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ON A SUM INVOLVING THE EULER TOTIENT FUNCTION

J. WU

ABSTRACT. In this short note, we prove that

$$\frac{4}{\pi^2}x \log x + O(x) \leq \sum_{n \leq x} \varphi\left(\left[\frac{x}{n}\right]\right) \leq \left(\frac{1}{3} + \frac{4}{\pi^2}\right)x \log x + O(x),$$

for $x \rightarrow \infty$, where $\varphi(n)$ is the Euler totient function and $[t]$ is the integral part of real t . This improves recent results of Bordellès-Heyman-Shparlinski and of Dai-Pan.

1. INTRODUCTION

As usual, denote by $\varphi(n)$ the Euler totient function and by $[t]$ the integral part of real t . Very recently, Bordellès, Heyman and Shparlinski [1] studied the asymptotic behaviour of the summatory function

$$(1.1) \quad S(x) := \sum_{n \leq x} \varphi\left(\left[\frac{x}{n}\right]\right)$$

and proved that

$$(1.2) \quad S(x) \leq \left(\frac{1}{2} + \frac{3}{\pi^2}\right)x \log x + 4x + \frac{\sqrt{x} \log x}{4} + \sqrt{x} \quad (x \geq 3),$$

$$(1.3) \quad S(x) \geq \left(\frac{2629}{4009} \cdot \frac{6}{\pi^2} + o(1)\right)x \log x \quad (x \rightarrow \infty).$$

They also posed a question: Is it true that

$$(1.4) \quad S(x) = \left(\frac{6}{\pi^2} + o(1)\right)x \log x$$

as $x \rightarrow \infty$? Some numerical evidences were given.

The aim of this short note is to improve Bordellès, Heyman and Shparlinski's (1.2) and (1.3), by refining their method. Our result is as follows.

Theorem 1. *We have*

$$(1.5) \quad \frac{4}{\pi^2}x \log x + O(x) \leq S(x) \leq \left(\frac{1}{3} + \frac{4}{\pi^2}\right)x \log x + O(x)$$

for $x \rightarrow \infty$.

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For comparison, we have

$$\begin{cases} \frac{1}{2} + \frac{3}{\pi^2} \approx 0.80396, \\ \frac{2629}{4009} \frac{6}{\pi^2} \approx 0.39841, \end{cases} \quad \text{and} \quad \begin{cases} \frac{1}{3} + \frac{4}{\pi^2} \approx 0.73861, \\ \frac{4}{\pi^2} \approx 0.40528. \end{cases}$$

As indicated by Bordellès, Heyman and Shparlinski in [1], the proof of (1.2) is rather elementary. But the lower bound (1.3) uses a much deeper approach relying on the theory of exponential pairs. In particular, to obtain the numerically stronger result, they used the recently discovered exponent pair $(\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon)$ by Bourgain [2]. We infer the reader to [5, Chapter 3, page 31] for the definition of the exponent pair.

Let $\psi(t) := t - [t] - \frac{1}{2}$ and $\delta \in \{0, 1\}$. For $x \geq 2$ and $1 \leq D \leq x$, define

$$(1.6) \quad \mathfrak{S}_\delta(x, D) := \sum_{D < d \leq 2D} \varphi(d) \psi\left(\frac{x}{d + \delta}\right).$$

The key of Bordellès, Heyman and Shparlinski's method is to find θ as large as possible such that

$$(1.7) \quad \mathfrak{S}_\delta(x, x^\theta) \ll x.$$

They shew that $\theta = \frac{2629}{4009} \approx 0.65577$ is admissible and obtained the constant $\theta \frac{6}{\pi^2}$ in the lower bound (1.3).

