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Long-distance asymptotic behaviour of multi-point correlation functions in massless quantum models

N. Kitanine¹, K. K. Kozłowski², J. M. Maillet³, V. Terras⁴

Abstract

We provide a microscopic model setting that allows us to readily access to the large-distance asymptotic behaviour of multi-point correlation functions in massless, one-dimensional, quantum models. The method of analysis we propose is based on the form factor expansion of the correlation functions and does not build on any field theory reasonings. It constitutes an extension of the restricted sum techniques leading to the large-distance asymptotic behaviour of two-point correlation functions obtained previously.

1 Introduction

Form factors expansions constitute natural means for studying correlation functions in the case of massive integrable quantum field theories in the infinite volume [16, 29]. Indeed, the presence of a gap in the spectrum leads both to regularity and well-ordering of the form factor series as an effective asymptotic series in the large-distance regime. However, these two crucial properties are no-longer satisfied for massless models; their lack constituted an obstruction to an efficient formulation or even handling of form factor series in the infinite volume limit of a model in the massless regime. The main problem in the case of massless models is that the form factors of local operators vanish as a non-integer power-law in the model's volume L . This originates from the operator's non-integer conformal dimension. When computing a form factor expansion, this non-integer power-law in L vanishing of individual form factors should be compensated by certain multi-dimensional sums (which contain a number of summands diverging in L) over appropriate sub-classes of low-lying excited states, hence making this setting rather intricate to tackle. Recently, however, the authors, in collaboration with N. Slavnov, have set forth a method of summing up, in the large-distance regime and for large but finite volume, the form factor expansion of two-point correlation functions in massless quantum integrable models [20]. The matter is that, as we have observed, in the large-distance regime, the sums over form factors associated to scanning the whole spectrum of

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the theory localise to so-called restricted sums. The latter correspond, in physical terms, to summing up over all low-lying energy (of the order $1/L$, with L being the volume assumed to be large but finite) excitations on the Fermi surface associated, in the thermodynamic limit, to different Umklapp excitation momenta. Although the number of terms in these multiple restricted sums tends to infinity with the volume, it has been shown in [20] that they can be evaluated exactly using purely combinatorial identities. Certain instances of these identities already appeared in the context of harmonic analysis on the infinite dimensional symmetric group [17] (see [26] for a first practical application to z -measures on Young diagrams).

In this paper we continue developing the technique of summation, in the large-distance regime, of form factor expansions, now for general n -point correlation functions. More precisely, we start from a model in *large but finite* volume L and make some technical assumptions on its spectrum and form factors. These properties are verified for quantum integrable models but we do trust that they should also hold for a much wider class of models. Building on these assumptions we develop summation techniques for form factor expansions that allow us to access to the large-distance regime of general multi-point correlation functions. The corresponding restricted sums generalize those obtained for the two-point case [20]. In fact, in the case of general n -point correlation functions (with $n \geq 3$) the corresponding sums can be interpreted as multiple restricted sums of the previous type (obtained for two-point functions) however highly coupled between themselves hence rendering the needed summation identity quite non-intuitive. The derivation of this identity for the restricted sums corresponding to general n -point correlation functions follows from the two possible representations for the large-size asymptotic behaviour of a Toeplitz determinant generated by Fisher–Hartwig symbols: the one corresponding to the Fisher–Hartwig asymptotic behaviour formula [8] and the one issuing from a form factor like expansion of this same Toeplitz determinant.

The large-distance asymptotic summation of form factors method we develop does reproduce, within a microscopic approach to the model, the conformal field theory/Luttinger-Liquid predictions for the asymptotics of correlation functions in massless models. As an example, the large-distance behaviour of four point functions in the XXZ spin-1/2 chain is obtained as

$$\langle \sigma_{m_1}^x \cdot \sigma_{m_2}^x \cdot \sigma_{m_3}^x \cdot \sigma_{m_4}^x \rangle \simeq 2|\mathcal{F}^+|^4 \left\{ \left| \frac{(m_2 - m_1) \cdot (m_4 - m_3)}{(m_3 - m_1) \cdot (m_4 - m_1) \cdot (m_3 - m_2) \cdot (m_4 - m_2)} \right|^\theta + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} + \dots,$$

matching the predictions given in [10, 25]. Above, θ is an explicit exponent that is expressed in terms of quantities parametrizing the low-lying excitations of the XXZ spin-1/2 chain and the dots refer to higher order (sub-leading) terms. The amplitude \mathcal{F}^+ corresponds to a properly normalized in the length L of the chain form factor of the σ^+ operator taken between the model's overall ground state $|\Psi_g\rangle$ and the local ground state in the sector with one spin being flipped down $|\Psi_g^{(-1)}\rangle$,

$$\mathcal{F}^+ = \lim_{L \rightarrow +\infty} \left\{ \left(\frac{L}{2\pi} \right)^\theta \langle \Psi_g | \sigma^+ | \Psi_g^{(-1)} \rangle \right\}.$$

The critical exponent θ was already present in the field theory predictions [10, 25]. We do stress however that our method goes beyond in that it gives a direct access and interpretation of the amplitudes arising in front of the algebraically decaying terms (along with the numerical pre-factor of 2). It allows one readily to extract the large-distance asymptotic behaviour of any multipoint correlation function. Moreover, our method is solely based on a genuine microscopic formulation of the model and at no stage of our analysis we do invoke a conjectural correspondence of the discrete model with some continuous field theory. Furthermore, with minor modifications, it should open a possibility to treat the case of time dependent correlation functions as it was done for two-point functions in [22].

We would like to stress that the general dependence in the distance we obtain is similar to the one for correlation functions of exponents of free fields, cf [6, 7]. In fact, such a form already appeared in the late 60's paper of

Kadanoff [15]. Recently, the same structure of the large-distance asymptotic behaviour has been proven, for the Ising correlation functions at the critical point, by Palmer [27] essentially based on the Sato, Miwa, Jimbo formalism [12, 13, 14]. It is in fact in these pioneering works [12, 13, 14] that the first rigorous approach to characterizing and computing multi-point correlation functions in free fermion model were given. The analysis developed there allowed these authors to provide, in particular, a characterization of multi-point correlation functions in a massless model for the case of the one-dimensional impenetrable Bose gas [11] and to extract explicitly the large-distance asymptotic behaviour of the so-called one-particle reduced density matrix in that model. Multi-point correlation functions in free fermion models have been also investigated through Pfaffian or integrable integral operators methods. In [1], Abraham, Barouch and McCoy derived a Pfaffian representation for four-point functions that allowed them to access to the time-dependent spin-spin correlation functions. In [2, 3] Bariev derived the large-distance asymptotic behaviour of three and then multi-point correlation functions in the off-critical Ising model. In [28] Slavnov obtained a system of partial differential equation characterizing the n -point off-diagonal correlation functions in the one-dimensional impenetrable Bose gas.

The paper is organized as follows. In Section 2 we provide a definition of the microscopic model and of all the quantities of interest to the problem. These definitions provide the general setting that allows for an application of our form factor summation method. It surely does hold for quantum integrable models such as the XXZ spin-1/2 chain or the quantum non-linear Schrödinger model, but should also be valid for some non-integrable models as well. Then, in Section 3, we outline the main steps of our method and describe the overall features behind its philosophy. In particular we present the restricted sums of interest which are at the root of our approach to the large-distance asymptotic behaviour of the multi-point correlation function. Finally, we apply our general result to the case of certain integrable models in Section 4. Technical aspects related to the derivation of the multidimensional restricted sums relevant for the present analysis are presented in Appendix A.

2 Overall setting for the analysis of multi-point correlation functions

This paper aims at introducing a method of analysis of the large-distance asymptotic behaviour of r -point ground state correlation functions of the type

$$C(\mathbf{x}_r; \mathbf{o}_r) = \langle \Psi_g | O_1(x_1) \dots O_r(x_r) | \Psi_g \rangle, \quad (2.1)$$

where we agree to represent r -dimensional vectors as

$$\mathbf{x}_r = (x_1, \dots, x_r) \quad \text{and} \quad \mathbf{o}_r = (o_1, \dots, o_r). \quad (2.2)$$

On the level of (2.1), the correlation function $C(\mathbf{x}_r; \mathbf{o}_r)$ is defined for a model in finite volume L . In this formula, $|\Psi_g\rangle$ represents the ground state of the model in finite volume whereas $O_a(x)$ are some elementary (in the sense defined below) local operators associated with the model and located at position x . Depending on the nature of the model (continuous or discrete), the position variables can be continuous or discrete. Throughout the paper, we shall build on the assumption that the states of the model are constructed out of a fixed number of pseudo-particles, this number possibly changing from one state to another. Within such a setting, we assume that one can associate a fixed integer o_a to each operator $O_a(x)$. These integers refer to the jump in the pseudo-particle number of the states connected by the operator. More precisely, we assume that the operator $O_a(x)$ is an elementary local operator that connects only states having $N + o_a$ and N pseudo-particles, N being arbitrary but o_a fixed. In other words, the only non-zero form factors of the operator $O_a(x)$ are given by the matrix elements

$$\langle \Phi_N | O_a(x) | \Psi_{N+o_a} \rangle \quad (2.3)$$

in which $|\Psi_{N+o_a}\rangle$ is some $N+o_a$ pseudo-particle eigenstate of the model's Hamiltonian whereas $|\Phi_N\rangle$ corresponds to an eigenstate built out of N pseudo-particles. Any local operator can be decomposed in terms of a finite number of such elementary operators $O_a(x)$. Finally, we focus solely on the case of translation invariant models (periodic boundary conditions) meaning that one can explicitly factor out the x -dependence out of (2.3) in terms of the relative momentum of the states $|\Psi_{N+o_a}\rangle$ and $|\Phi_N\rangle$.

Our aim is to extract the large-distance asymptotic behaviour of the correlation function, in the infinite volume limit, $L \rightarrow +\infty$, by using its form factor expansion obtained by inserting the closure relation in between the operators $O_a(x)$. The matter is that the latter can be re-summed in the appropriate regimes of interest. In fact, we shall focus on the following regime of parameters

$$1 \ll |x_l - x_k| \cdot p_F \quad \text{where} \quad l, k = 1, \dots, r-1, \quad l \neq k, \quad L \gg x_k \quad k = 1, \dots, r, \quad (2.4)$$

and p_F denotes the Fermi momentum associated with the model.

In the following, we describe a general framework such that, if the spectrum, eigenstates and form factors of a model fit within its grasp, then the present analysis holds. As we shall argue, this setting is verified for a number of quantum integrable models such as the non-linear Schrödinger model or the XXZ spin-1/2 chain in their massless phase. However, we are deeply convinced that the setting is, in fact, much more general and, in particular, encompasses also numerous instances of non-exactly solvable models.

2.1 The spectrum

We assume that, in the large- L limit, the spectrum of the model is of particle-hole type ¹ and that the ground state $|\Psi_g\rangle$ is constructed within an N pseudo-particle sector. We shall focus on states located in nearby sectors containing N_s pseudo-particles,

$$N_s = N + \bar{o}_s, \quad \bar{o}_s = \sum_{a=1}^s o_a. \quad (2.5)$$

Within our setting, the states of the system in an N_s pseudo-particle sector are labelled by two sets of n integers associated to so-called particles and holes excitations, $n = 0, 1, \dots, N_s$:

$$p_1^{(s)} < \dots < p_n^{(s)} \quad \text{and} \quad h_1^{(s)} < \dots < h_n^{(s)} \quad \text{with} \quad \begin{cases} p_a^{(s)} \in \llbracket -M_L^{(1)}; M_L^{(2)} \rrbracket \setminus \llbracket 1; N_s \rrbracket, \\ h_a^{(s)} \in \llbracket 1; N_s \rrbracket. \end{cases} \quad (2.6)$$

The values taken by the integers $M_L^{(a)}$, $a = 1, 2$, strongly depend on the model. Typically, for models having no upper bound on their energy, one has $M_L^{(a)} = +\infty$, while for model having an upper bound, $M_L^{(1)}, M_L^{(2)}$ are both finite but such that $M_L^{(a)} - N$, $a = 1, 2$, both go to $+\infty$ sufficiently fast with L . Also note that throughout this paper we shall adopt, for definiteness, the convention that the integers arising in the parametrization of a particle-hole excited state in the N_s pseudo-particle sector always bear a superscript (s) .

