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THE SELBERG-DELANGE METHOD IN SHORT INTERVALS WITH SOME APPLICATIONS

Z. CUI, G.-S. LÜ & J. WU

ABSTRACT. In this paper, we establish a quite general mean value result of arithmetic functions over short intervals with the Selberg-Delange method and give some applications. In particular, we generalize Selberg's result on the distribution of integers with a given number of prime factors and Deshouillers-Dress-Tenenbaum's arcsin law on divisors to the short interval case.

1. INTRODUCTION

This is the second paper of our series on the Selberg-Delange method for short intervals [1]. The method was initially introduced by Selberg [19] to study the distribution of integers having a given number of prime factors, and subsequently further developed by Delange [2, 3]. Roughly speaking it applies to evaluate mean values of arithmetic functions whose associated Dirichlet series are close to *complex* powers of the Riemann ζ -function. An excellent exposition of the theory and applications can be found in [20, Chapters II.5 and II.6]. Recently Cui & Wu [1] generalized this method to short interval when the power is *positive real*. In this paper we shall consider the complex power case which cannot be plainly treated with the method in [1]. Our aim is two-fold. First, we establish a quite general mean value result of arithmetic functions over short intervals, which generalize and improve the main result of [1]. Second, we provide five arithmetic applications of our mean value result on :

- Distribution of integers having a given number of prime factors in short intervals,
- Deshouillers-Dress-Tenenbaum arcsin law on divisors in short intervals,
- Divisor problem for $\tau_k(n)$ in short intervals.
- Mean values of $1/\tau_k(n)$ over short intervals.

We shall proceed along the same line of argument as in [1]. Its origin can be found in [20, Chapters II.5 and II.6].

1.1. Statement of main results.

Let $f(n)$ be an arithmetic function and let its Dirichlet series be defined by

$$(1.1) \quad \mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}.$$

Let $z \in \mathbb{C}$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $C > 0$, $M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s)$ defined as in (1.1) is said to be of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$ if the following conditions are verified:

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(a) for any $\varepsilon > 0$ we have

$$(1.2) \quad |f(n)| \ll_{\varepsilon} Mn^{\varepsilon} \quad (n \geq 1),$$

where the implied constant depends only on ε ;

(b) we have

$$\sum_{n=1}^{\infty} |f(n)|n^{-\sigma} \leq M(\sigma - 1)^{-\alpha} \quad (\sigma > 1);$$

(c) the Dirichlet series

$$(1.3) \quad \mathcal{G}(s; z, w) := \mathcal{F}(s)\zeta(s)^{-z}\zeta(2s)^{-w}$$

can be analytically continued to a holomorphic function in (some open set containing $\sigma \geq \frac{1}{2}$ and, in this region, $\mathcal{G}(s; z, w)$ satisfies the bound

$$(1.4) \quad |\mathcal{G}(s; z, w)| \leq M(|\tau| + 1)^{\max\{\delta(1-\sigma), 0\}} \log^A(|\tau| + 1)$$

uniformly for $|z| \leq B$ and $|w| \leq C$, where and in the sequel we implicitly define the real numbers σ and τ by the relation $s = \sigma + i\tau$ and choose the principal value of the complex logarithm.

Our first aim of this paper is to establish, under the previous assumptions, an asymptotic formula of

$$(1.5) \quad \sum_{x < n \leq x + x^{\theta}} f(n)$$

with $\theta \in (0, 1]$ as small as possible. In view of the zero-free region of Vinogradov for $\zeta(s)$ (see [20, page 161]), it seems rather difficult to prove such a result. One of our principal tools is Huxley's estimation on the zero density of the Riemann ζ -function. As usual, we denote by $N(\sigma, T)$ the number of zeros of $\zeta(s)$ in the region $\Re s \geq \sigma$ and $|\Im s| \leq T$. It is well known that there are two constants ψ and η such that

$$(1.6) \quad N(\sigma, T) \ll T^{\psi(1-\sigma)}(\log T)^{\eta}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 2$. Huxley [7] showed that

$$(1.7) \quad \psi = \frac{12}{5} \quad \text{and} \quad \eta = 9$$

are admissible. The zero density hypothesis is stated as

$$(1.8) \quad \psi = 2.$$

Combining (1.7) with the explicit formula (see [20, page 177]), Huxley derived his well known prime number theorem in short intervals [7] : for any $\theta \in (\frac{7}{12}, 1]$ and $y = x^{\theta}$, the asymptotic formula

$$(1.9) \quad \sum_{x < p \leq x+y} 1 \sim \frac{y}{\log x}$$

holds as $x \rightarrow \infty$. Corresponding to (1.9), Motohashi [13] proved the following result for the Möbius function $\mu(n)$: For any $\theta > \frac{7}{12}$ and $y = x^{\theta}$, the inequality

$$(1.10) \quad \sum_{x < n \leq x+y} \mu(n) = o(y)$$

holds as $x \rightarrow \infty$. Independently Ramachandra [14] obtained a better result :

$$(1.11) \quad \sum_{x < n \leq x+y} \mu(n) \ll_{A,\theta} \frac{y}{(\log x)^A}$$

For each $A > 0$. Their methods are similar. Our approach is a generalization and refinement of Motohashi's method [13]. The first key of this method is to construct a contour \mathcal{M}_T (see Section 2 below for its precise definition) in the critical strip such that for any $\varepsilon > 0$ we have

$$(1.12) \quad (|\tau| + 1)^{-\varepsilon(1-\sigma)} \ll_{\varepsilon} |\zeta(s)| \ll_{\varepsilon} (|\tau| + 1)^{\varepsilon(1-\sigma)}$$

for $s \in \mathcal{M}_T$. The second key is a very good bound for the density of “*small value points*” (i.e. satisfying (2.6) below), which was established by adapting Montgomery's new method to study the zero-densities of the Riemann ζ -function and of the Dirichlet L -functions [12]. With these two nice ideas and Huxley's zero density estimation, we establish a general asymptotic formula for the summatory function (1.5), see Theorem 1.1 below. It is worthy to point out that our Theorem 1.1 allows us to unify the treatment of (1.9) and (1.10); indeed the latter is a particular case of the former.

In order to state our main result, it is necessary to introduce some more notation. From [20, Theorem II.5.1], the function* $Z(s; z) := \{(s-1)\zeta(s)\}^z$ ($z \in \mathbb{C}$) is holomorphic in the disc $|s-1| < 1$, and admits, in the same disc, the Taylor series expansion

$$Z(s; z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z)}{j!} (s-1)^j,$$

where the $\gamma_j(z)$'s are entire functions of z satisfying the estimate

$$(1.13) \quad \frac{\gamma_j(z)}{j!} \ll_{B,\varepsilon} (1+\varepsilon)^j \quad (j \geq 0, |z| \leq B)$$

for all $B > 0$ and $\varepsilon > 0$. Under our hypothesis, the function $\mathcal{G}(s; z, w)\zeta(2s)^w Z(s; z)$ is holomorphic in the disc $|s-1| < \frac{1}{2}$ and

$$(1.14) \quad |\mathcal{G}(s; z, w)\zeta(2s)^w Z(s; z)| \ll_{A,B,C,\delta,\varepsilon} M$$

for $|s-1| \leq \frac{1}{2} - \varepsilon$, $|z| \leq B$ and $|w| \leq C$. Thus for $|s-1| < \frac{1}{2}$, we can write

$$(1.15) \quad \mathcal{G}(s; z, w)\zeta(2s)^w Z(s; z) = \sum_{\ell=0}^{\infty} g_{\ell}(z, w)(s-1)^{\ell},$$

where

$$(1.16) \quad g_{\ell}(z, w) := \frac{1}{\ell!} \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\partial^{\ell-j}(\mathcal{G}(s; z, w)\zeta(2s)^w)}{\partial s^{\ell-j}} \Big|_{s=1} \gamma_j(z).$$

The main result of this paper is as follows.

*In [20], $Z(s; z)$ is defined as $s^{-1}\{(s-1)\zeta(s)\}^z$ but obviously the argument of the proof there works for our $Z(s; z)$.

Theorem 1.1. *Let $z \in \mathbb{C}$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $C > 0$, $M > 0$ be some constants. Suppose that the Dirichlet series*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$$

is of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$. Then for any $\varepsilon > 0$, we have

$$(1.17) \quad \sum_{x < n \leq x+y} f(n) = y(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_{\ell}(z, w)}{(\log x)^{\ell}} + O(MR_N(x, y)) \right\}$$

uniformly for $x \geq 3$, $x^{1-1/(\psi+\delta)+\varepsilon} \leq y \leq x$, $N \geq 0$, $|z| \leq B$ and $|w| \leq C$, where $\lambda_{\ell}(z, w) := g_{\ell}(z, w)/\Gamma(z - \ell)$ and

$$(1.18) \quad R_N(x, y) := \frac{(c_1 N + 1)^{N+1}}{(\log x)^{N+1} + e^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}}$$

for some constants $c_1 > 0$ and $c_2 > 0$ depending only on B, C, δ and ε . The implied constant in the O -term depends only on A, B, C, α, δ and ε . In particular $\psi = \frac{12}{5}$ is admissible.

The admissible length of short intervals in Theorem 1.1 depends only on the zero density constant ψ of $\zeta(s)$ and δ in (1.4) (for which we take $\delta = 0$ in most applications). Its independence from the power z of $\zeta(s)$ in the representation of $\mathcal{F}(s)$ seems interesting. Theorem 1.1 generalizes and improves [1, Theorem 1] to the case of complex powers and intervals of shorter length.

Taking $N = 0$ in Theorem 1.1, we obtain readily the following corollary.

Corollary 1.2. *Under the conditions of Theorem 1.1, for any $\varepsilon > 0$, we have*

$$(1.19) \quad \sum_{x < n \leq x+y} f(n) = y(\log x)^{z-1} \left\{ \lambda_0(z, w) + O\left(\frac{M}{\log x}\right) \right\}$$

uniformly for $x \geq 2$, $x^{1-1/(\psi+\delta)+\varepsilon} \leq y \leq x$, $|z| \leq B$ and $|w| \leq C$, where

$$\lambda_0(z, w) := \frac{\mathcal{G}(1; z, w)\zeta(2)^w}{\Gamma(z)}$$

and the implied constant in the O -term depends only on A, B, α, δ and ε . Note that $\psi = \frac{12}{5}$ is admissible.

