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# GEODESIC STRETCH, PRESSURE METRIC AND MARKED LENGTH SPECTRUM RIGIDITY

COLIN GUILLARMOU, GERHARD KNEIPER, AND THIBAUT LEFEUVRE

ABSTRACT. We refine the recent local rigidity result for the marked length spectrum obtained by the first and third author in [GL19] and give an alternative proof using the geodesic stretch between two Anosov flows and some uniform estimate on the variance appearing in the central limit theorem for Anosov geodesic flows. In turn, we also introduce a new pressure metric on the space of isometry classes, that reduces to the Weil-Peterson metric in the case of Teichmüller space and is related to the works of [MM08, BCLS15].

## 1. INTRODUCTION

Let  $M$  be a smooth closed  $n$ -dimensional manifold. We denote by  $\mathcal{M}$  the Fréchet manifold consisting of smooth metrics on  $M$ . We denote by  $\mathcal{M}^{k,\alpha}$  the set of metrics with regularity  $C^{k,\alpha}$ ,  $k \in \mathbb{N}, \alpha \in (0, 1)$ . We fix a smooth metric  $g_0 \in \mathcal{M}$  with Anosov geodesic flow and define the unit tangent bundle by  $S_{g_0}M := \{(x, v) \in TM \mid |v|_{g_0} = 1\}$ . Recall that being Anosov means that there exists a flow-invariant continuous splitting

$$T(S_{g_0}M) = \mathbb{R}X \oplus E_s \oplus E_u,$$

such that

$$\begin{aligned} \|d\varphi_t^{g_0}(w)\| &\leq Ce^{-\lambda t}\|w\|, \quad \forall w \in E_s, \forall t \geq 0, \\ \|d\varphi_t^{g_0}(w)\| &\leq Ce^{-\lambda|t|}\|w\|, \quad \forall w \in E_u, \forall t \leq 0, \end{aligned}$$

where the constants  $C, \lambda > 0$  are uniform and the norm here is the one induced by the Sasaki metric of  $g_0$ . Such a property is satisfied in negative curvature.

**1.1. Geodesic stretch and marked length spectrum rigidity.** The set of primitive free homotopy classes  $\mathcal{C}$  of  $M$  is in one-to-one correspondance with the primitive conjugacy classes of  $\pi_1(M, x_0)$  (where  $x_0 \in M$  is arbitrary). When  $g_0$  is Anosov, there exists a unique closed geodesic  $\gamma_{g_0}(c)$  in each primitive free homotopy class  $c \in \mathcal{C}$  (see [Kli74]). This allows to define the *marked length spectrum* of the metric  $g_0$  by:

$$L_{g_0} : \mathcal{C} \rightarrow \mathbb{R}_+, \quad L_{g_0}(c) = \ell_{g_0}(\gamma_{g_0}(c)),$$

where  $\ell_{g_0}(\gamma)$  denotes the  $g_0$ -length of a curve  $\gamma \subset M$  computed with respect to  $g_0$ . The marked length spectrum can alternatively be defined for the whole set of free homotopy classes but it is obviously an equivalent definition.

It was conjectured by Burns-Katok [BK85] that the marked length spectrum of negatively curved manifolds determine the metric up to isometry in the sense that two negatively curved metric  $g$  and  $g_0$  with same marked length spectrum (namely  $L_g = L_{g_0}$ ) should be isometric. Although the conjecture was proved for surfaces by Croke and Otal [Ota90, Cro90]) and in some particular cases in higher dimension (for conformal metrics by Katok [Kat88] and when  $(M, g_0)$  is a locally symmetric space by the work of Hamenstädt and Besson-Courtois-Gallot [BCG95, Ham99]), it is still open in dimension higher or equal to 3 and open even in dimension 2 in the more general setting of Riemannian metrics with Anosov geodesic flows. The same type of problems can also be asked for billiards and we mention recent results on this problem by Avila-De Simoi-Kaloshin [ADSK16] and De Simoi-Kaloshin-Wei [DSKW17] for convex domains close to ellipses (although the Anosov case would rather correspond to the case of hyperbolic billiards). Recently, the first and last author obtained the following result on Burns-Katok conjecture:

**Theorem** (Guillarmou-Lefeuvre [GL19]). *Let  $(M, g_0)$  be a smooth Riemannian manifold with Anosov geodesic flow and further assume that its curvature is nonpositive if  $\dim M \geq 3$ . Then there exists  $k \in \mathbb{N}$  depending only on  $\dim M$  and  $\varepsilon > 0$  small enough depending on  $g_0$  such that the following holds: if  $g \in \mathcal{M}$  is such that  $\|g - g_0\|_{C^k} \leq \varepsilon$  and  $L_g = L_{g_0}$ , then  $g$  is isometric to  $g_0$ .*

One of the aims of this paper is to further investigate this result from different perspectives: new stability estimates and a refined characterization of the condition under which the isometry may hold. More precisely, our first result is that we can locally relax the assumption that the two marked length spectra of  $g$  and  $g_0$  exactly coincide to the weaker assumption that they "coincide at infinity" (in some sense that is made precise below) and still obtain the isometry. Given  $c \in \mathcal{C}$ , we also write  $\delta_{g_0}(c)$  to denote the probability Dirac measure carried by the unique  $g_0$ -geodesic  $\gamma_{g_0}(c) \in c$ .

In the following, we will say that  $L_g/L_{g_0} \rightarrow 1$  when

$$\lim_{j \rightarrow +\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} = 1, \quad (1.1)$$

for any sequence  $(c_j)_{j \in \mathbb{N}}$  of primitive free homotopy classes such that  $\lim_{j \rightarrow \infty} L_{g_0}(c_j) = +\infty$ , or equivalently  $\lim_{j \rightarrow \infty} L_g(c_j)/L_{g_0}(c_j) = 1$ , if  $\mathcal{C} = (c_j)_{j \in \mathbb{N}}$  is ordered by the increasing lengths  $L_{g_0}(c_j)$ . We notice that it is important to consider only the set of *primitive* closed geodesics in (1.1) in order to not imply directly that  $L_g = L_{g_0}$ .

**Theorem 1.1.** *Let  $(M, g_0)$  be a smooth Anosov Riemannian  $n$ -dimensional manifold and further assume that its curvature is nonpositive if  $n \geq 3$ . There exists  $k \in \mathbb{N}$  depending only on  $n$ ,  $\varepsilon > 0$  depending on  $g_0$  such that for each  $\alpha \in (0, 1)$  the following holds: let  $g \in \mathcal{M}^{k, \alpha}$  such that  $\|g - g_0\|_{C^{k, \alpha}} \leq \varepsilon$  and assume that  $L_g/L_{g_0} \rightarrow 1$ . Then  $g$  is isometric to  $g_0$ .*

We develop a new strategy of proof, different from [GL19], which relies on the introduction of the *geodesic stretch* between two metrics. This quantity was first introduced by Croke-Fathi [CF90] and further studied by the second author [Kni95]. If  $g$  is close enough to  $g_0$ , then by Anosov structural stability, the geodesic flows  $\varphi^{g_0}$  and  $\varphi^g$  are orbit conjugate via a homeomorphism  $\psi$ , i.e. they are conjugate up to a time reparametrization. The *infinitesimal stretch* is the infinitesimal function of time reparametrization  $a_g$ : it satisfies  $d\psi_g(z)X_{g_0}(z) = a_g(z)X_g(\psi_g(z))$  where  $z \in S_{g_0}M$  and  $X_{g_0}$  (resp.  $X_g$ ) denotes the geodesic vector field of  $g_0$  (resp.  $g$ ). The *geodesic stretch between  $g$  and  $g_0$  with respect to the Liouville<sup>1</sup> measure  $\mu_{g_0}^L$*  of  $g_0$  is then defined by

$$I_{\mu_{g_0}^L}(g_0, g) := \int_{S_{g_0}M} a_g d\mu_{g_0}^L.$$

It turns out to be equal to

$$I_{\mu_{g_0}^L}(g_0, g) = \lim_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)},$$

if  $(c_j)_{j \in \mathbb{N}} \subset \mathcal{C}$  is a sequence so that the uniform probability measures  $(\delta_{g_0}(c_j))_{j \in \mathbb{N}}$  supported on the closed geodesics of  $g_0$  in the class  $c_j$  converge to  $\mu_{g_0}^L$  in the weak sense of measures. While it has an interest on its own, it turns out that this method involving the geodesic stretch provides a new estimate which quantifies locally the distance between isometry classes in terms of this geodesic stretch functional (below  $H^{-\frac{1}{2}}(M)$  denotes the  $L^2$ -based Sobolev space of order  $-1/2$  and  $\alpha \in (0, 1)$  is any fixed exponent).

**Theorem 1.2.** *Let  $(M, g_0)$  be a smooth Riemannian  $n$ -dimensional manifold with Anosov geodesic flow and further assume that its curvature is nonpositive if  $n \geq 3$ . There exists  $k \in \mathbb{N}$  large enough depending only on  $n$ , some positive constants  $C, C', \varepsilon$  depending on  $g_0$  such that for all  $\alpha \in (0, 1)$ , the following holds: for each  $g \in \mathcal{M}^{k, \alpha}$  with  $\|g - g_0\|_{C^{k, \alpha}(M)} \leq \varepsilon$ , there exists a  $C^{k+1, \alpha}$ -diffeomorphism  $\psi : M \rightarrow M$  such that*

$$\begin{aligned} \|\psi^*g - g_0\|_{H^{-\frac{1}{2}}(M)} &\leq C \left( |1 - I_{\mu_{g_0}^L}(g_0, g)|^{\frac{1}{2}} + |\mathbf{P}(-J_{g_0}^u + a_g - 1)|^{\frac{1}{2}} \right) \\ &\leq C' \left( |\mathcal{L}_+(g)|^{\frac{1}{2}} + |\mathcal{L}_-(g)|^{\frac{1}{2}} \right) \end{aligned}$$

where  $J_{g_0}^u$  is the unstable Jacobian of  $\varphi^{g_0}$ ,  $\mathbf{P}$  denotes the topological pressure for the  $\varphi^{g_0}$  flow,  $a_g$  is the reparameterization coefficient relating  $\varphi^{g_0}$  and  $\varphi^g$  defined above, and

$$\mathcal{L}_+(g) := \limsup_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1, \quad \mathcal{L}_-(g) := \liminf_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1.$$

Note that Theorem 1.1 is a direct consequence of Theorem 1.2 (we have  $\mathbf{P}(-J_{g_0}^u + a_g - 1) = 0$  if  $L_g/L_{g_0} \rightarrow 1$ ), we will thus directly prove that stronger result. This result is an improvement over the Hölder stability result [GL19, Theorem 3] as it only involves the

<sup>1</sup>normalized with total mass 1

asymptotic behaviour of  $L_g/L_{g_0}$ . We emphasize however that one of the key facts to prove this theorem still boils down to some elliptic estimate on some variance operator acting on symmetric 2-tensors, denoted by  $\Pi_2^{g_0}$  in [GL19, Gui17]: indeed, we show that the combination of the Hessians of the geodesic stretch at  $g_0$  and of the pressure functional can be expressed in terms of this variance operator, which enjoys uniform lower bounds  $C_{g_0}\|\psi^*g - g_0\|_{H^{-1/2}}$  for some  $C_{g_0} > 0$ , at least once we have factored out the gauge (the diffeomorphism action by pull-back on metrics).

We also notice that in Theorem 1.2, although the  $H^{-\frac{1}{2}}(M)$  norm is a weak norm, a straightforward interpolation argument using that  $\|g\|_{C^{k,\alpha}} \leq \|g_0\|_{C^{k,\alpha}} + \varepsilon$  is uniformly bounded shows that an estimate of the form

$$\|\psi^*g - g_0\|_{C^{k'}} \leq C(|\mathcal{L}_+(g)| + |\mathcal{L}_-(g)|)^\delta$$

holds for any  $k' < k - n/2$  and some explicit  $\delta \in (0, 1/2)$  depending on  $k, k'$  ( $C > 0$  depending only on  $g_0$ ).

**1.2. Variance and pressure metric.** The variance operator appearing in the proof of Theorem 1.1 and 1.2 can be defined for  $h_1, h_2 \in C^\infty(M; S^2T^*M)$  satisfying  $\langle h_1, g_0 \rangle_{L^2} = \int_{SM} \text{Tr}_{g_0}(h_1) d\text{vol}_{g_0} = 0$  by

$$\langle \Pi_2^{g_0} h_1, h_2 \rangle := \int_{\mathbb{R}} \int_{SM} \langle e^{tX_{g_0}} h_1, h_2 \rangle_{g_0} d\mu_{g_0}^L dt,$$

where, for  $z = (x, v) \in SM$ ,  $(e^{tX_{g_0}} h)(z) := h_{\gamma_z(t)}(\dot{\gamma}_z(t), \dot{\gamma}_z(t))$  if  $\gamma_z(t)$  is the  $g_0$ -geodesic with initial condition  $(\gamma_z(0), \dot{\gamma}_z(0)) = (x, v)$ , and the product  $\langle \cdot, \cdot \rangle_{g_0}$  is the natural scalar product on symmetric tensors induced by  $g_0$ . The integral defining  $\Pi_2^{g_0}$  converges thanks to the exponential mixing of  $\varphi^{g_0}$  (see [Liv04]). The operator  $\Pi_2^{g_0}$  is a pseudodifferential operator of order  $-1$  that is elliptic on divergence-free tensors (see [Gui17, GL19, GL]). As a consequence, it satisfies good coercive estimates on all Sobolev or Hölder spaces: there is  $C_{g_0} > 0$  such that for all  $h \in H^{-\frac{1}{2}}(M; S^2T^*M)$  which is divergence-free (i.e.  $\text{Tr}_{g_0}(\nabla^{g_0} h) = 0$ )

$$\langle \Pi_2^{g_0} h, h \rangle \geq C_{g_0} \|h\|_{H^{-\frac{1}{2}}(M)}^2, \tag{1.2}$$

provided  $g_0$  is Anosov with non-positive curvature (or simply Anosov if  $\dim M = 2$ ). We show in Proposition 4.1 that  $g \mapsto \Pi_2^g$  is continuous with values in  $\Psi^{-1}(M)$  and this implies that for  $g_0$  a smooth Anosov metric (with non-positive curvature if  $\dim M > 2$ ), (1.2) holds uniformly if we replace  $g_0$  by any metric  $g$  in a small  $C^\infty$ -neighborhood of  $g_0$ . This allows to obtain a more uniform version of Theorems 1.1 and 1.2.

**Theorem 1.3.** *Let  $(M, g_0)$  be a smooth Riemannian  $n$ -dimensional manifold with Anosov geodesic flow and further assume that its curvature is nonpositive if  $n \geq 3$ . Then there exists  $k \in \mathbb{N}$ ,  $\varepsilon > 0$  and  $C_{g_0}$  depending on  $g_0$  such that for all  $g_1, g_2 \in \mathcal{M}$  such that*

$\|g_1 - g_0\|_{C^k} \leq \varepsilon$ ,  $\|g_2 - g_0\|_{C^k} \leq \varepsilon$ , there is a  $C^k$ -diffeomorphism  $\psi : M \rightarrow M$  such that

$$\|\psi^* g_2 - g_1\|_{H^{-\frac{1}{2}}(M)} \leq C_{g_0} (|\mathcal{L}_+(g_1, g_2)|^{\frac{1}{2}} + |\mathcal{L}_+(g_2, g_1)|^{\frac{1}{2}})$$

with

$$\mathcal{L}_+(g_1, g_2) := \limsup_{j \rightarrow \infty} \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} - 1.$$

In particular if  $L_{g_1} = L_{g_2}$ , then  $g_2$  is isometric to  $g_1$ .

This result suggests to define a distance on isometry classes metrics from the marked length spectrum by setting for  $g_1, g_2$  two  $C^{k,\alpha}$  metrics

$$d_L(g_1, g_2) := \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} \right|^{\frac{1}{2}} + \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} \right|^{\frac{1}{2}}.$$

We have as a corollary of Theorem 1.3

**Corollary 1.4.** *The map  $d_L$  descends to the space of isometry classes of Anosov non-positively curved metrics and defines a distance near the diagonal.*

We also define the *Thurston asymmetric distance* by

$$d_T(g_1, g_2) := \limsup_{j \rightarrow \infty} \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)}$$

and show that this is a distance on isometry classes of metrics with topological entropy equal to 1, see Proposition 4.9. This distance was introduced in Teichmüller theory by Thurston in [Thu98].

The coercive estimate (1.2) allows also to define a *pressure metric* on the open set consisting of isometry classes of Anosov non-positively curved metric (contained in  $\mathcal{M}/\mathcal{D}_0$  if  $\mathcal{D}_0$  is the group of smooth diffeomorphisms isotopic to the identity) by setting for  $h_1, h_2 \in T_{g_0}(\mathcal{M}/\mathcal{D}_0) \subset C^\infty(M; S^2 T^* M)$

$$G_{g_0}(h_1, h_2) := \langle \Pi_2^{g_0} h_1, h_2 \rangle_{L^2(M, d \text{vol}_{g_0})}.$$

We show in Section 3.3.1 that this metric is well-defined and restricts to (a multiple of) the Weil-Petersson metric on Teichmüller space if  $\dim M = 2$ : it is related to the construction of Bridgeman-Canary-Labourie-Sambarino [BCLS15, BCS18] and Mc Mullen [MM08], but with the difference that we work here in the setting of variable negative curvature and the space of metrics considered here is infinite dimensional.