A given choice of integers

$$\mathcal{I}_n^{(s)} = \left\{ \{p_a^{(s)}\}_1^n ; \quad \{h_a^{(s)}\}_1^n \right\} \quad (2.7)$$

¹Such an assumptions is not very restrictive as there are indications that the bound state part of the spectrum does not contribute to the algebraic part of the large-distance asymptotic behaviour of correlation functions. For instance, this is supported by exact calculations for the large-distance asymptotic behaviour of two-point functions in the XXZ spin-1/2 chain [18]. This is however most probably not the case for large-distance and large-time asymptotic behaviour.

for an excited state in the N_s pseudo-particle sector gives rise to a set of rapidities $\{\widehat{\mu}_{p_a}^{(s)}\}_1^n$ for the particle and $\{\widehat{\mu}_{h_a}^{(s)}\}_1^n$ for the holes associated with this state. The former and latter are computed from the integers $\mathcal{I}_n^{(s)}$ by the use of a so-called counting function $\widehat{\xi}_{\mathcal{I}_n^{(s)}}$ as the unique solutions to

$$\widehat{\xi}_{\mathcal{I}_n^{(s)}}(\widehat{\mu}_{p_a}^{(s)}) = \frac{p_a^{(s)}}{L} \quad \text{and} \quad \widehat{\xi}_{\mathcal{I}_n^{(s)}}(\widehat{\mu}_{h_a}^{(s)}) = \frac{h_a^{(s)}}{L}. \quad (2.8)$$

We do stress that the counting function depends on the choice of the excited state, *ie* on the choice of integers $\mathcal{I}_n^{(s)}$. Hence, (2.8) is, in fact, a quite complicated set of equations. We assume that all counting functions admit the large volume L asymptotic expansions

$$\widehat{\xi}_{\mathcal{I}_n^{(s)}}(\omega) = \xi(\omega) + \frac{1}{L}\xi_{-1}(\omega) - \frac{1}{L}F_{\mathcal{R}_n^{(s)}}(\omega) + \mathcal{O}\left(\frac{1}{L^2}\right). \quad (2.9)$$

The above asymptotic expansion involves three functions ξ , ξ_{-1} and $F_{\mathcal{R}_n^{(s)}}$.

- ξ is the asymptotic counting function. It is the same for *all* excited states and defines a set of rapidities $\{\mu_a\}_{a \in \mathbb{Z}}$ as the unique solutions to the equation

$$\xi(\mu_a) = \frac{a}{L} \quad \text{so that} \quad \widehat{\mu}_{p_a}^{(s)} \simeq \mu_{p_a}^{(s)} \quad \text{and} \quad \widehat{\mu}_{h_a}^{(s)} \simeq \mu_{h_a}^{(s)}, \quad (2.10)$$

at leading order in L .

- $F_{\mathcal{R}_n^{(s)}}$ is called the shift function (of the given excited state in respect to the model's ground state). It is a function of the macroscopic rapidities

$$\mathcal{R}_n^{(s)} = \left\{ \{\mu_{p_a}^{(s)}\}_1^n ; \{\mu_{h_a}^{(s)}\}_1^n \right\} \quad (2.11)$$

and also depends on the deviation $N_s - N = \overline{\theta}_s$ of the number N_s of pseudo-particles in the excited state in respect to N , the one for the ground state.

- ξ_{-1} represents the $1/L$ corrections to the ground state's counting functions. It is this quantity that drives the non-trivial part of the first sub-leading corrections to the ground state's energy. It appears for normalisation purposes so that the shift function for the ground state vanishes, *ie* $F_{\mathcal{R}_0^{(0)}} = 0$.

In the large- L limit and within such a setting, the rapidities for the ground state form a dense distribution on $[-q; q]$ – the so-called Fermi zone of the model – with density ξ' . The endpoints $\pm q$ are called the Fermi boundaries. Further, the observables of the model such as the relative momentum and energy are parametrized by the particle-hole rapidities

$$\Delta \mathcal{E}(\mathcal{I}_n^{(s)}) \equiv \mathcal{E}(\mathcal{I}_n^{(s)}) - \mathcal{E}(\mathcal{I}_0^{(0)}) = \sum_{a=1}^n \left(\varepsilon(\mu_{p_a}^{(s)}) - \varepsilon(\mu_{h_a}^{(s)}) \right) + \mathcal{O}\left(\frac{1}{L}\right), \quad (2.12)$$

$$\Delta \mathcal{P}(\mathcal{I}_n^{(s)}) \equiv \mathcal{P}(\mathcal{I}_n^{(s)}) - \mathcal{P}(\mathcal{I}_0^{(0)}) = \sum_{a=1}^n \left(p(\mu_{p_a}^{(s)}) - p(\mu_{h_a}^{(s)}) \right) + \mathcal{O}\left(\frac{1}{L}\right). \quad (2.13)$$

Above, $\mathcal{I}_0^{(0)} = \{\emptyset; \emptyset\}$ refers to the set of integers which parametrizes the ground state of the model and $\mathcal{P}(\mathcal{I}_n^{(s)})$ and $\mathcal{E}(\mathcal{I}_n^{(s)})$ are respectively the momentum and energy of the state parametrized by the set of integers $\mathcal{I}_n^{(s)}$. Furthermore, p is the so-called dressed momentum and similarly ε is the dressed energy. Typically, for quantum

integrable models, these functions are given as solutions to linear integral equations [9]. Their construction, for more complex, non-integrable models, is *a priori* much more complicated but carry direct physical meaning. Note that the particle-hole interpretation of the spectrum is manifest at the level of (2.12)-(2.13): the excitations consist in adding "particles" with rapidities $\mu_{p_1^{(s)}}, \dots, \mu_{p_n^{(s)}}$ and associated energies $\varepsilon(\mu_{p_1^{(s)}}), \dots, \varepsilon(\mu_{p_n^{(s)}})$ along with creating "holes" in the Fermi zone, with rapidities $\mu_{h_1^{(s)}}, \dots, \mu_{h_n^{(s)}}$ and associated energies $-\varepsilon(\mu_{h_1^{(s)}}), \dots, -\varepsilon(\mu_{h_n^{(s)}})$. The superscript (s) refers to the fact that the excitation labelled by $\mathcal{I}_n^{(s)}$ takes place above the lowest lying energy level in the $N_s = N + \overline{\mathcal{O}}_s$ quasi-particle sector. We henceforth shall assume that the only roots on \mathbb{R} of the dressed energy ε are at $\pm q$ and that the dressed momentum p is a strictly monotonous function.

There is however a huge degeneracy in what concerns the $O(1)$ part of the excitation energy $\Delta\mathcal{E}(\mathcal{I}_n^{(s)})$: numerous choices of the set $\mathcal{I}_n^{(s)}$ will lead to the *same* $O(1)$ part of $\Delta\mathcal{E}(\mathcal{I}_n^{(s)})$. For instance, it follows from (2.9)-(2.10) that, given any integer ℓ ,

$$\mu_k - \mu_{k+\ell} = O\left(\frac{\ell}{L}\right), \quad \text{so that} \quad \mathcal{I}_n^{(s)} = \{p_a\}_1^n; \{h_a\}_1^n \quad \text{and} \quad \widetilde{\mathcal{I}}_n^{(s)} = \{p_a + k_a\}_1^n; \{h_a + t_a\}_1^n \quad (2.14)$$

where k_a, t_a are some L -independent integers, will lead to the same value for the $O(1)$ part of $\Delta\mathcal{E}$, *ie*

$$\Delta\mathcal{E}(\mathcal{I}_n^{(s)}) - \Delta\mathcal{E}(\widetilde{\mathcal{I}}_n^{(s)}) = O\left(\frac{1}{L}\right). \quad (2.15)$$

2.2 The form factors

In the analysis of the multipoint correlation function $C(\mathbf{x}_r; \mathbf{o}_r)$ (2.1) through its form factor expansion, one needs to access to the form factors of the local operators \mathcal{O}_s , $s = 1, \dots, r$, between two excited states of the model. According to our assumptions on the structure of the model's spectrum, these will be labelled by two sets of multi-indices $\mathcal{I}_m^{(s-1)}$ and $\mathcal{I}_n^{(s)}$ corresponding to the outgoing (bra) and ingoing (ket) states. In fact, since we imposed periodic boundary conditions on the model and required translation invariance, the form factors will satisfy

$$\langle \Psi(\mathcal{I}_m^{(s-1)}) | \mathcal{O}_s(x) | \Psi(\mathcal{I}_n^{(s)}) \rangle = e^{ix(\Delta\mathcal{P})_{s-1}^s} \langle \Psi(\mathcal{I}_m^{(s-1)}) | \mathcal{O}_s(0) | \Psi(\mathcal{I}_n^{(s)}) \rangle, \quad (\Delta\mathcal{P})_{s-1}^s = \mathcal{P}(\mathcal{I}_m^{(s-1)}) - \mathcal{P}(\mathcal{I}_n^{(s)}). \quad (2.16)$$

Scalar observables introduced in the last subsection were parametrized, in the large- L limit, solely by the macroscopic set of rapidities $\mathcal{R}_n^{(s)}$ subordinate to the set of multi-indices $\mathcal{I}_n^{(s)}$. This is no longer the case for form factors as we demonstrated in our previous work [21]. Within our setting, the latter are parametrized, in the large- L limit, *not only* by the sets of macroscopic rapidities $\mathcal{R}_n^{(s)}, \mathcal{R}_m^{(s-1)}$, but also by the sets of *discrete* integers $\mathcal{I}_n^{(s)}, \mathcal{I}_m^{(s-1)}$. Namely, for properly normalized states $|\Psi(\mathcal{I}_n^{(s)})\rangle$ and their duals $\langle \Psi(\mathcal{I}_m^{(s-1)})|$, the form factors take the form

$$\begin{aligned} \mathcal{F}_{\mathcal{O}_s}(\mathcal{I}_m^{(s-1)} | \mathcal{I}_n^{(s)}) &= \langle \Psi(\mathcal{I}_m^{(s-1)}) | \mathcal{O}_s(0) | \Psi(\mathcal{I}_n^{(s)}) \rangle \\ &= \mathcal{S}^{(\mathcal{O}_s)}(\mathcal{R}_m^{(s-1)}; \mathcal{R}_n^{(s)}) \cdot \mathcal{D}^{(s)}(\mathcal{R}_m^{(s-1)}; \mathcal{R}_n^{(s)} | \mathcal{I}_m^{(s-1)}; \mathcal{I}_n^{(s)}) \cdot \left(1 + O\left(\frac{\ln L}{L}\right)\right). \end{aligned} \quad (2.17)$$

In this decomposition

- $\mathcal{S}^{(\mathcal{O}_s)}$ is called the smooth part since it only depends on the macroscopic rapidities $\mathcal{R}_n^{(s)}, \mathcal{R}_m^{(s-1)}$, and this in a smooth manner. As a consequence, a small (of the order $O(1)$) change in the value of the integers parametrizing the state, say $p_a \mapsto p_a + \kappa$, will not change the value of the smooth part (up to $1/L$ corrections). Furthermore, being a set function, it is invariant under permutation of the particle or hole rapidities.
- The smooth part enjoys hypergeometric-like reduction properties. Namely, if within a given set of macroscopic rapidities a particle's rapidity coincides with a hole rapidity, then this dependence simply disappears.

More precisely

$$\mathcal{S}^{(O_s)}(\mathcal{R}_m^{(s-1)}; \mathcal{R}_n^{(s)})_{\mu_{p_k^{(s)}} = \mu_{h_\ell^{(s)}}} = \mathcal{S}^{(O_s)}(\mathcal{R}_m^{(s-1)}; \widehat{\mathcal{R}}_n^{(s)}) \quad \text{with} \quad \widehat{\mathcal{R}}_n^{(s)} = \{\mu_{p_a^{(s)}}\}_{1;\neq k}^n; \{\mu_{h_a^{(s)}}\}_{1;\neq \ell}^n\}.$$

The same type of reduction holds as well for particle-hole rapidities associated with the excited states in the N_{s-1} quasi-particle sector.

- $\mathcal{D}^{(s)}$ is called the discrete part since it depends not only on the macroscopic rapidities $\mathcal{R}_n^{(s)}, \mathcal{R}_m^{(s-1)}$ but also *explicitly* (ie not through the parametrization by the macroscopic rapidities) on the sets of integers labelling the excited states $\mathcal{I}_n^{(s)}, \mathcal{I}_m^{(s-1)}$. The main effect of such a dependence is that a small change in the value of the integers parametrizing the state does imply a significant change (of the order of $O(1)$) in the value of $\mathcal{D}^{(s)}$. The discrete part thus keeps track of the microscopic details of the different excited states.

The smooth part represents, in fact, a non-universal part of the model's form factors. Its explicit expressions not only depends on the operator O_s but also varies strongly from one model to another, see [21, 24] for examples issuing from quantum integrable models. However, the part $\mathcal{D}^{(s)}$ is entirely universal within the present setting of the description of the model's spectrum. It solely depends on the values of the pseudo-particle sectors that the operator connects, *viz.* the integer $o_s = N_s - N_{s-1}$. Its general explicit expression plays no role in our analysis, in the sense that we shall only need the expression for specific excited states, namely the one belonging to the so-called ℓ_s -critical classes that we define below.

We refer the interested reader to [21, 20] for a thorougher discussion relative to the origin of the discrete $\mathcal{D}^{(s)}$ and smooth $\mathcal{S}^{(O_s)}$ parts in the framework of Bethe Ansatz solvable models.