Taking $f(n) = \mu(n)$ in Theorem 1.1, we have $z = -1$, $w = 0$ and $\mathcal{G}(s; z, w) \equiv 1$, $\delta = 0$, $\psi = \frac{12}{5}$, $\lambda(-1, \ell) = 0$ for all integers $\ell \geq 0$. Thus we can choose $N = \lceil c'(\log x)^{1/3}(\log_2 x)^{-4/3} \rceil$ with some small constant $c' > 0$ to obtain an improvement of Motohashi's result (1.10): For any $\theta > \frac{7}{12}$, we have

$$\sum_{x < n \leq x+y} \mu(n) \ll y e^{-c(\log x)^{1/3}(\log_2 x)^{-1/3}}$$

uniformly for $x \geq 2$ and $x^{\theta} \leq y \leq x$, where $c > 0$ is a constant depending on θ .

1.2. Integers having a fixed number of prime factors.

Denote by $\omega(n)$ (resp. $\Omega(n)$) the number of distinct (resp. all) prime factors of n . For each positive integer $k \geq 1$, consider

$$(1.20) \quad \pi_k(x) := |\{n \leq x : \omega(n) = k\}|$$

$$(1.21) \quad N_k(x) := |\{n \leq x : \Omega(n) = k\}|.$$

In 1909, Landau [11] proved by induction that for each *fixed* positive integer k , the following asymptotic formulas

$$\pi_k(x), N_k(x) \sim \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \quad (x \rightarrow \infty)$$

hold, where \log_ℓ denotes the ℓ -fold iterated logarithm. However, if we allow k to grow with x , the method by induction will become too technical (see [17, 18]). In [19], Selberg proposed a new and very elegant approach to attack this problem – identifying $\pi_k(x)$ with the coefficient of z^k in the expression $\sum_{n \leq x} z^{\omega(n)}$ and then applying Cauchy's integral formula. Through a detailed study of the sum over z , he proved that for any fixed constant $B > 0$ the asymptotic formula

$$(1.22) \quad \pi_k(x) = \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \lambda\left(\frac{k-1}{\log_2 x}\right) + O_B\left(\frac{k}{(\log_2 x)^2}\right) \right\}$$

holds uniformly for $x \geq 3$ and $1 \leq k \leq B \log_2 x$, where

$$(1.23) \quad \lambda(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z$$

and the implied constant depends only on B . In the same fashion, Selberg also proved that for any $\delta \in (0, 1)$, the asymptotic formula

$$(1.24) \quad N_k(x) = \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \nu\left(\frac{k-1}{\log_2 x}\right) + O_\delta\left(\frac{k}{(\log_2 x)^2}\right) \right\}$$

holds uniformly for $x \geq 3$ and $1 \leq k \leq (2 - \delta) \log_2 x$, where

$$(1.25) \quad \nu(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z$$

and the implied constant depends only on δ .

As the first application of Theorem 1.1, we shall generalize Selberg's results (1.22) and (1.24) to the short interval case.

Theorem 1.3. *Let $B > 0$. There exist positive constants $c_1 = c_1(B)$ and $c_2 = c_2(B)$ such that for any $\varepsilon > 0$, we have*

$$(1.26) \quad \pi_k(x+y) - \pi_k(x) = \frac{y}{\log x} \left\{ \sum_{j=0}^N \frac{P_{j,k}(\log_2 x)}{(\log x)^j} + O_{B,\varepsilon} \left(\frac{(\log_2 x)^k}{k!} R_N(x,y) \right) \right\}$$

uniformly for $x \geq 3$, $x^{1-1/\psi+\varepsilon} \leq y \leq x$ and $1 \leq k \leq B \log_2 x$, where $P_{j,k}(X)$ is a polynomial of degree at most $k-1$ and $R_N(x,y)$ is defined as in (1.18). Here c_1, c_2

in the definition of $R_N(x, y)$ and the implied constant depend on B and ε only. In particular, we have

$$P_{0,k}(X) = \sum_{m+\ell=k-1} \frac{\lambda^{(m)}(0)}{\ell!m!} X^\ell.$$

Moreover, under the same conditions, we have

$$(1.27) \quad \pi_k(x+y) - \pi_k(x) = \frac{y}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \lambda\left(\frac{k-1}{\log_2 x}\right) + O\left(\frac{k}{(\log_2 x)^2}\right) \right\}.$$

In particular $\psi = \frac{12}{5}$ is admissible in both assertions (1.26) and (1.27).

Theorem 1.4. *There exist absolute positive constants c_1 and c_2 such that for any $\varepsilon > 0$, we have*

$$(1.28) \quad N_k(x+y) - N_k(x) = \frac{y}{\log x} \left\{ \sum_{j=0}^N \frac{Q_{j,k}(\log_2 x)}{(\log x)^j} + O_{B,\varepsilon} \left(\frac{(\log_2 x)^k}{k!} R_N(x, y) \right) \right\}$$

uniformly for $x \geq 3$, $x^{1-1/\psi+\varepsilon} \leq y \leq x$ and $1 \leq k \leq \log_2 x$, where $Q_{j,k}(X)$ is a polynomial of degree at most $k-1$ and $R_N(x, y)$ is defined as in (1.18). Here c_1, c_2 in the definition of $R_N(x, y)$ and the implied constant depend on ε only. In particular, we have

$$Q_{0,k}(X) = \sum_{m+\ell=k-1} \frac{\nu^{(m)}(0)}{\ell!m!} X^\ell.$$

Moreover, under the same conditions, we have

$$(1.29) \quad N_k(x+y) - N_k(x) = \frac{y}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ \nu\left(\frac{k-1}{\log_2 x}\right) + O\left(\frac{k}{(\log_2 x)^2}\right) \right\}.$$

In particular $\psi = \frac{12}{5}$ is admissible in both assertions (1.28) and (1.29).

Remark 1. Kátai [9] applied Ramachandra's theorem [14] to obtain

$$\pi_k(x+y) - \pi_k(x) = \{1 + o(1)\} \frac{y}{\log x} \frac{(\log_2)^{k-1}}{(k-1)!}$$

uniformly for any $k \leq \log_2 x + c_x \sqrt{\log_2 x}$, where $c_x \rightarrow \infty$ sufficiently slowly, and $y \geq x^{1-1/\psi+\varepsilon}$. Clearly Theorem 1.3 improves Kátai's result in two directions: get a more precise asymptotic formula and extend domain of k .

Taking $k = 1$, we obtain Huxley's well known prime number theorem in short intervals (1.9).

1.3. The Deshouillers-Dress-Tenenbaum arcsin law on divisors.

For each positive integer n , denote by $\tau(n)$ the number of divisors of n and define the random variable D_n which takes the value $(\log d)/\log n$, as d runs through the set of the $\tau(n)$ divisors of n , with the uniform probability $1/\tau(n)$. The distribution function F_n of D_n is given by

$$F_n(t) = \text{Prob}(D_n \leq t) = \frac{1}{\tau(n)} \sum_{d|n, d \leq n^t} 1 \quad (0 \leq t \leq 1).$$

It is clear that the sequence $\{F_n\}_{n \geq 1}$ does not converge pointwisely on $[0, 1]$. However Deshouillers, Dress & Tenenbaum ([4] or [20, Theorem II.6.7]) proved that its Cesàro mean converges uniformly to the arcsin law. More precisely, they showed that the asymptotic formula

$$\frac{1}{x} \sum_{n \leq x} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right)$$

holds uniformly for $x \geq 2$ and $0 \leq t \leq 1$, and that the error term is optimal. Very recently Cui & Wu [1, Theorem 2] established a short interval version of this result: For $\varepsilon > 0$, we have

$$(1.30) \quad \frac{1}{y} \sum_{x < n \leq x+y} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O_\varepsilon\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $0 \leq t \leq 1$, $x \geq 2$ and $x^{62/77+\varepsilon} \leq y \leq x$, where the implied constant depends only on ε .

Our third application of Theorem 1.1 is to improve the exponent in (1.30).

Theorem 1.5. *For any $\varepsilon > 0$, the asymptotic formula (1.30) holds uniformly for $0 \leq t \leq 1$, $x \geq 2$ and $x^{19/24+\varepsilon} \leq y \leq x$, where the implied constant depends on ε only.*

For comparison, we have $\frac{62}{77} = 0.805\dots$ and $\frac{19}{24} = 0.791\dots$

1.4. Divisor problem for $\tau_k(n)$ on short intervals.

As usual, denote by $\Delta_k(x)$ the error term in the asymptotic formula for the k -dimension divisor problem:

$$D_k(x) := \sum_{n \leq x} \tau_k(n) = xP_{k-1}(\log x) + \Delta_k(x),$$

where $P_{k-1}(t)$ is a polynomial of degree $k-1$ with leading coefficient $1/(k-1)!$. The best known result for $\Delta_k(x)$ for $k \geq 4$ is as follows :

$$(1.31) \quad \Delta_4(x) \ll x^{1/2}(\log x)^5, \quad \Delta_k(x) \ll_{k,\varepsilon} x^{\theta_k+\varepsilon} \quad (k \geq 5)$$

with $\theta_k = \frac{3}{4} - \frac{1}{k}$ ($5 \leq k \leq 8$), $\theta_9 = \frac{35}{54}$, $\theta_{10} = \frac{41}{60}$, $\theta_{11} = \frac{7}{10}$, $\theta_k = \frac{k-1}{k+2}$ ($12 \leq k \leq 25$), $\theta_k = \frac{k-1}{k+4}$ ($26 \leq k \leq 50$), $\theta_k = \frac{31k-98}{32k}$ ($51 \leq k \leq 57$), $\theta_k = \frac{7k-34}{7k}$ ($k \geq 58$), where ε is an arbitrarily small positive number (see [5, (1.3)] for $k=4$ and [8, Theorem 12.3] for $k \geq 5$.) In 2006, Garaev, Luca and Nowak [5] considered the divisor problem for $\tau_4(n)$ in short intervals and proved that

$$\sum_{x < n \leq x+y} \tau_4(n) = \frac{1}{6}y(\log x)^3 \left\{ 1 + O\left(\left(\frac{\sqrt{x} \log x}{y}\right)^{2/3}\right) \right\}$$

for $x \geq 3$ and $x^{1/2} \log x \leq y \leq x^{1/2}(\log x)^{5/2}$. They also emphasized that for no other dimension $k \neq 4$ short interval results are known for the sum over $\tau_k(n)$ that are sharper than what is immediate from the (“long interval”) asymptotics for $D_k(x)$ (see [5, Remark]). The next theorem gives a such result for all integers $k \geq 7$.