Our improvements come from two simple observations:

- (a) Firstly we give a more careful treatment for the related exponential sum (see Proposition 2.1 below). This allows us to show that $\theta = \frac{2}{3} \approx 0.66666$ is admissible for (1.7) and to get a better lower bound. It is worth to note that our argument is simpler and that we do not need to use Bourgain's new exponent pair. In fact we only use the simplest exponent pair $(\frac{1}{2}, \frac{1}{2})$, i.e. van der Corput inequality.
- (b) Our argument about upper bound part is different from [1]. In order to improve upper bound (1.2), we introduce exponential sum technique to show that the constant in the upper bound can be improved to $1 - \theta + \theta \frac{6}{\pi^2}$ with $\theta = \frac{2}{3}$.

It is clear that Bordellès-Heyman-Shparlinski's conjecture is equivalent to (1.7) with $\theta = 1$. This seems rather difficult to prove, since this requires a square root cancellation in $\mathfrak{S}_\delta(x, x)$. (By (3.4) below, the trivial upper bound of $\mathfrak{S}_\delta(x, x)$ is $O(x^2)$.) However it should be possible to further improve slightly the constants in Theorem 1 by applying combinatorial identity on the Möbius function and more sophistic methods of multiple exponential sums (see [4, 6]).

When we redact our manuscript, we note that Lixia Dai & Hao Pan [3] also have remarked (b) above and obtained a better upper bound constant $\frac{1380}{4009} + \frac{2629}{4009} \frac{6}{\pi^2} \approx 0.74288$ than (1.2) by using Bordellès, Heyman and Shparlinski's $\theta = \frac{2629}{4009}$ value.

2. A KEY ESTIMATE

The aim of this section is to prove the following bound for $\mathfrak{S}_\delta(x, D)$, which plays a key role in the proof of Theorem 1.

Proposition 2.1. *Under the previous notation, we have*

$$(2.1) \quad \mathfrak{S}_\delta(x, D) \ll (x^\kappa D^{1+\lambda})^{1/(\kappa+1)} + x^\kappa D^{1-2\kappa+\lambda} \log x + x^{-1} D^3,$$

where (κ, λ) is an exponent pair.

2.1. Two preliminary lemmas.

In order to prove Proposition 2.1, we need two standard lemmas in the exponential sum theory. The first one is due to Vaaler (see [5, Theorem A.6] or [1, Lemma 4.1]).

Lemma 2.2. *For $x \geq 1$ and $H \geq 1$, we have*

$$\psi(x) = - \sum_{1 \leq |h| \leq H} \Phi\left(\frac{h}{H+1}\right) \frac{e(hx)}{2\pi i h} + R_H(x),$$

where $e(t) := e^{2\pi i t}$, $\Phi(t) := \pi t(1 - |t|) \cot(\pi t) + |t|$ and the error term $R_H(x)$ satisfies

$$|R_H(x)| \leq \frac{1}{2H+2} \sum_{0 \leq |h| \leq H} \left(1 - \frac{|h|}{H+1}\right) e(hx).$$

The second lemma is [5, Lemma 2.4].

Lemma 2.3. *Let $L(Q) := \sum_{j=1}^J C_j Q^{c_j} + \sum_{k=1}^K D_k Q^{-d_k}$, where $C_j, c_j, D_k, d_k > 0$. For any $Q \geq Q' > 0$, there exists some $Q_1 \in [Q', Q]$ such that*

$$L(Q_1) \ll \sum_{j=1}^J \sum_{k=1}^K (C_j^{d_k} D_k^{c_j})^{1/(c_j+d_k)} + \sum_{j=1}^J C_j Q'^{c_j} + \sum_{k=1}^K D_k Q^{-d_k}.$$

2.2. Proof of Proposition 2.1.

As usual, denote by $\mu(n)$ the Möbius function. By using the relation

$$\varphi(d) = \sum_{mn=d} \mu(m)n$$

and by a simple partial summation, we can write

$$\begin{aligned} \mathfrak{S}_\delta(x, D) &= \sum_{m \leq 2D} \mu(m) \sum_{D/m < n \leq 2D/m} n \psi\left(\frac{x}{mn + \delta}\right) \\ (2.2) \quad &\ll \sum_{m \leq 2D} \frac{D}{m} \max_{D/m < N \leq 2D/m} |\mathfrak{S}_\delta(x, D, m, N)|, \end{aligned}$$

where

$$\mathfrak{S}_\delta(x, D, m, N) := \sum_{D/m < n \leq N} \psi\left(\frac{x}{mn + \delta}\right).$$

