λ_0 is equivalent to having $\text{Res}_X^j(\lambda_0) \neq \text{Res}_X^1(\lambda_0)$ for some $j > 1$. Let us define for $j \geq 1, m \geq 0$ the subspace

$$V_m^j(\lambda_0) := \{u \in \text{Res}_X^j(\lambda_0); U_-^{m+1}u = 0\}. \quad (3.3)$$

Obviously $V_m^j(\lambda_0) \subset V_{m+1}^j(\lambda_0)$ for each $m \geq 0, j \geq 1$. The spaces $V_0^j(\lambda_0)$ are spanned by the generalized resonant states which are invariant under the unstable horocycle flow. First, following [DFG], we have the

Lemma 3.2. *For each $\lambda_0 \in \mathbb{C}$, there exists $m \geq 0$ such that $\text{Res}_X^1(\lambda_0) = V_m^1(\lambda_0)$.*

Proof. By the commutation relation (2.8), for $u \in \text{Res}_X^1(\lambda_0)$ we have for each $m \geq 0$

$$(-X - \lambda_0 - m)U_-^m u = 0$$

thus for $m > -\text{Re}(\lambda_0)$, we get $U_-^m u = 0$ since we know there is no Ruelle resonance in $\text{Re}(\lambda) > 0$ by Proposition 3.1 and $\text{WF}(U_-^m u) \subset \text{WF}(u) \subset E_u^*$. \square

Next we analyze the generalized resonant states that are in $\ker U_-$, i.e. the spaces $V_0^j(\lambda_0)$ for $j \geq 1$. This part is essentially contained in [FIFo] (even though they do not consider the problem from the point of view of Ruelle resonances) but we provide a more geometric method by using the Poisson operator, with the advantage that this approach extends to the convex co-compact setting. Our proof does not use the representation theory of $\text{SL}_2(\mathbb{R})$ at all.

Theorem 3.3. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a smooth oriented compact hyperbolic surface and let SM be its unit tangent bundle.*

1) *For each $\lambda_0 \in \mathbb{C} \setminus (-\frac{1}{2} \cup -\mathbb{N})$ the pushforward map $\pi_{0*} : \mathcal{D}'(SM) \rightarrow \mathcal{D}'(M)$ restricts to a linear isomorphism of complex vector spaces*

$$\pi_{0*} : V_0^1(\lambda_0) \rightarrow \ker(\Delta_M + \lambda_0(1 + \lambda_0)) \quad (3.4)$$

where Δ_M is the Laplacian on M acting on functions, and there are no Jordan blocks, i.e. $V_0^j(\lambda_0) = V_0^1(\lambda_0)$ for $j > 1$.

2) *For $\lambda_0 = -\frac{1}{2}$, the Jordan blocks are of order 1, i.e. $V_0^j(-\frac{1}{2}) = V_0^1(-\frac{1}{2})$ for $j > 2$, the map*

$$\pi_{0*} : V_0^1(-\frac{1}{2}) \rightarrow \ker(\Delta_M - \frac{1}{4}) \quad (3.5)$$

is a linear isomorphism of complex vector spaces and

$$\pi_{0*} : V_0^2(-\frac{1}{2}) \rightarrow \ker(\Delta_M - \frac{1}{4}) \quad (3.6)$$

has a kernel of dimension $\dim \ker(\Delta_M - \frac{1}{4})$.

3) *For $\lambda_0 = -n \in -\mathbb{N}$, there are no Jordan blocks, i.e. $V_0^j(-n) = V_0^1(-n)$ if $j > 1$. The following map is an isomorphism of complex vector spaces*

$$\pi_{n*} \oplus \pi_{-n*} : V_0^1(-n) \rightarrow H_n(M) \oplus H_{-n}(M) \quad (3.7)$$

with $H_n(M) := \{u \in C^\infty(M; \mathcal{K}^n); \bar{\partial}u = 0\}$, $H_{-n}(M) := \{u \in C^\infty(M; \mathcal{K}^{-n}); \partial u = 0\}$.

Proof. Case $\lambda_0 \notin -1/2 \cup -\mathbb{N}$, injectivity. Since $X + \lambda_0$ is nilpotent on $V_0^2(\lambda_0)$, we can decompose this space into Jordan blocks and there is a non-trivial Jordan block if and only if $X + \lambda_0$ is not identically 0 on $V_0^2(\lambda_0)$. Assume λ_0 is a Ruelle resonance and let $u^{(0)} \in V_0^1(\lambda_0) \setminus \{0\}$ be a resonant state. If there is a Jordan block associated to $u^{(0)}$, there is $u^{(1)} \in V_0^2(\lambda_0)$ so that

$$(X + \lambda_0)u^{(0)} = 0, \quad (X + \lambda_0)u^{(1)} = u^{(0)}.$$

We will show that in fact this is not possible. The wave front set of $u^{(j)}$ is contained in E_u^* . We lift $u^{(0)}$ and $u^{(1)}$ on $S\mathbb{H}^2$ by $\pi_\Gamma : S\mathbb{H}^2 \rightarrow SM$ to $\tilde{u}^{(0)}, \tilde{u}^{(1)}$ so that $\gamma^* \tilde{u}^{(j)} = \tilde{u}^{(j)}$ for all $\gamma \in \Gamma$, $j = 0, 1$. Take the distribution $v^{(0)} := \Phi_-^{-\lambda_0} \tilde{u}^{(0)}$ on $S\mathbb{H}^2$ where Φ_- is defined by (2.3). It satisfies $Xv^{(0)} = 0$ as $X\Phi_- = -\Phi_-$. Then by (2.12) there exists $\omega^{(0)} \in \mathcal{D}'(\mathbb{S}^1)$ so that $\mathcal{Q}_-\omega^{(0)} = v^{(0)}$. Using $\gamma^* \tilde{u}^{(0)} = \tilde{u}^{(0)}$ together with (2.5), we have that for any $\gamma \in \Gamma$

$$\gamma^* \omega^{(0)} = N_\gamma^{-\lambda_0} \omega^{(0)} \text{ with } N_\gamma := |d\gamma|^{-1}.$$

Next we compute, using (2.11), that for $v^{(1)} := \Phi_-^{-\lambda_0} (\tilde{u}^{(1)} + \log(\Phi_-) \tilde{u}^{(0)})$,

$$Xv^{(1)} = 0, \quad U_-v^{(1)} = 0$$

and thus by (2.12) there is $\omega^{(1)} \in \mathcal{D}'(\mathbb{S}^1)$ such that $\mathcal{Q}_-\omega^{(1)} = v^{(1)}$. Since $\gamma^* \tilde{u}^{(1)} = \tilde{u}^{(1)}$ for all $\gamma \in \Gamma$, we get

$$\gamma^* v^{(1)} - N_\gamma^{-\lambda_0} v^{(1)} = N_\gamma^{-\lambda_0} \log(N_\gamma) v^{(0)}, \quad \gamma^* \omega^{(1)} - N_\gamma^{-\lambda_0} \omega^{(1)} = N_\gamma^{-\lambda_0} \log(N_\gamma) \omega^{(0)}. \quad (3.8)$$

We apply the Poisson operator \mathcal{P}_{λ_0} to $\omega^{(0)}$ and $\omega^{(1)}$, where \mathcal{P}_λ is defined in Lemma 2.1. Using the same lemma, we get that $\tilde{\varphi}_0 := \mathcal{P}_{\lambda_0}(\omega^{(0)}) = \pi_{0*}\tilde{u}^{(0)} \neq 0$ is Γ -invariant and satisfies

$$(\Delta_{\mathbb{H}^2} + \lambda_0(1 + \lambda_0))\tilde{\varphi}_0 = 0$$

on \mathbb{H}^2 , and thus descend to $\varphi_0 := \pi_{0*}u^{(0)}$ on M as a non-zero eigenfunction of Δ_M with eigenvalue $\lambda_0(1 + \lambda_0)$ and this has to be a smooth function by ellipticity. Since Δ_M is self-adjoint on $L^2(M)$ this implies $\lambda_0(1 + \lambda_0) \in \mathbb{R}$.

Now set $\psi := \mathcal{P}_{\lambda_0}(\omega^{(1)}) \in C^\infty(\mathbb{H}^2)$, we also get from (2.5), (3.8)

$$\gamma^*\psi - \psi = \mathcal{P}_{\lambda_0}(\log(N_\gamma)\omega^{(0)}), \quad (\Delta + \lambda_0(1 + \lambda_0))\psi = 0.$$

Now, for arbitrary $\omega \in \mathcal{D}'(\mathbb{S}^1)$, we make the observation, by Taylor expanding the equation $(\Delta_{\mathbb{H}^2} + \lambda(1 + \lambda))\mathcal{P}_\lambda(\omega) = 0$ with respect to λ at λ_0 that for $k \geq 1$

$$(\Delta_{\mathbb{H}^2} + \lambda_0(1 + \lambda_0))\frac{\partial_\lambda^k \mathcal{P}_{\lambda_0}(\omega)}{k!} + (2\lambda_0 + 1)\frac{\partial_\lambda^{k-1} \mathcal{P}_{\lambda_0}(\omega)}{(k-1)!} + \frac{\partial_\lambda^{k-2} \mathcal{P}_{\lambda_0}(\omega)}{(k-2)!} = 0, \quad (3.9)$$

$$\partial_\lambda^k \mathcal{P}_\lambda(\omega)(x) = \pi_{0*}(\Phi_-^\lambda(\log \Phi_-)^k \mathcal{Q}_-(\omega)),$$

and for each $\gamma \in \Gamma$, $\gamma^*(\partial_\lambda \mathcal{P}_{\lambda_0}(\omega^{(0)})) = \partial_\lambda \mathcal{P}_{\lambda_0}(\omega^{(0)}) + \mathcal{P}_{\lambda_0}(\log(N_\gamma)\omega^{(0)})$. Thus, since $\lambda_0 \neq -\frac{1}{2}$, we can set

$$\tilde{\varphi}_1 := -(1 + 2\lambda_0)^{-1}(\partial_\lambda \mathcal{P}_{\lambda_0}(\omega^{(0)}) - \psi)$$

and we get on \mathbb{H}^2

$$(\Delta_{\mathbb{H}^2} + \lambda_0(1 + \lambda_0))\tilde{\varphi}_1 = \tilde{\varphi}_0, \quad \text{and } \forall \gamma \in \Gamma, \gamma^*\tilde{\varphi}_1 = \tilde{\varphi}_1.$$

This means that $\tilde{\varphi}_1$ descends to a smooth function φ_1 on M , which satisfies the equation

$$(\Delta_M + \lambda_0(1 + \lambda_0))\varphi_1 = \varphi_0. \quad (3.10)$$

In fact, since

$$\psi = \pi_{0*}(\Phi_-^{\lambda_0} \mathcal{Q}_-(\omega^{(1)})) = \pi_{0*}(\tilde{u}^{(1)} + \log(\Phi_-)\tilde{u}^{(0)}) = \pi_{0*}(\tilde{u}^{(1)}) + \partial_\lambda \mathcal{P}_{\lambda_0}(\omega^{(0)}),$$

we notice that $\varphi_1 = (1 + 2\lambda_0)^{-1}\pi_{0*}u^{(1)}$. If M is compact (3.10) can only hold if $\varphi_0 = 0$, since $\lambda_0(1 - \lambda_0) \in \mathbb{R}$ and thus

$$\|\varphi_0\|_{L^2(M)}^2 = \langle (\Delta_M + \lambda_0(1 + \lambda_0))\varphi_1, \varphi_0 \rangle = 0.$$

But $\varphi_0 = 0$ implies $\omega^{(0)} = 0$ by injectivity of \mathcal{P}_{λ_0} , and thus $u^{(0)} = 0$, which leads to a contradiction and thus $u^{(1)}$ does not exist. We have thus proved the non-existence of Jordan blocks if $\lambda_0 \neq -\frac{1}{2}$. It is also clear that, assuming now that there is no Jordan block, we can take the argument above and ignore the $u^{(1)}, v^{(1)}, \omega^{(1)}, \tilde{\varphi}_1$ terms, it shows that $\varphi_0 = \pi_{0*}u^{(0)}$ is in $\ker(\Delta_M + \lambda_0(1 + \lambda_0))$ and is nonzero if $u^{(0)} \neq 0$, and the map (3.4) is then injective.

Case $\lambda_0 \notin -1/2 \cup -\mathbb{N}$, surjectivity. To construct the reciprocal map, we proceed as follows: if $(\Delta_M + \lambda_0(1 + \lambda_0))\varphi_0 = 0$ with $\varphi_0 \neq 0$, then by the surjectivity of \mathcal{P}_{λ_0} in

Lemma 2.1, there is $\omega^{(0)} \in \mathcal{D}'(\mathbb{S}^1)$ such that $\mathcal{P}_{\lambda_0}(\omega^{(0)}) = \tilde{\varphi}_0$ where $\tilde{\varphi}_0$ is the lift of φ_0 to \mathbb{H}^2 . Using injectivity of \mathcal{P}_{λ_0} and $\gamma^*\tilde{\varphi}_0 = \tilde{\varphi}_0$ for all $\gamma \in \Gamma$, we get $\gamma^*\omega^{(0)} = N_\gamma^{-\lambda_0}\omega^{(0)}$. This implies that $\tilde{u}^{(0)} := \Phi_-^{\lambda_0}\mathcal{Q}_-(\omega^{(0)})$ is Γ invariant by (2.5) and that $(X + \lambda_0)\tilde{u}^{(0)} = 0$ and $U_-\tilde{u}^{(0)} = 0$ by (2.12). Then $\tilde{u}^{(0)}$ descends to an element $u^{(0)} \in \mathcal{D}'(SM)$ satisfying $(X + \lambda_0)u^{(0)} = 0$ and $U_-u^{(0)} = 0$. Now the differential operator U_- is elliptic outside $E_u^* \oplus E_0^*$ and $(X + \lambda_0)$ is elliptic outside $E_u^* \oplus E_s^*$, thus $\text{WF}(u^{(0)}) \subset E_u^*$. Using the characterization (3.1) this implies that $u^{(0)}$ is a Ruelle resonant state.