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## 2. PRELIMINARIES

On a closed manifold  $M$ , we will denote by  $\Psi^m(M)$  the space of classical pseudo-differential operators of order  $m \in \mathbb{R}$ . We recall that this is a Fréchet space when equipped with semi-norms of its full symbols in local charts. We also denote by  $H^s(M)$  the  $L^2$ -based Sobolev space of order  $s \in \mathbb{R}$ , with norm given by fixing a Riemannian metric on  $M$ . For  $k \in \mathbb{N}, \alpha \in (0, 1)$ , the spaces  $C^{k,\alpha}(M)$  are the usual Hölder spaces and  $\mathcal{D}'(M)$  will denote the space of distributions, dual to  $C^\infty(M)$ .

**2.1. Symmetric tensors and X-ray transform.** In this paragraph, we assume that the metric  $g$  is fixed and that its geodesic flow  $\varphi_t^g$  is Anosov on the unit tangent bundle  $SM$  of  $g$ . We denote by  $\mu^L$  the Liouville measure, normalized to be a probability measure on  $SM$ . For the sake of simplicity, we drop the index  $g$  in the notations. Given an integer  $m \in \mathbb{N}$ , we denote by  $\otimes^m T^*M \rightarrow M$ ,  $S^m T^*M \rightarrow M$  the respective vector bundle of  $m$ -tensors and symmetric  $m$ -tensors on  $M$ . Given  $f \in C^\infty(M; S^m T^*M)$ , we denote by  $\pi_m^* f \in C^\infty(SM)$  the canonical morphism  $\pi_m^* f : (x, v) \mapsto f_x(v, \dots, v)$ . The natural derivation of symmetric tensors is  $D := \sigma \circ \nabla$ , where  $\nabla$  is the Levi-Civita connexion and  $\sigma : \otimes^m T^*M \rightarrow S^m T^*M$  is the operation of symmetrization. The operator  $D$  is elliptic [GL, Lemma 2.4] with trivial kernel when  $m$  is odd and 1-dimensional kernel when  $m$  is even, given by the Killing tensors  $c\sigma(g^{\otimes m/2}), c \in \mathbb{R}$ . We denote by  $\langle \cdot, \cdot \rangle$  the scalar product on  $C^\infty(M; S^m T^*M)$  induced by the metric  $g_0$  (see [GL, Section 2] for further details). The formal adjoint of  $D$  with respect to this scalar product is  $D^* = -\text{Tr} \circ \nabla$ . We also denote by the same  $\langle \cdot, \cdot \rangle$  the natural  $L^2$  scalar product on  $C^\infty(SM)$  induced by the Liouville measure  $\mu^L$ . The formal adjoint of  $\pi_m^*$  with respect to these two scalar products is denoted by

$$\pi_{m*} : \mathcal{D}'(SM) \rightarrow \mathcal{D}'(M; S^m T^*M)$$

where  $\mathcal{D}'$  denotes the space of distributions, dual to  $C^\infty$ .

We recall that  $\mathcal{C}$ , the set of free homotopy classes in  $M$ , is in one-to-one correspondance with the set of conjugacy classes of  $\pi_1(M, x_0)$  for some arbitrary choice of  $x_0 \in M$  (see [Kli74]) and for each  $c \in \mathcal{C}$  there exists a unique closed geodesic  $\gamma(c) \in c$ . We denote its Riemannian length with respect to  $g$  by  $L(c) = \ell_g(\gamma(c))$ . The *X-ray transform* on  $SM$  is the operator defined by:

$$I : C^0(SM) \rightarrow \ell^\infty(\mathcal{C}), \quad If(c) = \frac{1}{L(c)} \int_0^{L(c)} f(\varphi_t(z)) dt,$$

where  $z \in \gamma(c)$  is any point. This is a continuous linear operator when  $\ell^\infty(\mathcal{C})$  is endowed with the sup norm on the sequences. Then, the X-ray transform  $I_m$  of symmetric  $m$ -tensors is simply defined by  $I_m := I \circ \pi_m^*$ . If  $X$  denotes the geodesic vector field on  $SM$ , a straightforward computation gives that  $X\pi_m^* = \pi_{m+1}^*D$ . This immediately implies that

$$\{Dp \mid p \in C^\infty(M; S^{m-1}T^*M)\} \subset \ker I_m \cap C^\infty(M; S^mT^*M). \quad (2.1)$$

Using the ellipticity of  $D$ , any tensor  $f \in C^\infty(M; S^mT^*M)$  can be decomposed uniquely as a sum

$$f = Dp + h, \quad (2.2)$$

with  $p \in C^\infty(M; S^{m-1}T^*M)$  and  $h \in C^\infty(M; S^mT^*M)$  is such that  $D^*h = 0$ . We call  $Dp$  the *potential part* of  $f$  and  $h$  the *solenoidal part*. The same decomposition holds in Sobolev regularity  $H^s(M)$ ,  $s \in \mathbb{R}$ , and in the  $C^{k,\alpha}(M)$  regularity,  $k \in \mathbb{N}, \alpha \in (0, 1)$ . We will write  $h = \pi_{\ker D^*}f$  and the solenoidal projection  $\pi_{\ker D^*}$  is a pseudodifferential operator of order 0 [GL, Lemma 2.6]. The X-ray transform is said to be *solenoidal injective* (or s-injective in short) if (2.1) is an equality. It is conjectured that  $I_m$  is s-injective as long as the metric is Anosov but it is only known in the following cases:

- for  $m = 0, 1$  [DS10],
- for any  $m \in \mathbb{N}$  in dimension 2 [PSU14, Gui17],
- for any  $m \in \mathbb{N}$ , in any dimension in non-positive curvature [CS98].

We also observe that the coercive estimate (1.2) and the continuity of  $g \mapsto \Pi_2^g \in \Psi^{-1}(M)$  implies the s-injectivity of  $I_2$  (and actually of  $I_m$  for any  $m \in \mathbb{N}$ , the generalization being straightforward) for any metric  $g$  in a  $C^\infty$ -neighborhood of a fixed Anosov metric  $g_0$  which is non-positively curved (see Proposition 2.3). In particular, this allows Anosov metrics with “some” positive curvature.

The direct study of the analytic properties of  $I_m$  is difficult as this operator involves integrals over the set of closed orbits, which is not a manifold. Nevertheless, in [Gui17], the second author introduced an operator  $\Pi_m$  that involves a sort of integration of tensors over “all orbits” and this space is essentially the manifold  $SM$ . The construction of  $\Pi_m : C^\infty(M; S^mT^*M) \rightarrow \mathcal{D}'(M; S^mT^*M)$  relies on microlocal tools coming from [FS11, DZ16] but a simpler definition that uses the fast mixing of the flow  $\varphi_t$  is given by

$$\begin{aligned} \Pi_m &:= \pi_{m*}(\Pi + \langle \cdot, 1 \rangle)\pi_m^* \text{ with} \\ \Pi : C^\infty(SM) &\rightarrow \mathcal{D}'(SM), \quad \langle \Pi f, f' \rangle := \lim_{T \rightarrow \infty} \int_{-T}^T \langle e^{tX} f, f' \rangle dt \end{aligned} \quad (2.3)$$

if  $\langle f, 1 \rangle = \int_{SM} f d\mu^L = 0$  and  $\Pi(1) := 0$ . The convergence of the integral as  $T \rightarrow \infty$  is insured by the exponential decay of correlations [Liv04] (in fact a polynomial decay is sufficient). We can thus write for  $\langle f, 1 \rangle = 0$

$$\langle \Pi f, f' \rangle = \int_{\mathbb{R}} \langle f \circ \varphi_t^{g_0}, f' \rangle_{L^2(SM)} dt.$$

We note the following useful properties of  $\Pi$ , proved in [Gui17, Theorem 1.1]:

- $\Pi : H^s(SM) \rightarrow H^{-s}(SM)$  is bounded for all  $s > 0$
- if  $f \in H^s(SM)$  with  $s > 0$ ,  $X\Pi f = 0$
- if  $f$  and  $Xf$  belong to  $H^s(SM)$  for  $s > 0$ , then  $\Pi Xf = 0$ .

We can make a link between  $\Pi$  and the variance in the central limit theorem for Anosov geodesic flows. The *variance* of  $\varphi_t$  with respect to the Liouville measure  $\mu^L$  is defined for  $u \in C^\alpha(SM)$ ,  $\alpha \in (0, 1)$  real-valued by:

$$\text{Var}_{\mu^L}(u) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_{SM} \left( \int_0^T u(\varphi_t(z)) dt \right)^2 d\mu^L(z), \quad (2.4)$$

under the condition that  $\int_{SM} u d\mu^L = 0$ . We observe, since  $\varphi_t$  preserves  $\mu^L$ , that

$$\begin{aligned} \text{Var}_{\mu^L}(u) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{S_{g_0}M} \int_0^T \int_0^T u(\varphi_{t-s}(z))u(z) dt ds d\mu^L(z) \\ &= \lim_{T \rightarrow \infty} \int_0^1 \int_{\mathbb{R}} \mathbf{1}_{[(t-1)T, tT]}(r) \langle u \circ \varphi_r, u \rangle_{L^2} dr dt. \end{aligned}$$

where the  $L^2$  pairing is with respect to  $\mu^L$ . By exponential decay of correlations [Liv04], we have for  $|r|$  large

$$|\langle u \circ \varphi_r, u \rangle_{L^2}| \leq C e^{-\nu|r|} \|u\|_{C^\alpha}^2$$

for some  $\alpha > 0$ ,  $\nu > 0$ ,  $C > 0$  independent of  $u$ . Thus, by Lebesgue theorem,

$$\text{Var}_{\mu^L}(u) = \langle \Pi u, u \rangle, \quad (2.5)$$

if  $\langle u, 1 \rangle = 0$ , showing that our operator  $\Pi$  is nothing more than the variance.

In the papers [Gui17, GL19, GL], the properties of  $\Pi_m$  are investigated:

- $\Pi_m$  is a positive selfadjoint pseudodifferential operator of order  $-1$ ,
- $\Pi_m D = 0$  and  $D^* \Pi_m = 0$ ,
- $\Pi_m$  is elliptic on solenoidal tensors; under the additional assumption that  $I_m$  is  $s$ -injective, it is invertible on solenoidal tensors in the sense that there exists a pseudodifferential operator  $Q$  of order 1 such that  $Q\Pi_m = \pi_{\ker D^*}$ .

In particular, using the spectral theorem, there is a bounded self-adjoint operator  $\sqrt{\Pi_m}$  on  $L^2$  such that  $\sqrt{\Pi_m} \sqrt{\Pi_m} = \Pi_m$ . We add the following property which will be used in this article:

**Lemma 2.1.** *If  $(M, g)$  has Anosov geodesic flow and  $I_2$  is  $s$ -injective, there exists a constant  $C > 0$  such that for all tensors  $h \in H^{-1/2}(M; S^2 T^* M)$ ,*

$$\langle \Pi_2 h, h \rangle \geq C \|\pi_{\ker D^*} h\|_{H^{-1/2}(M)}^2.$$

*Proof.* In [GL], the principal symbol of  $\Pi_2$  was computed and turned out to be

$$\sigma_2 := \sigma(\Pi_2) : (x, \xi) \mapsto |\xi|^{-1} A_2^2 \pi_{\ker i_\xi},$$

for some positive definite diagonal endomorphism  $A_2$  which is constant on both subspaces  $S_0^2 T^* M := \{h \in S^2 T^* M \mid \text{Tr}_g(h) = 0\}$  and  $\mathbb{R}g = \{\lambda g \in S^2 T^* M; \lambda \in \mathbb{R}\}$ . Here  $i_\xi$  is the interior product with the dual  $\xi^\sharp \in T_x M$  of  $\xi$  with respect to the metric. We introduce the symbol  $b \in C^\infty(T^* M)$  of order  $-1/2$  defined by  $b : (x, \xi) \mapsto \chi(x, \xi) |\xi|^{-1/2} A_2$ , where  $\chi \in C^\infty(T^* M)$  vanishes near the 0 section in  $T^* M$  and equal to 1 for  $|\xi| > 1$  and define  $B := \text{Op}(b) \in \Psi^{-1/2}(M; S^2 T^* M)$ , where  $\text{Op}$  is a quantization on  $M$ . Using that the principal symbol of  $\pi_{\ker D^*}$  is  $i_\xi$ , we observe that  $\Pi_2 = \pi_{\ker D^*} B^* B \pi_{\ker D^*} + R$ , where  $R \in \Psi^{-2}(M; S^2 T^* M)$ . Thus, given  $h \in H^{-1/2}(M, S^2 T^* M)$ :

$$\langle \Pi_2 h, h \rangle_{L^2} = \|B \pi_{\ker D^*} h\|_{L^2}^2 + \langle Rh, h \rangle_{L^2} \quad (2.6)$$

By ellipticity of  $B$ , there exists a pseudodifferential operator  $Q$  of order  $1/2$  such that  $QB \pi_{\ker D^*} = \pi_{\ker D^*} + R'$ , where  $R' \in \Psi^{-\infty}(M; S^2 T^* M)$  is smoothing. Thus there is  $C > 0$  such that for each  $h \in C^\infty(M; S^2 T^* M)$

$$\|\pi_{\ker D^*} h\|_{H^{-1/2}}^2 \leq \|QB \pi_{\ker D^*} h\|_{H^{-1/2}}^2 + \|R' h\|_{H^{-1/2}}^2 \leq C \|B \pi_{\ker D^*} h\|_{L^2}^2 + \|R' h\|_{H^{-1/2}}^2.$$

Since Lemma 2.1 is trivial on potential tensors, we can already assume that  $h$  is solenoidal, that is  $\pi_{\ker D^*} h = h$ . Reporting in (2.6), we obtain that

$$\begin{aligned} \|h\|_{H^{-1/2}}^2 &\leq C \langle \Pi_2 h, h \rangle_{L^2} - C \langle Rh, h \rangle_{L^2} + \|R' h\|_{H^{-1/2}}^2 \\ &\leq C \langle \Pi_2 h, h \rangle_{L^2} + C \|Rh\|_{H^{1/2}} \|h\|_{H^{-1/2}} + \|R' h\|_{H^{-1/2}}^2. \end{aligned} \quad (2.7)$$

Now, assume by contradiction that the statement in Lemma 2.1 does not hold, that is we can find a sequence of tensors  $f_n \in C^\infty(M; S^2 T^* M)$  such that  $\|f_n\|_{H^{-1/2}} = 1$  with  $D^* f_n = 0$  and

$$\|\sqrt{\Pi_2} f_n\|_{L^2}^2 = \langle \Pi_2 f_n, f_n \rangle_{L^2} \leq \frac{1}{n} \|f_n\|_{H^{-1/2}}^2 = \frac{1}{n} \rightarrow 0.$$

Up to extraction, and since  $R$  is of order  $-2$ , we can assume that  $R f_n \rightarrow v_1$  in  $H^{1/2}$  for some  $v_1$ , and  $R' f_n \rightarrow v_2$  in  $H^{-1/2}$ . Then, using (2.7), we obtain that  $(f_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $H^{-1/2}$  which thus converges to an element  $v_3 \in H^{-1/2}$  such that  $\|v_3\|_{H^{-1/2}} = 1$  and  $D^* v_3 = 0$ . By continuity,  $\Pi_2 f_n \rightarrow \Pi_2 v_3$  in  $H^{1/2}$  and thus  $\langle \Pi_2 v_3, v_3 \rangle = 0$ . Since  $v_3$  is solenoidal, we get  $\sqrt{\Pi_2} v_3 = 0$ , thus  $\Pi_2 v_3 = 0$ . Since we assumed  $I_2$  s-injective,  $\Pi_2$  is also injective by [GL, Lemma 4.6]. This implies that  $v_3 \equiv 0$ , thus contradicting  $\|v_3\|_{H^{-1/2}} = 1$ .  $\square$

We note that the same proof also works for tensors of any order  $m \in \mathbb{N}$ . In fact we can even get a uniform estimate:

**Lemma 2.2.** *Let  $(M, g_0)$  be a smooth compact Anosov Riemannian manifold with  $I_2^{g_0}$  being  $s$ -injective. There exists a  $C^\infty$  neighborhood  $\mathcal{U}_{g_0}$  of  $g_0$  and a constant  $C > 0$  such that for all  $g \in \mathcal{U}_{g_0}$  and all tensors  $h \in H^{-1/2}(M; S^2T^*M)$ ,*

$$\langle \Pi_2^g h, h \rangle_{L^2} \geq C \|\pi_{\ker D_g^*} h\|_{H^{-1/2}(M)}^2.$$

*Proof.* First, let  $g_0$  be fixed Anosov metric with  $I_2^{g_0}$   $s$ -injective (in particular it is the case if it has non-positive curvature). Proposition 4.1 (which will be proved later) shows that the operator  $\Pi_2 = \Pi_2^g$  is a continuous family as a map

$$g \in \mathcal{U}_{g_0} \mapsto \Pi_2^g \in \Psi^{-1}(M; S^2T^*M)$$

where  $\mathcal{U}_{g_0} \subset C^\infty(M; S^2T^*M)$  is a  $C^\infty$ -neighborhood of  $g_0$  and  $\Psi^{-1}(M; S^2T^*M)$  is equipped with its Fréchet topology given by the semi-norms of the symbols. Let  $h \in \ker D_g^*$  be a solenoidal (with respect to  $g$ ) symmetric 2-tensor, then  $h = \pi_{\ker D_g^*} h$ . Let  $C_{g_0} > 0$  be the constant provided by Lemma 2.1 applied to the metric  $g_0$ . We choose  $\mathcal{U}_{g_0}$  small enough so that  $\|\Pi_2^g - \Pi_2^{g_0}\|_{H^{-1/2} \rightarrow H^{1/2}} \leq C_{g_0}/3$  (this is made possible by the continuity of  $g \mapsto \Pi_2^g \in \Psi^{-1}$ ). Then:

$$\langle \Pi_2^g h, h \rangle = \langle (\Pi_2^g - \Pi_2^{g_0})h, h \rangle + \langle \Pi_2^{g_0} h, h \rangle \geq C_{g_0} \|\pi_{\ker D_{g_0}^*} h\|_{H^{-1/2}}^2 - C_{g_0}/3 \|h\|_{H^{-1/2}}^2$$

