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# TWO GENERALISATIONS OF THE TITCHMARSH DIVISOR PROBLEM

JIE WU

ABSTRACT. In this paper, we considered two generalisations of the classical Titchmarsh divisor problem: friable variant and short intervals.

## 1. INTRODUCTION

As usual, denote by  $\tau(n)$  the number of divisors of the integer  $n$  and by  $\varphi(n)$  the Euler totient function. The letter  $p$  is used to denote primes. The Titchmarsh divisor problem consists to evaluate the asymptotic behaviour of the counting function  $\sum_{p \leq x} \tau(p - a)$  for  $x \rightarrow \infty$ , where  $a \in \mathbb{Z}^*$  is a fixed non-zero integer. This problem was studied initially by Titchmarsh [15], who showed, under the Generalised Riemann Hypothesis (GRH) for the Dirichlet  $L$ -functions,

$$(1.1) \quad \sum_{p \leq x} \tau(p - a) \sim C_a x$$

for  $x \rightarrow \infty$ , where

$$C_a := \frac{\varphi(a)}{a} \prod_{p \nmid a} \left(1 + \frac{1}{p(p-1)}\right).$$

Linnik [8] removed GRH by his dispersion method. The best known result is due to Fouvry [3] and Bombieri-Friedlander-Iwaniec [1], who proved, independently, for any  $A > 1$ ,

$$\sum_{p \leq x} \tau(p - a) = C_a x + D_a \int_2^x \frac{dt}{\log t} + O_A\left(\frac{x}{(\log x)^A}\right),$$

where  $D_a$  is certain constant depending on  $a$ . Diverse generalisations were studied by different authors. Here we only present two such generalisations:

(i) Very recently Xi [18] considered a quadratic analogue of (1.1) and established the following inequalities

$$(1.2) \quad \{0.5 - o(1)\} \frac{6}{\pi^2} x \leq \sum_{p \leq x} \tau(p^2 + 1) \leq 3.496 \cdot \frac{6}{\pi^2} x$$

for  $x \rightarrow \infty$ . The expected asymptotic formula is

$$\sum_{p \leq x} \tau(p^2 + 1) \underset{x \rightarrow \infty}{\sim} \frac{6}{\pi^2} x.$$

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(ii) Pollack [11] studied analogues of (1.1) for elliptic curves and proved the following two results:

- For the CM elliptic curve  $E : y^2 = x^3 - x$ , there is an explicit constant  $C$  such that

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \tau(|E(\mathbb{F}_p)|) \sim Cx,$$

as  $x \rightarrow \infty$ , where  $|E(\mathbb{F}_p)|$  is the order of the group of points of the reduction at  $p$  of  $E$ .

- Let  $E$  be a non-CM elliptic curve over  $\mathbb{Q}$ . Assume the GRH for Dedekind zeta-functions. Then for  $x \rightarrow \infty$ , we have

$$\sum_{p \leq x}^* \tau(|E(\mathbb{F}_p)|) \asymp_E x,$$

where  $\sum^*$  means that the sum is restricted to primes of good reduction and the implied constants depend on  $E$ .

In this paper, we shall consider two other generalisations of the classic Titchmarsh divisor problem: friable variant and short intervals.

Firstly let us fix some notation. As usual, denote by  $P^+(n)$  the largest prime factor of integer  $n$  with the convention  $P^+(1) = 1$ . We say that an integer  $n$  is  $y$ -friable if  $P^+(n) \leq y$ . Let  $\rho(u)$  be the Dickman function, which is defined as the unique continuous solution of the differential-difference equation

$$\begin{cases} \rho(u) = 1 & (0 \leq u \leq 1), \\ u\rho'(u) = -\rho(u-1) & (u > 1). \end{cases}$$

For  $x \geq 1$  and  $y > 1$ , we use systematically the notation

$$(1.3) \quad u := \frac{\log x}{\log y}.$$

For  $a \in \mathbb{Z}^*$ ,  $q \in \mathbb{N}$  such that  $(a, q) = 1$  and  $x \geq 1$ ,  $y > 1$ , define

$$\pi(x, y; d, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{d} \\ P^+((p-a)/d) \leq y}} 1,$$

which counts the number of primes  $p \leq x$  in the *arithmetic progression with friable indices*  $\{a + md\}_{m \text{ } y\text{-friable}}$ . A key point to study friable variant of the Titchmarsh divisor problem (1.1) is that we need a theorem of Bombieri-Vinogradov type for  $\pi(x, y; d, a)$ . It seems that this is a very interesting and rather difficult question, since Pomerance's conjecture (see [12])