2.3 The critical ℓ_s class

As we shall argue in the following, only a very specific class of excited states will play an effective role in our analysis – the so-called critical states. These excited states are characterized by the fact that, in the $L \rightarrow +\infty$ limit, *all* macroscopic rapidities describing the particle and hole excitations "collapse" on the model's Fermi boundary:

$$\mu_{p_a^{(s)}} \simeq \pm q \quad \text{and} \quad \mu_{h_a^{(s)}} \simeq \pm q. \quad (2.18)$$

There, the \pm sign depends on whether the particle or hole's rapidity collapses on the right or left Fermi boundary.

A set of integers $\mathcal{I}_n^{(s)}$ is said to parametrize a critical excited state if the associated particle-hole integers $\{p_a^{(s)}\}_1^n$ and $\{h_a^{(s)}\}_1^n$ can be represented as

$$\{p_a^{(s)}\}_1^n = \{N_s + p_{a;+}\}_1^{n_{p;+}} \cup \{1 - p_{a;-}\}_1^{n_{p;-}} \quad \text{and} \quad \{h_a^{(s)}\}_1^n = \{1 + N_s - h_{a;+}\}_1^{n_{h;+}} \cup \{h_{a;-}\}_1^{n_{h;-}}, \quad (2.19)$$

where the integers $p_{a;\pm}, h_{a;\pm} \in \mathbb{N}^*$ are "small" compared to L , *ie*

$$\lim_{L \rightarrow +\infty} \frac{p_{a;\pm}}{L} = \lim_{L \rightarrow +\infty} \frac{h_{a;\pm}}{L} = 0, \quad (2.20)$$

and the integers $n_{p/h;\pm}$ satisfy to the constraint

$$n_{p;+} + n_{p;-} = n_{h;+} + n_{h;-} = n. \quad (2.21)$$

Within this setting, one can readily check that the critical excited state described above will have $n_{p;+/-}$ particles, resp. $n_{h;+/-}$ holes, on the right/left end of the Fermi zone $[-q; q]$ associated with the N_s pseudo-particle sector.

In fact, one can distinguish between various critical states by organizing them into so-called ℓ_s -critical classes. This classification takes its origin in the fact that all such states have a vanishing excitation energy (up to $O(1/L)$)

corrections) but can be gathered into classes depending on the value of their macroscopic momenta $2\ell_s p_F$, where $p_F = p(q)$ is the so-called Fermi momentum and

$$\ell_s = n_{p;+} - n_{h;+} = n_{h;-} - n_{p;-} . \quad (2.22)$$

The index s in ℓ_s is there so as to keep track of the N_s pseudo-particle sector with which the critical class is associated. However, for the sake of simplifying the notations in formulae, we chose not to emphasize the s dependence in the left or right particle/hole numbers $n_{p/h;\pm}$.

In fact, one has the following large- L expansion for the relative excitation momentum (2.13) associated with the ℓ_s -critical class excited state described above

$$\Delta\mathcal{P}(\mathcal{I}_n^{(s)}) = 2\ell_s p_F + \frac{2\pi}{L} \left\{ \sum_{a=1}^{n_{p;+}} p_{a;+} + \sum_{a=1}^{n_{h;+}} (h_{a;+} - 1) \right\} - \frac{2\pi}{L} \left\{ \sum_{a=1}^{n_{p;-}} (p_{a;-} - 1) + \sum_{a=1}^{n_{h;-}} h_{a;-} \right\} + \dots \quad (2.23)$$

where we *do stress* that all the terms included in the dots are either of the order of $O(1/L)$ *but do not* depend on the integers $p_{a;\pm}$ and $h_{a;\pm}$ or they depend on these integers *but* are of the order of $O(1/L^2)$.

It is convenient, for such ℓ_s -critical excited states, to use a reparametrisation of the sets of integers labelling the excitations. It is readily seen that the excited states of an ℓ_s -critical class are described by two sets of integers $\mathcal{J}_{n_{p;\pm};n_{h;\pm}}^{(s)}$, each containing all the informations on the local integers on the right (+) or left (-) Fermi boundaries. Here, we agree upon

$$\mathcal{J}_{n;m}^{(s)} = \{ \{p_a^{(s)}\}_1^n ; \{h_a^{(s)}\}_1^m \} . \quad (2.24)$$

Note that the value of ℓ_s is encoded in the very notation $\mathcal{J}_{n_{p;\pm};n_{h;\pm}}^{(s)}$ by means the identification (2.22). Since there is a one-to-one correspondence between the set $\mathcal{I}_n^{(s)}$ and $\mathcal{J}_{n_{p;+};n_{h;+}}^{(s)} \cup \mathcal{J}_{n_{p;-};n_{h;-}}^{(s)}$, we shall identify the two sets. We do stress that the parameter s does play a role in this correspondence, *cf* (2.19).

2.4 Large- L expansion of form factors connecting critical states

As we have already mentioned, solely critical states enter in the analysis of the form factor expansion at large spacial separation between the operators. Thus, we now present the explicit expression for the form factor taken between such states. Given two excited states

$$\mathcal{I}_m^{(s-1)} \equiv \mathcal{J}_{m_{p;+};m_{h;+}}^{(s-1)} \cup \mathcal{J}_{m_{p;-};m_{h;-}}^{(s-1)} \quad \text{and} \quad \mathcal{I}_n^{(s)} \equiv \mathcal{J}_{n_{p;+};n_{h;+}}^{(s)} \cup \mathcal{J}_{n_{p;-};n_{h;-}}^{(s)} \quad (2.25)$$

belonging respectively to the

$$\ell_{s-1} = m_{p;+} - m_{h;+} = m_{h;-} - m_{p;-} \quad \text{and} \quad \ell_s = n_{p;+} - n_{h;+} = n_{h;-} - n_{p;-} \quad (2.26)$$

critical classes, we shall assume that the form factors of local operators take the form²

$$\begin{aligned} \mathcal{F}_{O_s} \left(\mathcal{I}_m^{(s-1)} \middle| \mathcal{I}_n^{(s)} \right) &= \mathcal{F}_{O_s}(\ell_{s-1}, \ell_s) \cdot C^{(\ell_{s-1}; \ell_s)}(v_s^+, v_s^-) \\ &\times \mathcal{F}^{(+)} \left[\mathcal{J}_{m_{p;+};m_{h;+}}^{(s-1)} ; \mathcal{J}_{n_{p;+};n_{h;+}}^{(s)} \middle| v_s^+ \right] \cdot \mathcal{F}^{(-)} \left[\mathcal{J}_{m_{p;-};m_{h;-}}^{(s-1)} ; \mathcal{J}_{n_{p;-};n_{h;-}}^{(s)} \middle| v_s^- \right] . \end{aligned} \quad (2.27)$$

²A priori, this formula could be modified by the presence of a sign factor $(-1)^{\sigma_s + \sigma_{s-1}}$ originating from the determination of the square root one chooses for the norms. The integer σ_s depends, *a priori*, on $\mathcal{I}_n^{(s)}$. However, the very structure of such a sign factor makes its contribution to the form factor expansion irrelevant as it always appears twice. We have therefore disregarded its presence from the very beginning. One can also allow for the multiplication of (2.27) by a L -dependent phase factor of the form $e^{inL(\phi_s - \phi_{s-1})}$, with ϕ_s depending on $\mathcal{I}_n^{(s)}$. Once again, the appearance of such a phase factor has no influence on the form factor expansion (A.33), hence we simply omit it here (note however that the presence of such a phase factor should be taken into account in the definition (2.30) of $\mathcal{F}_{O_s}(\ell_{s-1}, \ell_s)$).

The constituents of the above formula are parametrized by the values

$$\nu_s^+ = \nu_s(q) - o_s \quad \text{and} \quad \nu_s^- = \nu_s(-q) \quad (2.28)$$

that the relative shift function between the ℓ_s, ℓ_{s-1} critical states,

$$\nu_s(\lambda) = F_{s-1}(\lambda) - F_s(\lambda), \quad (2.29)$$

takes on the right/left endpoints of the Fermi zone, up to subtracting the level o_s of the operator O_s in the case of the right endpoint. In formula (2.27), all the pre-factors but $\mathcal{F}_{O_s}(\ell_{s-1}, \ell_s)$ issue from the "relevant" part of the discrete term. Their explicit expressions are given below. Even though these may appear as slightly complicated, the physical interpretation of each factor in (2.27) is crystal clear. Indeed, the quantity $\mathcal{F}_{O_s}(\ell_{s-1}, \ell_s)$ represents the properly normalized finite and non-universal (*ie* model and operator dependent) part of the large- L behaviour of the form factor of the operator O_s taken between fundamental representatives of the ℓ_s and ℓ_{s-1} critical classes. More precisely, it is defined as

$$\mathcal{F}_{O_s}(\ell_{s-1}, \ell_s) = \lim_{L \rightarrow +\infty} \left\{ \left(\frac{L}{2\pi} \right)^{\rho_s(\nu_s^+) + \rho_s(\nu_s^-)} \langle \Psi(\mathcal{L}_{\ell_{s-1}}^{(s-1)}) | O_s(0) | \Psi(\mathcal{L}_{\ell_s}^{(s)}) \rangle \right\} \quad (2.30)$$

in which the sets of integers $\mathcal{L}_{\ell_{s-1}}^{(s-1)}$ and $\mathcal{L}_{\ell_s}^{(s)}$ parametrizing the excited states correspond to the fundamental representatives of the ℓ_{s-1} and ℓ_s critical classes. Namely, $\mathcal{L}_{\ell_s}^{(s)}$ is the set of particle-hole integers living on the Fermi boundary in the N_s pseudo-particle sector such that

$$\mathcal{L}_{\ell_s}^{(s)} = \begin{cases} \{p_{a;+} = a\}_1^{\ell_s}; \{\emptyset\} \cup \{\emptyset\}; \{h_{a;-} = a\}_1^{\ell_s} & \text{if } \ell_s \geq 0 \\ \{\emptyset\}; \{h_{a;+} = a\}_1^{-\ell_s} \cup \{p_{a;-} = a\}_1^{-\ell_s}; \{\emptyset\} & \text{if } \ell_s \leq 0 \end{cases} \quad (2.31)$$

The power of the volume L arising in (2.30) involves the right $\rho_s(\nu_s^+)$ and left $\rho_s(\nu_s^-)$ scaling dimensions whose generic expression reads

$$\rho_s(\nu) = \frac{1}{2}(\ell_s - \ell_{s-1})^2 + \frac{1}{2}\nu^2 - (\ell_s - \ell_{s-1})\nu. \quad (2.32)$$

The factor $\mathcal{F}^{(+)}$ corresponds to the contributions of the excitations on the right Fermi boundary of the model. It is a function of the value taken on the right Fermi boundary by the relative shift function ν_s associated with the ℓ_s class of interest. Also, $\mathcal{F}^{(+)}$ depends on the sets of integers $\mathcal{J}_{m_{p;+}, m_{h;+}}^{(s-1)}$ and $\mathcal{J}_{n_{p;+}, n_{h;+}}^{(s)}$ parametrizing the excitations on the right boundary for the $s-1$ and s excited states. Given two sets of integers

$$\mathcal{J}_{n_p; n_h} = \{p_a\}_1^{n_p}; \{h_a\}_1^{n_h} \quad \text{and} \quad \mathcal{J}_{n_k; n_t} = \{k_a\}_1^{n_k}; \{t_a\}_1^{n_t}, \quad (2.33)$$

the right Fermi boundary critical form factor reads

$$\begin{aligned} \mathcal{F}^{(+)}[\mathcal{J}_{n_p; n_h}; \mathcal{J}_{n_k; n_t} | \nu] &= \left(\frac{2\pi}{L} \right)^{\rho_s(\nu)} \cdot (-1)^{n_t} \cdot \left(\frac{\sin[\pi\nu]}{\pi} \right)^{n_t + n_h} \cdot \varpi(\mathcal{J}_{n_p; n_h}; \mathcal{J}_{n_k; n_t} | \nu) \\ &\times \frac{\prod_{a>b}^{n_p} (p_a - p_b) \cdot \prod_{a<b}^{n_h} (h_a - h_b) \cdot \prod_{a>b}^{n_k} (k_a - k_b) \cdot \prod_{a<b}^{n_t} (t_a - t_b)}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1) \cdot \prod_{a=1}^{n_k} \prod_{b=1}^{n_t} (k_a + t_b - 1)} \cdot \Gamma \left(\begin{matrix} \{p_a + \nu\}_1^{n_p}, & \{h_a - \nu\}_1^{n_h}, & \{k_a - \nu\}_1^{n_k}, & \{t_a + \nu\}_1^{n_t} \\ \{p_a\}_1^{n_p}, & \{h_a\}_1^{n_h}, & \{k_a\}_1^{n_k}, & \{t_a\}_1^{n_t} \end{matrix} \right). \end{aligned}$$