With the help of Lemma 2.2 (separating the terms with $h = 0$ or $h \neq 0$ in $R_H(x)$) and noticing the fact that $0 < \Phi(t) < 1$ ($0 < |t| < 1$), we can derive

$$\mathfrak{S}_\delta(x, D, m, N) \ll \frac{N}{H} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{D/m < n \leq N} e\left(\frac{hx}{mn + \delta}\right) \right|$$

for $D/m < N \leq 2D/m$ and $1 \leq H \leq N$.

Applying the exponent pair (κ, λ) to the inner sum over n , we find that

$$\begin{aligned} \mathfrak{S}_\delta(x, D, m, N) &\ll \frac{N}{H} + \sum_{h \leq H} \frac{1}{h} \left\{ \left(\frac{hx}{mN^2} \right)^\kappa N^\lambda + \frac{mN^2}{hx} \right\} \\ &\ll H^{-1}N + x^\kappa H^\kappa m^{-\kappa} N^{-2\kappa+\lambda} + x^{-1}mN^2 \end{aligned}$$

for all $H \in [1, N]$. According to Lemma 2.3, there is a $H \in [1, N]$ such that

$$\begin{aligned} \mathfrak{S}_\delta(x, D, m, N) &\ll (x^\kappa m^{-\kappa} N^{-\kappa+\lambda})^{1/(\kappa+1)} + x^\kappa m^{-\kappa} N^{-2\kappa+\lambda} + x^{-1}mN^2 \\ &\ll (x^\kappa D^{-\kappa+\lambda} m^{-\lambda})^{1/(\kappa+1)} + x^\kappa D^{-2\kappa+\lambda} m^{\kappa-\lambda} + x^{-1}D^2 m^{-1}. \end{aligned}$$

Inserting it into (2.2), we obtain the required result.

3. PROOF OF THEOREM 1: UPPER BOUND

Let $1 \leq N < \sqrt{x}$ be a parameter to be chosen later. We write

$$(3.1) \quad S(x) = S_1(x) + S_2(x)$$

with

$$S_1(x) := \sum_{n \leq N} \varphi\left(\left[\frac{x}{n}\right]\right), \quad S_2(x) := \sum_{N < n \leq x} \varphi\left(\left[\frac{x}{n}\right]\right).$$

A. Upper bound of $S_1(x)$

We have trivially

$$(3.2) \quad S_1(x) \leq \sum_{n \leq N} \frac{x}{n} = x \log N + O(x).$$

B. Upper bound of $S_2(x)$

In order to bound $S_2(x)$, we put $d = [x/n]$. Then

$$x/n - 1 < d \leq x/n \quad \text{and} \quad x/(d+1) < n \leq x/d.$$

Thus

$$\begin{aligned} (3.3) \quad S_2(x) &= \sum_{d \leq x/N} \varphi(d) \sum_{x/(d+1) < n \leq x/d} 1 \\ &= \sum_{d \leq x/N} \varphi(d) \left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{x}{d+1} + \psi\left(\frac{x}{d+1}\right) \right) \\ &= x \sum_{d \leq x/N} \frac{\varphi(d)}{d(d+1)} + \sum_{d \leq x/N} \varphi(d) \left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right) \right). \end{aligned}$$

It is well known that

$$(3.4) \quad \sum_{n \leq x} \frac{\varphi(n)}{n^2} = \frac{6}{\pi^2} \log x + O(1), \quad \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

With the help of these, it follows that

$$(3.5) \quad \sum_{d \leq x} \frac{\varphi(d)}{d(d+1)} = \sum_{d \leq x} \frac{\varphi(d)}{d^2} \left\{ 1 + O\left(\frac{1}{d}\right) \right\} = \frac{6}{\pi^2} \log x + O(1)$$

and

$$(3.6) \quad \left| \sum_{d \leq \sqrt{x}} \varphi(d) \left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right) \right) \right| \leq 2 \sum_{d \leq \sqrt{x}} \varphi(d) \ll x.$$