$$(1.4) \quad \pi(x, y; 1, a) \sim \pi(x)\rho(u)$$

is still open, where  $u := (\log x)/\log y$  and  $\pi(x)$  is the number of primes  $\leq x$ . Granville [5, Section 5.3] announced that (1.4) follows from the Elliott-Halberstam conjecture without proof. Very recently Wang [16] gave a detailed proof of such result: Let  $a \in \mathbb{Z}^*$  be a fixed

non-zero integer. Assuming that for any  $A > 0$  and  $\varepsilon > 0$  there is a positive number  $\eta = \eta(A, \varepsilon)$  such that

$$(EH[\eta]) \quad \sum_{d \leq x^{1-\eta}} \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} 1 - \frac{\pi(x)}{\varphi(d)} \right| \ll_{a,A,\varepsilon} \frac{x}{(\log x)^A},$$

then we have

$$(1.5) \quad \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} 1 = \pi(x) \rho(u) \{1 + O_a(\varepsilon)\}$$

for  $x \geq y \geq x^\varepsilon$ , where the implied constant depends on  $a$  only.

Our result on the friable variant of the classic Titchmarsh divisor problem (1.1) is as follows, which is comparable with (1.5).

**Theorem 1.** *Let  $a \in \mathbb{Z}^*$  be a fixed non-zero integer. For any  $\varepsilon > 0$ , there exists a positive number  $\eta = \eta(\varepsilon)$  such that if we assume the Elliott-Halberstam conjecture  $EH[\eta]$ , we have*

$$(1.6) \quad \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} \tau(p-a) = C_a \pi(x) (\log y) (\rho * \rho)(u) \{1 + O_a(\varepsilon)\}$$

uniformly for  $x \geq 2$  and  $x \geq y \geq x^\varepsilon$ , where the convolution of  $\rho$  and  $\rho$  is defined by

$$(1.7) \quad (\rho * \rho)(u) := \int_0^u \rho(v) \rho(u-v) dv$$

and the implied constant depends on  $a$  only.

About the generalisation of (1.1) to short intervals, we have the following result.

**Theorem 2.** *Let  $a \in \mathbb{Z}^*$  a fixed non-zero integer and  $\frac{3}{5} < \theta \leq 1$ . Then for  $x \rightarrow \infty$  and  $y = x^\theta$ , we have*

$$(1.8) \quad C_a^\dagger(\theta) y \left\{ 1 + O_{a,\theta} \left( \frac{\log_2 x}{\log x} \right) \right\} \leq \sum_{x < p \leq x+y} \tau(p-a) \leq C_a^\sharp(\theta) y \left\{ 1 + O_{a,\theta} \left( \frac{\log_2 x}{\log x} \right) \right\},$$

where  $\log_k$  is the  $k$ -fold iterated logarithm,

$$\begin{aligned} C_a^\dagger(\theta) &:= (2\theta - 1)C_a, \\ C_a^\sharp(\theta) &:= (2\theta - 1 - 4\log(2\theta - 1))C_a, \end{aligned}$$

and the implied  $O$ -constants depend on  $a$  and  $\theta$  at most.

Since  $C_a^\dagger(1) = C_a^\sharp(1) = C_a$ , the inequalities (1.8) with  $\theta = 1$  become the classic asymptotic formula (1.1). The key tool for the proof of this theorem is a Bombieri-Vinogradov theorem in short intervals (see Lemma 5.1 below).

## 2. SOME PRELIMINARY LEMMAS

Firstly we prove a preliminary lemma, which will be needed later.

**Lemma 2.1.** *Let  $a \in \mathbb{Z}^*$  be a fixed non-zero integer. Then for any  $\varepsilon \in (0, 1)$ , we have*

$$(2.1) \quad \sum_{\substack{d \leq x, (a,d)=1 \\ P^+(d) \leq y}} \frac{1}{\varphi(d)} = C_a \pi(x) (\log y) \int_0^u \rho(v) dv \left\{ 1 + O_{a,\varepsilon} \left( \frac{\log(u+1)}{\log y} \right) \right\}$$

uniformly for

$$(\mathcal{H}_\varepsilon) \quad x \geq 3 \quad \text{and} \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \leq y \leq x,$$

where  $u := (\log x)/\log y$  and the implied constants depend on  $a$  and  $\varepsilon$ .

*Proof.* Define

$$f(d) := \begin{cases} d/\varphi(d) & \text{if } (a, d) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We shall apply Lemma 2.2 below to prove

$$(2.2) \quad \sum_{\substack{d \leq x \\ P^+(d) \leq y}} f(d) = C_a \rho(u) x \left\{ 1 + O_{a,\varepsilon} \left( \frac{\log(u+1)}{\log y} \right) \right\}$$

uniformly for  $(x, y) \in \mathcal{H}_\varepsilon$ . To this end, it is sufficient to verify that the function  $f$  verifies the conditions of Lemma 2.2.