Case $\lambda_0 = -\frac{1}{2}$, Jordan block and kernel. We will show that there can be a Jordan block of order 1 but no Jordan blocks of order 2. Assume there is a Jordan block of order 2, i.e.

$$(X + \lambda_0)u^{(0)} = 0, \quad (X + \lambda_0)u^{(1)} = u^{(0)}, \quad (X + \lambda_0)u^{(2)} = u^{(1)},$$

for some $u^{(j)} \in V_0^{j+1}(\lambda_0)$. Define $\tilde{u}^{(j)} = \pi_\Gamma^*u^{(j)}$ their lifts to $S\mathbb{H}^2$, let $v^{(0)} := \Phi_-^{-\lambda_0}\tilde{u}^{(0)}$ and $v^{(1)} := \Phi_-^{-\lambda_0}(\tilde{u}^{(1)} + \log(\Phi_-)\tilde{u}^{(0)})$ as before, and

$$v^{(2)} := \Phi_-^{-\lambda_0}(\tilde{u}^{(2)} + \log(\Phi_-)\tilde{u}^{(1)} + \frac{1}{2}(\log \Phi_-)^2\tilde{u}^{(0)}).$$

Then $Xv^{(j)} = 0$, $U_-v^{(j)} = 0$, and $\gamma^*v^{(j)} = N_\gamma^{-\lambda_0} \sum_{\ell=0}^j \frac{(\log N_\gamma)^\ell}{\ell!} v^{(j-\ell)}$ thus $v^{(j)} = \mathcal{Q}_-(\omega^{(j)})$ for some $\omega^{(j)} \in \mathcal{D}'(\mathbb{S}^1)$ satisfying $\gamma^*\omega^{(j)} = N_\gamma^{-\lambda_0} \sum_{\ell=0}^j \frac{(\log N_\gamma)^\ell}{\ell!} \omega^{(j-\ell)}$. Set

$$\begin{aligned} \tilde{\varphi}_0 &= \mathcal{P}_{\lambda_0}(\omega^{(0)}) = \pi_{0*}\tilde{u}^{(0)}, \quad \tilde{\varphi}_1 := \mathcal{P}_{\lambda_0}(\omega^{(1)}) - \partial_\lambda \mathcal{P}_{\lambda_0}(\omega^{(0)}) = \pi_{0*}\tilde{u}^{(1)}, \\ \tilde{\varphi}_2 &:= \mathcal{P}_{\lambda_0}(\omega^{(2)}) - \partial_\lambda \mathcal{P}_{\lambda_0}(\omega^{(1)}) + \frac{1}{2}\partial_\lambda^2 \mathcal{P}_{\lambda_0}(\omega^{(0)}) = \pi_{0*}\tilde{u}^{(2)}, \end{aligned}$$

then using (3.9) we get $(\Delta_{\mathbb{H}^2} - \frac{1}{4})\tilde{\varphi}_j = 0$ for $j = 0, 1$ and $\gamma^*\tilde{\varphi}_j = \tilde{\varphi}_j$ for all $\gamma \in \Gamma$. As for the case $\lambda_0 \neq -\frac{1}{2}$, $\tilde{\varphi}_j$ descend to smooth functions φ_j on M and, since $(\Delta_M - \frac{1}{4})\varphi_2 = \varphi_0$, we see that $\|\varphi_0\|_{L^2(M)} = 0$ if φ_2 is non-zero, leading to a contradiction. This shows that the Jordan block for X at $\lambda_0 = -\frac{1}{2}$ can be only of order 1, and from the injectivity of the Poisson-Helgason transformation we know that $\varphi_0 \neq 0$ is a non-zero element of $\ker(\Delta_M - \frac{1}{4})$. Thus the map (3.5) is injective and the map (3.6) has a kernel of dimension less or equal to $\dim(\ker(\Delta_M - \frac{1}{4}))$.

Case $\lambda_0 = -\frac{1}{2}$, surjectivity. Assume $\varphi^{(0)} \in \ker(\Delta_M - \frac{1}{4})$ is non zero, then by Lemma 2.1 there is $\omega^{(0)} \in \mathcal{D}'(SM)$ such that $\mathcal{P}_{-1/2}(\omega^{(0)}) = \tilde{\varphi}_0$ where $\tilde{\varphi}_0$ is the lift of φ_0 on \mathbb{H}^2 . We use (2.17) and have the expansion as $t \rightarrow 0$ for each $\chi \in C^\infty(\mathbb{S}^1)$

$$\int_{\mathbb{S}^1} \mathcal{P}_{-1/2}(\omega^{(0)})\left(\frac{2-t}{2+t}\nu\right)\chi(\nu)dS(\nu) = \sqrt{t}(-\sqrt{2}\log(t)\langle\omega^{(0)}, \chi\rangle_{\mathbb{S}^1} + \langle\omega^{(1)}, \chi\rangle_{\mathbb{S}^1} + \mathcal{O}(t\log t))$$

for some non-zero $\omega^{(1)} \in \mathcal{D}'(\mathbb{S}^1)$. Notice that if $t(x) := 2\frac{1-|x|}{1+|x|}$ on \mathbb{H}^2 , we have $\gamma^*t(|x|\nu) = |d\gamma(\nu)|t(x) + \mathcal{O}(t(x)^2)$ as $|x| \rightarrow 1$ for each isometry $\gamma \in G$ and $\nu \in \mathbb{S}^1$. Then, since $\gamma^*\tilde{\varphi}_0 = \tilde{\varphi}_0$ for each $\gamma \in \Gamma$, we get

$$\gamma^*\omega^{(0)} = N_\gamma^{1/2}\omega^{(0)}, \quad \gamma^*\omega^{(1)} = N_\gamma^{1/2}\omega^{(1)} - \sqrt{2}N_\gamma^{1/2}(\log N_\gamma)\omega^{(0)}$$

for each $\gamma \in \Gamma$. Define $\tilde{u}^{(0)} := \Phi_-^{-1/2} \mathcal{Q}_-(\omega^{(0)})$ and $\tilde{u}^{(1)} := \Phi_-^{-1/2} \mathcal{Q}_-(\omega^{(1)}) + \sqrt{2}(\log \Phi_-) \tilde{u}^{(0)}$, which are Γ -invariant and satisfy

$$(X - \frac{1}{2})\tilde{u}^{(0)} = 0, \quad (X - \frac{1}{2})\tilde{u}^{(1)} = \tilde{u}^{(0)}, \quad U_- \tilde{u}^{(0)} = U_- \tilde{u}^{(1)} = 0.$$

These distributions thus descend to SM and are non-zero resonant and generalized resonant states since they also have wave-front sets in E_u^* by the same argument as for $\lambda \neq -1/2$. In particular any element in $\ker(\Delta_M - \frac{1}{4})$ this produces a Jordan block of order 1 and consequently the map (3.6) has a kernel of dimension $\dim \ker(\Delta_M - \frac{1}{4})$.

Case $\lambda_0 \in -\mathbb{N}$, injectivity. Let $\lambda_0 = -n$ with $n \in \mathbb{N}$. First, if $u \in V_0^k(-n)$, we have $u_0 \in \ker(\Delta_M + n(n-1))$. From the positivity of the Laplacian we get $u_0 = 0$ except possibly if $n = -1$ where u_0 must be constant. Let us show that in fact $u_0 = 0$ if $n = -1$: if $u \in V_0^1(-1)$ satisfies $u_0 = 1$, we have $2\eta_- u_1 = 1$ by (2.29), but then

$$0 < \|u_0\|_{L^2}^2 = 2\langle \eta_- u_1, u_0 \rangle_{L^2} = -2\langle u_1, \eta_+ u_0 \rangle_{L^2} = 0$$

leading to a contradiction. Thus $u_0 = 0$ for all n if $u \in V_0^1(-n)$. Thus Proposition 2.4 implies that there are no Jordan Blocks for the Ruelle resonances. By the proof of Proposition 2.4, the Fourier components u_k of u in the fibers satisfy $u_k = 0$ for $|k| < n$. By (2.29) we have $\eta_- u_n = 0$ and $\eta_+ u_{-n} = 0$, and thus by (2.27) we get $u_n \in H_n(M)$ and $u_{-n} \in H_{-n}(M)$. Furthermore note that $u_n = 0$ implies after iteratively applying (2.29), that $u_k = 0$ for all $k > n$, and similarly if $u_{-n} = 0$. The map (3.7) is thus injective.

Case $\lambda_0 \in -\mathbb{N}$, surjectivity. Conversely, we want to prove that for each $u_n \in H_n(M)$, there is a $u \in \mathcal{D}'(SM)$ so that $(X - n)u = 0$ and $U_- u = 0$. We construct u as a formal sum $u = \sum_{k \geq n} u_k$ where u_k are in $\pi_k^*(C^\infty(M; \mathcal{K}^k))$, that we will thus freely identify with sections of \mathcal{K}^k . We set $u_k = 0$ for all $k < n$ and we define recursively for $k > n$

$$u_k := \frac{2}{n-k} \eta_+ u_{k-1} \tag{3.11}$$

and set $u_{-k} = \overline{u_k}$. Clearly u_k are smooth and in fact analytic since u_n is. First let us show that the formal series $u = \sum_{k \in \mathbb{Z}} u_k$ fulfills $(X - n)u = 0$ and $U_- u = 0$. According to Lemma 2.2 these conditions are equivalent to the fact that

$$2\eta_+ u_{k-1} = (n-k)u_k \tag{3.12}$$

$$2\eta_- u_{k+1} = (n+k)u_k \tag{3.13}$$

holds for all $k \in \mathbb{Z}$. We see, that (3.12) is already fulfilled for all $k \in \mathbb{Z}$ by our recursive definition of the Fourier modes via (3.11). For $k < n-1$ also the condition (3.13) is fulfilled, as we have set all Fourier modes to be identically zero. For $k = n-1$ (3.13) is true because we identified u_n with a holomorphic section in \mathcal{K}^n and the case $k \geq n$ follows by induction from the following fact: for each $\ell \in \mathbb{N}_0$, if

$2\eta_- u_{n+\ell} = (2n + \ell - 1)u_{n+\ell-1}$ holds, then $2\eta_- u_{n+\ell+1} = (2n + \ell)u_{n+\ell}$. This fact is a direct consequence of the commutation relation

$$[\eta_+, \eta_-] = -\frac{i}{2}V \quad (3.14)$$

since

$$\begin{aligned} 2\eta_- u_{n+\ell+1} &\stackrel{(3.11)}{=} -\frac{4}{\ell+1}\eta_- \eta_+ u_{n+\ell} \stackrel{(3.14)}{=} -\frac{4}{\ell+1}\left(\eta_+ \eta_- + \frac{i}{2}V\right)u_{n+\ell} \\ &= -\frac{4}{\ell+1}\left(\frac{-\ell(2n+\ell-1)}{4} - \frac{(n+\ell)}{2}\right)u_{n+\ell} = (2n+\ell)u_{n+\ell}. \end{aligned}$$

Next we need to show that the formal sum $u = \sum_k u_k$ defines a distribution, and it suffices to check that $\|u_k\|_{L^2} = \mathcal{O}(|k|^N)$ for some N as $|k| \rightarrow \infty$. Let us give an argument which is close to the approach of Flaminio-Forni [FFo]: let $k > n$ and consider

$$\begin{aligned} \|u_k\|_{L^2(SM)}^2 &= \langle u_k, u_k \rangle_{L^2} = \left(\frac{2}{n-k}\right) \langle u_k, \eta_+ u_{k-1} \rangle_{L^2} = -\left(\frac{2}{n-k}\right) \langle \eta_- u_k, u_{k-1} \rangle_{L^2} \\ &= -\left(\frac{k+n-1}{n-k}\right) \langle u_{k-1}, u_{k-1} \rangle_{L^2} = \left(\frac{k+n-1}{k-n}\right) \|u_{k-1}\|_{L^2}^2. \end{aligned}$$

Thus recursively we obtain for $\ell > 0$ (here Γ is the Euler Gamma function)

$$\|u_{n+\ell}\|_{L^2(SM)}^2 = \Pi_{\ell,n} \|u_n\|_{L^2(SM)}^2 \quad \text{where } \Pi_{\ell,n} = \frac{\Gamma(2n+\ell)}{\Gamma(\ell+1)\Gamma(2n)} = \prod_{r=1}^{\ell} \frac{2n-1+r}{r}.$$

Now it is direct to check that $\Pi_{\ell,n} = \mathcal{O}(\ell^N)$ for some N . The same argument works with u_{-n} . \square

Here notice that by Riemann-Roch theorem, the spaces $H_n(M)$ have complex dimension

$$\dim H_n = \begin{cases} \frac{1}{2}(2n-1)|\chi(M)| & \text{if } n > 1 \\ \frac{1}{2}|\chi(M)| + 1 & \text{if } n = 1 \end{cases} \quad (3.15)$$

where $\chi(M)$ is the Euler characteristic of M .

To conclude this section, we describe the full Ruelle resonance spectrum of X by using Theorem 3.3.

Corollary 3.4. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a smooth oriented compact hyperbolic surface and let SM be its unit tangent bundle. Then for each λ_0 with $\text{Re}(\lambda_0) \leq 0$, $k \in \mathbb{N}$, and $j \in \mathbb{N}$, the operator U_+^k is injective on $V_0^j(\lambda_0 + k)$ if $\lambda_0 + k \neq 0$ and*

$$V_k^j(\lambda_0) = U_+^k(V_0^j(\lambda_0 + k)) \oplus V_{k-1}^j(\lambda_0).$$

In other words, we get $V_k^j(\lambda_0) = \bigoplus_{\ell=0}^k U_+^\ell(V_0^j(\lambda_0 + \ell))$.

Proof. First, we have $U_+^k(V_0^j(\lambda_0 + k)) \subset V_k^j(\lambda_0)$ since

$$(X + \lambda_0)^j U_+^k = U_+(X + \lambda_0 + 1)^j U_+^{k-1} = \cdots = U_+^k(X + \lambda_0 + k)^j \quad (3.16)$$

and similarly U_-^k maps $V_k^j(\lambda_0)$ to $V_0^j(\lambda_0 + k)$ with kernel $V_{k-1}^j(\lambda_0)$, so that it remains to show that the map

$$U_-^k U_+^k : V_0^j(\lambda_0 + k) \rightarrow V_0^j(\lambda_0 + k)$$

is one-to-one. Of course, since $V_0^j(\lambda_0 + k) = 0$ if $\operatorname{Re}(\lambda_0) + k > 0$, it suffices to assume that $\operatorname{Re}(\lambda_0) \leq -k$. But a direct computation using $[U_+, U_-] = 2X$ gives

$$U_-^k U_+^k = k! \prod_{\ell=1}^k (-2X - k + \ell) \text{ on } V_0^j(\lambda_0 + k).$$

When $j = 1$, $X = -(\lambda_0 + k)\operatorname{Id}$ on $V_0^1(\lambda_0 + k)$, thus

$$\prod_{\ell=1}^k (-2X - k + \ell) = \prod_{\ell=1}^k (2\lambda_0 + k + \ell) \operatorname{Id} \text{ on } V_0^1(\lambda_0 + k).$$

This is invertible if $\lambda_0 \neq -k$ since we know that $\operatorname{Re}(\lambda_0) + k \leq 0$. Now we can do an induction for $j > 1$: assume that $\ker U_-^k U_+^k \cap V_0^{j-1}(\lambda_0 + k) = 0$, then using $(X + \lambda_0)U_-^k U_+^k = U_-^k U_+^k(X + \lambda_0)$, we see that if $u \in V_0^j(\lambda_0 + k) \cap \ker U_-^k U_+^k$, $(X + \lambda_0)u \in V_0^{j-1}(\lambda_0 + k) \cap \ker U_-^k U_+^k = 0$ and this completes the argument when $\lambda_0 + k \neq 0$. \square

This Theorem describes the full Ruelle resonance spectrum in terms of the resonances associated to resonant states in $\ker U_-$. Indeed the only case which is not obvious is when $\lambda_0 = -n \in -\mathbb{N}$, and when we want to compute $V_{n+k}^j(-n)$ for $k \geq 0$. If $u \in V_n^j(-n)$, then $w := U_-^n u$ is in $V_0^j(0)$ which is non trivial only if $j = 1$, in which case $w = \text{constant}$. But then $\|w\|_{L^2}^2 = (-1)^n \langle U_-^{2n} w, u \rangle = 0$ and we deduce that $V_n^j(-n) = V_{n-1}^j(-n)$ and $V_{n+k}^j(-n) = V_{n-1}^j(-n)$ for all $k \geq 0$.