Note that $\mathcal{F}^{(+)}$ decays as an algebraic power of the volume L with a scaling dimension $\rho_s(\nu)$. Likewise, the factor $\mathcal{F}^{(-)}$ corresponds to the contributions of the excitations on the left Fermi boundary of the model. It is a function of the value ν_s^- taken on the left Fermi boundary by the relative shift function ν_s and of the sets $\mathcal{J}_{m_{p;-}, m_{h;-}}^{(s-1)}$ and $\mathcal{J}_{n_{p;-}, n_{h;-}}^{(s)}$ of integers parametrizing the excitations on the left Fermi boundaries for the $s-1$ and s -excited states. Given sets as in (2.33) for the parametrization of the excitations on the left Fermi boundary, one has

$$\begin{aligned} \mathcal{F}^{(-)}[\mathcal{J}_{n_p; n_h}; \mathcal{J}_{n_k; n_t} | \nu] &= \left(\frac{2\pi}{L}\right)^{\rho_s(\nu)} \cdot (-1)^{n_h} \cdot \left(\frac{\sin[\pi\nu]}{\pi}\right)^{n_h + n_t} \cdot \varpi(\mathcal{J}_{n_p; n_h}; \mathcal{J}_{n_k; n_t} | -\nu) \\ &\times \frac{\prod_{a>b}^{n_p} (p_a - p_b) \cdot \prod_{a<b}^{n_h} (h_a - h_b)}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1)} \cdot \frac{\prod_{a>b}^{n_k} (k_a - k_b) \cdot \prod_{a<b}^{n_t} (t_a - t_b)}{\prod_{a=1}^{n_k} \prod_{b=1}^{n_t} (k_a + t_b - 1)} \cdot \Gamma\left(\begin{matrix} \{p_a - \nu\}, & \{h_a + \nu\}, & \{k_a + \nu\}, & \{t_a - \nu\} \\ \{p_a\}, & \{h_a\}, & \{k_a\}, & \{t_a\} \end{matrix}\right). \end{aligned}$$

Just as for the right Fermi boundary critical form factor, the left one decays algebraically with the volume of the model with a left scaling dimension $\rho_s(\nu)$.

The expressions for the right and left Fermi boundary factors involve products of Γ -functions written as

$$\Gamma\left(\begin{matrix} \{v_a\}_1^n \\ \{w_a\}_1^m \end{matrix}\right) = \Gamma\left(\begin{matrix} v_1, \dots, v_n \\ w_1, \dots, w_m \end{matrix}\right) = \frac{\prod_{a=1}^n \Gamma(v_a)}{\prod_{a=1}^m \Gamma(w_a)}. \quad (2.34)$$

The Γ functions encode the structure of excitations within a given (s) -sector of excitations on the Fermi zone. The expressions for the left and right local factors $\mathcal{F}^{(\pm)}$ also involve the function ϖ whose presence translates the interactions between integers parametrizing the neighbouring sectors $(s-1)$ and (s) that are connected by the operator \mathcal{O}_s . Taken two sets of integers as in (2.33), one has

$$\varpi(\mathcal{J}_{n_p; n_h}; \mathcal{J}_{n_k; n_t} | \nu) = \prod_{a=1}^{n_h} \left\{ \frac{\prod_{b=1}^{n_k} (1 - k_b - h_a + \nu)}{\prod_{b=1}^{n_t} (t_b - h_a + \nu)} \right\} \cdot \prod_{a=1}^{n_p} \left\{ \frac{\prod_{b=1}^{n_t} (p_a + t_b + \nu - 1)}{\prod_{b=1}^{n_k} (p_a - k_b + \nu)} \right\}. \quad (2.35)$$

Finally, the function $C^{(\ell_{s-1}; \ell_s)}(\nu_s^+, \nu_s^-)$ is a normalization constant. It provides the appropriate normalization of the expression for the general critical form factor. It is such that it precisely cancels out the L -independent contributions of the right and left critical form factors when focusing on the fundamental representative of the ℓ_s , ℓ_{s-1} critical classes. Its explicit form can be computed in terms of the Barnes G -function [4] and reads:

$$C^{(\ell_{s-1}; \ell_s)}(\nu_s^+, \nu_s^-) = G\left(\begin{matrix} 1 + \nu_s^-, 1 - \nu_s^+, 1 + \ell_s - \nu_s^-, 1 - \ell_s + \nu_s^+ \\ 1 - \ell_s + \nu_s^-, 1 + \ell_s - \nu_s^+, 1 - \ell_{s-1} + \ell_s - \nu_s^-, 1 + \ell_{s-1} - \ell_s + \nu_s^+ \end{matrix}\right). \quad (2.36)$$

Note that above, we have adopted similar product conventions as for Γ -functions, cf (2.34).

3 Asymptotic behaviour of multi-point correlation functions

3.1 Long-distance asymptotic expansion of multi-point functions

In this section, building on our previous assumptions on the structure of the spectrum and form factors of local operators, we derive the large-distance asymptotic behaviour of the correlator $C(\mathbf{x}_r; \mathbf{o}_r)$, cf. (2.1). As already

discussed, our method builds on a microscopic analysis of the form factor expansion for this correlator. Hence, in between the operators s and $s + 1$ we insert a decomposition of the identity. In principle, we should thus sum up over all the states of the model, namely the states generated by the particle-hole excitations along with the more complex states related to bound states. However, building on the assumption that states containing bound states solely generate exponentially small corrections to the large-distance regime in the $|x_{k+1} - x_k| \cdot p_F \gg 1$ limit, we only sum-up over the particle-hole states, as described in Section 2.1. Thus, for each s ($s = 1, \dots, r - 1$) we sum up over all possible choices of the sets $\{\mathcal{I}_{n^{(s)}}^{(s)}\}$ as given in (2.7), with $n^{(s)} = 0, 1, \dots$. In other words the multi-point correlator is recast as

$$C(\mathbf{x}_r; \mathbf{o}_r) = \prod_{s=1}^{r-1} \left\{ \sum_{\{\mathcal{I}_{n^{(s)}}^{(s)}\}} \right\} \cdot \prod_{s=1}^{r-1} \left\{ \exp \left[i(x_{s+1} - x_s) \cdot \Delta \mathcal{P}(\mathcal{I}_{n^{(s)}}^{(s)}) \right] \right\} \cdot \prod_{s=1}^r \mathcal{F}_{O_s}(\mathcal{I}_{n^{(s-1)}}^{(s-1)} | \mathcal{I}_{n^{(s)}}^{(s)}), \quad (3.1)$$

in which $\Delta \mathcal{P}(\mathcal{I}_{n^{(s)}}^{(s)})$ stands for the relative excitation momentum defined in (2.13).

We do stress that the equality (3.1) is to be understood in the sense of having neglected all the contributions from the bound-states to the form factor expansion.

We now follow exactly the reasoning outlined in [20]. The multidimensional sum in (3.1) contains quickly oscillating terms with speed measured by the magnitude of $|x_{k+1} - x_k| \cdot p_F$, $k = 1, \dots, r - 1$. Thus, by analogy with a multiple integral of oscillatory type, the leading asymptotic behaviour will issue from localising the sums at the endpoints of the summation domain or at the saddle-points of the oscillating exponent. However, due to monotonicity of the dressed momentum function p , the relative excitation momentum $\Delta \mathcal{P}(\mathcal{I}_{n^{(s)}}^{(s)})$ does not have saddle-points. Hence, the leading $|x_k - x_l| \cdot p_F \gg 1$, $k \neq l$, asymptotic behaviour will be given by localizing the summand on the states belonging to ℓ_s -critical classes, $s = 1, \dots, r - 1$. Upon inserting the local expression for the model's form factors, a straightforward calculation leads to the conclusion that

$$C(\mathbf{x}_r; \mathbf{o}_r) \simeq \sum_{\substack{\boldsymbol{\ell}_{r-1} \\ \in \mathbb{Z}^{r-1}}} \left(\frac{2\pi}{L} \right)^{\vartheta(\boldsymbol{\ell}_{r-1}, \mathbf{o}_r)} \cdot \prod_{s=1}^{r-1} \left\{ e^{2i\ell_s(x_{s+1} - x_s)p_F} \right\} \cdot \prod_{s=1}^r \left\{ C^{(\ell_{s-1}; \ell_s)}(v_s^+, v_s^-) \right\} \\ \times \prod_{s=1}^r \left\{ \mathcal{F}_{O_s}(\ell_{s-1}, \ell_s) \right\} \cdot \mathcal{S}_{\boldsymbol{\ell}_{r-1}}^-(\{ \frac{2\pi}{L}(x_{s+1} - x_s) \}_1^{r-1}, \{ v_s^-(\boldsymbol{\ell}_s) \}_1^r) \cdot \mathcal{S}_{\boldsymbol{\ell}_{r-1}}^+(\{ \frac{2\pi}{L}(x_{s+1} - x_s) \}_1^{r-1}, \{ v_s^+(\boldsymbol{\ell}_s) \}_1^r). \quad (3.2)$$

The summand at fixed $\boldsymbol{\ell}_{r-1} \in \mathbb{Z}^{r-1}$ is weighted by an algebraic factor in the volume L . The associated exponent $\vartheta(\boldsymbol{\ell}_{r-1}, \mathbf{o}_r)$ reads

$$\vartheta(\boldsymbol{\ell}_{r-1}, \mathbf{o}_r) = \frac{1}{2} \sum_{s=1}^r \left\{ (v_s^+)^2 + (v_s^-)^2 \right\} - \sum_{s=1}^{r-1} \left\{ (v_s^+ + v_s^- - v_{s+1}^+ - v_{s+1}^-) \ell_s - 2\ell_s^2 \right\} - 2 \sum_{s=2}^{r-1} \ell_s \ell_{s-1}. \quad (3.3)$$

We do insist that each relative shift function v_s does depend, *a priori* on the integers ℓ_{s-1}, ℓ_s labelling the critical class of the excited states being connected by the s^{th} -operator.

The form factor expansion has been recast with the help of multi-point restricted sums $\mathcal{S}_{\boldsymbol{\ell}_{r-1}}^\pm$. These refer to multi-dimensional sums that correspond to summing-up the whole set of contributions coming from excitations on Fermi surfaces, for the intermediate states labelled by $s = 1, \dots, r - 1$, and belonging to given critical classes labelled by the vector $\boldsymbol{\ell}_{r-1}$. The $+$, resp. the $-$, sum corresponds to the contributions stemming from the right, resp. left, Fermi boundaries.

The multi-point restricted sums of interest are given as the below multi-dimensional sums:

$$\mathcal{S}_{\ell_{r-1}}^{\pm}(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) = \prod_{s=1}^{r-1} \sum_{\substack{n_p^{(s)}, n_h^{(s)}=0 \\ n_p^{(s)} - n_h^{(s)} = \pm \ell_s}}^{+\infty} \sum_{\mathcal{J}_{n_p^{(s)}, n_h^{(s)}}^{(s)}} \prod_{s=1}^{r-1} \mathcal{R}^{\pm}(\mathcal{J}_{n_p^{(s)}, n_h^{(s)}}^{(s)} | \nu_s, \nu_{s+1}; t_s) \prod_{s=2}^{r-1} \varpi(\mathcal{J}_{n_p^{(s-1)}, n_h^{(s-1)}}^{(s-1)}; \mathcal{J}_{n_p^{(s)}, n_h^{(s)}}^{(s)} | \pm \nu_s). \quad (3.4)$$

The summation in the above formula runs through all the possible choices of the sets of integers

$$\mathcal{J}_{n_p^{(s)}, n_h^{(s)}}^{(s)} = \left\{ \{p_a^{(s)}\}_1^{n_p^{(s)}}; \{h_a^{(s)}\}_1^{n_h^{(s)}} \right\} \quad (3.5)$$

parametrizing the excitations on the relevant boundary (left for $-$ and right for $+$), provided that these leads to an ℓ_s -critical class. The summations are repeated for $s = 1, \dots, r-1$. In more explicit terms, the formula corresponds to a multiple sum over particle-like $p_a^{(s)}$ and hole-like $h_a^{(s)}$ integers with the number $n_p^{(s)}$ of particle-like variables, resp. $n_h^{(s)}$ of hole-like variables, taking all admissible values in \mathbb{N} that are compatible with the condition $n_p^{(s)} - n_h^{(s)} = \pm \ell_s$. These particle and hole integers satisfy to the constraints

$$p_1^{(s)} < \dots < p_{n_p^{(s)}}^{(s)} \quad \text{with } p_a^{(s)} \in \mathbb{N}^* \quad \text{and} \quad h_1^{(s)} < \dots < h_{n_h^{(s)}}^{(s)} \quad \text{with } h_a^{(s)} \in \mathbb{N}^*. \quad (3.6)$$

