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ON VALUES TAKEN BY THE LARGEST PRIME FACTOR OF SHIFTED PRIMES (II)

BIN CHEN & JIE WU

ABSTRACT. Denote by \mathbb{P} the set of all primes and by $P(n)$ the largest prime factor of integer $n \geq 1$ with the convention $P(1) = 1$. Let $\eta_0 \approx 2.1426$ be the unique positive zero of the equation $\eta - 1 - 4\eta \log(\eta - 1) = 0$ in $(1, \infty)$. Very recently Wu proved that for $\eta \in (\frac{32}{17}, \eta_0)$ there is a constant $c(\eta) > 1$ such that for each fixed non-zero integer $a \in \mathbb{Z}^*$ the set

$$(0.1) \quad \mathcal{P}_{a,c,\eta} := \{p \in \mathbb{P} : p = P(q - a) \text{ for some prime } q \text{ with } p^\eta < q \leq c(\eta)p^\eta\}$$

has relative density 1 in \mathbb{P} . In this short note, we shall further extend the domain of η at the cost of obtaining a lower bound in place of an asymptotic formula, by showing that for each $\eta \in [\eta_0, 1 + \sqrt[4]{e})$ the set $\mathcal{P}_{a,c,\eta}$ has relative positive density in \mathbb{P} .

1. INTRODUCTION

Denote by \mathbb{P} the set of all prime numbers and by $P(n)$ the largest prime factor of the positive integer $n \geq 1$ with the convention $P(1) = 1$. Banks & Shparlinski [1] proposed to estimate the number of primes p that occur as the largest prime factor of a shifted prime $q - a$ when $q \in \mathbb{P}$ lies in a certain interval determined by p . This question has applications in theoretical computer science and has been considered by Vishnoi [8].

For $a \in \mathbb{Z}^*$, $c > 1$ and $\eta > 0$, we put

$$\mathcal{P}_{a,c,\eta} := \{r \in \mathbb{P} : r = P(q - a) \text{ for some prime } q \text{ with } r^\eta < q \leq cr^\eta\}$$

and

$$\pi_{a,c,\eta}(x) := |\{r \leq x : r \in \mathcal{P}_{a,c,\eta}\}|, \quad \pi(x) := |\{r \leq x : r \in \mathbb{P}\}|.$$

Banks & Shparlinski [1, Theorem 1.1] proved that for each $\eta \in (\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$, there exists a constant $c = c(\eta) > 1$ such that the asymptotic formula

$$(1.1) \quad \pi_{a,c,\eta}(x) = \pi(x) + O_{A,a,c,\eta}\left(\frac{x}{(\log x)^A}\right) \quad (x \rightarrow \infty)$$

holds for every fixed non-zero integer $a \in \mathbb{Z}^*$ and any constant $A > 1$. Moreover for $2 \leq \eta < 1 + \frac{3}{4}\sqrt{2}$, this estimate holds for any constant $c > 1$. Very recently, Wu [9] extended Banks-Shparlinski's interval $(\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$ to $(\frac{32}{17}, \eta_0)$, where η_0 is the unique positive zero of the equation $\eta - 1 - 4\eta \log(\eta - 1) = 0$ in $(1, \infty)$.

In this short note, we shall further extend domain of η at the cost of obtaining a positive proportion in place of the density 1.

Our result is as follows.

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Theorem 1. *Let $\eta \in (2, 1 + \sqrt[4]{e})$ and $c > 1$. There is a small positive number $\delta = \delta(\eta, c)$ such that for every fixed non-zero integer $a \in \mathbb{Z}^*$, we have*

$$(1.2) \quad \pi_{a,c,\eta}(x) \geq \frac{(\log 2)(\eta - 1)}{4\eta} \left(1 - 4 \log(\eta - 1) - 8\delta \frac{\log(\eta - 1)}{c - 1} \right) \pi(x),$$

as $x \rightarrow \infty$.

For comparison, we have

$$1 + \frac{3}{4}\sqrt{2} \approx 2.0606, \quad \eta_0 \approx 2.142 \quad \text{and} \quad 1 + \sqrt[4]{e} \approx 2.284.$$

The improvement of Wu [9] comes from the following two simple observations:

- (i) In many arithmetic applications, the linear sieve is more powerful than the sieve of dimension 2;
- (ii) With the help of the Chen-Iwaniec switching principle [2, 3] and his theorem of Bombieri-Vinogradov type (see [9, Proposition 3.2] or (2.7) below), Wu can sieve the sequence of convolution:

$$\{k\ell r + a : k \leq 2cr^{\eta-2}, 2 \nmid (a+k), c_1y/(kr) < \ell \leq c_2y/(kr)\}$$

for each fixed $r \in (\frac{1}{2}x, x] \cap \mathbb{P}$ by the linear sieve, instead of fixing (k, r) and sieving $\{n(knr + a)\}_n$ by the sieve of dimension 2 as in [1].