But the map  $\mathcal{U}_{g_0} \ni g \mapsto \pi_{\ker D_g^*} = \mathbb{1} - D_g \Delta_g^{-1} D_g^* \in \Psi^0$  is continuous, where  $\Delta_g := D_g^* D_g$  is the Laplacian on 1-forms, and this implies that for  $g$  in a possibly smaller neighborhood  $\mathcal{U}_{g_0}$  of  $g_0$ , using  $h = \pi_{\ker D_g^*} h$ :

$$\langle \Pi_2^g h, h \rangle \geq C_{g_0} \|\pi_{\ker D_g^*} h\|_{H^{-1/2}}^2 - \frac{2C_{g_0}}{3} \|h\|_{H^{-1/2}}^2 = C_{g_0}/3 \|\pi_{\ker D_g^*} h\|_{H^{-1/2}}^2.$$

The proof is complete.  $\square$

We also observe that the generalization of the previous Lemma to tensors of any order is straightforward. As mentioned earlier, an immediate consequence of the previous lemma is the following

**Proposition 2.3.** *Let  $(M, g_0)$  be a smooth Riemannian  $n$ -dimensional Anosov manifold with  $I_m^{g_0}$   $s$ -injective. Then, there exists a  $C^\infty$ -neighborhood  $\mathcal{U}_{g_0}$  of  $g_0$  in  $\mathcal{M}$  such that for any  $g \in \mathcal{U}_{g_0}$ , for any  $m \in \mathbb{N}$ ,  $I_m^g$  is  $s$ -injective.*

*Proof.* By [GL], the  $s$ -injectivity of  $I_m^g$  is equivalent to that of  $\Pi_m^g$  and the previous Lemma allows to conclude.  $\square$

**2.2. The space of Riemannian metrics.** We fix a *smooth* metric  $g_0 \in \mathcal{M}$  and consider an integer  $k \geq 2$  and  $\alpha \in (0, 1)$ . We recall that the space  $\mathcal{M}$  of all smooth metrics is a Fréchet manifold. We denote by  $\mathcal{D}_0 := \text{Diff}_0(M)$  the group of smooth diffeomorphisms on

$M$  that are isotopic to the identity, this is a Fréchet Lie group in the sense of [Ham82, Section 4.6]. The right action

$$\mathcal{M} \times \mathcal{D}_0 \rightarrow \mathcal{M}, \quad (g, \psi) \mapsto \psi^*g$$

is smooth and proper [Ebi68, Ebi70]. Moreover, if  $g$  is a metric with Anosov geodesic flow, it is direct to see from ergodicity that there are no Killing vector fields and thus the isotropy subgroup  $\{\psi \in \mathcal{D}_0 \mid \psi^*g = g\}$  of  $g$  is finite. For negatively curved metrics it is shown in [Fra66] that the action is free, i.e. the isotropy group is trivial. One cannot apply the usual quotient theorem [Tro92, p.20] in the setting of Banach or Hilbert manifolds but rather smooth Fréchet manifolds instead (using Nash-Moser theorem). Thus, in the setting of the space  $\mathcal{M}^-$  of negatively curved smooth metrics, which is a Frechet manifold, the slice theorem says that there is a neighborhood  $\mathcal{U}$  of  $g_0$ , a neighborhood  $\mathcal{V}$  of  $\text{Id}$  in  $\mathcal{D}_0$  and a Frechet submanifold  $\mathcal{S}$  containing  $g_0$  so that

$$\mathcal{S} \times \mathcal{V} \rightarrow \mathcal{U}, \quad (g, \psi) \mapsto \psi^*g \tag{2.8}$$

is a diffeomorphism of Frechet manifolds and  $T_{g_0}\mathcal{S} = \{h \in T_{g_0}\mathcal{M} \mid D_{g_0}^*h = 0\}$ . Moreover  $\mathcal{S}$  parametrizes the set of orbits  $g \cdot \mathcal{D}_0$  for  $g$  near  $g_0$  and  $T_g\mathcal{S} \cap T(g \cdot \mathcal{D}_0) = 0$ .

On the other hand, if one considers  $\mathcal{M}^{k,\alpha}$ , the space of metrics with  $C^{k,\alpha}$  regularity and  $\mathcal{D}_0^{k+1,\alpha} := \text{Diff}_0^{k+1,\alpha}(M)$ , the group of diffeomorphisms with  $C^{k+1,\alpha}$  regularity, then both spaces are smooth Banach manifolds. However, the action of  $\mathcal{D}_0^{k+1,\alpha}$  on  $\mathcal{M}^{k,\alpha}$  is no longer smooth but only topological which also prevents us from applying the quotient theorem.

Nevertheless, recalling  $g_0$  is smooth, if we consider  $\mathcal{O}^{k,\alpha}(g_0) := g_0 \cdot \mathcal{D}_0^{k+1,\alpha} \subset \mathcal{M}^{k,\alpha}$ , then this is a smooth submanifold of  $\mathcal{M}^{k,\alpha}$  and

$$T_g\mathcal{O}^{k,\alpha}(g_0) = \{D_gp \mid p \in C^{k+1,\alpha}(M; T^*M)\}.$$

Notice that (2.2) in  $C^{k,\alpha}$  regularity exactly says that given  $g \in \mathcal{O}^{k,\alpha}(g_0)$ , one has the decomposition:

$$T_g\mathcal{M} = T_g\mathcal{O}^{k,\alpha}(g_0) \oplus \ker D_g^*|_{C^{k,\alpha}(M, S^2T^*M)}. \tag{2.9}$$

Thus, an infinitesimal perturbation of a metric  $g \in \mathcal{O}^{k,\alpha}(g_0)$  by a symmetric 2-tensor that is solenoidal with respect to  $g$  is actually an infinitesimal displacement *transversally to the orbit*  $\mathcal{O}^{k,\alpha}(g_0)$ .

We will need a stronger version of the previous decomposition (2.9) which can be understood as a slice theorem. Its knowledge goes back to [Ebi68, Ebi70], see also [GL19] for a short proof in the  $C^{k,\alpha}$  category. We recall that  $g_0$  is assumed to be smooth.

**Lemma 2.4.** *Let  $k$  be an integer  $\geq 2$  and  $\alpha \in (0,1)$ . There exists a neighborhood  $\mathcal{U} \subset \mathcal{M}^{k,\alpha}$  of  $g_0$  in the  $C^{k,\alpha}$ -topology such that for any  $g \in \mathcal{U}$ , there exists a unique  $C^{k+1,\alpha}$ -diffeomorphism  $\psi$  such that  $\psi^*g$  is solenoidal with respect to  $g_0$ . Moreover, the map  $C^{k,\alpha}(M; S^2T^*M) \ni g \mapsto \psi \in \mathcal{D}_0^{k+1,\alpha}(M)$  is smooth.*

*Remark 2.5.* This is where we see the need to resort to  $C^{k,\alpha}$  regularity for  $\alpha \neq 0, 1$ : the pseudodifferential operator  $D_{g_0}^* D_{g_0}$  that arises naturally in the proof of this lemma act on these spaces but on  $C^k$ , for  $k \in \mathbb{N}$ . Instead, one would have to resort to Zygmund spaces  $C_*^k$ . We refer to [Tay91, Appendix A] for further details.

**2.3. Thermodynamic formalism.** Let  $f$  be a Hölder-continuous function on  $S_{g_0}M$ . We recall that its *pressure* [Wal82, Theorem 9.10] is defined by:

$$\mathbf{P}(f) := \sup_{\mu \in \mathfrak{M}_{\text{inv}}} \left( \mathbf{h}_\mu(\varphi_1^{g_0}) + \int_{S_{g_0}M} f d\mu \right), \quad (2.10)$$

where  $\mathfrak{M}_{\text{inv}}$  denotes the set of invariant (by the flow  $\varphi^{g_0}$ ) Borel probability measures and  $\mathbf{h}_\mu(\varphi_1^{g_0})$  is the metric entropy of the flow  $\varphi_1^{g_0}$  at time 1. It is actually sufficient to restrict the sup to ergodic measures  $\mathfrak{M}_{\text{inv,erg}}$  [Wal82, Corollary 9.10.1]. Since the flow is Anosov, the supremum is always achieved for a unique invariant ergodic measure  $\mu_f$  [HF, Theorem 9.3.4] called the *equilibrium state* of  $f$ . The measure  $\mu_f$  is also mixing and positive on open sets which rules out the possibility of a finite combination of Dirac measures supported on a finite number of closed orbits. Moreover  $\mu_f$  can be written as an infinite weighted sum of Dirac masses  $\delta_{g_0}(c_j)$  supported over the geodesics  $\gamma_{g_0}(c_j)$ , where  $c_j \in \mathcal{C}$  are the primitive classes (see [Par88] for the case  $\mathbf{P}(f) \geq 0$  or [PPS15, Theorem 9.17] for the general case). For example when  $\mathbf{P}(f) \geq 0$ ,

$$\int u d\mu_f = \lim_{T \rightarrow \infty} \frac{1}{N(T, f)} \sum_{\{j | L_{g_0}(c_j) \in [T, T+1]\}} e^{\int_{\gamma_{g_0}(c_j)} f} \int_{\gamma_{g_0}(c_j)} u, \quad (2.11)$$

where  $N(T, f) := \sum_{j, L_{g_0}(c_j) \in [T, T+1]} L_{g_0}(c_j) e^{\int_{\gamma_{g_0}(c_j)} f}$ . When  $f = 0$ , this is the measure of maximal entropy, also called the *Bowen-Margulis measure*  $\mu_{g_0}^{\text{BM}}$ ; in that case  $\mathbf{P}(0) = \mathbf{h}_{\text{top}}(\varphi_1^{g_0})$  is the topological entropy of the flow. When  $f = -J_{g_0}^u$ , where  $J_{g_0}^u : x \mapsto \partial_t |\det d\varphi_t^g(x)|_{E_u(x)}|_{t=0}$  is the *geometric potential*, one obtains the Liouville measure  $\mu_{g_0}^L$  induced by the metric  $g_0$ ; in that case,  $\mathbf{P}(-J_{g_0}^u) = 0$ . If we fix an exponent of Hölder regularity  $\nu > 0$ , then the map  $C^\nu(S_{g_0}M) \ni f \mapsto \mathbf{P}(f)$  is real analytic [Rue04, Corollary 7.10].

**2.4. Geodesic stretch.** We refer to [CF90, Kni95] for the original definition of this notion.

**2.4.1. Structural stability and time reparametrization.** We fix a smooth metric  $g_0 \in \mathcal{M}$  with Anosov geodesic flow and we view the geodesic flow and vector fields of any metric  $g$  close to  $g_0$  as living on the unit tangent bundle  $S_{g_0}M$  of  $g_0$  by simply pulling them back by the diffeomorphism

$$(x, v) \in S_{g_0}M \rightarrow \left(x, \frac{v}{|v|_g}\right) \in S_gM.$$

We denote by  $2\nu$  the exponent of Hölder regularity of the stable/unstable bundles of  $g_0$ . We fix some constant  $k \geq 2$  and  $\alpha \in (0, 1)$ . There exists a neighborhood  $\mathcal{U} \subset \mathcal{M}^{k,\alpha}$  of  $g_0$  such that, by the structural stability theorem [dILMM86], for any  $g \in \mathcal{U}$ , there exists a Hölder homeomorphism  $\psi_g : S_{g_0}M \rightarrow S_gM$ , differentiable in the flow direction, which is an orbit conjugacy i.e. such that

$$d\psi_g(z)X_{g_0}(z) = a_g(z)X_g(\psi_g(z)), \quad \forall z \in S_{g_0}M, \quad (2.12)$$

where  $a_g$  is a Hölder-continuous function on  $S_{g_0}M$ . Moreover, the map

$$g \mapsto (a_g, \psi_g) \in C^\nu(S_{g_0}M) \times C^\nu(S_{g_0}M, S_gM)$$

is  $C^{k-2}$  [Con92, Proposition 1.1] and  $\psi_g$  is homotopic to Id. Note that neither  $a_g$  nor  $\psi_g$  are unique but  $a_g$  is unique up to a coboundary and in all the following paragraphs, adding a coboundary to  $a_g$  will not affect the results.

From (2.12), we obtain that for  $t \in \mathbb{R}$ ,  $z \in S_{g_0}M$ ,  $\varphi_{\kappa_{a_g}(z,t)}^g(\psi_g(z)) = \psi_g(\varphi_t^{g_0}(z))$  with:

$$\kappa_{a_g}(z, t) = \int_0^t a_g(\varphi_s^{g_0}(z)) ds. \quad (2.13)$$

If  $c \in \mathcal{C}$  is a free homotopy class, then one has:

$$L_g(c) = \int_0^{L_{g_0}(c)} a_g(\varphi_s^{g_0}(z)) ds, \quad (2.14)$$

for any  $z \in \gamma_{g_0}(c)$ , the unique  $g_0$ -closed geodesic in  $c$ .

**2.4.2. Definition of the geodesic stretch.** We denote by  $\widetilde{M}$  the universal cover of  $M$ . Given a metric  $g \in \mathcal{M}$  on  $M$ , we denote by  $\widetilde{g}$  its lift to the universal cover. Given two metrics  $g_1$  and  $g_2$  on  $M$ , there exists a constant  $c > 0$  such that  $c^{-1}g_1 \leq g_2 \leq cg_1$ . This implies that any  $\widetilde{g}_1$ -geodesic is a quasi-geodesic for  $\widetilde{g}_2$ . In particular, this implies that the *ideal* (or *visual*) boundary  $\partial_\infty \widetilde{M}$  is independent of the choice of  $g$  and is naturally endowed with the structure of a topological manifold. Note that when restricting to metrics with pinched sectional curvatures  $-a^2 \leq \kappa \leq -b^2$ , the regularity of the ideal boundary becomes Hölder (the Hölder regularity depending on  $a$  and  $b$ ). We refer to [BH99, Chapter H.3] for further details. We denote by  $\mathcal{G}_g := S_{\widetilde{g}}\widetilde{M} / \sim$  (where  $z \sim z'$  if and only if there exists a time  $t \in \mathbb{R}$  such that  $\varphi_t(z) = z'$ ) the set of  $g$ -geodesics on  $\widetilde{M}$ : this is smooth  $2n$ -dimensional manifold. Moreover, there exists a Hölder continuous homeomorphism  $\Phi_g : \mathcal{G}_g \rightarrow \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta$ , where  $\Delta$  is the diagonal in  $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ .

We now consider a fixed metric  $g_0$  on  $M$  and a metric  $g$  in a neighborhood of  $g_0$ . If  $\psi_g$  denotes an orbit-conjugacy between the two geodesic flows, then  $\psi_g$  induces a homeomorphism  $\Psi_g : \mathcal{G}_{g_0} \rightarrow \mathcal{G}_g$ . The map

$$\Phi_g \circ \Psi_g \circ \Phi_{g_0}^{-1} : \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta \rightarrow \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta$$

is nothing but the identity.

Given  $z = (x, v) \in S_{g_0}M$ , we denote by  $c_{g_0}(z) : t \mapsto c_{g_0}(z, t) \in M$  the unique geodesic<sup>2</sup> such that  $c_{g_0}(z, 0) = x, \dot{c}_{g_0}(z, 0) = v$ . We consider  $\tilde{c}_{g_0}(z)$ , a lift of  $c_{g_0}(z)$  to the universal cover  $\tilde{M}$  and introduce the function

$$b : S_{g_0}M \times \mathbb{R} \rightarrow \mathbb{R}, \quad b(z, t) := d_{\tilde{g}}(\tilde{c}_{g_0}(z, 0), \tilde{c}_{g_0}(z, t)),$$

which computes the  $\tilde{g}$ -distance between the endpoints of the  $\tilde{g}_0$ -geodesic joining  $\tilde{c}_{g_0}(z, 0)$  to  $\tilde{c}_{g_0}(z, t)$ . It is an immediate consequence of the triangular inequality that  $(z, t) \mapsto b(z, t)$  is a subadditive cocycle for the geodesic flow  $\varphi^{g_0}$ , that is:

$$b(z, t + s) \leq b(z, t) + b(\varphi_t^{g_0}(z), s), \quad \forall z \in S_{g_0}M, \forall t, s \in \mathbb{R}$$

As a consequence, by the subadditive ergodic theorem (see [Wal82, Theorem 10.1] for instance), we obtain the following

**Lemma 2.6.** *Let  $\mu$  be an invariant probability measure for the flow  $\varphi_t^{g_0}$ . Then, the quantity*

$$I_\mu(g_0, g, z) := \lim_{t \rightarrow +\infty} b(z, t)/t$$

*exists for  $\mu$ -almost every  $z \in S_{g_0}M$ ,  $I_\mu(g_0, g, \cdot) \in L^1(S_{g_0}M, d\mu)$  and this function is invariant by the flow  $\varphi_t^{g_0}$ .*

We define the *geodesic stretch of the metric  $g$ , relative to the metric  $g_0$ , with respect to the measure  $\mu$*  by:

$$I_\mu(g_0, g) := \int_{S_{g_0}M} I_\mu(g_0, g, z) d\mu(z).$$

When the measure  $\mu$  in the previous definition is ergodic, the function  $I_\mu(g_0, g, \cdot)$  is thus ( $\mu$ -almost everywhere) equal to the constant  $I_\mu(g_0, g)$ . We recall that  $\delta_{g_0}(c)$  is the normalized measure supported on  $\gamma_{g_0}(c)$ , that is:

$$\delta_{g_0}(c) : f \mapsto \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} f(\varphi_t^{g_0}(z)) dt.$$

We can actually describe the stretch using the time reparametrization  $a_g$ .