Theorem 1.6. *Let $k \geq 7$ be a positive integer and $\varepsilon > 0$ be an arbitrarily small positive number. Then there is a positive constant c depending on k and ε such that the asymptotic formula*

$$\sum_{x < n \leq x+y} \tau_k(n) = yQ_{k-1}(\log x) \left\{ 1 + O_{k,\varepsilon} \left(e^{-c(\log x)^{1/3}(\log_2 x)^{-1/3}} \right) \right\}$$

holds uniformly for $x \geq 2$ and $x^{1-1/\psi+\varepsilon} \leq y \leq x$, where $Q_{k-1}(t)$ is a polynomial of degree $k-1$ with leading coefficient $1/(k-1)!$ and the implied constant depends only on k and ε . In particular $\psi = \frac{12}{5}$ is admissible.

It is interesting to note that the exponent (1.31) tends to 1 as $k \rightarrow \infty$, and that the length of short intervals in Theorem 1.6 is independent of k .

1.5. The mean value of $1/\tau_k(n)$ on short intervals.

Recently Sedunova [16] considered mean values of the following arithmetic functions over short intervals: $\tau_k(n)^{-1}$, $\sigma(n)/\tau(n)$, $r(n)^{-1}$, where $\tau_k(n) := \sum_{d|n} \tau_{k-1}(d)$, $\sigma(n) := \sum_{d|n} d$ and $r(n) := |\{(n_1, n_2) \in \mathbb{Z}^2 : n_1^2 + n_2^2 = n\}|$. In particular she proved that for any fixed integer $N \geq 0$ the asymptotic formula

$$(1.32) \quad \sum_{x < n \leq x+y} \frac{1}{\tau_k(n)} = \frac{y}{\sqrt{\log x}} \left\{ \sum_{\ell=0}^N \frac{a_\ell(k)}{(\log x)^\ell} + O_{k,N} \left(\frac{1}{(\log x)^{N+1}} \right) \right\}$$

holds uniformly for $x \geq 3$ and $x^{(21k+5)/(36k+5)} e^{(\log x)^{0.1}} \leq y \leq x$, where the $a_\ell(k)$ are some constants depending on k (see [16, Theorem 1]).

The fourth application of Theorem 1.1 is the following result.

Theorem 1.7. *For any $\varepsilon > 0$, the asymptotic formula (1.32) holds uniformly for $x \geq 2$ and $x^{7/12+\varepsilon} \leq y \leq x$, where the implied constant depends only on k , N and ε .*

Since $\frac{287k+64}{492k+64} \rightarrow \frac{7}{12}$ decreasingly as $k \rightarrow \infty$, Theorem 1.7 improves Sedunova's (1.32) for all k . It is worthy to note that our exponent is independent of k . Clearly the other results in [16, Theorems 2-7] can also be improved by Theorem 1.1 or its method of proof.

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2. MOTOHASHI METHOD

This section is devoted to depict Motohashi's method [13]. His original presentation is rather sketchy. Some key estimations (see Lemma 2.1 and Proposition 2.4 below) are outlined without many details. Here we would give a complete and detailed presentation for the sake of readers' convenience and the importance of this method.

2.1. Hooley-Huxley-Motohashi's contour \mathcal{M}_T .

Let ε be an arbitrarily small positive constant and let $T_0 = T_0(\varepsilon)$ be a large constant depending on ε only. As in [13], for $T \geq T_0$, put

$$(2.1) \quad \delta_T := C_0(\log T)^{-2/3}(\log_2 T)^{-1/3},$$

where C_0 is a suitable positive constant such that

$$(2.2) \quad (\log |\tau|)^{-2/3}(\log_2 |\tau|)^{-1/3} \ll |\zeta(s)| \ll (\log |\tau|)^{2/3}(\log_2 |\tau|)^{1/3}$$

for $\sigma \geq 1 - 100\delta_T$ and $1 \leq |\tau| \leq 100T$ (see [20, page 162]).

For $T \geq T_0$, write

$$(2.3) \quad J_T := [(\frac{1}{2} - \delta_T) \log T] \quad \text{and} \quad K_T := [T(\log T)^{-1}].$$

For each pair of integers (j, k) with $0 \leq j \leq J_T$ and $|k| \leq K_T$, we define

$$(2.4) \quad \Delta_{j,k} := \{s = \sigma + i\tau : \sigma_j \leq \sigma < \sigma_{j+1} \text{ and } \tau_k \leq \tau < \tau_{k+1}\},$$

where

$$(2.5) \quad \sigma_j := \frac{1}{2} + j(\log T)^{-1} \quad \text{and} \quad \tau_k := k \log T.$$

We divide $\Delta_{j,k}$ into two classes (W) and (Y) as follows.

- $\sigma_j \leq 1 - \varepsilon$: Then $\Delta_{j,k} \in (W)$ if $\Delta_{j,k}$ contains at least one zero of $\zeta(s)$, and $\Delta_{j,k} \in (Y)$ otherwise;

- $1 - \varepsilon < \sigma_j \leq 1 - \delta_T$: $\Delta_{j,k} \in (W)$ if and only if \exists at least one $s \in \Delta_{j,k}$ such that

$$(2.6) \quad |\zeta(s)M_{N_j}(s)| < \frac{1}{2}$$

with

$$(2.7) \quad \begin{cases} A' := \text{a fix large integer} \\ N_j := (A'(\log T)^5 \max_{\sigma \geq 4\sigma_j - 3, 1 \leq |\tau| \leq 4T} |\zeta(s)|)^{1/2(1-\sigma_j)} \\ M_{N_j}(s) := \sum_{n \leq N_j} \mu(n)n^{-s} \end{cases}$$

and $\Delta_{j,k} \in (Y)$ if and only if for all $s \in \Delta_{j,k}$

$$(2.8) \quad |\zeta(s)M_{N_j}(s)| \geq \frac{1}{2}.$$

For each k , we define

$$j_k := \begin{cases} \max_{\Delta_{j,k} \in (W)} j & \text{if } \exists j \text{ such that } \Delta_{j,k} \in (W), \\ 0 & \text{otherwise.} \end{cases}$$

Put

$$(2.9) \quad \mathcal{D}' := \bigcup_{0 \leq k \leq K_T} \bigcup_{0 \leq j \leq j_k} \Delta_{j,k}, \quad \mathcal{D}_0 := \bigcup_{0 \leq k \leq K_T} \bigcup_{j_k < j \leq J_T} \Delta_{j,k}$$

Clearly \mathcal{D}_0 consists of $\Delta_{j,k}$ of type (Y) only.

Hooley-Huxley-Motohashi's contour \mathcal{M}_T is symmetric about the real axis. Its upper part is the path in \mathcal{D}_0 consisting of horizontal and vertical line segments whose distances away from \mathcal{D}' are respectively d_h and d_v , given by

$$(2.10) \quad d_h := \log_2 T, \quad d_v := \begin{cases} \varepsilon^2 & \text{if } \sigma \leq 1 - \varepsilon, \\ (\log T)^{-1} & \text{if } 1 - \varepsilon < \sigma < 1 - \delta_T. \end{cases}$$

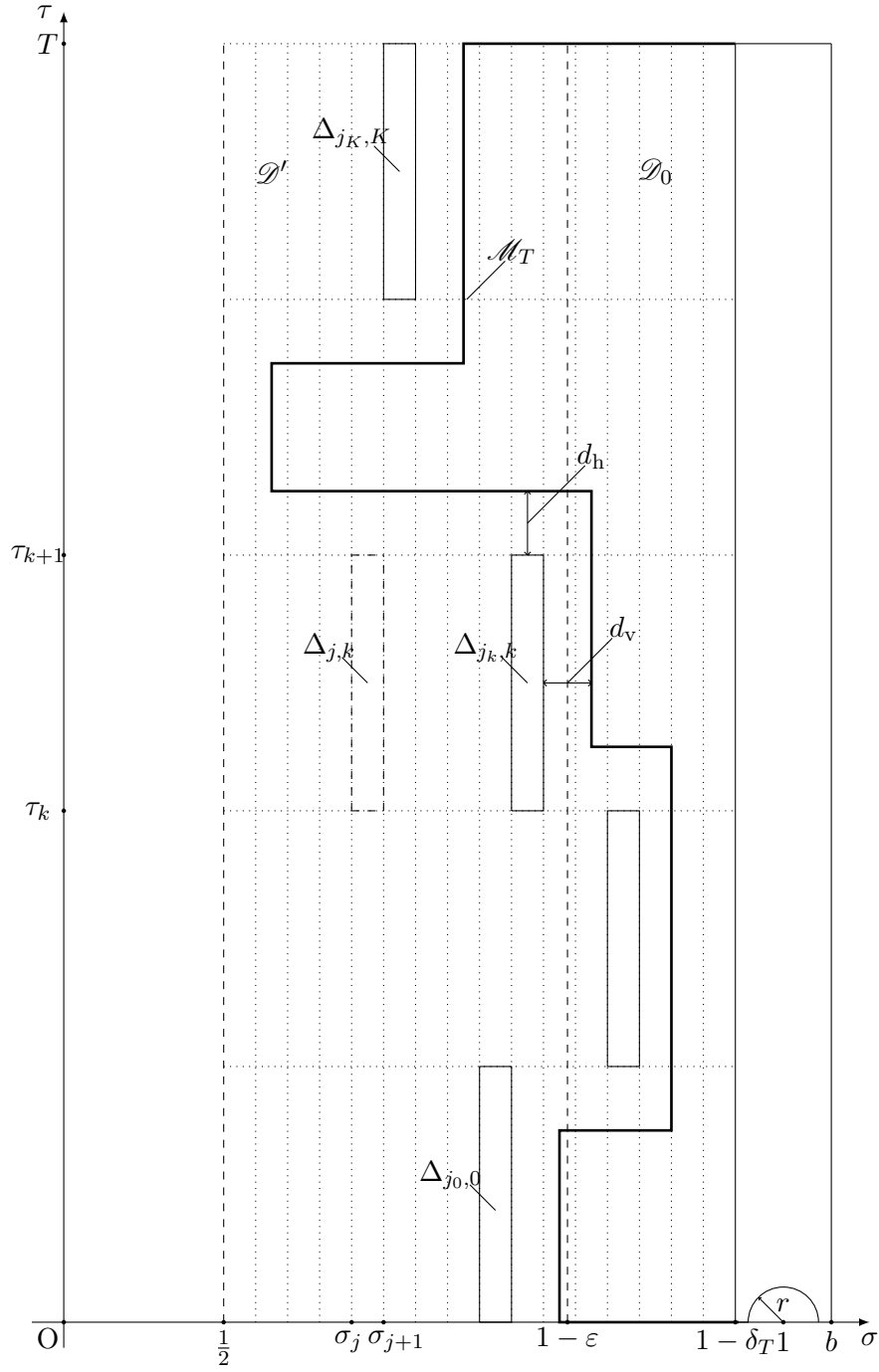


Figure 1 – Superior part of the contour \mathcal{M}_T

2.2. Lower and upper bounds of $\zeta(s)$ on \mathcal{M}_T .

In this subsection we give bounds to $\zeta(s)$ on \mathcal{M}_T . The next two lemmas are essentially due to Motohashi [13, page 478, lines 21–28]. For completeness we shall provide proofs.