4. RUELLE RESONANCES FOR CONVEX CO-COMPACT QUOTIENTS

In this section, we consider the case of a convex co-compact subgroup $\Gamma \subset G$ of isometries of \mathbb{H}^2 .

4.1. Geometry and dynamics of convex co-compact surfaces. The manifold $M = \Gamma \backslash \mathbb{H}^2$ is a non-compact complete smooth hyperbolic manifold with infinite volume but finitely many topological ends. Moreover, M can be compactified to the smooth manifold $\overline{M} = \Gamma \backslash (\mathbb{H}^2 \cup \Omega_\Gamma)$, if $\Omega_\Gamma \subset \mathbb{S}^1$ is the set of discontinuity of Γ . As in Section 2.3, we will denote by $\Lambda_\Gamma \subset \mathbb{S}^1$ the limit set of Γ . In fact, M is conformally compact in the sense of Mazzeo-Melrose [MaMe]: there is a smooth boundary defining function ρ such that $\bar{g} := \rho^2 g$ extends as a smooth metric on \overline{M} .

The group Γ is a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and acts on the Riemann sphere $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ as conformal transformations, it preserves the unit disk \mathbb{H}^2 and its complement $\overline{\mathbb{C}} \setminus \mathbb{H}^2$. Equivalently, by conjugating by $(z - i)/(z + i)$, Γ acts by conformal transformations on $\overline{\mathbb{C}}$ as a subgroup of $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$ and it preserves the half-planes $\mathbb{H}_\pm^2 := \{z \in \mathbb{C}; \pm \mathrm{Im}(z) > 0\}$. The half-planes are conformally equivalent through $z \mapsto \bar{z}$ if we put the opposite orientation on \mathbb{H}_+^2 and \mathbb{H}_-^2 . In this model the boundary is the compactified real line $\partial\mathbb{H}_\pm = \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and the limit set is a closed subset Λ_Γ of $\overline{\mathbb{R}}$, and its complement in $\overline{\mathbb{R}}$ is still denoted by Ω_Γ . Since $\overline{\gamma(\bar{z})} = \gamma(z)$ for each $\gamma \in \Gamma$, the quotients $M_+ := \Gamma \backslash (\mathbb{H}_+^2 \cup \Omega_\Gamma)$ and $M_- := \Gamma \backslash (\mathbb{H}_-^2 \cup \Omega_\Gamma)$ are smooth surface with boundaries, equipped with a natural conformal structure and M_+ is conformally equivalent to M_- . The surface $\Gamma \backslash (\overline{\mathbb{C}} \setminus \Lambda_\Gamma)$ is a compact surface diffeomorphic to the gluing $M_2 := M_+ \cup M_-$ of M_+ and M_- along their boundaries, moreover it is equipped with a smooth conformal structure which restricts to that of M_\pm .

We denote by $\mathcal{I} : M_2 \rightarrow M_2$ the involution fixing $\partial\overline{M}$ and derived from $z \mapsto \bar{z}$ when viewing Γ as acting in $\overline{\mathbb{C}} \setminus \Lambda_\Gamma$. The interior of M_+ and M_- are isometric if we put the hyperbolic metric $|dz|^2/(\mathrm{Im}(z))^2$ on \mathbb{H}_\pm^2 , and they are isometric to the hyperbolic surface we called M above. The conformal class of M_\pm corresponds to the conformal class of \bar{g} on \overline{M} as defined above. We identify M_+ with \overline{M} and define $H_{\pm n}(M)$ as the finite dimensional real vector spaces

$$\begin{aligned} H_n(M) &:= \{f|_{M_+}; f \in C^\infty(M_2; \mathcal{K}^n), \bar{\partial}f = 0, \mathcal{I}^*f = \bar{f}\}, \\ H_{-n}(M) &:= \{f|_{M_+}; f \in C^\infty(M_2; \mathcal{K}^{-n}), \partial f = 0, \mathcal{I}^*f = \bar{f}\}. \end{aligned} \tag{4.1}$$

Note that $f \in H_n(M)$ is equivalent to say that $\bar{\partial}f = 0$ with $\iota_{\partial\overline{M}}^* f$ real-valued, if $\iota_{\partial\overline{M}} : \partial\overline{M} \rightarrow \overline{M}$ is the inclusion map and $\iota_{\partial\overline{M}}^* f$ is the symmetric tensor on $\partial\overline{M}$ defined by

$$\forall(x, v) \in T\partial\overline{M}, \iota_{\partial\overline{M}}^* f(x)(\otimes^m v) = f(\iota_{\partial\overline{M}}(x))(\otimes^m d\iota_{\partial\overline{M}}.v)$$

where $T\partial\overline{M}$ is the real tangent space of $\partial\overline{M}$.

The dimension of $H_{\pm n}$ can be computed as follows: let

$$H_n(M_2) := \{f \in C^\infty(M_2; \mathcal{K}^n), \bar{\partial}f = 0\}$$

which can be viewed as complex and real vector space, with complex dimension that can be calculated by the Riemann-Roch theorem (3.15) and the fact that $\chi(M_2) = 2\chi(M)$. Now let $A : H_n(M_2) \rightarrow H_n(M_2)$ be the map $Af = \overline{\mathcal{I}^*f}$ satisfying $A^2 = \mathrm{Id}$. We have $H_n(M) = \ker(A - \mathrm{Id})$ as real vector spaces. The map $f \mapsto if$ is an isomorphism of real vector spaces from $H_n(M)$ to $\ker(A + \mathrm{Id}) \subset H_n(M_2)$ and thus we deduce that the real dimension of $H_n(M)$ equals the complex dimension of $H_n(M_2)$ and we get

$$\dim_{\mathbb{R}} H_n(M) = \begin{cases} (2n - 1)|\chi(M)| & \text{if } n > 1, \\ |\chi(M)| + 1 & \text{if } n = 1. \end{cases} \tag{4.2}$$

for a non-elementary group. For an elementary group, $\chi(M) = 0$ and $H_n(M)$ has real dimension equal to 1 (M_2 is a flat torus).

By [Gr], there exists a collar near $\partial\overline{M}$ and a diffeomorphism $\psi : [0, \epsilon)_r \times \partial\overline{M} \rightarrow \psi([0, \epsilon) \times \partial\overline{M}) \subset \overline{M}$ such that $\psi^*g = \frac{dr^2 + h(r)}{r^2}$ where $r \mapsto h(r)$ is a smooth family of metrics on $\partial\overline{M}$. We can choose $\rho = r \circ \psi^{-1}$ as boundary defining function. It is then clear that the hypersurfaces $\rho = \rho_0$ are strictly convex if $0 < \rho_0 < \epsilon$ is small enough, and therefore there exists a geodesically convex compact domain $Q \subset M$ with smooth boundary $\partial Q = \psi(\{\rho_0\} \times \partial\overline{M})$. Each geodesic $(x(t))_{t \in [0, t_0]}$ in Q with $x(t_0) \in \partial Q$ can be extended to a geodesic $(x(t))_{t \in [0, \infty)}$ so that $x(t) \in M \setminus Q$ for all $t > t_0$ and $x(t) \rightarrow \partial\overline{M}$ as $t \rightarrow +\infty$. The surface M is of the form $M = N \cup (\cup_{i=1}^{n_f} F_i)$ where N is a compact hyperbolic surface with geodesic boundary, called the *convex core*, and F_1, \dots, F_{n_f} are n_f ends isometric to funnels (see e.g. [Bo, Section 2.4])

$$(F_i, g) \simeq [0, \infty)_t \times (\mathbb{R}/\ell_i\mathbb{Z})_\theta \text{ with metric } dt^2 + \cosh(t)^2 d\theta^2 \quad (4.3)$$

for some $\ell_i > 0$ corresponding to the length of the geodesic of N where F_i is attached. The boundary of the compactification is $\partial\overline{M} = \cup_{i=1}^{n_e} S_i$ where $S_i := \mathbb{R}/\ell_i\mathbb{Z}$. We will now choose the function ρ to be equal to $\rho = 2e^{-t}$ in F_i so that $\overline{g}|_{S_i} = d\theta^2$. In the (ρ, θ) coordinates, the metric in F_i is given by $g = \rho^{-2}(d\rho^2 + (1 + \rho^2/4)^2 d\theta^2)$. Using the (ρ, θ) coordinates, we say that a function $f \in C^\infty(\overline{M})$ is *even* if in each F_i ,

$$\forall k \in \mathbb{N}_0, \partial_\rho^{2k+1} f|_{\rho=0} = 0$$

and we denote by $C_{\text{ev}}^\infty(\overline{M})$ the space of even functions. Note that

$$C_{\text{ev}}^\infty(\overline{M}) = \{f|_{M_+}; f \in C^\infty(M_2), \mathcal{I}^* f = f\} \quad (4.4)$$

after identifying \overline{M} and M_+ , and if \mathcal{I} is the involution on M_2 defined above. We refer to [Gu1] for detailed discussions about even metrics and functions on asymptotically hyperbolic manifolds.

The incoming (-) and outgoing (+) tails $\Lambda_\pm \subset SM$ of the flow are the sets

$$\Lambda_\pm := \{(x, v) \in SM; \exists t_0 \geq 0, \varphi_{\mp t}(x, v) \in Q, \forall t \in [t_0, \infty)\}.$$

In view of the property of Q , the set Λ_\pm is also the set of points (x, v) such that $\pi_0(\varphi_{\mp t}(x, v))$ does not tend to $\partial\overline{M}$ in \overline{M} as $t \rightarrow +\infty$. The trapped set $K \subset Q$ is the closed flow-invariant set

$$K := \Lambda_+ \cap \Lambda_-.$$

It is a direct observation that, if $B_\pm : S\mathbb{H}^2 \rightarrow \mathbb{S}^1$ are the endpoint maps defined in Section 2.1 and $\pi_\Gamma : S\mathbb{H}^2 \rightarrow SM$ the covering map defined in Section 2.3, then

$$\Lambda_\pm = \pi_\Gamma(\tilde{\Lambda}_\pm), \text{ where } \tilde{\Lambda}_\pm := B_\mp^{-1}(\Lambda_\Gamma). \quad (4.5)$$

Indeed, a point $(x, v) \in SM$ is in Γ_- if and only if it is trapped in the future, meaning that $\varphi_t(x, v) \not\rightarrow \partial\overline{M}$ as $t \rightarrow +\infty$. By lifting to the covering $S\mathbb{H}^2$ and taking $(\tilde{x}, \tilde{v}) \in$

$S\mathbb{H}^2$ so that $\pi_\Gamma(\tilde{x}, \tilde{v}) = (x, v)$, this means that the semi-geodesic starting at \tilde{x} tangent to \tilde{v} does not have endpoint in Ω_Γ , i.e, $B_+(\tilde{x}, \tilde{v}) \in \Lambda_\Gamma$. The bundle $T(SM)$ has a continuous (in fact smooth in our case) flow-invariant splitting (2.20) and we set E_0^*, E_u^*, E_s^* the dual splitting defined by (2.21). We define their restriction to the incoming/outgoing tails by

$$E_+^* := E_u^*|_{\Lambda_+}, \quad E_-^* := E_s^*|_{\Lambda_-}. \quad (4.6)$$

4.2. Ruelle resonances and generalized resonant states. To define Ruelle resonances and resonant states, we first need to recall the following result from [DyGu].

Theorem 4.1. *If $M = \Gamma \backslash \mathbb{H}^2$ is a convex co-compact hyperbolic surface, then the generator X of the geodesic flow on SM has a resolvent $R_X(\lambda) := (-X - \lambda)^{-1}$ that admits a meromorphic extension from $\{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) > 0\}$ to \mathbb{C} as a family of bounded operators $C_c^\infty(SM) \rightarrow \mathcal{D}'(SM)$. The resolvent $R_X(\lambda)$ has finite rank polar part at each pole λ_0 and the polar part is of the form*

$$-\sum_{j=1}^{J(\lambda_0)} \frac{(-X - \lambda_0)^{j-1} \Pi_{\lambda_0}^X}{(\lambda - \lambda_0)^j}, \quad J(\lambda_0) \in \mathbb{N}$$

for some finite rank projector $\Pi_{\lambda_0}^X$ commuting with X . Moreover $u \in \mathcal{D}'(SM)$ is in the range of $\Pi_{\lambda_0}^X$ if and only if $(X + \lambda_0)^{J(\lambda_0)} u = 0$ with u supported in Λ_+ and $\operatorname{WF}(u) \subset E_+^*$.

We define *Ruelle resonance*, *generalized Ruelle resonant state* and *Ruelle resonant state* as respectively a pole λ_0 of $R_X(\lambda)$, an element in $\operatorname{Im}(\Pi_{\lambda_0}^X)$ and an element in $\operatorname{Im}(\Pi_{\lambda_0}^X) \cap \ker(-X - \lambda_0)$. Define like in (3.2) and (3.3) the spaces

$$\operatorname{Res}_X^j(\lambda_0) := \{u \in \mathcal{D}'(SM); \operatorname{supp}(u) \subset \Lambda_+, \operatorname{WF}(u) \subset E_u^*, (-X - \lambda_0)^j u = 0\}, \quad (4.7)$$

$$V_m^j(\lambda_0) := \{u \in \operatorname{Res}_X^j(\lambda_0); U_-^{m+1} u = 0\}. \quad (4.8)$$

As in the compact case, the operator $(-X - \lambda_0)$ is nilpotent on the finite dimensional space $\operatorname{Res}_X(\lambda_0) := \cup_{j \geq 1} \operatorname{Res}_X^j(\lambda_0)$ and λ_0 is a Ruelle resonance if and only if $\operatorname{Res}_X^1(\lambda_0) \neq 0$. The presence of Jordan blocks for λ_0 is equivalent to having $\operatorname{Res}_X^k(\lambda_0) \neq \operatorname{Res}_X^1(\lambda_0)$ for some $k > 1$.

We make the important remark for what follows that when $\lambda_0 \in \mathbb{R}$, $V_m^j(\lambda_0)$ can be considered both as a real and as a complex vector space which admits a basis of real-valued distributions, and its dimension is the number of elements of the basis.