It remains to bound

$$E(x, N) := \sum_{\sqrt{x} < d \leq x/N} \varphi(d) \left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right) \right).$$

Let $D_k := x/(2^k N)$ and let K be the integer such that $D_{K+1} < \sqrt{x} \leq D_K$. By a simple dyadic split and by Proposition 2.1 with $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ (see [5, Theorem 3.10 or 2.2]), it follows that

$$(3.7) \quad \begin{aligned} |E(x, N)| &\leq \sum_{1 \leq k \leq K+1} (|\mathfrak{S}_0(x, D_k)| + |\mathfrak{S}_1(x, D_k)|) \\ &\ll \sum_{1 \leq k \leq K+1} (x^{1/3} D_k + x^{1/2} D_k^{1/2} \log x + x^{-1} D_k^3) \\ &\ll x^{1/3} (x/N) + x^{1/2} (x/N)^{1/2} \log x + x^{-1} (x/N)^3 \\ &\ll x^{4/3} N^{-1} + x N^{-1/2} \log x + x^2 N^{-3}. \end{aligned}$$

Inserting (3.5), (3.6) and (3.7) into (3.3), we find

$$(3.8) \quad S_2(x) \leq \frac{6}{\pi^2} x \log(x/N) + O(x^{4/3} N^{-1} + x N^{-1/2} \log x + x^2 N^{-3}).$$

C. *End of the proof*

Inserting (3.2) and (3.8) into (3.1), we find that

$$S(x) \leq x \log N + \frac{6}{\pi^2} x \log(x/N) + O(x^{4/3} N^{-1} + x N^{-1/2} \log x + x^2 N^{-3} + x),$$

which becomes, with the optimal choice of $N = x^{1/3}$,

$$S(x) \leq \left(\frac{1}{3} + \frac{4}{\pi^2} \right) x \log x + O(x).$$

4. PROOF OF THEOREM 1: LOWER BOUND

Similar to (3.3), we can write

$$S(x) = \sum_{d \leq x} \varphi(d) \sum_{x/(d+1) < n \leq x/d} 1.$$

Let $\sqrt{x} \leq D \leq x$ be a parameter to be chosen later. Thus

$$\begin{aligned}
 S(x) &\geq \sum_{d \leq D} \varphi(d) \sum_{x/(d+1) < n \leq x/d} 1 \\
 (4.1) \quad &= x \sum_{d \leq D} \frac{\varphi(d)}{d(d+1)} + \sum_{d \leq D} \varphi(d) \left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right) \right) \\
 &= \frac{6}{\pi^2} x \log D + O(x) + R(x, D),
 \end{aligned}$$

where

$$R(x, D) := \sum_{d \leq D} \varphi(d) \left(\psi\left(\frac{x}{d+1}\right) - \psi\left(\frac{x}{d}\right) \right).$$

Let $D_k := D/2^k$ and let K be the integer such that $D_{K+1} < 1 \leq D_K$. By a simple dyadic split and by Proposition 2.1 with $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$, it follows that

$$\begin{aligned}
 |R(x, D)| &\leq \sum_{1 \leq k \leq K+1} (|\mathfrak{S}_0(x, D_k)| + |\mathfrak{S}_1(x, D_k)|) \\
 (4.2) \quad &\ll \sum_{1 \leq k \leq K+1} (x^{1/3} D_k + x^{1/2} D_k^{1/2} \log x + x^{-1} D_k^3) \\
 &\ll x^{1/3} D + x^{1/2} D^{1/2} \log x + x^{-1} D^3.
 \end{aligned}$$

Inserting (4.2) into (4.1), we find

$$S(x) \geq \frac{6}{\pi^2} x \log D + O(x^{1/3} D + x^{1/2} D^{1/2} \log x + x^{-1} D^3).$$

The required result follows from the optimal choice of $D = x^{2/3}$.

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