With the help of the prime number theorem, it is easy to see that

$$\begin{aligned} \sum_{p \leq z} f(p) \log p &= \sum_{p \leq z, p \nmid a} \frac{p \log p}{p-1} = \sum_{p \leq z} \log p + O_a(\log z) \\ &= z + O_a(z e^{-c(\log z)^{3/5}(\log_2 z)^{-1/5}}) \end{aligned}$$

and

$$\sum_p \sum_{\nu \geq 2} \frac{f(p^\nu)}{p^\nu} \leq \sum_{p, \nu \geq 2} \frac{1}{p^\nu(1-p^{-1})} \leq \frac{2\pi^2}{3}.$$

This shows that  $f$  satisfies the conditions (2.3) and (2.4) of Lemma 2.2. Thus the asymptotic formula (2.2) is a particular case of this general lemma.

Now the required result (2.1) follows from (2.2) by a simple partial summation.  $\square$

Next we cite four lemmas, which are useful. The first one is a particular case of [14, Théorème 2.1] with  $\kappa = 1$  and  $R(z) = e^{c(\log z)^{3/5-\varepsilon}}$ , which has been used in the proof of Lemma 2.1 above.

**Lemma 2.2.** *Let  $A > 0$ ,  $C > 0$ ,  $\eta \in (0, \frac{1}{2})$  and  $\varepsilon > 0$ . Suppose that arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{R}^+$  verifies the following conditions*

$$(2.3) \quad \left| \sum_{p \leq z} f(p) \log p - z \right| \leq C z e^{-(\log z)^{3/5-\varepsilon}},$$

$$(2.4) \quad \sum_p \sum_{\nu \geq 2} \frac{f(p^\nu)}{p^{(1-\eta)\nu}} \leq A.$$

Then we have

$$(2.5) \quad \sum_{\substack{n \leq x \\ P^+(n) \leq y}} f(n) = C(f)x\rho(u) \left\{ 1 + O_\varepsilon \left( \frac{\log(u+1)}{\log y} \right) \right\}$$

uniformly for  $(x, y) \in (\mathcal{H}_\varepsilon)$ , where

$$C(f) := \prod_p (1 - 1/p) \sum_{\nu \geq 0} f(p^\nu)/p^\nu.$$

The second lemma is an elegant Brun-Titchmarsh inequality, due to Montgomery-Vaughan [10].

**Lemma 2.3.** *We have*

$$(2.6) \quad \sum_{\substack{x < p \leq x+y \\ p \equiv a \pmod{d}}} 1 < \frac{2y}{\varphi(d) \log(y/d)}$$

uniformly for  $1 < d < y \leq x$  and  $(a, d) = 1$ .

The third lemma is essentially due to Lachand and Tenenbaum [7]. \*

**Lemma 2.4.** *Let  $\mu(n)$  be the Möbius function and let  $P^-(n)$  be the largest prime factor with the convention  $P^-(1) = \infty$ . For any  $\varepsilon > 0$ , we have*

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{\mu(n)}{n} = \rho(u) + O_\varepsilon(e^{-(\log y)^{3/5-\varepsilon}})$$

uniformly in

$$(\mathcal{G}_\varepsilon) \quad x \geq 3 \quad \text{and} \quad \exp\{(\log x)^{2/5+\varepsilon}\} \leq y \leq x,$$

where  $u := \log x / \log y$  and the implied constant depends on  $\varepsilon$  only.

The last lemma is due to Iwaniec [6, Lemma 3] (see also [4, Lemme 4.1]).

**Lemma 2.5.** *Let  $D \geq 2$ . There are two sequences  $\{\lambda_d^\pm\}_{d \geq 1}$ , vanishing for  $d > D$  or  $\mu(d) = 0$ , verifying  $|\lambda_d^\pm| \leq 1$ , such that*

$$(2.7) \quad \sum_{d|n} \lambda_d^- \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda_d^+ \quad (n \geq 1)$$

and

$$(2.8) \quad \sum_{d|P(z)} \lambda_d^+ \frac{w(d)}{d} \leq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left( 1 - \frac{w(p)}{p} \right) \left\{ F(s) + O \left( \frac{e^{\sqrt{L-s}}}{\sqrt[3]{\log D}} \right) \right\}$$

$$(2.9) \quad \sum_{d|P(z)} \lambda_d^- \frac{w(d)}{d} \geq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left( 1 - \frac{w(p)}{p} \right) \left\{ f(s) + O \left( \frac{e^{\sqrt{L-s}}}{\sqrt[3]{\log D}} \right) \right\}$$

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\*In the original version of [7], there is a supplementary error term  $O_\varepsilon(\exp\{-(\log y)^{3/5-\varepsilon}\})$ . Recently de la Bretèche & Fiorilli [2] have succeeded to remove this superfluous error term.