In what concerns the functions composing the summand, ϖ has been defined in (2.35) whereas the functions \mathcal{R}^{\pm} read

$$\begin{aligned} \mathcal{R}^{-}(\mathcal{J}_{n_p, n_h} | \nu, \eta; t) &= \left(-\frac{\sin[\pi\nu]}{\pi} \cdot \frac{\sin[\pi\eta]}{\pi} \right)^{n_h} \cdot \frac{\prod_{a < b}^{n_p} (p_a - p_b)^2 \cdot \prod_{a < b}^{n_h} (h_a - h_b)^2}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1)^2} \\ &\quad \times \prod_{a=1}^{n_p} \left\{ e^{it(1-p_a)} \Gamma \left(\begin{matrix} p_a + \nu, p_a - \eta \\ p_a, p_a \end{matrix} \right) \right\} \cdot \prod_{a=1}^{n_h} \left\{ e^{-it h_a} \Gamma \left(\begin{matrix} h_a - \nu, h_a + \eta \\ h_a, h_a \end{matrix} \right) \right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}^{+}(\mathcal{J}_{n_p, n_h} | \nu, \eta; t) &= \left(-\frac{\sin[\pi\nu]}{\pi} \cdot \frac{\sin[\pi\eta]}{\pi} \right)^{n_h} \cdot \frac{\prod_{a < b}^{n_p} (p_a - p_b)^2 \cdot \prod_{a < b}^{n_h} (h_a - h_b)^2}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1)^2} \\ &\quad \times \prod_{a=1}^{n_p} \left\{ e^{it p_a} \Gamma \left(\begin{matrix} p_a - \nu, p_a + \eta \\ p_a, p_a \end{matrix} \right) \right\} \cdot \prod_{a=1}^{n_h} \left\{ e^{it(h_a-1)} \Gamma \left(\begin{matrix} h_a + \nu, h_a - \eta \\ h_a, h_a \end{matrix} \right) \right\}. \end{aligned}$$

Above, the set \mathcal{J}_{n_p, n_h} is as given in (2.33). Quite remarkably, the multi-particle restricted sums can be resummed in a quite compact way, generalising the results of [20, 26]

$$\begin{aligned} \mathcal{S}_{\ell_{r-1}}^{\pm}(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) &= \prod_{s=1}^{r-1} \left\{ e^{\pm i t_s \frac{\ell_s(\ell_s+1)}{2}} G \left(\begin{matrix} 1 \pm (\ell_s - \nu_s), 1 \pm (\ell_s + \nu_{s+1}) \\ 1 \mp \nu_s, 1 \pm \nu_{s+1} \end{matrix} \right) \right\} \\ &\quad \times \prod_{s=2}^{r-1} G \left(\begin{matrix} 1 \pm \nu_s, 1 \pm (\ell_{s-1} - \ell_s + \nu_s) \\ 1 \mp (\ell_s - \nu_s), 1 \pm (\ell_{s-1} + \nu_s) \end{matrix} \right) \cdot \prod_{b>a}^r \left(1 - e^{\pm i \sum_{s=a}^{b-1} t_s} \right)^{(\nu_a + \kappa_a)(\nu_b + \kappa_b)}. \quad (3.7) \end{aligned}$$

The product appearing at the bottom right of (3.7) involves integers κ_a , $a = 1, \dots, s$ which are defined in terms of the integers ℓ_a , $a = 1, \dots, s-1$, as

$$\kappa_s = \ell_{s-1} - \ell_s \quad \text{for } s = 1, \dots, r \quad \text{so that} \quad \sum_{a=1}^r \kappa_a = 0 \quad (3.8)$$

where, as above, we agree upon $\ell_0 = \ell_r = 0$.

In the present paper we do not pretend to prove the above summation formulae. We however give a formal derivation. These arguments does not lead, however, to a proof in that several justifications relative to the exchangeability of symbols and limits are omitted. We leave the rigorous proof of the summation formulae to some subsequence publication. Our present formal justification is based on the use of an auxiliary, "model", form factor series. The latter can be recast in two ways,

- as a Toeplitz determinant generated by a symbol with Fisher-Hartwig singularities, this by generalizing the techniques of form factor summations developed in [23];
- as an asymptotic series whose individual summands directly involve multi-particle restricted sums.

Comparisons of the large- N limit of both expressions yields the sought identities. We refer the reader to Appendix A for our formal proof.

Building on the above results allowing one to sum up the multi-point restricted sums and taking into account their most natural parametrization in terms of the κ_a integers, we change the summation variables to the more convenient κ -like reparametrisation:

$$\ell_s(\boldsymbol{\kappa}_r) = \sum_{a=s+1}^r \kappa_a \quad \text{for } s = 1, \dots, r-1. \quad (3.9)$$

It is then readily checked that, for any set of parameters $\{t_a\}$ and $\{\nu_a\}$,

$$\sum_{s=1}^{r-1} \ell_s(\boldsymbol{\kappa}_r) \cdot t_s = \sum_{s=2}^r \bar{t}_{s-1} \kappa_s, \quad (3.10)$$

and

$$2 \sum_{s=1}^r [\ell_s(\boldsymbol{\kappa}_r)]^2 - 2 \sum_{s=2}^{r-1} \ell_s(\boldsymbol{\kappa}_r) \ell_{s-1}(\boldsymbol{\kappa}_r) + 2 \sum_{s=1}^{r-1} \{(\nu_{s+1} - \nu_s) \ell_s(\boldsymbol{\kappa}_r)\} + \sum_{s=1}^r \nu_s^2 = \sum_{s=1}^r (\nu_s + \kappa_s)^2, \quad (3.11)$$

where in (3.10) we have used a similar notation as in (2.5). Upon using the explicit expressions for the multi-particle restricted sums, we are thus led to

$$\begin{aligned} C(\mathbf{x}_r; \mathbf{o}_r) &= \sum_{\substack{\boldsymbol{\kappa}_r \in \mathbb{Z}^r \\ \sum \kappa_a = 0}} \prod_{s=1}^r \left\{ e^{2ip_F \kappa_s x_s} \right\} \cdot \mathcal{F}(\{\kappa_a\}_1^r; \{o_a\}_1^r) \\ &\times \prod_{s=1}^r \left\{ \left(\frac{2\pi}{L} \right)^{\frac{1}{2}[\theta_s^+(\kappa_s)]^2 + \frac{1}{2}[\theta_s^-(\kappa_s)]^2} \right\} \cdot \prod_{b>a}^r \left\{ \left[1 - e^{\frac{2i\pi}{L}(x_b - x_a)} \right]^{\theta_b^+(\kappa_b) \theta_a^+(\kappa_a)} \cdot \left[1 - e^{-\frac{2i\pi}{L}(x_b - x_a)} \right]^{\theta_b^-(\kappa_b) \theta_a^-(\kappa_a)} \right\}. \end{aligned} \quad (3.12)$$

There, we have set

$$\theta_b^\pm(\kappa_b) = \nu_b^\pm + \kappa_b. \quad (3.13)$$

We do stress that θ_b^\pm has an explicit dependence on κ_b but also an *implicit* one through the relative shift function ν_b^\pm . Furthermore, we have made use of the fact that the relative shift functions ν_s^\pm *only* depend on the parameter κ_s . Finally, the asymptotic formula involves the factor

$$\mathcal{F}(\{\kappa_a\}_1^r; \{o_a\}_1^r) = \prod_{s=1}^r \mathcal{F}_{O_s}(\ell_{s-1}(\boldsymbol{\kappa}_r), \ell_s(\boldsymbol{\kappa}_r)) . \quad (3.14)$$

The above amplitude has a crystal clear interpretation: the pre-factor in front of the power-law decay of the multi-point correlation function associated with an excitation belonging to the $\ell_{r-1}(\boldsymbol{\kappa}_r)$ -critical classes is precisely given by the product of form factors of local operators taken between the typical representatives of the $\ell_{r-1}(\boldsymbol{\kappa}_r)$ -critical classes of interest. Note that, for a given $\boldsymbol{\kappa}_r$, the form of the L -dependence in (3.12) is already reminiscent of the expression for multi-point correlation functions in a conformal invariant theory of a strip of width L [5].

It is straightforward to take the $L \rightarrow +\infty$ limit in (3.12). This leads to

$$C(\mathbf{x}_r; \mathbf{o}_r) = \sum_{\substack{\boldsymbol{\kappa}_r \in \mathbb{Z}^r \\ \sum \kappa_a = 0}} \prod_{s=1}^r \left\{ e^{2ip_F \kappa_s x_s} \right\} \cdot \mathcal{F}(\{\kappa_a\}_1^r; \{o_a\}_1^r) \cdot \prod_{b>a}^r \left\{ [i(x_b - x_a)]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)} \cdot [-i(x_b - x_a)]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)} \right\} , \quad (3.15)$$

where we have used that

$$\sum_{s=1}^r \theta_s^\pm(\kappa_s) = 0 . \quad (3.16)$$

Note that the above asymptotic expansion provides one with an expression that is symmetric under a simultaneous permutation

$$(\mathbf{x}_r, \mathbf{o}_r) \mapsto (\mathbf{x}_r^\sigma, \mathbf{o}_r^\sigma) \quad \text{with} \quad \mathbf{x}_r^\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(r)}) \quad \sigma \in \mathfrak{S}_r . \quad (3.17)$$

This translates the fact that the local operators $O_r(x_r)$ commute at different distances and, in particular, in the long-distance regime.

4 Applications

4.1 The conformal regime

The asymptotic formula (3.15) is quite general in that it does not assume any specific order of magnitude for the spacing between the various "space"-parameters x_a . The sole constraint is that all have to be pairwise large, *viz.*

$$|x_k - x_\ell| \cdot p_F \gg 1 \quad \text{for} \quad k \neq \ell . \quad (4.1)$$

Thus, in such a general setting, determining the leading term of the asymptotics (3.15) leads to a very complex minimisation problem. However, the situation gets much simpler in the so-called conformal regime of the correlation functions. In the latter case, all distances scale with the same magnitude R ,

$$x_k = R \cdot z_k \quad 0 < \epsilon < |z_k - z_\ell| < \epsilon^{-1} \quad \text{for} \quad k \neq \ell \quad \text{and some} \quad \epsilon > 0 . \quad (4.2)$$

One then has

$$C(R \cdot \mathbf{z}_r; \mathbf{o}_r) \simeq \sum_{\substack{\boldsymbol{\kappa}_r \in \mathbb{Z}^r \\ \sum \kappa_a = 0}} \prod_{s=1}^r \left\{ \left(\frac{1}{R} \right)^{\frac{[\theta_s^+(\kappa_s)]^2}{2} + \frac{[\theta_s^-(\kappa_s)]^2}{2}} \right\} \prod_{s=1}^r \left\{ e^{2ip_F R \kappa_s z_s} \right\} \cdot \mathcal{F}(\{\kappa_a\}_1^r; \{o_a\}_1^r) \\ \times \prod_{b>a}^r \left\{ [i(z_b - z_a)]^{\theta_b^-(\kappa_b)\theta_a^-(\kappa_a)} \cdot [-i(z_b - z_a)]^{\theta_b^+(\kappa_b)\theta_a^+(\kappa_a)} \right\} \quad (4.3)$$

using again,

$$\sum_{s=1}^r \theta_s^\pm(\kappa_s) = 0. \quad (4.4)$$

The leading asymptotics are then obtained by choosing an integer vector $\boldsymbol{\kappa}_r \in \mathbb{Z}^r$ realising the minimum of

$$\boldsymbol{\kappa}_r \mapsto \sum_{s=1}^r [\theta_s^+(\kappa_s)]^2 + [\theta_s^-(\kappa_s)]^2. \quad (4.5)$$

In practical situations (such as Bethe Ansatz solvable models, *cf.* later on) the existence and uniqueness of the minimum, for generic values of the coupling constants, follows by mimicking the reasonings presented in [8].

4.2 Bethe Ansatz solvable models

In this subsection we specialise our results to the case of two Bethe Ansatz solvable models, the XXZ spin-1/2 chain and the non-linear Schrödinger model. The matter is that one has quite explicit expressions for the shift functions arising in the description of these models in terms of solutions to linear integral equations. The latter thus give access to the critical exponents.

The XXZ spin-1/2 chain corresponds to the Hamiltonian

$$\mathbf{H}_{XXZ} = \sum_{k=1}^L \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta (\sigma_k^z \sigma_{k+1}^z - 1) \right) - \frac{h}{2} \sum_{k=1}^L \sigma_k^z. \quad (4.6)$$

Here $\sigma_k^{x,y,z}$ are the spin operators (Pauli matrices) acting on the k -th site of the chain, h is an external magnetic field and the model is subject to periodic boundary conditions. The XXZ spin chain above exhibits different phases depending on the value of the anisotropy parameter Δ . We only focus on the critical regime $-1 < \Delta < 1$ where we set $\Delta = \cos \zeta$. It is known that, in its massless phase, the excitations in the XXZ-chain can be either of bound state nature (so-called string solutions) or particle-hole one. As argued previously, we solely focus on the particle-hole part of the spectrum.