4.3. Quantum resonances and scattering operator. Scattering theory on these surfaces has been largely developed by Guillopé-Zworski [GuZw1, GuZw2] and a comprehensive description is given in the book of Borthwick [Bo]. Quantum resonances are resonances of the Laplacian $\Delta_M = d^*d$ on $M = \Gamma \backslash \mathbb{H}^2$, and these are defined as poles of the meromorphic extension of the resolvent of Δ_M . The essential spectrum for

Δ_M is $[1/4, \infty)$ and the natural resolvent to consider is $R_\Delta(s) := (\Delta_M - s(1-s))^{-1}$ which is meromorphic in $\operatorname{Re}(s) > 1/2$ as a family of bounded operators on $L^2(M)$, with finitely many poles at

$$\sigma(\Delta_M) := \left\{s \in \left(\frac{1}{2}, 1\right); \ker_{L^2}(\Delta_M - s(1-s)) \neq 0\right\}$$

corresponding to the L^2 -eigenvalues in $(0, 1/4)$. By usual spectral theory, each pole at such s_0 is simple and the residue is

$$\operatorname{Res}_{s=s_0}(R_\Delta(s)) = \frac{\Pi_{s_0}^\Delta}{(2s_0 - 1)}$$

where $\Pi_{s_0}^\Delta$ the orthogonal projector on $\ker_{L^2}(\Delta_M - s_0(1-s_0))$; see [PaPe, Lemma 4.8] for example. The meromorphic extension of the resolvent was proved in [MaMe, GuZw1], we now recall this result:

Theorem 4.2. *If $M = \Gamma \backslash \mathbb{H}^2$ is a convex co-compact hyperbolic surface, then the non-negative Laplacian Δ_M on M has a resolvent $R_\Delta(s) := (\Delta_M - s(1-s))^{-1}$ that admits a meromorphic extension from $\{s \in \mathbb{C}; \operatorname{Re}(s) > 1/2\}$ to \mathbb{C} as a family of bounded operators $C_c^\infty(M) \rightarrow C^\infty(M)$. The resolvent $R_\Delta(s)$ has finite rank polar part at the poles and the polar part at $s_0 \neq 1/2$ is of the form*

$$\sum_{j=1}^{J(s_0)} \frac{(\Delta_M - s_0(1-s_0))^{j-1} \Pi_{s_0}^\Delta}{(s(1-s) - s_0(1-s_0))^j}, \quad J(s_0) \in \mathbb{N},$$

for some finite rank operator $\Pi_{s_0}^\Delta := (1 - 2s_0)\operatorname{Res}_{s_0}(R_\Delta(s))$ commuting with Δ_M . For $s_0 = 1/2$, $R_\Delta(s)$ has a pole of order at most 1 at s_0 , $\Pi_{s_0}^\Delta := \operatorname{Res}_{s_0}(R_\Delta(s))$ has finite rank and $(\Delta_M - 1/4)\Pi_{s_0}^\Delta = 0$. The operator $R_\Delta(s)$ is bounded as a map $\rho^N L^2(M) \rightarrow \rho^{-N} L^2(M)$ if $N > |\operatorname{Re}(s) - 1/2|$.

The main theorem of [MaMe] shows in addition that the Schwartz kernel $R_\Delta(s; x, x')$ of $R_\Delta(s)$ is of the form

$$(\rho(x)\rho(x'))^{-s} R_\Delta(s; x, x') \in C^\infty(\overline{M} \times \overline{M} \setminus \operatorname{diag}), \quad (4.9)$$

where diag denotes the diagonal of \overline{M} . Moreover, if $f \in C_c^\infty(M)$, $u_s := \rho^{-s} R_\Delta(s)f \in C^\infty(\overline{M})$ is a meromorphic family in $s \in \mathbb{C}$, and since the Laplacian Δ_M in each funnel is given by

$$\Delta_M = -(\rho\partial_\rho)^2 + \frac{1 - \rho^2/4}{1 + \rho^2/4} \rho\partial_\rho - \frac{\rho^2}{(1 + \rho^2/4)^2} \partial_\theta^2, \quad (4.10)$$

a series expansion of u_s in powers of ρ near $\partial\overline{M}$ directly shows that $u_s \in C_{\operatorname{ev}}^\infty(\overline{M})$. Therefore, for each pole s_0 of order $j \geq 1$, we get

$$\varphi \in \operatorname{Ran}(\Pi_{s_0}^\Delta) \Rightarrow \varphi \in \bigoplus_{k=0}^{j-1} \rho^{s_0} \log(\rho)^k C_{\operatorname{ev}}^\infty(\overline{M}). \quad (4.11)$$

Indeed, it suffices to consider the residue of $\rho^s u_s = R_\Delta(s)f$ at $s = s_0$ using Taylor expansion of ρ^s as $s = s_0$, when $f \in C_c^\infty(M)$ is arbitrary, and to use that u_s is even in ρ , and one gets (4.11).

We say that φ is a *generalized resonant state* for s_0 if $\varphi \in \text{Ran}(\Pi_{s_0}^\Delta)$, and that it is a *resonant state* if in addition $(\Delta_M - s_0(1 - s_0))\varphi = 0$. The multiplicity of a quantum resonance s_0 is defined to be the rank of $\Pi_{s_0}^\Delta$. We will define the generalized resonant states of order $j \geq 1$ at s_0 by

$$\text{Res}_\Delta^j(s_0) := \{\varphi \in \text{Ran}(\Pi_{s_0}^\Delta); (\Delta_M - s_0(1 - s_0))^j \varphi = 0\} \quad (4.12)$$

with $\Pi_{s_0}^\Delta := \text{Res}_{s=s_0}(R_\Delta(s))$.

We need a characterization of generalized resonant states of order j as solutions of

$$(\Delta_M - s_0(1 - s_0))^j u = 0$$

with very particular asymptotics for u at the boundary $\partial\overline{M}$ of the conformal compactification \overline{M} . For this purpose, we define the Poisson operator $\mathcal{E}_M(s)$ on M and the scattering operator $\mathcal{S}_M(s)$ by following the approach of Graham-Zworski [GrZw]; we shall refer to that paper for details. By [GrZw, Proposition 3.5], there is a meromorphic family of operators

$$\mathcal{E}_M(s) : C^\infty(\partial\overline{M}) \rightarrow C^\infty(M)$$

in $\text{Re}(s) \geq 1/2$, with only simple poles at $s \in \sigma(\Delta_M)$ and satisfying outside the poles

$$(\Delta_M - s(1 - s))\mathcal{E}_M(s)f = 0$$

for each $f \in C^\infty(\partial\overline{M})$ and with the property that this is the only solution such that there is $F_s, G_s \in C_{\text{ev}}^\infty(\overline{M})$ such that $F_s|_{\partial\overline{M}} = f$ and

$$\begin{aligned} \mathcal{E}_M(s)f &= \rho^{1-s}F_s + \rho^sG_s \text{ if } s \notin \frac{1}{2} + \mathbb{N}, \\ \mathcal{E}_M(s)f &= \rho^{1/2-k}F_{1/2+k} + \rho^{1/2+k}\log(\rho)G_{1/2+k} \text{ if } s = 1/2 + k, k \in \mathbb{N}. \end{aligned} \quad (4.13)$$

Here F_s, G_s are meromorphic with simple poles at $s \in \sigma(\Delta_M)$ and $\frac{1}{2} + \mathbb{N}$. The functions $F_{1/2+k}, G_{1/2+k}$ can be expressed in terms of residues of F_s, G_s at $1/2 + k$. Notice that in the case $M = \mathbb{H}^2$, $\mathcal{E}_{\mathbb{H}^2}(s) = \pi^{-\frac{1}{2}}2^{1-s}\frac{\Gamma(s)}{\Gamma(s-1/2)}\mathcal{P}_{s-1}$, where \mathcal{P}_s is the Poisson-Helgason transform of Lemma 2.1. By [GrZw, Proposition 3.9] the Schwartz kernel of $\mathcal{E}_M(s)$ is related to the Schwartz kernel of $R_\Delta(s)$ by

$$\mathcal{E}_M(s; x, \theta) = (2s - 1)[R_\Delta(s; x, x')\rho(x')^{-s}]|_{x'=\theta \in \partial\overline{M}}. \quad (4.14)$$

This operator thus admits a meromorphic continuation to $s \in \mathbb{C}$, as $R_\Delta(s)$ does. The scattering operator $\mathcal{S}_M(s)$ is a meromorphic family of operators acting on $C^\infty(\partial\overline{M})$ for $\text{Re}(s) \geq 1/2$, unitary on $\text{Re}(s) = \frac{1}{2}$, and defined by

$$\mathcal{S}_M(s)f := 2^{2s-1}\frac{\Gamma(s - \frac{1}{2})}{\Gamma(\frac{1}{2} - s)}G_s|_{\partial\overline{M}}. \quad (4.15)$$

This operator is holomorphic outside $\sigma(\Delta_M)$ and is a family of elliptic pseudo-differential operators of order $2s-1$, which is Fredholm of index 0 as a map $H^{2s-1}(\partial\overline{M}) \rightarrow L^2(\partial\overline{M})$, it extends meromorphically to \mathbb{C} and satisfies the functional equation

$$\mathcal{S}_M(s)^{-1} = \mathcal{S}_M(1-s). \quad (4.16)$$

By [GrZw], there are special points, namely $s = \frac{1}{2} + k$ with $k \in \mathbb{N}$, where $\mathcal{S}_M(s)$ is a differential operator on $\partial\overline{M} = \cup_{i=1}^{n_f} S_i$ which depends only on the metric $\bar{g}|_{\partial\overline{M}}$. Moreover we have

$$G_{1/2+k}|_{\partial\overline{M}} = c_k \mathcal{S}_M(1/2+k)f, \quad c_k \neq 0. \quad (4.17)$$

The computation of the scattering operator is done by Guillopé-Zworski [GuZw1, Appendix] for the hyperbolic cylinder $\mathcal{C}(\ell_i) := \mathbb{R}_t \times (\mathbb{R}/\ell_i\mathbb{Z})_\theta$ with metric $dt^2 + \cosh(t)^2 d\theta^2$, and their computation shows that $\ker \mathcal{S}_{\mathcal{C}(\ell_i)}(\frac{1}{2} + k) = 0$ for all $k \in \mathbb{N}$, which implies

$$\mathcal{S}_M(\frac{1}{2} + k) : C^\infty(\partial\overline{M}) \rightarrow C^\infty(\partial\overline{M}) \text{ is an isomorphism;} \quad (4.18)$$

note that this fact is also proved in [Bo, Lemma 8.6]. Finally, one has a functional equation similar to (2.15), which follows from the definition of $\mathcal{E}_M(s)$ and $\mathcal{S}_M(s)$

$$\mathcal{E}_M(s) = 2^{1-2s} \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(s - \frac{1}{2})} \mathcal{E}_M(1-s) \mathcal{S}_M(s) \quad (4.19)$$

We have the following result on quantum resonant states.

Proposition 4.3. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a convex co-compact surface. Then the following properties hold:*

- 1) *There is no quantum resonance at $\frac{1}{2} - \mathbb{N}$.*
- 2) *If $\operatorname{Re}(s_0) \geq 1/2$, the poles of $R_\Delta(s)$ at s_0 are simple and φ is a resonant state for s_0 if and only if $(\Delta_M - s_0(1-s_0))\varphi = 0$ with $\varphi \in \rho^{s_0} C^\infty(\overline{M})$.*
- 3) *If $\operatorname{Re}(s_0) < 1/2$ with $s_0 \notin 1/2 - \mathbb{N}$, a solution φ to $(\Delta_M - s_0(1-s_0))\varphi = 0$ is a resonant state for s_0 if and only if $\varphi \in \rho^{s_0} C_{\text{ev}}^\infty(\overline{M})$, and a function φ' is a generalized resonant state if and only if it is a resonant state or there is a resonant state φ so that $(\Delta_M - s_0(1-s_0))^j \varphi' = \varphi$ for some $j \geq 1$, and $\varphi' \in \bigoplus_{k=0}^j \rho^{s_0} \log(\rho)^k C_{\text{ev}}^\infty(\overline{M})$.*

Proof. To prove 1), we use [Gu2, Lemma 3.4] which says that if s_0 is a pole of $R_\Delta(s)$ then it is a pole of $S_M(s) := 2^{1-2s} \frac{\Gamma(1/2-s)}{\Gamma(s-1/2)} \mathcal{S}_M(s)$. But $S_M(s)$ has a pole of order 1 at $s_0 = 1/2 + k$ with residue $c_k \mathcal{S}_M(1/2+k)$ for some $c_k \neq 0$, and this operator has no kernel. Therefore, by expanding in Laurent series the functional equation $S_M(1-s)S_M(s) = \text{Id}$ at $s_0 = 1/2 - k$, we directly see that $S_M(s)$ must be of the form $S_M(s) = (s - s_0)L(s)$ for some holomorphic family of operator $L(s)$ near $s = s_0$, with $L(s_0)$ invertible on $C^\infty(\partial\overline{M})$. This shows 1).

Statement 2) is direct to see for $\operatorname{Re}(s_0) > 1/2$: the resonant states are of the desired form by [PaPe, Lemma 4.8] and conversely if $\varphi \in \rho^{s_0} C^\infty(\overline{M}) \cap \ker(\Delta_M - s_0(1-s_0))$, then $\varphi \in \ker_{L^2}(\Delta_M - s_0(1-s_0)) = \operatorname{Ran}(\Pi_{s_0}^\Delta)$. For $s_0 = 1/2$, the pole of the resolvent is

simple and the resonant states are in $\rho^{1/2}C^\infty(\overline{M})$ by [PaPe, Lemma 4.9]. The converse part will follow from the proof of 3).

Now we prove 3). By (4.11), if φ' is a generalized resonant state satisfying the equation $(\Delta_M - s_0(1 - s_0))^{j+1}\varphi' = 0$, then $\varphi' \in \bigoplus_{k=0}^J \rho^{s_0} \log(\rho)^k C_{\text{ev}}^\infty(\overline{M})$ for some J . Using the form of Δ_M in (4.10) and writing the formal expansions at $\rho = 0$ of that equation, it is direct to see that $J \leq j$.

We next prove the converse. Let $A_{s_0} := (\Delta_M - s_0(1 - s_0))$ and let φ be a solution of $A_{s_0}^{j+1}\varphi = 0$ with $\varphi \in \bigoplus_{\ell=0}^j \rho^{s_0} \log(\rho)^\ell C_{\text{ev}}^\infty(\overline{M})$ with $j \geq 0$. Define $\varphi_\ell := A_{s_0}^{j-\ell}\varphi$ for all $\ell \in 0, \dots, j$. It is easy to check that $\varphi_\ell \in \bigoplus_{k=0}^\ell \rho^{s_0} \log(\rho)^k C_{\text{ev}}^\infty(\overline{M})$. We first construct a holomorphic family $\phi(s) = \rho^s F(s)$ with $F(s) \in C_{\text{ev}}^\infty(\overline{M})$ such that

$$q(s) := (\Delta_M - s(1 - s))\phi(s) \in \rho^{s+2}C_{\text{ev}}^\infty(\overline{M}), \quad |q(s)| \leq C|s - s_0|^{j+1}\rho^{\text{Re}(s)+2}. \quad (4.20)$$

To construct $F(s)$, we will set $F(s) := \sum_{k=0}^j F_k(s - s_0)^k$ for some $F_k \in C_{\text{ev}}^\infty(\overline{M})$ well chosen. Taylor expanding at $s = s_0$

$$\phi(s) = \rho^s F(s) = \sum_{k=0}^j (s - s_0)^k \phi_k + \mathcal{O}((s - s_0)^{j+1}) \quad (4.21)$$

with $\phi_k := \rho^{s_0} \sum_{\ell=0}^k \frac{1}{\ell!} \log(\rho)^\ell F_{k-\ell}$, the equation

$$(\Delta_M - s(1 - s))\phi(s) = \mathcal{O}((s - s_0)^{j+1})$$

reduces to

$$A_{s_0}\phi_k + (2s_0 - 1)\phi_{k-1} + \phi_{k-2} = 0 \quad (4.22)$$

for all $k \leq j$, with the convention $\phi_{-1} = \phi_{-2} = 0$. The equation (4.22) can be solved by choosing ϕ_k to be a linear combination of the form

$$\phi_0 = \varphi_0, \quad \text{and } \phi_k = (1 - 2s_0)^k \varphi_k + \sum_{\ell=0}^{k-1} c_\ell(s_0) \varphi_\ell \quad \text{for } k \in [1, j]$$

for some polynomials $c_\ell(s_0)$ in s_0 , and we also note that $\phi_k \in \bigoplus_{\ell=0}^k \rho^{s_0} \log(\rho)^\ell C_{\text{ev}}^\infty(\overline{M})$. We can then define for $k \leq j$

$$F_k := \rho^{-s_0} \sum_{\ell=0}^k \frac{1}{\ell!} (-\log \rho)^\ell \phi_{k-\ell}.$$

We need to check that $F_k \in C_{\text{ev}}^\infty(\overline{M})$. To do this, we will show that

$$\rho^{-s_0} A_{s_0}(\rho^{s_0} F_k) \in \rho^2 C_{\text{ev}}^\infty(\overline{M}). \quad (4.23)$$

Indeed, since we know that $F_k \in \bigoplus_{\ell=0}^{2k} \log(\rho)^\ell C_{\text{ev}}^\infty(\overline{M})$, it is an easy exercise to check that (4.23) implies that $F_k \in C_{\text{ev}}^\infty(\overline{M})$ by using the expression (4.10) of Δ_M near $\partial\overline{M}$.