**Lemma 2.7.** *Let  $\mu_f$  be an equilibrium state for a Hölder function  $f$  with respect to the flow  $\varphi_t^{g_0}$ . Then:*

$$I_{\mu_f}(g_0, g) = \int_{SM_{g_0}} a_g d\mu_f = \lim_{j \rightarrow +\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)},$$

*where  $(c_j)_{j \geq 0} \in \mathcal{C}^{\mathbb{N}}$  is such that  $\delta_{g_0}(c_j) \rightharpoonup_{j \rightarrow +\infty} \mu_f$ .*

<sup>2</sup>For the sake of simplicity, we identify the geodesic and its arc-length parametrization.

*Proof.* We first prove the left equality. Let  $z \in S_{g_0}M$  and  $\tilde{c}_{g_0}(z)$  be a lift of  $c_{g_0}(z)$  to the universal cover. Let  $\tilde{c}_g(\psi_g(z))$  be the unique lift of  $c_g(\psi_g(z))$  with the same endpoints on the ideal boundary as  $\tilde{c}_{g_0}(z)$ . Then, there exists a constant  $C > 0$  such that for all  $t \geq 0$ :

$$|d_{\tilde{g}}(\tilde{c}_{g_0}(z, 0), \tilde{c}_{g_0}(z, t)) - \underbrace{d_{\tilde{g}}(\tilde{c}_g(\psi_g(z), 0), \tilde{c}_g(\psi_g(z), \kappa_{a_g}(t, z)))}_{=\kappa_{a_g}(t, z)}| \leq C.$$

This implies, using (2.13) that:

$$\lim_{t \rightarrow +\infty} b(z, t)/t = \lim_{t \rightarrow +\infty} \kappa_{a_g}(z, t)/t = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t a_g(\varphi_s^{g_0}(z)) ds = \int_{S_{g_0}M} a d\mu_f,$$

for  $\mu_f$ -almost every  $z \in S_{g_0}M$ , by the Birkhoff ergodic Theorem [Wal82, Theorem 1.14]. By (2.14) we also have

$$\int_{S_{g_0}M} a_g d\mu_f = \lim_{j \rightarrow \infty} \langle \delta_{g_0}(c_j), a_g \rangle = \lim_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)}$$

thus the proof is complete.  $\square$

As a consequence, we immediately obtain the

**Corollary 2.8.** *Let  $g$  belong to a fixed neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{M}^{k, \alpha}$ , and assume that for any sequence of primitive free homotopy classes  $(c_j)_{j \geq 0} \in \mathcal{C}^{\mathbb{N}}$  such that  $L_{g_0}(c_j) \rightarrow \infty$ , one has  $\lim_{j \rightarrow \infty} L_g(c_j)/L_{g_0}(c_j) = 1$ . Then, for any equilibrium state  $\mu_f$  with respect to  $\varphi_t^{g_0}$  associated to some Hölder function  $f$ , we have  $I_{\mu_f}(g_0, g) = 1$ .*

Combining this with the results of [GL19, Theorem 1], we also easily obtain:

**Theorem 2.9.** *Let  $(M, g_0)$  be a smooth Riemannian  $n$ -dimensional manifold with Anosov geodesic flow, topological entropy  $\mathbf{h}_{\text{top}}(g_0) = 1$  and assume that its curvature is nonpositive if  $n \geq 3$ . Then there exists  $k \in \mathbb{N}$  large enough depending only on  $n$ ,  $\varepsilon > 0$  small enough such that the following holds: there is  $C > 0$  depending on  $g_0$  so that for each  $g \in C^k(M; S^2T^*M)$  with  $\|g - g_0\|_{C^k} \leq \varepsilon$ , if*

$$\mathbf{h}_{\text{top}}(g) = 1, \quad \lim_{j \rightarrow +\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} = 1,$$

for some sequence  $(c_j)_{j \in \mathbb{N}}$  of primitive free homotopy classes such that  $\delta_{g_0}(c_j) \xrightarrow{j \rightarrow +\infty} \mu_{g_0}^{\text{BM}}$ , then  $g$  is isometric to  $g_0$ .

*Proof.* Given a metric  $g$ , one has by [Kni95, Theorem 1.2]<sup>3</sup> that

$$\mathbf{h}_{\text{top}}(g) \geq \frac{\mathbf{h}_{\text{top}}(g_0)}{I_{\mu_{g_0}^{\text{BM}}}(g_0, g)}, \quad (2.15)$$

<sup>3</sup>In [Kni95] the metric is assumed to be negatively curved, but the argument applies also for Anosov flows, as is shown in [BCLS15, Proposition 3.8]: it corresponds to Proposition 3.6 below in the case  $f := 1$  and  $f' = a_g$ .

with equality if and only if  $\varphi^{g_0}$  and  $\varphi^g$  are, up to a scaling, time-preserving conjugate, that is there exists homeomorphism  $\psi$  such that  $\psi \circ \varphi_{g_0}^{ct} = \varphi_g^t \circ \psi$  with  $c := \mathbf{h}_{\text{top}}(g)/\mathbf{h}_{\text{top}}(g_0)$ . In particular, restricting to metrics with entropy 1 one obtains that  $I_{\mu_{g_0}^{\text{BM}}}(g_0, g) \geq 1$  with equality if and only if the geodesic flows are conjugate, that is if and only if  $L_g = L_{g_0}$ . As a consequence, given  $g_0, g$  with entropy 1 such that  $L_g(c_j)/L_{g_0}(c_j) \rightarrow_{j \rightarrow +\infty} 1$  for some sequence  $\delta_{g_0}(c_j) \rightarrow_{j \rightarrow +\infty} \mu_{g_0}^{\text{BM}}$ , we obtain that  $I_{\mu_{g_0}^{\text{BM}}}(g_0, g) = 1$ , hence  $L_g = L_{g_0}$ . If  $k \in \mathbb{N}$  was chosen large enough at the beginning, we can then conclude by the local rigidity of the marked length spectrum [GL19, Theorem 1].  $\square$

It is of no harm to assume that  $g_0$  has entropy 1: indeed, considering  $\lambda g_0$  for some constant  $\lambda > 0$ , the entropy scales as  $\mathbf{h}_{\text{top}}(\lambda g) = \mathbf{h}_{\text{top}}(g)/\sqrt{\lambda}$  [Pat99, Lemma 3.23]. In particular, this also implies the local rigidity of the marked length spectrum because the topological entropy is determined by the marked length spectrum since  $s = \mathbf{h}_{\text{top}}(g)$  is the first pole of the Ruelle zeta function

$$\zeta_g(s) := \prod_{c \in \mathcal{C}} (1 - e^{-sL_g(c)}).$$

We will provide a more direct alternate proof of this fact in the next section without using the proof of [GL19].

### 3. A FUNCTIONAL ON THE SPACE OF METRICS

**3.1. A submanifold of the space of metrics.** Recall that  $2\nu$  is the exponent of Hölder continuity of the stable/unstable vector bundles for the fixed metric  $g_0$ . Given a metric  $g$  in a  $C^{k,\alpha}$ -neighborhood of  $g_0$ , we define the potential

$$V_g := J_{g_0}^u + a_g - 1 \in C^\nu(S_{g_0}M). \quad (3.1)$$

Remark that  $g \mapsto V_g \in C^\nu(S_{g_0}M)$  is  $C^{k-2}$  and for  $g = g_0$ ,  $V_{g_0} = J_{g_0}^u$ . We introduce the spaces

$$\mathcal{N}^{k,\alpha} := \{g \in \mathcal{M}^{k,\alpha} \mid \mathbf{P}(-V_g) = 0\}, \quad (3.2)$$

and  $\mathcal{N}_{\text{sol}}^{k,\alpha} := \mathcal{N}^{k,\alpha} \cap \ker D_{g_0}^*$ . In particular,  $g_0 \in \mathcal{N}_{\text{sol}}^{k,\alpha}$ . Given  $g \in \mathcal{N}^{k,\alpha}$ , we denote by  $m_g$  the unique equilibrium state for the potential  $V_g$ . We will also denote  $\mathcal{N}$  for the case where  $k = \infty$ .

**Lemma 3.1.** *There exists a neighborhood  $\mathcal{U} \subset \mathcal{M}^{k,\alpha}$  of  $g_0$  such that  $\mathcal{N}^{k,\alpha} \cap \mathcal{U}$  is a codimension one  $C^{k-2}$ -submanifold of  $\mathcal{U}$  and  $\mathcal{N}_{\text{sol}}^{k,\alpha} \cap \mathcal{U}$  is a  $C^{k-2}$ -submanifold of  $\mathcal{U}$ . Similarly, there is  $\mathcal{U} \subset \mathcal{M}$  an open neighborhood so that  $\mathcal{N} \cap \mathcal{U}$  is a Fréchet submanifold of  $\mathcal{M}$ .*

*Proof.* To prove this lemma, we will use the notion of differential calculus on Banach manifolds as it is stated in [Zei88, Chapter 73]. Note that  $\mathcal{M}^{k,\alpha}$  is a smooth Banach manifold and  $\mathcal{N}^{k,\alpha} \subset \mathcal{M}^{k,\alpha}$  is defined by the implicit equation  $F(g) = 0$  for

$$F : g \mapsto \mathbf{P}(-V_g) \in \mathbb{R}. \quad (3.3)$$

The map  $F$  being  $C^{k-2}$  (see §2.3, the pressure is real analytic so  $F$  inherits the regularity of  $g \mapsto a_g$ ), we only need to prove that  $dF_{g_0}$  does not vanish by [Zei88, Theorem 73.C]. This will immediately give that  $T_{g_0}\mathcal{N}^{k,\alpha} = \ker dF_{g_0}$ . In order to do so, we need a deformation lemma. For the sake of simplicity, we write the objects  $\cdot_\lambda$  instead of  $\cdot_{g_\lambda}$ .

**Lemma 3.2.** *Consider a smooth deformation  $(g_\lambda)_{\lambda \in (-1,1)}$  of  $g_0$  inside  $\mathcal{M}^{k,\alpha}$ . Then, there exists a Hölder-continuous function  $f : S_{g_0}M \rightarrow \mathbb{R}$  such that*

$$\pi_2^*(\partial_\lambda g_\lambda|_{\lambda=0}) - 2\partial_\lambda a_\lambda|_{\lambda=0} = X_{g_0}f.$$

*Proof.* Let  $c$  be a fixed free homotopy class,  $\gamma_0 \in c$  be the unique closed  $g_0$ -geodesic in the class  $c$ , which we parametrize by unit-speed  $z_0 : [0, \ell_{g_0}(\gamma_0)] \rightarrow S_{g_0}M$ . We define  $z_\lambda(s) = \psi_\lambda(z_0(s)) = (\alpha_\lambda(s), \dot{\alpha}_\lambda(s))$  (the dot is the derivative with respect to  $s$ ) where  $\psi_\lambda$  is the conjugation between  $g_\lambda$  and  $g_0$  : this gives a non-unit-speed parametrization of  $\gamma_\lambda$ , the unique closed  $g_\lambda$ -geodesic in  $c$ . We recall that  $\pi : TM \rightarrow M$  is the projection. We obtain using (2.12)

$$\begin{aligned} \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\dot{\alpha}_\lambda(s), \dot{\alpha}_\lambda(s)) ds &= \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\partial_s(\pi \circ z_\lambda(s)), \partial_s(\pi \circ z_\lambda(s))) ds \\ &= \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\partial_s(\pi \circ \psi_\lambda \circ z_0(s)), \partial_s(\pi \circ \psi_\lambda \circ z_0(s))) ds \\ &= \int_0^{\ell_{g_0}(\gamma_0)} a_\lambda^2(z_0(s)) \underbrace{g_\lambda(d\pi(X_{g_\lambda}(z_\lambda(s))), d\pi(X_{g_\lambda}(z_\lambda(s))))}_{=1} ds \\ &= \int_0^{\ell_{g_0}(\gamma_0)} a_\lambda^2(z_0(s)) ds. \end{aligned}$$

Since  $s \mapsto \alpha_0(s)$  is a unit-speed geodesic for  $g_0$ , it is a critical point of the energy functional (with respect to  $g_0$ ). Thus, by differentiating the previous identity with respect to  $\lambda$  and evaluating at  $\lambda = 0$ , one obtains:

$$\int_0^{\ell_{g_0}(\gamma_0)} \partial_\lambda g_\lambda|_{\lambda=0}(\dot{\alpha}_0(s), \dot{\alpha}_0(s)) ds = 2 \int_0^{\ell_{g_0}(\gamma_0)} \partial_\lambda a_\lambda|_{\lambda=0}(z_0(s)) ds.$$

As a consequence,  $\pi_2^*(\partial_\lambda g_\lambda|_{\lambda=0}) - 2\partial_\lambda a_\lambda|_{\lambda=0}$  is a Hölder-continuous function in the kernel of the X-ray transform: by the usual Livsic theorem, there exists a function  $f$  (with the same Hölder regularity), differentiable in the flow direction, such that  $\pi_2^*(\partial_\lambda g_\lambda|_{\lambda=0}) - 2\partial_\lambda a_\lambda|_{\lambda=0} = X_{g_0}f$ .  $\square$

We can now complete the proof of Lemma 3.1. We first prove the first part concerning  $\mathcal{N}^{k,\alpha}$ . By [PP90, Proposition 4.10], we have for  $h \in T_g\mathcal{N}^{k,\alpha}$ :

$$dF_g.h = - \int_{S_{g_0}M} da_g.h dm_g$$

where  $m_g$  is the equilibrium measure of  $-V_g$ . In particular, observe that for  $g = g_0$ , one has:

$$dF_{g_0} \cdot h = - \int_{S_{g_0}M} da_{g_0} \cdot h d\mu_{g_0}^L,$$

since  $m_{g_0} = \mu_{g_0}^L$ . Then, using Lemma 3.2, one obtains

$$dF_{g_0} \cdot h = - \int_{S_{g_0}M} da_{g_0} \cdot h d\mu_{g_0}^L = -\frac{1}{2} \int_{S_{g_0}M} \pi_2^* h d\mu_{g_0}^L = -c_2 \langle h, g_0 \rangle_{L^2}, \quad (3.4)$$

for some constant  $c_2 > 0$ . This is obviously surjective and we also obtain:

$$T_{g_0} \mathcal{N}^{k,\alpha} = \ker dF_{g_0} = \{h \in C^{k,\alpha}(M; S^2 T^* M) \mid \langle h, g_0 \rangle_{L^2} = 0\} = (\mathbb{R}g_0)^\perp,$$

where the orthogonal is understood with respect to the  $L^2$ -scalar product.

We now deal with  $\mathcal{N}_{\text{sol}}^{k,\alpha}$ . First observe that  $\ker D_{g_0}^*$  is a closed linear subspace of  $\mathcal{M}^{k,\alpha}$  and thus a smooth submanifold of  $\mathcal{M}^{k,\alpha}$ . By [Zei88, Corollary 73.50], it is sufficient to prove that  $\ker D_{g_0}^*$  and  $\mathcal{N}^{k,\alpha}$  are transverse at  $g_0$ . But observe that  $g_0 \in \ker D_{g_0}^* \simeq T_{g_0} \ker D_{g_0}^*$  and thus

$$T_{g_0} \ker D_{g_0}^* + T_{g_0} \mathcal{N}^{k,\alpha} = T_{g_0} \mathcal{M}^{k,\alpha}$$

showing transversality.

The case of  $\mathcal{N}$  follows directly from Nash-Moser theorem since  $F$  is a smooth tame map from a Fréchet space to  $\mathbb{R}$ , with a right inverse  $H_g$  for  $dF_g$  that is continuous in  $g$ : just take  $H_g 1 := g$ .  $\square$

We next show that metrics with the same marked length spectrum at infinity belong to  $\mathcal{N}^{k,\alpha}$ .

**Lemma 3.3.** *Let  $g \in \mathcal{U}$ . If for any sequence of primitive free homotopy classes  $(c_j)_{j \geq 0} \in \mathcal{C}^{\mathbb{N}}$  such that  $L_{g_0}(c_j) \rightarrow \infty$ ,  $\lim_{j \rightarrow \infty} L_g(c_j)/L_{g_0}(c_j) = 1$ , then  $g \in \mathcal{N}^{k,\alpha}$ .*

*Proof.* By §2.3, one has:

$$\mathbf{P}(-V_g) = \sup_{\mu \in \mathfrak{M}_{\text{inv,erg}}} \left( \mathbf{h}_\mu(\varphi^{g_0}) - \int_{S_{g_0}M} (J_{g_0}^u + a_g - 1) d\mu \right).$$

Note that by Corollary 2.8, for the equilibrium state  $m_g$  of  $-V_g$ , one has  $\int_{S_{g_0}M} (a_g - 1) dm_g = I_{m_g}(g_0, g) - 1 = 0$ . Thus:

$$\mathbf{P}(-V_g) = \mathbf{P}(-J_{g_0}^u) = \mathbf{P}(-V_{g_0}) = 0,$$

proving the claim.  $\square$

Note that in the proof, one actually just needs  $\lim_{j \rightarrow \infty} L_g(c_j)/L_{g_0}(c_j) = 1$  for a sequence  $c_j$  so that  $\delta_g(c_j) \rightarrow m_g$  to obtain that  $g \in \mathcal{N}^{k,\alpha}$ .