For every prime $r \in [\frac{1}{2}x, x]$, we write

$$(1.3) \quad \mathcal{Q}_1(r) := \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}}} 1 = \sum_{\substack{y < q \leq cy \\ P(q-a)=r}} 1 - \mathcal{Q}_2(r),$$

where

$$(1.4) \quad \mathcal{Q}_2(r) := \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}, P(q-a) > r}} 1.$$

The observations above allowed Wu to obtain a better upper bound for $\mathcal{Q}_2(r)$ than [1, Formula (9) or Page 143, line 2] of Banks & Shparlinski.

We shall prove Theorem 1 by introducing a new idea into Wu's refinement on Banks-Shparlinski's argument. Our key point is Proposition 3.1 below, which gives a good upper bound for the mean value of the counting function $\mathcal{Q}_2(r)$:

$$\sum_{x/2 < r \leq x} \mathcal{Q}_2(r).$$

For this, we shall sieve the sequence of convolution defined as in (4.1) below by the Rosser-Iwaniec linear sieve. The lower bound in Theorem 1 can be increased with more work, but this does not improve the point when the bound becomes trivial.

2. SOME PRELIMINARY LEMMAS

In this section, we cite three lemmas, which will be useful later.

2.1. The Rosser-Iwaniec linear sieve.

The first lemma is due to Iwaniec [4, 5].

Lemma 2.1. *Let $D \geq 2$ and let $\mu(n)$ be the Möbius function. Then there are two sequences $\{\lambda_d^\pm\}_{d \geq 1}$, vanishing for $d > D$ or $\mu(d) = 0$, verifying $|\lambda_d^\pm| \leq 1$, such that*

$$(2.1) \quad \sum_{d|n} \lambda_d^- \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda_d^+ \quad (n \geq 1)$$

and

$$(2.2) \quad \sum_{d|P_{\mathcal{P}}(z)} \lambda_d^+ \frac{w(d)}{d} \leq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ F(s) + O\left(\frac{e^{\sqrt{L-s}}}{\sqrt[3]{\log D}}\right) \right\}$$

$$(2.3) \quad \sum_{d|P_{\mathcal{P}}(z)} \lambda_d^- \frac{w(d)}{d} \geq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ f(s) + O\left(\frac{e^{\sqrt{L-s}}}{\sqrt[3]{\log D}}\right) \right\}$$

for any $z \in [2, D]$, $s = (\log D)/\log z$, set of prime numbers \mathcal{P} and multiplicative function w satisfying

$$(2.4) \quad 0 < w(p) < p \quad (p \in \mathcal{P}),$$

$$(2.5) \quad \prod_{u < p \leq v, p \in \mathcal{P}} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{L}{\log u}\right) \quad (2 \leq u \leq v),$$

where $P_{\mathcal{P}}(z) := \prod_{p \leq z, p \in \mathcal{P}} p$ and the implied O -constants are absolute. Here F, f are defined by the continuous solutions to the system

$$\begin{cases} (sF(s))' = f(s-1) & (s > 2) \\ (sf(s))' = F(s-1) & (s > 2) \end{cases}$$

with the initial condition

$$\begin{cases} sF(s) = 2e^\gamma & (1 \leq s \leq 2) \\ sf(s) = 0 & (0 < s \leq 2) \end{cases}$$

where γ is the Euler constant.

2.2. A mean value theorem of Bombieri-Vinogradov type.

As usual, for $(a, d) = 1$ define

$$\pi(x; d, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} 1.$$

The Bombieri-Vinogradov theorem can be stated as follows: For any $A > 0$, there exists a constant $B = B(A) > 0$ such that

$$(2.6) \quad \sum_{d \leq x^{1/2}(\log x)^{-B}} \max_{z \leq x} \max_{(a,d)=1} \left| \pi(z; d, a) - \frac{\pi(z)}{\varphi(d)} \right| \ll_A \frac{x}{(\log x)^A}$$

for all $x \geq 2$, where $\varphi(n)$ denotes the Euler totient function and the implied constant depends on A only.

The following proposition is a mean value theorem of Bombieri-Vinogradov type and will play a key role in the proof of Proposition 3.1 below.