Lemma 2.1. *Under the previous notation, we have*

$$(2.11) \quad e^{-(\log T)^{1-\varepsilon^2}} \ll |\zeta(s)| \ll e^{(\log T)^{1-\varepsilon^2}}$$

for $s \in \mathcal{M}_T$ with $\sigma \leq 1 - \varepsilon$, or s (with $1 - \varepsilon < \sigma \leq 1 - \varepsilon + \varepsilon^2$) on the horizontal segments in \mathcal{M}_T that intersect the vertical line $\Re s = 1 - \varepsilon$. Here the implied constant depends only on ε .

Proof. Let $s = \sigma + i\tau$ satisfy the conditions in this lemma. Without loss of generality, we can suppose that $\tau \geq T_0(\varepsilon)$. Let us consider the four circles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 , all centered at $s_0 = \log_2 \tau + i\tau$, with radii

$$\begin{aligned} r_1 &:= \log_2 \tau - 1 - \eta, \\ r_2 &:= \log_2 \tau - \sigma, \\ r_3 &:= \log_2 \tau - \sigma + \frac{1}{2}\varepsilon^2, \\ r_4 &:= \log_2 \tau - \sigma + \varepsilon^2, \end{aligned}$$

respectively. Here $\eta > 0$ is a parameter to be chosen later. We note that these four circles pass through the points $1 + \eta + i\tau$, $\sigma + i\tau$, $\sigma - \frac{1}{2}\varepsilon^2 + i\tau$ and $\sigma - \varepsilon^2 + i\tau$, respectively.

Clearly $\zeta(s) \neq 0$ in a region containing the disc $|s - s_0| \leq r_4$. Thus we can unambiguously define $\log \zeta(s)$ in this region. We fix a branch of the logarithm throughout the remaining discussion.

Let M_i denote the maximum of $|\log \zeta(s)|$ on \mathcal{C}_i relative to this branch. By using Hadamard's three circle theorem and the fact that $s = \sigma + i\tau$ is on \mathcal{C}_2 , we have

$$(2.12) \quad |\log \zeta(s)| \leq M_2 \leq M_1^{1-a} M_3^a,$$

where

$$\begin{aligned} a &= \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \\ &= \frac{\log(1 + (1 + \eta - \sigma)/(\log_2 \tau - 1 - \eta))}{\log(1 + (1 + \eta - \sigma + \frac{1}{2}\varepsilon^2)/(\log_2 \tau - 1 - \eta))} \\ &= \frac{1 + \eta - \sigma}{1 + \eta - \sigma + \frac{1}{2}\varepsilon^2} + O((\log_2 \tau)^{-1}). \end{aligned}$$

On taking $\eta = \sigma - \frac{1}{2} - \frac{1}{2}\varepsilon^2 - \frac{\varepsilon^3}{2(1+\varepsilon^3)}$ ($\eta \geq \frac{1}{4}\varepsilon^2$, since $\sigma \geq \frac{1}{2} + \varepsilon^2$), we have

$$(2.13) \quad a = 1 - \varepsilon^2 - \varepsilon^5 + O((\log_2 \tau)^{-1}).$$

On the circle \mathcal{C}_1 , we have

$$(2.14) \quad M_1 \leq \max_{\Re s \geq 1+\eta} \sum_{n=2}^{\infty} \left| \frac{\Lambda(n)}{n^s \log n} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n^{1+\eta}} \ll \frac{1}{\eta},$$

where $\Lambda(n)$ is the von Mangoldt function.

In order to bound M_3 , we shall apply the Borel-Carathéodory theorem to the function $\log \zeta(s)$ on the circles $\mathcal{C}_3, \mathcal{C}_4$. On the circle \mathcal{C}_4 , it is well known that

$$\Re(\log \zeta(s)) = \log |\zeta(s)| \ll \log \tau.$$

Hence the Borel-Carathéodory theorem gives

$$\begin{aligned} (2.15) \quad M_3 &\leq \frac{2r_3}{r_4 - r_3} \max_{|s-s_0| \leq r_4} \log |\zeta(s)| + \frac{r_4 + r_3}{r_4 - r_3} |\log \zeta(s_0)| \\ &\ll \frac{2(\log_2 \tau - \sigma + \frac{1}{2}\varepsilon^2)}{\frac{1}{2}\varepsilon^2} \log \tau + \frac{2\log_2 \tau - 2\sigma + \frac{1}{2}\varepsilon^2}{\frac{1}{2}\varepsilon^2} |\log \zeta(2 + i\tau)| \\ &\ll (\log_2 \tau) \log \tau. \end{aligned}$$

From (2.12), (2.13), (2.14) and (2.15), we deduce that

$$\begin{aligned} |\log \zeta(s)| &\ll (\eta^{-1})^{1-a} (\log_2 \tau \log \tau)^a \\ &\ll_{\varepsilon} (\log_2 \tau \log \tau)^{1-\varepsilon^2-\varepsilon^5} \\ &\leq (\log \tau)^{1-\varepsilon^2}. \end{aligned}$$

This leads to the required estimates. \square

Lemma 2.2. *Under the previous notation, we have*

$$(2.16) \quad T^{-400(1-\sigma_j)^{3/2}} (\log T)^{-4} \ll |\zeta(s)| \ll T^{100(1-\sigma_j)^{3/2}} (\log T)^4$$

for $s \in \mathcal{M}_T$ with $1 - \varepsilon < \sigma_j \leq \sigma < \sigma_{j+1}$. Here the implied constants are absolute. In particular we have

$$(2.17) \quad T^{-400\sqrt{\varepsilon}(1-\sigma_j)} (\log T)^{-4} \ll |\zeta(s)| \ll T^{100\sqrt{\varepsilon}(1-\sigma_j)} (\log T)^4$$

for $s \in \mathcal{M}_T$ with $1 - \varepsilon < \sigma_j \leq \sigma < \sigma_{j+1}$. All the implied constants are absolute.

Proof. According to [15, page 98], we have

$$(2.18) \quad |\zeta(s)| \ll \tau^{100(1-\sigma)^{3/2}} (\log \tau)^{2/3} \quad \left(\frac{1}{2} \leq \sigma \leq 1, \tau \geq 2\right).$$

This immediately implies the upper bound of (2.16) and

$$\begin{aligned} (2.19) \quad N_j^{1-\sigma_j} &= (A'(\log T)^5 \max_{\substack{\sigma \geq 4\sigma_j - 3 \\ 1 \leq |\tau| \leq 4T}} |\zeta(s)|)^{1/2} \\ &\ll T^{400(1-\sigma_j)^{3/2}} (\log T)^3. \end{aligned}$$

Next we consider the lower bound. Let $s \in \mathcal{M}_T$ with $1 - \varepsilon < \sigma_j \leq \sigma < \sigma_{j+1}$. Then there is an integer k such that $s \in \Delta_{j,k}$. According to the definition of \mathcal{M}_T , this $\Delta_{j,k}$ must be in (Y) and (2.8) holds for all s of this $\Delta_{j,k}$. On the other hand, (2.19) allows us to deduce that for $\sigma_j \leq \sigma < \sigma_{j+1}$

$$\begin{aligned} |M_{N_j}(s)| &\leq \sum_{n \leq N_j} n^{-\sigma_j} \ll (1 - \sigma_j)^{-1} N_j^{1-\sigma_j} \\ &\ll T^{400(1-\sigma_j)^{3/2}} (\log T)^4. \end{aligned}$$

Combining this with (2.8) immediately yields

$$|\zeta(s)| \geq (2|M_{N_j}(s)|)^{-1} \gg T^{-400(1-\sigma_j)^{3/2}} (\log T)^{-4}$$

for $s \in \mathcal{M}_T$ with $1 - \varepsilon < \sigma_j \leq \sigma < \sigma_{j+1}$.

Finally we note (2.17) is a simple consequence of (2.16) since $1 - \varepsilon < \sigma_j$ implies that $(1 - \sigma_j)^{1/2} \leq \sqrt{\varepsilon}$. \square

Proposition 2.3. *Under the previous notation, we have*

$$(2.20) \quad T^{-400\sqrt{\varepsilon}(1-\sigma)}(\log T)^{-4} \ll |\zeta(s)| \ll T^{400\sqrt{\varepsilon}(1-\sigma)}(\log T)^4.$$

for all $s \in \mathcal{M}_T$, where the implied constants depend only on ε .

Proof. Let $s \in \mathcal{M}_T$. Then there is a j such that $\sigma_j \leq \sigma < \sigma_{j+1}$. We consider the three possibilities.

- *The case of $1 - \varepsilon < \sigma_j$.*

The inequality (2.20) follows immediately from (2.17) of Lemma 2.2.

- *The case of $\sigma_j \leq \sigma \leq 1 - \varepsilon$.*

In this case, the first part of Lemma 2.1 shows that (2.20) holds again since $\sqrt{\varepsilon}(1 - \sigma) \geq \varepsilon^{3/2} \geq (\log T)^{-\varepsilon^2}$ for $T \geq T_0(\varepsilon)$.

- *The case of $\sigma_j \leq 1 - \varepsilon < \sigma$.*

In this case, s must be on the horizontal segment in \mathcal{M}_T , because the vertical segment keeps the distance ε^2 from the line $\Re s = \sigma_j$ and $\sigma_j < \sigma < \sigma_{j+1}$. Thus we can apply the second part of Lemma 2.1 to get (2.20) as before. \square

2.3. Montgomery's method and Huxley's zero-density estimation.

In [12], Montgomery developed a new method for studying zero-densities of the Riemann ζ -function and of the Dirichlet L -functions. Subsequently by modifying this method, Huxley [7] established his zero-density estimation (1.7) (see (2.21) below). In [13], Motohashi noted that Montgomery's method can be adapted to estimate the density of "small value points" (characterized by (2.6)). The estimation (2.22) below is due to Motohashi [13, (5)].