The non-linear Schrödinger model corresponds to the Hamiltonian

$$\mathbf{H}_{NLS} = \int_0^L \left\{ \partial_y \Phi^\dagger(y) \partial_y \Phi(y) + c \Phi^\dagger(y) \Phi^\dagger(y) \Phi(y) \Phi(y) - \mu \Phi^\dagger(y) \Phi(y) \right\} dy. \quad (4.7)$$

The model is defined on a circle of length L , so that the canonical Bose fields Φ, Φ^\dagger are subject to L -periodic boundary conditions. We solely focus on the repulsive regime $c > 0$ in the presence of a positive chemical potential $\mu > 0$. In this model, one can show that the excitations are only given by particles and holes.

One can show using Bethe Ansatz methods that, for both models, the general shift functions – in the sense of (2.9) – take the form

$$F_{\mathcal{R}_n,s}(\lambda) = -\bar{\theta}_s\left(\frac{Z(\lambda)}{2} + \nu\phi(\lambda, q)\right) - \nu \sum_{a=1}^n [\phi(\lambda, \mu_{p_a}) - \phi(\lambda, \mu_{h_a})], \quad (4.8)$$

in which the rapidities are the unique solutions to

$$\xi(\mu_a) = \frac{a}{L} \quad \text{with} \quad \xi(\omega) = \frac{p(\omega)}{2\pi} + \frac{D}{2}, \quad (4.9)$$

with p representing the dressed momentum associated with the model and $\bar{\theta}_s$ has been defined in (2.5). Although the form for ξ and $F_{\mathcal{R}_n,s}$ is similar for both models, the definition of the functions arising in their expressions differ. Also, one should set

- $\nu = 1$ for the non-linear Schrödinger model
- $\nu = -1$ for the XXZ chain.

The function ϕ is the dressed phase and Z the dressed charge. Setting

$$\theta(\lambda) = i \ln\left(\frac{ic + \lambda}{ic - \lambda}\right) \quad \text{for NLSM} \quad \text{and} \quad \theta(\lambda) = i \ln\left(\frac{\sinh(i\zeta + \lambda)}{\sinh(i\zeta - \lambda)}\right) \quad \text{for XXZ}, \quad (4.10)$$

one has that p solves the integro-differential equation under the requirement that $p(\lambda) = -p(-\lambda)$:

$$p(\lambda) - \nu \int_{-q}^q \theta(\lambda - \mu) \cdot p'(\mu) \cdot \frac{d\mu}{2\pi} = p_0(\lambda) \quad \text{with} \quad p_0(\lambda) = \begin{cases} \lambda & \text{for NLSM} \\ i \ln\left(\frac{\sinh(i\zeta/2 + \lambda)}{\sinh(i\zeta/2 - \lambda)}\right) & \text{for XXZ} \end{cases}. \quad (4.11)$$

The functions Z and ϕ solve the Lieb integral equations

$$Z(\lambda) - \nu \int_{-q}^q \theta'(\lambda - \mu) Z(\mu) \cdot \frac{d\mu}{2\pi} = 1 \quad \phi(\lambda, \nu) - \nu \int_{-q}^q \theta'(\lambda - \mu) \phi(\mu, \nu) \cdot \frac{d\mu}{2\pi} = \frac{\theta(\lambda - \nu)}{2\pi}. \quad (4.12)$$

As a consequence, the relative shift function ν_s between critical excited states belonging to the ℓ_s and ℓ_{s-1} classes and having a change in the pseudo-particle number of o_s takes the form

$$\nu_s(\lambda) = o_s\left(\frac{Z(\lambda)}{2} + \nu\phi(\lambda, q)\right) + (\ell_{s-1} - \ell_s)(Z(\lambda) - 1). \quad (4.13)$$

In particular, then, the critical exponents takes the explicit expressions

$$\theta_s^\pm(\kappa) = \kappa Z(q) \mp \frac{o_s}{2} Z^{-1}(q). \quad (4.14)$$

These data are already enough so as to extract, in the large-distance conformal limit, the asymptotic behaviour of multi-point correlation functions of the XXZ spin-1/2 chain. We shall discuss the example of the correlator

$$C_{xxxx} = \langle \Psi_g | \sigma_{x_1}^x \cdot \sigma_{x_2}^x \cdot \sigma_{x_3}^x \cdot \sigma_{x_4}^x | \Psi_g \rangle. \quad (4.15)$$

In order to apply the method developed in the present paper, one ought to decompose the operators $\sigma_{x_a}^x$ onto operators which change the particle number in a definite way, namely the $\sigma_{x_a}^\pm$ operators. Due to the conservation of the longitudinal total spin, such a decomposition leads to

$$C_{xxxx} = \sum_{\substack{\epsilon_a = \pm \\ \sum \epsilon_a = 0}} \langle \Psi_g | \sigma_{x_1}^{\epsilon_1} \cdot \sigma_{x_2}^{\epsilon_2} \cdot \sigma_{x_3}^{\epsilon_3} \cdot \sigma_{x_4}^{\epsilon_4} | \Psi_g \rangle. \quad (4.16)$$

We shall consider the large- x asymptotic behaviour of C_{xxxx} in the conformal scaling regime where $x_a = R \cdot z'_a$, R is the large parameter and the z'_a are fixed and pairwise distinct. Then, it follows from the previous calculations that

$$\begin{aligned} \langle \Psi_g | \sigma_{x_1}^{\epsilon_1} \cdot \sigma_{x_2}^{\epsilon_2} \cdot \sigma_{x_3}^{\epsilon_3} \cdot \sigma_{x_4}^{\epsilon_4} | \Psi_g \rangle &= \sum_{\substack{\kappa_4 \in \mathbb{Z}^4 \\ \sum \kappa_a = 0}} \prod_{s=1}^4 \left\{ \left(\frac{1}{R} \right)^{\frac{[\vartheta_{\epsilon_s}(\kappa_s)]^2}{2} + \frac{[\vartheta_{-\epsilon_s}(\kappa_s)]^2}{2}} \right\} \prod_{s=1}^4 \left\{ e^{2ip_F R \kappa_s z_s} \right\} \cdot \mathcal{F}(\{\kappa_a\}_1^r; \{\epsilon_a\}_1^r) \\ &\quad \times \prod_{b>a}^r \left\{ [i(z_b - z_a)]^{\vartheta_{-\epsilon_b}(\kappa_b) \vartheta_{-\epsilon_a}(\kappa_a)} \cdot [-i(z_b - z_a)]^{\vartheta_{\epsilon_b}(\kappa_b) \vartheta_{\epsilon_a}(\kappa_a)} \right\}. \end{aligned} \quad (4.17)$$

Above $\mathcal{F}(\{\kappa_a\}_1^r; \{\epsilon_a\}_1^r)$ represents the properly normalized in the volume L product of form factors of operators σ^{ϵ_a} . Also, we agree upon

$$\vartheta_\epsilon(\kappa) = \kappa Z(q) - \frac{\epsilon}{2Z(q)}. \quad (4.18)$$

In order to access to the leading asymptotics in the distance one should choose the combination of integers $\kappa_4 = (\kappa_1, \dots, \kappa_4)$ that minimises the exponent of R . In fact, independently of the operator (*ie* the sign of ϵ_a) one has

$$[\vartheta_{\epsilon_s}(\kappa_s)]^2 + [\vartheta_{-\epsilon_s}(\kappa_s)]^2 = 2(\kappa_s Z(q))^2 + \frac{1}{2Z^2(q)}. \quad (4.19)$$

As a consequence, independently of the choice of the ϵ_a 's, the leading asymptotics are given by the choice $\kappa_4 = 0$. It is then straightforward to see that

$$C_{xxxx} = 2\mathcal{F}_{\sigma^+}^2(0,0)\mathcal{F}_{\sigma^-}^2(0,0) \cdot \left\{ \left| \frac{(x_2 - x_1) \cdot (x_4 - x_3)}{(x_3 - x_1) \cdot (x_4 - x_1) \cdot (x_3 - x_2) \cdot (x_4 - x_2)} \right|^{\frac{1}{2Z^2(q)}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} + \dots \quad (4.20)$$

Above, $\mathcal{F}_{\sigma^\pm}(0,0)$ are the aforesaid form factors of the σ^\pm operators between appropriate states of the 0-critical class.

Although we have focused in the present example on ultra-local elementary operators acting on a single site, our approach would works just as good for higher composite operators such as finite products of local operators on adjacent site, such as

$$O(x) = \prod_{a=1}^p \sigma_{x+a}^{\epsilon_a}. \quad (4.21)$$

Conclusion

In this paper, we have generalized to the multi-point case the restricted sum formalism developed in [20] for the study of the form factor expansion of two-point correlation functions in massless models. In this framework, the computation of the form factor series, for large but finite volume and in the large distance regime, reduces to the computation of multi-dimensional sums over particular classes of low-energy excited states. Our formalism naturally applies to quantum integrable models such as the XXZ spin-1/2 chain in the massless regime or the quantum non-linear Schrödinger model in the repulsive regime. It also applies, upon quite reasonable assumptions on the way of parametrizing the model's spectrum and the structure of the model's form factors, to more general massless one-dimensional quantum models. Let us finally mention that this approach to multi-point correlation functions, developed here in the static case, can be extended, as in [22], to the time-dependent case, *ie* to the study of the large-distance and long-time asymptotic behaviour of multi-point dynamical correlation functions.

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A The summation identity

In order to establish the summation identities of interest, we shall compute the large- N behaviour of the sum

$$\mathcal{S}_N^{(r)} = \prod_{k=1}^{r-1} \left\{ \sum_{\substack{\ell_a^{(k)} \in \mathbb{Z} \\ \ell_1^{(k)} < \dots < \ell_N^{(k)}}} \right\} \cdot \left(\frac{\sin[\pi v_r]}{\pi} \right)^N \cdot \prod_{s=1}^{r-1} \prod_{a=1}^N \left\{ \frac{\sin[\pi v_s]}{\pi} \exp \{ i t_s \lambda_{\ell_a^{(s)}}^{(s)} \} \right\} \cdot \prod_{k=1}^r \left\{ \det_N \left[\frac{1}{\lambda_{\ell_a^{(k-1)}}^{(k-1)} - \lambda_{\ell_b^{(k)}}^{(k)}} \right] \right\} \quad (\text{A.1})$$

in two ways.

The definition of $\mathcal{S}_N^{(r)}$ involves the numbers $\lambda_a^{(k)}$ defined as

$$\lambda_a^{(0)} = a - \frac{N+1}{2} \quad \text{and} \quad \lambda_a^{(k)} = a - \frac{N+1}{2} - \bar{v}_k \quad \text{for } k = 1, \dots, r. \quad (\text{A.2})$$

Above, we have adopted the convenient shorthand notation: for a vector $\boldsymbol{\eta}_k$ we denote

$$\bar{\boldsymbol{\eta}}_k = \sum_{s=1}^k \boldsymbol{\eta}_s. \quad (\text{A.3})$$

Finally, in (A.1), v_s and t_s are some auxiliary complex numbers such that

$$\Re(v_s) \notin \frac{1}{2} + \mathbb{Z} \quad \text{and} \quad \bar{t}_s \in [0; 2\pi], \quad s = 1, \dots, r-1. \quad (\text{A.4})$$

Furthermore, the two boundary sequences of integers read

$$\ell_a^{(0)} = \ell_a^{(r)} = a. \quad (\text{A.5})$$

The $(r-1) \times N$ -fold sums (A.1) defining $\mathcal{S}_N^{(r)}$ can be recast in two different ways. On the one hand, one can relate it to a Toeplitz matrix. On the other hand, one can represent it as a sum over particle-hole like excitations associated with each intermediate state arising in the expansion. The latter generates the multidimensional restricted sums (3.4) that arise in the context of form factor expansions of multi-point correlation functions.