Proposition 2.2. *Let $\kappa_1(m)$ and $\kappa_2(m)$ be the characteristic functions of the odd integers and of even integers, respectively. Then for any $A > 0$, there is a constant $B = B(A) > 0$ such that the inequalities*

$$(2.7) \quad \sum_{\substack{d \leq x^{1/2}(\log x)^{-B} \\ 2 \nmid d}} \max_{z \leq x} \max_{(a,d)=1} \left| \sum_{\substack{mp \leq z \\ mp \equiv a \pmod{d}}} \kappa_i(m) - \frac{1}{\varphi(d)} \sum_{\substack{mp \leq z \\ (d,mp)=1}} \kappa_i(m) \right| \ll_A \frac{x}{(\log x)^A}$$

$$(2.8) \quad \sum_{\substack{d \leq x^{1/2}(\log x)^{-B} \\ 2 \nmid d}} \max_{z \leq x} \max_{(a,d)=1} \left| \sum_{\substack{mp_1 p_2 \leq z \\ mp_1 p_2 \equiv a \pmod{d}}} \kappa_i(m) - \frac{1}{\varphi(d)} \sum_{\substack{mp_1 p_2 \leq z \\ (d,mp_1 p_2)=1}} \kappa_i(m) \right| \ll_A \frac{x}{(\log x)^A}$$

hold for all $x \geq 3$.

Proof. The first inequality is Proposition 3.2 of [9]. With the help of Motohashi's convolution argument [7], (2.8) follows from (2.7) and the classical Bombieri-Vinogradov theorem (see (2.6) below). \square

2.3. An asymptotic formula.

The next lemma is [9, Lemma 3.3].

Lemma 2.3. *For each positive integer $n \geq 1$, define*

$$(2.9) \quad \psi(n) := \prod_{2 < p \mid n} \frac{p-1}{p-2}.$$

Then for $x \geq 2$, we have

$$\sum_{n \leq x} \psi(n) = \frac{x}{\Xi} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\},$$

where

$$(2.10) \quad \Xi := \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2} \right).$$

3. BANKS-SHPARLINSKI'S ARGUMENT AND SKETCH OF THE PROOF OF THEOREM 1

As in [9], the letters p, q, r and ℓ are always used to denote prime numbers, and d, m , and n always denote positive integers. In what follows, let $a \in \mathbb{Z}^*$ and $\eta \in [\eta_0, 1 + \sqrt[4]{e}]$. Let δ be a sufficiently small positive constant and let $c > 1$ be a parameter to be chosen later. Let $x_0(A, a, c, \eta, \delta)$ be a large constant depending on A, a, c, η, δ at most. For $x \geq x_0(A, a, c, \eta, \delta)$ and $r \in (\frac{1}{2}x, x]$, put $y := r^\eta$.

For every prime $r \in [\frac{1}{2}x, x]$, recall the definition of

$$(3.1) \quad \mathcal{Q}_1(r) := \sum_{\substack{y < q \leq cy \\ P(q-a)=r}} 1 \quad \text{and} \quad \mathcal{Q}_2(r) := \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}, P(q-a) > r}} 1.$$

Since $\eta > 2$, we have $x \leq 2r = 2y^{1/\eta} \leq y^{1/2}(\log y)^{-B}$. Thus the Bombieri-Vinogradov theorem (2.6) allows us to write

$$(3.2) \quad \begin{aligned} \sum_{x/2 < r \leq x} \mathcal{Q}_1(r) &= \sum_{x/2 < r \leq x} \left(\sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}}} 1 - \mathcal{Q}_2(r) \right) \\ &= (c-1) \sum_{x/2 < r \leq x} \frac{\pi(y)}{\varphi(r)} + O\left(\frac{y}{(\log y)^A}\right) - \sum_{x/2 < r \leq x} \mathcal{Q}_2(r). \end{aligned}$$

The following result constitutes the key to prove Theorem 1.

Proposition 3.1. *Under the previous notation, we have*

$$(3.3) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq 4(c-1+2\delta) \log(\eta-1) \sum_{x/2 < r \leq x} \frac{\pi(y)}{\varphi(r)} \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\}$$

for $x \rightarrow \infty$, where the implied constant depends on c, η and δ .

In Section 4, we shall prove this proposition. Now we suppose this proposition and complete the proof of Theorem 1.