We already know that $F_0 \in C_{\text{ev}}^\infty(\overline{M})$ and that (4.10) is true for $k = 0$. Now to prove (4.23), we write

$$A_{s_0}(\rho^{s_0} F_k) = \sum_{\ell=0}^k \frac{(-1)^\ell}{\ell!} \left(A_{s_0}(\phi_{k-\ell})(\log \rho)^\ell + \Delta_M((\log \rho)^\ell) \phi_{k-\ell} - 2\ell(\log \rho)^{\ell-1} N \phi_{k-\ell} \right)$$

where $N := \nabla(\log \rho)$ denotes the gradient of $\log \rho$ with respect to g . By (4.22), we get

$$\sum_{\ell=0}^k \frac{(-1)^\ell}{\ell!} A_{s_0}(\phi_{k-\ell})(\log \rho)^\ell = \rho^{s_0}(-F_{k-2} + (1 - 2s_0)F_{k-1}).$$

Next we have

$$\begin{aligned} & \sum_{\ell=1}^k \frac{(-1)^\ell}{\ell!} \ell(\log \rho)^{\ell-1} N \phi_{k-\ell} = \\ & \sum_{\ell=1}^k \frac{(-1)^\ell}{(\ell-1)!} N((\log \rho)^{\ell-1} \phi_{k-\ell}) - |N|_g^2 \sum_{\ell=2}^k \frac{(-1)^\ell}{(\ell-2)!} (\log \rho)^{\ell-2} \phi_{k-\ell} = \\ & -\rho^{s_0}((N + s_0|N|_g^2)F_{k-1} + |N|_g^2 F_{k-2}). \end{aligned}$$

and

$$\begin{aligned} & \sum_{\ell=0}^k \frac{(-1)^\ell}{\ell!} \Delta_M((\log \rho)^\ell) \phi_{k-\ell} = \\ & -|N|_g^2 \sum_{\ell=2}^k \frac{(-1)^\ell}{(\ell-2)!} (\log \rho)^{\ell-2} \phi_{k-\ell} + \sum_{\ell=1}^k \frac{(-1)^\ell}{(\ell-1)!} (\log \rho)^{\ell-1} \phi_{k-\ell} \Delta_M(\log \rho) = \\ & -|N|_g^2 \rho^{s_0} F_{k-2} - \Delta_M(\log \rho) \rho^{s_0} F_{k-1}. \end{aligned}$$

Consequently we get

$$\rho^{-s_0} A_{s_0}(\rho^{s_0} F_k) = (|N|_g^2 - 1)F_{k-2} + (2N + 1 - \Delta_M(\log \rho) + 2s_0(|N|_g^2 - 1))F_{k-1}. \quad (4.24)$$

By using (4.10), a direct computation gives that $\Delta_M(\log \rho) - 1 \in \rho^2 C_{\text{ev}}^\infty(\overline{M})$ and we also have $|N|_g^2 = 1$ near $\partial \overline{M}$, so $|N|_g^2 \in C_{\text{ev}}^\infty(\overline{M})$. Moreover N maps $C_{\text{ev}}^\infty(\overline{M})$ to $\rho^2 C_{\text{ev}}^\infty(\overline{M})$. An induction in k with (4.24) then shows (4.23). We directly get that the function $q(s)$ defined by (4.20) is a holomorphic family in $\rho^{s+2} C_{\text{ev}}^\infty(\overline{M})$, and by Taylor expanding the term $(\Delta_M - s(1-s))\phi(s)$ at $s = s_0$, we also have (by construction)

$$|\rho^{-s} q(s)| = \mathcal{O}(|s - s_0|^{j+1} \rho^2). \quad (4.25)$$

Using Green's formula in the region $\rho \geq \epsilon$ for some small $\epsilon > 0$, we see that for $z \in M$ fixed and s near s_0

$$\begin{aligned} (R_\Delta(1-s)q(s))(z) &= \phi(s; z) - \int_{\rho=\epsilon} \rho \partial_\rho R_\Delta(1-s; z, \rho, \theta) \phi(s; \rho, \theta) \frac{d\theta}{\rho} \\ &\quad + \int_{\rho=\epsilon} R_\Delta(1-s; z, \rho, \theta) \rho \partial_\rho \phi(s; \rho, \theta) \frac{d\theta}{\rho}. \end{aligned}$$

Now $\rho \partial_\rho \phi(s; \rho, \theta) = s \rho^s F(s; 0, \theta) + \mathcal{O}(\rho^{\operatorname{Re}(s)+1})$ and, using (4.14),

$$\rho \partial_\rho R_\Delta(1-s; z, \rho, \theta) = \frac{1-s}{1-2s} \mathcal{E}_M(1-s; z, \theta) + \mathcal{O}(\rho^{2-\operatorname{Re}(s)}).$$

Thus we obtain

$$R_\Delta(1-s)q(s) = \phi(s) - \mathcal{E}_M(1-s)F(s)|_{\partial\bar{M}}.$$

By (4.25), we also have $|R_\Delta(1-s)q(s)| = \mathcal{O}(|s-s_0|^{j+1})$ uniformly on compact sets of M , and therefore

$$\phi(s) = \mathcal{E}_M(1-s)F(s)|_{\partial\bar{M}} + \mathcal{O}(|s-s_0|^{j+1})$$

uniformly on compact sets. We define $f(s) := F(s)|_{\partial\bar{M}}$ and differentiate for $\ell \leq j$

$$\partial_s^\ell(\phi(s))|_{s=s_0} = \partial_s^\ell(\mathcal{E}_M(1-s)f(s))|_{s=s_0} = \rho^{s_0} \sum_{i=0}^{\ell} \log(\rho)^i H_i^\ell + \rho^{1-s_0} \sum_{i=0}^{\ell} \log(\rho)^i G_i^\ell$$

where $H_i^\ell, G_i^\ell \in C_{\text{ev}}^\infty(\bar{M})$ and $G_0^\ell|_{\partial\bar{M}} = \partial_s^\ell(\mathcal{S}_M(1-s)f(s))|_{s=s_0}$. But from (4.21) and the fact that $\phi_k \in \bigoplus_{\ell=0}^k \rho^{s_0} \log(\rho)^\ell C_{\text{ev}}^\infty(\bar{M})$, we see that $\partial_s^\ell(\mathcal{S}_M(1-s)f(s))|_{s=s_0} = 0$ for all $\ell \leq j$ and thus $\mathcal{S}_M(1-s)f(s) = \mathcal{O}(|s-s_0|^{j+1})$. Therefore $\mathcal{S}_M(1-s)f(s) = (s-s_0)^{j+1}r(s)$ for some holomorphic family $r(s) \in C^\infty(\partial\bar{M})$. We write $\mathcal{E}_M(s) = \sum_{\ell=1}^N (s-s_0)^{-\ell} Q_\ell + H(s)$ for some holomorphic operator family $H(s)$ near s_0 and some operators $Q_\ell : C^\infty(\partial\bar{M}) \rightarrow C^\infty(M)$. By using (4.19), we get

$$\mathcal{E}_M(1-s)f(s) = (s-s_0)^{j+1} \mathcal{E}_M(s)r(s) = \sum_{\ell=1}^N (s-s_0)^{-\ell+j+1} Q_\ell r(s) + (s-s_0)^{j+1} H(s)r(s)$$

which implies that $j+1 \leq N$ and $\partial_s^k(\phi(s))|_{s=s_0} \in \bigoplus_{\ell=1}^N \operatorname{Ran}(Q_\ell)$ for each $k \leq j$. By (4.14), we have $\bigoplus_{\ell=1}^N \operatorname{Ran}(Q_\ell) \subset \operatorname{Ran}(\Pi_{s_0}^\Delta)$ since the singular part of the Laurent expansion of $R_\Delta(s)$ is a finite rank operator with range $\operatorname{Ran}(\Pi_{s_0}^\Delta)$. Using again (4.21), this shows that ϕ_k , and therefore φ_k , are generalized resonant states for $k \leq j$. This completes the proof. \square

Remark 1. *The proof of 2) and 3) in Proposition 4.3 also applies in the more general setting of even asymptotically hyperbolic manifolds in the sense of [Gul], in any dimension $n+1$, by replacing $(\Delta_M - s_0(1-s_0))$ by $(\Delta_M - s_0(n-s_0))$ and $s_0 \notin 1/2 - \mathbb{N}$ by $s_0 \notin n/2 - \mathbb{N}$.*

Lemma 4.4. *For any $\varphi \in \rho^{s_0} C^\infty(\overline{M})$ satisfying $(\Delta_M - s_0(1-s_0))\varphi = 0$ with $s_0 \notin -\mathbb{N}_0$, there exists a distribution $\omega \in \mathcal{D}'(\mathbb{S}^1)$ supported in Λ_Γ such that $\pi_\Gamma^* \varphi = \mathcal{P}_{s_0-1}(\omega)$ and for each $\gamma \in \Gamma$, $\gamma^* \omega = N_\gamma^{-s_0+1} \omega$.*

Proof. Let $\tilde{\varphi} = \pi_\Gamma^* \varphi$ be the lift of φ to \mathbb{H}^2 . Then we have $(\Delta_{\mathbb{H}^2} - s_0(1-s_0))\tilde{\varphi} = 0$ on \mathbb{H}^2 and we claim that $\tilde{\varphi}$ is tempered on \mathbb{H}^2 . Indeed, if $T > 0$ and 0 denotes the center the unit ball in \mathbb{R}^2 (representing \mathbb{H}^2), consider $m(T) := \sup_{d_{\mathbb{H}^2}(x,0) \leq T} |\tilde{\varphi}(x)|$ where $d_{\mathbb{H}^2}(\cdot, \cdot)$ denotes the hyperbolic distance. For $\epsilon > 0$ small enough $M_\epsilon := \{x \in M; \rho(x) \geq \epsilon\}$ is a geodesically convex set and it is easy to see that there exists $C > 0$ so that for all $T > 0$ each point $x \in \mathbb{H}^2$ with $d_{\mathbb{H}^2}(x, 0) \leq T$ projects by the covering map π_Γ to the region M_ϵ for $\epsilon = Ce^{-T}$. Then $m(T) \leq \sup_{x \in M_\epsilon} (|\varphi(x)|) \leq C_{s_0} e^{\max(-\operatorname{Re}(s_0), 0)T}$ for some constant C_{s_0} depending on $\operatorname{Re}(s_0)$. Here the last inequality follows from $\varphi \in \rho^{s_0} C^\infty(\overline{M})$ and this estimate shows that $\tilde{\varphi}$ is tempered on \mathbb{H}^2 . By the surjectivity of the Poisson-Helgason transform, there exists a distribution $\omega \in \mathcal{D}'(\mathbb{S}^1)$ so that $\tilde{\varphi} = \mathcal{P}_{s_0-1}(\omega)$. By Lemma 2.1 and the discussion that follows, for any $\chi \in C^\infty(\mathbb{S}^1)$ one has for $t \in (0, \epsilon)$ with $\epsilon > 0$ small

$$\int_{\mathbb{S}^1} \mathcal{P}_{s_0-1}(\omega) \left(\frac{2-t}{2+t} \nu \right) \chi(\nu) d\nu = \begin{cases} t^{1-s_0} F_-(t) + t^{s_0} F_+(t) & \text{if } s_0 \notin 1/2 + \mathbb{Z} \\ t^{1-s_0} F_-(t) + t^{s_0} \log(t) F_+(t) & \text{if } s_0 \in 1/2 + \mathbb{N} \\ t^{1-s_0} \log(t) F_-(t) + t^{s_0} F_+(t) & \text{if } s_0 \in 1/2 - \mathbb{N}_0 \end{cases} \quad (4.26)$$

for some $F_\pm \in C^\infty([0, \epsilon))$ and $F_-(0) = C(s_0) \langle \omega, \chi \rangle$ where $C(s_0) \neq 0$ because $s_0 \notin -\mathbb{N}_0$. On the other hand, since $\pi_\Gamma^* \rho$ is a boundary defining function of Ω_Γ in $\mathbb{H}^2 \cup \Omega_\Gamma$, in a small neighborhood V_p of any point $p \in \Omega_\Gamma$ in $\mathbb{H}^2 \cup \Omega_\Gamma$, the function $\tilde{\varphi}$ has an asymptotic expansion as $t \rightarrow 0$

$$\tilde{\varphi} \left(\frac{2-t}{2+t} \nu \right) \sim \sum_{k=0}^{\infty} t^{s_0+k} \alpha_k(\nu)$$

for some $\alpha_k \in C^\infty(V_p)$, therefore if $\chi \in C_c^\infty(\Omega_\Gamma)$ is supported in V_p , we have

$$\int_{\mathbb{S}^1} \tilde{\varphi} \left(\frac{2-t}{2+t} \nu \right) \chi(\nu) d\nu \sim \sum_{k=0}^{\infty} t^{s_0+k} \langle \alpha_k, \chi \rangle$$

and thus from (4.26) we deduce that $\langle \omega, \chi \rangle = 0$. This shows that ω is supported in Λ_Γ . Let $\gamma \in \Gamma$, we have

$$\mathcal{P}_{s_0-1}(N_\gamma^{s_0-1} \gamma^* \omega) = \gamma^* \mathcal{P}_{s_0-1}(\omega)$$

which is also equal to $\mathcal{P}_{s_0-1}(\omega)$ since $\tilde{\varphi}$ is Γ -automorphic. By the injectivity of the Poisson-Helgason transform [DFG, Corollary 6.9], we thus deduce that $\gamma^* \omega = N_\gamma^{-s_0+1} \omega$. \square

We show the following

Theorem 4.5. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a smooth oriented convex co-compact hyperbolic surface and let SM be its unit tangent bundle.*

1) *For each $\lambda_0 \in \mathbb{C} \setminus (-\frac{1}{2} - \frac{1}{2}\mathbb{N}_0)$ the pushforward map $\pi_{0*} : \mathcal{D}'(SM) \rightarrow \mathcal{D}'(M)$ restricts to a linear isomorphism of complex vector spaces for each $j \geq 1$*

$$\pi_{0*} : V_0^j(\lambda_0) \rightarrow \text{Res}_{\Delta}^j(\lambda_0 + 1) \quad (4.27)$$

where Δ_M is the Laplacian on M acting on functions.