Proposition 2.4. *Under the previous notation, for $j = 0, 1, \dots, J_T$ we have*

$$(2.21) \quad |\{k \leq K_T : \Delta_{j,k} \in (W)\}| \ll T^{\psi(1-\sigma_j)}(\log T)^\eta$$

if $\sigma_j \leq 1 - \varepsilon$; and

$$(2.22) \quad |\{k \leq K_T : \Delta_{j,k} \in (W)\}| \ll T^{170(1-\sigma_j)^{3/2}}(\log T)^{13}$$

if $1 - \varepsilon \leq \sigma_j \leq 1 - \delta_T$. Here $(\psi, \eta) = (\frac{12}{5}, 9)$ is admissible.

Proof. The case of $\sigma_j \leq 1 - \varepsilon$ is very simple, because the number of (W) does not exceed the number of non-trivial zeros of $\zeta(s)$.

Next we suppose $1 - \varepsilon \leq \sigma_j \leq 1 - \delta_T$.

Let $\mathcal{K}_j(T)$ be a subset of the set

$$\{\log T \leq k \leq K_T : \Delta_{j,k} \in (W)\}$$

such that the difference of two distinct integers of $\mathcal{K}_j(T)$ is at least $3A'$, where A' is the large integer specified in (2.7). Obviously

$$|\{(\log T)^2 \leq k \leq K_T : \Delta_{j,k} \in (W)\}| \leq 3A'|\mathcal{K}_j(T)|.$$

Therefore it suffices to show that

$$(2.23) \quad |\mathcal{K}_j(T)| \ll_\varepsilon T^{170(1-\sigma_j)^{3/2}} (\log T)^{13}$$

for $T \geq T_0(\varepsilon)$, where the implied constant and the constant $T_0(\varepsilon)$ depend only on ε .

Let $M_x(s)$ be defined as in (2.7) and let $a_{n,x}$ be the n th coefficient of the Dirichlet series $\zeta(s)M_x(s)$. Then

$$(2.24) \quad a_{n,x} = \sum_{d|n, d \leq x} \mu(d).$$

By the Perron formula [22, Lemma, page 151], we can write

$$\sum_{n \geq 1} \frac{a_{n,x}}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(w+s) M_x(w+s) \Gamma(w) y^w dw$$

for $y > x \geq 3$ and $s = \sigma + i\tau \in \mathbb{C}$ with $\frac{1}{2} < \sigma < 1$. We take the contour to the line $\Re w = \alpha - \sigma < 0$ with $\alpha := 4\sigma_j - 3 \geq 1 - 4\varepsilon$, and in doing so we pass two simple poles at $w = 0$ and $w = 1 - s$. Our equation becomes

$$\sum_{n \geq 1} \frac{a_{n,x}}{n^s} e^{-n/y} = \zeta(s) M_x(s) + M_x(1) \Gamma(1-s) y^{1-s} + I(s; x, y),$$

where

$$I(s; x, y) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \zeta(\alpha + i\tau + iu) M_x(\alpha + i\tau + iu) \Gamma(\alpha - \sigma + iu) y^{\alpha - \sigma + iu} du.$$

Obviously the formula (2.24) implies that $a_{1,x} = 1$, $a_{n,x} = 0$ for $2 \leq n \leq x$ and $|a_{n,x}| \leq \tau(n)$ for $n > x$. With the classical estimate $\sum_{n \leq t} \tau(n) \ll t \log t$ and a simple partial integration, we obtain

$$\begin{aligned} \left| \sum_{n > y^2} \frac{a_{n,x}}{n^s} e^{-n/y} \right| &\leq \int_{y^2}^{\infty} t^{-\sigma} e^{-t/y} d\left(\sum_{n \leq t} \tau(n) \right) \\ &\ll e^{-y} y^{2-2\sigma} \log y + y^{-1} \int_{y^2}^{\infty} e^{-t/y} t^{1-\sigma} (\log t) dt \\ &\ll e^{-y/2} \end{aligned}$$

for $\sigma > \frac{1}{2}$. Inserting it into the precedent relation, we find that

$$(2.25) \quad \begin{aligned} e^{-1/y} + \sum_{x < n \leq y^2} \frac{a_{n,x}}{n^s} e^{-n/y} + O(e^{-y/2}) \\ = \zeta(s) M_x(s) + M_x(1) \Gamma(1-s) y^{1-s} + I(s; x, y) \end{aligned}$$

for $s \in \mathbb{C}$ with $\frac{1}{2} < \sigma < 1$ and $y > x \geq 3$.

If $k \in \mathcal{K}_j(T)$, then there is at least an $s_k := v_k + it_k \in \Delta_{j,k}$ such that

$$(2.26) \quad |\zeta(s_k) M_{N_j}(s_k)| \leq \frac{1}{2},$$

where $M_{N_j}(s) = M(s, N_j)$ is defined as in (2.7). By the definition of $\mathcal{K}_j(T)$, we have

$$\sigma_j \leq v_k \leq \sigma_{j+1}, \quad (\log T)^2 \leq t_k \leq T \quad \text{and} \quad |t_{k_1} - t_{k_2}| \geq 3A' \log T \quad (k_1 \neq k_2).$$

By the Stirling formula [21, page 151], we have

$$(2.27) \quad |\Gamma(s)| = \sqrt{2\pi} e^{-(\pi/2)|\tau|} |\tau|^{\sigma-1/2} \left\{ 1 + O\left(\frac{|\tan(\frac{\vartheta}{2})|}{|\tau|} + \frac{|a|^2 + |b|^2}{|\tau|^2} + \frac{|a|^3 + |b|^3}{|\tau|^3} \right) \right\}$$

uniformly for $a, b \in \mathbb{R}$ with $a < b$, $a \leq \sigma \leq b$ and $|\tau| \geq 1$, where $\vartheta := \arg s$ and the implied O -constant is absolute.

Since $|t_k| \geq (\log T)^2$, the Stirling formula allows us to deduce

$$(2.28) \quad |M_x(1)\Gamma(1-s_k)y^{1-s_k}| \ll (\log x)y^{1-v_k}e^{-(\pi/2)|t_k|}|t_k|^{1/2-v_k} \leq \frac{1}{10}$$

for all $3 \leq x \leq y \leq T^{100}$.

Similarly, using the estimates

$$\begin{aligned} \zeta(\alpha + it_k + iu) &\ll T + |u|, \\ M_x(\alpha + it_k + iu) &\ll x^{1-\alpha} \log x \ll T, \end{aligned}$$

and the Stirling formula (2.27), we derive that

$$(2.29) \quad \int_{|u| \geq A' \log T} |\zeta(\alpha + it_k + iu)M_x(\alpha + it_k + iu)\Gamma(\alpha - v_k + iu)| y^{\alpha-v_k} du \leq \frac{1}{10}$$

for all $3 \leq x \leq y \leq T^{100}$.

Taking $(s, x) = (s_k, N_j)$ in (2.25) and combining with (2.26), (2.28) and (2.29), we easily see that

$$(2.30) \quad \left| \sum_{N_j < n \leq y^2} \frac{a_{n, N_j}}{n^{s_k}} e^{-n/y} \right| \geq \frac{1}{6}$$

or

$$(2.31) \quad \left| \int_{-A' \log T}^{A' \log T} \zeta(\alpha + it_k + iu)M_{N_j}(\alpha + it_k + iu)\Gamma(\alpha - v_k + iu)y^{\alpha-v_k+iu} du \right| \geq \frac{1}{6}$$

or both.

Let $\mathcal{K}'_j(T)$ and $\mathcal{K}''_j(T)$ be the subsets of $\mathcal{K}_j(T)$ for which (2.30) and (2.31) hold respectively. Then

$$(2.32) \quad |\mathcal{K}_j(T)| \leq |\mathcal{K}'_j(T)| + |\mathcal{K}''_j(T)|.$$

First we bound $|\mathcal{K}'_j(T)|$. By a dyadic argument, there is a $U \in [N_j, y^2]$ such that

$$(2.33) \quad \left| \sum_{U < n \leq 2U} \frac{a_{n, N_j}}{n^{s_k}} e^{-n/y} \right| \geq (18 \log y)^{-1}$$

holds for $\gg |\mathcal{K}'_j(T)|(\log y)^{-1}$ integers $k \in \mathcal{K}'_j(T)$. Let \mathcal{S}' be the set of corresponding points s_k . Using [12, Theorem 8.4] with $\theta = \alpha := 4\sigma_j - 3$ and the bound

$$\sum_{U < n \leq 2U} \frac{\tau(n)^2}{n^{2\sigma_j}} e^{-2n/y} \ll U^{1-2\sigma_j} (\log T)^3 e^{-2U/y},$$

it follows that

$$\begin{aligned}
(2.34) \quad & \sum_{s_k \in \mathcal{S}'} \left| \sum_{U < n \leq 2U} \frac{a_{n, N_j}}{n^{s_k}} e^{-n/y} \right|^2 \\
& \ll \left(U + |\mathcal{S}'| \max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 4T}} |\zeta(s)| U^\alpha \right) U^{1-2\sigma_j} (\log T)^3 e^{-2U/y} \\
& \ll U^{2(1-\sigma_j)} (\log T)^3 e^{-2U/y} + |\mathcal{S}'| \max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 4T}} |\zeta(s)| U^{-2(1-\sigma_j)} (\log T)^3 e^{-2U/y}.
\end{aligned}$$

Since $U \geq N_j$, we have

$$\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 4T}} |\zeta(s)| U^{-2(1-\sigma_j)} (\log T)^3 \leq A'^{-1} (\log T)^{-2}.$$

On the other hand, the inequality (2.33) implies that the member on the left-hand side of (2.34) is

$$\geq |\mathcal{S}'| (18 \log y)^{-2} \geq |\mathcal{S}'| (1800 \log T)^{-2}.$$

Since A' is a fixed large integer, the last term on the right-hand side of (2.34) is smaller than this lower bound. Thus it can be simplified as

$$|\mathcal{S}'| (\log T)^{-2} \ll U^{2(1-\sigma_j)} (\log T)^3 e^{-2U/y}$$

for all $N_j \leq y \leq T^{100}$ and some $U \in [N_j, y^2]$. Noticing that

$$|\mathcal{S}'| \gg |\mathcal{K}'_j(T)| (\log T)^{-1},$$

we obtain

$$(2.35) \quad |\mathcal{K}'_j(T)| \ll y^{2(1-\sigma_j)} (\log T)^6$$

for all $N_j \leq y \leq T^{100}$.