for any  $z \in [2, D]$ ,  $s = (\log D)/\log z$ , the set  $\mathcal{P}$  of primes and the multiplicative function  $w$  satisfying

$$(2.10) \quad 0 < w(p) < p \quad (p \in \mathcal{P}),$$

$$(2.11) \quad \prod_{\substack{u < p \leq v \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{L}{\log u}\right) \quad (2 \leq u \leq v),$$

where  $P(z) := \prod_{p \leq z, p \in \mathcal{P}} p$ , the implied  $O$ -constants are absolute and  $F, f$  are defined by the continuous solutions to the system

$$\begin{cases} sF(s) = 2e^\gamma & (1 \leq s \leq 2), \\ sf(s) = 0 & (0 < s \leq 2), \\ (sF(s))' = f(s-1) & (s > 2), \\ (sf(s))' = F(s-1) & (s > 2). \end{cases}$$

Here  $\gamma$  is the Euler constant.

### 3. A VARIANT OF BOMBIERI-VINOGRADOV THEOREM

The asymptotic behaviours of  $\pi(x, y; d, a)$  should be an interesting new subject in the prime number theory. An initial study on this counting function can be found in a recent work of Liu, Wu & Xi [9]. The aim of this section is to establish (3.1) below by following their argument. We will see that it will play a key role in the proof of Theorem 1.

**Proposition 3.1.** *Let  $a \in \mathbb{Z}^*$  be a fixed non-zero integer,  $\eta \in (0, 1)$ ,  $\varepsilon > 0$  and  $A > 0$ . Assuming the Elliott-Halberstam conjecture  $\text{EH}[\eta]$ , the following estimate*

$$(3.1) \quad \sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} \left| \pi(x, y; d, a) - \frac{\pi(x)}{\varphi(d)} \rho\left(\frac{\log(x/d)}{\log y}\right) \right| \ll_{a, A, \varepsilon} \frac{x}{(\log x)^A} + \pi(x)(\log y)\eta u \int_0^{u/2} \rho(v) dv$$

holds uniformly for  $\eta \in (0, 1)$  and  $(x, y) \in \mathcal{G}_\varepsilon$ , where the implied constant depends on  $a, A$  and  $\varepsilon$  only.

*Proof.* Let  $\mathfrak{P} := \prod_{y < p \leq (x-a)/d} p$ . The Möbius inversion formula allows us to write

$$(3.2) \quad \begin{aligned} \pi(x, y; d, a) &= \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} \sum_{\ell | (\mathfrak{P}, (p-a)/d)} \mu(\ell) = \sum_{\substack{\ell \leq (x-a)/d \\ P^-(\ell) > y}} \mu(\ell) \pi(x; d\ell, a) \\ &= \pi_1(x, y; d, a) + \pi_2(x, y; d, a), \end{aligned}$$

where

$$\begin{aligned}\pi_1(x, y; d, a) &:= \sum_{\substack{\ell \leq x^{1-\eta}/d \\ P^-(\ell) > y}} \mu(\ell) \pi(x; d\ell, a), \\ \pi_2(x, y; d, a) &:= \sum_{\substack{x^{1-\eta}/d < \ell \leq (x-a)/d \\ P^-(\ell) > y}} \mu(\ell) \pi(x; d\ell, a).\end{aligned}$$

Firstly we treat  $\pi_1(x, y; d, a)$ . We note, via writing  $q = d\ell$ ,

$$(3.3) \quad \sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} \left| \pi_1(x, y; d, a) - \sum_{\substack{\ell \leq x^{1-\eta}/d \\ P^-(\ell) > y}} \frac{\mu(\ell)}{\varphi(d\ell)} \pi(x) \right| \leq \sum_{\substack{q \leq x^{1-\eta} \\ (a, q) = 1}} \tau(q) \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right|.$$

Trivially, for all  $q \leq x^{1-\eta}$ , we have

$$\pi(x; q, a) + \frac{\pi(x)}{\varphi(q)} \ll \frac{x}{q}.$$

Hence

$$\sum_{\substack{q \leq x^{1-\eta} \\ (a, q) = 1}} \tau(q)^2 \left| \pi(x; q, a) - \frac{\pi(x)}{\varphi(q)} \right| \ll x \sum_{q \leq x^{1-\eta}} \frac{\tau(q)^2}{q} \ll x(\log x)^4.$$

By (3.3) and the Cauchy-Schwarz inequality, this bound and the Elliott–Halberstam conjecture EH[ $\eta$ ] with  $2A + 4$  in place of  $A$  allow us to derive that

$$(3.4) \quad \sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} \left| \pi_1(x, y; d, a) - \sum_{\substack{\ell \leq x^{1-\eta}/d \\ P^-(\ell) > y}} \frac{\mu(\ell)}{\varphi(d\ell)} \pi(x) \right| \ll_{a, A} \frac{x}{(\log x)^A}$$

for all  $x \geq 1$  and  $y > 1$ .