A.1 Toeplitz determinant representation

Let $\mathcal{S}_{N;\ell_N^{(n)}}^{(n)}$ satisfy the induction

$$\mathcal{S}_{N;\ell_N^{(n+1)}}^{(n+1)} = \sum_{\substack{\ell_a^{(n)} \in \mathbb{Z} \\ \ell_1^{(n)} < \dots < \ell_N^{(n)}}} \prod_{a=1}^N \left\{ \frac{\sin[\pi \nu_{n+1}]}{\pi} e^{it_n(\ell_a^{(n)} - \frac{N+1}{2} - \bar{\nu}_n)} \right\} \cdot \det_N \left[\frac{1}{\ell_b^{(n)} - \ell_a^{(n+1)} + \nu_{n+1}} \right] \cdot \mathcal{S}_{N;\ell_N^{(n)}}^{(n)}, \quad (\text{A.6})$$

and be subject to the initialization condition

$$\mathcal{S}_{N;\ell_N^{(1)}}^{(1)} = \left(\frac{\sin[\pi \nu_1]}{\pi} \right)^N \cdot \det_N \left[\frac{1}{b - \ell_a^{(1)} + \nu_1} \right]. \quad (\text{A.7})$$

It is then readily seen that with $\ell_N^{(r)} = (1, \dots, N)$

$$\mathcal{S}_{N;\ell_N^{(r)}}^{(r)} = \mathcal{S}_N^{(r)}. \quad (\text{A.8})$$

Furthermore, we shall establish by induction on n that

$$\mathcal{S}_{N;\ell_N^{(n+1)}}^{(n+1)} = \prod_{s=1}^n \left\{ e^{-iN\bar{\nu}_s t_s} \right\} \cdot \prod_{a=1}^N \left\{ e^{i\bar{t}_n(\ell_a^{(n+1)} - \frac{N+1}{2})} \right\} \cdot \det_N \left[c_{\ell_a^{(n+1)} - b}[\chi_{n+1}] \right]. \quad (\text{A.9})$$

Above, given a function f on $[0; 2\pi]$, we have set

$$c_k[f] = \int_0^{2\pi} e^{-ik\theta} f(\theta) \cdot \frac{d\theta}{2\pi}. \quad (\text{A.10})$$

The function χ_n is expressed in terms of elementary jump-like Fisher–Hartwig symbols

$$\chi_n = \chi_{\nu_1, 0} \cdot \prod_{s=1}^{n-1} \chi_{\nu_{s+1}, \bar{t}_s} \quad (\text{A.11})$$

where

$$\chi_{\delta, \varphi}(\theta) = e^{i(\theta - \varphi + \pi)\delta} \left\{ \mathbf{1}_{[0; \varphi[} + e^{-2i\pi\delta} \mathbf{1}_{[\varphi; 2\pi[} \right\}. \quad (\text{A.12})$$

One can recast $\mathcal{S}_{N;\ell_N^{(1)}}^{(1)}$ in terms of a Toeplitz determinant. Indeed, one has

$$c_j[\chi_{\nu_1, 0}] = e^{-i\pi\nu_1} \int_0^{2\pi} e^{i(\nu_1 - j)\theta} \cdot \frac{d\theta}{2\pi} = \frac{e^{i\pi\nu_1} - e^{-i\pi\nu_1}}{i2\pi(\nu_1 - j)} = \frac{\sin[\pi\nu_1]}{\pi(\nu_1 - j)}. \quad (\text{A.13})$$

Thus, all in all,

$$\mathcal{S}_{N;\ell_N}^{(1)} = \det_N \left[c_{\ell_a^{(1)}-b} [\chi_{\nu_1,0}] \right], \quad (\text{A.14})$$

so that the induction hypothesis does indeed hold for $n = 1$. Then, assume it holds for some n . The antisymmetry of the determinants allows one to replace the summation over a fundamental simplex by one over the whole of \mathbb{Z}^N normalized by $1/N!$. Then, using the antisymmetry of the determinants, one can replace one of the determinants by $N!$ times the product of its diagonal entries. This ultimately yields:

$$\begin{aligned} \mathcal{S}_{N;\ell_N}^{(n+1)} &= \prod_{s=1}^n \left\{ e^{-iN\bar{\nu}_s t_s} \right\} \cdot \prod_{a=1}^N \left\{ e^{i\bar{t}_n(\ell_a^{(n+1)} - \frac{N+1}{2})} \right\} \\ &\quad \times \sum_{\ell_N^{(n)} \in \mathbb{Z}^N} \prod_{a=1}^N \left\{ \frac{\sin[\pi \nu_{n+1}]}{\pi} e^{i\bar{t}_n(\ell_a^{(n)} - \ell_a^{(n+1)})} \right\} \cdot \det_N \left[\frac{c_{\ell_b^{(n)}-b} [\chi_n]}{\ell_b^{(n)} - \ell_a^{(n+1)} + \nu_{n+1}} \right]. \end{aligned} \quad (\text{A.15})$$

Entering the sums into the lines of the determinant, one gets the representation:

$$\mathcal{S}_{N;\ell_N}^{(n+1)} = \prod_{s=1}^n \left\{ e^{-iN\bar{\nu}_s t_s} \right\} \cdot \prod_{a=1}^N \left\{ e^{i\bar{t}_n(\ell_a^{(n+1)} - \frac{N+1}{2})} \right\} \cdot \det_N \left[M_{\ell_a^{(n+1)}b}^{(n)} \right] \quad (\text{A.16})$$

where we have set

$$M_{ab}^{(n)} = \sum_{\ell \in \mathbb{Z}} \frac{\sin[\pi \nu_{n+1}]}{\pi} \cdot e^{i\bar{t}_n(\ell-a)} \cdot \frac{c_{\ell-b} [\chi_n]}{\ell-a+\nu_{n+1}} = \frac{\sin[\pi \nu_{n+1}]}{\pi} \cdot e^{-i\bar{t}_n a} \int_0^{2\pi} e^{ib\theta} \chi_n(\theta) \sum_{\ell \in \mathbb{Z}} \frac{e^{i\ell(\bar{t}_n-\theta)}}{\ell-a+\nu_{n+1}} \cdot \frac{d\theta}{2\pi}.$$

Using that, for $0 < t < 2\pi$

$$\sum_{\ell \in \mathbb{Z}} \frac{e^{i\ell t}}{\ell+a} = \frac{2i\pi e^{-iat}}{1-e^{-2i\pi a}}, \quad (\text{A.17})$$

one gets

$$\sum_{\ell \in \mathbb{Z}} \frac{e^{i\ell(\bar{t}_n-\theta)}}{\ell-a+\nu_{n+1}} = \frac{\pi e^{i\pi(\nu_{n+1}-a)}}{\sin[\pi(\nu_{n+1}-a)]} \cdot e^{-i(\nu_{n+1}-a)(\bar{t}_n-\theta)} \cdot \left\{ \mathbf{1}_{[0;\bar{t}_n]}(\theta) + e^{-2i\pi\nu_{n+1}} \mathbf{1}_{[\bar{t}_n;2\pi]}(\theta) \right\} \quad (\text{A.18})$$

leading to

$$M_{ab}^{(n)} = c_{a-b} [\chi_{n+1}], \quad (\text{A.19})$$

hence establishing the induction hypothesis at $n+1$. Then, the large- N asymptotic behavior of $\mathcal{S}_N^{(r)}$ can be obtained, say, from the results established in [8]:

$$\begin{aligned} \mathcal{S}_N^{(r)} &= \prod_{s=1}^{r-1} \left\{ e^{-iN t_s \bar{\nu}_s} \cdot e^{iN \kappa_{s+1} \bar{t}_s} \right\} \cdot \prod_{s=1}^r \left\{ \frac{G(1+\nu_s+\kappa_s, 1-\nu_s-\kappa_s)}{N(\nu_s+\kappa_s)^2} \right\} \\ &\quad \times \prod_{a \neq b}^r \left(1 - e^{i(\bar{t}_{a-1} - \bar{t}_{b-1})} \right)^{(\nu_a+\kappa_a)(\nu_b+\kappa_b)} \cdot (1 + o(1)), \end{aligned} \quad (\text{A.20})$$

where G is the Barnes function. Furthermore, $\kappa_r \in \mathbb{Z}^r$ is an integer valued vector such that κ_r maximises

$$\kappa_r \mapsto - \sum_{s=1}^r (\nu_s + \kappa_s)^2, \quad (\text{A.21})$$

under the constraint $\bar{\kappa}_r = 0$. Due to the hypothesis on ν_a , $a = 1, \dots, r$, the maximizer exists and is unique.

A.2 The form factor expansion representation

In order to get the form factor-like expansion, we relabel the integers $\ell_a^{(s)}$ in terms of the particle-hole like integers $\{p_a^{(s)}\}_1^{n^{(s)}}$ and $\{h_a^{(s)}\}_1^{n^{(s)}}$, for some $n^{(s)} = 0, \dots, N$. Namely, for any sequence $\ell_1^{(s)} < \dots < \ell_N^{(s)}$ we define the integer $n^{(s)}$ and the integers

$$p_1^{(s)} < \dots < p_{n^{(s)}}^{(s)} \quad \text{with} \quad p_a^{(s)} \in \mathbb{Z} \setminus \llbracket 1; N \rrbracket \quad \text{and} \quad h_1^{(s)} < \dots < h_{n^{(s)}}^{(s)} \quad \text{with} \quad h_a^{(s)} \in \llbracket 1; N \rrbracket \quad (\text{A.22})$$

as

$$\ell_a^{(s)} = a \quad \text{for } a \in \llbracket 1; N \rrbracket \setminus \{h_1^{(s)}, \dots, h_{n^{(s)}}^{(s)}\} \quad \text{and} \quad \ell_{h_a^{(s)}}^{(s)} = p_a^{(s)} \quad \text{for } a \in \llbracket 1; n^{(s)} \rrbracket. \quad (\text{A.23})$$

Then, after some algebra, one gets that

$$\begin{aligned} \det_N \left[\frac{1}{\lambda_{\ell_a^{(s-1)}}^{(s-1)} - \lambda_{\ell_b^{(s)}}^{(s)}} \right] \cdot \det_N^{-1} \left[\frac{1}{\lambda_a^{(s-1)} - \lambda_b^{(s)}} \right] &= \det_{n^{(s-1)}} \left[\frac{1}{\lambda_{h_a^{(s-1)}}^{(s-1)} - \lambda_{p_b^{(s-1)}}^{(s-1)}} \right] \cdot \det_{n^{(s)}} \left[\frac{1}{\lambda_{h_a^{(s)}}^{(s)} - \lambda_{p_b^{(s)}}^{(s)}} \right] \\ &\times \prod_{b=1}^{n^{(s-1)}} \prod_{a=1}^{n^{(s)}} \left\{ \frac{(p_b^{(s-1)} - h_a^{(s)} + \nu_s)(h_b^{(s-1)} - p_a^{(s)} + \nu_s)}{(h_b^{(s-1)} - h_a^{(s)} + \nu_s)(p_b^{(s-1)} - p_a^{(s)} + \nu_s)} \right\} \\ &\times \prod_{a=1}^{n^{(s)}} \left\{ \frac{\sin[\pi \nu_s]}{\pi} \Gamma \left(\begin{matrix} N+1-h_a^{(s)} + \nu_s, h_a^{(s)} - \nu_s, 1-p_a^{(s)} + \nu_s, N+1-p_a^{(s)} + i0^+ \\ N+1-h_a^{(s)}, h_a^{(s)}, 1-p_a^{(s)} + i0^+, N+1-p_a^{(s)} + \nu_s \end{matrix} \right) \right\} \\ &\times \prod_{a=1}^{n^{(s-1)}} \left\{ -\frac{\sin[\pi \nu_s]}{\pi} \Gamma \left(\begin{matrix} N+1-h_a^{(s-1)} - \nu_s, h_a^{(s-1)} + \nu_s, 1-p_a^{(s-1)} - \nu_s, N+1-p_a^{(s-1)} + i0^+ \\ N+1-h_a^{(s-1)}, h_a^{(s-1)}, 1-p_a^{(s-1)} + i0^+, N+1-p_a^{(s-1)} - \nu_s \end{matrix} \right) \right\}. \quad (\text{A.24}) \end{aligned}$$

Thus, the series expansion for \mathcal{S}_N takes the form

$$\begin{aligned} \mathcal{S}_N^{(r)} &= G_N(\{\nu_s\}_1^r; \{t_s\}_1^{r-1}) \cdot \prod_{a=1}^{r-1} \left\{ \sum_{n^{(s)}=0}^N \sum_{\substack{h_1^{(s)} < \dots < h_{n^{(s)}}^{(s)} \\ h_a^{(s)} \in \llbracket 1; N \rrbracket}} \sum_{\substack{p_1^{(s)} < \dots < p_{n^{(s)}}^{(s)} \\ p_a^{(s)} \in \mathbb{Z} \setminus \llbracket 1; N \rrbracket}} \right\} \prod_{s=1}^{r-1} \mathcal{H}_N(\{p_a^{(s)}\}_1^{n^{(s)}}; \{h_a^{(s)}\}_1^{n^{(s)}} \mid \nu_s, \nu_{s+1}; t_s) \\ &\times \prod_{s=1}^r \prod_{b=1}^{n^{(s-1)}} \prod_{a=1}^{n^{(s)}} \left\{ \frac{(p_b^{(s-1)} - h_a^{(s)} + \nu_s)(h_b^{(s-1)} - p_a^{(s)} + \nu_s)}{(h_b^{(s-1)} - h_a^{(s)} + \nu_s)(p_b^{(s-1)} - p_a^{(s)} + \nu_s)} \right\}, \quad (\text{A.25}) \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}_N(\{p_a\}_1^n; \{h_a\}_1^n \mid \nu, \eta; t) &= (-1)^n \cdot \left(\frac{\sin[\pi \nu]}{\pi} \cdot \frac{\sin[\pi \eta]}{\pi} \right)^n \det_n^2 \left[\frac{1}{p_a - h_b} \right] \prod_{a=1}^n \{e^{it(p_a - h_a)}\} \\ &\times \prod_{a=1}^n \Gamma \left(\begin{matrix} N+1-h_a + \nu, h_a - \nu, 1-p_a + \nu, N+1-p_a + i0^+ \\ N+1-h_a, h_a, 1-p_a + i0^+, N+1-p_a + \nu \end{matrix} \right) \\ &\times \prod_{a=1}^n \Gamma \left(\begin{matrix} N+1-h_a - \eta, h_a + \eta, 1-p_a - \eta, N+1-p_a + i0^+ \\ N+1-h_a, h_a, 1-p_a + i0^+, N+1-p_a - \eta \end{matrix} \right). \quad (\text{A.26}) \end{aligned}$$