2) *For each $\lambda_0 = -\frac{1}{2} - k$ with $k \in \mathbb{N}$, $V_0^j(\lambda_0) = 0$ and $\text{Res}_{\Delta_M}^j(\lambda_0 + 1) = 0$ for all $j \in \mathbb{N}$.*

3) *For $\lambda_0 = -\frac{1}{2}$, there are no Jordan blocks, i.e. $V_0^j(-\frac{1}{2}) = 0$ for $j > 1$, and the map*

$$\pi_{0*} : V_0^1(-\frac{1}{2}) \rightarrow \text{Res}_{\Delta}^1(1/2) \quad (4.28)$$

is a linear isomorphism of complex vector spaces.

4) *For $\lambda_0 = -n \in -\mathbb{N}$, if Γ is non-elementary, there are no Jordan blocks, i.e. $V_0^j(-n) = 0$ if $j > 1$, and the following map is an isomorphism of real vector spaces*

$$i^{n+1}\pi_{n*} : V_0^1(-n) \rightarrow H_n(M) \quad (4.29)$$

where $H_n(M)$ is defined by (4.1).

Proof. Case $\lambda_0 \in \mathbb{C} \setminus (-\frac{1}{2} - \frac{1}{2}\mathbb{N}_0)$, injectivity. The map $X + \lambda_0$ is a linear nilpotent map preserving the finite dimensional vector space $V_0(\lambda_0) := \bigoplus_{j \geq 1} V_0^j(\lambda_0)$ of generalized Ruelle resonant states in $\ker U_-$. Thus there is a decomposition into Jordan blocks for X on $V_0(\lambda_0)$: for each Jordan block of size j , one has a $u^{(0)} \in V_0^1(\lambda_0)$ and some $u^{(k)} \in V_0^{k+1}(\lambda_0)$ for $k \in [1, j]$ satisfying $(X + \lambda_0)u^{(k)} = u^{(k-1)}$. We lift each $u^{(k)} \in \mathcal{D}'(SM)$ to $S\mathbb{H}^2$ and get $\tilde{u}^{(k)} \in \mathcal{D}'(S\mathbb{H}^2)$ so that $\gamma^*\tilde{u}^{(k)} = \tilde{u}^{(k)}$ for all $\gamma \in \Gamma$, and $\tilde{u}^{(k)}$ is supported in $\tilde{\Lambda}_+$, where $\tilde{\Lambda}_+$ is defined by (4.5). Define for $k \geq 0$

$$\varphi_k := \pi_{0*}u^{(k)}, \quad \tilde{\varphi}_k := \pi_{0*}\tilde{u}^{(k)}. \quad (4.30)$$

From $u^{(0)} \in V_0^1(\lambda_0)$ we have that $(X + \lambda_0)u^{(0)} = 0$, $U_-u^{(0)} = 0$, and $u^{(0)}$ is supported in Λ_+ . The same equations hold for $\tilde{u}^{(0)}$ on \mathbb{H}^2 . Take the distribution $v^{(0)} := \Phi_-^{-\lambda_0}\tilde{u}^{(0)}$ satisfying $Xv^{(0)} = 0$. Then there exists $\omega^{(0)} \in \mathcal{D}'(\mathbb{S}^1)$ so that $\mathcal{Q}_-\omega^{(0)} = v^{(0)}$ by (2.12). Since $\text{supp}(\tilde{u}^{(0)}) \subset \tilde{\Lambda}_+$ and $\mathcal{Q}_-\omega^{(0)} = B_-^*\omega^{(0)}$, using (4.5) we directly get that $\text{supp}(\omega^{(0)}) \subset \Lambda_\Gamma$. Using $\gamma^*\tilde{u}^{(0)} = \tilde{u}^{(0)}$ together with (2.5), we have that for any $\gamma \in \Gamma$

$$\gamma^*\omega^{(0)} = N_\gamma^{-\lambda_0}\omega^{(0)} \text{ with } N_\gamma(\nu) = |d\gamma(\nu)|^{-1}.$$

We get that $\tilde{\varphi}_0 = \mathcal{P}_{\lambda_0}(\omega^{(0)}) = \pi_{0*}\tilde{u}^{(0)}$ satisfies

$$(\Delta_{\mathbb{H}^2} + \lambda_0(1 + \lambda_0))\tilde{\varphi}_0 = 0$$

on \mathbb{H}^2 . Now by Lemma 2.1, $\tilde{\varphi}_0 \neq 0$ if $\lambda_0 \notin -\mathbb{N}_0$ and $u^{(0)} \neq 0$, thus φ_0 is a non-zero solution on M of

$$(\Delta_M + \lambda_0(1 + \lambda_0))\varphi_0 = 0.$$

To prove that $s_0 = \lambda_0 + 1$ is a quantum resonance, we will use Proposition 4.3, and for that it is sufficient to prove that $\varphi_0 \in \rho^{s_0}C_{\text{ev}}^\infty(\overline{M})$, and in fact $\varphi_0 \in \rho^{s_0}C^\infty(\overline{M})$ is

sufficient since we assumed $\lambda_0 \notin -\mathbb{N}$. Take a point $p \in \partial\overline{M}$, and consider $\tilde{p} \in \Omega_\Gamma$ a lift of p to $\mathbb{H}^2 \cup \Omega_\Gamma$. To prove the desired statement, we take the boundary defining function $\rho_0(x) := 2(1 - |x|)/(1 + |x|)$ in the closed ball $\overline{\mathbb{H}^2}$ and we will show that $\rho_0^{-s_0}\tilde{\varphi}_0$ is a smooth function near the boundary of $\mathbb{H}^2 \cup \Omega_\Gamma$. Note that $\rho_0(x)^{-1}P(x, \nu)$ is smooth outside the subset $\{(x, \nu) \in \overline{\mathbb{H}^2} \times \mathbb{S}^1; x \neq \nu\}$ and since $\omega^{(0)}$ is supported in Λ_Γ , we deduce directly that $\rho_0^{-s_0}\mathcal{P}_{\lambda_0}(\omega^{(0)})$ is smooth in a neighbourhood of \tilde{p} in $\mathbb{H}^2 \cup \Omega_\Gamma$. We have proved that φ_0 is a quantum resonant state which in addition has asymptotic behaviour given in terms of the distribution $\omega^{(0)}$: there is an explicit constant $C(s_0) \neq 0$ so that

$$\pi_\Gamma^*([\rho^{-s_0}\varphi_0]|_{\partial\overline{M}}) = C(s_0)\eta^{s_0}\mathcal{S}(s_0)(\omega^{(0)}) \quad (4.31)$$

where $\eta \in C^\infty(\Omega_\Gamma)$ is defined by $\pi_\Gamma^*(\rho)\eta = \rho_0 + \mathcal{O}(\rho_0^2)$ near Ω_Γ . For the generalized resonant states, we proceed like in the compact case: define for $k \leq j$

$$v^{(k)} := \Phi_-^{-\lambda_0} \sum_{\ell=0}^k \frac{(\log \Phi_-)^{k-\ell}}{(k-\ell)!} \tilde{u}^{(\ell)}$$

which satisfies $Xv^{(k)} = 0$ and $U_-v^{(k)} = 0$. Thus there is a distribution $\omega^{(k)}$ supported in Λ_Γ such that $\mathcal{Q}_-\omega^{(k)} = v^{(k)}$, and using that $\gamma^*\tilde{u}^{(k)} = \tilde{u}^{(k)}$ for all $\gamma \in \Gamma$, we get

$$\gamma^*v^{(k)} = N_\gamma^{-\lambda_0} \sum_{\ell=0}^k \frac{(\log N_\gamma)^\ell}{\ell!} v^{(k-\ell)}, \quad \gamma^*\omega^{(k)} = N_\gamma^{-\lambda_0} \sum_{\ell=0}^k \frac{(\log N_\gamma)^\ell}{\ell!} \omega^{(k-\ell)}. \quad (4.32)$$

Writing $\tilde{u}^{(k)}$ in terms of the $v^{(\ell)}$ and using (3.9), we have

$$\tilde{u}^{(k)} = \Phi_-^{\lambda_0} \sum_{\ell=0}^k \frac{(-\log \Phi_-)^\ell v^{(k-\ell)}}{\ell!}, \quad \tilde{\varphi}_k = \sum_{\ell=0}^k \frac{(-1)^\ell \partial_\lambda^\ell \mathcal{P}_{\lambda_0}(\omega^{(k-\ell)})}{\ell!}.$$

By (3.9) or the proof of Lemma 2.3 we deduce that

$$(\Delta_{\mathbb{H}^2} + \lambda_0(1 + \lambda_0))\tilde{\varphi}_k = (1 + 2\lambda_0)\tilde{\varphi}_{k-1} - \tilde{\varphi}_{k-2}. \quad (4.33)$$

(with the convention $\tilde{\varphi}_i = 0$ for $i < 0$). This implies the following identities on M

$$(\Delta_{\mathbb{H}^2} + \lambda_0(1 + \lambda_0)) \sum_{\ell=1}^k \frac{\varphi_\ell}{(1 + 2\lambda_0)^{k+1-\ell}} = \varphi_{k-1}.$$

Using that $\partial_\lambda^k((P(x, \nu))^{\lambda+1})|_{\lambda_0}$ is of the form $\rho_0^{\lambda_0+1} \sum_{\ell=0}^k (\log(\rho_0(x)))^\ell H_k(x, \nu)$ for some functions H_k smooth in $\{(x, \nu) \in \overline{\mathbb{H}^2} \times \mathbb{S}^1; x \neq \nu\}$, we deduce like we did for φ_0 that $\varphi_k \in \bigoplus_{\ell=0}^k \rho^{s_0} \log(\rho)^\ell C^\infty(\overline{M})$ with $s_0 = \lambda_0 + 1$. Then, by Proposition 4.3, the function φ_k is a quantum generalized resonant state in $\text{Res}_\Delta^{k+1}(s_0)$ for each $k = 0, \dots, j$, and $\pi_{0*} : V_0^k(\lambda_0) \rightarrow \text{Res}_\Delta^k(\lambda_0 + 1)$ is injective.

Case $\lambda_0 \in \mathbb{C} \setminus (-\frac{1}{2} - \frac{1}{2}\mathbb{N}_0)$, surjectivity. Next we will show that this map is also surjective: let $s_0 \notin \frac{1}{2} - \frac{1}{2}\mathbb{N}_0$ be a pole of $R_\Delta(s)$ and denote by $\Pi_{s_0}^\Delta$ its residue. Recall

that $F_{s_0} := \text{Ran}(\Pi_{s_0}^\Delta)$ is finite dimensional and that $A_{s_0} := (\Delta_M - s_0(1 - s_0))|_{F_{s_0}}$ is a linear nilpotent operator preserving this finite dimensional space. Thus there is a decomposition into Jordan blocks for A_{s_0} on F_{s_0} : for each Jordan block, there is a $\phi_0 \in F_{s_0}$ so that $A_{s_0}\phi_0 = 0$ and some $\phi_k \in F_{\lambda_0}$ for $k \leq j$ so that $A_{s_0}\phi_k = \phi_{k-1}$. Note that by definition (4.12) we have $\phi_k \in \text{Res}_{\Delta_M}^{k+1}(s_0)$. We lift ϕ_k to \mathbb{H}^2 , we get $\tilde{\phi}_k := \pi_\Gamma^*\phi_k \in C^\infty(\mathbb{H}^2)$. Using $A_{s_0}\phi_k = \phi_{k-1}$ for each $k \geq 1$, we see that there exist $\tilde{\varphi}_k \in C^\infty(\mathbb{H}^2)$ so that $\tilde{\varphi}_0 = \tilde{\phi}_0$ and satisfying (4.33): $\tilde{\varphi}_k$ are linear combinations of $(\tilde{\phi}_\ell)_{\ell=0, \dots, k}$ and are thus Γ -invariant and descend to some function φ_k . We will then show that there exist $\omega^{(k)} \in \mathcal{D}'(\mathbb{S}^1)$ supported in Λ_Γ so that for all $\gamma \in \Gamma$ and $k \leq j$

$$\tilde{\varphi}_k = \sum_{\ell=0}^k \frac{(-1)^\ell \partial_\lambda^\ell \mathcal{P}_{\lambda_0}(\omega^{(k-\ell)})}{\ell!}, \quad \gamma^* \omega^{(k)} = N_\gamma^{-\lambda_0} \sum_{\ell=0}^k \frac{(\log N_\gamma)^\ell}{\ell!} \omega^{(k-\ell)} \quad (4.34)$$

where $\lambda_0 = s_0 - 1$. We prove this by induction on k . For $k = 0$, this is a consequence of Lemma 4.4. Suppose that (4.34) is satisfied with k replaced by m for all $m \leq k$, and we will show that the same hold at order $k + 1$. We set

$$\psi_{k+1} := \tilde{\varphi}_{k+1} + \sum_{\ell=0}^k \frac{(-1)^\ell \partial_\lambda^{\ell+1} \mathcal{P}_{\lambda_0}(\omega^{(k-\ell)})}{(\ell+1)!}$$

and using (4.33) and (3.9)

$$\begin{aligned} A_{s_0} \psi_{k+1} &= - (1 + 2\lambda_0) \sum_{\ell=0}^k \frac{(-1)^\ell \partial_\lambda^\ell \mathcal{P}_{\lambda_0}(\omega^{(k-\ell)})}{\ell!} - \sum_{\ell=1}^k \frac{(-1)^\ell \partial_\lambda^{\ell-1} \mathcal{P}_{\lambda_0}(\omega^{(k-\ell)})}{(\ell-1)!} \\ &\quad + (1 + 2\lambda_0) \tilde{\varphi}_k - \tilde{\varphi}_{k-1} = 0 \end{aligned}$$

where the last equality follows by using the first equation of (4.34) at order $k - 1$ and k . The surjectivity of the Poisson-Helgason transform in Lemma 2.1 implies that there exists $\omega^{(k+1)} \in \mathcal{D}'(\mathbb{S}^1)$ such that $\psi_{k+1} = \mathcal{P}_{\lambda_0}(\omega^{(k+1)})$. Now by definition of ψ_{k+1} , we have near each point $p \in \Omega_\Gamma$ that $\psi_{k+1} \in \bigoplus_{\ell=0}^{k+1} (\log \rho_0)^\ell \rho^{s_0} C^\infty(\mathbb{H}^2 \cup \Omega_\Gamma)$. Since $(\Delta_{\mathbb{H}^2} - s_0(1 - s_0))\psi_{k+1} = 0$, in fact one has $\psi_{k+1} \in \rho^{s_0} C^\infty(\mathbb{H}^2 \cup \Omega_\Gamma)$. Then the same arguments as in the proof of Lemma 4.4 imply that $\omega^{(k+1)}$ is supported in Λ_Γ . This shows the first equation of (4.34) at order $k + 1$. Using that $\gamma^* \tilde{\varphi}_{k+1} = \tilde{\varphi}_{k+1}$ for all $\gamma \in \Gamma$, the induction assumption (4.34) implies that

$$\gamma^* \psi_{k+1} - \psi_{k+1} = \mathcal{P}_{\lambda_0} \left(\sum_{\ell=1}^{k+1} \frac{(\log N_\gamma)^\ell \omega^{(k+1-\ell)}}{\ell!} \right)$$

but this is also equal to $\mathcal{P}_{\lambda_0}(N_\gamma^{\lambda_0} \gamma^* \omega^{(k+1)} - \omega^{(k+1)})$, which by injectivity of the Poisson-Helgason transform implies

$$N_\gamma^{\lambda_0} \gamma^* \omega^{(k+1)} - \omega^{(k+1)} = \sum_{\ell=1}^{k+1} \frac{(\log N_\gamma)^\ell \omega^{(k+1-\ell)}}{\ell!}$$

which is exactly (4.34) for $\omega^{(k+1)}$. Now we define the distributions on $S\mathbb{H}^2$

$$v^{(k)} := \mathcal{Q}_-\omega^{(k)}, \quad \tilde{u}^{(k)} := \Phi_-^{\lambda_0} \sum_{\ell=0}^k \frac{(-\log \Phi_-)^\ell v^{(k-\ell)}}{\ell!}.$$

By construction we have, for each $k \geq 0$, $(X + \lambda_0)\tilde{u}^{(k)} = \tilde{u}^{(k-1)}$ and $U_-\tilde{u}^{(k)} = 0$ (with the convention that $\tilde{u}^{(-1)} = 0$) and $\tilde{u}^{(k)}$ is supported in $\tilde{\Lambda}_+$. By a direct application of (4.34) and (2.5), we have $\gamma^*\tilde{u}^{(k)} = \tilde{u}^{(k)}$ for all k and $\gamma \in \Gamma$, which implies that the distributions $\tilde{u}^{(k)}$ descend to distributions $u^{(k)}$ on SM supported in Λ_+ and satisfying $(X + \lambda_0)u^{(k)} = u^{(k-1)}$ and $U_-u^{(k)} = 0$. Finally, from the equation $(X + \lambda_0)u^{(k)} = u^{(k-1)}$ the wave-front set of $u^{(k)}$ is contained in the annulator of $E_0 = \mathbb{R}X$ by elliptic regularity, and similarly from $U_-u^{(k)} = 0$ it is also contained in the annulator of U_- , thus it has to be contained in E_u^* (which is E_+^* over Λ_+). Notice also that $\pi_{0*}u^{(k)} = \varphi_k$ for each k . This implies that $u^{(k)} \in V_0^k(\lambda_0)$ and the map $\pi_{0*} : V_0^k(\lambda_0) \rightarrow \text{Res}_\Delta^k(\lambda_0 + 1)$ is surjective.