Noticing that  $(d, \ell) = 1$  and using Lemma 2.4, we can write

$$\begin{aligned}\frac{\pi(x)}{\varphi(d)} \sum_{\substack{\ell \leq x^{1-\eta}/d \\ P^-(\ell) > y}} \frac{\mu(\ell)}{\varphi(\ell)} &= \frac{\pi(x)}{\varphi(d)} \sum_{\substack{\ell \leq x^{1-\eta}/d \\ P^-(\ell) > y}} \frac{\mu(\ell)}{\ell} \left\{ 1 + O\left(\frac{1}{y}\right) \right\} \\ &= \frac{\pi(x)}{\varphi(d)} \rho\left(\frac{\log(x^{1-\eta}/d)}{\log y}\right) \left\{ 1 + O_\varepsilon\left(\frac{\log(u+1)}{\log y}\right) \right\}.\end{aligned}$$

for all  $(x, y) \in \mathcal{G}_\varepsilon$  and  $d \leq \sqrt{x}$ . On the other hand, for  $d \leq \sqrt{x}$  we have ([13, Corollaries III.5.8.3–8.4])

$$\begin{aligned}\rho\left(\frac{\log(x/d)}{\log y}\right) - \rho\left(\frac{\log(x^{1-\eta}/d)}{\log y}\right) &\ll \eta u \left| \rho'\left(\frac{\log(x/d)}{\log y}\right) \right| \\ &\ll \rho\left(\frac{\log(x/d)}{\log y}\right) \eta u \log(u+1).\end{aligned}$$



Inserting it into the proceeding relation, it follows that

$$(3.5) \quad \frac{\pi(x)}{\varphi(d)} \sum_{\substack{\ell \leq x^{1-\eta}/d \\ P^-(\ell) > y}} \frac{\mu(\ell)}{\varphi(\ell)} = \frac{\pi(x)}{\varphi(d)} \rho\left(\frac{\log(x/d)}{\log y}\right) \{1 + O(\eta u \log(u+1))\}$$

for all  $(x, y) \in \mathcal{G}_\varepsilon$  and  $d \leq \sqrt{x}$ .

With the help of (2.1), a simple partial integration gives us

$$(3.6) \quad \begin{aligned} & \sum_{\substack{d \leq \sqrt{x} \\ (a,d)=1 \\ P^+(d) \leq y}} \frac{1}{\varphi(d)} \rho\left(\frac{\log(x/d)}{\log y}\right) \\ &= \pi(x) \int_{1-}^{\sqrt{x-a}} \rho\left(\frac{\log(x/t)}{\log y}\right) d\left(\sum_{\substack{d \leq t, (a,d)=1 \\ P^+(d) \leq y}} \frac{1}{\varphi(d)}\right) \\ &= C_a \pi(x) (\log y) \int_0^{u/2} \rho(v) \rho(u-v) dv \left\{1 + O_{a,\varepsilon}\left(\frac{\log(u+1)}{\log y}\right)\right\} \\ &= \frac{C_a}{2} \pi(x) (\log y) (\rho * \rho)(u) \left\{1 + O_{a,\varepsilon}\left(\frac{\log(u+1)}{\log y}\right)\right\} \end{aligned}$$

for  $(x, y) \in \mathcal{H}_\varepsilon$ , we have used the trivial relation

$$\int_0^{u/2} \rho(v) \rho(u-v) dv = \frac{1}{2} \int_0^u \rho(v) \rho(u-v) dv = \frac{1}{2} (\rho * \rho)(u).$$

Combining (3.5) with (3.4) and using (3.6) to bound the contribution of the error term in (3.5), we can find

$$(3.7) \quad \sum_{\substack{d \leq \sqrt{x} \\ (a,d)=1 \\ P^+(d) \leq y}} \left| \pi_1(x, y; d, a) - \frac{\pi(x)}{\varphi(d)} \rho\left(\frac{\log(x/d)}{\log y}\right) \right| \ll_{a,A,\varepsilon} \frac{x}{(\log x)^A} + \pi(x) (\log y) (\rho * \rho)(u) \eta u \log(u+1)$$

for all  $(x, y) \in \mathcal{G}_\varepsilon$  and  $\eta \in (0, 1)$ , provided we assume EH $[\eta]$ .