Next we bound $|\mathcal{K}''_j(T)|$. Let $u_k \in [-A' \log T, A' \log T]$ such that

$$\zeta(s'_k) M_{N_j}(s'_k) = \max_{|u| \leq A \log T} |\zeta(\alpha + it_k + iu) M_{N_j}(\alpha + it_k + iu)|$$

where $s'_k := \alpha + it'_k$ and $t'_k := t_k + u_k$. Thus from (2.31) we deduce that

$$\begin{aligned}
\frac{1}{6} & \leq \left| \int_{-A' \log T}^{A' \log T} \zeta(\alpha + it_k + iu) M_{N_j}(\alpha + it_k + iu) \Gamma(\alpha - v_k + iu) y^{\alpha - v_k + iu} du \right| \\
& \leq y^{\alpha - v_k} |\zeta(s'_k) M_{N_j}(s'_k)| \int_{-A' \log T}^{A' \log T} |\Gamma(\alpha - v_k + iu)| du.
\end{aligned}$$

Since $\Gamma(s)$ has a simple pole at $s = 0$ and $|\alpha - v_k| \gg (\log T)^{-1}$, we can derive, via (2.27), that

$$\int_{-A' \log T}^{A' \log T} |\Gamma(\alpha - v_k + iu)| du \ll \log T$$

and thus

$$1 \ll y^{\alpha - \sigma_j} |M_{N_j}(s'_k)| \max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \log T,$$

or equivalently

$$|M_{N_j}(s'_k)| \gg y^{\sigma_j - \alpha} \left(\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \log T \right)^{-1}.$$

Hence there is a $V \in [1, N_j]$ such that

$$\left| \sum_{V < n \leq 2V} \mu(n) n^{-s'_k} \right| \gg y^{\sigma_j - \alpha} \left(\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \right)^{-1} (\log T)^{-2}$$

holds for $\gg |\mathcal{K}_j''(T)| (\log T)^{-1}$ integers $k \in \mathcal{K}_j''(T)$. Let \mathcal{S}'' be the corresponding set of points s'_k . We note $|t'_k| \leq 2T$ and

$$|t'_{k_1} - t'_{k_2}| \geq |t_{k_1} - t_{k_2}| - |u_{k_1} - u_{k_2}| \geq A' \log T.$$

Using [12, Theorem 8.4] with $\theta = \alpha = 4\sigma_j - 3$ and the bound

$$\sum_{V < n \leq 2V} n^{-2\alpha} \ll V^{1-2\alpha} \ll V^{7-8\sigma_j},$$

it follows that

$$(2.36) \quad \sum_{s'_k \in \mathcal{S}''} \left| \sum_{V < n \leq 2V} \mu(n) n^{-s'_k} \right|^2 \ll \left(V + |\mathcal{S}''| \max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| V^{4\sigma_j - 3} \right) V^{7-8\sigma_j} \\ \ll V^{8(1-\sigma_j)} + |\mathcal{S}''| \max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| V^{4(1-\sigma_j)}.$$

Take y such that

$$(2.37) \quad y^{2(\sigma_j - \alpha)} = A' N_j^{4(1-\sigma_j)} \left(\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \right)^3 (\log T)^4.$$

The left-hand side of (2.36) is

$$\geq |\mathcal{S}''| y^{2(\sigma_j - \alpha)} \left(\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \right)^{-2} (\log T)^{-4}.$$

Hence the inequality (2.36) can be simplified as

$$|\mathcal{S}''| y^{2(\sigma_j - \alpha)} \left(\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \right)^{-2} (\log T)^{-4} \ll N_j^{8(1-\sigma_j)}.$$

With

$$|\mathcal{S}''| \gg |\mathcal{K}_j''(T)| (\log T)^{-1},$$

we deduce that

$$(2.38) \quad |\mathcal{K}_j''(T)| \ll N_j^{8(1-\sigma_j)} y^{2(\alpha - \sigma_j)} \left(\max_{\substack{\sigma \geq \alpha \\ 1 \leq |\tau| \leq 8T}} |\zeta(s)| \right)^2 (\log T)^5.$$

On combining (2.32), (2.35), (2.38) and (2.37), it follows that

$$|\mathcal{K}_j(T)| \ll N_j^{(10/3)(1-\sigma_j)} (\log T)^3.$$

Now the required inequality follows from (2.19). This completes the proof. \square

3. PROOF OF THEOREM 1.1

We shall conserve the notation of Section 2. First we prove a lemma.

Lemma 3.1. *Let $z \in \mathbb{C}$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $C > 0$, $M > 0$ be some constants. Suppose that the Dirichlet series*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$$

is of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$. Then there is an absolute positive constant D such that we have

$$(3.1) \quad \mathcal{F}(s) \ll MD^B T^{(100B\sqrt{\varepsilon}+\delta)(1-\sigma)} (\log T)^{A+4B}$$

for all $s \in \mathcal{M}_T$, where the implied constant depends only on ε .

Proof. Since we have chosen the principal value of complex logarithm, we can write

$$(3.2) \quad |\zeta(s)^z| = |\zeta(s)|^{\Re z} e^{-(\Im z) \arg \zeta(s)} \leq e^{\pi B} |\zeta(s)|^{\Re z}$$

for all $s \in \mathbb{C}$ such that $\zeta(s) \neq 0$.

Invoking Proposition 2.3, we see that there is a suitable absolute constant D such that

$$(3.3) \quad |\zeta(s)^z| \ll_{\varepsilon} D^B T^{100B\sqrt{\varepsilon}(1-\sigma)} (\log T)^{4B}$$

for all $s \in \mathcal{M}_T$, where the implied constant depends only on ε .

Finally the required bound (3.1) follows from (3.3), the hypothesis (1.4) and the trivial bound $|\zeta(2s)| \asymp 1$ for $s \in \mathcal{M}_T$. \square

Now we are ready to prove Theorem 1.1.

Since the Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$, we can apply Corollary II.2.2.1 of [20] with the choice of parameters $\sigma_a = 1$, $\alpha = \alpha$, $\sigma = 0$ to write

$$\sum_{x < n \leq x+y} f(n) = \frac{1}{2\pi i} \int_{b-iT'}^{b+iT'} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds + O_{\varepsilon} \left(M \frac{x^{1+\varepsilon}}{T} \right),$$

where $b = 1 + 1/\log x$, $e^{\sqrt{\log x}} \leq T \leq x$ is a parameter to be chosen later and $T' = K_T \log T \sim T$.

Denote by Γ_T the path formed from the circle $|s-1| = r := 1/(2\log x)$ excluding the point $s = 1 - r$, together with the segment $[1 - \delta_T, 1 - r]$ traced out twice with respective arguments $+\pi$ and $-\pi$. By the residue theorem, the path $[b - iT', b + iT']$ is deformed into

$$\Gamma_T \cup [1 - \delta_T - iT', 1 - \delta_T + iT'] \cup [1 - \delta_T \pm iT', b \pm iT'].$$

In view of (2.2) and the hypothesis (1.4), the function $\mathcal{F}(s)$ is analytic in the interior of this contour and in the same domain

$$(3.4) \quad \mathcal{F}(s) \ll MD^C T^{\max\{\delta(1-\sigma), 0\}} (\log T)^{A+B},$$

where the implied constant and the constant D are absolute. The integral over the horizontal segments $[1 - \delta_T \pm iT', b \pm iT']$ is

$$\begin{aligned} \int_{1-\delta_T \pm iT'}^{b \pm iT'} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds &\ll \frac{MD^C (\log T)^{A+B}}{T} \int_{1-\delta_T}^b T^{\max\{\delta(1-\sigma), 0\}} x^\sigma d\sigma \\ &\ll MD^C \frac{x}{T} (\log T)^{A+B} \left(\int_{1-\delta_T}^1 \left(\frac{x}{T^\delta} \right)^{\sigma-1} d\sigma + 1 \right) \\ &\ll MD^C \frac{x}{T} (\log T)^{A+B-1}. \end{aligned}$$

Thus

$$(3.5) \quad \sum_{x < n \leq x+y} f(n) = I + O\left(MD^C \frac{x^{1+\varepsilon}}{T} \right),$$

where the implied constant depends on ε only and

$$I := \frac{1}{2\pi i} \int_{\Gamma_T \cup [1-\delta_T - iT', 1-\delta_T + iT']} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds.$$

Let \mathcal{M}_T be the Motohashi contour defined as in Section 2. Consider the two symmetric simply connected regions bounded by \mathcal{M}_T , the segment $[1 - \delta_T - iT', 1 - \delta_T + iT']$ and the two line segments $[\sigma_{j_0+1} + d_v, 1 - \delta_T]$ with respective arguments $+\pi$ and $-\pi$ measured from the real axis on the right of $1 - \delta_T$. It is clear that $\mathcal{F}(s)$ is analytic in these two simply connected regions. Denote by Γ_T^* the path joining (the two endpoints of) Γ_T with the two line segments $[\sigma_{j_0+1} + d_v, 1 - \delta_T]$ of the symmetric regions. Thanks to the residue theorem, we can write

$$(3.6) \quad I = I_1 + I_2,$$

with

$$\begin{aligned} I_1 &:= \frac{1}{2\pi i} \int_{\Gamma_T^*} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds, \\ I_2 &:= \frac{1}{2\pi i} \int_{\mathcal{M}_T} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds. \end{aligned}$$

A. Evaluation of I_1

According to our hypothesis, $\mathcal{G}(s; \kappa, w) \zeta(2s)^w Z(s; \kappa)$ is holomorphic and $O(M)$ in the disc $|s - 1| \leq \frac{1}{2} - \varepsilon^3 =: c$; the Cauchy integral formula implies that

$$(3.7) \quad g_\ell(\kappa, w) \ll Mc^{-\ell} \quad (\ell \geq 0, |z| \leq B, |w| \leq C),$$

where $g_\ell(\kappa, w)$ is defined as in (1.16). From this and (1.15), we deduce that for any integer $N \geq 0$ and $|s - 1| \leq \frac{1}{2} - \varepsilon^2$,

$$\mathcal{G}(s; \kappa, w) \zeta(2s)^w Z(s; \kappa) = \sum_{\ell=0}^N g_\ell(\kappa, w) (s-1)^\ell + O(M(|s-1|/c)^{N+1}).$$