By Lemma 2.7, we know that

$$I_{\mu_{g_0}^L}(g_0, g) = \int_{S_{g_0}M} a_g d\mu_{g_0}^L.$$

We introduce the functional

$$\Phi : \mathcal{N}_{\text{sol}}^{k,\alpha} \rightarrow \mathbb{R}, \quad \Phi(g) := I_{\mu_{g_0}^L}(g_0, g). \quad (3.5)$$

Note that  $\Phi$  is  $C^{k-2}$ , its regularity being limited by that of  $\mathcal{N}_{\text{sol}}^{k,\alpha}$ . Given  $h \in T_{g_0}\mathcal{N}_{\text{sol}}^{k,\alpha}$ , we thus obtain

$$d\Phi_{g_0}.h = \int_{S_{g_0}M} da_{g_0}.h d\mu_{g_0}^L = 0,$$

that is,  $g_0$  is a critical point of the functional  $\Phi$  on  $\mathcal{N}_{\text{sol}}^{k,\alpha}$ . We can extend  $\Phi$  to

$$\tilde{\Phi} : \mathcal{M}^{k,\alpha} \rightarrow \mathbb{R}, \quad \tilde{\Phi}(g) = I_{\mu_{g_0}^L}(g_0, g),$$

and we note that

$$d\tilde{\Phi}_{g_0}.h = \int_{S_{g_0}M} da_{g_0}.h d\mu_{g_0}^L = -dF_{g_0}.h = -C_n \langle h, g_0 \rangle \quad (3.6)$$

for some constant  $C_n > 0$  depending only on  $n = \dim(M)$  and  $\langle h, g_0 \rangle = \int_M \text{Tr}_{g_0}(h) d\text{vol}_{g_0}$ .

**Lemma 3.4.** *The map  $\Phi : \mathcal{N}_{\text{sol}}^{k,\alpha} \rightarrow \mathbb{R}$  is strictly convex at  $g_0$  and there is  $C > 0$  such that*

$$d^2\Phi_{g_0}(h, h) = \frac{1}{4} \langle \Pi_2^{g_0} h, h \rangle \geq C \|h\|_{H^{-\frac{1}{2}}(M)}^2$$

for all  $h \in T_{g_0}\mathcal{N}_{\text{sol}}^{k,\alpha}$ .

*Proof.* Since  $g_0$  is a critical point of  $\Phi$ , we have  $d^2\Phi_{g_0}(h, h) = \partial_\lambda^2 \Phi(g_\lambda)|_{\lambda=0}$  where  $g_\lambda := g_0 + \lambda h + \mathcal{O}(\lambda^2)$  is a smooth curve of metrics in  $\mathcal{N}^{k,\alpha}$ , and we write  $a_\lambda := a_{g_\lambda}$ ,  $V_\lambda := V_{g_\lambda}$  and denote by  $\dot{x}$  and  $\ddot{x}$  the derivatives with respect to  $\lambda$ . By Lemma 2.7, we have

$$\partial_\lambda^2 \Phi(g_\lambda)|_{\lambda=0} = \int_{S_{g_0}M} \ddot{a}_0 d\mu_{g_0}^L. \quad (3.7)$$

But we also know that  $\mathbf{P}(-V_\lambda) = 0$ , thus if we differentiate twice, we obtain

$$d^2\mathbf{P}_{-V_0}(\dot{V}_0, \dot{V}_0) - d\mathbf{P}_{-V_0}(\ddot{V}_0) = 0. \quad (3.8)$$

By (3.4), we have

$$\int_{S_{g_0}M} \dot{V}_0 d\mu_{g_0}^L = \int_{S_{g_0}M} \dot{a}_0 d\mu_{g_0}^L = -dF_{g_0}.h = 0,$$

thus we obtain by [PP90, Proposition 4.11] that

$$\begin{aligned} d^2\mathbf{P}_{-V_0}(\dot{V}_0, \dot{V}_0) &= \text{Var}_{\mu_{g_0}^L}(\dot{V}_0) = \langle \Pi^{g_0} \dot{V}_0, \dot{V}_0 \rangle, \\ d\mathbf{P}_{-V_0}(\ddot{V}_0) &= \int_{S_{g_0}M} \ddot{V}_0(z) d\mu_{g_0}^L \end{aligned} \quad (3.9)$$

where  $\text{Var}_{\mu_{g_0}^L}(h)$  is the variance defined in (2.4), equal to  $\langle \Pi^{g_0} h, h \rangle$  by (2.5). Also note that  $\dot{V}_0 = \dot{a}_0 = \frac{1}{2} \pi_2^* \dot{g}_0 + X_{g_0} f$  for some  $f \in C^\nu(S_{g_0}M)$ ,  $\nu > 0$ , by Lemma 3.2. We also have  $\ddot{V}_0 = \ddot{a}_0$ . As a consequence, we get from (3.7), (3.8) and (3.9)

$$\begin{aligned} d^2 \Phi_{g_0}(h, h) &= \int_{S_{g_0}M} \ddot{a}_0 d\mu_{g_0}^L = \langle \Pi^{g_0} \dot{V}_0, \dot{V}_0 \rangle = \frac{1}{4} \langle \Pi^{g_0} \pi_2^* \dot{g}_0, \pi_2^* \dot{g}_0 \rangle \\ &= \frac{1}{4} (\langle \Pi_2^{g_0} \dot{g}_0, \dot{g}_0 \rangle - \langle \pi_2^* \dot{g}_0, 1 \rangle^2) = \frac{1}{4} \langle \Pi_2^{g_0} \dot{g}_0, \dot{g}_0 \rangle, \end{aligned}$$

where we used in the third identity that  $\Pi^{g_0} X_{g_0} f = 0 = X_{g_0} \Pi^{g_0} f$  if both  $f$  and  $X_{g_0} f$  are in  $C^\nu(S_{g_0}M)$  for some  $\nu > 0$ . In the last equality,  $\langle \pi_2^* \dot{g}_0, 1 \rangle^2 = c_2 \langle \dot{g}_0, g_0 \rangle^2 = 0$ , since  $T_{g_0} \mathcal{N}^{k, \alpha} = (\mathbb{R}g_0)^\perp$ . Since  $h \in T_{g_0} \mathcal{N}_{\text{sol}}^{k, \alpha}$ ,  $h$  is divergence-free with respect to  $g_0$ . The result then follows from Lemma 2.1.  $\square$

Remark that since the Hessian of  $\Phi$  at  $g_0$  controls only a weaker norm than the  $C^{k, \alpha}$ , it is not completely obvious that there is a neighborhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{N}^{k, \alpha}$  (with its natural topology) so that  $\Phi(g) = 0$  for  $g \in \mathcal{U}$  if and only if  $g = g_0$ . We will discuss that point in the proof of Theorem 1.2.

**3.2. Stability estimate involving the stretch: proof of Theorem 1.2.** We now prove Theorem 1.2 as Theorem 1.1 is a direct consequence of it.

*Proof.* From now,  $C_{g_0}, C'_{g_0}$  will denote positive constants depending only on  $g_0$ , and whose value may change from line to line. Let us pick  $g \in \mathcal{M}^{k, \alpha}$  with regularity  $k \geq 5$  large enough (which will be determined in the end). Here, we need  $k \geq 5$  because we need  $\Phi$  to be at least  $C^3$  (where  $3 = k - 2 = 5 - 2$ ) in order to use a Taylor expansion of order 3. We denote by  $g' = \psi^* g \in \ker D_{g_0}^*$ , with  $\psi \in \mathcal{D}_0^{k+1, \alpha}$ , the  $g_0$ -solenoidal metric obtained by applying Lemma 2.4. We now work in the space  $\mathcal{M}^{5, \alpha}$  and write the Taylor expansion of both  $F(g') = \mathbf{P}(-V_{g'})$  and  $\tilde{\Phi}(g') = I_{\mu_{g_0}^L}(g_0, g')$  at  $g_0$  using (3.6):

$$\begin{aligned} \tilde{\Phi}(g') &= 1 + d\tilde{\Phi}_{g_0}(g' - g_0) + \frac{1}{2} d^2 \tilde{\Phi}_{g_0}(g' - g_0, g' - g_0) + \mathcal{O}(\|g' - g_0\|_{C^{5, \alpha}}^3) \\ F(g') &= -d\tilde{\Phi}_{g_0}(g' - g_0) + \frac{1}{2} d^2 F_{g_0}(g' - g_0, g' - g_0) + \mathcal{O}(\|g' - g_0\|_{C^{5, \alpha}}^3). \end{aligned}$$

As in (3.8) and using (3.9), we have for  $h \in C^{5, \alpha}(M; S^2 T^* M)$

$$\begin{aligned} d^2 F_{g_0}(h, h) &= d^2 \mathbf{P}_{-V_0}(da_{g_0} h, da_{g_0} h) - d\mathbf{P}_{-V_0} \cdot d^2 a_{g_0}(h, h) \\ &= d^2 \mathbf{P}_{-V_0}(da_{g_0} h, da_{g_0} h) - d^2 \tilde{\Phi}_{g_0}(h, h), \end{aligned}$$

thus we get

$$\tilde{\Phi}(g') - 1 + F(g') = \frac{1}{2} d^2 \mathbf{P}_{-V_0}(da_{g_0}(g' - g_0), da_{g_0}(g' - g_0)) + \mathcal{O}(\|g' - g_0\|_{C^{5, \alpha}}^3). \quad (3.10)$$

Using [PP90, Proposition 4.11], we get for each  $u \in C^\nu(S_{g_0}M)$

$$d^2\mathbf{P}_{-V_0}(u, u) = \langle \Pi^{g_0}(u - \langle u, 1 \rangle), (u - \langle u, 1 \rangle) \rangle = \langle \Pi^{g_0}u, u \rangle,$$

because  $\Pi^{g_0}(1) = 0$  (and where  $\langle u, 1 \rangle = \int_{S_{g_0}M} u d\mu_{g_0}^L$ ). By Lemma 3.2,  $da_{g_0}(g' - g_0) = \frac{1}{2}\pi_2^*(g' - g_0) + X_{g_0}f$  for some  $f \in C^\nu(S_{g_0}M)$ , which then yields, for  $h := g' - g_0$ :

$$d^2\mathbf{P}_{-V_0}(da_{g_0}h, da_{g_0}h) = \frac{1}{4}(\langle \Pi_2^{g_0}h, h \rangle - \langle h, g_0 \rangle^2) - d^2\tilde{\Phi}_{g_0}(h, h).$$

Here, our normalization convention is  $\langle g_0, g_0 \rangle = 1$ . Combining with (3.10), we obtain

$$|\tilde{\Phi}(g') - 1| + |F(g')| \geq \frac{1}{8}(\langle \Pi_2^{g_0}h, h \rangle - \langle h, g_0 \rangle^2) - C_{g_0}\|h\|_{C^{5,\alpha}}^3.$$

By Lemma 2.1 and the fact that  $D_{g_0}^*h = 0$ , we deduce that

$$|\tilde{\Phi}(g') - 1| + |F(g')| \geq C_{g_0}\|h\|_{H^{-\frac{1}{2}}(M)}^2 - \frac{1}{8}\langle h, g_0 \rangle^2 - C'_{g_0}\|h\|_{C^{5,\alpha}}^3.$$

But since by (3.6), there is  $C_n > 0$  depending only on  $n$  so that for  $\|g - g_0\|_{C^{5,\alpha}}$  small enough

$$|\langle h, g_0 \rangle|^2 = C_n^{-2}|d\tilde{\Phi}_{g_0}h|^2 \leq 2C_n^{-2}|\tilde{\Phi}(g') - 1|^2 + C_{g_0}\|h\|_{C^{5,\alpha}}^4$$

we conclude that

$$|\tilde{\Phi}(g) - 1| + |F(g)| \geq C_{g_0}\|h\|_{H^{-\frac{1}{2}}(M)}^2 - C'_{g_0}\|h\|_{C^{5,\alpha}}^3.$$

Using Sobolev embedding and interpolation estimates, we get

$$\|g' - g_0\|_{C^{5,\alpha}}^3 \leq C_{g_0}\|g' - g_0\|_{H^{\frac{3}{2}+5+\alpha'}}^3 \leq C'_{g_0}\|g' - g_0\|_{H^{-1/2}}^2\|g' - g_0\|_{H^k},$$

with  $k > \frac{3}{2}n + 16 + 3\alpha$  and  $\alpha' > \alpha$ . Thus assuming that  $\|g - g_0\|_{C^{k,\alpha}}$  is small enough depending on  $C_{g_0}$ , we obtain

$$\|g' - g_0\|_{H^{-1/2}}^2 \leq C_{g_0} \left( |\tilde{\Phi}(g) - 1| + |F(g)| \right). \quad (3.11)$$

We also recall (see [PPS15, Corollary 9.17]) that

$$\begin{aligned} \mathbf{P}(-V_g) &= \lim_{T \rightarrow \infty} \frac{1}{T} \log \sum_{c \in \mathcal{C}, L_{g_0}(c) \in [T, T+1]} e^{-\int_{\gamma_{g_0}(c)} V_g} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \log \sum_{c \in \mathcal{C}, L_{g_0}(c) \in [T, T+1]} e^{-\int_{\gamma_{g_0}(c)} J_{g_0}^u e^{L_{g_0}(c) - L_g(c)}.} \end{aligned}$$

Thus, if we order  $\mathcal{C} = (c_j)_{j \in \mathbb{N}}$  by the lengths (i.e.  $L_{g_0}(c_j) \geq L_{g_0}(c_{j-1})$ ), and we define

$$\mathcal{L}_+(g) := \limsup_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1, \quad \mathcal{L}_-(g) := \liminf_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1,$$

we see that for all  $\delta > 0$  small, there is  $T_0 > 0$  large so that for all  $j$  with  $L_{g_0}(c_j) \in [T, T+1]$  with  $T \geq T_0$ :

$$e^{\min((T+1)(-\mathcal{L}_+(g)-\delta), T(-\mathcal{L}_+(g)-\delta))} \leq e^{L_{g_0}(c_j)-L_g(c_j)} \leq e^{\max((T+1)(-\mathcal{L}_-(g)+\delta), T(-\mathcal{L}_-(g)+\delta))}.$$

We deduce, using  $\mathbf{P}(-V_{g_0}) = 0$ ,

$$-\mathcal{L}_+(g) - \delta \leq \mathbf{P}(-V_g) \leq -\mathcal{L}_-(g) + \delta.$$

Since  $\delta > 0$  is arbitrarily small, we obtain  $|\mathbf{P}(-V_g)| \leq \max(|\mathcal{L}_+(g)|, |\mathcal{L}_-(g)|)$  and combining with (3.11) and Lemma 2.7, we get the announced result.  $\square$

We remark that the proof above (using (3.10)) also shows that if we work on the slice of metrics

$$\{g \in \mathcal{M}^{k,\alpha} \mid D_{g_0}^* g = 0, \int_M \text{Tr}_{g_0}(h) d\text{vol}_{g_0} = 0\}$$

then there is  $C_{g_0} > 0, \varepsilon > 0$  such that if  $\|g - g_0\|_{C^{k,\alpha}} < \varepsilon$  with  $\varepsilon$  small enough

$$\|g - g_0\|_{H^{-\frac{1}{2}}(M)}^2 \leq C_{g_0} \left( \limsup_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - \liminf_{j \rightarrow \infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} \right).$$

### 3.3. The pressure metric on the space of negatively curved metrics.

3.3.1. *Definition of the pressure metric using the variance.* On  $\mathcal{M}^-$ , the cone of smooth negatively-curved metrics, we introduce the non-negative symmetric bilinear form

$$G_g(h_1, h_2) := \langle \Pi_2^g h_1, h_2 \rangle_{L^2(M, d\text{vol}_g)}, \quad (3.12)$$

defined for  $g \in \mathcal{M}$ ,  $h_j \in T_g \mathcal{M} \simeq C^\infty(M; S^2 T^* M)$ . It is nondegenerate on  $T_g \mathcal{M} \cap \ker D_g^*$ , namely  $G_g(h, h) \geq C_g \|h\|_{H^{-1/2}}^2$  by Lemma 2.2 and the constant  $C_g$  turns out to be locally uniform for  $g$  near a given metric  $g_0$ . Combining these facts, we obtain

**Proposition 3.5.** *Let  $g_0 \in \mathcal{M}^-$ , then the bilinear form  $G$  defined in (3.12) produces a Riemannian metric on the quotient space  $\mathcal{M}^-/\mathcal{D}_0$  near the class  $[g_0]$ , where  $\mathcal{M}^-/\mathcal{D}_0$  is identified with the slice  $\mathcal{S}$  passing through  $g_0$  as in (2.8).*

*Proof.* It suffices to show that  $G$  is non-degenerate on  $T\mathcal{S}$ . Let  $h \in T_g \mathcal{S}$  and assume that  $G_g(h, h) = 0$ . We can write  $h = L_V g + h'$  where  $D_g^* h' = 0$  and  $V$  is a smooth vector field and  $L_V$  the Lie derivative wth respect to  $V$ . By Lemma 2.1 we obtain  $0 = G_g(h, h) \geq C \|h'\|_{H^{-1/2}}$ . Thus  $h = L_V g$ , but we also know that  $T_g \mathcal{S} \cap \{L_V g \mid V \in C^\infty(M; T^* M)\} = \{0\}$  since  $\mathcal{S}$  is a slice. Therefore  $h = 0$ .  $\square$