We now turn to  $\pi_2(x, y; d, a)$ . For  $p-a = d\ell m$ , we have  $(a, m) = 1$  and  $m \leq x^\eta$  subject to the restrictions in  $\pi_2(x, y; d, a)$ . We have trivially

$$|\pi_2(x, y; d, a)| \leq \sum_{\substack{x^{1-\eta}/d < \ell \leq (x-a)/d \\ P^-(\ell) > y}} \pi(x; d\ell, a) \leq \sum_{\substack{m \leq x^\eta \\ (a,m)=1}} \sum_{\substack{p \leq x, p \equiv a \pmod{dm} \\ P^-((p-a)/dm) > y}} 1.$$

We are now in a position to apply sifting arguments subject to the target sequence

$$\mathcal{A}(x; dm, a) := \{(p-a)/dm : p \leq x \text{ and } p \equiv a \pmod{dm}\}.$$

Trivially we have, with the notation  $\mathfrak{P}_2(y) := \prod_{2 < p \leq y} p$ ,

$$|\pi_2(x, y; d, a)| \leq \sum_{\substack{m \leq x^\eta \\ (a,m)=1}} \sum_{\substack{n \in \mathcal{A}(x; dm, a) \\ (n, \mathfrak{P}_2(y))=1}} 1.$$

Let  $\{\lambda_q^+\}_{q \geq 1}$  be an upper bound sieve of level  $Q$  as in Lemma 2.5, so that

$$\begin{aligned} |\pi_2(x, y; d, a)| &\leq \sum_{\substack{m \leq x^\eta \\ (a, m) = 1}} \sum_{n \in \mathcal{A}(x; dm, a)} \sum_{q | (n, \mathfrak{P}_2(y))} \lambda_q^+ \\ &= \sum_{\substack{q \leq Q \\ q | \mathfrak{P}_2(y)}} \lambda_q^+ \sum_{\substack{m \leq x^\eta \\ (a, m) = 1}} \pi(x; dmq; a). \end{aligned}$$

We may take  $Q = x^{1/2-2\eta}$  such that  $\sqrt{x}Qx^\eta = x^{1-\eta}$ . As before, we may approximate  $\pi(x; dmq; a)$  on average over  $d, m, q$  and apply the Elliott-Halberstam conjecture  $\text{EH}[\eta]$ . Similar to (3.3), we can prove

$$\sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} \sum_{\substack{q \leq Q \\ q | \mathfrak{P}_2(y)}} |\lambda_q^+| \sum_{\substack{m \leq x^\eta \\ (a, m) = 1}} \left| \pi(x; dmq; a) - \frac{\pi(x)}{\varphi(dm)} \right| \ll_{a, A} \frac{x}{(\log x)^A}$$

for all  $x \geq 2$  and  $y > 1$ . It now follows that

$$\sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} |\pi_2(x, y; d, a)| \leq \pi(x) \sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} \sum_{\substack{m \leq x^\eta \\ (a, m) = 1}} \frac{1}{\varphi(dm)} \sum_{\substack{q \leq Q \\ q | \mathfrak{P}_2(y)}} \frac{\lambda_q^+}{\varphi(q)} + O_{a, A} \left( \frac{x}{(\log x)^A} \right).$$

From Lemma 2.5, Mertens' formula and the inequality  $\varphi(dm) \geq \varphi(d)\varphi(m)$ , we derive

$$\begin{aligned} \sum_{\substack{d \leq \sqrt{x} \\ (a, d) = 1 \\ P^+(d) \leq y}} |\pi_2(x, y; d, a)| &\ll_{a, A} \pi(x) \sum_{\substack{d \leq \sqrt{x} \\ P^+(d) \leq y}} \sum_{m \leq x^\eta} \frac{1}{\varphi(dm)} \prod_{2 < p < y} \frac{p-2}{p-1} + \frac{x}{(\log x)^A} \\ (3.8) \quad &\ll_{a, A} \frac{\pi(x)}{\log y} \sum_{\substack{d \leq \sqrt{x} \\ P^+(d) \leq y}} \frac{1}{\varphi(d)} \sum_{m \leq x^\eta} \frac{1}{\varphi(m)} + \frac{x}{(\log x)^A} \\ &\ll_{a, A, \varepsilon} \pi(x)(\log y)\eta u \int_0^{u/2} \rho(v) dv + \frac{x}{(\log x)^A} \end{aligned}$$

for all  $(x, y) \in \mathcal{H}_\varepsilon$  and  $\eta \in (0, 1)$ . Now the required result follows from (3.7)–(3.8) and the trivial inequality

$$(3.9) \quad (\rho * \rho)(u) \leq 2\rho\left(\frac{u}{2}\right) \int_0^{u/2} \rho(v) dv \ll \frac{1}{\log(u+1)} \int_0^{u/2} \rho(v) dv.$$