From (3.2), (3.3) and the following simple asymptotic formula

$$\sum_{x/2 < r \leq x} \frac{1}{\varphi(r)} = \frac{\log 2}{\log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\},$$

we deduce that

$$(3.4) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_1(r) \geq C_1(c, \eta, \delta) \frac{\pi(y)}{\log x} \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\},$$

where

$$(3.5) \quad C_1(c, \eta, \delta) := (\log 2)(c-1) \left(1 - 4 \log(\eta-1) - \delta \frac{\log(\eta-1)^8}{c-1} \right).$$

On the other hand, the Brun-Titchmarsh inequality (see [6]) give us

$$\begin{aligned} \mathcal{Q}_1(r) &\leq \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}}} 1 \leq \frac{2(c-1)y}{\varphi(r) \log((c-1)y/r)} \\ &\leq \frac{4(c-1)y}{(\eta-1)x \log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\} \end{aligned}$$

for all primes $r \in (\frac{1}{2}x, x]$. This implies that

$$(3.6) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_1(r) \leq \frac{4(c-1)y}{(\eta-1)x \log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\} \sum_{\substack{x/2 < r \leq x \\ \mathcal{Q}_1(r) \neq 0}} 1.$$

Combining (3.4) and (3.6), it follows that

$$\sum_{\substack{x/2 < r \leq x \\ \mathcal{Q}_1(r) \neq 0}} 1 \geq C_2(c, \eta, \delta) \pi(x) \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\}$$

with

$$C_2(c, \eta, \delta) := \frac{(\log 2)(\eta - 1)}{4\eta} \left(1 - 4 \log(\eta - 1) - \delta \frac{\log(\eta - 1)^8}{c - 1} \right).$$

This implies the required inequality (1.2). The proof of Theorem 1 is completed assuming Proposition 3.1.

4. PROOF OF PROPOSITION 3.1

For simplicity of notation, we put

$$c_1 := 1 - \delta \quad \text{and} \quad c_2 := c + \delta.$$

If a prime number q is counted in $\mathcal{Q}_2(r)$, then we can write $q - a = k\ell r$, where ℓ is the largest prime factor of $q - a$. Since $\ell > r \in (\frac{1}{2}x, x]$ and $y < q \leq cy$, we have $k \leq (cy - a)/(\ell r) \leq 2cr^{\eta-2}$. On the other hand, noticing that ℓ, r and $q = k\ell r + a$ are odd, we must have $2 \nmid (a + k)$. By the Chen-Iwaniec switching principle (see [2, 3]), we see that $\sum_{x/2 < r \leq x} \mathcal{Q}_2(r)$ does not exceed the number of primes in the sequence

$$(4.1) \quad \left\{ k\ell r + a : \frac{1}{2}x < r \leq x, k \leq 2cr^{\eta-2}, 2 \nmid (a + k), c_1 y / (kr) < \ell \leq c_2 y / (kr) \right\}.$$

We shall sieve this sequence by the set of primes $\mathcal{P}_2 := \{p \in \mathbb{P} : p > 2\}$. Define

$$P_m(z) := \prod_{p < z, p \nmid m} p \quad \text{with} \quad z := y^{1/4} (\log y)^{-B(3)/2} < r.$$

The inversion formula of Möbius allows us to write that

$$\begin{aligned} \sum_{x/2 < r \leq x} \mathcal{Q}_2(r) &\leq \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ k\ell r + a \text{ is prime}}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1 \\ &\leq \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r + a, P_2(z)) = 1}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1 \\ &= \sum_{x/2 < r \leq x} \sum_{k \leq 2cr^{\eta-2}, 2 \nmid (a+k)} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} \sum_{d | (k\ell r + a, P_2(z))} \mu(d). \end{aligned}$$

Using Lemma 2.1, it follows that

$$\sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq \sum_{x/2 < r \leq x} \sum_{k \leq 2cr^{\eta-2}, 2 \nmid (a+k)} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} \sum_{d | (k\ell r + a, P_2(z))} \lambda_d^+.$$

By inversion of summations, we can write

$$\sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq \sum_{d | P_2(z)} \lambda_d^+ \mathcal{K}(y; d, -a),$$

where

$$\mathcal{K}(y; d, b) := \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ k\ell r \equiv b \pmod{d}}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1.$$

Introducing the notation

$$\mathcal{K}_d(y) := \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r, d)=1}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1$$

and

$$E(t; d, b) := \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ k\ell r \equiv b \pmod{d}}} \sum_{\ell \leq t / (kr)} 1 - \frac{1}{\varphi(d)} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r, d)=1}} \sum_{\ell \leq t / (kr)} 1,$$

we write

$$\mathcal{K}(y; d, -a) = \frac{1}{\varphi(d)} \mathcal{K}_d(y) + E(c_2 y; d, -a) - E(c_1 y; d, -a).$$