Case $\lambda_0 \in -\frac{1}{2} - \mathbb{N}$. Note that $\text{Res}_\Delta^j(1/2 - k) = 0$ has been already shown in Proposition 4.4. In order to see $V_0^j(-1/2 - k) = 0$, we use a similar argument as above: If $u \in V_0^1(\lambda_0)$ with $\lambda_0 = -1/2 - k$ for $k \in \mathbb{N}$, then $\varphi := \pi_{0*}u$ solves $(\Delta_M - s_0(1 - s_0))\varphi = 0$ with $s_0 = \lambda_0 + 1$. Furthermore, by Lemma 2.1, there is $\omega \in \mathcal{D}(\mathbb{S}^1)$ supported in Λ_Γ so that $\gamma^*\omega = N_\gamma^{\lambda_0}\omega$ for all $\gamma \in \Gamma$ and $\tilde{\varphi} := \pi_\Gamma^*\varphi = \mathcal{P}_{\lambda_0}(\omega)$. Using (2.13) and (2.18), we see that $\tilde{\varphi} = \rho_0^{1/2-k}F_1 + \rho_0^{1/2+k}\log(\rho_0)F_2$ near each point $\tilde{p} \in \Omega_\Gamma$, where F_j are smooth functions on $\mathbb{H}^2 \cup \Omega_\Gamma$ near \tilde{p} . This implies that $\varphi \in \rho^{1/2-k}C^\infty(\overline{M}) \oplus \rho^{1/2+k}\log(\rho)C^\infty(\overline{M})$. But we also have $\varphi = \mathcal{E}_M(1/2 + k)f$ if $f := [\rho^{-1/2+k}\varphi]|_{\partial\overline{M}}$ by the properties of $\mathcal{E}_M(s)$ (see (4.13)). As in the proof of claim 1), the fact that $\text{supp}\omega \subset \Lambda_\Gamma$ implies the vanishing of the $\rho^{-1/2+k}\log(\rho)$ terms in the asymptotic of φ . This implies by (4.17) that $\mathcal{S}_M(1/2 + k)f = 0$, which by (4.18) shows that $f = 0$, and thus $\varphi = 0$, proving the claim 2).

Case $\lambda_0 = -1/2$. The arguments above show that each Ruelle resonant state $u \in V_0^1(-1/2)$ produces a quantum resonance $\varphi = \pi_{0*}u$ whose lift to \mathbb{H}^2 is $\mathcal{P}_{\lambda_0}(\omega) \in \rho^{1/2}C^\infty(\overline{M})$ at $s_0 = 1/2$ for some $\omega \in \mathcal{D}'(\mathbb{S}^1)$ supported in Λ_Γ and satisfying $\gamma^*\omega = N_\gamma^{1/2}\omega$ for all $\gamma \in \Gamma$. If there is an element $u' \in V_0^2(\lambda_0)$, then like in Theorem 3.3 we have $\varphi' := \pi_{0*}u'$ satisfying $(\Delta_M - 1/4)\varphi' = (2\lambda_0 + 1)\varphi = 0$, with $\pi_\Gamma^*(\varphi') =: \mathcal{P}_{\lambda_0}(\omega') - \partial_\lambda \mathcal{P}_{\lambda_0}(\omega)$ for some $\omega' \in \mathcal{D}'(\mathbb{S}^1)$ supported in Λ_Γ . We have

$$\partial_\lambda \mathcal{P}_{-1/2}(\omega)(x) = \log(1 - |x|^2)\mathcal{P}_{-1/2}(\omega)(x) + \int_{\mathbb{S}^1} P(x, \nu)^{1/2} \log(|x - \nu|^2)\omega(\nu)d\nu$$

and, using that $\text{supp}(\omega) \in \Lambda_\Gamma$ and (2.17), this implies directly that near a point $\tilde{p} \in \Omega_\Gamma$,

$$\partial_\lambda \mathcal{P}_{-1/2}(\omega)(x, \nu) = \sqrt{2}\log \rho_0(x)(\partial_\lambda \mathcal{S}(1/2)\omega)(\nu) + \mathcal{O}(1)$$

as $|x| \rightarrow 1$. Since $\mathcal{P}_{\lambda_0}(\omega') \in \rho_0^{1/2}C^\infty(\mathbb{H}^2 \cup \Omega_\Gamma)$, we get that $\varphi' = \log(\rho)f + \mathcal{O}(1)$ near $\partial\overline{M}$, for some non-zero $f \in C^\infty(\partial\overline{M})$. Therefore by 2) in Proposition 4.3 the function φ' is not a generalized resonant state. This shows that $-1/2$ is a pole of order at most

1 for $R_X(\lambda)$, there is no Jordan blocks and $\pi_{0*} : V_0^1(-1/2) \rightarrow \text{Res}_\Delta^1(-1/2)$ is injective. The surjectivity works as for the cases above.

Case $\lambda_0 \in -\mathbb{N}_0$, injectivity. By Proposition 2.4, among generalized states at $\lambda_0 = -n$ killed by U_- and π_{0*} , there can be only resonant states. Let u be a Ruelle resonant state satisfying $(X - n)u = 0$, $U_-u = 0$ and $\text{supp}(u) \subset \Lambda_+$. We can always assume that u is real valued: indeed, since the spectral parameter $-n$ is real valued, there is a basis of real-valued resonant states. The space $V_0(-n)$ can then be considered as a real vector space.

First, we assume that $\pi_{0*}u = 0$. Consider the Fourier components u_k in the fiber variables, then by the proof of Proposition 2.4, we have $u_k = 0$ for all $|k| < n$, and by (2.29) we also get $\eta_-u_n = 0$ and $\eta_+u_{-n} = 0$. In particular $\bar{\partial}u_n = 0$ and $\partial u_{-n} = 0$ when we view $u_{\pm n}$ as (distributional) sections of $\mathcal{K}^{\pm n}$. Notice by ellipticity that $u_{\pm n}$ are smooth and actually analytic. We will denote by $\tilde{u}_{\pm n} = \pi_\Gamma^*u_{\pm n}$ their lift to \mathbb{H}^2 , and in the ball model we can write $\tilde{u}_n = f_n dz^n$ and $\tilde{u}_{-n} = f_{-n} d\bar{z}^n$ for some holomorphic (resp. antiholomorphic) functions f_n (resp. f_{-n}) on \mathbb{H}^2 satisfying

$$\forall \gamma \in \Gamma, \quad \gamma^* f_n = (\partial_z \gamma)^{-n} f_n \quad \text{and} \quad \gamma^* f_{-n} = (\bar{\partial}_z \gamma)^{-n} f_{-n}.$$

Take the distribution $v := \Phi_-^n \tilde{u}$ where $\tilde{u} := \pi_\Gamma^*u$: we get $Xv = 0$ and $U_-v = 0$ thus there exists $\omega \in \mathcal{D}'(\mathbb{S}^1)$ so that $\mathcal{Q}_-\omega = v$ by (2.12). Since $\text{supp}(\tilde{u}) \subset B_-^{-1}(\Lambda_\Gamma)$, we get directly that $\text{supp}(\omega) \subset \Lambda_\Gamma$ and by (2.5), for any $\gamma \in \Gamma$, $\gamma^*\omega = N_\gamma^n \omega$. We want to write the Fourier mode \tilde{u}_k (in the fiber variable) in terms of ω : Therefore we write $z = x_1 + ix_2 \in \mathbb{H}^2$ and identify the unit tangent vector $\frac{1}{2}(1 - |z|^2)(\cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{x_2}) \in S_z\mathbb{H}^2$ with $e^{i\theta}$. We get

$$\begin{aligned} \tilde{u}_k(z) &= (2\pi)^{-1} \left(\int_{S_z\mathbb{H}^2} \tilde{u}(z, e^{i\theta}) e^{-ik\theta} d\theta \right) \left(\frac{2}{1 - |z|^2} \right)^k dz^k \\ &= 2^k (2\pi)^{-1} \left(\int_0^{2\pi} \omega(e^{i\alpha}) e^{-ik\alpha} |1 - ze^{-i\alpha}|^{2(n-1)} \frac{(\bar{z}e^{i\alpha} - 1)^k}{(1 - ze^{-i\alpha})^k} d\alpha \right) \frac{dz^k}{(1 - |z|^2)^{k+n-1}} \end{aligned}$$

where we used the change of variable $e^{i\theta} = B_z^{-1}(e^{i\alpha})$ defined by (2.2). For $|k| < n$ we have $\tilde{u}_k = 0$, thus by evaluating at $z = 0$ we get

$$\forall k \in (-n, n), \quad 0 = \int_0^{2\pi} \omega(e^{i\alpha}) e^{-ik\alpha} d\alpha. \quad (4.35)$$

For $k = n$, we know that $\tilde{u}_n = f_n dz^n$ with f_n holomorphic on \mathbb{H}^2 , thus we deduce that

$$f_n(z) = \frac{(-2)^n}{2\pi} \int_0^{2\pi} \omega(e^{i\alpha}) \frac{e^{-in\alpha}}{1 - ze^{-i\alpha}} d\alpha. \quad (4.36)$$

We deduce from this that $f_n(z)$ has a series expansion converging in $|z| < 1$ given by

$$f_n(z) = \frac{(-2)^n}{2\pi} \sum_{k=0}^{\infty} \omega_k z^k, \quad \omega_k := \langle \omega, e^{-i(n+k)\alpha} \rangle,$$

where here we notice that $|\omega_k| = \mathcal{O}((1+|k|)^N)$ for some N , since ω is in some Sobolev space on \mathbb{S}^1 (here and below, $\langle \cdot, \cdot \rangle$ denotes the bilinear distributional pairing on \mathbb{S}^1 with respect to the natural measure $d\alpha$ of mass 2π). From (4.36) and since $\text{supp}(\omega) \subset \Lambda_\Gamma$, we see that $f_n(z)$ extend holomorphically to $\mathbb{C} \setminus \Lambda_\Gamma$. The section $f_n(z)dz^n$ is holomorphic on $\mathbb{C} \setminus \Lambda_\Gamma$ and Γ equivariant, thus descend to a holomorphic section of \mathcal{K}^n on M_2 with the notation of Section 4.1. Similarly, we get (using that ω is real-valued)

$$f_{-n}(z) = \frac{(-2)^n}{2\pi} \int_0^{2\pi} \omega(e^{i\alpha}) \frac{e^{in\alpha}}{1 - \bar{z}e^{i\alpha}} d\alpha = \frac{(-2)^n}{2\pi} \sum_{k=0}^{\infty} \overline{\omega_k z^k} = \overline{f_n(z)}.$$

Now for each $\psi \in C^\infty(\mathbb{S}^1)$, write $\psi = \sum_{k \in \mathbb{Z}} \psi_k e^{ik\theta}$, then for $r < 1$ we get

$$\int_0^{2\pi} f_n(re^{i\theta}) \psi(\theta) d\theta = (-2)^n \sum_{k \geq 0} \omega_k r^k \psi_{-k}$$

and this converges to $(-2)^n \langle \Pi_+(\omega e^{-in\theta}), \psi \rangle$ as $r \rightarrow 1$ where Π_+ is the Szegő projector, i.e. the projector $\mathbb{1}_{[0, \infty)}(-i\partial_\theta)$ on the non-negative Fourier modes on \mathbb{S}^1 . By using (4.35), this means that f_n has a weak limit on \mathbb{S}^1 and $f_n|_{\mathbb{S}^1} = (-2)^n e^{-in\theta} \Pi_+(\omega)$ in the weak sense. Similarly, we have $f_{-n}|_{\mathbb{S}^1} = (-2)^n e^{in\theta} \Pi_-(\omega)$ if $\Pi_- := \mathbb{1}_{(-\infty, 0]}(-i\partial_\theta)$. Now by injectivity of the Poisson-Helgason transform at the spectral parameter 0 we obtain $C_n \in \mathbb{R}^*$ such that

$$\mathcal{P}_0(\omega) = \mathcal{P}_0(\Pi_+(\omega)) + \mathcal{P}_0(\Pi_-(\omega)) = C_n(z^n f_n + \overline{z^n f_n})$$

But since $\omega|_{\Omega_\Gamma} = 0$ the harmonic function $z^n f_n + \overline{z^n f_n}$ vanishes on Ω_Γ thus $e^{in\theta} f_n|_{\Omega_\Gamma} \in i\mathbb{R}$, which means that $i^{n+1} \iota_{\partial \overline{M}}^* u_n$ is a real-valued symmetric tensor on $\partial \overline{M}$. This is equivalent to say that $i^{n+1} u_n \in H_n(M)$. Moreover, the map $u \mapsto i^{n+1} u_n \in H_n(M)$ is injective since $u_n = 0$ implies $\omega = 0$ and thus $u = 0$.