This completes the proof of Proposition 3.1.  $\square$

#### 4. PROOF OF THEOREM 1

In view of the symmetry of divisors of integer  $n$ , we have

$$(4.1) \quad \tau(n) = 2 \sum_{\substack{d | n \\ d < \sqrt{n}}} 1 + \delta_\square(n)$$

with

$$\delta_{\square}(n) := \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus we can write

$$(4.2) \quad \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} \tau(p-a) = 2 \sum_{\substack{d \leq (x-a)^{1/2} \\ (a,d)=1, P^+(d) \leq y}} \sum_{\substack{p \leq x \\ p \equiv a \pmod{d} \\ P^+((p-a)/d) \leq y}} 1 + \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} \delta_{\square}(p-b) \\ = 2(\mathcal{M} + \mathcal{R}_1) + \mathcal{R}_2,$$

where

$$\begin{aligned} \mathcal{M} &:= \sum_{\substack{d \leq \sqrt{x-a} \\ (a,d)=1, P^+(d) \leq y}} \frac{\pi(x)}{\varphi(d)} \rho\left(\frac{\log(x/d)}{\log y}\right), \\ \mathcal{R}_1 &:= \sum_{\substack{d \leq \sqrt{x-a} \\ (a,d)=1, P^+(d) \leq y}} \left( \pi(x, y; d, a) - \frac{\pi(x)}{\varphi(d)} \rho\left(\frac{\log(x/d)}{\log y}\right) \right), \\ \mathcal{R}_2 &:= \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} \delta_{\square}(p-b). \end{aligned}$$

By noticing that  $\delta_{\square}(p-b) = 1 \Rightarrow p-b = n^2$ , we have trivially

$$(4.3) \quad \mathcal{R}_2 \leq 2\sqrt{x}$$

uniformly for  $x \geq 1$  and  $y > 1$ .

According to Proposition 3.1, we have

$$(4.4) \quad \mathcal{R}_1 \ll_{a,A,\varepsilon} \frac{x}{(\log x)^A} + x \rho\left(\frac{u}{2}\right) \eta u$$

uniformly for  $(x, y) \in \mathcal{G}_{\varepsilon}$ .

Finally according to (3.6), we have

$$(4.5) \quad \mathcal{M} = \frac{C_a}{2} \pi(x) (\log y) (\rho * \rho)(u) \left\{ 1 + O_{a,\varepsilon} \left( \frac{\log(u+1)}{\log y} \right) \right\}$$

for  $(x, y) \in \mathcal{H}_{\varepsilon}$ .

Now inserting (4.4), (4.3) and (4.5) into (4.2) and using (3.9), we find that

$$(4.6) \quad \sum_{\substack{p \leq x \\ P^+(p-a) \leq y}} \tau(p-a) = C_a \pi(x) (\log y) (\rho * \rho)(u) \left\{ 1 + O_{a,\varepsilon} \left( \frac{\log(u+1)}{\log y} \right) \right\} \\ + O_{a,A,\varepsilon} \left( \frac{x}{(\log x)^A} + \pi(x) (\log y) \eta u \int_0^{u/2} \rho(v) dv \right)$$

uniformly for  $(x, y) \in \mathcal{G}_{\varepsilon}$  and  $\eta \in (0, 1)$ . Clearly (4.6) implies the required result (1.6)  $\square$

## 5. PROOF OF THEOREM 2

**5.1. Lower bound.** Firstly we prove the lower bound in (1.6). Our principal tool is a mean value theorem of Bombieri-Vinogradov type in short intervals, due to Wu [17, Théorème 1.2].

**Lemma 5.1.** *Let  $g(\ell)$  be an arithmetic function satisfying the condition*

$$(5.1) \quad \sum_{\ell \leq x} |g(\ell)|^2 / \ell \ll (\log x)^\lambda \quad (x \geq 2)$$

for some constant  $\lambda > 0$ . Define

$$H(z, h, d, a, \ell) := \sum_{\substack{\ell p \leq z+h \\ \ell p \equiv a \pmod{d}}} 1 - \sum_{\substack{\ell p \leq z \\ \ell p \equiv a \pmod{d}}} 1 - \frac{1}{\varphi(d)} \int_{z/\ell}^{(z+h)/\ell} \frac{dt}{\log t}.$$

Then for any  $\frac{3}{5} < \theta \leq 1$  and  $A > 0$ , there exists a constant  $B = B(\theta, A) > 0$  such that the inequality

$$(5.2) \quad \sum_{d \leq D} \max_{(a,d)=1} \max_{h \leq y} \max_{x/2 < z \leq x} \left| \sum_{\ell \leq L, (\ell,d)=1} g(\ell) H(z, h, d, a, \ell) \right| \ll_{\theta, \varepsilon} \frac{y}{(\log x)^A}$$

holds for  $x \geq 3$ ,  $D := x^{\theta-1/2}/(\log x)^B$  and  $L = x^{(5\theta-3)/2}$ .