Thus we have

$$(3.8) \quad I_1 = \sum_{\ell=0}^N g_\ell(\kappa, w) M_\ell(x, y) + O(Mc^{-N} E_N(x, y)),$$

where

$$M_\ell(x, y) := \frac{1}{2\pi i} \int_{\Gamma_T^*} (s-1)^{\ell-z} \frac{(x+y)^s - x^s}{s} ds,$$

$$E_N(x, y) := \int_{\Gamma_T^*} \left| (s-1)^{N+1-z} \frac{(x+y)^s - x^s}{s} \right| |ds|.$$

Firstly we evaluate $M_\ell(x, y)$. Using the formula

$$(3.9) \quad \frac{(x+y)^s - x^s}{s} = \int_x^{x+y} t^{s-1} dt$$

and Corollary II.5.2.1 of [20], we write

$$M_\ell(x, y) = \int_x^{x+y} \left(\frac{1}{2\pi i} \int_{\Gamma_T^*} (s-1)^{\ell-z} t^{s-1} ds \right) dt$$

$$= \int_x^{x+y} (\log t)^{z-1-\ell} \left\{ \frac{1}{\Gamma(\kappa-\ell)} + O\left(\frac{(c_1\ell+1)^\ell}{t^{\delta_T/2}} \right) \right\} dt,$$

where we have used the following inequality

$$47^{|z-\ell|} \Gamma(1+|z-\ell|) \ll_B (c_1\ell+1)^\ell \quad (\ell \geq 0, |z| \leq B).$$

The constant c_1 and the implied constant depend at most on B . Besides for $|z| \leq B$,

$$\int_x^{x+y} (\log t)^{z-1-\ell} dt = \int_0^y \log^{z-1-\ell}(x+t) dt$$

$$= y(\log x)^{z-1-\ell} \left\{ 1 + O_B\left(\frac{(\ell+1)y}{x \log x} \right) \right\}.$$

Inserting this into the preceding formula, we obtain

$$(3.10) \quad M_\ell(x, y) = y(\log x)^{z-1-\ell} \left\{ \frac{1}{\Gamma(z-\ell)} + O_B\left(\frac{(\ell+1)y}{\Gamma(z-\ell)x \log x} + \frac{(c_1\ell+1)^\ell}{x^{\delta_T/2}} \right) \right\}$$

for $\ell \geq 0$ and $|z| \leq B$.

Next we estimate $E_N(x, y)$. In view of the trivial inequality

$$(3.11) \quad \left| \frac{(x+y)^s - x^s}{s} \right| \ll yx^{\sigma-1},$$

we deduce that

$$(3.12) \quad E_N(x, y) \ll \int_{1/2+\varepsilon^2}^{1-1/\log x} (1-\sigma)^{N+1-\Re e z} x^{\sigma-1} y d\sigma + \frac{y}{(\log x)^{N+2-\Re e z}}$$

$$\ll \frac{y}{(\log x)^{N+2-\Re e z}} \left(\int_1^\infty t^{N+1-\Re e z} e^{-t} dt + 1 \right)$$

$$\ll y(\log x)^{\Re e z-1} \left(\frac{c_1 N+1}{\log x} \right)^{N+1}$$

uniformly for $x \geq y \geq 2$, $N \geq 0$ and $|z| \leq B$, where the constant $c_1 > 0$ and the implied constant depends only on B .

Inserting (3.10) and (3.12) into (3.8) and using (3.7), we find that

$$(3.13) \quad I_1 = y(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(z, w)}{(\log x)^\ell} + O_B(E_N^*(x, y)) \right\},$$

where

$$E_N^*(x, y) := \frac{y}{x} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(z, w)|}{(\log x)^\ell} + \frac{(c_1 N + 1)^{N+1}}{x^{\delta_T/2}} + M \left(\frac{c_1 N + 1}{\log x} \right)^{N+1}.$$

B. Evaluation of I_2

Let \mathcal{M}'_T be the union of those vertical line segments of \mathcal{M}_T whose real part is equal to $\frac{1}{2} + \varepsilon^2$ (i.e. corresponding to those k such that $j_k = 0$) and $\mathcal{M}''_T := \mathcal{M}_T \setminus \mathcal{M}'_T$. Denote by I'_2 and I''_2 the contribution of \mathcal{M}'_T and \mathcal{M}''_T to I_2 , respectively. Using the trivial inequality

$$\left| \frac{(x+y)^s - x^s}{s} \right| \ll \frac{x^{1/2+\varepsilon^2}}{|\tau|+1} \quad (s \in \mathcal{M}'_T)$$

and Lemma 3.1, we can deduce

$$(3.14) \quad \begin{aligned} I'_2 &\ll MD^B x^{1/2+\varepsilon^2} T^{(\delta+100B\sqrt{\varepsilon})(1/2-\varepsilon^2)} (\log T)^{A+4B+1} \\ &\ll Mx^{1/2+\delta/(2\psi+2\delta)+\sqrt{\varepsilon}} \\ &\ll Mx^{1-1/(\psi+\delta)+\sqrt{\varepsilon}} \end{aligned}$$

with the value of T given by (3.16) below and $\psi \geq 2$.

Next we bound I''_2 . In view of (3.11), we can write that

$$(3.15) \quad \begin{aligned} I''_2 &\ll y \int_{\mathcal{M}''_T} |\mathcal{F}(s)| x^{\sigma-1} |ds| \\ &\ll y \sum_{0 \leq j \leq J_T} \sum_{\substack{0 \leq k \leq K_T \\ \Delta_{j,k} \in (W)}} \int_{\mathcal{M}_T(j,k)} |\mathcal{F}(s)| x^{\sigma-1} |ds|, \end{aligned}$$

where $\mathcal{M}_T(j, k)$ is the vertical line segment of \mathcal{M}''_T around $\Delta_{j,k}$ and the horizontal line segments with $\sigma \leq \sigma_j + d_v$. Clearly the length of $\mathcal{M}_T(j, k)$ is $\ll \log T$. Thus with the help of Lemma 3.1, it is easy to see that

$$\int_{\mathcal{M}_T(j,k)} |\mathcal{F}(s)| x^{\sigma-1} |ds| \ll MD^B (\log T)^{A+4B+1} T^{(\delta+100B\sqrt{\varepsilon})(1-\sigma_j-d_v)} x^{\sigma_j+d_v-1}$$

for all $0 \leq k \leq K_T$. Inserting it into (3.15) and using Proposition 2.4, we can deduce, with the notation $J_{T,0} := \lfloor (\frac{1}{2} - \varepsilon) \log T \rfloor$, that

$$I''_2 \ll MD^B y (\log T)^{A+4B+18+\eta} (I''_{2,*} + I''_{2,\dagger}),$$

where

$$I''_{2,*} := \sum_{0 \leq j \leq J_{T,0}} T^{(\delta+100B\sqrt{\varepsilon})(1-\sigma_j-d_v)} x^{\sigma_j+d_v-1} \cdot T^{\psi(1-\sigma_j)},$$

$$I''_{2,\dagger} := \sum_{J_{T,0} < j \leq J_T} T^{(\delta+100B\sqrt{\varepsilon})(1-\sigma_j-d_v)} x^{\sigma_j+d_v-1} \cdot T^{100\sqrt{\varepsilon}(1-\sigma_j)}.$$

Taking

$$(3.16) \quad T := x^{(1-\sqrt{\varepsilon})/(\psi+\delta+100B\sqrt{\varepsilon})}$$

and in view of (2.10), it is easy to check that

$$I''_{2,*} \ll x^{\varepsilon^2} \sum_{0 \leq j \leq J_{T,0}} \left(x/T^{\psi+\delta+100B\sqrt{\varepsilon}} \right)^{-(1-\sigma_j)} \log x \ll x^{\varepsilon^2-\varepsilon^{3/2}} \log x \ll x^{-\varepsilon^2}$$

and

$$I''_{2,\dagger} \ll \sum_{J_{T,0} < j \leq J_T} \left(x/T^{\delta+100(B+1)\sqrt{\varepsilon}} \right)^{-(1-\sigma_j)} \ll e^{-2c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}$$

Inserting it into the preceding estimate for I''_2 , we conclude that

$$(3.17) \quad I''_2 \ll_B M y e^{-c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}.$$

Now from (3.5), (3.6), (3.13), (3.14) and (3.17), we deduce that

$$\sum_{x < n \leq x+y} f(n) = y(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(z, w)}{(\log x)^\ell} + O_{A,B,C,\alpha,\delta,\varepsilon}(R_N^*(x, y)) \right\}$$

uniformly for $x \geq 3$, $x^{1-1/(\psi+\delta)+\varepsilon} \leq y \leq x$, $N \geq 0$, $|z| \leq B$ and $|w| \leq C$, where

$$R_N^*(x, y) := \frac{y}{x} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(z, w)|}{(\log x)^\ell} + M \left\{ \left(\frac{c_1 N + 1}{\log x} \right)^{N+1} + \frac{(c_1 N + 1)^{N+1}}{e^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}} \right\}$$

for some constants $c_1 > 0$ and $c_2 > 0$ depending only on B , C , δ and ε .

It remains to prove that the first term on the right-hand side can be absorbed by the third. In view of (1.14), the Cauchy formula allows us to write

$$g_\ell(z, w) \ll_{A,B,C,\delta} M 3^\ell$$

for $|z| \leq B$, $|w| \leq C$ and $\ell \geq 1$. Combining this with the Stirling formula, we easily derive $\lambda_\ell(z, w) \ll_{A,B,C,\delta} M(9/\ell)^\ell$ for $|z| \leq B$, $|w| \leq C$ and $\ell \geq 1$. This implies that

$$\frac{y}{x} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(z, w)|}{(\log x)^\ell} \ll_{A,B,C,\delta} M \frac{y}{x} \ll_{A,B,C,\delta,\varepsilon} \frac{M(c_1 N + 1)^{N+1}}{e^{c_2(\log x)^{1/3}(\log_2 x)^{-1/3}}}$$

holds uniformly for $x \geq 3$, $x^{1-1/(\psi+\delta)+\varepsilon} \leq y \leq x$, $N \geq 0$, $|z| \leq B$ and $|w| \leq C$. This completes the proof.