In (A.25), $G_N(\{\nu_s\}_1^r; \{t_s\}_1^{r-1})$ is built out of products of so-called background form factor. These have been considered in [19]. In this free-fermionic setting, each such form factor can be computed explicitly in terms of Barnes

functions. More explicitly, one has

$$\begin{aligned}
G_N(\{\nu_s\}_1^r; \{t_s\}_1^{r-1}) &\equiv \prod_{s=1}^{r-1} \left\{ e^{-iNt_s \bar{\nu}_s} \right\} \prod_{s=1}^r \left\{ \left(\frac{\sin[\pi \nu_s]}{\pi} \right)^N \det_N \left[\frac{1}{a-b+\nu_s} \right] \right\} \\
&= \prod_{s=1}^{r-1} \left\{ e^{-iNt_s \bar{\nu}_s} \right\} \cdot \prod_{s=1}^r \left\{ G(1+\nu_s, 1-\nu_s) \right\} \cdot \prod_{s=1}^r \left\{ G \left(\begin{matrix} N+1, N+1 \\ N+1-\nu_s, N+1+\nu_s \end{matrix} \right) \right\} \\
&= \frac{\prod_{s=1}^{r-1} \left\{ e^{-iNt_s \bar{\nu}_s} \right\}}{\prod_{s=1}^r N \nu_s^2} \cdot \prod_{s=1}^{r-1} \left\{ G(1+\nu_s, 1-\nu_s) \right\} \cdot (1+o(1)) . \quad (\text{A.27})
\end{aligned}$$

Now we focus in more details on the sum over the integers $\{p_a^{(s)}\}_1^{n^{(s)}}$ and $\{h_a^{(s)}\}_1^{n^{(s)}}$. We will reorganize the sum into one over excitations of Umklapp type. This will allow us to identify clearly the leading oscillating power, hence making a comparison with the Fischer–Hartwig asymptotics. In the following, we agree upon

$$H_L = \left\{ 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor \right\} \quad , \quad H_R = \left\{ \left\lfloor \frac{N}{2} \right\rfloor + 1, \dots, N \right\} \quad , \quad P_L = -\mathbb{N}^* \quad \text{and} \quad P_R = \mathbb{N} + N . \quad (\text{A.28})$$

Above $[\cdot]$ stands for the floor function. It then follows that one can recast the various sums defining $\mathcal{S}_N^{(r)}$ as

$$\mathcal{S}_N^{(r)} = G_N(\{\nu_s\}_1^r; \{t_s\}_1^{r-1}) \prod_{s=1}^{r-1} \left\{ \sum_{n^{(s)}=0}^N \sum_{m^{(s)}=0}^{n^{(s)}} \sum_{r^{(s)}=0}^{n^{(s)}} \right\} f(\{m^{(s)}\}_1^{r-1}; \{r^{(s)}\}_1^{r-1}; \{n^{(s)}\}_1^{r-1}), \quad (\text{A.29})$$

where

$$\begin{aligned}
f(\{m^{(s)}\}_1^{r-1}; \{r^{(s)}\}_1^{r-1}; \{n^{(s)}\}_1^{r-1}) &= \prod_{s=1}^{r-1} \left\{ \sum_{\substack{h_1^{(s)} < \dots < h_{m^{(s)}}^{(s)} \\ h_a^{(s)} \in H_L}} \sum_{\substack{h_{m^{(s)}+1}^{(s)} < \dots < h_{n^{(s)}}^{(s)} \\ h_a^{(s)} \in H_R}} \sum_{\substack{p_1^{(s)} < \dots < p_{r^{(s)}}^{(s)} \\ p_a^{(s)} \in P_L}} \sum_{\substack{p_{r^{(s)}+1}^{(s)} < \dots < p_{n^{(s)}}^{(s)} \\ p_a^{(s)} \in P_R}} \right\} \\
&\times \prod_{s=1}^{r-1} \left\{ \mathcal{H}_N(\{p_a^{(s)}\}_1^{n^{(s)}}; \{h_a^{(s)}\}_1^{n^{(s)}} \mid \nu_s, \nu_{s+1}; t_s) \right\} \cdot \prod_{s=1}^r \prod_{b=1}^{n^{(s-1)}} \prod_{a=1}^{n^{(s)}} \left\{ \frac{(p_b^{(s-1)} - h_a^{(s)} + \nu_s)(h_b^{(s-1)} - p_a^{(s)} + \nu_s)}{(h_b^{(s-1)} - h_a^{(s)} + \nu_s)(p_b^{(s-1)} - p_a^{(s)} + \nu_s)} \right\} . \quad (\text{A.30})
\end{aligned}$$

It then remains to make several simplifications in each of these summands by using the fact that some terms, due to the largeness of N can be simplified. For this purpose, for each set of right or left type integers, we reparameterise the variables in terms of "small" ones by

$$p_a^{(s)} = 1 - p_{a;-}^{(s)} \quad \text{for} \quad a = 1, \dots, r^{(s)} \quad , \quad p_a^{(s)} = N + p_{a-r^{(s)};+}^{(s)} \quad \text{for} \quad a = r^{(s)} + 1, \dots, n^{(s)} \quad , \quad (\text{A.31})$$

and

$$h_a^{(s)} = h_{a;-}^{(s)} \quad \text{for} \quad a = 1, \dots, m^{(s)} \quad , \quad h_a^{(s)} = N - h_{a-m^{(s)};+}^{(s)} + 1 \quad \text{for} \quad a = 1 + m^{(s)}, \dots, n^{(s)} . \quad (\text{A.32})$$

Since we are solely interested in the leading $N \rightarrow +\infty$ asymptotics of each term subordinate to a choice of $n^{(s)}, r^{(s)}$ and $m^{(s)}$, we shall make several simplifying assumptions. First, we shall assume that we can extend the summation up to \mathbb{N}^* for all of the variables, this without altering the leading $N \rightarrow +\infty$ asymptotics. Then, we

shall work as if the series were convergent in a strong sense. This will allow us to treat the variables $\{p_{a;\pm}^{(s)}\}$ and $\{h_{a;\pm}^{(s)}\}$ as "small" in respect to N . Finally, expanding the summand into powers of N , we shall only keep the leading contribution.

A long but straightforward calculation then shows that $\mathcal{S}_N^{(r)}$ admits the following large- N behaviour

$$\mathcal{S}_N^{(r)} \simeq \prod_{s=1}^r \left\{ G(1 + \nu_s, 1 - \nu_s) \cdot N^{-\nu_s^2} \right\} \sum_{\substack{\ell_{r-1} \\ \in \mathbb{Z}^{r-1}}} \prod_{s=1}^{r-1} \left\{ e^{iN t_s (\ell_s - \bar{\nu}_s)} N^{-2\ell_s^2} N^{2(\nu_s - \nu_{s+1})\ell_s} \right\} \prod_{s=2}^{r-1} \left\{ N^{2\ell_s \ell_{s-1}} \right\} \\ \times \mathcal{S}_{\ell_{r-1}}^- (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) \cdot \mathcal{S}_{\ell_{r-1}}^+ (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r). \quad (\text{A.33})$$

Above, the functions $\mathcal{S}_{\ell_{r-1}}^\pm$ are as defined in (3.4).

A.3 Identification of the asymptotics and the summation formula

In order to identify the asymptotics one shall change the variables of summation in (A.33) according to (3.8)-(3.9). Upon using the identities (3.10) and (3.11), we are, all-in-all, led to

$$\mathcal{S}_N^{(r)} = \prod_{s=1}^{r-1} \left\{ e^{-iN t_s \bar{\nu}_s} \right\} \sum_{\substack{\kappa_r \in \mathbb{Z}^r \\ \sum_{s=1}^r \kappa_s = 0}} \prod_{s=1}^r \frac{G(1 + \nu_s, 1 - \nu_s)}{N^{(\nu_s + \kappa_s)^2}} \prod_{s=1}^{r-1} \left\{ e^{iN \kappa_{s+1} \bar{t}_s} \right\} \cdot \mathcal{S}_{\ell_{r-1}(\kappa_r)}^- (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) \cdot \mathcal{S}_{\ell_{r-1}(\kappa_r)}^+ (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r). \quad (\text{A.34})$$

The identification of the leading asymptotics of the Toeplitz determinant with the leading in N term of (A.33) yields

$$\mathcal{S}_{\ell_{r-1}(\kappa_r)}^- (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) \cdot \mathcal{S}_{\ell_{r-1}(\kappa_r)}^+ (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) = \prod_{s=1}^r \left\{ \frac{G(1 - \nu_s - \kappa_s) G(1 + \nu_s + \kappa_s)}{G(1 - \nu_s) G(1 + \nu_s)} \right\} \cdot \prod_{a \neq b}^r \left(1 - e^{i(\bar{t}_{a-1} - \bar{t}_{b-1})} \right)^{(\nu_a + \kappa_a)(\nu_b + \kappa_b)}.$$

The parts relative to $\mathcal{S}_{\ell_{r-1}(\kappa_r)}^- (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r)$ (resp. $\mathcal{S}_{\ell_{r-1}(\kappa_r)}^+ (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r)$) can be readily inferred by applying a Wiener-Hopf factorization on $\mathbb{C} \setminus \mathbb{R}$. In order to ensure uniqueness of such a factorization, one should fix the asymptotic behaviour of $\mathcal{S}_{\ell_{r-1}(\kappa_r)}^\pm (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r)$ at $t \rightarrow \infty$, this non-tangentially to \mathbb{R} and under the constraint $\pm \Im(t) > 0$. The latter are readily read-off from the explicit expressions (3.4) for $\mathcal{S}_{\ell_{r-1}(\kappa_r)}^\pm (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r)$. Namely

$$\mathcal{S}_{\ell_{r-1}(\kappa_r)}^+ (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) \underset{t_a \rightarrow +i\infty}{\sim} \prod_{s=1}^{r-1} \left\{ e^{it_s \frac{\ell_s(\ell_s+1)}{2}} G \left(\begin{matrix} 1 + \ell_s - \nu_s, 1 + \ell_s + \nu_{s+1} \\ 1 - \nu_s, 1 + \nu_{s+1} \end{matrix} \right) \right\} \cdot \prod_{s=2}^{r-1} G \left(\begin{matrix} 1 + \nu_s, 1 + \ell_{s-1} - \ell_s + \nu_s \\ 1 - \ell_s + \nu_s, 1 + \ell_{s-1} + \nu_s \end{matrix} \right)$$

and

$$\mathcal{S}_{\ell_{r-1}(\kappa_r)}^- (\{t_s\}_1^{r-1}, \{\nu_s\}_1^r) \underset{t_a \rightarrow -i\infty}{\sim} \prod_{s=1}^{r-1} \left\{ e^{-it_s \frac{\ell_s(\ell_s+1)}{2}} G \left(\begin{matrix} 1 - \ell_s + \nu_s, 1 - \ell_s - \nu_{s+1} \\ 1 + \nu_s, 1 - \nu_{s+1} \end{matrix} \right) \right\} \cdot \prod_{s=2}^{r-1} G \left(\begin{matrix} 1 - \nu_s, 1 - \ell_{s-1} + \ell_s - \nu_s \\ 1 - \ell_{s-1} - \nu_s, 1 + \ell_s - \nu_s \end{matrix} \right).$$

It is then readily checked that one has the decomposition

$$\prod_{s=1}^r \frac{G(1 - \nu_s - \kappa_s) G(1 + \nu_s + \kappa_s)}{G(1 - \nu_s) G(1 + \nu_s)} = \prod_{s=1}^{r-1} G \left(\begin{matrix} 1 + \ell_s - \nu_s, 1 + \ell_s + \nu_{s+1}, 1 - \ell_s + \nu_s, 1 - \ell_s - \nu_{s+1} \\ 1 - \nu_s, 1 + \nu_{s+1}, 1 + \nu_s, 1 - \nu_{s+1} \end{matrix} \right) \\ \times \prod_{s=2}^{r-1} G \left(\begin{matrix} 1 + \nu_s, 1 + \ell_{s-1} - \ell_s + \nu_s, 1 - \nu_s, 1 - \ell_{s-1} + \ell_s - \nu_s \\ 1 - \ell_s + \nu_s, 1 + \ell_{s-1} + \nu_s, 1 - \ell_{s-1} - \nu_s, 1 + \ell_s - \nu_s \end{matrix} \right). \quad (\text{A.35})$$

The latter was the last missing piece so as to obtain the formulae (3.7). ■

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