3.3.2. *Definition using the intersection number.* Let us assume that  $g$  is in a fixed  $C^2$ -neighborhood of  $g_0$ . Since  $J_{g_0}^u > 0$ , we obtain that  $V_g = J_{g_0}^u + a_g - 1 > 0$  if  $g$  is close enough to  $g_0$ . By [Sam14, Lemma 2.4], there exists a unique constant  $\mathbf{h}_{V_g} \in \mathbb{R}$  such that  $\mathbf{P}(-\mathbf{h}_{V_g}V_g) = 0$ . In particular,  $\mathcal{N}$  coincides in a neighborhood of  $g_0$  with the set  $\{g \in \mathcal{M} \mid \mathbf{h}_{V_g} = 1\}$ . One can express the constant  $\mathbf{h}_{V_g}$  as  $\mathbf{h}_{V_g} = \mathbf{h}_{\text{top}}(\varphi_t^{g_0, V_g})$ , where  $\varphi_t^{g_0, V_g}$  is a time-reparametrization of the geodesic flow of  $g_0$  (see [BCLS15, Section 3.1.1]). More precisely, given  $f \in C^\nu(S_{g_0}M)$  a Hölder-continuous positive function on  $S_{g_0}M$ , we introduce  $\mathbf{h}_f$  to be the unique real number such that  $\mathbf{P}(-\mathbf{h}_f f) = 0$  and we set:

$$S_{g_0}M \times \mathbb{R} \ni (z, t) \mapsto \kappa_f(z, t) := \int_0^t f(\varphi_s^{g_0}(z)) ds.$$

For a fixed  $z \in S_{g_0}M$ , this is a homeomorphism on  $\mathbb{R}$  and thus allows to define:

$$\varphi_{\kappa_f(z, t)}^{g_0, f}(z) := \varphi_t^{g_0}(z). \quad (3.13)$$

We now follow the approach of [BCLS15, Section 3.4.1]. Given two Hölder-continuous functions  $f, f' \in C^\nu(S_{g_0}M)$  such that  $f > 0$ , one can define an *intersection number* [BCLS15, Eq. (13)]

$$\mathbf{I}_{g_0}(f, f') := \frac{\int_{S_{g_0}M} f' d\mu_{-\mathbf{h}_f f}}{\int_{S_{g_0}M} f d\mu_{-\mathbf{h}_f f}}$$

where  $d\mu_{-\mathbf{h}_f f}$  is the equilibrium measure for the potential  $-\mathbf{h}_f f$ . We have the following result, which follows from [BCLS15, Proposition 3.8] stated for Anosov flows on compact metric spaces:

**Proposition 3.6** (Bridgeman-Canary-Labourie-Sambarino [BCLS15]). *Let  $f, f' : S_{g_0}M \rightarrow \mathbb{R}_+$  be two Hölder-continuous positive functions. Then:*

$$\mathbf{J}_{g_0}(f, f') := \frac{\mathbf{h}_{f'}}{\mathbf{h}_f} \mathbf{I}_{g_0}(f, f') \geq 1$$

*with equality if and only if  $\mathbf{h}_f f$  and  $\mathbf{h}_{f'} f'$  are cohomologous for the geodesic flow  $\varphi_t^{g_0}$  of  $g_0$ . The quantity  $\mathbf{J}_{g_0}(f, f')$  is called the renormalized intersection number.*

We apply the previous proposition with  $f := J_{g_0}^u$  (then  $\mathbf{h}_{J_{g_0}^u} = 1$ ) and  $f' := V_g$ . Without assuming that  $g \in \mathcal{N}$  (that is we do not necessarily assume that  $\mathbf{h}_{V_g} = 1$ ), we have

$$\begin{aligned} \mathbf{J}_{g_0}(J_{g_0}^u, V_g) &= \mathbf{h}_{V_g} \mathbf{I}_{g_0}(J_{g_0}^u, V_g) = \mathbf{h}_{V_g} \frac{\int_{S_{g_0}M} (J_{g_0}^u + a_g - \mathbf{1}) d\mu_{g_0}^L}{\int_{S_{g_0}M} J_{g_0}^u d\mu_{g_0}^L} \\ &= \mathbf{h}_{V_g} \frac{\mathbf{h}_L(g_0) + I_{\mu_{g_0}^L}(g_0, g) - 1}{\mathbf{h}_L(g_0)} \geq 1 \end{aligned}$$

where  $\mathbf{h}_L(g_0)$  is the entropy of Liouville measure for  $g_0$ . In the specific case where  $g \in \mathcal{N}$ ,  $\mathbf{h}_{V_g} = 1$  and we find that  $I_{\mu_{g_0}^L}(g_0, g) \geq 1$  with equality if and only if  $a_g$  is cohomologous

to 1, that is if and only if  $L_g = L_{g_0}$ , or alternatively if and only if  $\varphi^g$  and  $\varphi^{g_0}$  are time-preserving conjugate. This computation holds as long as  $J_{g_0}^u + a_g - 1 > 0$  (which is true in a  $C^2$ -neighborhood of  $g_0$ ).

In particular, on  $\mathcal{N}$ , we have the linear relation

$$\mathbf{J}_{g_0}(J_{g_0}^u, V_g) = 1 + \frac{I_{\mu_{g_0}^L}(g_0, g) - 1}{\mathbf{h}_L(g_0)}.$$

In the notations of [BCLS15, Proposition 3.11], the second derivative computed for the family  $(g_\lambda)_{\lambda \in (-1,1)} \in \mathcal{N}$  is

$$\partial_\lambda^2 \mathbf{J}_{g_0}(J_{g_0}^u, V_{g_\lambda})|_{\lambda=0} = \frac{1}{\mathbf{h}_L(g_0)} \partial_\lambda^2 I_{\mu_{g_0}^L}(g_0, g_\lambda)|_{\lambda=0} = \frac{\langle \Pi_2^{g_0} \dot{g}_0, \dot{g}_0 \rangle}{4\mathbf{h}_L(g_0)} \quad (3.14)$$

and is called the *pressure form*. When considering a slice transverse to the  $\mathcal{D}_0$  action on  $\mathcal{N}$ , it induces a metric called the *pressure metric* by Lemma 2.1. To summarize:

**Lemma 3.7.** *Given a smooth metric  $g_0$ , the metric  $G_{g_0}$  restricted to  $\mathcal{N}$  can be obtained from the renormalized intersection number by*

$$G_{g_0}(h, h) = 4\mathbf{h}_L(g_0) \partial_\lambda^2 \mathbf{J}_{g_0}(J_{g_0}^u, V_{g_\lambda})|_{\lambda=0}$$

where  $(g_\lambda)_{\lambda \in (-1,1)}$  is any family of metrics such that  $g_\lambda \in \mathcal{N}$  and  $\dot{g}_0 = h \in T_{g_0}\mathcal{N}$ .

**3.3.3. Link with the Weil-Petersson metric.** We now assume that  $M = S$  is an orientable surface of genus  $\geq 2$  and let  $\mathcal{T}(S)$  be the Teichmüller space of  $S$ . We fix a hyperbolic metric  $g_0$ . Given  $\eta, \rho \in \mathcal{T}(S)$ , the intersection number is defined as

$$\mathbf{I}(\eta, \rho) := \mathbf{I}_{g_0}(a_{g_\eta}, a_{g_\rho}) = \frac{\int_{S_{g_0}M} a_{g_\rho} d\mu_\eta}{\int_{S_{g_0}M} a_{g_\eta} d\mu_\eta}$$

where  $[g_\eta] = \eta$ ,  $[g_\rho] = \rho$  and  $\mu_\eta$  is the equilibrium state of  $-h_{a_{g_\eta}} a_{g_\eta}$ . Note that  $\mathbf{h}_{a_{g_\eta}} = \mathbf{h}_{\text{top}}(\varphi_t^{g_0, a_\eta}) = 1$  since  $\varphi^{g_0, a_\eta}$  is conjugate to the geodesic flow of  $g_\eta$ , which in turn has constant curvature, and by [Sam14, Lemma 2.4],  $a_{g_\eta} d\mu_\eta / \int_{S_{g_0}M} a_{g_\eta} d\mu_\eta$  is the measure of maximal entropy of the flow  $\varphi_t^{g_0, a_\eta}$ , thus also the normalized Liouville measure of  $g_\eta$  (viewed on  $S_{g_0}M$ ). This number  $\mathbf{I}(\eta, \rho)$  is in fact *independent of  $g_0$*  as it can alternatively be written

$$\mathbf{I}(\eta, \rho) = \lim_{T \rightarrow \infty} \frac{1}{N_T(\eta)} \sum_{c \in \mathcal{C}, L_{g_\eta}(c) \leq T} \frac{L_{g_\rho}(c)}{L_{g_\eta}(c)}$$

where  $N_T = \#\{c \in \mathcal{C} \mid L_{g_\eta}(c) \leq T\}$  (see [BCS18, Proof of Th. 4.3]). In particular, taking  $g_0 = g_\eta$ , one has

$$\mathbf{I}(\eta, \rho) = I_{\mu_{g_\eta}^L}(g_\eta, g_\rho).$$

As explained in [BCS18, Theorem 4.3], up to a normalization constant  $c_0$  depending on the genus only, the Weil-Peterson metric on  $\mathcal{T}(S)$  is equal to

$$\|h\|_{\text{WP}}^2 = c_0 \partial_\lambda^2 \mathbf{I}(\eta, \eta_\lambda)|_{\lambda=0} = c_0 \partial_\lambda^2 I_{\mu_{g_\eta}^L}(g_\eta, g_{\eta_\lambda})|_{\lambda=0}, \quad (3.15)$$

where  $\dot{\eta}_0 = h$  and  $(g_{\eta_\lambda})_{\lambda \in (-1,1)}$  is a family of hyperbolic metrics such that  $[g_{\eta_\lambda}] = \eta_\lambda$ ,  $\eta = \eta_0 = [g_0]$ . This fact follows from combined works of Thurston, Wolpert [Wol86] and Mc Mullen [MM08]: the length of a random geodesic  $\gamma$  on  $(S, g_0)$  with respect to  $g_{\eta_\lambda}$  has a local minimum at  $\lambda = 0$  and the Hessian is positive definite (Thurston), equals to the Weil-Peterson norm squared of  $\dot{g}$  (Wolpert [Wol86]) and given by a variance (Mc Mullen [MM08]); here random means equidistributed with respect to the Liouville measure of  $g_0$ . We can check that the metric  $G$  also corresponds to this metric:

**Proposition 3.8.** *The metric  $G$  on  $\mathcal{T}(S)$  is a multiple of the Weil-Peterson metric.*

*Proof.* This follows directly from (3.14), (3.15) and the fact that  $\mathbf{h}_L(g_\eta) = 1$  if  $g_\eta$  has curvature  $-1$ .  $\square$

*Remark 3.9.* We notice that the positivity of the metric in the case of Teichmüller space follows only from some convexity argument in finite dimension. In the case of general metrics with negative curvature, the coercive estimate of Lemma 2.1 on the variance is much less obvious due to the infinite dimensionality of the space. As it turns out, this is the key for the local rigidity in Theorem 1.1.

#### 4. UNIFORM ELLIPTIC ESTIMATES ON $\Pi_2$

In this section, we prove that the operator  $\Pi_2^g \in \Psi^{-1}$  depends continuously on  $g$ . Let  $\mathcal{M}^{\text{An}}$  be the space of smooth Riemannian metrics with Anosov geodesic flow.

**Proposition 4.1.** *The map  $\mathcal{M}^{\text{An}} \ni g \mapsto \Pi_2^g \in \Psi^{-1}(M)$  is continuous, when  $\Psi^{-1}(M)$  is equipped with its canonical topology of Fréchet spaces.*

We fix a metric  $g_0$  and we work in a neighborhood  $\mathcal{U}$  of  $g_0$  in the  $C^\infty$  topology. In particular, we will always assume that this neighborhood  $\mathcal{U}$  is small enough so that any  $g \in \mathcal{U}$  has an Anosov geodesic flow that is orbit-conjugated to that of  $g_0$  by structural stability. We will also see the geodesic flows  $(\varphi_t^g)_{t \in \mathbb{R}}$  as acting on the unit bundle  $SM := S_{g_0}M$  for  $g_0$  by using the natural identification  $S_gM \rightarrow S_{g_0}M$  obtained by scaling in the fibers. The operator  $\pi_2^*$  associated to  $g$  becomes: for  $(x, v) \in S_{g_0}M$

$$(\pi_2^*h)(x, v) = h_x(v, v)|v|_g^{-2},$$

if  $h \in C^\infty(M; S^2T^*M)$ , it is just a scaling times the  $\pi_2^*$  associated to  $g_0$ .

**4.1. The resolvents of  $X_g$  and anisotropic spaces.** We first recall the construction of resolvents of  $X_g$  from Faure-Sjöstrand [FS11] (see also [DZ16]) and in particular the version used in Dang-Guillarmou-Rivière-Shen [DGRS] that deals with the continuity with respect to the flow  $X_g$ . There are two resolvents bounded on  $L^2$  for  $X_g$  defined for  $\operatorname{Re}(\lambda) > 0$  by

$$R_g^\pm(\lambda) := \pm \int_0^\infty e^{-\lambda t} e^{\pm t X_g} f dt$$

for  $f \in L^2(SM, d\mu_g^L)$ . They solve  $(-X_g \pm \lambda)R_g^\pm(\lambda) = \operatorname{Id}$  on  $L^2$ . The following results are proved in [FS11], and we use here the presentation of [DGRS, Sections 3.2 and 3.3] due to the need of uniformity with respect to  $g$ : there is  $c_0 > 0$  depending only on  $g$ , locally uniform with respect to  $g$  ( $c_0$  depends only on the Anosov exponents of contraction/dilation of  $d\varphi_1^g$ ), such that for each  $N_0 > 0, N_1 > 16N_0$ ,  $R_g^\pm(\lambda)$  admits a meromorphic extension in  $\operatorname{Re}(\lambda) > -c_0 N_0$  as a bounded operator

$$R_g^-(\lambda) : \mathcal{H}^{m_g^{N_0, N_1}} \rightarrow \mathcal{H}^{m_g^{N_0, N_1}}, \quad R_g^+(\lambda) : \mathcal{H}^{-m_g^{N_0, N_1}} \rightarrow \mathcal{H}^{-m_g^{N_0, N_1}} \quad (4.1)$$

where  $\mathcal{H}^{\pm m_g^{N_0, N_1}}$  are Hilbert spaces depending on  $N_0 > 0, N_1 > 0$  satisfying the following properties:

$$H^{2N_1}(SM) \subset \mathcal{H}^{m_g^{N_0, N_1}} \subset H^{-2N_0}(SM), \quad H^{2N_0}(SM) \subset \mathcal{H}^{-m_g^{N_0, N_1}} \subset H^{-2N_1}(SM)$$

and defined by

$$\mathcal{H}^{\pm m_g^{N_0, N_1}} = (A_{m_g^{N_0, N_1}})^{\mp 1} L^2(SM), \quad A_{m_g^{N_0, N_1}} := \operatorname{Op}(e^{m_g^{N_0, N_1} \log f})$$

and  $A_{m_g^{N_0, N_1}}$  is an invertible pseudo-differential operator with inverse having principal symbol  $e^{-m_g^{N_0, N_1} \log f}$ . Here  $\operatorname{Op}$  denotes a quantization, while  $m_g^{N_0, N_1} \in S^0(T^*(SM))$ ,  $f \in S^1(T^*(SM), [1, \infty))$  (the usual classes of symbols) are homogeneous of respective degree 0 and 1 in  $|\xi| > R$ , for some  $R > 1$  independent of  $g$ , and constructed from the lifted flow  $\Phi_t^g = ((d\varphi_t^g)^{-1})^T$  acting on  $T^*(SM)$ . The function  $f$  can be taken depending only on  $g_0$  for  $g$  in a small enough  $C^\infty$  neighborhood  $\mathcal{U}$  of  $g_0$ . Moreover there are small conic neighborhoods  $C_u(g_0)$  and  $C_s(g_0)$  of  $E_u^*(g_0)$  and  $E_s^*(g_0)$  such that for any smaller open conic neighborhood  $C'_u(g_0) \subset C_u(g_0)$  of  $E_u^*(g_0)$  and  $C'_s(g_0) \subset C_s(g_0)$  of  $E_s^*(g_0)$ ,  $m_g^{N_0, N_1}$  satisfies:

$$\begin{cases} m_g^{N_0, N_1}(z, \xi) \geq N_1, & (z, \xi) \in C'_s(g_0), \\ m_g^{N_0, N_1}(z, \xi) \geq N_1/8 & (z, \xi) \notin C_u(g_0), \\ m_g^{N_0, N_1}(z, \xi) \leq -N_0 & (z, \xi) \in C'_u(g_0), \end{cases} \quad (4.2)$$

and  $m_g(x, \xi) \in [-2N_0, 2N_1]$  for all  $(z, \xi) \in T^*(SM)$ . We note that [DGRS, Lemma 3.3] shows that  $m_g^{N_0, N_1}$  is smooth with respect to the metric  $g$  and that  $f$  can be taken to be independent of  $g$  for  $g$  close enough to  $g_0$ . The spaces  $\mathcal{H}^{m_g^{N_0, N_1}}$  are called *anisotropic Sobolev spaces*. The pseudodifferential operators  $A_{m_g^{N_0, N_1}}$  belong to the class  $\Psi^{2N_1}(SM)$  but also to

some anisotropic subclass denoted  $\Psi^{m_g^{N_0, N_1}}(SM)$  admitting composition formulas; we refer to [FRS08, FS11] for details.