4. PROOFS OF THEOREMS 1.3 AND 1.4

Since the proofs of Theorems 1.3 and 1.4 are very similar, we shall only prove the former. For $z \in \mathbb{C}$ and $\sigma > 1$, we can write

$$\begin{aligned} \mathcal{F}_1(s; z) &:= \sum_{n \geq 1} z^{\omega(n)} n^{-s} = \prod_p (1 + z(p^s - 1)^{-1}) \\ &= \zeta(s)^z \zeta(2s)^{z(1-z)/2} \mathcal{G}_1(s; z, \frac{z(1-z)}{2}), \end{aligned}$$

where

$$\mathcal{G}_1(s; z, \frac{z(1-z)}{2}) := \prod_p \left(1 + \frac{z}{p^s - 1}\right) \left(1 - \frac{1}{p^s}\right)^z \left(1 - \frac{1}{p^{2s}}\right)^{z(1-z)/2}.$$

We expand $\mathcal{G}_1(s; z, \frac{z(1-z)}{2})$ into the Dirichlet series

$$\mathcal{G}_1(s; z, \frac{z(1-z)}{2}) = \sum_{n \geq 1} b_{1z}(n) n^{-s},$$

then $b_{1z}(n)$ is the multiplicative function whose values on prime powers are determined by the identity

$$1 + \sum_{\nu \geq 1} b_{1z}(p^\nu) \xi^\nu = \left(1 + \frac{z\xi}{1-\xi}\right) (1-\xi)^z (1-\xi^2)^{z(1-z)/2} \quad (|\xi| < 1).$$

In particular $b_{1z}(p) = b_{1z}(p^2) = 0$ and the Cauchy integral formula gives

$$|b_{1z}(p^\nu)| \leq M(B) 2^{\nu/2} \quad (\nu \geq 3, |z| \leq B),$$

where

$$M(B) := \sup_{|z| \leq B, |\xi| \leq 1/\sqrt{2}} \left| \left(1 + \frac{z\xi}{1-\xi}\right) (1-\xi)^z (1-\xi^2)^{z(1-z)/2} \right|.$$

From these we deduce that for $\sigma > \frac{1}{3}$,

$$\sum_p \sum_{\nu \geq 1} \frac{|b_{1z}(p^\nu)|}{p^{\nu\sigma}} \leq \sum_p \sum_{\nu \geq 3} \frac{M(B)}{(p^\sigma/\sqrt{2})^\nu} \leq \sum_p \frac{2^{3/2} M(B)}{p^{2\sigma}(p^\sigma - \sqrt{2})} \ll_B \frac{1}{3\sigma - 1}.$$

So the Dirichlet series $\sum_{n=1}^{\infty} z^{\omega(n)} n^{-s}$ is of type $\mathcal{P}(z, \frac{z(1-z)}{2}, B, 0, 0, B, C(B), M(B))$, where $C(B)$ is a positive constant depending on B .

Define $g_\ell(z)$ by

$$\begin{aligned} \mathcal{F}_1(s)(s-1)^z &= Z(s; z) \zeta(2s)^{z(1-z)/2} \mathcal{G}_1(s; z, \frac{z(1-z)}{2}) \\ (4.1) \quad &= \sum_{\ell=0}^{\infty} g_\ell(z) (s-1)^\ell \quad (|s-1| < \frac{1}{6}). \end{aligned}$$

Applying Theorem 1.1 to the Dirichlet series $\sum_{n=1}^{\infty} z^{\omega(n)} n^{-s}$, we obtain the following result.

Lemma 4.1. *Let $B > 0$ be a constant. For any $\varepsilon > 0$, we have*

$$(4.2) \quad \sum_{x < n \leq x+y} z^{\omega(n)} = y(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(z)}{(\log x)^\ell} + O_{B,\varepsilon}(MR_N(x, y)) \right\}$$

uniformly for

$$x \geq 3, \quad x \geq y \geq x^{1-1/\psi+\varepsilon}, \quad |z| \leq B, \quad N \geq 0,$$

where $\lambda_\ell(z) := g_\ell(z)/\Gamma(z - \ell)$ and $R_N(x, y)$ is defined as in (1.18). The constants c_1, c_2 in $R_N(x, y)$ and the implied constant depends only on B and ε .

Lemma 4.1 improves Theorem 3 of [10] in two directions: get a more precise asymptotic formula and extend the domain $x^{7/12+\varepsilon} \leq y \leq x^{2/3-\varepsilon}$ to $x^{7/12+\varepsilon} \leq y \leq x$.

The next lemma is a short interval version of the asymptotic formula (13) of [20, Theorem II.6.3]. We omit the proof as it is very similar.

Lemma 4.2. *Let $B > 0$ and $0 < \theta \leq 1$ be two positive constants. For each integer $n \geq 1$, let*

$$a_z(n) = \sum_{k=0}^{\infty} c_k(n) z^k$$

be a holomorphic function for $|z| \leq B$. Let $N \geq 0$ be a non-negative integer. Suppose that there exist $N + 1$ holomorphic functions $h_0(z), \dots, h_N(z)$ for $|z| \leq B$ and a quantity $R_N(x, y)$ independent of z such that

$$(4.3) \quad \sum_{x < n \leq x+y} a_z(n) = y(\log x)^{z-1} \left\{ \sum_{\ell=0}^N \frac{z h_\ell(z)}{(\log x)^\ell} + O_{B,\theta}(R_N(x, y)) \right\}$$

holds uniformly for $x \geq 3$, $x \geq y \geq x^\theta$ and $|z| \leq B$. Then we have

$$(4.4) \quad \sum_{x < n \leq x+y} c_k(n) = \frac{y}{\log x} \left\{ \sum_{j=0}^N \frac{R_{j,k}(\log_2 x)}{(\log x)^j} + O_{B,\theta} \left(\frac{(\log_2 x)^k}{k!} R_N(x, y) \right) \right\}$$

uniformly for $x \geq 3$, $x \geq y \geq x^\theta$ and $1 \leq k \leq B \log_2 x$, where

$$(4.5) \quad R_{j,k}(X) := \sum_{\ell+m=k-1} \frac{h_j^{(m)}(0)}{\ell! m!} X^\ell$$

and the implied constants depend only on B and θ .

If, in addition, we suppose that $|h_0''(z)| \leq D$ ($|z| \leq B$), then we have

$$\sum_{x < n \leq x+y} c_k(n) = \frac{y}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!} \left\{ h_0 \left(\frac{k-1}{\log_2 x} \right) + O_{B,\theta} \left(\frac{D(k-1)}{(\log_2 x)^2} + \frac{\log_2 x}{k} R_0(x, y) \right) \right\}$$

uniformly for $x \geq 3$, $x \geq y \geq x^\theta$ and $1 \leq k \leq B \log_2 x$. Here the implied constants depend only on B and θ .

Now we are ready to finish the proof of Theorem 1.3. According to Lemma 4.1, the condition (4.3) of Lemma 4.2 is satisfied with the following choices:

$$a_z(n) = z^{\omega(n)}, \quad z h_\ell(z) = \lambda_\ell(z), \quad \theta = 1 - 1/\psi + \varepsilon,$$

$\lambda_\ell(z)$ and $R_N(x, y)$ are defined as in Lemma 4.1, and $c_k(n)$ is the characteristic function on the set of integers n such that $\omega(n) = k$. Thus the assertion (a) is an immediate consequence of this lemma.

5. PROOFS OF THEOREMS 1.5 AND 1.6

The proof of Theorem 1.5 will be proceeded exactly as in [1]. The only difference is the use of Corollary 1.2 in place of [1, Theorem 1].

Since $\sum_{n \geq 1} \tau_k(n)n^{-s} = \zeta(s)^k$ for $\sigma > 1$, we can apply Theorem 1.1 with $z = k$, $w = 0$, $\mathcal{G}(s; k, 0) \equiv 1$ and $A = \delta = 0$. Taking $N = [c'(\log x)^{1/3}(\log_2 x)^{-4/3}]$ with some small constant c' and noticing that $\lambda_\ell(k, 0) = 0$ for all $\ell \geq k$, we obtain the result of Theorem 1.6.

6. PROOF OF THEOREM 1.7

Since the function $\tau_k(n)$ is multiplicative and

$$\tau_k(p^\nu) = \binom{k + \nu - 1}{\nu} = \frac{1}{\nu!} \prod_{j=0}^{\nu-1} (k + j),$$

we can write, for $\sigma > 1$,

$$\begin{aligned} \sum_{n \geq 1} \tau_k(n)^{-1} n^{-s} &= \prod_p \left(1 + \sum_{\nu \geq 1} \binom{k + \nu - 1}{\nu}^{-1} p^{-\nu s} \right) \\ &= \zeta(s)^{\frac{1}{k}} \zeta(2s)^{-\frac{2k^3+2k^2+2k+1}{k^2}} \mathcal{G}_3\left(s; \frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}\right), \end{aligned}$$

where

$$\mathcal{G}_3(s; z, w) := \prod_p \left(\sum_{\nu \geq 0} \binom{k + \nu - 1}{\nu}^{-1} \frac{1}{p^{\nu s}} \right) \left(1 - \frac{1}{p^s} \right)^z \left(1 - \frac{1}{p^{2s}} \right)^w.$$

As before, we expand $\mathcal{G}_3\left(s; \frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}\right)$ as a Dirichlet series:

$$\mathcal{G}_3\left(s; \frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}\right) = \sum_{n \geq 1} b_{3k}(n) n^{-s}$$

where $b_{3k}(n)$ is the multiplicative function for which the values on prime powers are determined by the identity

$$1 + \sum_{\nu \geq 1} b_{3k}(p^\nu) \xi^\nu = \left(\sum_{\nu \geq 0} \binom{k + \nu - 1}{\nu}^{-1} \xi^\nu \right) (1 - \xi)^{\frac{1}{k}} (1 - \xi^2)^{-\frac{2k^3+2k^2+2k+1}{k^2}}.$$

It is easy to see that the right-hand side is an analytic function in $|\xi| < 1$ and $b_{3k}(p) = b_{3k}(p^2) = 0$. Again the Cauchy integral formula yields

$$|b_{3k}(p^\nu)| \ll_k 2^{\nu/2} \quad (\nu \geq 3), \quad \mathcal{G}_3\left(s; \frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}\right) \ll_{k,\sigma} 1 \quad (\sigma > \frac{1}{3}).$$

This shows that the Dirichlet series associated to $\tau_k(n)^{-1}$ is of type

$$\mathcal{P}\left(\frac{1}{k}, -\frac{2k^3+2k^2+2k+1}{k^2}, \frac{1}{k}, 0, \frac{1}{k}, \frac{2k^3+2k^2+2k+1}{k^2}, M(k)\right),$$

where $M(k)$ is a positive constant depending on k . Therefore the required result follows immediately from Theorem 1.1 with any fixed positive integer N .

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