Case $\lambda_0 \in -\mathbb{N}$, surjectivity. Conversely, let $u_n \in i^{-n-1} H_n(M)$ and consider its lift $\tilde{u}_n = f_n dz^n$ to \mathbb{H}^2 . The holomorphic function f_n satisfies $f_n(\gamma(z)) = (\partial_z \gamma(z))^{-n} f_n(z)$ for all $\gamma \in \Gamma$, or equivalently $\gamma^* \tilde{u}_n = \tilde{u}_n$, and we can assume that $e^{in\theta} f_n|_{\Omega_\Gamma} \in i\mathbb{R}$. The tensor u_n is bounded on \overline{M} with respect to the hyperbolic metric thus $|f_n(z) dz^n|_{g_{\mathbb{H}^2}} \in L^\infty(\mathbb{H}^2)$, and since $|2dz^n / (1 - |z|^2)^n|_{g_{\mathbb{H}^2}} = 1$, we deduce that $z \mapsto f_n(z)(1 - |z|^2)^n$ is bounded in the unit disk and therefore f_n is tempered. In particular there exists $\omega_\pm \in \mathcal{D}'(\mathbb{S}^1)$ so that $\mathcal{P}_0(\omega_+) = z^n f_n$ and $\mathcal{P}_0(\omega_-) = \overline{z^n f_n}$, and in fact $\omega_- = \overline{\omega_+}$. We have $\omega := \omega_+ + \omega_-$ which is supported on Λ_Γ since ω is the boundary value of $\text{Re}(z^n f_n)$ to \mathbb{S}^1 in the weak sense. Next we want to describe the covariance of ω with respect to each $\gamma \in \Gamma$: write $\gamma(e^{i\alpha}) = e^{i\mu(\alpha)}$ for the action on \mathbb{S}^1 , then $|d\gamma(e^{i\alpha})| = \mu'(\alpha)$ we have $\gamma^*(z^n f_n(z)) = (z \frac{\partial_z \gamma(z)}{\gamma(z)})^{-n} f_n(z)$, which when restricted on \mathbb{S}^1 gives

$$\gamma^* \omega = \left(\frac{-i\partial_\alpha(\gamma(e^{i\alpha}))}{\gamma(e^{i\alpha})} \right)^{-n} \omega = |d\gamma|^{-n} \omega. \quad (4.37)$$

Thus $\tilde{u} = \Phi_-^{-n} \mathcal{Q}_-(\omega) \in \mathcal{D}'(\mathbb{H}^2)$ solves $(X + n)\tilde{u} = 0$ and $U_- \tilde{u} = 0$, it is Γ -invariant by using (4.37) and has support in $\tilde{\Lambda}_+ = B_-^{-1}(\Lambda_\Gamma)$. This implies that \tilde{u} descends to a Ruelle resonance u , with support in Λ_- and with $\text{WF}(u) \subset E_+^*$ by ellipticity arguments as before. The map $u_n \mapsto u$ is injective since $u_n \rightarrow \omega$ is injective. Moreover by construction ω has vanishing k -Fourier components for all $|k| < n$ on \mathbb{S}^1 , thus it is in the kernel of \mathcal{P}_{-n} , which means that $\pi_{0*} u = 0$, and thus $\pi_{k*} u = 0$ for all $|k| < n$ by Lemma 2.2 (just like in the proof of Proposition 2.4).

To conclude the proof we have to prove that a Ruelle resonant state $u \in V_0^1(-n)$ satisfying $u_0 = \pi_{0*} u \neq 0$ does not exist. If $u_0 \neq 0$, we have by Lemma 2.1 that $(\Delta_M - n(1 - n))u_0 = 0$. The lift $\tilde{u}_0 = \pi_\Gamma^* u_0$ to \mathbb{H}^2 must be in $\text{Ran}(\mathcal{P}_{-n})$, which by (2.19) implies that $\tilde{u}_0 \in \rho_0^{1-n} C^\infty(\overline{\mathbb{H}^2})$ is an element of $\ker(\Delta_{\mathbb{H}^2} - n(1 - n))$ of the form

$$\tilde{u}_0(z) = (1 - |z|^2)^{1-n} L_n(z, \bar{z})$$

for some polynomial L_n of degree $2n - 2$. Since for each $\gamma \in \Gamma$ we have $(1 - |\gamma(z)|^2) = (1 - |z|^2)|\gamma'(z)|$, we deduce from $\gamma^* \tilde{u} = \tilde{u}$ that

$$L_n(\gamma(z), \overline{\gamma(z)}) = |\gamma'(z)|^{n-1} L_n(z, \bar{z}).$$

Taking $z = z_\pm$ to be the two fixed points of γ , we deduce that $L_n(z_\pm, \bar{z}_\pm) = 0$ since $|\gamma'(z_\pm)| \neq 1$. Therefore L_n is a polynomial in (z, \bar{z}) vanishing on the limit set Λ_Γ , and thus it vanishes on the whole \mathbb{S}^1 by analyticity, if Γ is non-elementary. We deduce that $\tilde{u}_0 = \mathcal{O}(\rho_0^{2-n})$ at $\mathbb{S} = \partial\mathbb{H}^2$, and thus $u \in \rho^{2-n} C^\infty(\overline{M})$. From the form of Δ_M near $\rho = 0$ given by (4.10), a Taylor expansion in $\rho = 0$ of the equation $(\Delta_M - n(1 - n))u_0 = 0$ implies that actually $u_0 \in \rho^n C^\infty(\overline{M}) \subset L^2(M)$, and therefore $u_0 = 0$ since $n(1 - n) \leq 0$, leading to a contradiction. \square

The case of an elementary group. Let us briefly discuss the case of an elementary group generated by one transformation $\gamma \in \text{PSL}_2(\mathbb{R})$, and we only need to focus on the spectral points $-n \in -\mathbb{N}$. We can assume that the two fixed points of γ in the disk model are ± 1 (with -1 being the repulsive one), and mapping the disk to the upper half-plane so that ∓ 1 is mapped to 0 and ± 1 to ∞ , the transformation γ is conjugated to $\gamma_\mp : z \mapsto e^{\pm \ell} z$ for some $\ell > 0$. By the discussion in the proof of Theorem 4.5, the Ruelle resonant states at $-n \in -\mathbb{N}$ are in correspondence with the $\omega \in \mathcal{D}'(\mathbb{S}^1)$ supported in $\{-1, 1\}$ and satisfying $\gamma^* \omega = |N_\gamma|^n \omega$. There can only be Dirac masses and its derivatives at ± 1 . To analyse the restriction of ω on a small neighborhood $O_\mp \subset \mathbb{S}^1$ of ∓ 1 , we use the upper half-plane model, $\omega|_{O_\mp}$ becomes (after conjugation) a distribution $\omega_\mp \in \mathcal{D}'(\mathbb{R})$ supported in $\{0\}$ and satisfying $\gamma_\mp^* \omega_\mp = e^{\mp n \ell} \omega_\mp$. The only solutions are of the form $c_\mp \delta_0^{(n-1)}$ for $c_\mp \in \mathbb{C}$ if $\delta_0^{(j)}$ denotes the j -th derivative of the Dirac mass δ_0 at 0 in \mathbb{R} . The same argument shows that there is no generalised resonant state (no Jordan blocks) for the operator X , i.e. $V_0^j(-n) = V_0^1(-n)$ for $j > 1$, and the space of Ruelle resonant states at $-n$ is exactly of dimension 2. The Poisson

kernel in the half space model is $P(x, y, x') = \frac{y}{y^2 + |x - x'|^2}$ where $z = x + iy \in \mathbb{H}^2$ and $x' \in \mathbb{R}$ is the point at infinity. We can just compute that in the upper half-space model where the repulsive fixed point $-1 \in \mathbb{S}^1$ is mapped to 0 and the repulsive $+1 \in \mathbb{S}^1$ to ∞ , we have

$$\mathcal{P}_{-n}(\omega_-) = c_- \langle \delta_{x'=0}^{(n-1)}, \frac{(y^2 + |x - x'|^2)^{n-1}}{y^{n-1}} \rangle = c_- Q(x, y) / y^{n-1}$$

where Q is a homogeneous polynomial of degree $n - 1$ in \mathbb{C} , satisfying $Q(-x, y) = (-1)^{n-1} Q(x, y)$. The same argument shows that in the half-plane model where $-1 \in \mathbb{S}^1$ is mapped to ∞ and $+1 \in \mathbb{S}^1$ to 0 shows that $\mathcal{P}_{-n}(\omega_+) = c_+ Q(x, y) / y^{n-1}$. On the other hand, the mapping from one half-plane model to the other can be chosen to be $A : z \mapsto -1/z$ and, due to the parity property of $Q(x, y)$ in x , we easily see that $Q(A(z)) / (\text{Im}(Az))^{n-1} = (-1)^{n-1} Q(z) / (\text{Im}(z))^{n-1}$. This shows that π_{0*} maps $V_0^1(-n)$ to a 1-dimensional space $\text{Res}_\Delta^1(-n + 1)$ of quantum resonant states. The elements in $\ker \pi_{0*} \cap V_0^1(-n)$ are in correspondence with $H_n(M)$, as we discussed in the proof of Theorem 4.5, and this space has real dimension equal to 1. The quantum resonances at $s = -n + 1$ are computed in [Bo, Chapters 5.1 and 8.2]: there are poles of order 2, there is a 1-dimensional space of quantum resonant states and the multiplicity is equal to 2 due to the Jordan block. The correspondence between classical and quantum resonant states is thus different from the non-elementary group case for these particular points.

Finally exactly the same proof as Corollary 3.4 gives the full Ruelle resonance spectrum.

Corollary 4.6. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a smooth oriented convex co-compact hyperbolic surface and let SM be its unit tangent bundle. Then for each $\lambda_0 \in \mathbb{C}$ with $\text{Re}(\lambda_0) \leq 0$, $k \in \mathbb{N}_0$, and $j \in \mathbb{N}$, the operator U_+^k is injective on $V_0^j(\lambda_0 + k)$ and we get*

$$V_k^j(\lambda_0) = \bigoplus_{\ell=0}^k U_+^\ell (V_0^j(\lambda_0 + \ell)).$$

5. ZETA FUNCTIONS

The zeta function of the flow is defined by

$$Z_X(\lambda) = \exp \left(- \sum_{\gamma_0} \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{-\lambda k \ell(\gamma_0)}}{|\det(1 - P(\gamma_0)^k)|} \right) \quad (5.1)$$

where γ_0 are primitive closed geodesics and $P(\gamma_0)$ is the linearized Poincaré map of the geodesic flow on this geodesic. The function converges for $\text{Re}(\lambda) > \delta_\Gamma$ where $\delta_\Gamma < 1$ is the Hausdorff dimension of the limit set Λ_Γ (see [Pa]). By [DyGu, Theorem 4],

the function $Z_X(\lambda)$ admits a holomorphic extension to $\lambda \in \mathbb{C}$ with zeros at Ruelle resonances and the order of a Ruelle resonance λ_0 as a zero of $Z_X(\lambda)$ is given by

$$\text{ord}_{\lambda_0}(Z_X(\lambda)) = \text{Rank}(\Pi_{\lambda_0}^X) \quad (5.2)$$

where $\Pi_{\lambda_0}^X = -\text{Res}_{\lambda_0} R_X(\lambda)$ is the projector on generalized Ruelle resonant states. In particular we deduce from (5.2) and Theorem 4.6 the

Proposition 5.1. *The order of λ_0 as a zero of $Z_X(\lambda)$ is given by*

$$\text{ord}_{\lambda_0}(Z_X(\lambda)) = \dim \text{Res}_X(\lambda_0) = \sum_{p \in \mathbb{N}_0} \dim(\text{Res}_X(\lambda_0 + p) \cap \ker U_-).$$

where $\text{Res}_X(\lambda_0) = \cup_{j \geq 1} \text{Res}_X^j(\lambda_0)$ is the space of generalized resonant states at λ_0 , with $\text{Res}_X^j(\lambda_0)$ defined in (4.7).

The Selberg zeta function $Z_S(\lambda)$ is defined by

$$Z_S(\lambda) := \exp \left(- \sum_{\gamma_0} \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{-\lambda k \ell(\gamma_0)}}{\det(1 - P_s(\gamma_0)^k)} \right) \quad (5.3)$$

where the sum is over all primitive closed geodesics and $P_s(\gamma_0) = P(\gamma_0)|_{E_s}$ is the contracting part of $P(\gamma_0)$. For each closed geodesic γ on $M = \Gamma \backslash \mathbb{H}^2$, there is an associated conjugacy class in Γ , with a representative that we still denote by $\gamma \in \Gamma$ and whose axis in \mathbb{H}^2 descends to the geodesic γ ; the linear Poincaré map along this closed geodesic is easy to compute since γ is conjugated to $z \mapsto e^{\ell(\gamma)} z$ in the upper half-space model of \mathbb{H}^2 . Using this expression we get $P(\gamma_0)|_{E_s} = e^{-\ell(\gamma_0)} \text{Id}$ and $P(\gamma_0)|_{E_u} = e^{\ell(\gamma_0)} \text{Id}$ thus

$$e^{-\frac{1}{2}k\ell(\gamma_0)} \det(1 - P_s(\gamma_0)^k)^{-1} = e^{-\frac{1}{2}k\ell(\gamma_0)} (1 - e^{-k\ell(\gamma_0)})^{-1} = \sum_{p=0}^{\infty} e^{-k\ell(\gamma_0)(1/2+p)} \quad (5.4)$$

and $|\det(1 - P(\gamma_0)^k)| = e^{k\ell(\gamma_0)} \det(1 - P_s(\gamma_0)^k)^2$. This implies the formula

$$Z_X(\lambda) = \prod_{p=1}^{\infty} Z_S(\lambda + p), \quad Z_S(\lambda) = \frac{Z_X(\lambda - 1)}{Z_X(\lambda)}. \quad (5.5)$$

By combining (5.5) with Proposition 5.1 and Theorem 4.5, we obtain

Corollary 5.2. *Let $M = \Gamma \backslash \mathbb{H}^2$ be a convex co-compact oriented hyperbolic smooth surface and assume Γ is non-elementary. Then its Selberg zeta function $Z_S(s)$ is holomorphic with zeros given by:*

1) quantum resonances $s_0 \notin -\mathbb{N}_0$ with order

$$\text{ord}_{s_0}(Z_S(s)) = \text{Rank}(\text{Res}_{s_0} R_{\Delta}(s))$$

2) *negative integers* $-n \in -\mathbb{N}_0$ *with order*

$$\text{ord}_{-n}(Z_S(s)) = \dim_{\mathbb{R}} H_{n+1}(M) = \begin{cases} (2n+1)|\chi(M)| & \text{if } n > 0, \\ |\chi(M)| + 1 & \text{if } n = 0 \end{cases}$$

where $H_{n+1}(M)$ are defined by (4.1).

The description of the zeros of $Z_{\Gamma}(s)$ in this setting was also done by Borthwick-Judge-Perry [BJP] (including the case with cusps). We remark that in [BJP] the topological contribution in the order of $Z_S(s)$ at $s = 0$ is $|\chi(M)|$. Furthermore there is a spectral contribution coming from the multiplicity of 0 as a quantum resonance. This multiplicity is exactly 1 by the proof of Theorem 1.2 in [GuGu], the resonant states being the constants (see the discussion after the proof of [GuGu, Theorem 1.2]), which matches with part 2) in Corollary 5.2. For the points $s = -n$ with $n \in \mathbb{N}$, [BJP] obtain a zero of order $(2n+1)|\chi(M)| + \nu(-n)$ where $\nu(-n) = \text{Rank}(\text{Res}_{-n}R_{\Delta}(s))$ is the order of $-n$ as a resonance, and thus 2) of Corollary 5.2 implies that $\nu(-n) = 0$ for non-elementary groups Γ .

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