Eventually, [DGRS, Proposition 6.1] shows that there is a small open neighborhood  $W_\delta$  of the circle  $\{\lambda \in \mathbb{C} \mid |\lambda| = \delta\}$  for some small  $\delta > 0$  so that

$$\mathcal{U} \times W_\delta \ni (g, \lambda) \mapsto A_{m_g^{N_0, N_1}} R_g^-(\lambda) (A_{m_g^{N_0, N_1}})^{-1} \in \mathcal{L}(H^1(SM), L^2(SM)) \quad (4.3)$$

is continuous.<sup>4</sup>

**4.2. The operator  $\Pi_2^g$  in terms of resolvents.** Following [Gui17], the link between  $\Pi^g$  and the resolvent is given by the Laurent expansion

$$\Pi^g = R_g^+(0) - R_g^-(0)$$

where  $R_g^+(\lambda)$  has a pole of order 1 and  $R_g^\pm(0)$  is defined by

$$R_g^\pm(\lambda) = \pm \lambda^{-1} \langle \cdot, 1 \rangle + R_g^\pm(0) + \mathcal{O}(\lambda)$$

and  $R_g^-(0) = -(R_g^+(0))^*$  where the adjoint is with respect to the Liouville measure.

**Lemma 4.2.** *Let  $\chi \in C_c^\infty(\mathbb{R})$  be even and equal to 1 in  $[-T, T]$  and supported in the interval  $(-T-1, T+1)$ . Then we have*

$$\Pi^g = \int_{\mathbb{R}} \chi(t) e^{tX_g} dt - R_g^+(0) \int_0^{+\infty} \chi'(t) e^{tX_g} dt + R_g^-(0) \int_0^{+\infty} \chi'(t) e^{-tX_g} dt - \langle \cdot, 1 \rangle \int_{\mathbb{R}} \chi. \quad (4.4)$$

*Proof.* For  $\text{Re}(\lambda) > 0$ , we can write by integration by parts

$$\begin{aligned} R_g^\pm(\lambda) &= \pm \int_0^\infty \chi(t) e^{-t(\lambda \mp X_g)} dt \pm \int_0^\infty (1 - \chi(t)) e^{-t(\lambda \mp X_g)} dt \\ &= \pm \int_0^\infty \chi(t) e^{-t(\lambda \mp X_g)} dt - R_g^\pm(\lambda) \int_0^\infty \chi'(t) e^{t(\pm X_g - \lambda)} dt. \end{aligned}$$

Then taking the limit as  $\lambda \rightarrow 0$ , we obtain

$$R_g^\pm(0) = \pm \int_0^\infty \chi(t) e^{\pm tX_g} dt - R_g^\pm(0) \int_0^\infty \chi'(t) e^{\pm tX_g} dt \mp \int_0^\infty \chi(t) dt \langle \cdot, 1 \rangle$$

and summing gives the result.  $\square$

Next, we remark that, using that  $\varphi_t^g(x, -v) = -\varphi_{-t}^g(x, v)$  (where multiplication by  $-1$  is the symmetry in the fibers of  $SM$ ), it is straightforward to check that for all  $t \in \mathbb{R}$

$$\pi_{2*} e^{tX_g} \pi_2^* = \pi_{2*} e^{-tX_g} \pi_2^*,$$

<sup>4</sup>In [DGRS, Proposition 6.1], there is a small semi-classical parameter  $h > 0$  appearing: we can just fix this parameter small enough. It does not play any role here except in the quantization procedure  $\text{Op}$ . We also add that in [DGRS, Proposition 6.1],  $N_1$  is chosen to be equal to  $20N_0$  for notational convenience, but the proof does not use that fact.

which also implies that  $\pi_{2*}R_g^+(0)e^{tX_g}\pi_2^* = -\pi_{2*}R_g^-(0)e^{-tX_g}\pi_2^*$  and thus

$$\Pi_2^g = 2\pi_{2*} \int_0^\infty \chi(t)e^{-tX_g} dt \pi_2^* + 2\pi_{2*}R_g^-(0) \int_0^{+\infty} \chi'(t)e^{-tX_g} dt \pi_2^* + \left(1 - \int_{\mathbb{R}} \chi\right) \langle \cdot, 1 \rangle. \quad (4.5)$$

We are going to prove that these three terms depend continuously on  $g$ . Note that

$$\left(1 - \int_{\mathbb{R}} \chi\right) \langle f, 1 \rangle = \left(1 - \int_{\mathbb{R}} \chi\right) \int_{SM} f(z) d\mu_g^L(z)$$

and thus the  $g$ -continuity of this term is immediate. Now, we claim the following:

**Lemma 4.3.** *There is  $T > 0$  large enough and a neighborhood  $\mathcal{U}' \subset \mathcal{U}$  of  $g_0$  in  $\mathcal{M}^{\text{An}}$  so that for all  $x \in M$  and all  $g \in \mathcal{U}'$  the exponential map of  $g$  in the universal cover  $\widetilde{M}$*

$$\exp_x^{\widetilde{g}} : \{v \in T_x \widetilde{M}; |v|_g \leq T\} \rightarrow \widetilde{M}$$

*is a diffeomorphism onto its image and  $\Phi_t^g(V^*) \subset C'_u(g_0)$  for all  $t \geq T$ , if  $\Phi_t^g := ((d\varphi_t^g)^{-1})^T$  is the symplectic lift of  $\varphi_t^g$  and  $V^* \subset T^*(SM)$  is the annihilator of the vertical bundle  $V = \ker d\pi_0 \subset T(SM)$ .*

*Proof.* By Lemma 3.1 of [DGRS], the cone  $C'_u(g_0)$  can be chosen so that there is  $T > 0$  and  $\mathcal{U}'$  such that for all  $t \geq T$  and all  $g \in \mathcal{U}'$ ,  $\Phi_t^g(C'_u(g_0)) \subset C'_u(g_0)$ . We also know that  $\Phi_{T_0}^{g_0}(V^*) \subset C'_u(g_0)$  for some  $T_0 > T$  by hyperbolicity of  $g_0$ , but by continuity of  $g \mapsto \Phi_{T_0}^g$ , the same holds for all  $g$  in some possible smaller neighborhood  $\mathcal{U}'' \subset \mathcal{U}'$ , thus for all  $t \geq T_0$  and all  $g \in \mathcal{U}''$ ,  $\Phi_t^g(V^*) \subset C'_u(g_0)$ . Now, we claim that, up to choosing  $\mathcal{U}''$  even smaller, the exponential map is a diffeomorphism on  $|v|_g \leq T$  in the universal cover: indeed, Anosov geodesic flows have no pair of conjugate points.  $\square$

**4.3. Proof or Proposition 4.1.** Let us define

$$\Omega_1^g := \pi_{2*} \int_0^\infty \chi(t)e^{-tX_g} dt \pi_2^*, \quad \Omega_2^g := \pi_{2*}R_g^-(0) \int_0^{+\infty} \chi'(t)e^{-tX_g} dt \pi_2^*$$

Proposition 4.1 is a consequence of the following two Lemmas.

**Lemma 4.4.** *For each  $g \in \mathcal{U}'$ ,  $\Omega_1^g \in \Psi^{-1}(M)$  with principal symbol*

$$\sigma(\Omega_1^g)(x, \xi) = c_n |\xi|^{-1} \sigma(\pi_{\ker D_g^*}) = c_n |\xi|^{-1} \pi_{\ker i_\xi}$$

*for some  $c_n > 0$  depending only on  $n = \dim M$  and the map  $g \mapsto \Omega_1^g$  is continuous with respect to the smooth topology on  $\mathcal{U}'$  and the usual Fréchet topology on  $\Psi^{-1}(M)$ .*

*Proof.* The fact that, for each  $g \in \mathcal{M}^{\text{An}}$ , the operator  $\Omega_1^g \in \Psi^{-1}(M)$  is proved in [Gui17, Theorem 3.5], the computation of the principal symbol follows from the computation [SSU05] and is done in details in our setting in [GL, Theorem 4.4.]. We need to check the continuity with respect to  $g$  in the  $\Psi^{-1}(M)$  topology and we can proceed as in [PU05,

Lemma 3.1]. For  $h \in C^\infty(M; S^2T^*M)$ , we can write explicitly in  $(x_i)_i$  coordinates in the universal cover  $\widetilde{M}$  near a point  $p \in \widetilde{M}$

$$(\Omega_1^g h(x))_{ij} = \int_{S_x \widetilde{M}} \int_0^\infty \chi(t) \widetilde{h}_{\exp_x^{\widetilde{g}}(tv)}(\partial_t \exp_x^{\widetilde{g}}(tv), \partial_t \exp_x^{\widetilde{g}}(tv)) p_{ij}(x, v) dt dS_x(v)$$

where  $p_{ij}(x, v)$  are polynomials of order 2 in the  $v$  variable,  $\widetilde{h} \in C^\infty(\widetilde{M}; S^2T^*M)$  is the lift of  $h$  to the universal cover  $\widetilde{M}$ ,  $dS_x$  is the natural measure on the sphere  $S_x \widetilde{M}$ . Using Lemma 4.3, we can perform the change of coordinates  $(t, v) \in (0, T) \times S_x \widetilde{M} \mapsto y := \exp_x^{\widetilde{g}}(tv) \in \widetilde{M}$ , we get  $t = d_{\widetilde{g}}(x, y)$  the distance in  $\widetilde{M}$ , and

$$dt dv = \frac{J_x^g(y)}{(d_{\widetilde{g}}(x, y))^{n-1}} d\text{vol}_g(y), \quad v = \frac{(\exp_x^{\widetilde{g}})^{-1}(y)}{d_{\widetilde{g}}(x, y)}, \quad \partial_t \exp_x^{\widetilde{g}}(tv) = (\nabla^g \exp_x^{\widetilde{g}})((\exp_x^{\widetilde{g}})^{-1}(y)),$$

for some  $J_x^g(y)$  smooth in  $x, y, g$ . This implies that

$$\Omega_1^g h(x) = \int_M K_g(x, y) h(y) d\text{vol}_g(y)$$

for some  $K_g(x, y)$  which is smooth in  $(g, x, y)$  outside the diagonal  $x = y$  and, near the diagonal, it has the form

$$K_g(x, y) = d_{\widetilde{g}}(x, y)^{-n+1} F\left(g, x, \frac{x-y}{|x-y|}, |x-y|\right)$$

with  $F$  smooth in all its variables, due to the smoothness of the exponential map and the distance function in  $\widetilde{M}$  with respect to  $g \in C^\infty$ . Since the local full symbols are Fourier transforms of the integral kernel  $K_g$  in polar coordinates around the diagonal, we deduce the continuity (and indeed, smoothness) of  $\Omega_1^g$  as an element of  $\Psi^{-1}(M)$  with respect to the metric  $g$ .  $\square$

**Lemma 4.5.** *The operator  $\Omega_2^g$  has a smooth Schwartz kernel for each  $g \in \mathcal{U}'$  and the map*

$$g \in \mathcal{U}' \mapsto \Omega_2^g \in C^\infty(M \times M; S^2T^*M \otimes (S^2T^*M)^*)$$

*is smooth if we identify  $\Omega_2^g$  with its Schwartz kernel.*

*Proof.* First we observe that if  $B \in \Psi^0(SM)$  is chosen, independently of  $g$ , so that  $B^* = B$  and  $B$  microsupported in a small conic neighborhood of  $V^*$  not intersecting  $\mathcal{C}_u(g_0)$  and equal microlocally to the identity in a slightly smaller conic neighborhood of  $V^*$ , then

$$\pi_2^* = B\pi_2^* + S_g, \quad \pi_{2*} = \pi_{2*}B + S_g^*$$

with  $S_g$  a continuous family of smoothing operators. We will show that the operator

$$\Omega_3^g := BR_g^-(0) \int_T^{T+1} \chi'(t) e^{-tX_g} B dt$$

is a continuous family (with respect to  $g$ ) of smoothing operators. We need to show that for each  $N > 0$ ,  $\Omega_3^g : H^{-N}(SM) \rightarrow H^N(SM)$  is a continuous family with respect to  $g$  of bounded operators. To study  $R_g^+(0)$ , it suffices to write it under the form

$$R_g^-(0) = \frac{1}{2\pi i} \int_{|\lambda|=\delta} \frac{R_g^-(\lambda)}{\lambda} d\lambda \quad (4.6)$$

with  $\delta$  small enough so that the only pole of  $R_g^-(\lambda)$  in  $|\lambda| \leq \delta$  is  $\lambda = 0$ , and thus it amounts to analyze  $R_g^+(\lambda)$  on  $\{|\lambda| = \delta\}$ . We let  $B_T \in \Psi^0(M)$  be microsupported in a conic neighborhood of  $\cup_{t \in [T, T+1]} \Phi_t^g(\text{WF}(B))$ , so that by Egorov (or simply the formula of composition of  $\Psi^0(SM)$  with diffeomorphisms of  $SM$ )

$$\forall t \in [T, T+1], \quad e^{-tX_g} B = B_T e^{-tX_g} B + S'_g(t)$$

for some continuous family  $(g, t) \mapsto S'_g(t)$  of smoothing operators. We note that by taking  $\mathcal{U}'$  small enough and  $\text{WF}(B)$  close enough to  $V^*$ , Lemma 4.3 insures that we can choose  $B_T$  depending only on  $T$  (thus uniform in  $g \in \mathcal{U}'$ ) so that  $\text{WF}(B_T) \subset C'_u(g_0)$ . Thus

$$\int_T^{T+1} \chi'(t) e^{-tX_g} B dt = B_T \int_T^{T+1} \chi'(t) e^{-tX_g} B dt + S''_g$$

for some continuous family  $g \mapsto S''_g$  of smoothing operators. Next we use (4.1) with the choice  $N_0 = N + 1$  and  $N_1/8 = N + 1$ . Since by (4.2)

$$m_g(z, \xi) \leq -N - 1 \text{ for all } (z, \xi) \in \text{WF}(B_T),$$

we obtain, using the composition properties in [FRS08, Theorem 8] that  $A_{m_g^{N_0, N_1}} B_T \in \Psi^{-N-1}(SM)$  is uniformly bounded with respect to  $g$  and continuous as a map  $g \in \mathcal{U}' \mapsto A_{m_g^{N_0, N_1}} B_T \in \mathcal{L}(H^{-N}(SM), H^1(SM))$ . In particular

$$\mathcal{U}' \ni g \mapsto A_{m_g^{N_0, N_1}} \int_T^{T+1} \chi'(t) e^{-tX_g} B dt \in \mathcal{L}(H^{-N}(SM), H^1(SM))$$

is continuous. We can then use these facts together with (4.3) and (4.6) to deduce that

$$\mathcal{U}' \ni g \mapsto A_{m_g^{N_0, N_1}} R_g^-(0) (A_{m_g^{N_0, N_1}})^{-1} A_{m_g^{N_0, N_1}} \int_T^{T+1} \chi'(t) e^{-tX_g} B dt \in \mathcal{L}(H^{-N}(SM), L^2(SM))$$

is continuous. Finally, using that  $\text{WF}(B) \cap C_u(g_0) = \emptyset$  and  $-m_g^{N_0, N_1} \leq -N - 1$  outside  $C_u(g_0)$  by (4.2), we have that  $B(A_{m_g^{N_0, N_1}})^{-1} \in \Psi^{-N}(SM)$  uniformly in  $g$  (using again [FRS08, Theorem 8]) and the following map is continuous

$$\mathcal{U}' \ni g \mapsto B(A_{m_g^{N_0, N_1}})^{-1} \in \mathcal{L}(L^2(SM), H^N(SM)).$$

This shows that  $\mathcal{U}' \ni g \mapsto \Omega_3^g \in \mathcal{L}(H^{-N}(SM), H^N(SM))$  is continuous. The terms involving the smoothing remainders  $S_g$  appearing in the difference between  $\Omega_3^g$  and  $\Omega_2^g$  can be dealt using the same argument, and indeed are even simpler to consider. The proof is then complete.  $\square$

The proof of Proposition 4.1 is simply the combination of Lemma 4.5 and Lemma 4.4.  $\square$

As a corollary we prove Theorem 1.3.