In view of (4.1), we have trivially

$$\tau(n) = \sum_{d|n} 1 \geq 2 \sum_{\substack{d|n \\ d < n^{\theta-1/2}}} 1.$$

Thus we can write

$$(5.3) \quad \begin{aligned} \sum_{x < p \leq x+y} \tau(p-a) &\geq 2 \sum_{x < p \leq x+y} \sum_{\substack{d|(p-a) \\ d < (p-a)^{\theta-1/2}}} 1 \\ &\geq 2 \sum_{\substack{d \leq x^{\theta-1/2}/(\log x)^B \\ (a,d)=1}} \sum_{\substack{x < p \leq x+y \\ p \equiv a \pmod{d}}} 1 \\ &= 2(\mathcal{M} + \mathcal{R}), \end{aligned}$$

where

$$(5.4) \quad \mathcal{M} := \sum_{\substack{d \leq x^{\theta-1/2}/(\log x)^B \\ (a,d)=1}} \frac{1}{\varphi(d)} \int_x^{x+y} \frac{dt}{\log t},$$

$$(5.5) \quad \mathcal{R} := \sum_{\substack{d \leq x^{\theta-1/2}/(\log x)^B \\ (a,d)=1}} \left( \sum_{\substack{x < p \leq x+y \\ p \equiv a \pmod{d}}} 1 - \frac{1}{\varphi(d)} \int_x^{x+y} \frac{dt}{\log t} \right).$$

In Lemma 5.1, taking

$$g(\ell) = \begin{cases} 1 & \text{if } \ell = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we can obtain

$$(5.6) \quad |\mathcal{R}| \leq \sum_{\substack{d \leq x^{\theta-1/2}/(\log x)^B \\ (a,d)=1}} \left| \sum_{\substack{x < p \leq x+y \\ p \equiv a \pmod{d}}} 1 - \frac{1}{\varphi(d)} \int_x^{x+y} \frac{dt}{\log t} \right| \ll \frac{y}{\log x}.$$

By using (2.1) with  $y = x$ , it follows that

$$(5.7) \quad \begin{aligned} \mathcal{M} &= \left\{ \left( \theta - \frac{1}{2} \right) C_a \log x + O(\log_2 x) \right\} \int_x^{x+y} \frac{dt}{\log t} \\ &= \left( \theta - \frac{1}{2} \right) C_a y \left\{ 1 + O\left( \frac{\log_2 x}{\log x} \right) \right\}. \end{aligned}$$

Inserting (5.7) and (5.6) into (5.3), we obtain the lower bound in (1.6).

**5.2. Upper bound.** Now we prove the bound in (1.6). By (4.1), we write

$$(5.8) \quad \begin{aligned} \sum_{x < p \leq x+y} \tau(p-a) &\leq 2 \sum_{x < p \leq x+y} \sum_{\substack{d|(p-a) \\ d < \sqrt{p-a}}} 1 + 2\sqrt{x} \\ &\leq 2(\mathcal{M} + \mathcal{R} + \mathcal{S} + \sqrt{x}), \end{aligned}$$

where  $\mathcal{M}$  and  $\mathcal{R}$  are defined as in (5.4)-(5.5) and

$$\mathcal{S} := \sum_{x^{\theta-1/2}/(\log x)^B < d \leq \sqrt{3x}} \sum_{\substack{x < p \leq x+y \\ p \equiv a \pmod{d}}} 1.$$

In the preceding subsection, we have proved that

$$(5.9) \quad \mathcal{M} + \mathcal{R} = \left( \theta - \frac{1}{2} \right) C_a y \left\{ 1 + O\left( \frac{\log_2 x}{\log x} \right) \right\}.$$

On the other hand, by the Brun-Titchmarsh inequality (2.6), it follows that

$$\begin{aligned} \mathcal{S} &< \sum_{x^{\theta-1/2}/(\log x)^B < d \leq \sqrt{3x}} \frac{2y}{\varphi(d) \log(y/d)} \\ &= \frac{2y}{\log y} \sum_{x^{\theta-1/2}/(\log x)^B < d \leq \sqrt{3x}} \frac{1}{\varphi(d) (1 - (\log d)/\log y)}. \end{aligned}$$

In view of (2.1) with  $y = x$ , a simple partial integration allows us to deduce that

$$(5.10) \quad \begin{aligned} \mathcal{S} &= \frac{2y}{\log y} \int_{x^{\theta-1/2}/(\log x)^B}^{\sqrt{3x}} \frac{1}{1 - (\log t)/\log y} d \sum_{d \leq t} \frac{1}{\varphi(d)} \\ &= \left\{ 1 + O\left( \frac{\log_2 x}{\log x} \right) \right\} 2C_a y \int_{1-1/(2\theta)}^{1/(2\theta)} \frac{1}{1-v} dv \\ &\leq \left\{ 1 + O\left( \frac{\log_2 x}{\log x} \right) \right\} 2 \log(2\theta - 1)^{-1} C_a y. \end{aligned}$$

New the upper bound in (1.6) follows from (5.9), (5.10) and (5.8). This completes the proof of Theorem 2.

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