Inserting into the preceding formula, it follows that

$$(4.2) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq \mathcal{M} + \mathcal{E},$$

where

$$\mathcal{M} := \sum_{d|P_2(z)} \frac{\lambda_d^+}{\varphi(d)} \mathcal{K}_d(y), \quad \mathcal{E} := \sum_{d|P_2(z)} \lambda_d^+ (E(c_2 y, d, -a) - E(c_1 y, d, -a)).$$

With the help of (2.8) of Proposition 2.2 with $D := z^2 = y^{1/2}(\log y)^{-B(3)}$, we can derive

$$(4.3) \quad |\mathcal{E}| \leq \sum_{d \leq y^{1/2}(\log y)^{-B(3)}} (|E(c_2 y, d, -a)| + |E(c_1 y, d, -a)|) \ll \frac{y}{(\log y)^3}.$$

It remains to evaluate \mathcal{M} . By inversion of summations, we deduce that

$$\begin{aligned} \mathcal{M} &= \sum_{d|P_2(z)} \frac{\lambda_d^+}{\varphi(d)} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r, d)=1}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1 \\ &= \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ c_1 y / (kr) < \ell \leq c_2 y / (kr)}} \sum_{d|P_2 k \ell r(z)} \sum_{\varphi(d)} \frac{\lambda_d^+}{\varphi(d)}. \end{aligned}$$

Using Lemma 2.1 with $w(d) = d/\varphi(d)$, $D = z^2$ and $\mathcal{P} = \{p \in \mathbb{P} : p \nmid 2k\ell r\}$, it follows that

$$\mathcal{M} \leq \left\{ F(2) + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \sum_{\substack{c_1 y / (kr) < \ell \leq c_2 y / (kr)}} \prod_{\substack{p \leq z \\ p \nmid 2k\ell r}} \left(1 - \frac{1}{p-1}\right).$$

On the other hand, the Mertens formula allows us to deduce that

$$\begin{aligned} \prod_{p \leq z, p \nmid 2m} \left(1 - \frac{1}{p-1}\right) &= \prod_{2 < p \leq z} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2 < p|m} \frac{p-1}{p-2} \prod_{2 < p \leq z} \left(1 - \frac{1}{p}\right) \\ &= \frac{2\Xi\psi(m)e^{-\gamma}}{\log z} \left\{ 1 + O\left(\frac{1}{\log z}\right) \right\}, \end{aligned}$$

where Ξ and $\psi(m)$ are defined as in (2.10) and (2.9), respectively. Inserting this into the preceding relation and using the fact that $F(2) = e^\gamma$, we find

$$\mathcal{M} \leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{2\Xi}{\log z} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \sum_{c_1 y/(kr) < \ell \leq c_2 y/(kr)} \psi(k\ell r).$$

Noticing that $\ell > r > \frac{1}{2}x$ and that ℓ and r are primes, we have

$$\psi(k\ell r) \leq \psi(k)\psi(\ell)\psi(r) = \{1 + O(x^{-1})\}\psi(k).$$

Thus

$$\begin{aligned} \mathcal{M} &\leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{2\Xi}{\log z} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \psi(k) \sum_{c_1 y/(kr) < \ell \leq c_2 y/(kr)} 1 \\ &= \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{2\Xi(c_2 - c_1)y}{\log z} \sum_{x/2 < r \leq x} \frac{\mathcal{S}(r)}{r \log(y/r)}, \end{aligned}$$

where

$$\mathcal{S}(r) := \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \frac{\psi(k)}{k(1 - (\log k)/\log(y/r))}.$$

With the help of Lemma 2.3, a simple partial integration leads to

$$\begin{aligned} \mathcal{S}(r) &= \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\} \frac{1}{2} \sum_{k \leq cr^{\eta-2}} \frac{\psi(k)}{k(1 - (\log k)/\log(y/r))} \\ &= \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\} \frac{1}{2\Xi} \int_1^{cr^{\eta-2}} \frac{dt}{t(1 - (\log t)/\log(y/r))} \\ &= \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\} \frac{\log(\eta - 1)}{2\Xi} \log(y/r), \end{aligned}$$

where Ξ is defined as in (2.10). Combining it with the preceding formula, it follows that

$$\begin{aligned} (4.4) \quad \mathcal{M} &\leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \sum_{x/2 < r \leq x} \frac{(c_2 - c_1)y}{r \log z} \log(\eta - 1) \\ &\leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} 4(c - 1 + 2\delta) \log(\eta - 1) \sum_{x/2 < r \leq x} \frac{\pi(y)}{\varphi(r)}. \end{aligned}$$

Inserting (4.4) and (4.3) into (4.2), we obtain the required inequality (3.3).

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