**4.4. Proof of Theorem 1.3.** Let  $g_0 \in \mathcal{M}^{\text{An}}$  and assume  $g_0$  has non-positive curvature if  $n \geq 3$ . Using Lemma 2.4 (or its proof in [GL19]), for  $g_1, g_2 \in \mathcal{M}$  close enough to  $g_0$  in  $C^{k,\alpha}$  norm, we can find  $\psi \in \mathcal{D}_0^{k+1,\alpha}$  (with  $k \geq 5$  to be chosen later) such that  $D_{g_1}^*(\psi^*g_2) = 0$ . Moreover  $g'_2 = \psi^*g_2$  satisfies

$$\|g'_2 - g_1\|_{C^{k,\alpha}} \leq C(\|g_1 - g_0\|_{C^{k,\alpha}} + \|g_2 - g_0\|_{C^{k,\alpha}})$$

for some  $C$  depending only on  $g_0$ . We can then rewrite the proof of Theorem 1.2 but by replacing  $g_0$  by  $g_1$ . This gives that for  $g_1, g_2$  close enough to  $g_0$  in  $\mathcal{M}^{5,\alpha}$

$$|\tilde{\Phi}_{g_1}(g_2) - 1| + |F_{g_1}(g_2)| \geq C_n \langle \Pi_2^{g_1}(g_2 - g'_1), (g'_2 - g_1) \rangle - C'_{g_1} \|g_2 - g_1\|_{C^{5,\alpha}}^3$$

where  $C_n$  depends only on  $n = \dim M$  and  $C_{g_1}$  depends on  $\|g_1\|_{C^{5,\alpha}}$ , and  $F_{g_1}(g_2) := \mathbf{P}(-J_{g_1}^u - a_{g_1,g_2} + 1)$  while  $\tilde{\Phi}_{g_1}(g_2) = I_{\mu_{g_1}^L}(g_1, g_2)$ , where  $a_{g_1,g_2}$  is the time reparameterization coefficient in the conjugation between the flows  $\varphi^{g_1}$  and  $\varphi^{g_2}$  and the pressure and the stretch are taken with respect to the flow  $\varphi^{g_1}$ . Combining Proposition 4.1 and Lemma 2.2, we deduce that there is  $C_{g_0}, C'_{g_0} > 0$  depending only on  $g_0$  so that for  $g_1, g_2 \in \mathcal{M}$  in a small enough neighborhood of  $g_0$  in the  $C^\infty$  topology,

$$|\tilde{\Phi}_{g_1}(g_2) - 1| + |F_{g_1}(g_2)| \geq C_{g_0} \|g'_2 - g_1\|_{H^{-\frac{1}{2}}(M)} - C'_{g_0} \|g_2 - g_1\|_{C^{5,\alpha}}^3.$$

This means that there is  $\varepsilon > 0$  depending on  $g_0$  and  $k$  large enough so that for all  $g_1, g_2 \in \mathcal{M}$  smooth satisfying  $\|g_j - g_0\|_{C^{k,\alpha}(M)} \leq \varepsilon$  the estimate above hold. Reasoning like in the proof of Theorem 1.2, we obtain the result.  $\square$

**4.5. Distances from the marked length spectrum.** In this paragraph, we discuss different notions of distances involving the marked length spectrum on the space of isometry classes of negatively-curved metrics.

4.5.1. *Length distance.* We define the following map:

**Definition 4.6.** Let  $k$  be as in Theorem 1.3. We define the marked length distance map  $d_L : \mathcal{M}^{k,\alpha} \times \mathcal{M}^{k,\alpha} \rightarrow \mathbb{R}^+$  by

$$d_L(g_1, g_2) := \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} \right|^{\frac{1}{2}} + \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} \right|^{\frac{1}{2}}.$$

We get as a Corollary of Theorem 1.3:

**Corollary 4.7.** *The map  $d_L$  descends to the set of isometry classes near  $g_0$  and defines a distance in a small  $C^{k,\alpha}$ -neighborhood of the isometry class of  $g_0$ .*

*Proof.* It is clear that  $d_L$  is invariant by action of diffeomorphisms homotopic to Identity since  $L_g = L_{\psi^*g}$  for such diffeomorphisms  $\psi$ . Now let  $g_1, g_2, g_3$  three metrics. We have

$$\begin{aligned} \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} \right|^{\frac{1}{2}} &= \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_3}(c_j)} \frac{L_{g_3}(c_j)}{L_{g_2}(c_j)} \right|^{\frac{1}{2}} \\ &\leq \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_3}(c_j)} \right|^{\frac{1}{2}} + \limsup_{j \rightarrow \infty} \left| \log \frac{L_{g_3}(c_j)}{L_{g_2}(c_j)} \right|^{\frac{1}{2}}. \end{aligned}$$

thus  $d_L$  satisfies the triangular inequality. Finally, By Theorem 1.3, if  $d_L(g_1, g_2) = 0$  with  $g_1, g_2$  in the  $C^{k,\alpha}$  neighborhood  $\mathcal{U}_{g_0}$  of Theorem 1.3, we have  $g_1$  isometric to  $g_2$ , showing that  $d_L$  produces a distance on the quotient of  $\mathcal{U}_{g_0}$  by diffeomorphisms.  $\square$

We also note that Theorem 1.3 states that there is  $C_{g_0} > 0$  such that for each  $g_1, g_2 \in C^{k,\alpha}(M; S^2T^*M)$  close to  $g_0$  there is a diffeomorphism such that

$$d_L(g_1, g_2) \geq C_{g_0} \|\psi^*g_1 - g_2\|_{H^{-1/2}}$$

showing that the pressure norm is controlled by the  $d_L$  distance.

4.5.2. *Thurston distance.* We also introduce the Thurston distance on metrics with topological entropy 1, generalizing the distance introduced by Thurston in [Thu98] for surfaces on Teichmüller space (all hyperbolic metrics on surface have topological entropy equal to 1). We denote by  $\mathcal{E}$  (resp.  $\mathcal{E}^{k,\alpha}$ ) the space of metrics in  $\mathcal{M}$  (resp. in  $\mathcal{M}^{k,\alpha}$ ) with topological entropy  $\mathbf{h}_{\text{top}} = 1$ . With the same arguments than in Lemma 3.1, this is a codimension 1 submanifold of  $\mathcal{M}$  and if  $g_0 \in \mathcal{E}^{k,\alpha}$ , one has:

$$T_{g_0}\mathcal{E}^{k,\alpha} := \left\{ h \in C^{k,\alpha}(M; S^2T^*M) \mid \int_{S_{g_0}M} \pi_2^*h \, d\mu_{g_0}^{\text{BM}} = 0 \right\} \quad (4.7)$$

**Definition 4.8.** We define the Thurston non-symmetric distance map  $d_T : \mathcal{E}^{k,\alpha} \times \mathcal{E}^{k,\alpha} \rightarrow \mathbb{R}^+$  by

$$d_T(g_1, g_2) := \limsup_{j \rightarrow \infty} \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)}.$$

We will prove the

**Proposition 4.9.** *The map  $d_T$  descends to the set of isometry classes of metrics in  $\mathcal{E}^{k,\alpha}$  (for  $k \in \mathbb{N}$  large enough,  $\alpha \in (0, 1)$ ) with topological entropy equal to 1 and defines a non-symmetric distance in a small  $C^{k,\alpha}$ -neighborhood of the diagonal.*

Moreover, this distance is non-symmetric in the pair  $(g_1, g_2)$  which is also the case of the original distance introduced by Thurston [Thu98] but this is just an artificial limitation<sup>5</sup>: “It would be easy to replace  $L$ <sup>6</sup> by its symmetrization  $1/2(L(g, h) + L(h, g))$ , but it seems

<sup>5</sup>Thurston, [Thu98].

<sup>6</sup>In the notations of Thurston,  $L(g, h) = \limsup_{j \rightarrow \infty} \log \frac{L_g(c_j)}{L_h(c_j)}$ .

that, because of its direct geometric interpretations,  $L$  is more useful just as it is.” In order to justify that this is a distance, we start with the

**Lemma 4.10.** *Let  $g_1, g_2 \in \mathcal{M}$ . Then:*

$$\limsup_{j \rightarrow \infty} \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} = \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} I_m(g_1, g_2)$$

Here  $m$  is seen as an invariant ergodic measure for the flow  $\varphi_t^{g_1}$  living on  $S_{g_1}M$ . However, writing  $M = \Gamma \backslash \widetilde{M}$  with  $\Gamma \simeq \pi_1(M, x_0)$  for  $x_0 \in M$ , it can also be identified with a geodesic current on  $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta$ , that is a  $\Gamma$ -invariant Borel measure, also invariant by the flip  $(\xi, \eta) \mapsto (\eta, \xi)$  on  $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta$ . This point of view has the advantage of being independent of  $g_1$  (see [ST18]).

*Proof.* First of all, we claim that

$$\sup_{m \in \mathfrak{M}_{\text{inv,erg}}} I_m(g_1, g_2) = \sup_{m \in \mathfrak{M}_{\text{inv}}} I_m(g_1, g_2).$$

Of course, it is clear that  $\sup_{m \in \mathfrak{M}_{\text{inv,erg}}} I_m(g_1, g_2) \leq \sup_{m \in \mathfrak{M}_{\text{inv}}} I_m(g_1, g_2)$  and thus we are left to prove the reverse inequality. By compactness, we can consider a measure  $m_0 \in \mathfrak{M}_{\text{inv}}$  realizing  $\sup_{m \in \mathfrak{M}_{\text{inv}}} I_m(g_1, g_2)$ . By Choquet representation Theorem (see [Wal82, pp. 153]), there exists a (unique) probability measure  $\tau$  on  $\mathfrak{M}_{\text{inv,erg}}$  such that  $m_0$  admits the ergodic decomposition  $m_0 = \int_{\mathfrak{M}_{\text{inv,erg}}} m \, d\tau(m)$ . Thus:

$$\begin{aligned} I_{m_0}(g_1, g_2) &= \int_{S_{g_1}M} a_{g_1, g_2} \, dm_0 \\ &= \int_{\mathfrak{M}_{\text{inv,erg}}} \int_{S_{g_1}M} a_{g_1, g_2} \, dm \, d\tau(m) \\ &\leq \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} \int_{S_{g_1}M} a_{g_1, g_2} \, dm \int_{\mathfrak{M}_{\text{inv,erg}}} d\tau(m) = \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} I_m(g_1, g_2), \end{aligned}$$

which eventually proves the claim.

Let  $(c_j)_{j \in \mathbb{N}}$  be a subsequence such that  $\lim_{j \rightarrow +\infty} L_{g_2}(c_j)/L_{g_1}(c_j)$  realizes the lim sup. Then, by compactness, we can extract a subsequence such that  $\delta_{g_1}(c_j) \rightarrow m \in \mathfrak{M}_{\text{inv}}$ . Thus:

$$L_{g_2}(c_j)/L_{g_1}(c_j) = \langle \delta_{g_1}(c_j), a_{g_1, g_2} \rangle \rightarrow_{j \rightarrow +\infty} \langle m, a_{g_1, g_2} \rangle = I_m(g_1, g_2),$$

which proves, using our preliminary remark, that

$$\limsup_{j \rightarrow +\infty} L_{g_2}(c_j)/L_{g_1}(c_j) \leq \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} I_m(g_1, g_2).$$

To prove the reverse inequality, we consider a measure  $m_0 \in \mathfrak{M}_{\text{inv,erg}}$  such that  $I_{m_0}(g_1, g_2) = \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} I_m(g_1, g_2)$  (which is always possible by compactness). Since  $m_0$  is invariant and

ergodic, there exists a sequence of free homotopy classes  $(c_j)_{j \in \mathbb{N}}$  such that  $\delta_{g_1}(c_j) \rightarrow m_0$ . Then, like previously, one has

$$I_{m_0}(g_1, g_2) = \lim_{j \rightarrow +\infty} L_{g_2}(c_j)/L_{g_1}(c_j) \leq \limsup_{j \rightarrow +\infty} L_{g_2}(c_j)/L_{g_1}(c_j),$$

which provides the reverse inequality.  $\square$

We can now prove Proposition 4.9.

*Proof of Proposition 4.9.* By (2.15), for  $g_1, g_2 \in \mathcal{E}^{k, \alpha}$ , we have that  $I_{\mu_{g_1}^{\text{BM}}}(g_1, g_2) \geq 1$  and thus by Lemma 4.10, we obtain that  $d_T(g_1, g_2) \geq 0$  (note that  $g_1$  and  $g_2$  do not need to be close for this property to hold). Moreover, triangular inequality is immediate for this distance. Eventually, if  $d_T(g_1, g_2) = 0$ , then  $0 \leq \log I_{\mu_{g_1}^{\text{BM}}}(g_1, g_2) \leq d_T(g_1, g_2) = 0$ , that is  $I_{\mu_{g_1}^{\text{BM}}}(g_1, g_2) = 1$  and by Theorem 2.9, it implies that  $g_1$  is isometric to  $g_2$  if  $g_2$  is close enough to  $g_1$  in the  $C^{k, \alpha}$ -topology (note that this neighborhood depends on  $g_1$ ).  $\square$

We now investigate with more details the structure of the distance  $d_T$ . A consequence of Lemma 4.10 is the following expression of the Thurston Finsler norm:

**Lemma 4.11.** *Let  $g_0 \in \mathcal{E}^{k, \alpha}$  and  $(g_t)_{t \in [0, \varepsilon]}$  be a smooth family of metrics and let  $f := \partial_t g_t|_{t=0}$ . Then:*

$$\|f\|_T := \left. \frac{d}{dt} d_T(g_0, g_t) \right|_{t=0} = \frac{1}{2} \sup_{m \in \mathfrak{M}_{\text{inv, erg}}} \int_{S_{g_0} M} \pi_2^* f \, dm \quad (4.8)$$

The norm  $\|\cdot\|_T$  is a Finsler norm on  $T_{g_0} \mathcal{E}^{k, \alpha} \cap \ker D_{g_0}^*$

*Proof.* We introduce  $u(t) := e^{d_T(g_0, g_t)}$  and write  $a_t := a_{g_0, g_t}$  for the time reparametrization (as in (2.12)). Then:

$$\begin{aligned} u'(0) &= \left. \frac{d}{dt} d_T(g_0, g_t) \right|_{t=0} = \lim_{t \rightarrow 0} \sup_{m \in \mathfrak{M}_{\text{inv, erg}}} \int_{S_{g_0} M} \frac{a_t - 1}{t} \, dm \\ &= \sup_{m \in \mathfrak{M}_{\text{inv, erg}}} \int_{S_{g_0} M} \dot{a}_0 \, dm = \frac{1}{2} \sup_{m \in \mathfrak{M}_{\text{inv, erg}}} \int_{S_{g_0} M} \pi_2^* f \, dm, \end{aligned}$$

since  $\dot{a}_0 = \partial_t a_t|_{t=0}$  and  $\pi_2^* f$  are cohomologous by Lemma 3.2.

We now prove that this is a Finsler norm in a neighborhood of the diagonal. We fix  $g_0 \in \mathcal{E}^{k, \alpha}$ . By Lemma 2.4, isometry classes near  $g_0$  can be represented by solenoidal tensors, namely there exists a  $C^{k, \alpha}$ -neighborhood  $\mathcal{U}$  of  $g_0$  such that for any  $g \in \mathcal{U}$ , there exists a (unique)  $\psi \in \mathcal{D}_0^{k+1, \alpha}$  such that  $D_{g_0}^* \psi^* g = 0$ . Moreover, if  $g \in \mathcal{E}^{k, \alpha}$ , then  $\psi^* g \in \mathcal{E}^{k, \alpha}$ . As a consequence, using (4.7), the statement now boils down to proving that (4.8) is a norm for solenoidal tensors  $f \in C^{k, \alpha}(M; S^2 T^* M)$  such that  $\int_{S_{g_0} M} \pi_2^* f \, d\mu_{g_0}^{\text{BM}} = 0$ . Since triangular

inequality,  $\mathbb{R}_+$ -scaling and non-negativity are immediate, we simply need to show that  $\|f\|_T = 0$  implies  $f = 0$ . Now, for such a tensor  $f$ , we have

$$\begin{aligned} \mathbf{P}(\pi_2^* f) &= \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} \mathbf{h}_m(\varphi_1^{g_0}) + \int_{S_{g_0} M} \pi_2^* f dm \\ &\leq \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} \mathbf{h}_m(\varphi_1^{g_0}) + \sup_{m \in \mathfrak{M}_{\text{inv,erg}}} \int_{S_{g_0} M} \pi_2^* f dm = \underbrace{\mathbf{h}_{\text{top}}(\varphi_1^{g_0})}_{=1} + 0 \end{aligned}$$

and this supremum is achieved for  $m = \mu_{g_0}^{\text{BM}}$  and  $\mathbf{P}(\pi_2^* f) = 1$ . As a consequence, the equilibrium state associated to the potential  $\pi_2^* f$  is the Bowen-Margulis measure  $\mu_{g_0}^{\text{BM}}$  (the equilibrium state associated to the potential 0) and thus  $\pi_2^* f$  is cohomologous to a constant  $c \in \mathbb{R}$  (see [HF, Theorem 9.3.16]) which has to be  $c = 0$  since the average of  $\pi_2^* f$  with respect to Bowen-Margulis is equal to 0, that is there exists a Hölder-continuous function  $u$  such that  $\pi_2^* f = Xu$ . Since  $f \in \ker D_{g_0}^*$ , the s-injectivity of the X-ray transform  $I_2^{g_0}$  implies that  $f \equiv 0$ .  $\square$

The asymmetric Finsler norm  $\|\cdot\|_T$  induces a distance  $d_F$  between isometry classes namely

$$d_F(g_1, g_2) = \inf_{\gamma: [0,1] \rightarrow \mathcal{E}, \gamma(0)=g_1, \gamma(1)=g_2} \int_0^1 \|\dot{\gamma}(t)\|_T dt$$

It is easy to prove that  $d_T(g_1, g_2) \leq d_F(g_1, g_2)$ . Indeed, consider a  $C^1$ -path  $\gamma : [0, 1] \rightarrow \mathcal{E}$  such that  $\gamma(0) = g_1, \gamma(1) = g_2$ . Then, considering  $N \in \mathbb{N}, t_i := i/N$ , we have by triangular inequality

$$\begin{aligned} d_T(g_1, g_2) &\leq \sum_{i=0}^{N-1} d_T(\gamma(t_i), \gamma(t_{i+1})) = \sum_{i=0}^{N-1} \|\dot{\gamma}(t_i)\|_T (t_{i+1} - t_i) + \mathcal{O}(|t_{i+1} - t_i|^2) \\ &\rightarrow_{N \rightarrow +\infty} \int_0^1 \|\dot{\gamma}(t)\|_T dt, \end{aligned}$$

which proves the claim. In [Thu98], Thurston proves that, in restriction to Teichmüller space, the asymmetric Finsler norm induces the distance  $d_T$ , that is  $d_T = d_F$ . We make the following conjecture, which would imply the marked length spectrum rigidity:

**Conjecture 4.12.** *The distances  $d_T$  coincide with  $d_F$  for isometry classes of negatively curved metrics with topological entropy equal to 1.*

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